Swap vega in BGM: pitfalls and alternatives

Swap Vega in BGM: Pitfall and Alternative

Raoul Pietersz[1] and Antoon Pelsser[2]

This paper contains a warning for practitioners who are developing the Libor BGM model for risk management of a swap-based interest rate derivative. Namely for certain volatility functions the estimate of swap vega may be poor. An explanation for this phenomenon is given and a simple alternative method is developed that estimates vega with clarity at a low number of simulation paths for all volatility functions.

Introduction

The Libor BGM interest rate model was introduced by Brace, Gątarek, Musiela (1997) and others. This model is presently most popular amongst both academics and practitioners alike. To name a reason, the Libor BGM model has the potential of risk managing exotic interest rate derivatives that depend on both the cap and swaption markets. BGM has the potential to become the *central interest rate model*. It features lognormal Libor rates and almost lognormal swap rates and consequently also the market standard Black formula for caps and swaptions.

The choice of the volatility function allows for future volatility modeling. In this paper we show however that this introduces a pitfall when calculating swap vega. The swap vega is the sensitivity of a derivative with respect to an underlying swaption volatility. In combination with certain volatility functions BGM may[3] produce poorly estimated swap vega when these are calculated by re-calibration and with a low number of simulation paths, say 10,000. Incorrect swap vega leave practitioners with unknowingly taking on large uncovered positions and thus increasing the variance of profit and loss (P&L). Unstable vega lead to large and unnecessary transaction costs when rebalancing the hedging portfolio based on fluctuations of vega that are not really material.

Re-calibration approach

A common and usually very successful method for calculating a Greek in a model equipped with a calibration algorithm is to perturb market input, re-calibrate and then re-value the option. The difference in value divided by the perturbation size is then an estimate for the Greek. If however this technique is applied to the calculation of swap vega in the Libor BGM model, then it may (depending on the volatility function) yield estimates with large uncertainty. In other words, the

[1] Erasmus Research Institute of Management, Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands (email: pietersz@few.eur.nl, tel: +31 10 4088932, fax: +31 10 4089136) and Market Risk - Modelling and Product Analysis (HQ1059), ABN AMRO Bank, P.O. Box 283, 1000 EA Amsterdam, The Netherlands
[2] Econometric Institute, Faculty of Economics, Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands (email: pelsser@few.eur.nl, tel: +31 10 4081270, fax: +31 10 4089169) and Nationale Nederlanden, P.O. Box 796, 3000 AT Rotterdam, The Netherlands
[3] This does not happen with for example constant swap rate volatility.

standard error of the vega is relatively high. The uncertainty disappears of course by increasing the number of simulation paths, but the number required for clarity can by far supersede 10,000, which is probably the maximum in a practical environment. This large uncertainty in vega has been illustrated in the exhibit, for three volatility function cases.

[INSERT EXHIBIT SOMEWHERE HERE]

We see that for a constant volatility calibration the vega is estimated with low uncertainty. The number of simulation paths needed for clarity of vega thus severely depends on the chosen calibration. To understand the cause, we look at swap vega in the swap market model.

Swap vega and the swap market model

A swap market model features log-normally distributed swap rates. The implied swaption volatility in terms of instantaneous volatility is given by

(1) $$\sigma_{\text{Implied}} = \sqrt{\frac{1}{T}\int_0^T |\sigma(s)|^2 \, ds}$$

The swap vega is defined as the sensitivity of the value V of a derivative with respect to the implied swaption volatility:

$$\text{vega} = \frac{\partial V}{\partial \sigma_{\text{Implied}}}.$$

As may be seen from equation (1), there are an uncountable number of perturbations in the swap rate instantaneous volatility to obtain the very same perturbation in the Black implied swaption volatility. There is however a natural 1-dimensional parameterized perturbation, namely a simple proportional increment. This has been illustrated in Figure 1.

