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Bounded and almost periodic solvability of nonautonomous quasilinear hyperbolic systems

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Abstract. The paper concerns boundary value problems for general nonautonomous first-order quasilinear hyperbolic systems in a strip. We construct small global classical solutions, assuming that the right-hand sides are small. In the case that all data of the quasilinear problem are almost periodic, we prove that the bounded solution is also almost periodic. For the nonhomogeneous version of a linearized problem, we provide stable dissipativity conditions ensuring a unique bounded continuous solution for any smooth right-hand sides. In the autonomous case, this solution is two times continuously differentiable. In the nonautonomous case, the continuous solution is differentiable under additional dissipativity conditions, which are essential. A crucial ingredient of our approach is a perturbation theorem for general linear hyperbolic systems. One of the technical complications we overcome is the "loss of smoothness" property of hyperbolic PDEs.

1. Introduction

1.1. Problem setting

We consider a first-order quasilinear hyperbolic system

$$\partial_t V + A(x,t,V)\partial_x V + B(x,t,V)V = f(x,t), \quad x \in (0,1), t \in \mathbb{R}, \tag{1.1}$$

where $V = (V_1, \dots, V_n)$ and $f = (f_1, \dots, f_n)$ are vectors of real-valued functions, and $A = (A_{jk})$ and $B = (B_{jk})$ are $n \times n$-matrices of real-valued functions. The matrix A is supposed to have n real eigenvalues $A_j(x,t,V)$ in a neighborhood of $V = 0$ in $\mathbb{R}^n$ such that

$$A_1(x,t,V) > \dots > A_m(x,t,V) > 0 > A_{m+1}(x,t,V) > \dots > A_n(x,t,V)$$

for some integer $0 \le m \le n$. These assumptions imply that there exists a smooth and nondegenerate $n \times n$-matrix $Q(x,t,V) = \big(Q_{jk}(x,t,V)\big)$ such that

$$Q^{-1}(x,t,V)A(x,t,V)Q(x,t,V) = \mathrm{diag}(A_1(x,t,V), \dots, A_n(x,t,V)).$$

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We supplement the system (1.1) with the boundary conditions

$$\begin{aligned} U_j(0,t) &= (RZ)_j(t) + h_j(t), \quad 1 \le j \le m, \ t \in \mathbb{R}, \\ U_j(1,t) &= (RZ)_j(t) + h_j(t), \quad m < j \le n, \ t \in \mathbb{R}, \end{aligned} \tag{1.2}$$

where R is a (time-dependent) bounded linear operator,

$$Z(t) = \left(U_1(1,t), \ldots, U_m(1,t), U_{m+1}(0,t), \ldots, U_n(0,t)\right),$$

and

$$U(x,t) = Q^{-1}(x,t,V)V(x,t). \tag{1.3}$$

The purpose of the paper is to establish conditions on the coefficients A, B, f, and h and the boundary operator R ensuring that the problem (1.1)–(1.3) has a unique small global classical solution, which is two times continuously differentiable. If the data in (1.1) and (1.2) are almost periodic (respectively, periodic) in t, we prove that the bounded solution is almost periodic (respectively, periodic) in t also.

Let

$$\Pi = \{(x,t) \in \mathbb{R}^2 \, : \, 0 \le x \le 1\}$$

and $BC(\Pi; \mathbb{R}^n)$ be the Banach space of all continuous and bounded maps $u : \Pi \to \mathbb{R}^n$ with the usual sup-norm

$$\|u\|_{BC} = \sup\left\{|u_j(x,t)| \, : \, (x,t) \in \Pi, \, j \le n\right\}.$$

Moreover, $BC^k(\Pi; \mathbb{R}^n)$ denotes the space of k times continuously differentiable and bounded maps $u : \Pi \to \mathbb{R}^n$, with norm

$$\|u\|_{BC^k} = \sum_{0 \le i+j \le k} \left\| \partial_x^i \partial_t^j u \right\|_{BC}.$$

We also use the spaces $BC_t^k(\Pi; \mathbb{R}^n)$ of functions $u \in BC(\Pi; \mathbb{R}^n)$ such that $\partial_t u, \ldots, \partial_t^k u \in BC(\Pi; \mathbb{R}^n)$, with norm

$$\|u\|_{BC_t^k} = \sum_{j=0}^{k} \|\partial_t^j u\|_{BC}.$$

Similarly, $BC^k(\mathbb{R}; \mathbb{R}^n)$ denotes the space of k times continuously differentiable and bounded maps $u : \mathbb{R} \to \mathbb{R}^n$. If $n = 1$, we will simply write $BC^k(\mathbb{R})$ for $BC^k(\mathbb{R}; \mathbb{R})$, and likewise for all the spaces introduced above.

Given two Banach spaces X and Y, the space of all bounded linear operators $A : X \to Y$ is denoted by $\mathcal{L}(X,Y)$, with the operator norm $\|A\|_{\mathcal{L}(X,Y)} = \sup\{\|Au\|_Y : u \in X, \|u\|_X \le 1\}$. We will use also the usual notation $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$.

Let $\|\cdot\|$ denote the norm in $\mathbb{R}^n$ defined by $\|y\| = \max_{j \le n} |y_j|$. We suppose that the data of the problem (1.1)–(1.3) satisfy the following conditions.

(A1) There exists $\delta_0 > 0$ such that

– the entries of the matrices $A(x,t,V)$, $B(x,t,V)$, and $Q(x,t,V)$ have bounded and continuous partial derivatives up to the second order in $(x,t) \in \Pi$ and in $V \in \mathbb{R}^n$ with $\|V\| \le \delta_0$,

– there exists $\Lambda_0 > 0$ such that

$$\begin{aligned}
&\inf\left\{A_j(x,t,V) : (x,t) \in \Pi,\ \|V\| \le \delta_0, 1 \le j \le m\right\} \ge \Lambda_0,\\
&\sup\left\{A_j(x,t,V) : (x,t) \in \Pi,\ \|V\| \le \delta_0, m < j \le n\right\} \le -\Lambda_0,\\
&\inf\left\{|A_j(x,t,V) - A_k(x,t,V)| : (x,t) \in \Pi,\ \|V\| \le \delta_0, 1 \le j \ne k \le n\right\} \ge \Lambda_0,\\
&\inf\{|\det Q(x,t,V)| : (x,t) \in \Pi,\ \|V\| \le \delta_0\} \ge \Lambda_0.
\end{aligned}$$

(A2) $f \in BC_t^2(\Pi;\mathbb{R}^n)$, $\partial_x f \in BC_t^1(\Pi;\mathbb{R}^n)$, and $h \in BC^2(\mathbb{R};\mathbb{R}^n)$.

(A3) R is a bounded linear operator on $BC(\mathbb{R};\mathbb{R}^n)$. The restriction of R to $BC^1(\mathbb{R};\mathbb{R}^n)$ (respectively, to $BC^2(\mathbb{R};\mathbb{R}^n)$) is a bounded linear operator on $BC^1(\mathbb{R};\mathbb{R}^n)$ (respectively, on $BC^2(\mathbb{R};\mathbb{R}^n)$). Moreover, for $v \in BC^1(\mathbb{R};\mathbb{R}^n)$ it holds

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(Rv)_j(t) &= \left(R'v\right)_j(t) + \left(\widetilde{R}v'\right)_j(t),\\
\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{R}v)_j(t) &= \left(\widetilde{R}'v\right)_j(t) + \left(\widehat{R}v'\right)_j(t),
\end{aligned} \tag{1.4}$$

where $v'(t) = \frac{\mathrm{d}}{\mathrm{d}t}v(t)$ and $R', \widetilde{R}, \widetilde{R}', \widehat{R} : BC(\mathbb{R};\mathbb{R}^n) \to BC(\mathbb{R};\mathbb{R}^n)$ are some bounded linear operators.

Notation and further assumptions Set

$$\begin{aligned}
a(x,t) &= \operatorname{diag}\left(A_1(x,t,0),\dots,A_n(x,t,0)\right),\\
b(x,t) &= Q^{-1}(x,t,0)\Big(B(x,t,0)Q(x,t,0) + \partial_t Q(x,t,0) + A(x,t,0)\partial_x Q(x,t,0)\Big),
\end{aligned}$$

and

$$\gamma_j = \inf_{x,t} \frac{b_{jj}(x,t)}{|a_j(x,t)|}, \quad \tilde{\gamma}_j = \inf_{x,t}\left|\frac{b_{jj}(x,t)}{a_j(x,t)}\right|, \quad \beta_j = \sup_{x,t}\sum_{k\ne j}\left|\frac{b_{jk}(x,t)}{a_j(x,t)}\right|.$$

In Sect. 2, we will consider a linearized version of the system (1.1); see (2.8). The characteristics of this linear system, which we need already now, are defined as follows. For given $j \le n$, $x \in [0,1]$, and $t \in \mathbb{R}$, the jth characteristic passing through the point $(x,t) \in \Pi$ is defined as the solution $\xi \in [0,1] \mapsto \omega_j(\xi) = \omega_j(\xi,x,t) \in \mathbb{R}$ of the initial value problem

$$\partial_\xi \omega_j(\xi,x,t) = \frac{1}{a_j(\xi,\omega_j(\xi,x,t))}, \quad \omega_j(x,x,t) = t. \tag{1.5}$$

Due to the assumption **(A1)**, the characteristic curve $\tau = \omega_j(\xi)$ reaches the boundary of Π in two points with distinct ordinates. Let x_j denote the abscissa of that point whose ordinate is smaller. Specifically,

$$x_j = \begin{cases} 0 \text{ if } 1 \le j \le m,\\ 1 \text{ if } m < j \le n.\end{cases}$$

Write

$$c_j^l(\xi, x, t) = \exp \int_x^\xi \left[\frac{b_{jj}}{a_j} - l \frac{\partial_t a_j}{a_j^2} \right] (\eta, \omega_j(\eta))\, \mathrm{d}\eta \tag{1.6}$$

and introduce operators $G_1, G_2, H_1, H_2 \in \mathcal{L}(BC(\mathbb{R}, \mathbb{R}^n))$ by

$$\begin{aligned} (G_1\psi)_j(t) &= c_j^1(x_j, 1-x_j, t)(\widetilde{R}\psi)_j(\omega_j(x_j, 1-x_j, t)),\\ (G_2\psi)_j(t) &= c_j^2(x_j, 1-x_j, t)(\widehat{R}\psi)_j(\omega_j(x_j, 1-x_j, t)),\\ \left(H_l\psi\right)_j(t) &= c_j^l(x_j, 1-x_j, t)\psi_j(\omega_j(x_j, 1-x_j, t)) \text{ if } \inf_{x,t} b_{jj} > 0,\\ \left(H_l\psi\right)_j(t) &= c_j^l(1-x_j, x_j, t)\psi_j(\omega_j(1-x_j, x_j, t)) \text{ if } \sup_{x,t} b_{jj} < 0. \end{aligned} \tag{1.7}$$

In what follows, we will use the simplified notations

$$\begin{aligned} \|R_j\| &= \sup\left\{ \|(Ru)_j\|_{BC(\mathbb{R})} \,:\, \|u\|_{BC(\Pi;\mathbb{R}^n)} = 1 \right\},\\ \|R\| &= \|R\|_{\mathcal{L}(BC(\mathbb{R};\mathbb{R}^n))} = \max_{j\le n} \|R_j\|. \end{aligned}$$

We consider two sets of stable conditions on the data of the original problem.

(B1) For each $j \le n$, it holds

$$\begin{aligned} \|R_j\| + \tfrac{\beta_j}{\gamma_j}\left(1 - e^{-\gamma_j}\right) &< 1 \quad \text{if } \inf_{x,t} b_{jj} > 0,\\ e^{-\gamma_j}\|R_j\| + \tfrac{\beta_j}{\gamma_j}\left(1 - e^{-\gamma_j}\right) &< 1 \quad \text{if } \inf_{x,t} b_{jj} < 0,\\ \|R_j\| + \beta_j &< 1 \quad \text{if } \inf_{x,t} b_{jj} = 0. \end{aligned}$$

(B2) For each $j \le n$, it holds

$$\begin{aligned} \inf_{x,t} b_{jj} > 0, \quad e^{-\gamma_j}\|R_j\| < 1,\\ \left(1 + \|R\| \left[1 - \max_{i\le n}\left\{e^{-\gamma_i}\|R_i\|\right\}\right]^{-1}\right) \frac{\beta_j}{\gamma_j}\left(1 - e^{-\gamma_j}\right) < 1. \end{aligned}$$

Moreover, in the particular case of periodic boundary conditions $(Rz)_j = z_j$ or, the same, in the case

$$u_j(0, t) = u_j(1, t) \text{ for all } j \le n, \tag{1.8}$$

we consider yet another set of conditions.

(B3) For each $j \le n$, it holds

$$\inf_{x,t} |b_{jj}| \ne 0 \quad \text{and} \quad \frac{\beta_j}{\tilde{\gamma}_j}\left(2 - e^{-\tilde{\gamma}_j}\right) < 1.$$

Note that, if $\inf_{x,t} b_{jj} > 0$, then the conditions (**B1**) and (**B2**) differ at least in the restrictions imposed on the boundary operator R. More precisely, since the constants γ_j are positive for all $j \le n$, the condition (**B2**) allows for $\|R_j\| \ge 1$, what is not allowed by (**B1**).

1.2. Main result

A continuous function $w(x,t,v)$ defined on $[0,1]\times\mathbb{R}\times[-\delta_0,\delta_0]^n$ is a *Bohr almost periodic in t uniformly in x and v* (see [6, p. 55]) if for every $\mu>0$ there exists a relatively dense set of μ-almost periods of w, i.e., for every $\mu>0$ there exists a positive number l such that every interval of length l on $\mathbb{R}$ contains a number τ such that

$$|w(x,t+\tau,v)-w(x,t,v)|<\mu \quad \text{for all } (x,t)\in\Pi \text{ and } \|v\|\le\delta_0.$$

Let $AP(\mathbb{R},\mathbb{R}^n)$ be the space of Bohr almost periodic vector functions. Analogously, let $AP(\Pi,\mathbb{R}^n)$ be the space of Bohr almost periodic vector functions in t uniformly in x. By $C_T(\mathbb{R},\mathbb{R}^n)$ and $C_T(\Pi,\mathbb{R}^n)$, we denote the spaces of continuous and T-periodic in t vector functions, defined on $\mathbb{R}$ and Π, respectively.

The main result of the paper is stated in the following two theorems.

Theorem 1. *Let the conditions* (**A1**)–(**A3**) *and at least one of the conditions* (**B1**) *and* (**B2**) *be fulfilled. If the inequalities*

$$\|G_i\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))}<1 \tag{1.9}$$

are satisfied for both $i=1$ and $i=2$, then the following is true:

1. *There exist $\varepsilon>0$ and $\delta>0$ such that, if $\|f\|_{BC_t^2}+\|h\|_{BC^2}\le\varepsilon$, then the problem (1.1)–(1.3) has a unique classical solution $V^*\in BC^2(\Pi,\mathbb{R}^n)$ such that $\|V^*\|_{BC_t^2}+\|\partial_x V^*\|_{BC_t^1}\le\delta$. Furthermore, there exist $\varepsilon>0$ and $\delta>0$ such that, if $\|f\|_{BC_t^2}+\|\partial_x f\|_{BC_t^1}+\|h\|_{BC^2}\le\varepsilon$, then $\|V^*\|_{BC^2}\le\delta$.*

2. *Suppose that A, B, Q, f, and h are Bohr almost periodic in t uniformly in $x\in[0,1]$ and $V\in[-\delta_0,\delta_0]^n$ (resp., A, B, Q, f, and h are T-periodic in t). Moreover, suppose that the restriction of the boundary operator R to $AP(\mathbb{R};\mathbb{R}^n)$ (resp., to $C_T(\mathbb{R},\mathbb{R}^n)$) is a bounded linear operator on $AP(\mathbb{R};\mathbb{R}^n)$ (resp., on $C_T(\mathbb{R},\mathbb{R}^n)$). Then the bounded classical solution V^* to the problem (1.1)–(1.3) belongs to $AP(\Pi,\mathbb{R}^n)$ (resp., to $C_T(\Pi,\mathbb{R}^n)$).*

Theorem 2. *Let $(Rz)_j=z_j$ for each $j\le n$ and the conditions* (**A1**), (**A2**), *and* **(B3)** *be fulfilled. If the inequalities*

$$\|H_i\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))}<1 \tag{1.10}$$

are satisfied for both $i=1$ and $i=2$, then Parts 1 and 2 of Theorem 1 are true for the problem (1.1), (1.8), (1.3).

The paper is organized as follows. In Sect. 2, we formulate statements of independent interest for general linear first-order nonautonomous boundary value problems related to solving the original quasilinear problem. In Sect. 3, we comment on the problem (1.1)–(1.3) and on our main assumptions. In particular, we give an example showing that in the nonautonomous setting the conditions (1.9) and (1.10) are essential for

C^2-regularity of the bounded continuous solutions. Section 4.1 is devoted to bounded continuous solvability of the linear boundary value problems (including the linearized version of the original problem). In Sect. 4.2, we prove C^2-regularity of the bounded continuous solutions. A crucial point in our approach is a perturbation theorem for the general linear problem (2.1), (2.5), (2.7). This result, Theorem 6, is proved in Sect. 4.3. Our main result, Theorems 1 and 2, is proved in Sect. 5.

2. Relevant linear problems

Setting Our approach to the quasilinear problem (1.1)–(1.3) is based on a thorough analysis of a linearized problem. As we will see later, the main reason behind global classical solvability of the quasilinear problem (1.1)–(1.3) lies in the fact that the corresponding nonhomogeneous linear problem has a unique smooth bounded solution for any smooth right-hand side. We therefore first establish stable sufficient conditions ensuring the last property. To this end, consider the following general nonhomogeneous linear system

$$\partial_t v + a^*(x,t)\partial_x v + b^*(x,t)v = g(x,t), \quad x \in (0,1), t \in \mathbb{R}, \tag{2.1}$$

where $g = (g_1, \ldots, g_n)$ is a vector of real-valued functions, $a^* = (a^*_{jk})$ and $b^* = (b^*_{jk})$ are $n \times n$-matrices of real-valued functions. Note that, if $a^*(x,t) = A(x,t,0)$ and $b^*(x,t) = B(x,t,0)$, then (2.1) is a nonhomogeneous version of the linearized system (1.1). This is a reason why we use the same notation for the general linear problem and for the linearized version of the original quasilinear problem.

