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Fair and consistent prize allocation in competitions

Citation for published version (APA):

Dietzenbacher, B. J., & Kondratev, A. Y. (2023). Fair and consistent prize allocation in competitions. *Management Science*, *69*(6), 3319-3339. https://doi.org/10.1287/mnsc.2022.4520

Document status and date:
Published: 01/06/2023

DOI:
10.1287/mnsc.2022.4520

Document Version:
Publisher's PDF, also known as Version of record

Document license:
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Publisher: Institute for Operations Research and the Management Sciences (INFORMS)
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Management Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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To cite this article:
Bas J. Dietzenbacher, Aleksei Y. Kondratev (2022) Fair and Consistent Prize Allocation in Competitions. Management Science

Published online in Articles in Advance 07 Sep 2022

. https://doi.org/10.1287/mnsc.2022.4520

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MANAGEMENT SCIENCE
Articles in Advance, pp. 1–21
ISSN 0025-1909 (print), ISSN 1526-5501 (online)

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Fair and Consistent Prize Allocation in Competitions

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Received: July 31, 2021
Revised: March 11, 2022; June 7, 2022
Accepted: July 4, 2022
Published Online in Articles in Advance: September 7, 2022

https://doi.org/10.1287/mnsc.2022.4520

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Abstract. Given the ranking of competitors, how should the prize endowment be allocated? This paper introduces and axiomatically studies the prize allocation problem. We focus on consistent prize allocation rules satisfying elementary solidarity and fairness principles. In particular, we derive several families of rules satisfying anonymity, order preservation, and endowment monotonicity, which all fall between the equal division rule and the winner-takes-all rule. Our results may help organizers to select the most suitable prize allocation rule for rank-order competitions.

History: Accepted by Manel Baucells, behavioral economics and decision analysis.
Funding: A. Y. Kondratev is supported by HSE University [Basic Research Program and "Priority 2030" Program].

Keywords: fair allocation • rank-order tournament • prize structure • tournament design • axiomatic analysis • consistency

1. Introduction

Innovation and crowdsourcing competitions, sales competitions in companies, and sporting events often take the form of rank-order tournaments.[1] Each such tournament is held once, and the participants know the rules and the prize structure in advance. The absolute performance of a participant can be determined by, for example, the volume of sales in a sales competition, the number of strokes in golf, or the time of elimination from a poker tournament. However, a feature of the ranking format is that participants receive prizes according to their relative performance.

To distribute prizes, the organizers need to know three things: the list of the participants, their ranking in the tournament, and the size of the prize endowment. For each such triple, the *prize allocation rule* used should uniquely distribute the endowment among the participants. Examples of prize structures are presented in Table 1. The Professional Golfers' Association Tour (PGA TOUR) conducts many regular golf tournaments with different purses but the same rules for distributing prizes. Table 1 shows examples of two such tournaments with purses of \$9.3 million (The Genesis Invitational) and \$6.6 million (Safeway Open). The winner of the tournament receives 18% of the purse, the runner-up 10.9%, and so on. The World Championship of Online Poker (WCOOP) has a comparable purse but follows a different prize money distribution scheme.

How did the organizers design the prize allocation rules for these tournaments? More generally, is there a reliable systematic way to design such rules? For solving this problem, the current paper proposes an axiomatic approach. In this approach, rather than choosing a rule directly, an organizer needs to select some principles the rule has to satisfy. Then, it leads to an impossibility, a single rule, or a family of rules that all satisfy the principles. Thus, on the one hand, the axiomatic approach simplifies the decision of the organizers, whereas on the other hand, it gives them enough freedom to select desirable principles.

Let us introduce the three basic principles. Consistent with the principle of *anonymity* is that the participants' rewards depend only on their position in the competition. That is, the organizers award a prize for a position regardless of which participant takes that position. According to the principle of *order preservation*, a higher position does not correspond to a lower reward. This creates the right incentives for participants, as the efforts made during preparation for and participation in the tournament help the competitor to end up higher in the ranking and receive a more valuable reward. Third, *endowment monotonicity* is satisfied when an increase in the endowment does not decrease any reward.[2] This creates uniform incentives for all participants, as both strong and weak competitors are interested in increasing the endowment of the tournament. For instance, in some poker tournaments, players can rebuy and add on chips, which increase the purse. Also, during some e-sports tournaments, viewers and sponsors can add to the purse. The three principles of anonymity, order preservation, and endowment monotonicity are so undemanding that

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Table 1. Examples of Single Rank-Order Tournaments

Position	1	2	3	4	5	6	7	8	9	10	Top 10	Purse
Panel A: PGA TOUR 2019/2020 golf tournaments												
Genesis	1,674	1,014	642	456	381	337	314	291	272	253	5,633	9,300
Safeway	1,188	719	455	323	271	239	223	206	193	180	3,998	6,600
Share	18.0	10.9	6.9	4.9	4.1	3.63	3.38	3.13	2.93	2.73	60.6	100
Panel B: WCOOP 2019 main event poker tournament												
Prize	1,666	1,188	847	603	430	307	219	156	111	79	5,605	11,180
Share	14.9	10.6	7.57	5.40	3.85	2.74	1.96	1.39	0.99	0.71	50.1	100

Notes. The purse, the total prize for top 10 positions, and the prize for each position from 1 to 10 are given in thousands of dollars. The shares are given in percentages of the purse.

even together they do not exclude any visible rule. Therefore, we impose these properties throughout this paper.

The universal principle of *consistency* is often applied in fair allocation problems (Balinski and Young 1982, Thomson 2012). We apply this principle to the prize allocation problem as follows. Suppose that the participants have some ranking in a competition and receive the corresponding prizes. Then, some participants leave with their prizes, and the remaining endowment is redistributed among the remaining participants, taking into account their modified composition. A rule is consistent if such a redistribution does not change the amount that the participants receive. Thus, consistent rules may help to stabilize the prize allocation in a competition.

Although consistency with respect to single rank-order tournaments is demanding, it is often observed in reality. For example, the *equal division* (ED) rule and the *winner-takes-all* (WTA) rule are consistent. As a more interesting example, suppose that in a competition with 100 participants, the organizer awards a number of fixed-size cash prizes of $2,000. The limited endowment provides a certain number of equal prizes to the participants with the highest positions, and any remaining balance goes to the next participant. So, if the fund is $6,000, then the three best participants A, B, and C win $2,000 each. If the fund grows to $11,000, then the top five participants A, B, C, D, and E win $2,000 each, and the participant in sixth position, F, wins $1,000. (Prizes are usually distributed in that way in satellites of poker tournaments.) Consistency requires that if the organizer has already transferred the prize money to participants B, C, and F, then reapplying the rule to the competition without participants B, C, and F and with $6,000 endowment will leave everything as before; participants A, D, and E receive $2,000 each because they are the best among the remaining participants.

What do *consistent* prize distribution rules look like? The equal division rule, the winner-takes-all rule, and the aforementioned rule illustrate the main feature of such rules. Our first main result shows that all consistent rules are some combination of these three rules, and we call them *interval rules* (Theorem 1). Because this is still a large family, we can strengthen its properties.

In particular, we can strengthen order preservation. We say that the rule satisfies *strict order preservation* if prizes for positions from the first to the last form a strictly decreasing sequence. Unfortunately, no consistent rule satisfies strict order preservation. On the other hand, the winner-takes-all rule is the only consistent rule in which the prize for the first position always exceeds the prize for the last position (Corollary 1).

We can also strengthen endowment monotonicity. To do this, we examine three more stringent properties. The first strengthening is *strict endowment monotonicity*, which requires that with an increase in the endowment, each prize strictly increases. The equal division rule is the only consistent rule that satisfies strict endowment monotonicity (Corollary 2).

The second strengthening is *winner strict endowment monotonicity*. In addition to endowment monotonicity, this property requires that with an increase in the endowment, the prize for the first position strictly increases. This leads to the following subfamily of rules. Each such rule sets a maximum size of an individual prize, which all competitors receive equally regardless of their position, whereas all the excess goes to the winner (Corollary 3). Therefore, we call this a *winner-takes-surplus* (WTS) rule. For example, the size of a laboratory's premium fund is often recognized only at the end of the year. The head of the laboratory with 10 employees can consider a fair premium of $2,000. Then, with a fund size of $10,000, each employee receives $1,000. However, if the size of the fund is $24,000, then the best worker gets $6,000, and each of the others receives $2,000.

The third strengthening of endowment monotonicity is also suggested by the examples in Table 1. The prize structures from golf and poker are *scale invariant*; that is, when the fund is increased k times, each prize also increases k times. We show that the only two consistent and scale-invariant rules for the distribution of prizes are the equal division rule and the winner-

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takes-all rule (Corollary 4). For this reason, the prize structures from poker and golf are not consistent.

Because consistency can be a demanding requirement, we formulate two relaxations. Imagine that a tournament is held among participants of different skill levels. The organizer could distribute the fund according to the general ranking of all participants, or the organizer could divide the fund into two parts and distribute the prizes separately among the participants of high and low levels. If both methods lead to the same distribution of prizes, then the rule satisfies *local consistency*.[3] If the prize distribution is the same for high-level participants, then the rule satisfies *top consistency*.[4] Such rules are particularly desirable in competitions where losers take their prizes first and leave the competition one by one (as in poker) or in groups (as in golf). In addition, the fact that adding or deleting low-level competitors is much easier than adding or deleting high-level competitors could explain why most organizers of real competitions prefer top consistent rules to "bottom" consistent rules. For example, Table 1 shows that the top 10 poker tournament participants receive a total of \$5.605 million of a purse of \$11.18 million. Local consistency and top consistency require that if you apply the rule separately to the top 10 participants and a purse of \$5.605 million, then they will receive the same prizes.

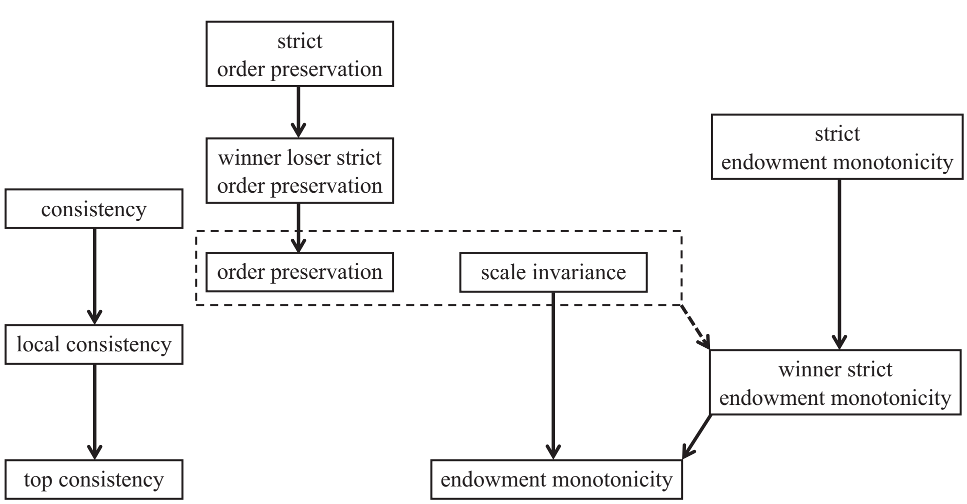
Our second main result describes the family of *locally consistent* rules that satisfy winner strict endowment monotonicity. We call them *single-parametric rules*. Each such rule is generated by some continuous and nondecreasing function f as follows. The winner of the competition receives some prize x, the runner-up receives a prize $f(x)$, the third position receives a prize $f(f(x))$, and so on. The value of x is unambiguously determined from the condition that the organizer distributes the whole endowment (Theorem 2 and Corollary 5). We further show that the prize structure is locally consistent and scale invariant if and only if the prizes form a *geometric* sequence with factor λ; that is, the prize for position $r+1$ is the fraction λ of the prize for position r (Corollary 6). For example, the prizes in the final stage of the poker tournament from Table 1 form a geometric sequence with $\lambda = 0.713$ and therefore, are locally consistent. The International Biathlon Union World Cup is another example where the prizes are almost geometrical (Kondratev et al. 2022). The prizes in golf tournaments, however, do not form a geometric sequence and therefore, are not even locally consistent.

Our third main result describes the family of *top consistent* rules that satisfy winner strict endowment monotonicity. Each such *parametric rule* is generated by some sequence of continuous and nondecreasing functions $f_2, f_3, \ldots$ as follows. The winner of the competition receives some prize x, the runner-up receives a prize $f_2(x)$, the third position receives a prize $f_3(x)$, and so on. The value of x is unambiguously determined from the condition that the organizer distributes the whole endowment (Theorem 3 and Corollary 7). Finally, we show that the prize structure is top consistent and scale invariant if and only if there exists some sequence $\lambda_1, \lambda_2, \ldots$ such that the prize for position r is the fraction λ_r/λ_1 of the prize for the winner (Corollary 8). We call them *proportional rules*. For example, the prizes in the golf tournaments from Table 1 are distributed according to a proportional rule with $\lambda_1 = 18.0, \lambda_2 = 10.9, \lambda_3 = 6.9$, and so on.

The relations of the properties are presented in Figure 1. Note that no rule satisfies all properties considered. However, the geometric rules with factor λ such that $0 < \lambda < 1$ satisfy all properties except for consistency.

The practical benefits of this paper are as follows.

Figure 1. Properties of Prize Allocation Rules



Note. The solid arrows indicate direct implications, and the dashed arrow indicates a joint implication.