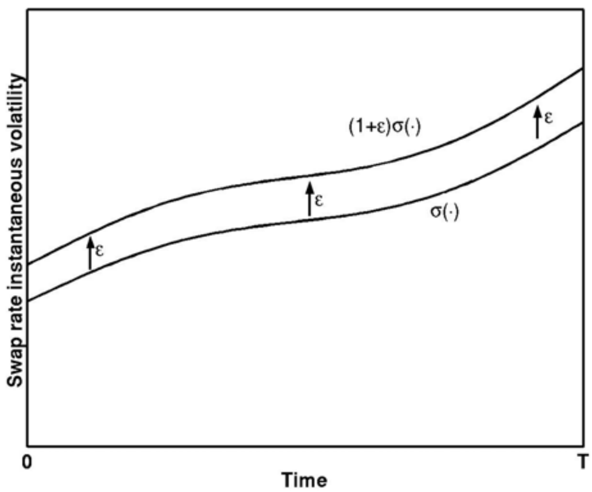

Figure 1: Natural increment of Black implied swaption volatility through proportional increment of the swap rate instantaneous volatility.

In terms of an equation,

(2) $\{\sigma(\cdot)\}_{\text{perturbed}} = (1+\varepsilon)\sigma(\cdot)$.

It may be shown that the above perturbation leads to:

- An increase in the implied volatility for the relevant swaption bucket.
- All other swaption volatilities remain unchanged.
- The swap rate correlation is not changed.

An alternative method

The method is based on a perturbation in the forward rate volatility to match a constant swap rate volatility increment. The method is briefly hinted at in section 10.6.3 of Rebonato, 2002, in terms of covariance matrices. In comparison, our derivation below is written more directly in terms of the volatility vectors.

In Hull and White, 2000, it is shown that the swap rate volatility vector is a weighted average of forward rate volatility vectors;

$$\overline{\sigma}_{i:N}(t) = \sum_{j=1}^{N} w_j^{i:N}(t)\overline{\sigma}_j(t).$$

and that a high-quality approximating formula for European swaption prices can be obtained by evaluating the weights at time zero. Write $w_j^{i:N} := w_j^{i:N}(0)$ and make the convention that $\overline{\sigma}_i(t) = \overline{\sigma}_{i:N}(t) = 0, t > T_i$, then these volatility vectors can be jointly related through the matrix equation

$$[\overline{\sigma}_{\bullet:N}(t)] = W[\overline{\sigma}_{\bullet}(t)].$$

The perturbation in swap rate volatility for the kth bucket prescribed by equation (2) is

$$[\overline{\sigma}_{\bullet:N}(t)] \to [\overline{\sigma}_{\bullet:N}(t)] + \varepsilon[0 \quad \cdots \quad 0 \quad \overline{\sigma}_{k:N}(t) \quad 0 \quad \cdots \quad 0]^{\mathrm{T}}.$$

The corresponding perturbation in the Libor volatility vectors is given by

$$[\overline{\sigma}_{\bullet}(t)] \to [\overline{\sigma}_{\bullet}(t)] + \varepsilon W^{-1}[0 \quad \cdots \quad 0 \quad \overline{\sigma}_{k:N}(t) \quad 0 \quad \cdots \quad 0]^{\mathrm{T}}$$

With the new Libor volatility vectors, prices can be recomputed in the BGM model and the vegas calculated. The vega calculated with the alternative approach for the deal of the exhibit have been displayed in Figure 2.

