University of Helsinki
Faculty of Science
Department of Mathematics and Statistics

Master's thesis

Descriptive combinatorics and distributed algorithms

Joona Särkijärvi

Supervisor: Tapani Hyttinen

April 9, 2023

HELSINGIN YLIOPISTO — HELSINGFORS UNIVERSITET — UNIVERSITY OF HELSINKI

Tiedekunta/Osasto — Fakultet/Sektion — Faculty

Faculty of Science

Laitos — Institution — Department

Department of Mathematics and Statistics

Tekijä — Författare — Author

Joona Särkijärvi

Työn nimi — Arbetets titel — Title

Descriptive combinatorics and distributed algorithms

Oppiaine — Läroämne — Subject

Mathematics

Työn laji — Arbetets art — Level	Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Master's thesis	April 2023	53

Tiivistelmä — Referat — Abstract

Both descriptive combinatorics and distributed algorithms are interested in solving graph problems with certain local constraints. This connection is not just superficial, as Bernshteyn showed in his seminal 2020 paper. This thesis focuses on that connection by restating the results of Bernshteyn. This work shows that a common theory of locality connects these fields. We also restate the results that connect these findings to continuous dynamics, where they found that solving a colouring problem on the free part of the subshift ${}^{\Gamma}2$ is equivalent to there being a fast LOCAL algorithm solving this problem on finite sections of the Cayley graph of Γ.

We also restate the result on the continuous version of Lovász Local Lemma by Bernshteyn. The LLL is a powerful probabilistic tool used throughout combinatorics and distributed computing. They proved a version of the lemma that, under certain topological constraints, produces continuous solutions.

Avainsanat — Nyckelord — Keywords

Descriptive combinatorics, Distributed computing, LOCAL model, Lovász Local Lemma

Säilytyspaikka — Förvaringsställe — Where deposited

Kumpulan tiedekirjasto

Muita tietoja — Övriga uppgifter — Additional information

HELSINGIN YLIOPISTO — HELSINGFORS UNIVERSITET — UNIVERSITY OF HELSINKI

Tiedekunta/Osasto — Fakultet/Sektion — Faculty

Matemaattis-luonnontieteellinen

Laitos — Institution — Department

Matematiikan ja tilastotieteen laitos

Tekijä — Författare — Author

Joona Särkijärvi

Työn nimi — Arbetets titel — Title

Descriptive combinatorics and distributed algorithms

Oppiaine — Läroämne — Subject

Matematiikka

Työn laji — Arbetets art — Level	Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Pro gradu -tutkielma	Huhtikuu 2023	53

Tiivistelmä — Referat — Abstract

Sekä deskriptiivisessä kombinatoriikassa että hajautetussa laskennassa tutkitaan verkko-ongelmia, joissa esiintyy paikallisia rajoituksia. Kuten Anton Bernshteyn osoitti uraauurtavassa artikkelissaan vuonna 2020, tämä yhteys ei ole pelkästään pinnallinen. Tämä työ esittelee Bernshteynin tuloksia, joissa hän kehitti alojen välisiä yhteyksiä, sekä yhtenäisti ongelmien paikallisuuden teoriaa. Nämä tulokset yhtyvät jatkuvien dynaamisten järjestelmien teoriaan. Työssä osoitetaan, että tietynlaiset väritysongelmat ovat jatkuvasti ratkaistavissa avaruuden $^{\Gamma}2$ vapaassa osassa, jos ja vain jos on olemassa nopea LOCAL-algoritmi joka ratkaisee kyseenomaisen ongelman Γ:n Cayley-verkossa. Lisäksi työssä esitellään Bernshteynin versio Lovászin lokaalista lemmasta. Kyseistä lemmaa käytetään jatkuvasti kombinatoriikan ja hajautetun laskennan tutkimuksessa, missä se tuottaa eksistenssituloksia. Tämä uusi LLL:n versio takaa näiden tulosten jatkuvuuden, mikäli tietyt topologiset ehdot täyttyvät.