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• To design a prize allocation rule, the axiomatic approach requires far less information compared with other approaches. In addition to the set of competitors, their ranking, and the prize endowment, we only need the desirable properties our rule has to satisfy. Thus, we do not need any information on the space of efforts, the costs of efforts, how efforts transform into a ranking, utility functions of the competitors, and the objective function of the organizer.

• Decisions based on the axiomatic approach can be easier to justify. Each competition has many stakeholders, and achieving an agreement among them might be as difficult as winning the competition. We provide tools to convert an agreement on principles and axioms into an agreement on a specific rule or a family of rules.

• If an organizer wants to choose a rule and accepts some set of axioms, then the problem is reduced to choosing a single rule from the resulting family. On the one hand, this simplifies the organizer's decision. For example, if anonymity, order preservation, scale invariance, and local consistency are required, then the prize sequence has to be geometric. On the other hand, because the families are broad enough, the organizer is free to choose the rule within the selected family that maximizes some other desirable goals.

This paper proceeds as follows. Section 2 provides a literature overview. Section 3 introduces the model and characterizes consistent prize allocation rules. Section 4 characterizes locally consistent rules and top consistent rules. Section 5 concludes. The proofs are contained in the appendix.

2. Literature Overview

In this brief literature overview, we highlight other papers related to our key topics: fairness, consistency, rank-order tournaments, prize allocation, and axiomatic characterization.

Francis Galton was perhaps the first to write on how the total prize money should be divided into prizes for each ranking position in a rank-order tournament (Galton 1902). Using a probabilistic approach with order statistics, Galton concluded that when only two prizes are given, the first prize should be approximately three times the value of the second. Because of poor economic motivation, this approach has not received much attention in the literature.

The seminal paper of Lazear and Rosen (1981) analyzed rank-order tournaments from an economic perspective. In their model, a firm assigns a certain prize to each ranking position regardless of the identity of the worker who occupies the position. Then, each worker chooses a level of effort that leads to an output. The relative ranking of the outputs determines the prizes for workers. Lazear and Rosen (1981) found the optimal prize structure that maximizes each worker's utility in equilibrium. The subsequent literature proposed similar models and studied optimal prize structures that maximize different goals.[5] In this literature, like in the current paper, anonymity and order preservation are standard properties for prize sequences (see also O'Keeffe et al. 1984). Olszewski and Siegel (2020) found that total effort maximization, concave prize valuations, and convex effort costs even call for the strict order preservation of prizes.

In this paper, the prize fund can vary, and we axiomatically characterize the WTS and geometric rules. When the fund is fixed in advance, similar prize structures appeared in the literature for other reasons. Moldovanu et al. (2007) introduced contests for status and found that when the fund is high enough, the WTS prize sequence maximizes the total effort of competitors. Gershkov et al. (2009) showed that under some assumptions, the WTS prize sequence is an efficient redistribution of output among partners within teams. Newman and Tafkov (2014) provided an experiment and showed that relative performance information has a negative effect on performance of competitors if the WTS prize sequence is used. Xiao (2016) studied geometric prize sequences because it guarantees the existence and uniqueness of a Nash equilibrium under some assumptions, which is hard to prove for general prize sequences.

To apply the aforementioned approach, which has become dominant in the economic literature, an organizer needs information on the effort costs and the utility functions of the competitors and how the efforts transform into a ranking. This information may not be available to the organizer in practice. In contrast, the axiomatic approach only requires information on the desirable principles the prize allocation has to satisfy.

The consistency principle for allocation rules was motivated by Balinski and Young (2001, p. 141) as "every part of a fair division should be fair." Later, Thomson (2012, p. 418) argued that this should be restated as "every part of every socially desirable allocation should be socially desirable." As in the current paper, consistency plays a key role in many allocation problems with infinitely divisible goods.[6] For a comprehensive introduction to the consistency principle, we refer to Thomson (2011).

In particular, our model and results share characteristic features with the following studies. First, although motivated from a different context, a similar model is studied by Hougaard et al. (2017), where hierarchically ordered agents redistribute their individually generated revenues, so the endowment is endogenous instead of exogenous as in our model. Using the lowest-rank consistency axiom similar to our top consistency, their main results characterize families of geometric rules. Although the families share

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the same name, their rules are different from the geometric prize allocation rules obtained as a special case in the current paper. Osório (2017) did obtain sharing rules analogous to the geometric rules in our framework by applying some form of consistency to operational problems (such as river-sharing problems, sequential allocation, and rationing problems). Second, the construction of our interval rules is somewhat reminiscent of the construction of generalized equal sacrifice rules for taxation problems as introduced by Chambers and Moreno-Ternero (2017). For the interval rules, the prizes of the competitors are increased from the lower bound to the upper bound of the interval in the order of the ranking, and the equal division rule is applied outside the intervals. Generalized equal sacrifice rules apply constrained equal sacrifice rules based on strictly increasing functions inside each interval and the leveling tax outside the intervals. Remarkably, the characterization of the family of generalized equal sacrifice rules is based on a composition axiom, which does not have an analogy for prize allocation rules. Third, the parametric prize allocation rules, where the prize of each position is described by a continuous and nondecreasing function, closely resemble the parametric taxation methods obtained by Young (1987), where the tax assessed on each claim level is described by such a function. Remarkably, Young's family is obtained by full consistency, whereas ours is obtained only by top consistency.

We are aware of two applications of the consistency axiom in the context of competitions. First, for competitions in which participants put in effort to increase their probability of winning a single prize, Skaperdas (1996) characterized the additive contest success functions, which provide each player's probability of winning as a function of all players' efforts. Although this model is mathematically equivalent to prize allocation based on the cardinal performance of competitors, we are not aware of real examples of competitions using such schemes. Second, Flores-Szwagrzak and Treibich (2020) characterized measures of individual productivity when team membership and team production are observable but individual contributions to team production are not. The authors illustrated their approach with data from the National Basketball League.

For developments in the literature on tournament design, we refer to the survey articles of Wright (2014), Kendall and Lenten (2017), and Lenten and Kendall (2021) and an excellent book of Csató (2021). Recently, Bergantiños and Moreno-Ternero (2020a; b, 2021, 2022a; b; c) initiated axiomatic studies on broadcasting revenue sharing, which is another essential part of money sharing in competitions (e.g., in soccer leagues) using a different informational basis.

The literature on prize allocation in rank-order competitions has not taken an axiomatic approach, and the fair allocation literature has not focused on competitions. We connect these two fields by applying the axiomatic approach to prize allocation in rank-order competitions. Surprisingly, the literature on this topic is scarce. A rare example is Petróczy and Csató (2021), who proposed a prize distribution rule based on pairwise comparisons in a championship consisting of a series of single rank-order competitions and illustrated their approach with the data from the Formula One World Championship (racing). However, the authors did not provide axiomatic characterizations. To our knowledge, our paper is the first that develops and motivates prize allocation rules for rank-order competitions directly from axioms.

3. Consistent Prize Allocation Rules

3.1. Model

Let U be a countable set of at least three potential *competitors*. On each given occasion, a finite subset $N \subseteq U$ participates in a competition. This competition results in a strict *ranking*, a bijection $\mathcal{R}: N \to \{1, \ldots, |N|\}$ assigning to each competitor a unique *position*. Here, competitor $i \in N$ is ranked higher than competitor $j \in N$ if $\mathcal{R}(i) < \mathcal{R}(j)$. The *endowment* $E \in \mathbb{R}_+$ is the amount of prize money to be allocated among the competitors. Thus, a *competition* is a triple $(N, \mathcal{R}, E)$.

Assuming that more money is better for each competitor, the preferences of the competitors over the feasible prize allocations are in conflict. A *prize allocation rule* φ (or simply a *rule*) assigns to each competition $(N, \mathcal{R}, E)$ an allocation of the endowment among the competitors; that is, $\varphi(N, \mathcal{R}, E) \in \mathbb{R}^N_+$ is such that

$$\sum_{i \in N} \varphi_i(N, \mathcal{R}, E) = E.$$

Throughout this paper, φ denotes a generic rule.

Reflecting the opposing principles of egalitarianism and elitism, two extreme and elementary rules are the equal division rule and the winner-takes-all rule. The equal division rule divides the prize money equally among the competitors. The winner-takes-all rule allocates all the prize money to the competitor ranked first.

ED Rule. For each competition $(N, \mathcal{R}, E)$ and each competitor $i \in N$,

$$\mathrm{ED}_i(N, \mathcal{R}, E) = \frac{E}{|N|}.$$

WTA Rule. For each competition $(N, \mathcal{R}, E)$ and each competitor $i \in N$,

$$\mathrm{WTA}_i(N, \mathcal{R}, E) = \begin{cases} E & \text{if } \mathcal{R}(i) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

The ED rule is desirable from an egalitarian perspective because it treats each competitor equally; the

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WTA rule is desirable from an elitist perspective because it rewards the winner for achieving the highest position in the ranking.

We take an axiomatic approach to study the fundamental differences between prize allocation rules. This means that we formulate some desirable properties of rules and analyze their implications when imposed in various combinations. An elementary property imposing a form of equal treatment of competitors is *anonymity*, which requires that the allocation does not depend on the identities of the competitors but only on the number of competitors, their ranking, and the endowment.

Anonymity. For each pair of competitions $(N, \mathcal{R}, E)$ and $(N', \mathcal{R}', E)$ with equal numbers of competitors $|N| = |N'|$ and each pair of competitors $i \in N$ and $j \in N'$ with equal positions $\mathcal{R}(i) = \mathcal{R}'(j)$,

$$\varphi_i(N, \mathcal{R}, E) = \varphi_j(N', \mathcal{R}', E).$$

Imposing the allocation to reflect the ranking, *order preservation* requires that the prize of a competitor ranked higher is at least the prize of a competitor ranked lower.

Order Preservation. For each competition $(N, \mathcal{R}, E)$ and each pair of competitors $i, j \in N$, if i is ranked higher than j, then

$$\varphi_i(N, \mathcal{R}, E) \geq \varphi_j(N, \mathcal{R}, E).$$

The solidarity property *endowment monotonicity* requires that no competitor is worse off when the endowment increases.

Endowment Monotonicity. For each pair of competitions $(N, \mathcal{R}, E)$ and $(N, \mathcal{R}, E')$ with $E < E'$ and each competitor $i \in N$,

$$\varphi_i(N, \mathcal{R}, E) \leq \varphi_i(N, \mathcal{R}, E').$$

3.2. Joint Characterization

In principle, prize allocation may depend on the number of competitors. A criterion for evaluating whether a rule prescribes coherent allocations for competitions with different numbers of competitors is *consistency*. Consider an arbitrary competition $(N, \mathcal{R}, E)$, and suppose that some competitors $S \subseteq N$ redistribute their accumulated prizes. This redistribution is based on their *subranking*, the bijection $\mathcal{R}_S : S \to \{1, \ldots, |S|\}$ such that for each pair of competitors $i, j \in S$, we have $\mathcal{R}_S(i) < \mathcal{R}_S(j)$ if and only if $\mathcal{R}(i) < \mathcal{R}(j)$. A rule is consistent if it allocates to each competitor $i \in S$ in the corresponding reduced competition the same prize as in the original competition.[7]

Consistency. For each competition $(N, \mathcal{R}, E)$, each nonempty subset of competitors $S \subseteq N$, and each competitor $i \in S$,

$$\varphi_i(N, \mathcal{R}, E) = \varphi_i\left(S, \mathcal{R}_S, \sum_{j \in S} \varphi_j(N, \mathcal{R}, E)\right).$$

The equal division rule and the winner-takes-all rule both satisfy anonymity, order preservation, endowment monotonicity, and consistency, but these rules are not the only ones. Let a rule be defined by allocating the endowment in the following way. For a fixed $\Delta > 0$, up to the first Δ dollars are allocated to the winner, the surplus up to the next Δ dollars is allocated to the competitor ranked second, and so on until each competitor has been allocated Δ dollars. If there is still money left, the first additional Δ dollars are allocated to the winner, the next additional Δ dollars to the competitor ranked second, and so on. This procedure continues until the full endowment is allocated among the competitors. We call such rules *step rules*.

Step Rules. There exists $\Delta > 0$ such that for each competition $(N, \mathcal{R}, E)$ and each competitor $i \in N$,

$$\phi_i^\Delta(N, \mathcal{R}, E) = \begin{cases} 0 & \text{if} \quad 0 \leq E \leq (\mathcal{R}(i) - 1)\Delta; \\ E - (\mathcal{R}(i) - 1)\Delta & \\ & \text{if} \quad (\mathcal{R}(i) - 1)\Delta \leq E \leq \mathcal{R}(i)\Delta; \\ \Delta & \text{if} \quad \mathcal{R}(i)\Delta \leq E \leq |N|\,\Delta; \\ k\Delta + \phi_i^\Delta(N, \mathcal{R}, E - |N|\,k\Delta) & \\ & \text{if} \quad |N|\,k\Delta \leq E \leq |N|\,(k+1)\Delta. \end{cases}$$

Tables 2–4 illustrate how the endowment from one to six dollars is allocated among two, three, and four competitors, respectively, for the step rule with $\Delta = 1$. The step rules also satisfy anonymity, order preservation, endowment monotonicity, and consistency.