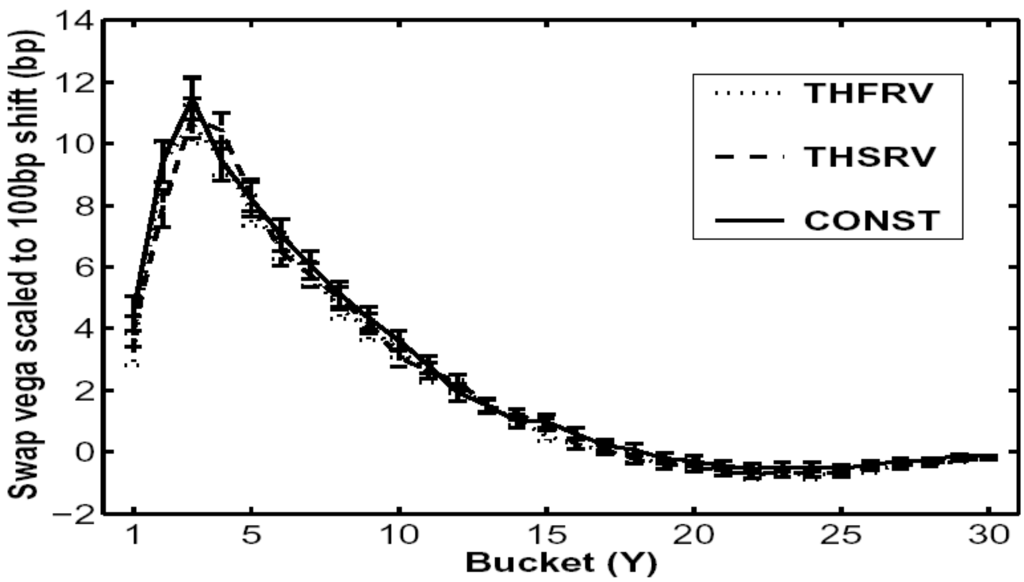

Figure 2: Vega calculated with the alternative approach. Error bars denote a 95% confidence bound based on twice the standard error.

Reason for poor estimation of vega for certain volatility functions

In this section an explanation is given for the poorly estimated vega of the re-calibration approach for certain volatility functions. To check whether the true swap rate dynamics are captured we simply have to verify that the swap rate volatility is perturbed as prescribed by equation (2). This test was performed for the THFRV deal of the exhibit. Results show that for the re-calibration approach the swap rate volatility increment (in the limit) is completely different than prescribed by equation (2). This holds for all buckets. For illustration we restrict to the exhibit in Figure 3.

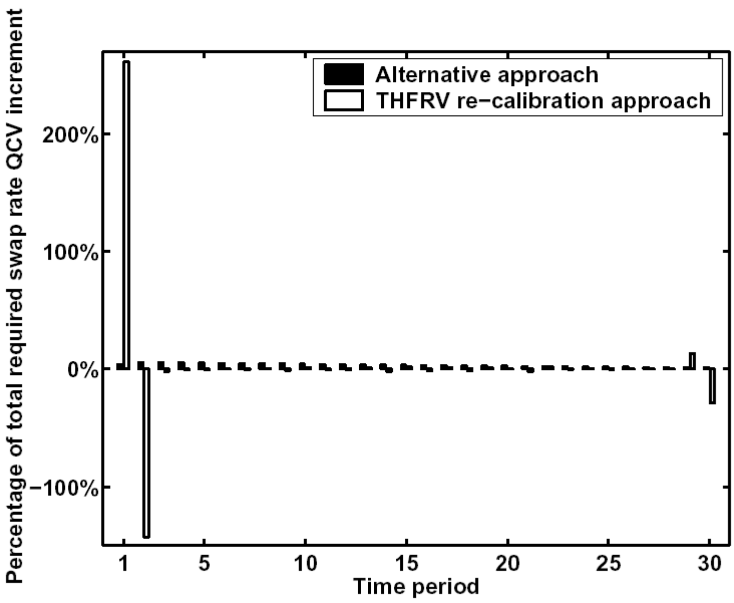

Figure 3: **Swap rate volatility dynamics for both the re-calibration and alternative approach in the setup of the exhibit. Concern here is the calculation of vega corresponding to bucket 30. To accomplish this, the price differential has to be computed in the limit of the 30 into 1 swaption**

implied volatility perturbation $\Delta\sigma$ tending to zero. This implies a swap rate quadratic variation increment of 30 $\Delta\sigma^2$. This total variation increment has to be distributed over all time periods. The first data set displays the constant volatility way of distributing the variation increment, namely proportional to the swap rate instantaneous volatility over the period. The second data set displays the distribution as implied by re-calibrating the THFRV model to the swaption volatilities. Note that for each data set the sum of the variation increments equals 100%.