Avainsanat — Nyckelord — Keywords

Deskriptiivinen kombinatoriikka, Hajautettu laskenta, LOCAL-malli, Lovászin lokaali lemma

Säilytyspaikka — Förvaringsställe — Where deposited

Kumpulan tiedekirjasto

Muita tietoja — Övriga uppgifter — Additional information

Contents

Chapter 1

Introduction

Descriptive combinatorics is a fairly new field of mathematics that sits at the intersection between *descriptive set theory* and *combinatorics*. It answers questions about definable graphs in topological spaces, and it has many connections to different fields of mathematics. The aspect that we are interested in is the study of certain problems defined on graphs and asks when those problems have solutions that respect topological constraints. On the other hand, the LOCAL model of distributed computation is used to analyze distributed algorithms. The model concentrates solely on the locality of these algorithms, dismissing any memory and message constraints that are common in distributed models.

This thesis focuses on the first connections that were made between these two fields by Bernshteyn in his seminal 2020 paper [2]. This connection has been further explored in many papers since, see e.g. [10, 9, 4]. These findings seem to paint a picture of a common theory of locality. We restate the results by Bernshteyn in [2, 3], where they made the connection between the infinite graphs that admit solutions that respect topological constraints and those finite graphs that have a fast LOCAL algorithm solving the same problem. These results were further connected to the field of continuous dynamics, where they showed that one can show the existence of a continuous solution to a problem in any zero-dimensional Polish Γ-space just by finding a solution in an explicit family of finite graphs.

These results culminate in the following theorem.

Theorem (Theorem 4.26, Theorem 4.37 and Theorem 4.39)**.** *Let $S \subset \Gamma$ be a finite set and let $\mathcal{P}$ be a finite set of S-connected k-patterns. Then there is a family of finite graphs $\mathcal{H}_S$ such that the following statements are equivalent:*

(I) There is a continuous $\mathcal{P}$-avoiding k-colouring of $\mathrm{Free}({}^{\Gamma}2)$*.*

(II) Every free zero-dimensional Polish Γ-space admits a continuous $\mathcal{P}$-avoiding k-colouring.

(III) There is a graph in $\mathcal{H}_S$ that admits a $\mathcal{P}$-avoiding k-colouring.

(IV) All but finitely many graphs in $\mathcal{H}_S$ admit a $\mathcal{P}$-avoiding k-colouring.

(V) There is a deterministic distributed algorithm in the LOCAL *model that, given an n-vertex S-labeled subgraph G of the Cayley graph $G(\Gamma, S)$, outputs a $\mathcal{P}$-avoiding k-colouring of G in $O(\log^* n)$ rounds.*

Additionally, we restate the continuous version of the Lovász Local Lemma by Bernshteyn in [3]. The Lovász Local Lemma is a very powerful probabilistic method used to produce existence results in combinatorics and distributed computing.

Chapter 2

Preliminaries

In this chapter, we introduce the definitions and terminology used throughout this article. First, we will introduce some basic set theory and combinatorics. We will also provide some topological terminology and a classical theorem from descriptive set theory. Later on, we provide two definitions for the LOCAL model of distributed computation. One is the classical definition that is used in theoretical computer science, and the other is a more mathematical definition of the model. We will use the latter in our proofs, but the first is much more intuitive, so we show it as well.

2.A Descriptive combinatorics

Our approach to this subject was from the set-theoretic side, so our notation primarily follows the notation from set theory, see e.g. book by Kunen [13]. For graph theoretic parts, our notations are standard, see for example the book by Diestel [6]. When it comes to topological spaces, we will state the topology when the space is first introduced. As a general rule, every countable set, finite, or infinite, will have discrete topology.

Definition 2.1 (Set notation)**.** The set of natural numbers has discrete topology and is denoted as ω. If A and B are sets, with ${}^{A}B$ we denote the set of functions from A to B. The set of finite sequences of elements of A is denoted as

$$A^{<\omega} = \bigcup_{i\in\omega} A^i.$$

If A is topological space, each A^i has the product topology and the set $A^{<\omega}$ has the disjoint union topology.