Suppose that

$$a^*_{jk} \in BC^1(\Pi) \text{ and } b^*_{jk} \in BC(\Pi) \quad \text{for all } j,k \le n \tag{2.2}$$

and the matrix a^* has n real eigenvalues $a_1(x,t), \ldots, a_n(x,t)$ such that $a_1(x,t) > \cdots > a_m(x,t) > 0 > a_{m+1}(x,t) > \cdots > a_n(x,t)$. Let $q(x,t) = \big(q_{jk}(x,t)\big)$ be a nondegenerate $n \times n$-matrix such that $q_{jk} \in BC^1(\Pi)$ and

$$a(x,t) = q^{-1}(x,t)a^*(x,t)q(x,t) = \operatorname{diag}(a_1(x,t), \ldots, a_n(x,t)). \tag{2.3}$$

The existence of such a matrix follows from the assumptions on a^*. Note that, if (2.1) is a linearized version of (1.1), then the matrix q is defined by $q(x,t) = Q(x,t,0)$. Let λ_0 be a positive real such that

$$\begin{aligned}
&\inf\left\{a_j(x,t) \,:\, (x,t) \in \Pi, 1 \le j \le m\right\} > \lambda_0,\\
&\sup\left\{a_j(x,t) \,:\, (x,t) \in \Pi, m < j \le n\right\} < -\lambda_0,\\
&\inf\left\{|a_j(x,t) - a_k(x,t)| \,:\, (x,t) \in \Pi, 1 \le j \ne k \le n\right\} > \lambda_0,\\
&\inf\left\{|\det q(x,t)| \,:\, (x,t) \in \Pi\right\} > \lambda_0.
\end{aligned} \tag{2.4}$$

We subject the system (2.1) to the boundary conditions

$$\begin{aligned}
u_j(0,t) &= (Rz)_j(t) + h_j(t), \quad 1 \le j \le m, \ t \in \mathbb{R},\\
u_j(1,t) &= (Rz)_j(t) + h_j(t), \quad m < j \le n, \ t \in \mathbb{R},
\end{aligned} \tag{2.5}$$

where

$$z(t) = (u_1(1,t), \ldots, u_m(1,t), u_{m+1}(0,t), \ldots, u_n(0,t)) \tag{2.6}$$

and

$$u = q^{-1}(x,t)v. \tag{2.7}$$

The system (2.1) with respect to u reads

$$\partial_t u + a(x,t)\partial_x u + b(x,t)u = g(x,t), \quad x \in (0,1),\ t \in \mathbb{R}, \tag{2.8}$$

where $b(x,t) = q^{-1}\left(b^* q + \partial_t q + a^* \partial_x q\right)$. It is evident that the problems (2.1), (2.5), (2.7) and (2.8), (2.5) are equivalent.

An operator representation

Let

$$c_j(\xi,x,t) = \exp \int_x^\xi \left[\frac{b_{jj}}{a_j}\right](\eta, \omega_j(\eta))\, \mathrm{d}\eta, \quad d_j(\xi,x,t) = \frac{c_j(\xi,x,t)}{a_j(\xi,\omega_j(\xi))}. \tag{2.9}$$

Suppose that g and h are sufficiently smooth and bounded functions. As usual, a function $u \in BC^1(\Pi;\mathbb{R}^n)$ is called a *bounded classical* solution to (2.8), (2.5) if it satisfies (2.8), (2.5) pointwise. Similarly, a function $v \in BC^1(\Pi;\mathbb{R}^n)$ is called a *bounded classical* solution to the problem (2.1), (2.5), (2.7) if it satisfies (2.1), (2.5), (2.7) pointwise. It is straightforward to show that a function $u \in BC^1(\Pi;\mathbb{R}^n)$ is the bounded classical solution to (2.8), (2.5) if and only if u satisfies the system of integral equations

$$\begin{aligned} u_j(x,t) &= c_j(x_j,x,t)(Rz)_j(\omega_j(x_j)) + c_j(x_j,x,t)h_j(\omega_j(x_j)) \\ &- \int_{x_j}^x d_j(\xi,x,t)\left(\sum_{k\neq j} b_{jk}(\xi,\omega_j(\xi))u_k(\xi,\omega_j(\xi)) - g_j(\xi,\omega_j(\xi))\right)\mathrm{d}\xi, \quad j \le n, \end{aligned} \tag{2.10}$$

pointwise. This motivates the following definitions. A function $u \in BC(\Pi;\mathbb{R}^n)$ is called a *bounded continuous* solution to (2.8), (2.5) if it satisfies (2.10) pointwise. A function $v \in BC(\Pi;\mathbb{R}^n))$ is called a *bounded continuous* solution to (2.1), (2.5), (2.7) if the function $u = q^{-1}v$ satisfies (2.10) pointwise.

Let us introduce operators $C, D \in \mathcal{L}(BC(\Pi;\mathbb{R}^n))$ and an operator $F \in \mathcal{L}\left(BC(\Pi;\mathbb{R}^{2n}); BC(\Pi;\mathbb{R}^n)\right)$ by

$$\begin{aligned} &(Cu)_j(x,t) = c_j(x_j,x,t)(Rz)_j(\omega_j(x_j,x,t)), \\ &(Du)_j(x,t) = -\int_{x_j}^x d_j(\xi,x,t)\sum_{k\neq j} b_{jk}(\xi,\omega_j(\xi,x,t))u_k(\xi,\omega_j(\xi,x,t))\mathrm{d}\xi, \\ &(F(g,h))_j(x,t) = \int_{x_j}^x d_j(\xi,x,t)g_j(\xi,\omega_j(\xi,x,t))\mathrm{d}\xi + c_j(x_j,x,t)h_j(\omega_j(x_j,x,t)). \end{aligned} \tag{2.11}$$

Then the system (2.10) can be written in the operator form

$$u = Cu + Du + F(g, h). \tag{2.12}$$

BC solutions

Theorems 3 and 4 give *stable* sufficient conditions for BC solvability of the linear problem (2.1), (2.5), (2.7). If the data of the problem are sufficiently smooth, in the autonomous case these conditions even ensure BC^2-regularity. In the nonautonomous case, we need an additional condition to ensure BC^1-regularity and yet another condition to ensure BC^2-regularity. These additional conditions, which are stated in Theorem 5, turn out to be essential; see Sect. 3.6. This seems to be a new interesting phenomenon for nonautonomous hyperbolic PDEs.

Theorem 3. *Let $R \in \mathcal{L}(BC(\mathbb{R};\mathbb{R}^n))$. Suppose that the conditions (2.2)–(2.3) and one of the conditions* (**B1**) *and* (**B2**) *are fulfilled. Then, for any $g \in BC(\Pi;\mathbb{R}^n)$ and $h \in BC(\mathbb{R};\mathbb{R}^n)$, the problem (2.1), (2.5), (2.7) has a unique bounded continuous solution v. Moreover, the a priori estimate*

$$\|v\|_{BC} \le K(\|g\|_{BC} + \|h\|_{BC}) \tag{2.13}$$

is fulfilled for a constant $K > 0$ not depending on g and h.

Theorem 4. *Let $(Rz)_j = z_j$ for each $j \le n$. Suppose that the conditions (2.2)–(2.3) and* (**B3**) *are fulfilled. Then, for any $g \in BC(\Pi;\mathbb{R}^n)$, the problem (2.1), (1.8), (2.7) has a unique bounded continuous solution v. Moreover, the estimate (2.13) is fulfilled with $h = 0$ and with a positive constant K not depending on g.*

Higher regularity of bounded continuous solutions is the subject of the next theorem.

Theorem 5. *Assume that the assumptions of Theorem 3 (resp., Theorem 4) are fulfilled.*

1. *Let $a^*_{jk}, q_{jk} \in BC^2_t(\Pi)$, $\partial_x a^*_{jk}, \partial_x q_{jk}, b^*_{jk}, g_j \in BC^1_t(\Pi)$, and $h_j \in BC^1(\mathbb{R})$ for all $j, k \le n$. Suppose that the restriction of R to $BC^1(\mathbb{R};\mathbb{R}^n)$ is a bounded linear operator on $BC^1(\mathbb{R};\mathbb{R}^n)$ satisfying (1.4). If the inequality (1.9) for $i = 1$ (resp., the inequality (1.10) for $i = 1$) is true, then the bounded continuous solution v to the problem (2.1), (2.5), (2.7) (resp., to the problem (2.1), (1.8), (2.7)) belongs to $BC^1(\Pi,\mathbb{R}^n)$. Moreover, the a priori estimate*

$$\|v\|_{BC^1} \le K_1\big(\|g\|_{BC^1_t} + \|h\|_{BC^1}\big) \ \big(\text{resp., } \|v\|_{BC^1} \le K_1\|g\|_{BC^1_t}\big) \tag{2.14}$$

is fulfilled for a constant $K_1 > 0$ not depending on g and h.

2. *Let, additionally, $b^*_{jk}, g_j \in BC^2_t(\Pi)$ and $h_j \in BC^2(\mathbb{R})$ for all $j, k \le n$ and the restriction of R to $BC^2(\mathbb{R};\mathbb{R}^n)$ be a bounded linear operator on $BC^2(\mathbb{R};\mathbb{R}^n)$. If the inequality (1.9) for $i = 2$ (resp., the inequality (1.10) for $i = 2$) is true, then $v \in BC^2_t(\Pi,\mathbb{R}^n)$ and $\partial_x v \in BC^1_t(\Pi,\mathbb{R}^n)$. Moreover, the a priori estimate*

$$\begin{gathered}\|v\|_{BC^2_t} + \|\partial_x v\|_{BC^1_t} \le K_2\big(\|g\|_{BC^2_t} + \|h\|_{BC^2}\big)\\ \big(\text{resp., } \|v\|_{BC^2_t} + \|\partial_x v\|_{BC^1_t} \le K_2\|g\|_{BC^2_t}\big)\end{gathered} \tag{2.15}$$

is fulfilled for a constant $K_2 > 0$ not depending on g and h.

A perturbation theorem Along with the system (2.1), we consider its perturbed version

$$\partial_t v + \tilde{a}^*(x,t)\partial_x v + \tilde{b}^*(x,t)v = g(x,t), \tag{2.16}$$

where $\tilde{a}^* = \left(\tilde{a}^*_{jk}\right)$ and $\tilde{b}^* = \left(\tilde{b}^*_{jk}\right)$ are $n \times n$-matrices of real-valued functions with $\tilde{a}^*_{jk} \in BC^1(\Pi)$ and $\tilde{b}^*_{jk} \in BC(\Pi)$. The matrix $\tilde{a}^*$ is supposed to have n real eigenvalues $\tilde{a}_1(x,t), \dots, \tilde{a}_n(x,t)$ such that $\tilde{a}_1(x,t) > \cdots > \tilde{a}_m(x,t) > 0 > \tilde{a}_{m+1}(x,t) > \cdots > \tilde{a}_n(x,t)$. Let $\tilde{q}(x,t) = \left(\tilde{q}_{jk}(x,t)\right)$ be a nondegenerate $n \times n$-matrix such that $\tilde{q}_{jk} \in BC^1(\Pi)$ and

$$\tilde{a}(x,t) = \tilde{q}^{-1}(x,t)\tilde{a}^*(x,t)\tilde{q}(x,t) = \operatorname{diag}(\tilde{a}_1(x,t), \dots, \tilde{a}_n(x,t)). \tag{2.17}$$

Due to the assumptions on a^*, we can fix $\varepsilon_0 > 0$ such that, whenever $\|\tilde{a}^* - a^*\|_{BC} \le \varepsilon_0$ and $\|\tilde{q} - q\|_{BC} \le \varepsilon_0$, the condition (2.4) is fulfilled with $\tilde{a}_j$ and $\tilde{q}$ in place of a_j and q, respectively.

Theorem 6. 1. *If the assumptions of Part 1 of Theorem 5 are fulfilled, then there exists $\varepsilon_1 \le \varepsilon_0$ such that, for all $\tilde{a}^*_{jk}$, $\tilde{b}^*_{jk}$, and $\tilde{q}_{jk}$ satisfying the conditions*

$$\begin{aligned}
&\tilde{a}^*_{jk}, \tilde{q}_{jk} \in BC^2_t(\Pi),\ \partial_x\tilde{a}^*_{jk}, \partial_x\tilde{q}_{jk} \in BC^1_t(\Pi),\ \tilde{b}^*_{jk} \in BC^1_t(\Pi),\\
&\|\tilde{a}^* - a^*\|_{BC^2_t} + \|\partial_x\tilde{a}^* - \partial_x a^*\|_{BC^1_t} \le \varepsilon_1,\ \|\tilde{b}^* - b^*\|_{BC^1_t} \le \varepsilon_1,\\
&\|\tilde{q} - q\|_{BC^2_t} + \|\partial_x\tilde{q} - \partial_x q\|_{BC^1_t} \le \varepsilon_1,
\end{aligned} \tag{2.18}$$

the system (2.16), (2.5), (2.7) with $\tilde{q}$ in place of q has a unique bounded classical solution $\tilde{v} \in BC^1(\Pi; \mathbb{R}^n)$. Moreover, $\tilde{v}$ satisfies the a priori estimate (2.14) with $\tilde{v}$ in place of v for a constant K_1 not depending on $\tilde{a}^*$, $\tilde{b}^*$, $\tilde{q}$, g, and h.

2. *If the assumptions of Part 2 of Theorem 5 are fulfilled, then there exists $\varepsilon_1 \le \varepsilon_0$ such that, for all $\tilde{a}^*_{jk}$ and $\tilde{b}^*_{jk}$ satisfying the conditions (2.18) and the stronger conditions*

$$\tilde{b}^*_{jk} \in BC^2_t(\Pi) \quad \text{and} \quad \|\tilde{b}^* - b^*\|_{BC^2_t} \le \varepsilon_1,$$

the system (2.16), (2.5), (2.7) with $\tilde{q}$ in place of q has a unique bounded classical solution $\tilde{v} \in BC^2_t(\Pi; \mathbb{R}^n)$ with $\partial_x\tilde{v} \in BC^1_t(\Pi; \mathbb{R}^n)$. Moreover, $\tilde{v}$ satisfies the a priori estimate (2.15) with $\tilde{v}$ in place of v for a constant K_2 not depending on $\tilde{a}^*$, $\tilde{b}^*$, $\tilde{q}$, g, and h.

3. Comments on the problem and the assumptions

3.1. About the quasilinear system (1.1)

It is well known that quasilinear hyperbolic PDEs are accompanied by various singularities as shocks and blow-ups. Since the characteristic curves are controlled by unknown functions, the characteristics of the same family intersect each other in general and, therefore, bring different values of the corresponding unknown functions into the intersection points (appearance of shocks). The nonlinearities in $B(x, t, u)$ often lead to infinite increase of solutions in a finite time (appearance of blow-ups). When speaking about global classical solutions, one needs to provide conditions preventing the singular behavior.

Certain classes of nonlinearities ensuring a nonsingular behavior for autonomous quasilinear systems are described in [13,22]. Some monotonicity and sign preserving conditions on the coefficients of the nonautonomous quasilinear hyperbolic systems are imposed in [1,25]. In the present paper, we study nonautonomous quasilinear hyperbolic systems with lower-order terms and use a different approach focusing on small solutions only. We do not need any of the above constraints. Instead, we assume a regular behavior of the linearized system and smallness of the right-hand sides. Small periodic classical solutions for autonomous quasilinear hyperbolic systems without lower-order terms and with small nondiagonal elements of the matrix $A = A(V)$ for $V \approx 0$ were investigated in [27]. In our setting, the nondiagonal entries of the matrix $A = A(x, t, V)$ are not necessarily small and the lower-order coefficients $B(x, t, V)$ are not necessarily zero. Our dissipativity conditions depend both on the boundary operator and on the coefficients of the hyperbolic system.

In Sect. 3.6, we show that the additional dissipativity conditions (1.9) and (1.10) are in general necessary for C^2-regularity of continuous solutions, which is a notable fact in the context of nonautonomous hyperbolic problems.

3.2. About the boundary conditions (1.2)

The boundary operator R covers different kinds of reflections and delays, in particular,

$$(RZ)_j(t) = \sum_{k=1}^{n} \left[r_{jk}(t) Z_k(t - \theta_{jk}(t)) + \int_0^{\vartheta_{jk}(t)} p_{jk}(t, \tau) Z_k(t - \tau)\, d\tau \right], \quad j \le n,$$

where r_{jk}, p_{jk}, θ_{jk}, and ϑ_{jk} are known BC^1-functions. The boundary operators R' and $\widetilde{R}$ introduced in (1.4) are in this case computed by the formulas

$$\begin{aligned}
(R'Z)_j(t) &= \sum_{k=1}^{n} \Big[r'_{jk}(t) Z_k(t - \theta_{jk}(t)) + p_{jk}(t, \vartheta_{jk}(t)) Z_k(t - \vartheta_{jk}(t)) \vartheta'_{jk}(t) \\
&\quad + \int_0^{\vartheta_{jk}(t)} \partial_t p_{jk}(t, \tau) Z_k(t - \tau)\, d\tau \Big], \\
(\widetilde{R}Z)_j(t) &= \sum_{k=1}^{n} \Big[r_{jk}(t) Z_k(t - \theta_{jk}(t))(1 - \theta'_{jk}(t)) + \int_0^{\vartheta_{jk}(t)} p_{jk}(t, \tau) Z_k(t - \tau)\, d\tau \Big].
\end{aligned}$$

Boundary conditions of the reflection type appear, in particular, in semiconductor laser modeling [21,29] and in boundary feedback control problems [2,7,10,26], while integral boundary conditions (with delays [23]) appear, for instance, in hyperbolic age-structured models [5,14].

3.3. Nilpotency of the operator C

Theorem 3 can be extended if the operator C is nilpotent. This is the case of the so-called smoothing boundary conditions, see e.g. [16]. The smoothing property allowed us in [20] to solve the problem (1.1)–(1.3) where the boundary conditions (1.2) are specified to be of the reflection type, without the requirement of the smallness of $\|D\|_{\mathcal{L}(BC(\Pi,\mathbb{R}^n))}$. In [20], we used the assumption that the evolution family generated by a linearized problem has exponential dichotomy on $\mathbb{R}$ and proved that the dichotomy survives under small perturbations in the coefficients of the hyperbolic system. For more general boundary conditions (in particular, for (1.2)) when the operator C is not nilpotent, the issue of the robustness of exponential dichotomy for hyperbolic PDEs remains a challenging open problem.