It turns out that anonymity is implied by order preservation, endowment monotonicity, and consistency. This is captured by the following lemma.[8]

Lemma 1. *If a rule satisfies order preservation, endowment monotonicity, and consistency, then it satisfies anonymity.*

To have a full understanding of the joint implication of order preservation, endowment monotonicity, and consistency, we only need to focus on the structure of

Table 2. Step Rule with $\Delta = 1$ for Two Competitors

φ_1	φ_2	E
1	0	1
1	1	2
2	1	3
2	2	4
3	2	5
3	3	6

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Table 3. Step Rule with $\Delta = 1$ for Three Competitors

φ_1	φ_2	φ_3	E
1	0	0	1
1	1	0	2
1	1	1	3
2	1	1	4
2	2	1	5
2	2	2	6

the corresponding rules for competitions with two competitors. The following lemma shows that each such rule has at most one consistent extension.[9]

Lemma 2. *If two rules satisfying endowment monotonicity and consistency coincide for each competition with two competitors, then the two rules coincide for each competition with an arbitrary number of competitors.*

How do rules satisfying order preservation, endowment monotonicity, and consistency look for competitions with two competitors? To this end, it helps to graphically illustrate possible *allocation paths* (i.e., draw the allocations assigned to competitions with two competitors when the endowment increases from zero).

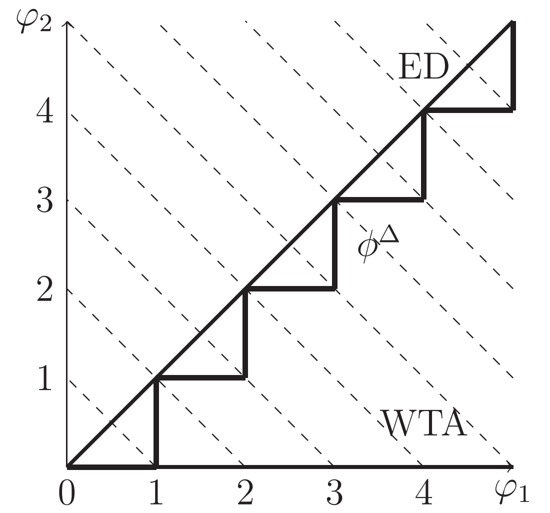
Let $(N, \mathcal{R}, E)$ with $|N| = 2$ be a competition with two competitors. Denote $N = \{1, 2\}$ such that $\mathcal{R}(1) = 1$ and $\mathcal{R}(2) = 2$. Let the endowment E gradually increase from zero. Then, the allocation paths of the ED rule, the WTA rule, and the step rule ϕ^{Δ} with $\Delta = 1$ are illustrated in Figure 2.

Each rule satisfying order preservation, endowment monotonicity, and consistency is a combination of the ED rule, the WTA rule, and the step rules in the following way. Let $N \subseteq U$ be a finite set of competitors, and consider a ranking and endowment. Then, there exist disjoint intervals (a_k, b_k) with $a_k \in \mathbb{R}_+$ and $b_k \in \mathbb{R}_+ \cup \{+\infty\}$ for each k such that the following holds. Each competitor is allocated a prize of a_k when the endowment equals $|N| a_k$. If the endowment is higher, first the prize of the winner is increased to b_k, then the prize of the competitor ranked second is increased to b_k, and so on. This means that each competitor is allocated a prize of b_k when the endowment equals $|N| b_k$. If the average endowment does not belong to one of these intervals, it is divided equally among the competitors. We call such rules *interval rules*.

Table 4. Step Rule with $\Delta = 1$ for Four Competitors

φ_1	φ_2	φ_3	φ_4	E
1	0	0	0	1
1	1	0	0	2
1	1	1	0	3
1	1	1	1	4
2	1	1	1	5
2	2	1	1	6

Figure 2. Path of the Step Rule with $\Delta = 1$



Notes. The horizontal axis depicts the prize of the competitor ranked first, and the vertical axis depicts the prize of the competitor ranked second. The dashed lines indicate different levels of the endowment.

Interval Rules. There exist disjoint intervals $(a_1, b_1), (a_2, b_2), \ldots$ with $a_1, a_2, \ldots \in \mathbb{R}_+$ and $b_1, b_2, \ldots \in \mathbb{R}_+ \cup \{+\infty\}$ such that for each competition $(N, \mathcal{R}, E)$ and each competitor $i \in N$,

$$\varphi_i(N, \mathcal{R}, E) = \begin{cases} a_k + \phi_i^{b_k - a_k}(N, \mathcal{R}, E - |N|\, a_k) & \text{if } \; |N|\, a_k \le E \le |N|\, b_k; \\ \dfrac{E}{|N|} & \text{otherwise.} \end{cases}$$

Note that the ED rule is an interval rule with $a_k = b_k = 0$ for each k. The WTA rule is an interval rule with $a_1 = 0$ and $b_1 = +\infty$. A step rule with representation Δ is an interval rule with $a_k = (k-1)\Delta$ and $b_k = k\Delta$ for each k.

Example 1. Let $(N, \mathcal{R}, E)$ with $|N| = 2$ be a competition with two competitors. Denote $N = \{1, 2\}$ such that $\mathcal{R}(1) = 1$ and $\mathcal{R}(2) = 2$. The allocation path of the interval rule φ with $a_1 = 1$, $b_1 = a_2 = 2\frac{1}{2}$, $b_2 = 3$, $a_3 = 3\frac{1}{2}$, and $b_3 = +\infty$ is illustrated in Figure 3.

Theorem 1. *A rule satisfies order preservation, endowment monotonicity, and consistency if and only if it is an interval rule.*

3.3. Strengthening Properties

Theorem 1 shows that many rules satisfy order preservation, endowment monotonicity, and consistency. This means that there is room for imposing additional requirements. In particular, it may be possible to

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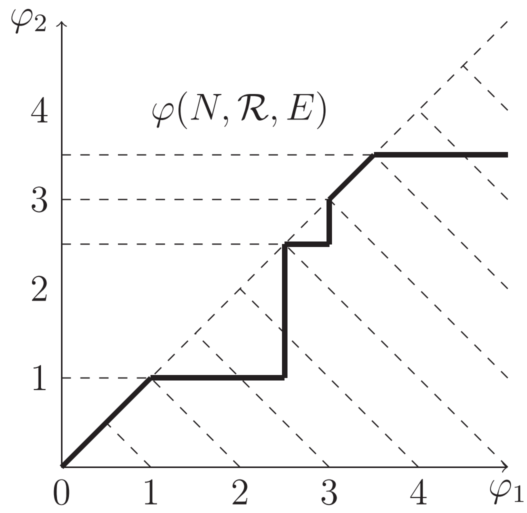
Figure 3. Path of the Interval Rule in Example 1



strengthen one of the properties. Unfortunately, order preservation cannot be strengthened to *strict order preservation*, which requires that the prize of a competitor ranked higher is more than the prize of a competitor ranked lower. However, it can be strengthened to the weaker property *winner-loser strict order preservation*, which requires in addition to order preservation, that the prize of the competitor ranked first is higher than the prize of the competitor ranked last. The winner-takes-all rule is the only interval rule satisfying winner-loser strict order preservation.

Strict Order Preservation. For each competition $(N, \mathcal{R}, E)$ with $E > 0$ and each pair of competitors $i, j \in N$, if i is ranked higher than j, then

$$\varphi_i(N, \mathcal{R}, E) > \varphi_j(N, \mathcal{R}, E).$$

Winner-Loser Strict Order Preservation. For each competition $(N, \mathcal{R}, E)$ with $E > 0$ and each pair of competitors $i, j \in N$, if i is ranked higher than j, then

$$\varphi_i(N, \mathcal{R}, E) \geq \varphi_j(N, \mathcal{R}, E),$$

and if i is ranked first and j is ranked last, then

$$\varphi_i(N, \mathcal{R}, E) > \varphi_j(N, \mathcal{R}, E).$$

Corollary 1.

i. *No rule satisfies strict order preservation, endowment monotonicity, and consistency.*

ii. *The unique rule satisfying winner-loser strict order preservation, endowment monotonicity, and consistency is the winner-takes-all rule.*

Instead of strengthening order preservation, it is also possible to strengthen endowment monotonicity. One possibility is *strict endowment monotonicity*, which requires that each prize increases when the endowment increases. Another less demanding possibility is *winner strict endowment monotonicity*, which requires in addition to endowment monotonicity, that the prize for the first position increases when the endowment increases.

Strict Endowment Monotonicity. For each pair of competitions $(N, \mathcal{R}, E)$ and $(N, \mathcal{R}, E')$ with $E < E'$ and each competitor $i \in N$,

$$\varphi_i(N, \mathcal{R}, E) < \varphi_i(N, \mathcal{R}, E').$$

Winner Strict Endowment Monotonicity. For each pair of competitions $(N, \mathcal{R}, E)$ and $(N, \mathcal{R}, E')$ with $E < E'$ and each competitor $i \in N$,

$$\varphi_i(N, \mathcal{R}, E) \leq \varphi_i(N, \mathcal{R}, E'),$$

and if i is ranked first, then

$$\varphi_i(N, \mathcal{R}, E) < \varphi_i(N, \mathcal{R}, E').$$

The equal division rule is the only interval rule that satisfies strict endowment monotonicity. Moreover, an interval rule satisfies winner strict endowment monotonicity if and only if it is a winner-takes-surplus rule (i.e., it coincides with the ED rule up to some value of the endowment, and it allocates the surplus according to the WTA rule).

Corollary 2. *The unique rule satisfying order preservation, strict endowment monotonicity, and consistency is the equal division rule.*

WTS Rules. There exists $a \in \mathbb{R}_+ \cup \{+\infty\}$ such that for each competition $(N, \mathcal{R}, E)$ and each competitor $i \in N$,

$$\varphi_i(N, \mathcal{R}, E) = \begin{cases} E - (|N| - 1)a & \text{if } E \geq |N|\, a \text{ and } \mathcal{R}(i) = 1; \\ a & \text{if } E \geq |N|\, a \text{ and } \mathcal{R}(i) \neq 1; \\ \dfrac{E}{|N|} & \text{otherwise.} \end{cases}$$

Note that the ED rule is a WTS rule with $a = +\infty$. The WTA rule is a WTS rule with $a = 0$. Tables 5–7 illustrate how a WTS rule φ allocates the endowment from a to $6a$ among two, three, and four competitors, respectively. The corresponding allocation path is illustrated in Figure 4.

Table 5. Winner-Takes-Surplus Rule for Two Competitors

φ_1	φ_2	E
$a/2$	$a/2$	a
a	a	$2a$
$2a$	a	$3a$
$3a$	a	$4a$
$4a$	a	$5a$
$5a$	a	$6a$

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Table 6. Winner-Takes-Surplus Rule for Three Competitors

φ_1	φ_2	φ_3	E
$a/3$	$a/3$	$a/3$	a
$2a/3$	$2a/3$	$2a/3$	$2a$
a	a	a	$3a$
$2a$	a	a	$4a$
$3a$	a	a	$5a$
$4a$	a	a	$6a$

Corollary 3. *A rule satisfies order preservation, winner strict endowment monotonicity, and consistency if and only if it is a winner-takes-surplus rule.*

Alternatively, endowment monotonicity can be strengthened to *endowment additivity*. Suppose that the endowment turns out to be larger than expected. Then, there are two ways of proceeding. First, the initial allocation is cancelled, and the rule is applied to the competition with the new endowment. Second, the rule is applied to the competition with the increment as the endowment, and the resulting allocation is added to the initial allocation. A rule satisfies endowment additivity if both ways of proceeding lead to the same allocation.

Endowment Additivity. For each pair of competitions $(N, \mathcal{R}, E)$ and $(N, \mathcal{R}, E')$ and each competitor $i \in N$,

$$\varphi_i(N, \mathcal{R}, E + E') = \varphi_i(N, \mathcal{R}, E) + \varphi_i(N, \mathcal{R}, E').$$

Endowment additivity is equivalent to *scale invariance*, which requires that each competitor is assigned a fixed share of the endowment.[10] Even when the endowment is not yet known, the competitors know to which share they are entitled. Among all interval rules, only the ED rule and the WTA rule satisfy endowment additivity (scale invariance).

Scale Invariance. For each competition $(N, \mathcal{R}, E)$ and each competitor $i \in N$,

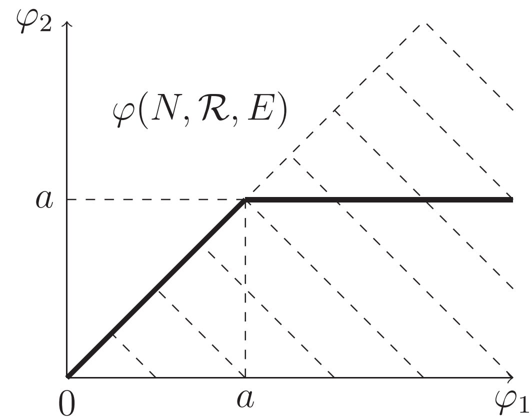
$$\varphi_i(N, \mathcal{R}, E) = E\varphi_i(N, \mathcal{R}, 1).$$

Lemma 3. *A rule satisfies endowment additivity if and only if it satisfies scale invariance.*

Table 7. Winner-Takes-Surplus Rule for Four Competitors

φ_1	φ_2	φ_3	φ_4	E
$a/4$	$a/4$	$a/4$	$a/4$	a
$a/2$	$a/2$	$a/2$	$a/2$	$2a$
$3a/4$	$3a/4$	$3a/4$	$3a/4$	$3a$
a	a	a	a	$4a$
$2a$	a	a	a	$5a$
$3a$	a	a	a	$6a$

Figure 4. Path of a Winner-Takes-Surplus Rule



Corollary 4. *The only two rules satisfying order preservation, endowment additivity (scale invariance), and consistency are the equal division rule and the winner-takes-all rule.*

The results of Section 3 are presented in Table 8.