To understand that the resulting vega is more hard to estimate for the THFRV/THSRV case, note that the vega is a multiple of an expectation of a difference in discounted payoffs in a model with either perturbed volatility or the original volatility,

$$\Psi_{i:N} = cE\left[P_{i:N} - P\right],$$

here $\Psi_{i:N}$ denotes the estimated vega for bucket i, c is the reciprocal of the perturbation size $\Delta\sigma_{i:N}$, and $P_{i:N}$ and P denote the discounted payoff in the perturbed and original model, respectively. The expectation under the risk neutral measure is denoted by $E[\bullet]$. The simulation variance of the vega is thus given by

$$\mathrm{Var}\left[\Psi_{\mathrm{i:N}}\right] = c^2 \mathrm{Var}\left[P_{i:N} - P\right] = c^2 \left\{ \mathrm{Var}\left[P_{i:N}\right] - 2\mathrm{Cov}\left[P_{i:N}, P\right] + \mathrm{Var}\left[P\right] \right\}$$

The vega standard error is thus minimized if the covariance between the discounted payoff in either the original and the perturbed model is largest. This occurs under small perturbations of volatility as implied by the constant volatility regime. In the presence of a perturbation such as the THFRV recalibration of Figure 3 however the stochasticity in the simulation basically is moved around to other *independent* stochastic increments, thereby decreasing the covariance. This leads to the large uncertainty in the vega.

Comparison with the swap market model

An approximate equivalence between the Libor and swap market models was established in Joshi and Theis, 2002. On the basis of this equivalence, vega calculated with the alternative approach for the Libor market model were shown to be largely similar to those obtained in a swap market model. The reader is referred to Pietersz and Pelsser, 2003, for this and further details on the whole topic described in this text.

Conclusions

We showed that care should be taken when calculating swap vega per bucket in the Libor BGM model by re-calibration, because the perturbation in instantaneous swap rate volatility is hidden and potentially unstable. If the method proposed in this text is applied however, then it is possible to obtain correct swap vega per bucket in the BGM model for any volatility function at a low number of simulation paths. ■

ACKNOWLEDGEMENTS

We are particularly grateful to Les Gulko and two anonymous referees who made insightful suggestions for improving the article. All errors are our own.

REFERENCES

Brace A, D Gątarek and M Musiela, 1997
The market model of interest rate dynamics
Mathematical Finance 7, pages 127-155

Hull J and A White, 2000
Forward rate volatilities; swap rate volatilities, and implementation of the Libor market model
Journal of Fixed Income 3, pages 46-62

Joshi, M and J Theis, 2002
Bounding Bermudan swaptions in a Swap-Rate Market Model
Quantitative Finance 2, pages 370-377

Pietersz R and A Pelsser, 2003
Risk managing Bermudan swaptions in the Libor BGM model
Rotterdam: Erasmus University Working Paper

Rebonato R, 2002
Modern Pricing of Interest-Rate Derivatives: The LIBOR Market Model and Beyond
New Jersey: Princeton University Press

EXHIBIT: Examples of swap vega based on re-calibration

BGM. Suppose given a tenor structure $0 =: T_0, T_1, \ldots, T_{N+1}$ with corresponding forward Libor rates L_i for borrowing/lending over the period $[T_i, T_{i+1}]$. Each forward rate is modeled as

$$\frac{dL_i(t)}{L_i(t)} = \sigma_i(t)\, dW^{(i+1)}(t) \, .$$

Here σ_i denotes forward rate volatility and $W^{(i+1)}$ denotes a Brownian motion under the ith forward measure. The correlation structure is modeled by

$$dW^{(i+1)}(t)\, dW^{(j+1)}(t) = \rho_{ij}(t) \, .$$

Calibrations. Three volatility calibrations are considered:

1. (THFRV) *Time homogeneous forward rate volatility.* The algorithm is based on Newton Rhapson. Maintains future cap volatility curve.
2. (THSRV) *Time homogeneous swap rate volatility.* The calibration algorithm is a two stage bootstrap algorithm. Maintains future swaption volatility curve.
3. (CONST) *Constant forward rate volatility.* Implies constant swap rate volatility. Distorts future volatility curves.