3.4. Space-periodic problems and exponential dichotomy

In the case of space-periodic boundary conditions (1.8), our assumption (**B3**) implies, according to [15], that the evolution family generated by the linearized problem has the exponential dichotomy on $\mathbb{R}$. For more general boundary conditions (2.5), one can expect the same dichotomy behavior of the evolution family whenever one of the two assumptions (**B1**) and (**B2**) is fulfilled, but this still remains a subject of future work.

3.5. Time-periodic problems and small divisors

Analysis of time-periodic solutions to hyperbolic PDEs usually encounters a complication known as the problem of small divisors. However, this obstacle does not appear in our setting due to the nonresonance assumptions (**B1**), (**B2**), and (**B3**). Similar conditions were discussed in [17,18].

The completely resonant case (closely related to small divisors) is qualitatively different. This case is discussed in a series of papers by Temple and Young (see, e.g., [30,31]) about time-periodic solutions to one-dimensional linear Euler equations with the periodic boundary conditions (1.8). In this case, one cannot expect any stable nonresonant conditions of our type. More precisely, in the setting of [30,31] it holds $b_{jj} = 0$ for all j, and hence, our condition (**B3**) is not satisfied. Therefore, the operator $I - C$ is not bijective, while the bijectivity property is a crucial point in our Theorems 3 and 4.

3.6. Conditions (1.9) and (1.10) are essential for higher regularity of bounded continuous solutions, in general

In the autonomous case, when the operator R and the coefficients in the hyperbolic system (2.8) do not depend on t, we have $R' = 0$, $\widetilde{R} = R$, and $c_j^l \equiv c_j$ for all $j \le n$ and $l = 1, 2$. Then the bounds (1.9) and (1.10) straightforwardly follow from the assumptions of any of Theorems 3 and 4. The higher regularity of solutions follows automatically. This means that we have to explicitly impose the conditions (1.9) and (1.10) only in the nonautonomous case.

We now show that in the nonautonomous case, if the estimate (1.9) is not fulfilled for $i = 1$, then Part 1 of Theorem 5 is not true in general. Similarly, if (1.9) is not fulfilled for $i = 2$, then one can show that Part 2 of Theorem 5 is not necessarily true.

Consider the following example, satisfying all the assumptions in Part 1 of Theorem 5 except (1.9) for $i = 1$:

$$\begin{aligned}&\partial_t u_1 + \frac{2}{4\pi - 1}\partial_x u_1 = 1, \quad \partial_t u_2 - (2 + \sin t)\partial_x u_2 = 0,\\&u_j(x, t + 2\pi) = u_j(x, t), \quad j = 1, 2,\\&u_1(0, t) = r_1(t)u_2(0, t), \quad u_2(1, t) = r_2 u_1(1, t),\end{aligned} \tag{3.1}$$

where a 2π-periodic and positive C^1-function $r_1(t)$ and a constant r_2 are such that

$$0 < \sup_{t\in\mathbb{R}} r_1(t) < 1, \quad 0 < r_2 < 1. \tag{3.2}$$

In this case, all assumptions of Theorem 3 are true since $\|R_1\| = \sup_{t\in\mathbb{R}} r_1(t) < 1$, $\|R_2\| = r_2 < 1$, and $b_{jk} = 0$ for all $j, k \leq 2$. The system (3.1) has a unique bounded continuous solution $u = (u_1, u_2) \in BC(\Pi, \mathbb{R}^2)$. Since all the coefficients of the problem are 2π-periodic in t, it is a simple matter to show that the solution u is 2π-periodic in t as well (sf. Sect. 5.3).

We have

$$\begin{aligned}&\omega_1(\xi, x, t) = \frac{4\pi - 1}{2}(\xi - x) + t,\\&\omega_2(\xi, x, t) = p^{-1}(p(t) + \xi - x) \text{ with } p(t) = -2t + \cos t,\end{aligned}$$

and

$$\begin{aligned}\partial_t\omega_2(\xi, x, t) &= \exp\int_\xi^x \left(\frac{a'}{a^2}\right)(\omega_2(\eta, x, t))\,\mathrm{d}\eta\\&= \exp\int_\xi^x \frac{\mathrm{d}}{\mathrm{d}\eta}\ln a(\omega_2(\eta, x, t))\,\mathrm{d}\eta = \frac{a(t)}{a(\omega_2(\xi, x, t))},\end{aligned} \tag{3.3}$$

where $a(t) = -2 - \sin t$. Then the system (2.10) reads

$$u_1(x, t) = r_1\left(t - \frac{4\pi - 1}{2}x\right)u_2\left(0, t - \frac{4\pi - 1}{2}x\right) + \frac{4\pi - 1}{2}x, \tag{3.4}$$

$$u_2(x, t) = r_2 u_1(1, p^{-1}(p(t) + 1 - x)). \tag{3.5}$$

Inserting (3.4) into (3.5), we get

$$\begin{aligned}u_2(0, t) = r_2 r_1&\left(p^{-1}(p(t) + 1) - \frac{4\pi - 1}{2}\right)\\&\times u_2\left(0, p^{-1}(p(t) + 1) - \frac{4\pi - 1}{2}\right) + r_2\frac{4\pi - 1}{2}.\end{aligned} \tag{3.6}$$

Using the 2π-periodicity of u_2 in t, we now find the values of t at which u_2 has the same argument in both sides of (3.6). This is the case if, for instance, $t - 2\pi = p^{-1}(p(t)+1) - (4\pi - 1)/2$. This equality is true if and only if $p(t) + 1 = p(t - \frac{1}{2})$ or, the same,

$$\cos t - \cos\left(t - \frac{1}{2}\right) = -2\sin\left(t - \frac{1}{4}\right)\sin\left(\frac{1}{4}\right) = 0.$$

The last equation has the solutions $1/4 + \pi k$, $k \in \mathbb{Z}$. Set $t_0 = 1/4$. Then, due to (3.2), the equation (3.6) yields

$$u_2(0, t_0) = r_2\,\frac{4\pi - 1}{2(1 - r_2 r_1(t_0))} \neq 0. \tag{3.7}$$

Notice the obvious identity $p^{-1}(p(t) + 1) = \omega_2(1, 0, t)$. If the derivative $\partial_t u_2(0, t_0)$ exists, then it is given by the formula

$$\partial_t u_2(0, t_0) = r_2 r_1(t_0)\partial_t\omega_2(1, 0, t_0)\partial_t u_2(0, t_0) + r_2 r_1'(t_0)\partial_t\omega_2(1, 0, t_0)u_2(0, t_0). \tag{3.8}$$

By (3.3), we have

$$\partial_t\omega_2(1, 0, t_0) = \frac{a(t_0)}{a(\omega_2(1, 0, t_0))} = \frac{-2 - \sin(1/4)}{-2 - \sin(-1/4)} > 1.$$

We can choose a constant r_2 and a smooth 2π-periodic function $r_1(t)$ such that, additionally to the condition (3.2), it holds

$$r_2 r_1(t_0)\partial_t\omega_2(1, 0, t_0) = 1 \text{ and } r_1'(t_0) \neq 0, \tag{3.9}$$

contradicting (3.7)–(3.8). This means that the continuous solution to (3.6) and, hence, also to (3.4)–(3.5) is not differentiable at $t = t_0$.

The violation of the condition (1.9) can be seen also directly. It suffices to note that, by (3.9), for $\psi \in BC(\mathbb{R}, \mathbb{R}^2)$ such that $\|\psi\|_{BC} = 1$ and $\psi_1(\omega_2(1, 0, t_0)) = 1$, we have

$$\begin{aligned}
&\|G_1\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^2))} \geq |(G_1\psi)_2(t_0)| = c_2^1(1, 0, t_0)|(\widetilde{R}\psi)_2(\omega_2(1, 0, t_0))|\\
&\quad = c_2^1(1, 0, t_0)r_2|\psi_1(\omega_2(1, 0, t_0))| = r_2 \exp\int_0^1 \left(-\frac{a'(\omega_2(\eta, 0, t_0))}{a(\omega_2(\eta, 0, t_0))^2}\right) d\eta\\
&\quad = r_2 \exp\int_1^0 \frac{d}{d\eta}\ln a(\omega_2(\eta, 0, t_0))\, d\eta = r_2\frac{a(t_0)}{a(\omega_2(1, 0, t_0))} = r_2\partial_t\omega_2(1, 0, t_0) > 1.
\end{aligned}$$

3.7. Quasilinear hyperbolic systems in applications

Quasilinear systems of the type (1.1) cover, in particular, the one-dimensional version of the classical Saint-Venant system for shallow water [28] and its generalizations (see, e.g. [3]), the water flow in open channels [12], and one-dimensional Euler equations [11,30,31,33]. They are also used to describe rate-type materials in viscoelasticity [8,9,24] and the interactions between heterogeneous cancer cell [4].

The behavior of unsteady flows in open horizontal and frictionless channel is described in [32] by the Saint-Venant system of the type

$$\partial_t A - \partial_x (AV) = 0, \quad \partial_t V - \partial_x S = 0, \qquad t \geq 0, x \in (0, L), \tag{3.10}$$

where L is the length of the channel, $A = A(t, x)$ is the area of the cross section occupied by the water at position x and time t, and $V = V(t, x)$ is the average velocity over the cross section. Furthermore, $S = \frac{1}{2} V^2 + gh(A)$, where $h(A)$ is the depth of the water. This system is subjected to flux boundary conditions. Note that in the smooth setting the system (3.10) is of our type (2.1). As described in [32], in a neighborhood of an equilibrium point the system (3.10) can be written in Riemann invariants in the diagonal form (2.8). The flux boundary conditions are then transformed into boundary conditions of the type (2.5).

The nonautonomous first-order quasilinear system

$$\partial_t u - \partial_x v = 0, \quad \partial_t v - \phi(t, v) \partial_x u = \psi(t, v),$$

is used to model the stress–strain law for metals [8,9,24]. Here v and u denote the stress and the Lagrangian velocity, while the functions ϕ and ψ measure, respectively, the noninstantaneous and the instantaneous response of the metal to an increment of the stress.

4. Linear system

4.1. Existence of bounded continuous solutions

4.1.1. Proof of Theorem 3

We first give the proof under the assumption (**B1**). We have to prove that $I - C - D$ is a bijective operator from $BC(\Pi; \mathbb{R}^n)$ to itself. It suffices to establish the estimate

$$\|C\|_{\mathcal{L}(BC(\Pi,\mathbb{R}^n))} + \|D\|_{\mathcal{L}(BC(\Pi,\mathbb{R}^n))} < 1. \tag{4.1}$$

Using (2.9), we have

$$\begin{aligned} c_j(x_j, x, t) &= \exp \left\{ \int_x^0 \left[\frac{b_{jj}}{a_j} \right] (\eta, \omega_j(\eta)) \, d\eta \right\} \leq e^{-\gamma_j x}, \qquad j \leq m, \\ c_j(x_j, x, t) &= \exp \left\{ \int_x^1 \left[\frac{b_{jj}}{a_j} \right] (\eta, \omega_j(\eta)) \, d\eta \right\} \leq e^{-\gamma_j (1-x)}, \ j > m. \end{aligned} \tag{4.2}$$

If $\inf\limits_{x,t} b_{jj} \geq 0$, then $\gamma_j \geq 0$, and if $\inf\limits_{x,t} b_{jj} < 0$, then $\gamma_j < 0$. Combining this with (4.2), we obtain that

$$\sup_{x,t} c_j(x_j, x, t) = 1 \quad \text{if} \quad \inf_{x,t} b_{jj} \geq 0,$$

$$\sup_{x,t} c_j(x_j, x, t) \le e^{-\gamma_j} \text{ if } \inf_{x,t} b_{jj} < 0.$$

By the definition (2.11) of the operator D, for all $u \in BC(\Pi; \mathbb{R}^n)$ with $\|u\|_{BC} = 1$ and all $(x,t) \in \Pi$ we have

$$\begin{aligned}
|(Du)_j(x,t)| &\le \beta_j \int_0^x \exp\left\{\int_x^\xi \left[\frac{b_{jj}}{a_j}\right](\eta, \omega_j(\eta))\,\mathrm{d}\eta\right\} \mathrm{d}\xi \le \beta_j \int_0^x e^{-\gamma_j(x-\xi)}\,\mathrm{d}\xi \\
&= \frac{\beta_j}{\gamma_j}\left(1 - e^{-\gamma_j x}\right) \le \frac{\beta_j}{\gamma_j}\left(1 - e^{-\gamma_j}\right) \quad \text{ if } j \le m,\ \gamma_j \ne 0, \\
|(Du)_j(x,t)| &\le \beta_j \int_x^1 \exp\left\{\int_x^\xi \left[\frac{b_{jj}}{a_j}\right](\eta, \omega_j(\eta))\,\mathrm{d}\eta\right\} \mathrm{d}\xi \\
&\le \frac{\beta_j}{\gamma_j}\left(1 - e^{-\gamma_j}\right) \quad \text{ if } j > m,\ \gamma_j \ne 0, \\
|(Du)_j(x,t)| &\le \beta_j \quad \text{ if } j \le n,\ \gamma_j = 0.
\end{aligned}$$

Note that $\gamma_j = 0$ iff $\inf_{x,t} b_{jj} = 0$. Using (**B1**), we immediately get the inequality (4.1). This implies that, for given $g \in BC(\Pi; \mathbb{R}^n)$ and $h \in BC(\mathbb{R}; \mathbb{R}^n)$, the equation (2.12) has the unique solution $u = (I - C - D)^{-1} F(g,h)$. Hence, $v = qu$ is the continuous solution to (2.1), (2.5), (2.7). The estimate (2.13) now easily follows. The proof of Theorem 3 under the assumption (**B1**) is complete.

Now we assume that the assumption (**B2**) is fulfilled. Our aim is to prove that the operator $I - C \in \mathcal{L}(BC(\Pi; \mathbb{R}^n))$ is bijective and that the following estimate is fulfilled:

$$\|(I - C)^{-1} D\|_{\mathcal{L}(BC(\Pi, \mathbb{R}^n))} < 1. \tag{4.3}$$

To prove the bijectivity of $I - C \in \mathcal{L}(BC(\Pi; \mathbb{R}^n))$, we consider the equation

$$u_j(x,t) = c_j(x_j, x, t)(Rz)_j(\omega_j(x_j, x, t)) + r_j(x,t), \quad j \le n, \tag{4.4}$$

with respect to $u \in BC(\Pi; \mathbb{R}^n)$, where r belongs to $BC(\Pi; \mathbb{R}^n)$ and z is given by (2.6). The operator $I - C \in \mathcal{L}(BC(\Pi; \mathbb{R}^n))$ is bijective iff the equation (4.4) is uniquely solvable for any $r \in BC(\Pi; \mathbb{R}^n)$. Putting $x = 0$ for $m < j \le n$ and $x = 1$ for $1 \le j \le m$, the system (4.4) reads as follows with respect to $z(t)$:

$$z_j(t) = c_j(x_j, 1 - x_j, t)(Rz)_j(\omega_j(x_j, 1 - x_j, t)) + r_j(1 - x_j, t), \quad j \le n. \tag{4.5}$$

Introduce operator $G_0 \in \mathcal{L}(BC(\mathbb{R}, \mathbb{R}^n))$ by

$$(G_0\psi)_j(t) = c_j(x_j, 1 - x_j, t)(R\psi)_j(\omega_j(x_j, 1 - x_j, t)), \quad j \le n. \tag{4.6}$$

Note that $(G_0 z)_j(t) = (Cu)_j(1 - x_j, t)$, $j \le n$. This implies that for all $u \in BC(\Pi; \mathbb{R}^n)$ with $\|u\|_{BC} = 1$ it holds

$$\begin{aligned}
\|(G_0 u)_j\|_{BC} &\le \|R_j\| \exp \int_0^1 \left[-\frac{b_{jj}}{a_j}\right](\eta, \omega_j(\eta, 1, t))\,\mathrm{d}\eta \le \|R_j\| e^{-\gamma_j},\ j \le m, \\
\|(G_0 u)_j\|_{BC} &\le \|R_j\| \exp \int_0^1 \left[\frac{b_{jj}}{a_j}\right](\eta, \omega_j(\eta, 0, t))\,\mathrm{d}\eta \le \|R_j\| e^{-\gamma_j}, \quad j > m.
\end{aligned} \tag{4.7}$$

Hence, the operator $I - G_0$ is bijective due to the assumption (**B2**). It should be noted that $\|C\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))} = 1$, while $\|G_0\|_{\mathcal{L}(BC(\mathbb{R};\mathbb{R}^n))} < 1$. We, therefore, can rewrite the system (4.5) in the form

$$z = (I - G_0)^{-1}\tilde{r}, \tag{4.8}$$

where $\tilde{r}(t) = (r_1(1,t), \dots, r_m(1,t), r_{m+1}(0,t), \dots, r_n(0,t))$. Substituting (4.8) into (4.4), we obtain

$$\begin{aligned} u_j(x,t) &= \left[(I - C)^{-1} r\right]_j (x,t) \\ &= c_j(x_j, x, t)\left[R(I - G_0)^{-1}\tilde{r}\right]_j (\omega_j(x_j, x, t)) + r_j(x,t), \ \ j \le n. \end{aligned} \tag{4.9}$$

The assumption that $\inf\limits_{x,t} b_{jj} > 0$ entails that $c_j(x_j, x, t) \le 1$ for all $(x,t) \in \Pi$ and all $j \le n$. Therefore,

$$\|(I - C)^{-1}\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))} \le \|R\|\|(I - G_0)^{-1}\|_{\mathcal{L}(BC(\mathbb{R};\mathbb{R}^n))} + 1.$$

Combining this with the second inequality in (**B2**), we arrive at the estimate (4.3) and, therefore, conclude that the formula $u = \left[I - (I - C)^{-1} D\right]^{-1} (I - C)^{-1} F(g,h)$ gives the solution to (2.12). Hence, $v = qu$ is the continuous solution to (2.1), (2.5), (2.7), and this solution satisfies the estimate (2.13). This completes the proof of Theorem 3 under the assumption (**B2**).