4. Weakening Consistency

4.1. Locally Consistent Prize Allocation Rules

We know that the equal division rule and the winner-takes-all rule are two members of a family of rules satisfying anonymity, order preservation, endowment monotonicity, and consistency. Strengthening order preservation or endowment monotonicity leads to impossibilities, uniqueness, or restricted families. However, consistency may be considered too strong a requirement because an arbitrary subranking may not reflect the results of the original competition well. Suppose, for instance, that two competitors of a competition with a large number of competitors reevaluate their prizes on the basis of their subranking. Then, the reduced competition tends to lose significant features of original competition (e.g., it does not take into account whether the two competitors originally ranked first and second or first and last).

In fact, it may be desirable to weaken consistency to *local consistency*, which requires only consistent allocations for reduced competitions where for each two participants, each other participant with an intermediate position is also involved. In other words, local consistency requires invariance under splitting up the full competition into smaller competitions, where the participants in the top segment of the ranking are put in a separate competition, the participants in the second segment of the ranking are put in a second competition, and so on.[11]

Local Consistency. For each competition $(N, \mathcal{R}, E)$, each nonempty subset of competitors $S \subseteq N$ with

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Table 8. Properties of Consistent Rules

	Interval rules Theorem 1 $(a_k, b_k),\ k = 1, 2, \ldots$	WTA Corollary 1 One rule	ED Corollary 2 One rule	WTS Corollary 3 $a \in [0, \infty]$	WTA and ED Corollary 4 Two rules
Anonymity	+	+	+	+	+
Order preservation	+[a]	+	+[a]	+[a]	+[a]
Winner-loser strict Order preservation	Only WTA	+[a]	–	Only WTA	Only WTA
Strict order preservation	–	–	–	–	–
Endowment monotonicity	+[a]	+[a]	+	+	+
Winner strict Endowment monotonicity	Only WTS	+	+	+[a]	+
Strict Endowment monotonicity	Only ED	–	+[a]	Only ED	Only ED
Endowment additivity (Scale invariance)	Only WTA, ED	+	+	Only WTA, ED	+[a]
Consistency	+[a]	+[a]	+[a]	+[a]	+[a]

Notes. Plus indicates that each rule in the family satisfies the property, and minus indicates that each rule in the family does not satisfy the property. A WTS rule with representation a is an interval rule with $a_1 = a$ and $b_1 = \infty$. The WTA rule is an interval rule with $a_1 = 0$, $b_1 = \infty$, and a WTS rule with $a = 0$. The ED rule is an interval rule with $a_k = b_k = 0$ for all k and a WTS rule with $a = \infty$.
[a]Axiomatic characterizations.

$|\mathcal{R}(i) - \mathcal{R}(j)| \le |S| - 1$ for all $i, j \in S$, and each competitor $i \in S$,

$$\varphi_i(N, \mathcal{R}, E) = \varphi_i\left(S, \mathcal{R}_S, \sum_{j \in S} \varphi_j(N, \mathcal{R}, E)\right).$$

In contrast to consistency, the following example shows that a rule satisfying order preservation, endowment monotonicity, and local consistency does not necessarily satisfy anonymity.

Example 2. Let $i, j \in U$ be two potential competitors. Let the rule φ be defined in the following way. If i is ranked first and j is ranked second, then φ divides the endowment equally among competitors i and j. Otherwise, φ divides the endowment according to the WTA rule. Formally, φ assigns to each competition $(N, \mathcal{R}, E)$ the allocation such that if $i, j \in N$, $\mathcal{R}(i) = 1$, and $\mathcal{R}(j) = 2$, then

$$\varphi_k(N, \mathcal{R}, E) = \begin{cases} \frac{1}{2}E & \text{if } k \in \{i, j\}; \\ 0 & \text{otherwise,} \end{cases}$$

and otherwise,

$$\varphi_k(N, \mathcal{R}, E) = \begin{cases} E & \text{if } \mathcal{R}(k) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then, φ satisfies order preservation, endowment monotonicity, and local consistency but not anonymity. By Lemma 1, φ does not satisfy consistency.

Needless to say, all interval rules satisfy anonymity, order preservation, endowment monotonicity, and local consistency, but these rules are not the only ones. The following example provides another rule satisfying these properties.

Example 3. Let a rule be defined by allocating the endowment in the following way. Up to the first dollar is allocated to the winner. The surplus is divided equally among the competitors ranked first and second until they are allocated two dollars and one dollar, respectively. Then, the surplus is divided equally among the competitors ranked first, second, and third until they are allocated three dollars, two dollars, and one dollar, respectively. This procedure continues until the competitor ranked last is allocated one dollar. If there is still money left, it is divided equally among all competitors.

Tables 9–11 illustrate how the endowment from one to eight dollars is allocated among two, three, and four competitors, respectively. The allocation paths of the ED rule, the WTA rule, and the rule φ described in this example are illustrated in Figure 5. This rule also satisfies anonymity, order preservation, endowment monotonicity, and local consistency.

Note that the arithmetic type of rule in Example 3 satisfies winner strict endowment monotonicity (i.e., the competitor ranked first is better off when the endowment

Table 9. Eight Dollars Allocated Among Two Competitors in Example 3

φ_1	φ_2	E
1	0	1
3/2	1/2	2
2	1	3
5/2	3/2	4
3	2	5
7/2	5/2	6
4	3	7
9/2	7/2	8

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Table 10. Eight Dollars Allocated Among Three Competitors in Example 3

φ_1	φ_2	φ_3	E
1	0	0	1
3/2	1/2	0	2
2	1	0	3
7/3	4/3	1/3	4
8/3	5/3	2/3	5
3	2	1	6
10/3	7/3	4/3	7
11/3	8/3	5/3	8

increases). Each rule satisfying anonymity, order preservation, winner strict endowment monotonicity, and local consistency admits a compact indirect description. For each such rule, there exists a continuous and nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(x) \le x$ for each x such that the assigned allocation for each competition can be described in the following way. The competitor ranked first is allocated a prize of $x \in \mathbb{R}_+$. The competitor ranked second is allocated a prize of $f(x)$. The competitor ranked third is allocated a prize of $f(f(x))$ and so on. In total, the full endowment is allocated among the competitors. We call such rules *single-parametric rules*.

Single-Parametric Rules. There exists a continuous and nondecreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(x) \le x$ for each x such that for each competition $(N, \mathcal{R}, E)$ and each competitor $i \in N$,

$$\varphi_i(N, \mathcal{R}, E) = f^{(\mathcal{R}(i)-1)}(x),$$

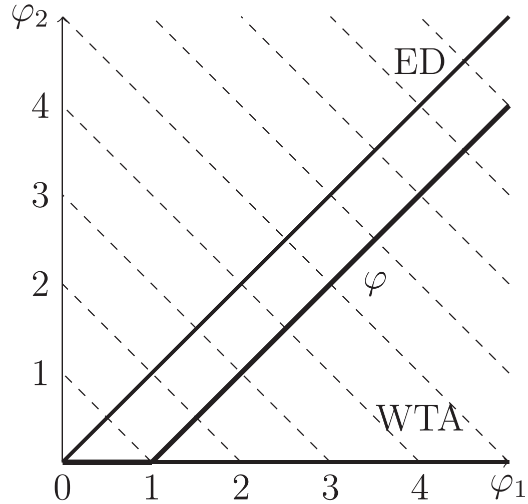
where we denote $f^{(0)}(x) = x$ and $f^{(k)}(x) = f(f^{(k-1)}(x))$ for each k and $x \in \mathbb{R}_+$ is chosen such that $\sum_{k=1}^{|N|} f^{(k-1)}(x) = E$.

Note that the ED rule is a single-parametric rule with $f(x) = x$ for each x. The WTA rule is a single-parametric rule with $f(x) = 0$ for each x. A winner-takes-surplus rule with representation a is a single-parametric rule with $f(x) = \min\{a, x\}$ for each x. The arithmetic type of rule from Example 3 is a single-parametric rule with $f(x) = \max\{0, x-1\}$ for each x.

Table 11. Eight Dollars Allocated Among Four Competitors in Example 3

φ_1	φ_2	φ_3	φ_4	E
1	0	0	0	1
3/2	1/2	0	0	2
2	1	0	0	3
7/3	4/3	1/3	0	4
8/3	5/3	2/3	0	5
3	2	1	0	6
13/4	9/4	5/4	1/4	7
7/2	5/2	3/2	1/2	8

Figure 5. Path of the Locally Consistent Rule in Example 3



Theorem 2. *A rule satisfies anonymity, order preservation, winner strict endowment monotonicity, and local consistency if and only if it is a single-parametric rule.*

Theorem 2 does not describe all rules satisfying anonymity, order preservation, endowment monotonicity, and local consistency. If such a rule does not satisfy winner strict endowment monotonicity, the prize of the competitor ranked second cannot be expressed as a function of the prize of the competitor ranked first. Moreover, as the following example shows, such a rule for competitions with two competitors does not necessarily have a unique locally consistent extension to competitions with more competitors.

Example 4. Let a rule be defined by allocating the endowment in the following way. Up to the first dollar is allocated to the winner. Subsequently, one dollar is allocated to the competitor ranked second, and then, another dollar is allocated to the winner. Thereafter, one dollar is allocated to the competitor ranked third, another dollar is allocated to the competitor ranked second, and then, another dollar is allocated to the winner. This procedure continues until the winner is allocated a number of dollars equal to the number of competitors. If there is still money left, first another dollar is allocated to the competitor ranked last, then another dollar is allocated to the competitor second to last, and so on. This continues until the full endowment is allocated among the competitors.

Tables 12–14 illustrate how the endowment from one to eight dollars is allocated among two, three, and four competitors, respectively. This rule satisfies anonymity, order preservation, endowment monotonicity, and local consistency, but it does not satisfy winner strict endowment monotonicity. Moreover, for competitions with two competitors, this rule is equal

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Table 12. Eight Dollars Allocated Among Two Competitors in Example 4

φ_1	φ_2	E
1	0	1
1	1	2
2	1	3
2	2	4
3	2	5
3	3	6
4	3	7
4	4	8

Table 14. Eight Dollars Allocated Among Four Competitors in Example 4

φ_1	φ_2	φ_3	φ_4	E
1	0	0	0	1
1	1	0	0	2
2	1	0	0	3
2	1	1	0	4
2	2	1	0	5
3	2	1	0	6
3	2	1	1	7
3	2	2	1	8

to the step rule with $\Delta = 1$ (see Figure 2 and Tables 2–4).

Subfamilies of single-parametric rules are obtained if order preservation is strengthened to strict order preservation or winner-loser strict order preservation or if winner strict endowment monotonicity is strengthened to strict endowment monotonicity. A single-parametric rule satisfies winner-loser strict order preservation if and only if it does not coincide with the ED rule for each positive endowment, it satisfies strict order preservation if and only if it does not coincide with the ED rule nor with the WTA rule for each positive endowment, and it satisfies strict endowment monotonicity if and only if it does not allocate any additional endowment according to the WTA rule.

Corollary 5. *Let φ be a single-parametric rule with representation f. Then, the following statements hold.*

i. *φ satisfies winner-loser strict order preservation if and only if $f(x) < x$ for each $x > 0$;*

ii. *φ satisfies strict order preservation if and only if $0 < f(x) < x$ for each $x > 0$; and*

iii. *φ satisfies strict endowment monotonicity if and only if f is increasing.*

If winner strict endowment monotonicity is strengthened to endowment additivity (scale invariance), then the interesting family of *geometric rules* is obtained. For each geometric rule, there exists $\lambda \in [0, 1]$ such that the assigned allocation for each competition can be described in the following way. The competitor ranked second is allocated a prize of λ times the prize of the competitor ranked first. The competitor ranked third is allocated a prize of λ times the prize of the competitor ranked second (i.e., λ^2 times the prize of the competitor ranked first). The competitor ranked fourth is allocated a prize of λ^3 times the prize of the competitor ranked first and so on. In total, the full endowment is allocated.

Table 13. Eight Dollars Allocated Among Three Competitors in Example 4

φ_1	φ_2	φ_3	E
1	0	0	1
1	1	0	2
2	1	0	3
2	1	1	4
2	2	1	5
3	2	1	6
3	2	2	7
3	3	2	8

Geometric Rules. There exists $\lambda \in [0, 1]$ such that for each competition $(N, \mathcal{R}, E)$ and each competitor $i \in N$,

$$\varphi_i(N, \mathcal{R}, E) = \frac{\lambda^{\mathcal{R}(i)-1}}{\sum_{k=1}^{|N|} \lambda^{k-1}} E.$$

Note that the ED rule is a geometric rule with $\lambda = 1$. The WTA rule is a geometric rule with $\lambda = 0$. (Here, we define $\lambda^0 = 1$ for each $\lambda \in [0, 1]$.) The allocation in the final stage of the poker tournament presented in Table 1 is a geometric rule with $\lambda = 0.713$. The allocation paths of the ED rule, the WTA rule, and the geometric rule φ in the poker tournament are illustrated in Figure 6.

Corollary 6. *A rule satisfies anonymity, order preservation, endowment additivity (scale invariance), and local consistency if and only if it is a geometric rule.*

4.2. Top Consistent Prize Allocation Rules

Theorem 2 shows that many rules satisfy anonymity, order preservation, winner strict endowment monotonicity, and local consistency. Yet, there are still several rules observed in practice that do not satisfy local consistency, such as the golf tournament in Table 1. Nevertheless, these rules may satisfy *top consistency*, which requires only consistent allocations for reduced competitions involving the highest-ranked participants. In other words, top consistency requires invariance for the participants in the top segment of the ranking when they are put in a separate competition. In competitions where participants are eliminated sequentially, top consistency can be interpreted as invariance under prize allocation at the moment of elimination.