We also consider each natural number $n \in \omega$ to be the canonical set with n elements, $\{0, 1, \ldots, n-1\}$. We use this to emphasize the cardinality of the set, as usually, we are not really interested in the exact elements.

Definition 2.2 (Graph notation)**.** A graph $G = (V, E)$ is a pair consisting of **vertices** $V(G)$, and **edges** $E(G) \subseteq V(G)^2$. An edge $(x, y) \in E(G)$ connects the **adjacent** vertices x and y. Two adjacent vertices are also called **neighbours**. A **rooted graph** (G, x) is a graph with a dedicated vertex x, which is called the **root**.

Most of our graphs are simple, where a vertex is never adjacent to itself. Also, unless stated otherwise, our graphs do not have directed edges, so we do not distinguish between edges (x, y) and (y, x).

Definition 2.3 (Paths and distances)**.** A path in G is a finite sequence of vertices $(v_0, v_1, \ldots, v_n)$, where each vertex is adjacent to the vertices before and after it. The length of a path is equal to one less than the number of vertices on it. The **distance** $\text{dist}_G(x, y)$ is defined as the length of the shortest path between x and y. If no such path exists, the distance is defined to be infinite. A graph where all pairwise distances are finite is called **connected**. Since (x) is a path with length 0, we have that $\text{dist}_G(x, x) = 0$ for all $x \in V(G)$.

Definition 2.4 (Graph neighbourhoods)**.** By $N_G(x)$ we denote the **neighbours** of x, i.e. the set of vertices adjacent to x. Notice that unless x is adjacent to itself, this set does not contain x. The **radius** r **neighbourhood** of x is the set $B_G(x, r) = \{y \in V(G) \mid \text{dist}_G(x, y) \leq r\}$. This set always contains x itself. The **degree** $\deg(x)$ of a vertex is the number of neighbours it has. The maximum degree of G is defined as $\Delta(G) = \sup\{\deg(x) \mid x \in V(G)\}$. A graph is **locally finite** if each vertex has only finitely many neighbours. A set of vertices $X \subseteq V(G)$ is **independent** if no two vertices in X are neighbours.

Definition 2.5 (Structured graph)**.** Let G be a graph. Now $\boldsymbol{G} = (G, \sigma)$ is a structured graph, where the structure map σ is a function on the finite sequences of $V(G)$,

$$\sigma : V(G)^{<\omega} \to \omega \cup \{\perp\}.$$

The function σ is used to add additional information about the graph. We could use any countable set as the label set, but for simplicity, we will just use $\omega \cup \{\perp\}$, where $\perp$ indicates null information.

Note that if σ and τ are two structure maps on G, we can form the function

$$\begin{aligned}&\pi : V(G)^{<\omega} \to (\omega \cup \{\perp\})^2, \\ &\pi(x) = (\sigma(x), \tau(x)).\end{aligned}$$

Now as $(\omega \cup \{\perp\})^2$ is countable, (G, π) is also a structure map. In essence, with some implicit encoding, we can add countably many different structures to a graph. To distinguish the graph $\boldsymbol{G}$ with additional structure map f, we use the notation $(\boldsymbol{G}, f)$ and $\boldsymbol{G}_f$ interchangeably.

Definition 2.6 (Graph isomorphism)**.** Let G_1, G_2 be graphs. An isomorphism φ between G_1 and G_2 is a bijection $\varphi : V(G_1) \to V(G_2)$ s.t.

$$\forall x, y \in V(G_1) \quad ((x, y) \in E(G_1) \iff (\varphi(x), \varphi(y)) \in E(G_2)).$$

If $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are structured graphs, we also require that

$$\forall x \in V(G_1)^{<\omega} : \sigma_1(x) = \sigma_2(\varphi(x)).$$

Also if (G_1, x_1) and (G_2, x_2) are rooted graphs, we require that $\varphi(x_1) = x_2$. If an isomorphism between G_1 and G_2 exists, we say that the graphs are **isomorphic**, $G_1 \cong G_2$. The **isomorphism class** $[G]$ is the set of all graphs isomorphic to G.