4.1.2. Proof of Theorem 4

We follow the proof of Theorem 3 under the assumption (**B2**), with the following changes. Since in the periodic case one can integrate in both forward and backward time directions, we use an appropriate integral analog of the problem (2.8), (1.8), namely

$$\begin{aligned} u_j(x,t) &= c_j(x_j, x, t) u_j(x_j, \omega_j(x_j)) + c_j(x_j, x, t) h_j(\omega_j(x_j)) \\ &\quad - \int_{x_j}^{x} d_j(\xi, x, t) \left(\sum_{k \ne j} b_{jk}(\xi, \omega_j(\xi)) u_k(\xi, \omega_j(\xi)) - g_j(\xi, \omega_j(\xi))\right) \mathrm{d}\xi \ \text{ if } b_{jj} > 0, \\ u_j(x,t) &= c_j(1 - x_j, x, t) u_j(1 - x_j, \omega_j(1 - x_j)) + c_j(1 - x_j, x, t) h_j(\omega_j(1 - x_j)) \\ &\quad - \int_{1 - x_j}^{x} d_j(\xi, x, t) \left(\sum_{k \ne j} b_{jk}(\xi, \omega_j(\xi)) u_k(\xi, \omega_j(\xi)) - g_j(\xi, \omega_j(\xi))\right) \mathrm{d}\xi \ \text{ if } b_{jj} < 0. \end{aligned}$$

Note that in the case of general boundary conditions (2.5) we could integrate only in the backward time direction where the boundary conditions are given. Now, instead of the system (4.5), we have the following decoupled system:

$$\begin{aligned} u_j(1 - x_j, t) &= c_j(x_j, 1 - x_j, t) u_j(x_j, \omega_j(x_j, 1 - x_j, t)) + r_j(1 - x_j, t) \text{ if } b_{jj} > 0, \\ u_j(x_j, t) &= c_j(1 - x_j, x_j, t) u_j(1 - x_j, \omega_j(1 - x_j, x_j, t)) + r_j(x_j, t) \text{ if } b_{jj} < 0. \end{aligned}$$

The analog of the operator G_0 introduced in (4.6), which we denote by H_0, reads

$$\begin{aligned} (H_0\psi)_j(t) &= c_j(x_j, 1 - x_j, t)\psi_j(\omega_j(x_j, 1 - x_j, t)) \ \text{ if } b_{jj} > 0, \\ (H_0\psi)_j(t) &= c_j(1 - x_j, x_j, t)\psi_j(\omega_j(1 - x_j, x_j, t)) \ \text{ if } b_{jj} < 0. \end{aligned} \tag{4.10}$$

One can easily see that $\|C\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))} = 1$, while $\|H_0\|_{\mathcal{L}(BC(\mathbb{R};\mathbb{R}^n))} < 1$. It follows that the operator $I - H_0$ and, hence, the operator $I - C$ is bijective, as desired. The rest of the proof goes similarly to the second part of the proof of Theorem 3.

4.2. Higher regularity of the bounded continuous solutions: Proof of Theorem 5

We divide the proof into a number of claims. Part 1 of the theorem follows from Claims 1–4, while Part 2 follows from Claims 5–6.

We give a proof under the assumptions of Theorem 3. The proof under the assumptions of Theorem 4 follows the same line, and we will point out only the differences.

We begin with Part 1. Let $u \in BC(\Pi, \mathbb{R}^n)$ be the bounded continuous solution to the problem (2.8), (2.5).

Claim 1. The generalized directional derivatives $(\partial_t + a_j\partial_x)u_j$ are continuous functions.

Proof of Claim. Take an arbitrary sequence of smooth functions $u^l : \Pi \to \mathbb{R}^n$ approaching u in $BC(\Pi, \mathbb{R}^n)$ and an arbitrary smooth function $\varphi : (0, 1) \times \mathbb{R} \to \mathbb{R}$ with compact support. Let $\langle \cdot, \cdot \rangle$ denote the scalar product in $L^2((0, 1) \times \mathbb{R})$. Using (2.10), for every $j \le n$ we have

$$\begin{aligned}
&\Big\langle (\partial_t + a_j\partial_x)u_j, \varphi\Big\rangle = \Big\langle u_j, -\partial_t\varphi - \partial_x(a_j\varphi)\Big\rangle = \lim_{l\to\infty} \Big\langle u_j^l, -\partial_t\varphi - \partial_x(a_j\varphi)\Big\rangle \\
&= \lim_{l\to\infty} \Big\langle c_j(x_j, x, t)\Big[(Rz^l)_j(\omega_j(x_j, x, t)) + h_j(\omega_j(x_j, x, t))\Big] \\
&\quad - \int_{x_j}^{x} d_j(\xi, x, t)\Big[\sum_{k\neq j}\Big[b_{jk}u_k^l\Big](\xi, \omega_j(\xi)) - g_j(\xi, \omega_j(\xi))\Big]d\xi, -\partial_t\varphi - \partial_x(a_j\varphi)\Big\rangle \\
&= \lim_{l\to\infty} \Big\langle -\sum_{k=1}^{n} b_{jk}(x, t)u_k^l + g_j(x, t), \varphi\Big\rangle = \Big\langle -\sum_{k=1}^{n} b_{jk}(x, t)u_k + g_j(x, t), \varphi\Big\rangle,
\end{aligned}$$

as desired. Here we used the notation

$$z^l(t) = \Big(u_1^l(1, t), \ldots, u_m^l(1, t), u_{m+1}^l(0, t), \ldots, u_n^l(0, t)\Big)$$

and the equality

$$(\partial_t + a_j\partial_x)\psi(\omega_j(\xi, x, t)) = 0, \tag{4.11}$$

being true for any $\psi \in C^1(\mathbb{R})$. □

We substitute (2.12) into itself and get

$$u = Cu + (DC + D^2)u + (I + D)F(g, h). \tag{4.12}$$

Claim 2. The operators DC and D^2 map continuously $BC(\Pi, \mathbb{R}^n)$ into $BC_t^1(\Pi, \mathbb{R}^n)$.

Proof of Claim. It suffices to show that there exists a positive constant K_{11} such that for all $u \in BC_t^1(\Pi, \mathbb{R}^n)$ we have

$$\left\| \partial_t \left[(DC + D^2)u \right] \right\|_{BC} \leq K_{11} \|u\|_{BC}. \tag{4.13}$$

Straightforward transformations give the representation

$$\begin{aligned} \partial_t \left[(DCu)_j(x,t) \right] &= -\partial_t \Bigg(\int_{x_j}^{x} d_j(\xi, x, t) \sum_{k \neq j} b_{jk}(\xi, \omega_j(\xi, x, t)) \\ &\qquad \times c_k(x_k, \xi, \omega_j(\xi))(Rz)_k(\omega_k(x_k, \xi, \omega_j(\xi))) \, d\xi \Bigg) \\ &= -\sum_{k \neq j} \int_{x_j}^{x} \partial_t d_{jk}(\xi, x, t)(Rz)_k(\omega_k(x_k, \xi, \omega_j(\xi))) \, d\xi \\ &\quad - \sum_{k \neq j} \int_{x_j}^{x} d_{jk}(\xi, x, t) \frac{d}{dt}(Rz)_k(\omega_k(x_k, \xi, \omega_j(\xi))) \, d\xi, \end{aligned} \tag{4.14}$$

where the functions

$$d_{jk}(\xi, x, t) = d_j(\xi, x, t) b_{jk}(\xi, \omega_j(\xi)) c_k(x_k, \xi, \omega_j(\xi))$$

are uniformly bounded and have continuous and uniformly bounded first-order derivatives in t. An upper bound as in (4.13) for the first sum in (4.14) follows directly from the regularity and the boundedness assumptions on the coefficients of the original problem.

The strict hyperbolicity assumption (2.4) admits the following representation formula:

$$\begin{aligned} &\frac{d}{dt}(Rz)_k(\omega_k(x_k, \xi, \omega_j(\xi, x, t))) \\ &\quad = \frac{d}{d\xi}(Rz)_k(\omega_k(x_k, \xi, \omega_j(\xi))) \frac{\partial_3 \omega_k(x_k, \xi, \omega_j(\xi)) \partial_t \omega_j(\xi)}{\partial_2 \omega_k(x_k, \xi, \omega_j(\xi)) + \partial_3 \omega_k(x_k, \xi, \omega_j(\xi)) \partial_\xi \omega_j(\xi)} \\ &\quad = \frac{d}{d\xi}(Rz)_k(\omega_k(x_k, \xi, \omega_j(\xi))) \frac{\partial_t \omega_j(\xi) a_j(\xi, \omega_j(\xi)) a_k(\xi, \omega_j(\xi))}{a_k(\xi, \omega_j(\xi)) - a_j(\xi, \omega_j(\xi))}. \end{aligned}$$

Here and in what follows ∂_j denotes the partial derivative with respect to the jth argument. Hence,

$$\begin{aligned} &\int_{x_j}^{x} d_{jk}^1(\xi, x, t) \frac{d}{d\xi}(Rz)_k(\omega_k(x_k, \xi, \omega_j(\xi))) \, d\xi \\ &\quad = d_{jk}^1(\xi, x, t)(Rz)_k(\omega_k(x_k, \xi, \omega_j(\xi))) \Big|_{\xi = x_j}^{x} \\ &\qquad - \int_{x_j}^{x} (Rz)_k(\omega_k(x_k, \xi, \omega_j(\xi))) \partial_\xi d_{jk}^1(\xi, x, t) \, d\xi, \end{aligned} \tag{4.15}$$

where

$$d_{jk}^1(\xi, x, t) = d_{jk}(\xi, x, t) \frac{\partial_t \omega_j(\xi) a_j(\xi, \omega_j(\xi)) a_k(\xi, \omega_j(\xi))}{a_k(\xi, \omega_j(\xi)) - a_j(\xi, \omega_j(\xi))}.$$

Combining (4.14) with (4.15), we conclude that $\partial_t(DC)$ is bounded as stated in (4.13).

Similarly,

$$\begin{aligned}
&\partial_t\left[(D^2u)_j(x,t)\right] \\
&\quad=\sum_{k\neq j}\sum_{l\neq k}\int_{x_j}^{x}\int_{x_k}^{\xi}\partial_t d_{jkl}(\xi,\xi_1,x,t)u_l(\xi_1,\omega_k(\xi_1,\xi,\omega_j(\xi,x,t)))\,\mathrm{d}\xi_1\mathrm{d}\xi \\
&\qquad+\sum_{k\neq j}\sum_{l\neq k}\int_{x_j}^{x}\int_{x_k}^{\xi}d_{jkl}(\xi,\xi_1,x,t)\partial_t u_l(\xi_1,\omega_k(\xi_1,\xi,\omega_j(\xi,x,t)))\,\mathrm{d}\xi_1\mathrm{d}\xi,
\end{aligned}$$

where

$$d_{jkl}(\xi,\xi_1,x,t)=d_j(\xi,x,t)b_{jk}(\xi,\omega_j(\xi))d_k(\xi_1,\xi,\omega_j(\xi))b_{kl}(\xi_1,\omega_k(\xi_1,\xi,\omega_j(\xi))).$$

A desired estimate for the first summand is obvious, while for the second summand follows from the following transformations. For definiteness, assume that $j,k\le m$ (the cases $j>m$ or $k>m$ are similar). Taking into account the identity

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}u_l(\xi_1,\omega_k(\xi_1,\xi,\omega_j(\xi))) \\
&\quad=\frac{\mathrm{d}}{\mathrm{d}\xi}u_l(\xi_1,\omega_k(\xi_1,\xi,\omega_j(\xi)))\frac{\partial_t\omega_j(\xi)a_j(\xi,\omega_j(\xi))a_k(\xi,\omega_j(\xi))}{a_k(\xi,\omega_j(\xi))-a_j(\xi,\omega_j(\xi))},
\end{aligned}$$

we get

$$\begin{aligned}
&\int_{x_j}^{x}\int_{x_k}^{\xi}d_{jkl}(\xi,\xi_1,x,t)\partial_t u_l(\xi_1,\omega_k(\xi_1,\xi,\omega_j(\xi)))\,\mathrm{d}\xi_1\mathrm{d}\xi \\
&\quad=\int_{x_j}^{x}\int_{x_k}^{\xi}d^1_{jkl}(\xi,\xi_1,x,t)\frac{\mathrm{d}}{\mathrm{d}\xi}u_l(\xi_1,\omega_k(\xi_1,\xi,\omega_j(\xi)))\,\mathrm{d}\xi_1\mathrm{d}\xi,
\end{aligned} \tag{4.16}$$

where

$$d^1_{jkl}(\xi,\xi_1,x,t)=d_{jkl}(\xi,\xi_1,x,t)\frac{\partial_t\omega_j(\xi)a_j(\xi,\omega_j(\xi))a_k(\xi,\omega_j(\xi))}{a_k(\xi,\omega_j(\xi))-a_j(\xi,\omega_j(\xi))}.$$

The right-hand side of (4.16) can be written as

$$\begin{aligned}
&\int_0^x\int_{\xi_1}^x d^1_{jkl}(\xi,\xi_1,x,t)\frac{\mathrm{d}}{\mathrm{d}\xi}u_l(\xi_1,\omega_k(\xi_1,\xi,\omega_j(\xi)))\,\mathrm{d}\xi\mathrm{d}\xi_1 \\
&\quad=\int_0^x d^1_{jkl}(\xi,\xi_1,x,t)u_l(\xi_1,\omega_k(\xi_1,\xi,\omega_j(\xi)))\Big|_{\xi=\xi_1}^{x}\,\mathrm{d}\xi_1 \\
&\qquad+\int_0^x\int_{\xi_1}^x u_l(\xi_1,\omega_k(\xi_1,\xi,\omega_j(\xi)))\frac{\mathrm{d}}{\mathrm{d}\xi}d^1_{jkl}(\xi,\xi_1,x,t)\,\mathrm{d}\xi\mathrm{d}\xi_1,
\end{aligned}$$

which implies an upper bound as in (4.13). □

Claim 3. $I-C$ is a bijective operator from $BC^1_t(\Pi,\mathbb{R}^n)$ to itself.

Proof of Claim. We are done if we show that the system (4.4) is uniquely solvable in $BC_t^1(\Pi, \mathbb{R}^n)$ for any $r \in BC_t^1(\Pi, \mathbb{R}^n)$. Obviously, this is true if and only if

$$I - G_0 \text{ is a bijective operator from } BC^1(\mathbb{R}, \mathbb{R}^n) \text{ to } BC^1(\mathbb{R}, \mathbb{R}^n), \tag{4.17}$$

where the operator $G_0 \in \mathcal{L}(BC(\mathbb{R}, \mathbb{R}^n))$ is given by (4.6). To prove (4.17), let us norm the space $BC^1(\mathbb{R}, \mathbb{R}^n)$ with

$$\|y\|_\sigma = \|y\|_{BC} + \sigma \|\partial_t y\|_{BC}, \tag{4.18}$$

where a positive constant σ will be defined later on. Note that the norms (4.18) are equivalent for all $\sigma > 0$. We therefore have to prove that there exist constants $\sigma < 1$ and $\gamma < 1$ such that

$$\|G_0 y\|_{BC} + \sigma \left\| \frac{\mathrm{d}}{\mathrm{d}t} G_0 y \right\|_{BC} \le \gamma \left(\|y\|_{BC} + \sigma \|y'\|_{BC} \right) \text{ for all } y \in BC^1(\mathbb{R}, \mathbb{R}^n).$$

Combining (2.9) with the formula

$$\partial_t \omega_j(\xi, x, t) = \exp \int_\xi^x \left[\frac{\partial_t a_j}{a_j^2} \right] (\eta, \omega_j(\eta, x, t))\, \mathrm{d}\eta, \tag{4.19}$$

we get that $c_j^1(\xi, x, t) = c_j(\xi, x, t) \partial_t \omega_j(\xi, x, t)$. Then for $y \in BC^1(\mathbb{R}, \mathbb{R}^n)$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(G_0 y)_j(t) &= \partial_t c_j(x_j, 1 - x_j, t)(Ry)_j(\omega_j(x_j, 1 - x_j, t)) \\ &\quad + c_j^1(x_j, 1 - x_j, t) \left[\left(R' y\right)_j + (\widetilde{R} y')_j \right] (\omega_j(x_j, 1 - x_j, t)), \ \ j \le n. \end{aligned}$$

Define an operator $W \in \mathcal{L}(BC(\mathbb{R}, \mathbb{R}^n))$ by

$$\begin{aligned} (Wy)_j(t) &= \partial_t c_j(x_j, 1 - x_j, t)(Ry)_j(\omega_j(x_j, 1 - x_j, t)) \\ &\quad + c_j^1(x_j, 1 - x_j, t) \left(R' y\right)_j (\omega_j(x_j, 1 - x_j, t)). \end{aligned} \tag{4.20}$$

On the account of (4.6) and (4.7), both assumptions (**B1**) and (**B2**) implies that $\|G_0\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} < 1$. Moreover, the assumption (1.9) for $i = 1$ of Theorem 5 yields $\|G_1\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} < 1$. Fix $\sigma < 1$ such that $\|G_0\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} + \sigma \|W\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} < 1$. Set

$$\gamma = \max \left\{ \|G_0\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} + \sigma \|W\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))}, \|G_1\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} \right\}.$$

Hence,

$$\|G_0 y\|_\sigma \le \|G_0 y\|_{BC} + \sigma \|Wy\|_{BC} + \sigma \left\|G_1 y'\right\|_{BC} \le \gamma \left(\|y\|_{BC} + \sigma \left\|y'\right\|_{BC} \right).$$

Furthermore, $\|(I - G_0)^{-1} y\|_\sigma \le (1 - \gamma)^{-1} \|y\|_\sigma$, which yields the bound

$$\|(I - G_0)^{-1} y\|_{BC_t^1} \le \frac{1}{\sigma} \|(I - G_0)^{-1} y\|_\sigma \le \frac{1}{\sigma(1 - \gamma)} \|y\|_\sigma \le \frac{1}{\sigma(1 - \gamma)} \|y\|_{BC_t^1}. \tag{4.21}$$

Finally, from (4.9) and (4.21) we obtain that

$$\|(I-C)^{-1}\|_{\mathcal{L}(BC_t^1(\Pi;\mathbb{R}^n))} \le 1+\frac{1}{\sigma(1-\gamma)}\|C\|_{\mathcal{L}(BC_t^1(\Pi;\mathbb{R}^n))}. \tag{4.22}$$

The proof of the claim under the assumptions of Theorem 3 is complete.

The proof under the assumptions of Theorem 4 follows the same line with this changes: We specify $(Rz)_j \equiv z_j$ for all $j \le n$ and replace the operator G_0 by the operator H_0 (see the formula (4.10)). Hence, $(R'z)_j \equiv 0$ and $(\widetilde{R}z)_j \equiv z_j$ for all $j \le n$ and all $z \in BC^1(\mathbb{R}, \mathbb{R}^n)$. □

Now, Claims 2 and 3 together with the equation (4.12) imply that the bounded continuous solution u to problem (2.8), (2.5) is the bounded classical solution. Then, by definition, the function $v = qu$ is the bounded classical solution to the problem (2.1), (2.5), (2.7).