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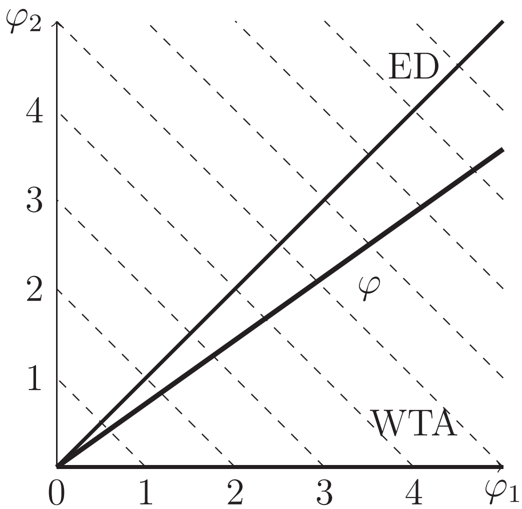
Figure 6. Path of a Geometric Rule



Top Consistency. For each competition $(N, \mathcal{R}, E)$, each nonempty subset of competitors $S \subseteq N$ with $\mathcal{R}(i) \le |S|$ for each $i \in S$, and each competitor $i \in S$,

$$\varphi_i(N,\mathcal{R},E) = \varphi_i\left(S,\mathcal{R}_S,\sum_{j\in S}\varphi_j(N,\mathcal{R},E)\right).$$

Needless to say, all single-parametric rules satisfy anonymity, order preservation, winner strict endowment monotonicity, and top consistency, but these rules are not the only ones. The following example provides another rule satisfying these properties.

Example 5. Let a rule be defined by allocating the endowment in the following way. Up to the first two dollars are allocated to the winner. The surplus is divided equally among the competitors ranked first and second until they are allocated three dollars and one dollar, respectively. Then, the surplus is divided equally among the competitors ranked first, second, and third until they are allocated four dollars, two dollars, and one dollar, respectively. This procedure continues until the competitor ranked last is allocated one dollar. If there is still money left, it is divided equally among all competitors.

Tables 15–17 illustrate how the endowment from one to eight dollars is allocated among two, three, and four competitors, respectively. This rule also satisfies anonymity, order preservation, winner strict endowment monotonicity, and top consistency.

Each rule satisfying anonymity, order preservation, winner strict endowment monotonicity, and top consistency admits a compact indirect description. For each such rule, there exist continuous and nondecreasing functions $f_1, f_2, \ldots : \mathbb{R}_+ \to \mathbb{R}_+$ with $f_1(x) = x$ for each x and $f_{k+1}(x) \le f_k(x)$ for each k such that the assigned allocation for each competition can be described in the following way. The competitor ranked first is allocated a prize of $f_1(x) = x$. The competitor ranked second is allocated a prize of $f_2(x)$. Generally, the prize for position k is $f_k(x)$. In total, the full endowment is allocated among the competitors. We call such rules *parametric rules*.[12]

Table 15. Eight Dollars Allocated Among Two Competitors in Example 5

φ_1	φ_2	E
1	0	1
2	0	2
5/2	1/2	3
3	1	4
7/2	3/2	5
4	2	6
9/2	5/2	7
5	3	8

Parametric Rules. There exist continuous and nondecreasing functions $f_1, f_2, \ldots : \mathbb{R}_+ \to \mathbb{R}_+$ with $f_1(x) = x$ for each x and $f_{k+1}(x) \le f_k(x)$ for each k such that for each competition $(N, \mathcal{R}, E)$ and each competitor $i \in N$,

$$\varphi_i(N,\mathcal{R},E) = f_{\mathcal{R}(i)}(x),$$

where $x \in \mathbb{R}_+$ is such that $\sum_{k=1}^{|N|} f_k(x) = E$.

Note that each single-parametric rule with representation f is a parametric rule with $f_k = f^{(k-1)}$ for each k. In particular, the ED rule is a parametric rule with $f_k(x) = x$ for each k. The WTA rule is a parametric rule with $f_1(x) = x$ and $f_k(x) = 0$ for each $k \ge 2$. A winner-takes-surplus rule with representation a is a parametric rule with $f_1(x) = x$ and $f_k(x) = \min\{a, x\}$ for each $k \ge 2$. A geometric rule with factor λ is a parametric rule with $f_k(x) = \lambda^{k-1}x$ for each k. The rule from Example 5 is a parametric rule with $f_1(x) = x$ and $f_k(x) = \max\{0, x-k\}$ for each $k \ge 2$.

Theorem 3. *A rule satisfies anonymity, order preservation, winner strict endowment monotonicity, and top consistency if and only if it is a parametric rule.*

Subfamilies of parametric rules are again directly obtained if order preservation is strengthened to strict

Table 16. Eight Dollars Allocated Among Three Competitors in Example 5

φ_1	φ_2	φ_3	E
1	0	0	1
2	0	0	2
5/2	1/2	0	3
3	1	0	4
10/3	4/3	1/3	5
11/3	5/3	2/3	6
4	2	1	7
13/3	7/3	4/3	8

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Table 17. Eight Dollars Allocated Among Four Competitors in Example 5

φ_1	φ_2	φ_3	φ_4	E
1	0	0	0	1
2	0	0	0	2
5/2	1/2	0	0	3
3	1	0	0	4
10/3	4/3	1/3	0	5
11/3	5/3	2/3	0	6
4	2	1	0	7
17/4	9/4	5/4	1/4	8

order preservation or winner-loser strict order preservation or if winner strict endowment monotonicity is strengthened to strict endowment monotonicity.

Corollary 7. *Let φ be a parametric rule with representation $f_1, f_2, \ldots$. Then, the following statements hold.*

i. *φ satisfies winner-loser strict order preservation if and only if $f_2(x) < x$ for each $x > 0$;*

ii. *φ satisfies strict order preservation if and only if $f_{k+1}(x) < f_k(x)$ for each k and each $x > 0$; and*

iii. *φ satisfies strict endowment monotonicity if and only if $f_k(x)$ is increasing for each k.*

If winner strict endowment monotonicity is strengthened to endowment additivity (scale invariance), then the family of widely used *proportional rules* is obtained. For each proportional rule, there exist $\lambda_1, \lambda_2, \ldots \in \mathbb{R}_+$ with $\lambda_1 > 0$ and $\lambda_{k+1} \leq \lambda_k$ for each k such that for each competition, the prize for position k is proportional to λ_k. In total, the full endowment is allocated.

Proportional Rules. There exist $\lambda_1, \lambda_2, \ldots \in \mathbb{R}_+$ with $\lambda_1 > 0$ and $\lambda_{k+1} \leq \lambda_k$ for each k such that for each competition $(N, \mathcal{R}, E)$ and each competitor $i \in N$,

$$\varphi_i(N, \mathcal{R}, E) = \frac{\lambda_{\mathcal{R}(i)}}{\sum_{k=1}^{|N|} \lambda_k} E.$$

Note that each geometric rule with factor λ is a proportional rule with $\lambda_k = \lambda^{k-1}$ for each k. In particular, the ED rule is a proportional rule with $\lambda_k = 1$ for each k. The WTA rule is a proportional rule with $\lambda_1 = 1$ and $\lambda_k = 0$ for each $k \geq 2$. The allocation rule in the golf tournament of Table 1 is a proportional rule with $\lambda_1 = 18.0$, $\lambda_2 = 10.9$, $\lambda_3 = 6.9$, and so on.

Corollary 8. *A rule satisfies anonymity, order preservation, endowment additivity (scale invariance), and top consistency if and only if it is a proportional rule.*

The results of Section 4 are presented in Table 18.

5. Concluding Remarks

We initiate here an axiomatic approach to study prize allocation rules in rank-order competitions. We introduce a model in which the competitors, their ranking, and the endowment are the primitives, and we axiomatically characterize three families of rules: interval rules (Theorem 1), single-parametric rules (Theorem 2), and parametric rules (Theorem 3). In the appendix, we show that the axioms used are logically independent. Moreover, we obtain several subfamilies: winner-takes-surplus rules (Corollary 3), geometric rules (Corollary 6), and proportional rules (Corollary 8). Each of these families and subfamilies includes the equal division rule and the winner-takes-all rule. The relations of the families are presented in Figure 7.

Our axiomatic framework can be further developed in many directions. A straightforward extension would incorporate tied positions. In that case, anonymity as well as order preservation implies equal prizes for tied competitors. In practice (e.g., in golf and poker), tied players get an equal share of the joint prize money for the positions they occupy. For instance, if two golf players in the PGA TOUR in Table 1 share the second position, then they each receive half of the second and the third prize, $(10.9 + 6.9)/2 = 8.9$, whereas the subsequent competitor gets the prize of the fourth position. Another possibility to accommodate ties is to generally allow for multiple competitors with the same position. Then, the two golf players would each get the second prize, 10.9, whereas the subsequent competitor would get the third prize. By adequately redefining all rules and axioms on the domain, including ties, both methods are compatible with the properties in our results.

For single rank-order tournaments, a natural direction is the study of other desirable axioms, such as population monotonicity and those formalizing meaningful lower bounds on the prizes to compensate for the participation costs. An alternative weakening of consistency that may be explored in our model is average consistency (Maschler and Owen 1989), which requires that the prize of each competitor equals their average prize in all reduced competitions.

A different type of extension arises particularly in situations where more data are available and more details can be taken into account for prize allocations. Many real-life organizers divide the endowment into an anonymous part and a nonanonymous part, where the latter is allocated according to fame, market power, historical results, participation experience, or any other characteristic of the competitors.[13] In some competitions, the ordinal ranking can be replaced by a cardinal ranking (e.g., using finish times, scores, or the volume of sales). Another possibility is to incorporate specific competition structures, such as knockout tournaments, round-robin tournaments, and multiple rank-order tournaments.

Multiple rank-order tournaments, consisting of a series of single rank-order tournaments, are interesting for the following reasons. In many real series, a competitor gets a prize and a number of points in

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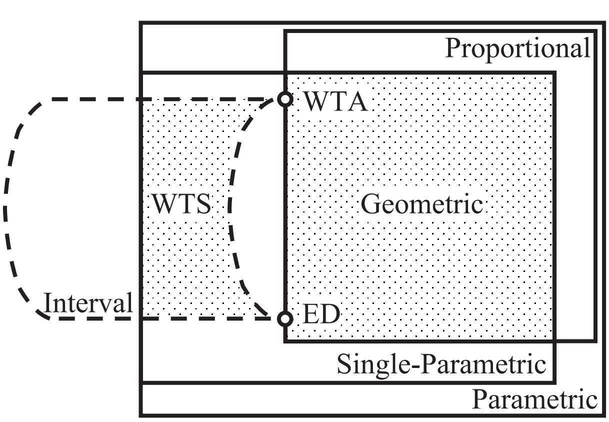
Table 18. Properties of Top and Locally Consistent Rules

	Single-parametric rules Theorem 2 and Corollary 5 Function $f: 0 \le f(x) \le x$	Geometric rules Corollary 6 $\lambda \in [0,1]$	Parametric rules Theorem 3 and Corollary 7 Functions f_k: $0 \le f_{k+1}(x) \le f_k(x) \le x$ $k = 1, 2, \ldots$	Proportional rules Corollary 8 $\lambda_k \in [0,\infty)$: $\lambda_{k+1} \le \lambda_k$ $k = 1, 2, \ldots$
Anonymity	+[a]	+[a]	+[a]	+[a]
Order preservation	+[a]	+[a]	+[a]	+[a]
Winner-loser strict order preservation	$f(x) < x$ for $x > 0$	All but ED	$f_2(x) < x$ for $x > 0$	$\lambda_2 < \lambda_1$
Strict order preservation	$0 < f(x) < x$ for $x > 0$	All but WTA, ED	$f_{k+1}(x) < f_k(x)$ for $x > 0$, for each k	$\lambda_{k+1} < \lambda_k$ for each k
Endowment monotonicity	+	+	+	+
Winner strict endowment monotonicity	+[a]	+	+[a]	+
Strict endowment monotonicity	f is increasing	All but WTA	f_k is increasing for each k	$\lambda_k > 0$ for each k
Endowment additivity (scale invariance)	Only geometric	+[a]	Only proportional	+[a]
Top consistency	+	+	+[a]	+[a]
Local consistency	+[a]	+[a]	Only single-parametric	Only geometric
Consistency	Only WTS	Only WTA, ED	Only WTS	Only WTA, ED

Notes. Plus indicates that each rule in the family satisfies the property. The WTA is a single-parametric rule with $f(x) = 0$ for each x, a geometric rule with $\lambda = 0$, a parametric rule with $f_1(x) = x$ and $f_k(x) = 0$ for each $k \ge 2$, and a proportional rule with $\lambda_1 = 1$ and $\lambda_k = 0$ for each $k \ge 2$. The ED rule is a single-parametric rule with $f(x) = x$ for each x, a geometric rule with $\lambda = 1$, a parametric rule with $f_k(x) = x$ for each k, and a proportional rule with $\lambda_k = 1$ for each k. The WTS rule with representation a is a single-parametric rule with $f(x) = \min\{a, x\}$ for each x and a parametric rule with $f_1(x) = x$ and $f_k(x) = \min\{a, x\}$ for each $k \ge 2$. A geometric rule with factor λ is a single-parametric rule with $f(x) = \lambda x$ for each x, a parametric rule with $f_k(x) = \lambda^{k-1} x$ for each k, and a proportional rule with $\lambda_k = \lambda^{k-1}$ for each k. A single-parametric rule with representation f is a parametric rule with $f_k = f^{(k-1)}$ for each k. A proportional rule with ratio λ_k is a parametric rule with $f_k(x) = \lambda_k x / \lambda_1$ for each k.
[a]Axiomatic characterizations.

each single tournament. Then, the sum of points determines the aggregate ranking and the bonuses for the entire series. A straightforward question is how to jointly choose a points system, a prize structure for each single tournament, and a bonus structure for the entire series. In particular, because Corollary 6 calls for geometric prize sequences and Kondratev et al. (2022) justify geometric point sequences, should we choose the same parameter for both geometric sequences?