The forward rate correlation was calibrated by means of a principal component analysis (PCA) (Hull and White, 2000).

Market and deal data. We considered a 31NC1 co-terminal Bermudan payer's swaption deal struck at 5% with annual compounding. The notation xNCy denotes an 'x non-call y' Bermudan option, which is exercisable into a swap with a maturity of x years from today and callable only after y years. The option is callable annually. The BGM tenor structure is 0 < 1 < 2 < … < 31. All initial forward rates are taken to equal 5%. The time-zero forward rate instantaneous correlation is assumed given by the form

$$\rho_{ij}(0) = \exp\left\{-\beta \,|\, T_i - T_j \,|\right\}, \;\; \beta = 0.05 \, .$$

The market European swaption volatilities were taken as displayed in Table A.

Table A: Swaption volatilities for re-calibration illustration.

Expiry	1	2	3	…	28	29	30
Tenor	30	29	28	…	3	2	1
Swaption Volatility	15.2%	15.2%	15.4%	…	20.4%	20.6%	20.8%

Numerical results. The numerical results have been displayed in Figure 4 and 5. The vega have been poorly estimated for the THFRV and THSRV cases, whereas these have been more accurately estimated by the constant volatility calibration. For the

THFRV case, the vega have been calculated at 1,000,000 simulation paths, see figure 6.

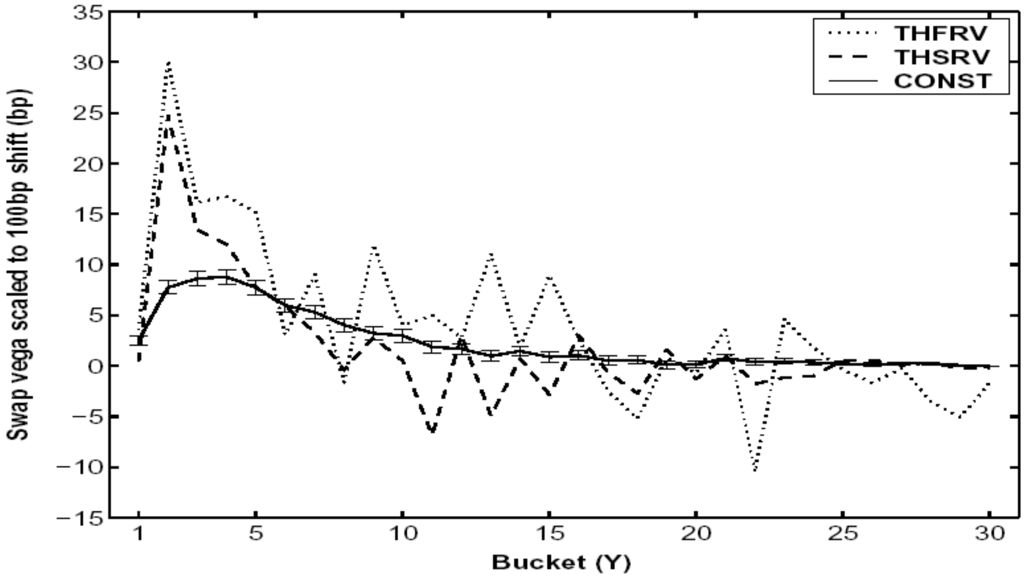

Figure 4: Re-calibration swap vega results for 10,000 simulation paths. Error bars around the CONST vegas denote a 95% confidence bound based on twice the standard error. The vega is a scaled numerical derivative and we verified that it is insensitive to the actual size of the small volatility perturbation used.