Claim 4. The bounded classical solution v to the problem (2.1), (2.5), (2.7) fulfills the estimate (2.14).

Proof of Claim. Combining the estimates (2.13), (4.13), and (4.22) with the equations (4.12) and (2.7), we obtain that

$$\begin{aligned}\|v\|_{BC_t^1} \le \|q\|_{BC_t^1}\|u\|_{BC_t^1} \le \|q\|_{BC_t^1}\left(1+\frac{1}{\sigma(1-\gamma)}\|C\|_{\mathcal{L}(BC_t^1(\Pi;\mathbb{R}^n))}\right)\\ \times\left\|(DC+D^2)u+(I+D)F(g,h)\right\|_{BC_t^1} \le K_{12}\left(\|g\|_{BC_t^1}+\|h\|_{BC^1}\right),\end{aligned}$$

where u is the bounded classical solution to (2.8), (2.5) and K_{12} is a positive constant not depending on g and h. Now, from (2.1) we get

$$\begin{aligned}\|\partial_x v\|_{BC} &\le \left\|(a^*)^{-1}\right\|_{BC}(\|g\|_{BC}+\|b^*v\|_{BC}+\|\partial_t v\|_{BC})\\ &\le K_{13}\left(\|g\|_{BC_t^1}+\|h\|_{BC^1}\right)\end{aligned}$$

for some $K_{13}>0$ not depending on g and h. The estimate (2.14) follows. □

The proof of Part 1 of the theorem is complete.

Now we prove Part 2. Formal differentiation of the system (2.1) in the distributional sense with respect to t and the boundary conditions (2.5) pointwise, we get, respectively,

$$\begin{aligned}(\partial_t+a^*\partial_x)\partial_t v+\left(b^*-\partial_t a^*(a^*)^{-1}\right)\partial_t v+\left(\partial_t b^*-\partial_t a^*(a^*)^{-1}b^*\right)v\\ =\partial_t g-\partial_t a^*(a^*)^{-1}g\end{aligned} \tag{4.23}$$

and

$$\partial_t u_j(x_j,t)=\frac{d}{dt}(Rz)_j(t)+h_j'(t)=\left(R'z\right)_j(t)+(\widetilde{R}z')_j(t)+h_j'(t),\quad j\le n. \tag{4.24}$$

Introduce a new variable $w=q^{-1}\partial_t v=\partial_t u+q^{-1}\partial_t qu$ and rewrite the problem (4.23)–(4.24) with respect to w as follows:

$$\partial_t w+a(x,t)\partial_x w+b^1(x,t)w=g^1(x,t,v), \tag{4.25}$$

$$w_j(x_j,t) = \frac{d}{dt}(Rz)_j(t) + h'_j(t) + \left[q^{-1}\partial_t q u\right]_j(x_j,t) = (\widetilde{R}y)_j(t) + h^1_j(t), \quad j \le n, \tag{4.26}$$

where

$$\begin{aligned} b^1(x,t) &= q^{-1}\left(b^* q - \partial_t a^* (a^*)^{-1} q + \partial_t q + a^* \partial_x q\right) = b - q^{-1}\partial_t a^* (a^*)^{-1} q, \\ g^1(x,t,v) &= -q^{-1}\left(\partial_t b^* - \partial_t a^* (a^*)^{-1} b^*\right) v + q^{-1}\left(\partial_t g - \partial_t a^* (a^*)^{-1} g\right), \\ h^1_j(t) &= \left(R'z\right)_j(t) + (\widetilde{R}\rho)_j(t) + h'_j(t) + \left[q^{-1}\partial_t q u\right]_j(x_j,t), \quad j \le n, \\ y(t) &= (w_1(1,t), \dots, w_m(1,t), w_{m+1}(0,t), \dots, w_n(0,t)), \\ \rho(t) &= \Big(\left[\partial_t q^{-1} v\right]_1(1,t), \dots, \left[\partial_t q^{-1} v\right]_m(1,t), \\ &\qquad \left[\partial_t q^{-1} v\right]_{m+1}(0,t), \dots, \left[\partial_t q^{-1} v\right]_n(0,t)\Big). \end{aligned}$$

Claim 5. The function $w \in BC(\Pi, \mathbb{R}^n)$ satisfies both (4.25) in the distributional sense and (4.26) pointwise if and only if w satisfies the following system pointwise for all $j \le n$:

$$\begin{aligned} w_j(x,t) &= c^1_j(x_j,x,t)\left((\widetilde{R}y)_j(\omega_j(x_j)) + h^1_j(\omega_j(x_j))\right) \\ &\quad - \int_{x_j}^x d^1_j(\xi,x,t)\Big(\sum_{k\ne j} b^1_{jk}(\xi,\omega_j(\xi))w_k(\xi,\omega_j(\xi)) - g^1_j(\xi,\omega_j(\xi),v(\xi,\omega_j(\xi)))\Big)d\xi. \end{aligned} \tag{4.27}$$

Proof of Claim. Set

$$d^1_j(\xi,x,t) = \frac{c^1_j(\xi,x,t)}{a_j(\xi,\omega_j(\xi))}.$$

To prove the *sufficiency*, take an arbitrary sequence $w^l \in BC^1(\Pi;\mathbb{R}^n)$ approaching w in $BC(\Pi;\mathbb{R}^n)$. Take an arbitrary smooth function $\varphi : (0,1)\times\mathbb{R} \to \mathbb{R}$ with compact support. On the account of (4.27), we have

$$\begin{aligned} \left\langle (\partial_t + a_j\partial_x)w_j, \varphi\right\rangle &= -\left\langle w_j, (\partial_t\varphi + \partial_x(a_j\varphi)\right\rangle = \lim_{l\to\infty}\left\langle w^l_j, -\partial_t\varphi - \partial_x(a_j\varphi)\right\rangle \\ &= \lim_{l\to\infty}\Big\langle -c^1_j(x_j,x,t)\left((\widetilde{R}y^l)_j(\omega_j(x_j)) + h^1_j(\omega_j(x_j))\right) \\ &\quad + \int_{x_j}^x d^1_j(\xi,x,t)\Big(\sum_{k\ne j} b^1_{jk}(\xi,\omega_j(\xi))w^l_k(\xi,\omega_j(\xi)) \\ &\quad - g^1_j(\xi,\omega_j(\xi),v(\xi,\omega_j(\xi)))\Big)d\xi, \partial_t\varphi + \partial_x(a_j\varphi)\Big\rangle \\ &= -\lim_{l\to\infty}\Big\langle\Big(b_{jj}(x,t) - \frac{\partial_t a_j(x,t)}{a_j(x,t)}\Big)w^l_j + \sum_{k\ne j} b^1_{jk}(x,t)w^l_k - g^1_j(x,t,v), \varphi\Big\rangle \end{aligned}$$

$$= -\left\langle \left(b_{jj}(x,t) - \frac{\partial_t a_j(x,t)}{a_j(x,t)} \right) w_j + \sum_{k \neq j} b^1_{jk}(x,t) w_k - g^1_j(x,t,v), \varphi \right\rangle,$$

where $y^l = \left(w^l_1(1,t), \ldots, w^l_m(1,t), w^l_{m+1}(0,t), \ldots, w^l_n(0,t)\right)$. It remains to note that for all $j \leq n$

$$b_{jj} - a_j^{-1} \partial_t a_j \equiv b^1_{jj}, \tag{4.28}$$

what easily follows from the diagonality of the matrix a and the identity

$$q^{-1} \partial_t a^* \, (a^*)^{-1} q = \left(\partial_t a + q^{-1} \partial_t q a - a q^{-1} \partial_t q \right) a^{-1}.$$

Moreover, putting $x = x_j$ in (4.27), we immediately get (4.26). The proof of the sufficiency is complete.

To prove the *necessity*, assume that the function w satisfies (4.25) in the distributional sense and (4.26) pointwise. On the account of (4.11), we rewrite the system (4.25) in the form

$$\begin{aligned} &(\partial_t + a_j(x,t)\partial_x) \left(c^1_j(x_j,x,t)^{-1} w_j \right) \\ &= c^1_j(x_j,x,t)^{-1} \left(- \sum_{k \neq j} b^1_{jk}(x,t) w_k + g^1_j(x,t,v) \right), \end{aligned} \tag{4.29}$$

without destroying the equalities in the sense of distributions. To prove that w satisfies (4.27) pointwise, we use the constancy theorem of distribution theory claiming that any distribution on an open set with zero generalized derivatives is a constant on any connected component of the set. By (4.29), this theorem implies that for each $j \leq n$ the expression

$$\begin{aligned} c^1_j(x_j,x,t)^{-1} \Bigg[w_j(x,t) + \int_{x_j}^x d^1_j(\xi,x,t) \Bigg(\sum_{k \neq j} \left[b^1_{jk} w_k \right] (\xi, \omega_j(\xi)) \\ - g^1_j \Big(\xi, \omega_j(\xi), v(\xi, \omega_j(\xi)) \Big) \Bigg) d\xi \Bigg] \end{aligned} \tag{4.30}$$

is constant along the characteristic curve $\omega_j(\xi,x,t)$. In other words, the distributional directional derivative $(\partial_t + a_j(x,t)\partial_x)$ of the function (4.30) is equal to zero. Since (4.30) is a continuous function, $c^1_j(x_j,x_j,t) = 1$, and the trace $w_j(x_j,t)$ is given by (4.26), it follows that w satisfies the system (4.27) pointwise, as desired. □

Claim 6. The bounded classical solution v to the problem (2.1), (2.5), (2.7) fulfills the estimate (2.15).

Proof of Claim. We rewrite the system (4.27) in the operator form

$$w = C_1 w + D_1 w + F_1(g^1, h^1), \tag{4.31}$$

where $C_1, D_1 \in \mathcal{L}(BC(\Pi;\mathbb{R}^n))$ and $F_1 \in \mathcal{L}(BC(\Pi;\mathbb{R}^{2n}), BC(\Pi;\mathbb{R}^n))$ are operators defined by

$$
\begin{aligned}
(C_1 w)_j(x,t) &= c_j^1(x_j,x,t)(\widetilde{R}y)_j(\omega_j(x_j)),\\
(D_1 w)_j(x,t) &= -\int_{x_j}^{x} d_j^1(\xi,x,t)\sum_{k\neq j} b_{jk}^1(\xi,\omega_j(\xi))w_k(\xi,\omega_j(\xi))d\xi,\\
\left[F_1(g^1,h^1)\right]_j(x,t) &= \int_{x_j}^{x} d_j^1(\xi,x,t)g_j^1\left(\xi,\omega_j(\xi),v(\xi,\omega_j(\xi))\right)d\xi\\
&\quad + c_j^1(x_j,x,t)h_j^1(\omega_j(x_j)).
\end{aligned}
\tag{4.32}
$$

Iterating (4.31), we obtain

$$
w = C_1 w + (D_1C_1 + D_1^2)w + (I + D_1)F_1(g^1,h^1). \tag{4.33}
$$

Using the same argument as in Claim 2, we conclude that the operators D_1C_1 and D_1^2 map continuously $BC(\Pi,\mathbb{R}^n)$ into $BC_t^1(\Pi,\mathbb{R}^n)$. Moreover, the following smoothing estimate is true:

$$
\left\|(D_1C_1 + D_1^2)w\right\|_{BC_t^1} \le K_{21}\|w\|_{BC} \tag{4.34}
$$

for some $K_{21} > 0$ not depending on $w \in BC(\Pi,\mathbb{R}^n)$.

Next, we prove that $I - C_1$ is a bijective operator from $BC_t^1(\Pi,\mathbb{R}^n)$ to itself. The proof is similar to the proof of Claim 3. We have to show that the system

$$
w_j(x,t) = c_j^1(x_j,x,t)(\widetilde{R}y)_j(\omega_j(x_j,x,t)) + \alpha_j(x,t), \quad j \le n,
$$

is uniquely solvable in $BC_t^1(\Pi,\mathbb{R}^n)$ for each $\alpha \in BC_t^1(\Pi,\mathbb{R}^n)$. It is sufficient to show that

$$
I - G_1 \text{ is a bijective operator from } BC^1(\mathbb{R},\mathbb{R}^n) \text{ to itself,} \tag{4.35}
$$

where the operator $G_1 \in \mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))$ is given by (1.7). To prove (4.35), we use the space $BC^1(\mathbb{R},\mathbb{R}^n)$ normed by (4.18). We are done if we prove that there exist constants $\sigma_1 < 1$ and $\gamma_1 < 1$ such that

$$
\|G_1\psi\|_{BC} + \sigma_1\left\|\frac{d}{dt}G_1\psi\right\|_{BC} \le \gamma_1\left(\|\psi\|_{BC} + \sigma_1\|\psi'\|_{BC}\right)
$$

for all $\psi \in BC^1(\mathbb{R},\mathbb{R}^n)$.

As it follows from (1.6) and (4.19), $c_j^2(\xi,x,t) = c_j^1(\xi,x,t)\partial_t\omega_j(\xi,x,t)$. Define operator $W_1 \in \mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))$ by

$$
\begin{aligned}
(W_1\psi)_j(t) &= \partial_t c_j^1(x_j,1-x_j,t)(\widetilde{R}\psi)_j(\omega_j(x_j,1-x_j,t))\\
&\quad + c_j^2(x_j,1-x_j,t)\left(\widetilde{R}'\psi\right)_j(\omega_j(x_j,1-x_j,t)), \; j \le n.
\end{aligned}
\tag{4.36}
$$

Taking into account (1.4) and (1.7), for given $\psi \in BC^1(\mathbb{R}, \mathbb{R}^n)$, it holds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[(G_1\psi)_j(t)\right] &= \partial_t c_j^1(x_j, 1-x_j, t)(\widetilde{R}\psi)_j(\omega_j(x_j, 1-x_j, t)) \\ &\quad + c_j^2(x_j, 1-x_j, t)\Big[(\widetilde{R}'\psi)_j + (\widehat{R}\psi')_j\Big](\omega_j(x_j, 1-x_j, t)) \\ &= (W_1\psi)_j(t) + (G_2\psi)_j(t), \quad j \le n.\end{aligned}$$

By the assumption (1.9), $\|G_1\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} < 1$ and $\|G_2\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} < 1$. Fix $\sigma_1 < 1$ such that $\|G_1\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} + \sigma_1 \|W_1\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} < 1$. Set

$$\gamma_1 = \max\left\{\|G_1\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} + \sigma_1 \|W_1\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))}, \|G_2\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))}\right\}.$$

It follows that

$$\begin{aligned}\|G_1\psi\|_{\sigma_1} &= \|G_1\psi\|_{BC} + \sigma_1 \left\|\frac{\mathrm{d}}{\mathrm{d}t}G_1\psi\right\|_{BC} \le \|G_1\psi\|_{BC} \\ &\quad + \sigma_1\|W_1\psi\|_{BC} + \sigma_1\left\|G_2\psi'\right\|_{BC} \le \gamma_1\left(\|\psi\|_{BC} + \sigma_1\left\|\psi'\right\|_{BC}\right),\end{aligned}$$

which proves (4.35).

Similarly to (4.22), the inverse to $I - C_1$ can be estimated from above as follows:

$$\|(I - C_1)^{-1}\|_{\mathcal{L}(BC_t^1(\Pi;\mathbb{R}^n))} \le 1 + \frac{1}{\sigma_1(1-\gamma_1)}\|C_1\|_{\mathcal{L}(BC_t^1(\Pi;\mathbb{R}^n))}.$$

Combining this estimate with (2.14), (4.33), and (4.34), we get

$$\begin{aligned}\|\partial_t v\|_{BC_t^1} &\le \|q\|_{BC_t^1}\|w\|_{BC_t^1} \\ &\le \|q\|_{BC_t^1}\left(1 + \frac{\|C_1\|_{\mathcal{L}(BC_t^1(\Pi;\mathbb{R}^n))}}{\sigma_1(1-\gamma_1)}\right)\left\|\left(D_1C_1 + D_1^2\right) + (I + D_1)\,F_1\left(g^1, h^1\right)\right\|_{BC_t^1} \\ &\le K_{22}\left(\|g^1\|_{BC_t^1} + \|h^1\|_{BC^1}\right) \le K_{23}\left(\|g\|_{BC_t^2} + \|h\|_{BC^2}\right),\end{aligned}$$

the constants K_{22} and K_{23} being independent of g and h. By (4.23), there exists a constant K_{24} not depending on g and h such that

$$\|\partial_x v\|_{BC_t^1} \le K_{24}\left(\|g\|_{BC_t^2} + \|h\|_{BC^2}\right),$$

which implies the estimate (2.15), as desired. □

4.3. A perturbation result: Proof of Theorem 6

In the new variable

$$u = \tilde{q}^{-1}v, \tag{4.37}$$

the perturbed system (2.16) reads

$$\partial_t u + \tilde{a}(x,t)\partial_x u + \tilde{b}(x,t)u = g(x,t), \tag{4.38}$$

where $\tilde{a}$ is defined by (2.17) and $\tilde{b}(x,t) = \tilde{q}^{-1}\left(\tilde{b}^*\tilde{q} + \partial_t\tilde{q} + \tilde{a}^*\partial_x\tilde{q}\right)$.