Figure 7. Families of Prize Allocation Rules



Another open question is how to apply for competitions the rules developed for ranking, voting, or budget allocation. For instance, Kreweras (1965) and Fishburn (1984) developed a probabilistic voting rule known as maximal lotteries. Brandl et al. (2016, p. 1843) noted that "the lotteries returned by probabilistic social choice functions do not necessarily have to be interpreted as probability distributions. They can, for instance, also be seen as fractional allocations of divisible objects such as time shares or monetary budgets." Airiau et al. (2019, p. 12) argued that "the maximal lotteries rule, while attractive according to consistency axioms, spends the entire budget on the Condorcet winner if it exists. This is often undesirable in a budgeting context." We can conclude from these arguments that any application of well-known ranking, voting, or allocation rules must be remotivated and rejustified in the context of competitions.

Acknowledgments

The authors thank three anonymous reviewers, the associate editor, Alexander Nesterov, Egor Ianovski, Hervé Moulin, and other colleagues from the International Laboratory of Game Theory and Decision Making for helpful comments. The authors give special thanks to William Thomson (the University of Rochester) for detailed suggestions. They also

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thank David Connolly (HSE University) for help with language editing.

Appendix.

Endowment monotonicity is a stronger property than *endowment continuity*, which requires that small changes in the endowment have a small impact on the assigned allocation. This is standard in the literature on fair allocation.

Endowment Continuity

For each set of competitors N and their ranking $\mathcal{R}$, we have $\varphi(N,\mathcal{R},E) \to \varphi(N,\mathcal{R},E')$ if $E \to E'$.

Lemma A.1. *If a rule satisfies endowment monotonicity, then it satisfies endowment continuity.*

Proof. Let φ be a rule satisfying endowment monotonicity. Let $(N,\mathcal{R},E)$ and $(N,\mathcal{R},E')$ be two competitions. Let $i \in N$. By endowment monotonicity,

$$\begin{aligned}|E-E'| &= \left|\sum_{j\in N}\varphi_j(N,\mathcal{R},E) - \sum_{j\in N}\varphi_j(N,\mathcal{R},E')\right| \\ &= \left|\sum_{j\in N}(\varphi_j(N,\mathcal{R},E) - \varphi_j(N,\mathcal{R},E'))\right| \\ &= \sum_{j\in N}\left|\varphi_j(N,\mathcal{R},E) - \varphi_j(N,\mathcal{R},E')\right| \\ &\geq |\varphi_i(N,\mathcal{R},E) - \varphi_i(N,\mathcal{R},E')|\,.\end{aligned}$$

This means that $\varphi_i(N,\mathcal{R},E) \to \varphi_i(N,\mathcal{R},E')$ if $E \to E'$. Hence, φ satisfies endowment continuity. □

Lemma 1. *If a rule satisfies order preservation, endowment monotonicity, and consistency, then it satisfies anonymity.*

Proof. Let φ be a rule satisfying order preservation, endowment monotonicity, and consistency. By Lemma A.1, φ satisfies endowment continuity.

Suppose for the sake of contradiction that φ does not satisfy anonymity. Then, there exist two competitions $(N,\mathcal{R},E)$ and $(N',\mathcal{R}',E)$ with equal numbers of competitors $|N|=|N'|$ and two competitors $i_1 \in N$ and $j_1 \in N'$ with equal positions $\mathcal{R}(i_1)=\mathcal{R}'(j_1)$ such that $\varphi_{i_1}(N,\mathcal{R},E) < \varphi_{j_1}(N',\mathcal{R}',E)$. Because

$$\sum_{k\in N}\varphi_k(N,\mathcal{R},E) = E = \sum_{k\in N'}\varphi_k(N',\mathcal{R}',E),$$

there exist two competitors $i_2 \in N$ and $j_2 \in N'$ with equal positions $\mathcal{R}(i_2)=\mathcal{R}'(j_2)$ such that $\varphi_{i_2}(N,\mathcal{R},E) > \varphi_{j_2}(N',\mathcal{R}',E)$. Suppose without loss of generality that $\mathcal{R}'(j_1)=\mathcal{R}(i_1) < \mathcal{R}(i_2)=\mathcal{R}'(j_2)$. Denote $x=\varphi(N,\mathcal{R},E)$ and $y=\varphi(N',\mathcal{R}',E)$. By consistency,

$$(x_{i_1},x_{i_2}) = \varphi(\{i_1,i_2\},\mathcal{R}_{\{i_1,i_2\}},x_{i_1}+x_{i_2})$$

and

$$(y_{j_1},y_{j_2}) = \varphi(\{j_1,j_2\},\mathcal{R}'_{\{j_1,j_2\}},y_{j_1}+y_{j_2}).$$

This is illustrated in the following way:

N	$\mathcal{R}$	φ
i_1	1	x_{i_1}
i_2	2	x_{i_2}

N	$\mathcal{R}$	φ
j_1	1	y_{j_1}
j_2	2	y_{j_2}.

(A.1)

By order preservation, $y_{j_2} < x_{i_2} \leq x_{i_1} < y_{j_1}$. One of the following six cases holds.

Case A.1. $i_1 = j_2$ and $i_2 = j_1$.

By (A.1),

N	$\mathcal{R}$	φ
$i_1 = j_2$	1	x_{i_1}
$i_2 = j_1$	2	x_{i_2}

N	$\mathcal{R}$	φ
j_1	1	y_{j_1}
j_2	2	y_{j_2}.

Then, there exists another competitor $k \in U\backslash\{i_1,i_2\}$. By order preservation, endowment continuity, and consistency, there exist endowments E' and E'' such that

N	$\mathcal{R}$	φ
i_1	1	x_{i_1}
i_2	2	x_{i_2}
k	3	$E'-x_{i_1}-x_{i_2}$

N	$\mathcal{R}$	φ
i_1	1	x_{i_1}
k	2	$E''-x_{i_1}-x_{i_2}$
i_2	3	x_{i_2}.

By order preservation, $E'-x_{i_1}-x_{i_2} \leq x_{i_2} \leq E''-x_{i_1}-x_{i_2}$. By consistency,

N	$\mathcal{R}$	φ
i_1	1	x_{i_1}
k	2	$E'-x_{i_1}-x_{i_2}$

N	$\mathcal{R}$	φ
i_1	1	x_{i_1}
k	2	$E''-x_{i_1}-x_{i_2}$.

By endowment monotonicity,

N	$\mathcal{R}$	φ
i_1	1	x_{i_1}
k	2	x_{i_2}.

By order preservation, endowment continuity, consistency, and (A.1), there exist endowments E''' and E'''' such that

N	$\mathcal{R}$	φ
i_1	1	x_{i_1}
i_2	2	$E'''-x_{i_1}-x_{i_2}$
k	3	x_{i_2}

N	$\mathcal{R}$	φ
j_1	1	y_{j_1}
j_2	2	y_{j_2}
k	3	$E''''-y_{j_1}-y_{j_2}$.

By order preservation, $E'''-x_{i_1}-x_{i_2} \leq x_{i_1}$ and $E''''-y_{j_1}-y_{j_2} \leq y_{j_2}$. By consistency,

N	$\mathcal{R}$	φ
i_2	1	$E'''-x_{i_1}-x_{i_2}$
k	2	x_{i_2}

N	$\mathcal{R}$	φ
$i_2 = j_1$	1	y_{j_1}
k	2	$E''''-y_{j_1}-y_{j_2}$.

Then, $E'''-x_{i_1}-x_{i_2} \leq x_{i_1} < y_{j_1}$ and $x_{i_2} > y_{j_2} \geq E''''-y_{j_1}-y_{j_2}$. This contradicts endowment monotonicity.

Case A.2. $i_1 = j_2$ and $i_2 \neq j_1$.

By order preservation, endowment continuity, consistency, and (A.1), there exists endowment E' such that

N	$\mathcal{R}$	φ
i_1	1	x_{i_1}
j_1	2	$E'-x_{i_1}-x_{i_2}$
i_2	3	x_{i_2}.

By order preservation, $E'-x_{i_1}-x_{i_2} \geq x_{i_2}$. By consistency,

N	$\mathcal{R}$	φ
$i_1 = j_2$	1	x_{i_1}
j_1	2	$E'-x_{i_1}-x_{i_2}$.

Then, $x_{i_1} < y_{j_1}$ and $E'-x_{i_1}-x_{i_2} \geq x_{i_2} > y_{j_2}$. By (A.1), a contradiction follows from Case A.1.

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Case A.3. $i_1 \neq j_2$ and $i_2 = j_1$.

By order preservation, endowment continuity, consistency, and (A.1), there exists endowment E' such that

N	$\mathcal{R}$	φ
i_1	1	x_{i_1}
j_2	2	$E' - x_{i_1} - x_{i_2}$
i_2	3	x_{i_2}.

By order preservation, $E' - x_{i_1} - x_{i_2} \leq x_{i_1}$. By consistency,

N	$\mathcal{R}$	φ
j_2	1	$E' - x_{i_1} - x_{i_2}$
$i_2 = j_1$	2	x_{i_2}.

Then, $E' - x_{i_1} - x_{i_2} \leq x_{i_1} < y_{j_1}$ and $x_{i_2} > y_{j_2}$. By (A.1), a contradiction follows from Case A.1.

Case A.4. $i_1 = j_1$ and $i_2 \neq j_2$.

By order preservation, endowment continuity, consistency, and (A.1), there exists endowment E' such that

N	$\mathcal{R}$	φ
i_1	1	x_{i_1}
j_2	2	$E' - x_{i_1} - x_{i_2}$
i_2	3	x_{i_2}.

By order preservation, $E' - x_{i_1} - x_{i_2} \geq x_{i_2}$. By consistency,

N	$\mathcal{R}$	φ
$i_1 = j_1$	1	x_{i_1}
j_2	2	$E' - x_{i_1} - x_{i_2}$.

Then, $x_{i_1} < y_{j_1}$ and $E' - x_{i_1} - x_{i_2} \geq x_{i_2} > y_{j_2}$. By (A.1), this contradicts endowment monotonicity.

Case A.5. $i_1 \neq j_1$ and $i_2 = j_2$.

By order preservation, endowment continuity, consistency, and (A.1), there exists endowment E' such that

N	$\mathcal{R}$	φ
i_1	1	x_{i_1}
j_1	2	$E' - x_{i_1} - x_{i_2}$
i_2	3	x_{i_2}.

By order preservation, $E' - x_{i_1} - x_{i_2} \leq x_{i_1}$. By consistency,

N	$\mathcal{R}$	φ
j_1	1	$E' - x_{i_1} - x_{i_2}$
$i_2 = j_2$	2	x_{i_2}.

Then, $E' - x_{i_1} - x_{i_2} \leq x_{i_1} < y_{j_1}$ and $x_{i_2} > y_{j_2}$. By (A.1), this contradicts endowment monotonicity.

Case A.6. $i_1 \neq j_1 \neq i_2$ and $i_1 \neq j_2 \neq i_2$.

By order preservation, endowment continuity, consistency, and (A.1), there exist z_{j_1} and z_{j_2} such that

N	$\mathcal{R}$	φ
i_1	1	x_{i_1}
j_1	2	z_{j_1}
j_2	3	z_{j_2}
i_2	4	x_{i_2}.

By order preservation, $z_{j_1} \leq x_{i_1}$ and $z_{j_2} \geq x_{i_2}$. By consistency,

N	$\mathcal{R}$	φ
j_1	1	z_{j_1}
j_2	2	z_{j_2}.

Then, $z_{j_1} \leq x_{i_1} < y_{j_1}$ and $z_{j_2} \geq x_{i_2} > y_{j_2}$. By (A.1), this contradicts endowment monotonicity. □

Lemma 2 and its proof are analogous to theorem A of Aumann and Maschler (1985) in the context of claims problems. This type of result is also known as an elevator lemma (Thomson 2011).