Definition 2.7 (Proper vertex and edge colourings)**.** Let G be a graph. An *arbitrary* function $f : V(G) \to k$ is called a **vertex colouring**, or just the colouring of G.

In a similar fashion, the function $f : E(G) \to k$ is an **edge colouring**. A vertex colouring is **proper** if

$$\forall x, y \in V(G) \quad ((x, y) \in E(G) \implies f(x) \neq f(y)).$$

An edge colouring is **proper** if

$$\forall x, y, z \in V(G) \quad ((x, y), (y, z) \in E(G) \implies f((x, y)) \neq f((y, z)))$$

Throughout this thesis, we assume that k is countable.

Definition 2.8 (Finite structured graph sets $\mathcal{FSG}$ and $\mathcal{FSG}_\cdot$)**.** We define $\mathcal{FSG}$ as the class of all isomorphism classes of finite structured graphs, and $\mathcal{FSG}_\cdot$ as the set of all isomorphism classes of finite structured rooted graphs.

Note that both $\mathcal{FSG}$ and $\mathcal{FSG}_\cdot$ are countable. We equip them with discrete topologies.

Definition 2.9 (Neighbourhood graphs)**.** For any graph G and distance $r \in \omega$, we can define the distance r neighbourhood graph G' as $V(G') = V(G)$ and

$$E(G') = \{(x, y) \mid \operatorname{dist}_G(x, y) \leq r\}$$

Next, we introduce some of the topological vocabulary that we use.

Definition 2.10 (Topological vocabulary)**.** Let X be a topological space. The set $Y \subseteq X$ is **Borel** if it can be formed from the open sets of X using complementation and countable union. We say that X is **zero-dimensional** if it has a countable base of clopen sets. X is a **Polish space** if it is a separable completely metrizable space. If (X, τ) is a Polish space, then the measurable space (X, Σ) is called **standard Borel space** if Σ is the set of Borel sets of (X, τ).

The following theorem is a classical theorem from descriptive set theory.

Theorem 2.11 (Kuratowski & Ryll-Nardzewski measurable selection theorem)**.** *Let $P \subseteq X \times Y$ be Borel and let the fibre*

$$P_x = \{y \in Y \mid (x, y) \in P\}$$

for every $x \in X$ be a countable union of compact sets. Now

$$\exists^Y P = \{x \in X \mid (x, y) \in P \text{ for some } y \in Y\}$$

is Borel, and there exists a Borel uniformization function $f \subseteq P$ such that $\operatorname{dom}(f) = \exists^Y P$.

2.B Distributed computing

We will now introduce our model of distributed computation. We will first show the definition that is used in theoretical computer science since this definition is simple and it provides us with good intuition about the model. We will then show a more mathematical definition of the model, which we will use in our proofs. These definitions are based on the introductory book by Barenboim, and Elkin [1] and the article by Bernshteyn [2].

Definition 2.12 (Deterministic LOCAL algorithm)**.** Let G be a graph, and $n = |V(G)|$. A deterministic LOCAL algorithm is trying to solve a problem on the graph G. Depending on the problem definition, the edges or vertices of G might already have some labels. Each vertex is a computer that is equipped with a unique identifier. The identifiers cannot be arbitrarily large, but they can be polynomial in n. Each edge represents a communication link between the respective computers, so $\boldsymbol{G}$ is both the problem graph and the communication graph.

The computation proceeds in synchronous rounds. Initially, each computer is aware of its own id and any labels on it given by the problem definition. Each computer is running the same algorithm. In each round, the vertices execute the following steps

1. Send unbounded messages to their neighbours.

2. Receive unbounded messages from their neighbours.

3. Do an unbounded amount of computation on the current information.

4. If ready, decide upon an output for this vertex.

The output of the LOCAL algorithm is the graph G where each vertex has an output label from some countable set, given by the problem definition. This label can be for example a colour (or a number) for the vertex, a colouring of the adjacent edges using some encoding, or then some combination of these.