We will use the following notation. The jth characteristic of (4.38) passing through the point $(x,t) \in \Pi$ is defined as the solution $\xi \in [0,1] \mapsto \tilde{\omega}_j(\xi) = \tilde{\omega}_j(\xi,x,t) \in \mathbb{R}$ of the initial value problem

$$\partial_\xi \tilde{\omega}_j(\xi,x,t) = \frac{1}{\tilde{a}_j(\xi,\tilde{\omega}_j(\xi,x,t))}, \quad \tilde{\omega}_j(x,x,t) = t.$$

Introduce notation

$$\tilde{c}_j(\xi,x,t) = \exp\int_x^\xi \left[\frac{\tilde{b}_{jj}}{\tilde{a}_j}\right](\eta,\tilde{\omega}_j(\eta))\,\mathrm{d}\eta, \quad \tilde{d}_j(\xi,x,t) = \frac{\tilde{c}_j(\xi,x,t)}{\tilde{a}_j(\xi,\tilde{\omega}_j(\xi))}$$

and operators $\widetilde{C}, \widetilde{D} \in \mathcal{L}(BC(\Pi;\mathbb{R}^n))$ and $\widetilde{F} \in \mathcal{L}\left(BC(\Pi;\mathbb{R}^{2n}); BC(\Pi;\mathbb{R}^n)\right)$ by

$$\begin{aligned}
(\widetilde{C}u)_j(x,t) &= \tilde{c}_j(x_j,x,t)(Ru)_j(\tilde{\omega}_j(x_j)),\\
(\widetilde{D}u)_j(x,t) &= -\int_{x_j}^x \tilde{d}_j(\xi,x,t)\sum_{k\neq j}\tilde{b}_{jk}(\xi,\tilde{\omega}_j(\xi))u_k(\xi,\tilde{\omega}_j(\xi))\mathrm{d}\xi,\\
(\widetilde{F}(g,h))_j(x,t) &= \int_{x_j}^x \tilde{d}_j(\xi,x,t)g_j(\xi,\tilde{\omega}_j(\xi))\mathrm{d}\xi + \tilde{c}_j(x_j,x,t)h_j(\tilde{\omega}_j(x_j)).
\end{aligned}$$

Consider the following operator representation of the perturbed problem (4.38), (2.5), (4.37):

$$u = \widetilde{C}u + \widetilde{D}u + \widetilde{F}(g,h). \tag{4.39}$$

Iterating this formula, we get

$$u = \widetilde{C}u + \left(\widetilde{D}\widetilde{C} + \widetilde{D}^2\right)u + \left(I + \widetilde{D}\right)\widetilde{F}(g,h). \tag{4.40}$$

Let $\widetilde{G}_0, \widetilde{G}_1, \widetilde{G}_2, \widetilde{W} \in \mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))$, $\widetilde{C}_1, \widetilde{D}_1 \in \mathcal{L}(BC(\Pi,\mathbb{R}^n))$, and $\widetilde{F}_1 \in \mathcal{L}(BC(\Pi,\mathbb{R}^{2n}))$ denote operators given by the right-hand sides of the formulas (4.6), (1.7), (4.20), and (4.32), respectively, where a, b, and ω_j are replaced by $\tilde{a}$, $\tilde{b}$, and $\tilde{\omega}_j$, respectively.

Assume that the condition (**B1**) is fulfilled. Similar argument works in the case of (**B2**) or (**B3**).

Proof of Part 1. Note that the assumptions (**B1**) and (1.9) of Theorem 5 are stable with respect to small perturbations of a and b. Since small perturbations of a^*, b^*, and q imply small perturbations of a and b, there exists $\varepsilon_{11} \le \varepsilon_0$ such that, for all $\tilde{a}^*$ and $\tilde{b}^*$ varying in the range

$$\begin{gathered}
\|\tilde{a}^* - a^*\|_{BC_t^2} + \|\partial_x\tilde{a}^* - \partial_x a^*\|_{BC_t^1} \le \varepsilon_{11}, \quad \|\tilde{b}^* - b^*\|_{BC_t^1} \le \varepsilon_{11},\\
\|\tilde{q} - q\|_{BC_t^2} + \|\partial_x\tilde{q} - \partial_x q\|_{BC_t^1} \le \varepsilon_{11},
\end{gathered} \tag{4.41}$$

the conditions (**B1**) and (1.9) for $i = 1$ remain to be true with $\tilde{a}$ and $\tilde{b}$ in place of a and b, respectively.

Due to Part 1 of Theorem 5, the system (4.38), (2.5), (4.37) has a unique bounded classical solution $\tilde{u} \in BC^1(\Pi;\mathbb{R}^n)$ for each fixed $\tilde{a}^*$, $\tilde{b}^*$, and $\tilde{q}$.

To derive the a priori estimate (2.14) with $\tilde{v}$ in place of v, note that the value of $\varepsilon_{11} > 0$ can be chosen so small that there exists a positive real $\nu_1 < 1$ such that the left-hand sides of (**B1**) and (1.9) for $i = 1$, calculated for the perturbed problem (4.38), (4.37), (2.5), are bounded from above by $1-\nu_1$. Due to the proof of Theorem 3, this implies the inequality

$$\|\widetilde{C}\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))} + \|\widetilde{D}\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))} \le 1-\nu_1,$$

which is uniform in $\tilde{a}^*$, $\tilde{b}^*$, and $\tilde{q}$. Combining this inequality with (4.39), we conclude that there exists a constant $\widetilde{K} > 0$ not depending on $\tilde{a}^*$, $\tilde{b}^*$, $\tilde{q}$, g, and h such that

$$\|\tilde{u}\|_{BC} \le \widetilde{K}\left(\|g\|_{BC} + \|h\|_{BC}\right). \tag{4.42}$$

We immediately see from (**B1**), (1.9) for $i = 1$, (4.7), and (4.20) that there exist constants $\widetilde{K}_1 > 0$ and $\nu_2 < 1$ such that

$$\|\widetilde{G}_0\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} \le 1-\nu_2,\ \|\widetilde{G}_1\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} \le 1-\nu_2,\ \|\widetilde{W}\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} \le \widetilde{K}_1,$$

uniformly in $\tilde{a}^*$, $\tilde{b}^*$, and $\tilde{q}$ fulfilling (4.41) with ε_{12} in place of ε_{11}.

Put $\gamma = 1-\nu_2+\sigma\widetilde{K}_1$ and fix $\sigma < 1$ such that $\gamma < 1$. Now we apply the argument used to prove the estimate (4.22) and get

$$\|(I-\widetilde{C})^{-1}\|_{\mathcal{L}(BC_t^1(\Pi;\mathbb{R}^n))} \le 1 + \frac{1}{\sigma(1-\gamma)}\|\widetilde{C}\|_{\mathcal{L}(BC_t^1(\Pi;\mathbb{R}^n))}. \tag{4.43}$$

Similarly to the proof of Claim 2 in Sect. 4.2, we show that the operators $\widetilde{D}\widetilde{C}$ and $\widetilde{D}^2$ are smoothing and map $BC(\Pi,\mathbb{R}^n)$ into $BC_t^1(\Pi,\mathbb{R}^n)$. Moreover, there exists a constant $\widetilde{K}_2$ such that, for all $\tilde{a}^*$, $\tilde{b}^*$, and $\tilde{q}$ fulfilling the inequalities (4.41) with ε_{12} in place of ε_{11}, it holds

$$\left\|\partial_t\left[\left(\widetilde{D}\widetilde{C}+\widetilde{D}^2\right)u\right]\right\|_{BC} \le \widetilde{K}_2\|u\|_{BC} \tag{4.44}$$

for all $u \in BC(\Pi,\mathbb{R}^n)$. Now we combine the estimates (4.42)–(4.44) with the equations (4.40). We conclude that there exist constants $\varepsilon_1 \le \varepsilon_{12}$ and $K_1 > 0$ such that, for all $\tilde{a}^*$, $\tilde{b}^*$, $\tilde{q}$, g, and h varying in the range (4.41) with ε_1 in place of ε_{11}, the estimate (2.14) is true with $\tilde{v}$ in place of v.

Proof of Part 2. Let ε_1 be a constant satisfying Part 1 of Theorem 6. Consider a perturbed version of the equation (4.33) where C_1, D_1, and F_1 are replaced by $\widetilde{C}_1$, $\widetilde{D}_1$, and $\widetilde{F}_1$, respectively. Proceeding similarly to Part 1, we use (4.36) and (1.9) for $i = 2$ and conclude that the constant ε_1 can be chosen so small that there exist positive reals $\nu_3 < 1$ and $\widetilde{K}_3$ fulfilling the bounds

$$\|\widetilde{G}_2\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} < 1-\nu_3, \quad \|\widetilde{W}_1\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} \le \widetilde{K}_3, \tag{4.45}$$

uniformly in $\tilde{a}^*$, $\tilde{b}^*$, and $\tilde{q}$ satisfying the estimates (4.41) with ε_1 in place of ε_{11} as well as the stronger estimate $\|\tilde{b}^* - b^*\|_{BC_t^2} \le \varepsilon_1$. The desired a priori estimate (2.15) for the ε_1-perturbed problem then easily follows from the perturbed versions of (4.33) and (4.34).

The proof of Theorem 6 is complete.

5. Quasilinear system: Proof of main result

5.1. Proof of Part 1 of Theorem 1: Bounded solutions

Let δ_0 be a constant satisfying Assumption **(A1)** and ε_1 be a constant satisfying Part 2 of Theorem 6. Since the functions A and B are C^2-smooth, there exists $\delta_1 \le \delta_0$ such that for all $\varphi \in BC^2(\Pi, \mathbb{R}^n)$ with

$$\|\varphi\|_{BC_t^2} + \|\partial_x \varphi\|_{BC_t^1} \le \delta_1 \tag{5.1}$$

we have

$$\|a^\varphi\|_{BC_t^2} + \|\partial_x a^\varphi\|_{BC_t^1} \le \varepsilon_1, \quad \|q^\varphi\|_{BC_t^2} + \|\partial_x q^\varphi\|_{BC_t^1} \le \varepsilon_1, \quad \|b^\varphi\|_{BC_t^2} \le \varepsilon_1, \tag{5.2}$$

where $a^\varphi(x,t) = A(x,t,\varphi(x,t)) - A(x,t,0)$, $q^\varphi(x,t) = Q(x,t,\varphi(x,t)) - Q(x,t,0)$, and $b^\varphi(x,t) = B(x,t,\varphi(x,t)) - B(x,t,0)$. Therefore, due to Theorem 6, for given $\varphi \in BC^2(\Pi;\mathbb{R}^n)$ satisfying (5.1), the system

$$\partial_t V + A(x,t,\varphi)\partial_x V + B(x,t,\varphi)V = f(x,t) \tag{5.3}$$

with the boundary conditions (1.2) and with $U(x,t) = Q^{-1}(x,t,\varphi)V(x,t)$ has a unique solution $V^\varphi \in BC_t^2(\Pi,\mathbb{R}^n)$ such that $\partial_x V^\varphi \in BC_t^1(\Pi,\mathbb{R}^n)$. Moreover, the estimate (2.15) holds with v replaced by V^φ and is uniform in φ obeying (5.1). Since V^φ satisfies (5.3), it belongs to $BC^2(\Pi,\mathbb{R}^n)$.

Put $V^0(x,t) = 0$. For a given nonnegative integer number k, construct the iteration $V^{k+1}(x,t)$ as the unique bounded classical solution to the linear system

$$\partial_t V^{k+1} + A(x,t,V^k)\partial_x V^{k+1} + B(x,t,V^k)V^{k+1} = f(x,t) \tag{5.4}$$

subjected to the boundary conditions

$$U_j^{k+1}(x_j,t) = (RZ^{k+1})_j(t) + h_j(t), \quad j \le n, \tag{5.5}$$

where

$$Z^{k+1}(t) = \left(U_1^{k+1}(1,t),\dots,U_m^{k+1}(1,t),U_{m+1}^{k+1}(0,t),\dots,U_n^{k+1}(0,t)\right)$$

and

$$U^{k+1}(x,t) = Q^{-1}(x,t,V^k)V^{k+1}(x,t). \tag{5.6}$$

The function U^{k+1} then satisfies the system

$$\partial_t U^{k+1} + \hat{a}(x,t,V^k)\partial_x U^{k+1} + \hat{b}(x,t,V^k)U^{k+1} = Q^{-1}(x,t,V^k)f(x,t),$$

where

$$\begin{aligned} \hat{a}(x,t,V^k) &= \operatorname{diag}\left(A_1(x,t,V^k),\dots,A_n(x,t,V^k)\right), \\ \hat{b}(x,t,V^k) &= (Q^k)^{-1}\left(B^k Q^k + \partial_t Q^k + A^k\partial_x Q^k\right). \end{aligned} \tag{5.7}$$

Here and below in this proof we also use the short notation A^k, B^k, Q^k, a^k, and b^k for $A(x,t,V^k)$, $B(x,t,V^k)$, $Q(x,t,V^k)$, $\hat{a}(x,t,V^k)$, and $\hat{b}(x,t,V^k)$, respectively.

We divide the proof into a number of claims.

Claim 7. Suppose that

$$\|f\|_{BC_t^2} + \|h\|_{BC^2} \le \delta_1/K_2, \tag{5.8}$$

where K_2 is the constant as in Part 2 of Theorem 6. Then there exists a sequence V^k of bounded classical solutions to (5.4)–(5.6) belonging to $BC^2(\Pi;\mathbb{R}^n)$ such that

$$\|V^k\|_{BC_t^2} + \|\partial_x V^k\|_{BC_t^1} \le \delta_1 \quad \text{for all } k. \tag{5.9}$$

Proof of Claim. Note that the first iteration V^1 satisfies the system (2.1) with $g = f$ and the boundary conditions (2.5), (2.7). Due to Theorem 5, there exists a unique bounded classical solution V^1 such that $V^1 \in BC_t^2(\Pi,\mathbb{R}^n)$ and $\partial_x V^1 \in BC_t^1(\Pi,\mathbb{R}^n)$. Since A^0, Q^0, and B^0 are continuously differentiable in x, from the system (5.4) differentiated in x it follows that $V^1 \in BC^2(\Pi,\mathbb{R}^n)$. Moreover, V^1 satisfies the bound (2.15) with v and g replaced by V^1 and f, respectively. Since f and h obey (5.8), the estimate (5.9) with $k=1$ follows. Due to (5.1)–(5.2), we then have

$$\begin{aligned} \|A^1 - A^0\|_{BC_t^2} + \|\partial_x A^1 - \partial_x A^0\|_{BC_t^1} \le \varepsilon_1, \quad \|B^1 - B^0\|_{BC_t^2} \le \varepsilon_1, \\ \|Q^1 - Q^0\|_{BC_t^2} + \|\partial_x Q^1 - \partial_x Q^0\|_{BC_t^1} \le \varepsilon_1. \end{aligned} \tag{5.10}$$

Theorem 6 now implies that there exists a unique bounded classical solution V^2 such that $V^2 \in BC_t^2(\Pi,\mathbb{R}^n)$ and $\partial_x V^2 \in BC_t^1(\Pi,\mathbb{R}^n)$. Similarly, $V^2(x,t)$ fulfills the bound (5.9) with $k=2$ and, due to (5.4), belongs to $BC^2(\Pi,\mathbb{R}^n)$. On the account of (5.1)–(5.2), we also have the estimates (5.10) with A^1, B^1, and Q^1 replaced by A^2, B^2, and Q^2, respectively.

Proceeding by induction, assume that the problem (5.4)–(5.6) has a unique bounded classical solution V^k belonging to $BC^2(\Pi,\mathbb{R}^n)$ and satisfying the estimate (5.9) and, hence the estimates

$$\begin{aligned} \|A^k - A^0\|_{BC_t^2} + \|\partial_x A^k - \partial_x A^0\|_{BC_t^1} \le \varepsilon_1, \quad \|B^k - B^0\|_{BC_t^2} \le \varepsilon_1, \\ \|Q^k - Q^0\|_{BC_t^2} + \|\partial_x Q^k - \partial_x Q^0\|_{BC_t^1} \le \varepsilon_1. \end{aligned} \tag{5.11}$$

Now, using Theorem 6 and the system (5.4) differentiated in x, we conclude that the problem (5.4)–(5.6) has a unique bounded classical solution $V^{k+1} \in BC^2(\Pi,\mathbb{R}^n)$.

Moreover, this solution fulfills the inequalities (5.9) and (5.11) with $k+1$ in place of k.

□

Claim 8. There exists $\varepsilon_2 \le \delta_1/K_2$ such that, if $\|f\|_{BC_t^2} + \|h\|_{BC^2} < \varepsilon_2$, then the sequence V^k of solutions to the problem (5.4)–(5.6) converges in $BC^1(\Pi;\mathbb{R}^n)$ to a classical solution to (1.1)–(1.3).

Proof of Claim. Set

$$w^{k+1} = V^{k+1} - V^k = Q^k U^{k+1} - Q^{k-1} U^k, \tag{5.12}$$

$$Y^{k+1} = \left(Q^k\right)^{-1} w^{k+1}. \tag{5.13}$$

Hence,

$$Y^{k+1} = U^{k+1} - U^k + \chi^k, \tag{5.14}$$

where $\chi^k = (Q^k)^{-1}\left(Q^k - Q^{k-1}\right)U^k$.