Lemma 2. *If two rules satisfying endowment monotonicity and consistency coincide for each competition with two competitors, then the two rules coincide for each competition with an arbitrary number of competitors.*

Proof. Let φ and φ' be two rules satisfying endowment monotonicity and consistency such that $\varphi(N, \mathcal{R}, E) = \varphi'(N, \mathcal{R}, E)$ for each competition with two competitors. Suppose for the sake of contradiction that there exists a competition $(N, \mathcal{R}, E)$ such that $\varphi(N, \mathcal{R}, E) \neq \varphi'(N, \mathcal{R}, E)$. Because

$$\sum_{i \in N} \varphi_i(N, \mathcal{R}, E) = E = \sum_{i \in N} \varphi'_i(N, \mathcal{R}, E),$$

there exist two competitors $i \in N$ and $j \in N$ such that

$$\varphi_i(N, \mathcal{R}, E) < \varphi'_i(N, \mathcal{R}, E) \text{ and } \varphi_j(N, \mathcal{R}, E) > \varphi'_j(N, \mathcal{R}, E). \tag{A.2}$$

By consistency,

$$(\varphi_i(N, \mathcal{R}, E), \varphi_j(N, \mathcal{R}, E)) = \varphi(\{i, j\}, \mathcal{R}_{\{i,j\}}, \varphi_i(N, \mathcal{R}, E) + \varphi_j(N, \mathcal{R}, E))$$

and

$$\begin{aligned}(\varphi'_i(N, \mathcal{R}, E), \varphi'_j(N, \mathcal{R}, E)) &= \varphi'(\{i, j\}, \mathcal{R}_{\{i,j\}}, \varphi'_i(N, \mathcal{R}, E) + \varphi'_j(N, \mathcal{R}, E)) \\ &= \varphi(\{i, j\}, \mathcal{R}_{\{i,j\}}, \varphi'_i(N, \mathcal{R}, E) + \varphi'_j(N, \mathcal{R}, E)).\end{aligned}$$

By endowment monotonicity,

$$\varphi_i(N, \mathcal{R}, E) \leq \varphi'_i(N, \mathcal{R}, E) \text{ and } \varphi_j(N, \mathcal{R}, E) \leq \varphi'_j(N, \mathcal{R}, E),$$

or

$$\varphi_i(N, \mathcal{R}, E) \geq \varphi'_i(N, \mathcal{R}, E) \text{ and } \varphi_j(N, \mathcal{R}, E) \geq \varphi'_j(N, \mathcal{R}, E).$$

This contradicts (A.2). Hence, $\varphi(N, \mathcal{R}, E) = \varphi'(N, \mathcal{R}, E)$ for each competition $(N, \mathcal{R}, E)$. □

Theorem 1. *A rule satisfies order preservation, endowment monotonicity, and consistency if and only if it is an interval rule.*

Proof. It is readily checked that each interval rule satisfies order preservation, endowment monotonicity, and consistency.

Let φ be a rule satisfying these properties. By Lemma A.1, φ satisfies endowment continuity. By Lemma 1, φ satisfies anonymity. By Lemma 2, we only need to show that φ is an interval rule for each competition with two competitors because each such rule has a unique consistent extension to competitions with more competitors. Let $N \subseteq U$ with $|N| = 2$, and let $\mathcal{R}$ be a ranking. Denote $N = \{1, 2\}$ such that $\mathcal{R}(1) = 1$ and $\mathcal{R}(2) = 2$. The proof consists of two steps.

Step A.1. For each endowment E, if $E = x_1 + x_2$ such that

$$\varphi(N, \mathcal{R}, x_1 + x_2) = (x_1, x_2), \tag{A.3}$$

then

$$\varphi(N, \mathcal{R}, x_1 + x_1) = (x_1, x_1) \text{ or } \varphi(N, \mathcal{R}, x_2 + x_2) = (x_2, x_2). \tag{A.4}$$

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Proof of Step A.1. Let E be an endowment, and denote $\varphi(N,\mathcal{R},E)=(x_1,x_2)$. Then, $E=x_1+x_2$, and (A.3) holds. By order preservation, $x_1\geq x_2$. If $x_1=x_2$, then (A.4) follows immediately from (A.3).

Suppose that $x_1>x_2$. Let $N'=\{1,2,3\}$, and let $\mathcal{R}'$ be a ranking of N' such that $\mathcal{R}'(1)=1$, $\mathcal{R}'(2)=2$, and $\mathcal{R}'(3)=3$. By order preservation and endowment continuity, there exists endowment E' such that $\varphi_1(N',\mathcal{R}',E')+\varphi_3(N',\mathcal{R}',E')=x_1+x_2$. By anonymity, consistency, and (A.1),

$$\varphi(N',\mathcal{R}',E')=(x_1,E'-x_1-x_2,x_2). \tag{A.5}$$

By order preservation, $x_1+x_2+x_2\leq E'\leq x_1+x_1+x_2$. By anonymity and consistency,

$$\varphi(N,\mathcal{R},E'-x_1)=(E'-x_1-x_2,x_2) \tag{A.6}$$

$$\text{and } \varphi(N,\mathcal{R},E'-x_2)=(x_1,E'-x_1-x_2). \tag{A.7}$$

If $E'=x_1+x_2+x_2$, then (A.4) follows immediately from (A.6). If $E'=x_1+x_1+x_2$, then (A.4) follows immediately from (A.7).

Suppose that $x_1+x_2+x_2<E'<x_1+x_1+x_2$. Then, $E'-x_1<x_1+x_2<E'-x_2$. By endowment monotonicity, (A.3), (A.6), and (A.7),

$$\varphi(N,\mathcal{R},E'')=\begin{cases}(E''-x_2,x_2) & \text{if } E'-x_1\leq E''\leq x_1+x_2;\\ (x_1,E''-x_1) & \text{if } x_1+x_2\leq E''\leq E'-x_2.\end{cases} \tag{A.8}$$

Denote $(y_1,y_2,y_3)=\varphi(N',\mathcal{R}',x_1+x_2+x_2)$ and $(z_1,z_2,z_3)=\varphi(N',\mathcal{R}',x_1+x_1+x_2)$. By endowment monotonicity and (A.5),

$$\begin{aligned}&y_1\leq x_1\leq z_1,\\ &y_2\leq E'-x_1-x_2\leq z_2,\\ &y_3\leq x_2\leq z_3.\end{aligned}$$

Then, $y_1+y_2\leq E'-x_2$ and $E'-x_1\leq z_2+z_3$. Because $y_3\leq x_2$ and $y_1+y_2+y_3=x_1+x_2+x_2$, we have $x_1+x_2\leq x_1+x_2+x_2-y_3=y_1+y_2$. Because $x_1\leq z_1$ and $z_1+z_2+z_3=x_1+x_1+x_2$, we have $z_2+z_3=x_1+x_1+x_2-z_1\leq x_1+x_2$. This means that

$$\begin{aligned}&x_1+x_2\leq y_1+y_2\leq E'-x_2,\\ \text{and}\quad &E'-x_1\leq z_2+z_3\leq x_1+x_2.\end{aligned}$$

By anonymity, consistency, and (A.8),

$$\begin{aligned}&\varphi_1(N,\mathcal{R},y_1+y_3)=y_1=\varphi_1(N,\mathcal{R},y_1+y_2)=x_1,\\ \text{and}\quad &\varphi_2(N,\mathcal{R},z_1+z_3)=z_3=\varphi_2(N,\mathcal{R},z_2+z_3)=x_2.\end{aligned}$$

Because $y_1\leq z_1$ and $y_3\leq z_3$, we have $y_1+y_3\leq z_1+z_3$. By endowment monotonicity and (A.8), for each E'' such that $\varphi_1(N,\mathcal{R},E'')\geq x_1$, we have $E''\geq x_1+x_2$. Because $\varphi_1(N,\mathcal{R},y_1+y_3)=x_1$, we have $y_1+y_3\geq x_1+x_2$. By endowment monotonicity and (A.8), for each E'' such that $\varphi_2(N,\mathcal{R},E'')\leq x_2$, we have $E''\leq x_1+x_2$. Because $\varphi_2(N,\mathcal{R},z_1+z_3)=x_2$, we have $z_1+z_3\leq x_1+x_2$. Hence,

$$x_1+x_2\leq y_1+y_3\leq z_1+z_3\leq x_1+x_2.$$

Then, $y_1+y_3=x_1+x_2$ and $z_1+z_3=x_1+x_2$. Because $y_1=x_1$ and $z_3=x_2$, we have $y_3=x_2$ and $z_1=x_1$. This means that $y=(x_1,x_2,x_2)$ and $z=(x_1,x_1,x_2)$. By anonymity and consistency, $\varphi(N,\mathcal{R},x_1+x_1)=(x_1,x_1)$ and $\varphi(N,\mathcal{R},x_2+x_2)=(x_2,x_2)$. Hence, (A.4) holds. □

Step A.2. There exist disjoint intervals $(a_1,b_1),(a_2,b_2),\ldots$ with $a_1,a_2,\ldots\in\mathbb{R}_+$ and $b_1,b_2,\ldots\in\mathbb{R}_+\cup\{+\infty\}$ such that

$$\varphi(N,\mathcal{R},E)=\begin{cases}(E-a_k,a_k) & \text{if } a_k+a_k\leq E\leq b_k+a_k;\\ (b_k,E-b_k) & \text{if } b_k+a_k\leq E\leq b_k+b_k;\\ \dfrac{E}{2} & \text{otherwise.}\end{cases} \tag{A.9}$$

Proof of Step A.2. If $\varphi_1(N,\mathcal{R},E)=\varphi_2(N,\mathcal{R},E)$ for each endowment E, then (A.9) follows immediately by defining $a_k=b_k=0$ for each k.

Let E be an endowment such that $\varphi_1(N,\mathcal{R},E)>\varphi_2(N,\mathcal{R},E)$. Let $(x_1,x_2)=\varphi(N,\mathcal{R},E)$. Then, $E=x_1+x_2$, $x_1>x_2$, and

$$\varphi(N,\mathcal{R},x_1+x_2)=(x_1,x_2).$$

By Step A.1, $\varphi(N,\mathcal{R},x_1+x_1)=(x_1,x_1)$ or $\varphi(N,\mathcal{R},x_2+x_2)=(x_2,x_2)$.

Suppose that $\varphi(N,\mathcal{R},x_1+x_1)=(x_1,x_1)$. Define

$$\begin{aligned}&b_E=x_1\\ \text{and}\quad &a_E=\min_{E'\in\mathbb{R}_+}\{\varphi_2(N,\mathcal{R},E')\mid\varphi_1(N,\mathcal{R},E')=b_E\}.\end{aligned} \tag{A.10}$$

Then, $a_E<b_E$ because $a_E\leq x_2<x_1=b_E$. Moreover, $\varphi(N,\mathcal{R},b_E+a_E)=(b_E,a_E)$ and $\varphi(N,\mathcal{R},b_E+b_E)=(b_E,b_E)$. By endowment monotonicity,

$$\varphi(N,\mathcal{R},E')=(b_E,E'-b_E)\quad\text{if}\quad b_E+a_E\leq E'\leq b_E+b_E. \tag{A.11}$$

This also means that

$$b_E+a_E<E'<b_E+b_E\quad\text{if}\quad a_E<\varphi_2(N,\mathcal{R},E')<b_E. \tag{A.12}$$

Let E' be an endowment with $a_E+a_E<E'<b_E+a_E$. Denote $(y_1,y_2)=\varphi(N,\mathcal{R},E')$. By endowment monotonicity and (A.10), $y_1<b_E$ and $y_2\leq a_E$. Because $a_E\leq a_E+a_E-y_2<E'-y_2=y_1$, then $a_E<y_1$. By Step A.1, $\varphi(N,\mathcal{R},y_1+y_1)=(y_1,y_1)$ or $\varphi(N,\mathcal{R},y_2+y_2)=(y_2,y_2)$. If $\varphi(N,\mathcal{R},y_1+y_1)=(y_1,y_1)$, then $a_E<y_1=\varphi_2(N,\mathcal{R},y_1+y_1)<b_E$, (A.12) implies $b_E+a_E<y_1+y_1<b_E+b_E$, and (A.11) implies $y_1=\varphi_1(N,\mathcal{R},y_1+y_1)=b_E$, which is a contradiction. This means that $\varphi(N,\mathcal{R},\varphi_2(N,\mathcal{R},E')+\varphi_2(N,\mathcal{R},E'))=(\varphi_2(N,\mathcal{R},E'),\varphi_2(N,\mathcal{R},E'))$ for each endowment E' with $a_E+a_E<E'<b_E+a_E$. Because $\varphi_2(N,\mathcal{R},b_E+a_E)=a_E$, by endowment continuity we have $\varphi(N,\mathcal{R},a_E+a_E)=(a_E,a_E)$. By endowment monotonicity,

$$\varphi(N,\mathcal{R},E')=(E'-a_E,a_E)\quad\text{if}\quad a_E+a_E\leq E'\leq b_E+a_E.$$

Suppose that $\varphi(N,\mathcal{R},x_2+x_2)=(x_2,x_2)$. Define

$$\begin{aligned}&a_E=x_2\\ \text{and } &b_E=\sup_{E'\in\mathbb{R}_+}\{\varphi_1(N,\mathcal{R},E')\mid\varphi_2(N,\mathcal{R},E')=a_E\}.\end{aligned} \tag{A.13}$$

Then, $a_E<b_E$ because $a_E=x_2<x_1\leq b_E$. Moreover, $\varphi(N,\mathcal{R},a_E+a_E)=(a_E,a_E)$ and $\varphi(N,\mathcal{R},b_E+a_E)=(b_E,a_E)$. By endowment monotonicity,

$$\varphi(N,\mathcal{R},E')=(E'-a_E,a_E)\quad\text{if}\quad a_E+a_E\leq E'\leq b_E+a_E. \tag{A.14}$$

This also means that

$$a_E+a_E<E'<b_E+a_E\quad\text{if}\quad a_E<\varphi_1(N,\mathcal{R},E')<b_E. \tag{A.15}$$

Let E' be an endowment with $b_E+a_E<E'<b_E+b_E$. Denote $(y_1,y_2)=\varphi(N,\mathcal{R},E')$. By endowment monotonicity and (A.13), $b_E\leq y_1$ and $a_E<y_2$. Then, $y_2<b_E$ because $y_2=E'-y_1<b_E+b_E-y_1\leq b_E$. By Step A.1, $\varphi(N,\mathcal{R},y_1+y_1)=(y_1,y_1)$ or $\varphi(N,\mathcal{R},y_2+y_2)=(y_2,y_2)$. If $\varphi(N,\mathcal{R},y_2+y_2)=(y_2,y_2)$, then $a_E<y_2=\varphi_1(N,\mathcal{R},y_2+y_2)<b_E$, (A.15) implies $a_E+a_E<y_2+y_2<b_E+a_E$, and (A.14) implies $y_2=\varphi_2(N,\mathcal{R},y_2+y_2)=a_E$, which is a contradiction. This means that $\varphi(N,\mathcal{R},\varphi_1(N,\mathcal{R},E')+\varphi_1(N,\mathcal{R},E'))=(\varphi_1(N,\mathcal{R},E'),\varphi_1(N,\mathcal{R},E'))$ for each endowment E' with $b_E+a_E<E'<b_E+b_E$. Because $\varphi_1(N,\mathcal{R},b_E+a_E)=b_E$,

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by endowment continuity we have $\varphi(N,\mathcal{R},b_E+b_E)=(b_E,b_E)$. By endowment monotonicity,

$$\varphi(N,\mathcal{R},E')=(b_E,E'-a_E)\quad\text{if}\quad b_E+a_E\le E'\le b_E+b_E.$$

The set of intervals $\{(a_E,b_E):\varphi_1(N,\mathcal{R},E)>\varphi_2(N,\mathcal{R},E)\}$ is countable because the intervals are disjoint, and we can construct a one-to-one correspondence between each interval and a rational number from the interval. Hence, (A.9) holds. □

By anonymity, (A.9) holds for each competition with two competitors. By Lemma 2, the description of the corresponding interval rule is the unique consistent extension of (A.9) to competitions with more competitors. □

Remark A.1. The axioms order preservation, endowment monotonicity, and consistency (used in Theorem 1) are logically independent.