We measure the computation in rounds. We say that the algorithm has halted in T rounds if at that point each computer has decided upon an output. We say that the deterministic algorithm A solves some problem on G in T rounds, if given *any* distribution of the identifiers on $V(G)$, A halts in T rounds and the resulting labeling is a solution to the problem.

From this definition, we can get a clear intuition about the LOCAL model. However, it is very different from the rest of our definitions, which makes any attempts to connect it to descriptive combinatorics complicated. We now make the following observation about deterministic LOCAL algorithms, which cannot use any randomness to their advantage.

Theorem 2.13 (LOCAL visibility argument)**.** *Let* A *be a deterministic* LOCAL *algorithm running on G. Then there exists a deterministic* LOCAL *algorithm* A′ *that produces the same output in the same amount of rounds, where each vertex always just sends everything it knows to its neighbours.*

Proof. Let $x, y \in V(G)$ be some adjacent vertices running the algorithm A. Since A is deterministic, the message that x sends to y during round t, $M(x, y, t)$, must be fully determined by the information x has at the beginning of that round. Let $I(x, t)$ be all that information. Now consider the case where x just sends $I(x, t)$ to y instead. In this case, the vertex y can just compute $M(x, y, t)$ on its own, since the message was a function of the information. It can then run the algorithm A normally, so changing the messages to include everything the vertex knows does not change anything else about the algorithm. □

This observation takes advantage of the fact that message size, memory, and computation are all unbounded in the LOCAL model. It also provides us with the visibility radius approach to LOCAL algorithms. Since at round 0, each vertex knows only its own starting situation, it can "see" its radius 0 neighbourhood in the graph G. In each round, this visibility radius increases by one. This approach yields us a more technical definition.

Definition 2.14 (Deterministic LOCAL algorithm, version 2)**.** A LOCAL algorithm A is a function $\mathrm{A} : \mathcal{FSG}. \to \omega$. Let $\boldsymbol{G}$ be a locally finite structured graph. The output of A on $\boldsymbol{G}$ after T rounds is defined as $\mathrm{A}(\boldsymbol{G}, T) : V(\boldsymbol{G}) \to \omega$,

$$\mathrm{A}(\boldsymbol{G}, T)(x) = \mathrm{A}([B_{\boldsymbol{G}}(x, T), x]).$$

For given $[\boldsymbol{N}, x] \in \mathcal{FSG}.$, the output of $\mathrm{A}([\boldsymbol{N}, x])$ using this definition is the output of the respective LOCAL algorithm running on x after it has "seen" the neighbourhood $\boldsymbol{N}$. Notice that the finite structured graph $\boldsymbol{N}$ does not necessarily have a vertex colouring that the algorithm could use as identifiers. In these cases, the algorithm is not required to do anything smart, so the function can output anything.

We can employ LOCAL algorithms to solve a multitude of graph problems, but the most interesting problems are those that are local in nature. We define the locally checkable labelings, which have been the main interest in research to the LOCAL model. To keep in line with our set-theoretic notation, we call these problems **locally verifiable colourings**.

Definition 2.15 (Locally verifiable colourings [14])**.** A locally verifiable colouring problem Π is a pair $\Pi = (t, P)$, where $t \in \omega$ is the validation radius, and $P : \mathcal{FSG}. \to \{0, 1\}$ is the validation function. We say that a vertex colouring $f : V(\boldsymbol{G}) \to \omega$ is a solution to Π iff the validation function returns 1 everywhere when run on the colored graph $\boldsymbol{G}_f$,

$$\forall x \in V(\boldsymbol{G}) \quad P(\boldsymbol{G}_f, t)(x) = 1.$$

These problems encompass most of the common graph problems. They include for example proper vertex colourings, proper edge colourings, maximal matchings, many edge orientation problems, etc.