First we derive a boundary value problem for w^{k+1}. To this end, introduce the following notation:

$$\begin{aligned}
\bar{\chi}^k(t) &= \left(\chi_1^k(1,t),\dots,\chi_m^k(1,t),\chi_{m+1}^k(0,t),\dots,\chi_n^k(0,t)\right),\\
\bar{Y}^{k+1}(t) &= Z^{k+1} - Z^k + \bar{\chi}^k\\
&= \left(Y_1^{k+1}(1,t),\dots,Y_m^{k+1}(1,t),Y_{m+1}^{k+1}(0,t),\dots,Y_n^{k+1}(0,t)\right),\\
\zeta_j^k(t) &= -\left[R\left(\bar{\chi}^k\right)\right]_j(t) + \chi_j^k(x_j,t),\ \ j\le n.
\end{aligned}$$

On the account of (5.13) and (5.14), the boundary conditions (5.5) with respect to Y^{k+1} can be written as follows:

$$Y_j^{k+1}(x_j,t) = \left[RZ^{k+1}\right]_j(t) - \left[RZ^k\right]_j(t) + \chi_j^k(x_j,t), \quad j\le n,$$

or, in the above notation, as

$$Y_j^{k+1}(x_j,t) = \left[R\left(\bar{Y}^{k+1}\right)\right]_j(t) + \zeta_j^k(t), \quad j\le n. \tag{5.15}$$

Therefore, the function w^{k+1} is the classical BC^2 solution to the system

$$\partial_t w^{k+1} + A(x,t,V^k)\partial_x w^{k+1} + B(x,t,V^k)w^{k+1} = f^k(x,t) \tag{5.16}$$

with the boundary conditions (5.13), (5.15), where

$$\begin{aligned}
f^k(x,t) &= -\left(B^k - B^{k-1}\right)V^k - \left(A^k - A^{k-1}\right)\partial_x V^k\\
&= -\int_0^1 \partial_3 B\left(x,t,\sigma V^k(x,t) + (1-\sigma)V^{k-1}(x,t)\right)\mathrm{d}\sigma\, w^k(x,t)V^k(x,t)
\end{aligned}$$

$$-\int_0^1 \partial_3 A\left(x,t,\sigma V^k(x,t)+(1-\sigma)V^{k-1}(x,t)\right)d\sigma\, w^k(x,t)\partial_x V^k(x,t).$$

Now we show that the sequence w^{k+1} converges to zero in $BC_t^1(\Pi;\mathbb{R}^n)$. By Claim 7, the functions V^k and V^{k-1} satisfy the same estimate (5.9). On the account of (5.1)–(5.2), there exists a constant N_1 not depending on V^k, V^{k-1}, and w^k such that

$$\begin{aligned}\|f^k\|_{BC_t^1} &\le N_1\left(\|V^k\|_{BC^1}+\|\partial_x V^k\|_{BC_t^1}\right)\|w^k\|_{BC_t^1}\\ &\le N_1K_2\left(\|f\|_{BC_t^2}+\|h\|_{BC^2}\right)\|w^k\|_{BC_t^1}.\end{aligned} \tag{5.17}$$

Similarly we obtain the bound

$$\|\zeta^k\|_{BC^1} \le N_1K_2\left(\|f\|_{BC_t^2}+\|h\|_{BC^2}\right)\|w^k\|_{BC_t^1}, \tag{5.18}$$

where the constant N_1 does not depend on V^k, V^{k-1}, and w^k and is chosen to satisfy both the inequalities (5.17) and (5.18). By Part 1 of Theorem 6, the solution w^{k+1} to the problem (5.16), (5.13), (5.15) satisfies the estimate (2.14) with v, f, and h replaced by w^{k+1}, f^k, and ζ^k, respectively. Combining this estimate with (5.17)–(5.18), we derive the inequality

$$\begin{aligned}\|w^{k+1}\|_{BC_t^1} &\le K_1\left(\|f^k\|_{BC_t^1}+\|\zeta^k\|_{BC^1}\right)\\ &\le K_1K_2N_1\left(\|f\|_{BC_t^2}+\|h\|_{BC^2}\right)\|w^k\|_{BC_t^1}.\end{aligned} \tag{5.19}$$

Set

$$\varepsilon_2=\min\left\{\delta_1/K_2,\,(K_1K_2N_1)^{-1}\right\}. \tag{5.20}$$

If $\|f\|_{BC_t^2}+\|h\|_{BC^2}<\varepsilon_2$, then, due to (5.19), the sequence w^k is strictly contracting in $BC_t^1(\Pi;\mathbb{R}^n)$ and, hence tends to zero in $BC_t^1(\Pi;\mathbb{R}^n)$.

By the inequality (5.9) and the assumptions of Theorem 6, the inverse $(A^k)^{-1}$ exists for every k and, moreover, is bounded in $BC(\Pi;\mathbb{R}^n)$ uniformly in k. Now, the equation (5.16) yields

$$\begin{aligned}&\|\partial_x w^{k+1}\|_{BC} \le \|(A^k)^{-1}\|_{BC}\left(\|f^k\|_{BC}+\|\partial_t w^{k+1}\|_{BC}+\|B^k\|_{BC}\|w^{k+1}\|_{BC}\right)\\ &\le K_2N_1\|(A^k)^{-1}\|_{BC}\left(1+K_1+K_2\|B^k\|_{BC}\right)\left(\|f\|_{BC_t^2}+\|h\|_{BC^2}\right)\|w^k\|_{BC_t^1},\end{aligned} \tag{5.21}$$

which together with (5.19) gives the convergence $\|w^{k+1}\|_{BC^1}\to 0$ as $k\to\infty$.

Finally, because of (5.12), the sequence V^k converges to some function V^* in $BC^1(\Pi;\mathbb{R}^n)$. It is a simple matter to show that V^* is a classical solution to the problem (1.1)–(1.3). The proof of the claim is complete. □

Claim 9. There exist positive constants $\varepsilon\le\varepsilon_2$ and $\delta\le\delta_1$ such that, if $\|f\|_{BC_t^2}+\|h\|_{BC^2}\le\varepsilon$, then the classical solution V^* belongs to $BC^2(\Pi;\mathbb{R}^n)$ and satisfies the estimate

$$\|V^*\|_{BC_t^2}+\|\partial_x V^*\|_{BC_t^1}\le\delta. \tag{5.22}$$

Proof of Claim. we start with proving that the sequence V^k converges in $BC^2(\Pi;\mathbb{R}^n)$. First show that the sequence $W^{k+1} = (Q^k)^{-1}\partial_t V^{k+1} = \partial_t U^{k+1} + (Q^k)^{-1}(\partial_t Q^k + \partial_3 Q^k Q^{k-1} W^k)U^{k+1}$ converges in $BC_t^1(\Pi;\mathbb{R}^n)$. To this end, we differentiate the problem (5.4)–(5.6) with respect to t and, similarly to (4.25) and (4.26), write down the resulting problem in the diagonal form with respect to W^{k+1}, as follows:

$$\begin{aligned}\partial_t W^{k+1} + \hat{a}(x,t,V^k)\partial_x W^{k+1} + b^1(x,t,V^k)W^{k+1} \\ = g^{1k}(x,t,W^k)W^{k+1} + g^{2k}(x,t,W^k),\end{aligned} \tag{5.23}$$

$$W_j^{k+1}(x_j,t) = (\widetilde{R}y^{k+1})_j(t) + h_j^k(t,W^k),\ \ j \le n, \tag{5.24}$$

where

$$\begin{aligned}
b^1(x,t,V^k) &= b^k - (Q^k)^{-1}\partial_t A^k (A^k)^{-1} Q^k,\\
g^{1k}(x,t,W^k) &= (Q^k)^{-1}\partial_3 A^k Q^{k-1} W^k (A^k)^{-1} Q^k - (Q^k)^{-1}\partial_3 Q^k Q^{k-1} W^k\\
&\quad -(Q^k)^{-1} A^k \partial_3 Q^k (A^{k-1})^{-1}\left(f - Q^{k-1}W^k - B^{k-1}V^k\right),\\
g^{2k}(x,t,W^k) &= (Q^k)^{-1}\Big(-\partial_t B^k V^{k+1} + \partial_t A^k\,(A^k)^{-1} B^k V^{k+1}\\
&\quad +\partial_t f - \partial_t A^k\,(A^k)^{-1} f - \partial_3 B^k Q^{k-1} W^k V^{k+1}\\
&\quad +\partial_3 A^k\,Q^{k-1} W^k (A^k)^{-1}(B^k V^{k+1} - f)\Big),\\
h_j^k(t,W^k) &= \left(R' Z^{k+1}\right)_j(t) - (\widetilde{R}\rho^k)_j(t) + h'_j(t) + \varrho_j^k(x_j,t),\ \ j\le n,\\
y^{k+1}(t) &= \left(W_1^{k+1}(1,t),\dots,W_m^{k+1}(1,t),W_{m+1}^{k+1}(0,t),\dots,W_n^{k+1}(0,t)\right),\\
\rho^k(t) &= \left(\varrho_1^k(1,t),\dots,\varrho_m^k(1,t),\varrho_{m+1}^k(0,t),\dots,\varrho_n^k(0,t)\right),\\
\varrho_j^k(x,t) &= \left[(Q^k)^{-1}(\partial_t Q^k + \partial_3 Q^k Q^{k-1} W^k)U^{k+1}\right]_j(x,t).
\end{aligned}$$

It is evident that the sequence W^{k+1} of solutions to the problem (5.23)–(5.26) converges in $BC_t^1(\Pi;\mathbb{R}^n)$ if and only if the sequence $W_t^{k+1} = \partial_t W^{k+1}$ converges in $BC(\Pi;\mathbb{R}^n)$. To prove the last statement, we differentiate the system (5.23) in the distributional sense and the boundary conditions (5.26) pointwise in t. We, therefore, obtain the following problem with respect to W_t^{k+1}:

$$\partial_t W_t^{k+1} + a^k\partial_x W_t^{k+1} + b^{2k} W_t^{k+1} = \mathcal{G}_1(k) W_t^{k+1} + \mathcal{G}_2(k) W_t^k + g^{3k}, \tag{5.25}$$

$$(W_t)_j^{k+1}(x_j,t) = (\widehat{R}(y^{k+1})')_j(t) + \left[\mathcal{H}(k)W_t^k\right]_j(t) + \tilde{h}_j^k(t),\ \ j\le n, \tag{5.26}$$

where

$$\begin{aligned}
b^{2k} &= b^{1k} - \partial_t a^k\,(a^k)^{-1},\\
g^{3k} &= \Big(\partial_t g^{1k} - \partial_3 b^{1k} Q^{k-1} W^k\\
&\quad + \left(\partial_3 a^k Q^{k-1} W^k (a^k)^{-1} + \partial_t a^k (a^k)^{-1}\right)(b^{1k} - g^{1k}) - \partial_t b^{1k}\Big) W^{k+1}
\end{aligned}$$

$$
\begin{aligned}
&+\partial_t g^{2k} - \partial_3 a^k Q^{k-1} W^k (a^k)^{-1} g^{2k} - \partial_t a^k (a^k)^{-1} g^{2k},\\
\tilde{h}_j^k(t) &= (\widetilde{R}' y^{k+1})_j(t) + \partial_t h_j^k(t, W^k).
\end{aligned}
$$

Moreover, b^{1k} is used to denote the function $b^1(x, t, V^k)$, while the operators $\mathcal{G}_1(k), \mathcal{G}_2(k) \in \mathcal{L}(BC(\Pi; \mathbb{R}^n))$ and $\mathcal{H}_j(k) \in \mathcal{L}(BC(\Pi; \mathbb{R}^n); BC(\mathbb{R}; \mathbb{R}^n))$ are defined by

$$
\begin{aligned}
\left[\mathcal{G}_1(k) W_t^{k+1}\right](x, t) &= \left(g^{1k} + \partial_3 a^k Q^{k-1} W^k (a^k)^{-1}\right) W_t^{k+1},\\
\left[\mathcal{G}_2(k) W_t^{k}\right](x, t) &= \partial_3 g^{1k} W_t^k W^{k+1} + \partial_3 g^{2k} W_t^k,\\
\left[\mathcal{H}(k) W_t^{k}\right]_j(t) &= \partial_2 h_j^k(t, W^k) W_t^k(x_j, t), \quad j \le n.
\end{aligned}
$$

Similarly to Claim 5 in the proof of Theorem 5, the function W_t^{k+1} satisfies (5.25) in the distributional sense and (5.26) pointwise if and only if it satisfies the following operator equation:

$$
\begin{aligned}
W_t^{k+1} &= C(k) W_t^{k+1} + D(k) W_t^{k+1}\\
&\quad + F(k)\left(\mathcal{G}_1(k) W_t^{k+1} + \mathcal{G}_2(k) W_t^k + g^{3k}, \mathcal{H}(k) W_t^k + \tilde{h}^k\right),
\end{aligned}
\tag{5.27}
$$

where the operators $C(k)$, $D(k)$, and $F(k)$ are defined by the right-hand sides of the corresponding formulas in (2.11) with a, b, and R replaced, respectively, by a^k, b^{2k}, and $\widehat{R}$. Moreover, the functions ω_j, c_j, and d_j are replaced appropriately by ω_j^k, c_j^k, and d_j^k. Note that computing $C(k) W_t^{k+1}$, we put $z = (y^{k+1})'$ in the right-hand side of the first formula in (2.11).

Iterating (5.27), we get

$$
\begin{aligned}
W_t^{k+1} &= C(k) W_t^{k+1} + \left(D(k) C(k) + D^2(k)\right) W_t^{k+1}\\
&\quad + (I + D(k)) F(k)\left(\mathcal{G}_1(k) W_t^{k+1} + \mathcal{G}_2(k) W_t^k + g^{3k}, \mathcal{H}(k) W_t^k + \tilde{h}^k\right).
\end{aligned}
\tag{5.28}
$$

Now we intend to show that there exists $\delta_2 \le \delta_1$ such that, given a nonnegative integer k and V^k satisfying the estimate (5.9) with δ_2 in place of δ_1, the formula (5.28) is equivalent to the following one:

$$
W_t^{k+1} = \mathcal{A}(k) W_t^k + X^k, \tag{5.29}
$$

where $\mathcal{A}(k) \in \mathcal{L}(BC(\Pi; \mathbb{R}^n))$ and $X^k \in BC(\Pi; \mathbb{R}^n)$ are given by

$$
\begin{aligned}
\mathcal{A}(k) W &= [I - C(k) - (I + D(k)) F(k) (\mathcal{G}_1(k), 0)]^{-1}\\
&\quad \times (I + D(k)) F(k) (\mathcal{G}_2(k) W, \mathcal{H}(k) W),\\
X^k &= [I - C(k) - (I + D(k)) F(k) (\mathcal{G}_1(k), 0)]^{-1} \left(D(k) C(k) + D^2(k)\right) W_t^{k+1}\\
&\quad + [I - C(k) - (I + D(k)) F(k) (\mathcal{G}_1(k), 0)]^{-1} (I + D(k)) F(k) \left(g^{3k}, \tilde{h}^k\right).
\end{aligned}
\tag{5.30}
$$

It suffices to show that, for every nonnegative integer k, the operator $I - C(k) - (I + D(k)) F(k) (\mathcal{G}_1(k), 0)$ is invertible and has a bounded inverse. Even more, we will show that the inverse is bounded uniformly in k. With this aim, denote by $G_0(k)$ operator defined by the right-hand sides of (4.6), where a_j, b_{jj}, and ω_j are replaced, respectively, by a_j^k, b_{jj}^{2k}, and ω_j^k. Moreover, denote by $G_2(k)$ operator defined by the right-hand sides of the second formula in (1.7), where a_j, b_{jj}, and ω_j are replaced, respectively, by a_j^k, b_{jj}^k, and ω_j^k.

Note that, similarly to (4.28), we have $b_{jj}^{1k} = b_{jj}^k - (a_j^k)^{-1}\partial_t a_j^k$. Then, accordingly to the notation introduced above, the function b_{jj}^{2k} is given by the formula $b_{jj}^{2k} = b_{jj}^k - 2(a_j^k)^{-1}\partial_t a_j^k$. This means that the operators $G_0(k)$ and $G_2(k)$ coincide.

Therefore, on the account of the estimates (5.9) and (4.45), the operators $G_2(k)$ fulfill the inequality $\|G_2(k)\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} < 1 - \nu_3$ for all $k \in \mathbb{N}$ and, hence, the inequality $\|G_0(k)\|_{\mathcal{L}(BC(\mathbb{R},\mathbb{R}^n))} < 1 - \nu_3$ for all $k \in \mathbb{N}$.

Finally, similarly to the proof of the invertibility of $I - C$ in Sect. 4.1.1, the invertibility of $I - C(k)$ follows from the invertibility of $I - G_0(k)$ (see the inequality (4.9)). Furthermore, the following estimate is true for all $k \in \mathbb{N}$:

$$\|(I - C(k))^{-1}\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))} \le 1 + \nu_3^{-1}\|C(k)\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))}.$$

As the operators $C(k)$ are bounded uniformly in k, the inverse operators $(I - C(k))^{-1}$ are bounded uniformly in k also. Taking into account that the set of all invertible operators whose inverses are bounded is open, our task is, therefore, reduced to show that the operator $(I + D(k))F(k)\,(\mathcal{G}_1(k), 0)$ is sufficiently small whenever δ_1 is sufficiently small. Note that Claim 7 is true with δ_2 in place of δ_1 for any $\delta_2 \le \delta_1$. This implies that for any $\sigma > 0$ there is δ_2 such that for all V^k fulfilling (5.9) with δ_2 in place of δ_1, we have $\|\mathcal{G}_1(k)\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))} = \|g^1(x, t, W^k(x, t))\|_{BC} \le \sigma$ for all k. Moreover, the operators $D(k)$ and $F(k)$ are bounded, uniformly in k. Consequently, if δ_2 is sufficiently small, then for all $k \in \mathbb{N}$ and all f and h satisfying (5.8) with δ_2 in place of δ_1, the operator $I - C(k) - (I + D(k))F(k)\,(\mathcal{G}_1(k), 0)$ is invertible and the inverse is bounded by a constant not depending on k. Fix δ_2 satisfying the last property. The equivalence of (5.28) and (5.29) is, therefore, proved.

Now, to prove that the sequence W_t^{k+1} converges in $BC(\Pi;\mathbb{R}^n)$ as $k \to \infty$, we apply to the equation (5.29) a linear version of the fiber contraction principle, see [19, Lemma A.1]. Accordingly to [19, Lemma A.1], we have to show that, first,

$$X^k \text{ converges in } BC(\Pi;\mathbb{R}^n) \text{ as } k \to \infty, \tag{5.31}$$

second, that there exists $c < 1$ such that for all $W \in BC(\Pi;\mathbb{R}^n)$ it holds

$$\|\mathcal{A}(k)W\|_{BC} \le c\|W\|_{BC}, \tag{5.32}$$

and, third, that

$$\mathcal{A}(k)W \text{ converges in } BC(\Pi;\mathbb{R}^n) \text{ as } k \to \infty. \tag{5.33}$$

To show (5.31), note that the operators $D(k)$ and $C(k)$ depend neither on W_t^k nor on W^k. Similarly to the proof of Claim 2 in Sect. 4.2, one can show that the operators

$D(k)C(k)$ and $D(k)^2$ are smoothing and map $BC(\Pi;\mathbb{R}^n)$ into $BC^1_t(\Pi;\mathbb{R}^n)$. This implies that $D(k)C(k)W_t^{k+1}$ and $D(k)^2W_t^{k+1}$, actually, do not depend on W_t^{k+1}, but on W^{k+1}. Moreover, using (5.9), we get the following estimate:

$$\left\|\left(D(k)C(k)+D(k)^2\right)W_t^{k+1}\right\|_{BC}\le \widehat{K}_{11}\|W^{k+1}\|_{BC}$$

for some $\widehat{K}_{11}$ not depending on k. It follows that the right-hand side of the second formula in (5.30) does not depend on W_t^k for all k, and therefore, the convergence (5.31) immediately follows from Claim 8.

Since all the operators in the right-hand side of the first formula in (5.30) do not depend on W_t^k for all k, the convergence (5.33) follows again from Claim 8.

It remains to prove (5.32). Due to Claim 7, for any $\sigma>0$ there exists $\delta\le\delta_2$ such that for all f and h satisfying the bound (5.8) with δ in place of δ_1 (and, hence for V^k satisfying the bound (5.9) with δ in place of δ_1) it holds $\|\mathcal{G}_2(k)\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))}+\|\mathcal{H}(k)\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))}\le\sigma$. Moreover, if $\delta\le\delta_2$ is sufficiently small, then all other operators in the right-hand side of the first equality in (5.30) are bounded uniformly in k. We, therefore, conclude that there exists $\delta\le\delta_2$ such that (5.32) is fulfilled.