Proof. The rule that allocates all the prize money to the competitor ranked last satisfies endowment monotonicity and consistency but not order preservation. The rule that coincides with the equal division rule for competitions with an endowment of at most one dollar and with the winner-takes-all rule for competitions with a higher endowment satisfies order preservation and consistency but not endowment monotonicity. The rule from Example 2 satisfies order preservation and endowment monotonicity but not consistency. □

Lemma 3. *A rule satisfies endowment additivity if and only if it satisfies scale invariance.*

Proof. Let φ be a rule satisfying scale invariance. Let $(N,\mathcal{R},E)$ and $(N,\mathcal{R},E')$ be two competitions, and let $i\in N$ be a competitor. Then,

$$\begin{aligned}\varphi_i(N,\mathcal{R},E+E')&=(E+E')\varphi_i(N,\mathcal{R},1)\\&=E\varphi_i(N,\mathcal{R},1)+E'\varphi_i(N,\mathcal{R},1)\\&=\varphi_i(N,\mathcal{R},E)+\varphi_i(N,\mathcal{R},E').\end{aligned}$$

Hence, φ satisfies endowment additivity.

Now, let φ be a rule satisfying endowment additivity. Then, φ satisfies endowment monotonicity. By Lemma A.1, φ satisfies endowment continuity. Let $(N,\mathcal{R},E)$ be a competition, and let $i\in N$ be a competitor. If E is a rational number, then there exist two natural numbers $p\in\mathbb{N}$ and $q\in\mathbb{N}$ such that $E=\frac{p}{q}$. By endowment additivity,

$$\begin{aligned}\varphi_i(N,\mathcal{R},E)&=\varphi_i\left(N,\mathcal{R},\frac{p}{q}\right)\\&=p\varphi_i\left(N,\mathcal{R},\frac{1}{q}\right)\\&=\frac{p}{q}q\varphi_i\left(N,\mathcal{R},\frac{1}{q}\right)\\&=\frac{p}{q}\varphi_i(N,\mathcal{R},1)\\&=E\varphi_i(N,\mathcal{R},1).\end{aligned}$$

By endowment continuity, $\varphi_i(N,\mathcal{R},E)=E\varphi_i(N,\mathcal{R},1)$ for each real number E. Hence, φ satisfies scale invariance. □

Theorem 2. *A rule satisfies anonymity, order preservation, winner strict endowment monotonicity, and local consistency if and only if it is a single-parametric rule.*

Proof. It is readily checked that each single-parametric rule satisfies anonymity, order preservation, winner strict endowment monotonicity, and local consistency.

Let φ be a rule satisfying these properties. Then, φ satisfies endowment monotonicity. By Lemma A.1, φ satisfies endowment continuity. Let $N\subseteq U$ with $|N|=2$, and let $\mathcal{R}$ be a ranking. Denote $N=\{1,2\}$ such that $\mathcal{R}(1)=1$ and $\mathcal{R}(2)=2$. For each $x\in\mathbb{R}_+$, define $f(x)=y$ if and only if $\varphi_1(N,\mathcal{R},x+y)=x$ and $\varphi_2(N,\mathcal{R},x+y)=y$. By winner strict endowment monotonicity, endowment continuity, and order preservation, $f:\mathbb{R}_+\to\mathbb{R}_+$ is a well-defined, continuous, and nondecreasing function with $f(x)\le x$ for each x. By anonymity, φ is a single-parametric rule with representation f for each competition with two competitors. Let φ' be a single-parametric rule with representation f for each competition with an arbitrary number of competitors.

Suppose for the sake of contradiction that there exists a competition $(N,\mathcal{R},E)$ such that $\varphi'(N,\mathcal{R},E)\ne\varphi(N,\mathcal{R},E)$. Denote $N=\{1,\dots,|N|\}$ such that $\mathcal{R}(k)=k$ for each $k\in N$. Let $i\in N$ be such that $\varphi_i'(N,\mathcal{R},E)\ne\varphi_i(N,\mathcal{R},E)$ and $\varphi_j'(N,\mathcal{R},E)=\varphi_j(N,\mathcal{R},E)$ for each $j\in N$ with $j<i$. Suppose without loss of generality that $\varphi_i'(N,\mathcal{R},E)>\varphi_i(N,\mathcal{R},E)$. By local consistency and anonymity,

$$\begin{aligned}\varphi_{i+1}'(N,\mathcal{R},E)&=\varphi_{i+1}'(\{i,i+1\},\mathcal{R}_{\{i,i+1\}},\varphi_i'(N,\mathcal{R},E)+\varphi_{i+1}'(N,\mathcal{R},E))\\&=f(\varphi_i'(N,\mathcal{R},E))\\&\ge f(\varphi_i(N,\mathcal{R},E))\\&=\varphi_{i+1}(\{i,i+1\},\mathcal{R}_{\{i,i+1\}},\varphi_i(N,\mathcal{R},E)+\varphi_{i+1}(N,\mathcal{R},E))\\&=\varphi_{i+1}(N,\mathcal{R},E).\end{aligned}$$

In a similar way, this implies that $\varphi_{i+2}'(N,\mathcal{R},E)\ge\varphi_{i+2}(N,\mathcal{R},E)$. Continuing this reasoning, $\varphi_j'(N,\mathcal{R},E)\ge\varphi_j(N,\mathcal{R},E)$ for each $j\in N$ with $j>i$. This means that

$$\sum_{j\in N}\varphi_j'(N,\mathcal{R},E)>\sum_{j\in N}\varphi_j(N,\mathcal{R},E)=E.$$

This is a contradiction. Hence, $\varphi'(N,\mathcal{R},E)=\varphi(N,\mathcal{R},E)$ for each competition $(N,\mathcal{R},E)$. □

Remark A.2. The axioms anonymity, order preservation, winner strict endowment monotonicity, and local consistency (used in Theorem 2) are logically independent.

Proof. The rule from Example 2 satisfies order preservation, winner strict endowment monotonicity, and local consistency but not anonymity. The geometric rule with factor $\lambda=2$ satisfies anonymity, winner strict endowment monotonicity, and local consistency but not order preservation. The step rules satisfy anonymity, order preservation, and local consistency but not winner strict endowment monotonicity. The rule that coincides with the ED rule for competitions with two competitors and with the WTA rule for competitions with more than two competitors satisfies anonymity, order preservation, and winner strict endowment monotonicity but not local consistency. □

Theorem 3. *A rule satisfies anonymity, order preservation, winner strict endowment monotonicity, and top consistency if and only if it is a parametric rule.*

Proof. It is readily checked that each parametric rule satisfies anonymity, order preservation, winner strict endowment monotonicity, and top consistency.

Let φ be a rule satisfying these properties. Then, φ satisfies endowment monotonicity. By Lemma A.1, φ satisfies endowment continuity. We show that φ is a parametric

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rule by induction on the number of competitors. Clearly, $\varphi_i(N,\mathcal{R},E)=E=f_1(x)$ holds for each competition $(N,\mathcal{R},E)$ with $|N|=1$ and each competitor $i\in N$. Let $n\in\mathbb{N}$, and assume that there exist continuous and nondecreasing functions $f_1,f_2,\dots,f_n:\mathbb{R}_+\to\mathbb{R}_+$ with $f_1(x)=x$ for each x and $f_{k+1}(x)\le f_k(x)$ for each k such that for each competition $(N,\mathcal{R},E)$ with $|N|\le n$ and each competitor $i\in N$, we have $\varphi_i(N,\mathcal{R},E)=f_{\mathcal{R}(i)}(x)$, where $x\in\mathbb{R}_+$ is such that $\sum_{k=1}^{|N|}f_k(x)=E$.

Let $N=\{1,\dots,n+1\}$ be a set of $n+1$ competitors, and let $\mathcal{R}$ be the ranking defined by $\mathcal{R}(i)=i$ for all $i\in N$. By order preservation, endowment continuity, and winner strict endowment monotonicity, $\varphi_1(N,\mathcal{R},E)$ is an unbounded, continuous, and increasing function of E. Hence, for each $x\in\mathbb{R}_+$, there is a unique E such that $\varphi_1(N,\mathcal{R},E)=x$, we can define $f_{n+1}(x)=\varphi_{n+1}(N,\mathcal{R},E)$, and by top consistency and the induction hypotheses, $\varphi_2(N,\mathcal{R},E)=f_2(x),\dots,\varphi_n(N,\mathcal{R},E)=f_n(x)$. By endowment continuity, winner strict endowment monotonicity, and order preservation, $f_{n+1}:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous and nondecreasing function with $f_{n+1}(x)\le f_n(x)$ for each x. By anonymity, φ is a parametric rule with representation $f_1,f_2,\dots,f_{n+1}$ for each competition with $n+1$ competitors. □

Remark A.3. The axioms anonymity, order preservation, winner strict endowment monotonicity, and top consistency (used in Theorem 3) are logically independent.

Proof. The rule from Example 2 satisfies order preservation, winner strict endowment monotonicity, and top consistency but not anonymity. The geometric rule with factor $\lambda=2$ satisfies anonymity, winner strict endowment monotonicity, and top consistency but not order preservation. The step rules satisfy anonymity, order preservation, and top consistency but not winner strict endowment monotonicity. The rule that coincides with the ED rule for competitions with two competitors and with the WTA rule for competitions with more than two competitors satisfies anonymity, order preservation, and winner strict endowment monotonicity but not top consistency. □

Endnotes

[1] Rank-order competitions have been studied for sales (Kalra and Shi 2001), sports (Szymanski 2003), innovation (Terwiesch and Xu 2008), and crowdsourcing (Archak and Sundararajan 2009).

[2] Endowment monotonicity is a standard principle in the fair allocation literature; we refer to resource monotonicity in the book by Moulin (2003).

[3] Our first weakening of consistency was inspired by the local stability of Young (1988).

[4] Our second weakening of consistency was inspired by the first-player consistency of Potters and Sudhölter (1999) and the lowest-rank consistency of Hougaard et al. (2017).

[5] Other optimization objectives for prize structures include the total output (Glazer and Hassin 1988), the revenue to the organizer (Barut and Kovenock 1998), the total effort of competitors (Moldovanu and Sela 2001), the highest effort among all competitors (Moldovanu and Sela 2006), the weighted total effort of the top k competitors (Archak and Sundararajan 2009), the number of participating competitors (Azmat and Möller 2009), and the number of participating talented competitors (Azmat and Möller 2018).

[6] Consistency has been applied in seminal papers on, for example, cost allocation problems (Moulin 1985), claims problems (Aumann and Maschler 1985, Young 1987, Moulin 2000), cooperative games (Peleg 1986, Hart and Mas-Colell 1989), bargaining problems (Lensberg 1987; 1988), exchange economies (Thomson 1988), atomless economies (Thomson and Zhou 1993), allocation problems with single-peaked preferences (Thomson 1994), and resource allocation problems (Moreno-Ternero and Roemer 2006).

[7] All results in Section 3 hold when consistency is weakened to *bilateral consistency*, requiring consistency only for reduced competitions with two competitors.

[8] Lemma 1 is somewhat reminiscent of the result of Chambers and Thomson (2002) stating that equal treatment of equals and consistency together imply anonymity in the context of claims problems. Note that equal treatment of equals has no merit in the prize allocation problem of this paper.

[9] Lemma 2 and its proof are analogous to theorem A of Aumann and Maschler (1985) in the context of claims problems. This type of result is also known as an elevator lemma (Thomson 2011).

[10] In the literature on fair division, the scale invariance axiom is also called *homogeneity*.

[11] All results in Section 4 hold when local consistency is weakened to *bilateral local consistency*, requiring local consistency only for reduced competitions with two participants.

[12] The family of parametric prize allocation rules resembles the family obtained by Young (1987) in the context of claims problems.

[13] Lack of anonymity can lead to unforeseen issues. Recently, Csató (2022) provided a real-life example, where the nonanonymity of the prize allocation rule led to improper incentives for the competitors.

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