Definition 2.16 (Deterministic round complexity of Π)**.** Let $\mathcal{G} \subseteq \mathcal{FSG}$, and let Π be a locally verifiable colouring that is defined on each $\boldsymbol{G}$ s.t. $[\boldsymbol{G}] \in \mathcal{G}$. For any function $T : \omega \to \omega$, we say that $\mathsf{Det}_{\Pi,\mathcal{G}}(n) \leq T(n)$ iff there exists a deterministic LOCAL algorithm A that solves Π on each n-vertex graph of $\mathcal{G}$ with arbitrary identifiers from n in $T(n)$ rounds. That is, given any $\boldsymbol{G}$ with $[\boldsymbol{G}] \in \mathcal{G}$, and any bijection $\mathsf{Id} : V(\boldsymbol{G}) \to n$, where $n = |V(\boldsymbol{G})|$, the colouring produced by $\mathrm{A}(\boldsymbol{G}_{\mathsf{Id}}, T)$ is a solution to Π. If this is the case, we say that the deterministic round complexity of Π on the family $\mathcal{G}$ is at most $T(n)$.

Notice that our definition requires the identifiers to not only be polynomial in $|V(\boldsymbol{G})|$, but be an exact match. This will only help our argument, as we use this definition only in selected proofs.

The round complexities that come up in the LOCAL model are familiar functions, except for one:

Definition 2.17 (Iterated logarithm)**.** The iterated logarithm $\log^*$ is a function on the positive real numbers. It is defined as

$$\log^*(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 + \log^*(\log_2(x)) & \text{if } x > 1. \end{cases}$$

In essence, this is the number of times you have to take the base 2 logarithm of x to get an answer ≤ 1. It is a *very* slowly growing function.

Chapter 3

Results from fast LOCAL algorithms

This chapter is solely focused on two quite straightforward theorems. We show that if a graph meets certain topological and local requirements, we can run fast LOCAL algorithms on it by tricking the algorithm to think that it is running on a finite graph that has unique small id:s. If we get the colouring for the id:s to be Borel or even continuous, this method provides us with a solution to any local colouring problem on such graphs, given that there is a fast deterministic LOCAL algorithm that can solve the problem in similar finite graphs.

3.A Borel solutions

The main problem in using LOCAL algorithms in infinite graphs is the requirement of unique id:s for the vertices. First of all, we cannot guarantee that a LOCAL algorithm does anything meaningful if it sees vertices with the same id. Second, if we want the solution to be Borel, any colouring that we could use for id:s must be also Borel. Therefore in this first section, we show that for certain graphs, we can construct locally unique Borel colourings that can be then used as id:s to trick a LOCAL algorithm. All of the results in this section are by Bernshteyn in [2], unless stated otherwise.

Definition 3.1 (Borel graph)**.** A **Borel graph** G is a graph where $V(G)$ is a standard Borel space, and $E(G)$ is a Borel subset of $V(G)^2$. A **Borel structured graph** is a Borel graph where additionally $\sigma^{-1}(n)$ is a Borel subset of $V(G)^{<\omega}$ for each $n \in \omega \cup \{\perp\}$.

We define our algorithms to work on some family of finite graphs. Thus we need to make sure that our infinite graphs look like they would belong to that family, at least when we restrict our scope to some finite section.

Definition 3.2 ((R,n)-local in $\mathcal{G}$)**.** Let $\mathcal{G} \subseteq \mathcal{FSG}$., and let $R, n \in \omega$. We say that a structured graph $\boldsymbol{G}$ is (R,n)**-locally** in $\mathcal{G}$ if for each $x \in V(\boldsymbol{G})$, there is some $\boldsymbol{H}$ and $y \in V(\boldsymbol{H})$ s.t. $[\boldsymbol{H}] \in \mathcal{G}$ and $[B_{\boldsymbol{G}}(x,R),x] = [B_{\boldsymbol{H}}(y,R),y]$. In other words, each radius R neighbourhood of $\boldsymbol{G}$ is isomorphic to some radius R neighbourhood of an n-vertex graph in $\mathcal{G}$.