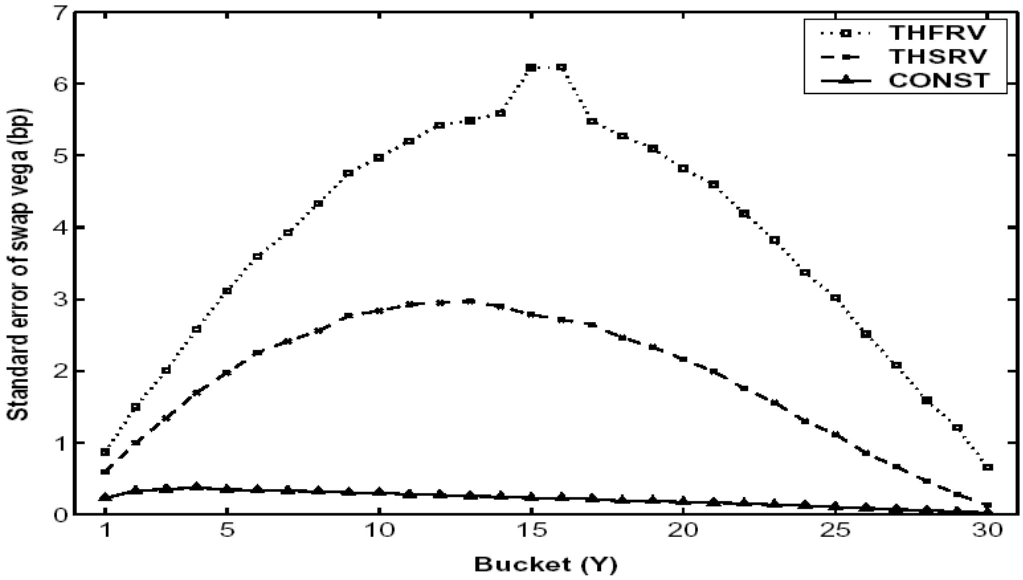

Figure 5: Empirical standard errors of the vega for 10,000 simulation paths.

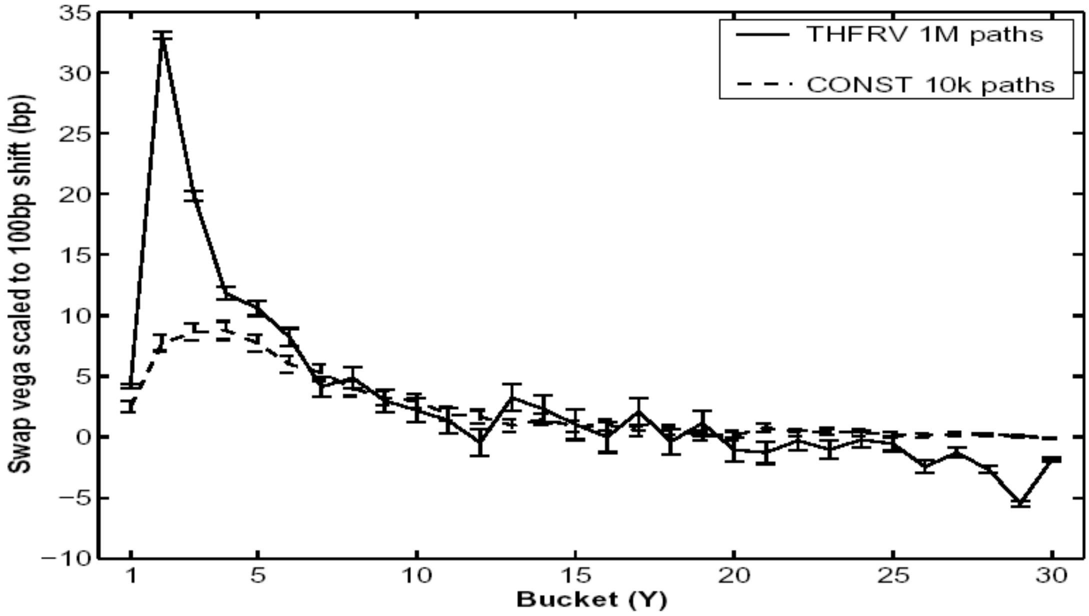

Figure 5: Re-calibration THFRV vega results for 1,000,000 simulation paths. Error bars denote a 95% confidence bound based on twice the standard error. The vega is a scaled numerical derivative and we verified that it is insensitive to the actual size of the small volatility perturbation used.