Set $\varepsilon=\min\left\{\delta/K_2,(K_1K_2N_1)^{-1}\right\}$ (see also (5.20)). Therefore, by Lemma [19, Lemma A.1], if $\|f\|_{BC_t^2}+\|h\|_{BC^2}<\varepsilon$, then the sequence W_t^{k+1} converges in $BC(\Pi;\mathbb{R}^n)$ as $k\to\infty$.

Finally, by Claim 8 and the equality $W^{k+1}=(Q^k)^{-1}\partial_tV^{k+1}$, we conclude that the second derivative of V^* in t exists and that the sequence $\partial_t^2V^k$ converges to $\partial_t^2V^*$ in $BC(\Pi;\mathbb{R}^n)$ as $k\to\infty$. Differentiating (5.4) first in t and then in x, we conclude that V^k converges to V^* in $BC^2(\Pi;\mathbb{R}^n)$ as $k\to\infty$.

The desired estimate (5.22) now easily follows from the bound (5.9) and the system (5.4) differentiated in x and t. □

Claim 10. Let ε and δ be as in Claim 9. Then there exists $\delta'=\delta'(\varepsilon,\delta)$ such that, if $\|f\|_{BC_t^2}+\|\partial_xf\|_{BC_t^1}+\|h\|_{BC^2}\le\varepsilon$, then $\|V^*\|_{BC^2}\le\delta'$.

Proof of Claim. Consider the system (1.1) with V replaced by V^*, differentiated in x. Using the fact that $A(x,t,V^*)$ has a bounded inverse and taking into account the estimate (5.22), we derive an upper bound for $\partial_x^2V^*$. One can easily see that this bound depends on ε and δ and that there exists $\delta'=\delta'(\varepsilon,\delta)$ such that $\|V^*\|_{BC^2}\le\delta'$, as desired. □

Claim 11. Let ε and δ be as in Claim 9. Then for any f and h such that $\|f\|_{BC_t^2}+\|h\|_{BC^2}\le\varepsilon$, the classical solution to the problem (1.1)–(1.3) fulfilling the estimate (5.22) is unique.

Proof of Claim. On the contrary, suppose that $\widetilde{V}$ is a classical solution to the problem (1.1)–(1.3) different from V^*, such that $\|\widetilde{V}\|_{BC_t^2}+\|\partial_x\widetilde{V}\|_{BC_t^1}\le\delta$. Then, due to (5.1)–(5.2), the functions $\widetilde{A}(x,t)=A(x,t,\tilde{V}(x,t))$, $\widetilde{Q}(x,t)=Q(x,t,\tilde{V}(x,t))$ and

$\widetilde{B}(x,t) = B(x,t,\tilde{V}(x,t))$ fulfill the inequalities

$$\|\widetilde{A} - A^0\|_{BC_t^2} + \|\partial_x \widetilde{A} - \partial_x A^0\|_{BC_t^1} \le \varepsilon_1, \quad \|\widetilde{B} - B^0\|_{BC_t^1} \le \varepsilon_1$$

$$\|\widetilde{Q} - Q^0\|_{BC_t^2} + \|\partial_x \widetilde{Q} - \partial_x Q^0\|_{BC_t^1} \le \varepsilon_1.$$

The difference $\tilde{w}^{k+1} = V^{k+1} - \widetilde{V}$ satisfies the system

$$\partial_t \tilde{w}^{k+1} + \widetilde{A}(x,t)\partial_x \tilde{w}^{k+1} + \widetilde{B}(x,t)\tilde{w}^{k+1} = \tilde{f}^k(x,t)$$

and the boundary conditions (5.13), (5.15) with w^{k+1} replaced by $\tilde{w}^{k+1}$ and with

$$Y^{k+1} = \widetilde{Q}^{-1}\tilde{w}^{k+1}, \quad \chi^k = \widetilde{Q}^{-1}(Q^k - \widetilde{Q})U^{k+1}.$$

Here $\widetilde{Q}(x,t) = Q\left(x,t,\widetilde{V}(x,t)\right)$ and

$$\tilde{f}^k(x,t) = \left(\widetilde{B}(x,t) - B^k(x,t)\right) V^{k+1}(x,t) + \left(\widetilde{A}(x,t) - A^k(x,t)\right) \partial_x V^{k+1}(x,t).$$

By the argument as in the proof of Claim 8, the functions $\tilde{f}^k(x,t)$ and $\zeta^k(x,t)$ are C^1-smooth in t and satisfy the upper bounds (5.17) and (5.18) with f^k and w^k replaced by $\tilde{f}^k$ and $\tilde{w}^k$, respectively.

Similarly to (5.19) and (5.21), we derive the bounds

$$\|\tilde{w}^{k+1}\|_{BC_t^1} \le K_1 K_2 N_1 \left(\|f\|_{BC_t^2} + \|h\|_{BC^2}\right) \|\tilde{w}^k\|_{BC_t^1},$$

$$\|\partial_x \tilde{w}^{k+1}\|_{BC} \le \frac{1}{\Lambda_0} K_2 N_1 \left(1 + K_1 + K_2 \|B^k\|_{BC}\right) \left(\|f\|_{BC_t^2} + \|h\|_{BC^2}\right) \|\tilde{w}^k\|_{BC_t^1}.$$

The desired convergence $\|\tilde{w}^k(t)\|_{BC^1} \to 0$ as $k \to \infty$ follows. This means that $\widetilde{V}(x,t) = V^*(x,t)$, contradicting to our assumption. □

5.2. Proof of Part 2 of Theorem 1: Almost periodic solutions

We have to prove that the constructed solution $V^*(x,t)$ is Bohr almost periodic in t. The proof uses the fact that the limit of a uniformly convergent sequence of Bohr almost periodic functions depending uniformly on parameters is almost periodic uniformly in parameters [6, p. 57]. Moreover, we will use the fact that, if a function $w(x,t)$ has bounded and continuous partial derivatives up to the second order in both $x \in [0,1]$ and in $t \in \mathbb{R}$ and is Bohr almost periodic in t uniformly in x (or, simply, almost periodic), the last property is true for $\partial_x w(x,t)$ and $\partial_t w(x,t)$ also. Specifically, the almost periodicity of $\partial_t w(x,t)$ follows from [6, Theorem 2.5], while the almost periodicity of $\partial_x w(x,t)$ is shown in [20, Section 5.2]. We are, therefore, reduced to showing that the approximating sequence V^k, constructed in Sect. 5.1, is a sequence of almost periodic functions.

We use the induction on k. Recall that $V^0 \equiv 0$. Assuming that the iteration $V^k(x,t)$ is Bohr almost periodic for an arbitrary fixed $k \in \mathbb{N}$, let us prove that

$V^{k+1}(x,t)$ is almost periodic also. By the assumptions of the theorem, the matrices $A(x,t,V^k(x,t))$, $B(x,t,V^k(x,t))$, $Q(x,t,V^k(x,t))$, $\partial_x Q(x,t,V^k(x,t))$, and $\partial_t Q(x,t,V^k(x,t))$ are almost periodic as compositions of almost periodic functions. Below we will use a slightly modified notation for $\hat{a}(x,t,V^k)$ and $\hat{b}(x,t,V^k)$ (see (5.7)), namely $a^k(x,t) = \hat{a}(x,t,V^k(x,t))$ and $b^k(x,t) = \hat{b}(x,t,V^k(x,t))$. Set $q^k(x,t) = Q(x,t,V^k(x,t))$. It follows that a^k and b^k are almost periodic. Fix $\mu > 0$ and let τ be a μ-almost period of the matrices a^k, q^k and b^k. Then the differences $\tilde{a}^k(x,t) = a^k(x,t+\tau) - a^k(x,t)$, $\tilde{b}^k(x,t) = b^k(x,t+\tau) - b^k(x,t)$, and $\tilde{q}^k(x,t) = q^k(x,t+\tau) - q^k(x,t)$ satisfy the inequalities

$$\|\tilde{a}^k\|_{BC} \le \mu, \quad \|\tilde{b}^k\|_{BC} \le \mu, \quad \|\tilde{q}^k\|_{BC} \le \mu \tag{5.34}$$

uniformly in x and t.

First derive a few simple estimates. Let $\omega_j^k(\xi,x,t)$ be the solution to the equation (1.5) where a_j is replaced by a_j^k. Then the following identity is true:

$$\frac{d}{d\eta}\left(\omega_j^k(\eta,x,t) + \tau - \omega_j^k(\eta,x,t+\tau)\right) = \frac{1}{a_j^k(\eta,\omega_j^k(\eta,x,t))} - \frac{1}{a_j^k(\eta,\omega_j^k(\eta,x,t+\tau))}.$$

Since $\omega_j^k(x,x,t) = t$ and $\omega_j^k(x,x,t+\tau) = t+\tau$, it holds

$$\begin{aligned}
&\omega_j^k(\eta,x,t) + \tau - \omega_j^k(\eta,x,t+\tau)\\
&\quad = \int_x^\eta \left(\frac{1}{a_j^k(\xi,\omega_j^k(\xi,x,t))} - \frac{1}{a_j^k(\xi,\omega_j^k(\xi,x,t+\tau))}\right) d\xi\\
&\quad = \int_x^\eta \frac{a_j^k(\xi,\omega_j^k(\xi,x,t+\tau)) - a_j^k(\xi,\omega_j^k(\xi,x,t)+\tau)}{a_j^k(\xi,\omega_j^k(\xi,x,t))a_j^k(\xi,\omega_j^k(\xi,x,t+\tau))} d\xi\\
&\qquad + \int_x^\eta \frac{a_j^k(\xi,\omega_j^k(\xi,x,t)+\tau) - a_j^k(\xi,\omega_j^k(\xi,x,t))}{a_j^k(\xi,\omega_j^k(\xi,x,t))a_j^k(\xi,\omega_j^k(\xi,x,t+\tau))} d\xi.
\end{aligned} \tag{5.35}$$

By (5.34),

$$|a_j^k(\xi,\omega_j^k(\xi,x,t)+\tau) - a_j^k(\xi,\omega_j^k(\xi,x,t))| \le \mu,$$

the estimate being uniform in ξ, x, t, and j. Due to the mean value theorem,

$$\begin{aligned}
&a_j^k(\xi,\omega_j^k(\xi,x,t+\tau)) - a_j^k(\xi,\omega_j^k(\xi,x,t)+\tau) = (\omega_j^k(\xi,x,t+\tau) - \omega_j^k(\xi,x,t) - \tau)\\
&\quad \times \int_0^1 \partial_2 a_j^k\left(\xi, \alpha\omega_j^k(\xi,x,t+\tau) + (1-\alpha)(\omega_j^k(\xi,x,t)+\tau\right) d\alpha.
\end{aligned}$$

Applying the Gronwall's inequality to the identity (5.35), we derive the estimate

$$\left|\omega_j^k(\eta,x,t) + \tau - \omega_j^k(\eta,x,t+\tau)\right| \le \frac{\mu}{\Lambda_0^2} \exp\left\{\frac{\|a_j^k\|_{BC_t^1}}{\Lambda_0^2}\right\} = L_1\mu, \tag{5.36}$$

the constant L_1 being independent of μ, η, x, t, and j.

Next we show that $a_j^k(\eta, \omega_j^k(\eta, x, t))$ and $b_{ji}^k(\eta, \omega_j^k(\eta, x, t))$ are almost periodic. For that, we use (5.36) and the fact that τ is a μ-almost period of a_j^k and b_{ji}^k. We get

$$\begin{aligned}
&\left|a_j^k(\eta, \omega_j^k(\eta, x, t)) - a_j^k(\eta, \omega_j^k(\eta, x, t+\tau))\right| \\
&\quad \le \left|a_j^k(\eta, \omega_j^k(\eta, x, t)) - a_j^k(\eta, \omega_j^k(\eta, x, t)+\tau)\right| \\
&\qquad + \left|a_j^k(\eta, \omega_j^k(\eta, x, t)+\tau) - a_j^k(\eta, \omega_j^k(\eta, x, t+\tau))\right| \\
&\quad \le \left(1 + L_1 \|\partial_t a_j^k\|_{BC}\right)\mu \le L_2\mu,
\end{aligned} \tag{5.37}$$

where L_2 does not depend on μ, η, x, t, and j. Similar estimates are true for b_{ji}^k and q_{ji}^k, namely

$$\begin{aligned}
&|b_{ji}^k(\eta, \omega_j^k(\eta, x, t)) - b_{ji}^k(\eta, \omega_j^k(\eta, x, t+\tau))| \le L_2\mu, \\
&|q_{ji}^k(\eta, \omega_j^k(\eta, x, t)) - q_{ji}^k(\eta, \omega_j^k(\eta, x, t+\tau))| \le L_2\mu,
\end{aligned} \tag{5.38}$$

where L_2 is chosen to be a common constant satisfying both (5.37) and (5.38).

Now we prove that

$$(Rv)_j(\omega_j^k(x_j, x, t)) \in AP(\Pi) \quad \text{for all} \quad v \in AP(\mathbb{R}, \mathbb{R}^n) \cap BC^1(\mathbb{R}, \mathbb{R}^n). \tag{5.39}$$

Fix an arbitrary $v \in AP(\mathbb{R}, \mathbb{R}^n) \cap BC^1(\mathbb{R}, \mathbb{R}^n)$. By the assumption, $(Rv)(t) \in AP(\mathbb{R}, \mathbb{R}^n)$. Let τ be a common μ-almost period of the functions $(Rv)(t)$ and $a^k(x, t)$. Applying the mean value theorem and using the assumption (**A3**) and the estimate (5.36), we get

$$\begin{aligned}
&\left|(Rv)_j(\omega_j^k(x_j, x, t)) - (Rv)_j(\omega_j^k(x_j, x, t+\tau))\right| \\
&\quad \le \left|(Rv)_j(\omega_j^k(x_j, x, t)) - (Rv)_j(\omega_j^k(x_j, x, t)+\tau)\right| \\
&\qquad + \left|(Rv)_j(\omega_j^k(x_j, x, t)+\tau) - (Rv)_j(\omega_j^k(x_j, x, t+\tau))\right| \\
&\quad \le \mu\left(1 + L_1 \sup_{t\in\mathbb{R}}\left|\frac{\mathrm{d}}{\mathrm{d}t}(Rv)_j(t)\right|\right),
\end{aligned}$$

which proves (5.39).

The estimates (5.37) and (5.38) imply that the functions in the right-hand sides of the equalities in (2.9) with a_j^k, b_{jj}^k, and ω_j^k in place of a_j, b_{jj}, and ω_j, respectively, are almost periodic for all $j \le n$, uniformly in $\xi, x \in [0, 1]$. Let $\widehat{C}(k)$, $\widehat{D}(k)$, and $\widehat{F}(k)$ be defined by the right-hand side of (2.11) with a_j, b_{jj}, and ω_j replaced by a_j^k, b_{jj}^k, and ω_j^k, respectively. Taking into account (5.39), we conclude that the operators $\widehat{C}(k)$, $\widehat{D}(k)$, and $\widehat{F}(k)$ map the space $AP(\Pi, \mathbb{R}^n) \cap BC_t^1(\Pi, \mathbb{R}^n)$ into itself.

Let the condition (**B1**) be fulfilled. Due to the proof of Theorem 3, this yields

$$\|\widehat{C}(k)\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))} + \|\widehat{D}(k)\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))} < 1.$$

Hence, the operator $I - \widehat{C}(k) - \widehat{D}(k)$ is bijective from $BC(\Pi; \mathbb{R}^n))$ into itself. As a consequence, the solution $U^{k+1} \in BC(\Pi; \mathbb{R}^n)$ to the equation $U^{k+1} = (\widehat{C}(k) + \widehat{D}(k))U^{k+1} + \widehat{F}(k)(f, h)$ is given by the (uniformly convergent) Neumann series

$$U^{k+1} = \left(I - \widehat{C}(k) - \widehat{D}(k)\right)^{-1} \widehat{F}(k)(f, h) = \sum_{j=0}^{\infty} \left(\widehat{C}(k) + \widehat{D}(k)\right)^{j} \widehat{F}(k)(f, h). \tag{5.40}$$

Since the functions f and h are continuously differentiable in t, the function $F(k)(f, h)$ belongs to $BC_t^1(\Pi, \mathbb{R}^n)$. Moreover,

$$\left(\widehat{C}(k) + \widehat{D}(k)\right)^{j} \text{ maps } AP(\Pi, \mathbb{R}^n) \cap BC_t^1(\Pi, \mathbb{R}^n) \text{ to } AP(\Pi, \mathbb{R}^n)$$

for each j. Therefore, the right-hand side and, hence, the left-hand side of (5.40) belong to $AP(\Pi, \mathbb{R}^n)$. This means that that the function $V^{k+1} = Q^k U^{k+1}$ belongs to $AP(\Pi, \mathbb{R}^n)$, as desired.

If the condition (**B2**) (resp., (**B3**)) is fulfilled, then we use a similar argument. More precisely, we consider the formula (4.9) with C and G_0 (resp., with C and H_0) replaced appropriately by $\widehat{C}(k)$ and $\widehat{G}_0(k)$ (resp., by $\widehat{C}(k)$ and $\widehat{H}_0(k)$). Taking into account the inequalities $\|\widehat{G}_0(k)\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))} < 1$ (resp., $\|\widehat{H}_0(k)\|_{\mathcal{L}(BC(\Pi;\mathbb{R}^n))} < 1$), we use the Neumann series representation for the operator $(I - \widehat{G}_0(k))^{-1}$ (resp., $(I - \widehat{H}_0(k))^{-1}$) to conclude that the iterated solution U^{k+1} and, hence, $V^{k+1} = Q^k U^{k+1}$ belong to $AP(\Pi, \mathbb{R}^n)$. The proof is therefore complete.

5.3. Proof of Part 2 of Theorem 1: Periodic solutions

We follow the proof of the almost periodic case in Sect. 5.2, on each step referring to the periodicity instead of the almost periodicity. Obvious simplifications in the proof are caused by the identity $\omega_j^k(\eta, x, t) + T = \omega_j^k(\eta, x, t + T)$.

5.4. Proof of Theorem 2: Bounded solutions for space-periodic problems

The proof of Theorem 2 repeats the proof of Theorem 1, with the only difference being that we need to refer to Theorem 4 instead of Theorem 3.

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