Lemma 3.3. *If G is a locally finite Borel graph, then the distance d neighbourhood graph G' is Borel for all $d \in \omega$.*

Proof. Let G be a Borel graph, and let G' be the distance d neighbourhood graph of G. We prove that $E(G')$ is Borel by induction. Cases $d = 0$ and $d = 1$ are clear, since $\emptyset$ and $E(G)$ are Borel.

Now suppose the distance r neighbourhood graph G' is Borel, and let G'' be the distance $r + 1$ neighbourhood graph.

Let

$$A = \{X \subseteq V(G)^2 \mid X \times V(G) \text{ is Borel in } V(G)^3\}, \text{ and}$$
$$B = \{X \subseteq V(G)^2 \mid V(G) \times X \text{ is Borel in } V(G)^3\}.$$

We can see that A and B are σ-algebras and that they contain all open sets of $V(G)^2$. Thus they contain all Borel sets of $V(G)^2$, so $E(G') \in A$ and $E(G) \in B$. Now the sets

$$(E(G') \times V(G))$$
$$(V(G) \times E(G))$$

are Borel in $V(G)^3$, and therefore

$$\begin{aligned} D &= (E(G') \times V(G)) \cap (V(G) \times E(G)) \\ &= \{(x, y, z) \mid (x, y) \in E(G'),\ (y, z) \in E(G)\} \end{aligned}$$

is Borel.

Notice that as G is locally finite, the fibres $\{(x, z) \in V(G)^2 \mid (x, y, z) \in D\}$ are finite. Now by Theorem 2.11, the projection along the second coordinate

$$\begin{aligned} &\left\{(x, z) \in V(G)^2 \mid (x, y, z) \in D\right\} \\ =&\left\{(x, z) \in V(G)^2 \mid \mathrm{dist}_G(x, z) \leq r + 1\right\} \\ =&E(G'') \end{aligned}$$

is Borel, so therefore the distance $r + 1$ neighbourhood graph of G is Borel. □

Lemma 3.4. *Let $\boldsymbol{G}$ be a locally finite Borel structured graph, $r, n \in \omega$, and $|B_{\boldsymbol{G}}(x, r)| \leq n$ for all $x \in V(\boldsymbol{G})$. Now $f : V(\boldsymbol{G}) \to \mathcal{FSG}_\cdot$, $f(x) = [B_{\boldsymbol{G}}(x, r), x]$ is a Borel function.*

Proof. Fix some $[\boldsymbol{N}] \in \mathcal{FSG}_\cdot$, and let $n' = |V(\boldsymbol{N})|$. We will first construct a relation $D_{\boldsymbol{N}} \subseteq V(\boldsymbol{G})^{n'}$ where we list the vertices of all r-neighbourhoods isomorphic to $\boldsymbol{N}$. We will then use Theorem 2.11 to single out the root of those neighbourhoods, and thus show that $f^{-1}([\boldsymbol{N}])$ is Borel.

Let $(y_i)_{i \in |V(\boldsymbol{N})|}$ be an enumeration of the elements of $V(\boldsymbol{N})$, where y_0 is the root of $\boldsymbol{N}$ and the remaining vertices are in an arbitrary order. Then for any fixed n'-tuple $x = (x_0, \ldots, x_{n'-1}) \in V(\boldsymbol{G})^{n'}$, let us use the notation $\varphi_x^y : V(\boldsymbol{N}) \to x$ for the map $y_i \mapsto x_i$. Now let

$$D_{\boldsymbol{N}} = \left\{(x_0, x_1, \ldots, x_{n'-1}) \in V(\boldsymbol{G})^{n'} \;\middle|\; \varphi_x^y : \boldsymbol{N} \xrightarrow{\cong} (B_{\boldsymbol{G}}(x_0, r), x_0)\right\}.$$

Note that for each vertex x_0 s.t. $(B_{\boldsymbol{G}}(x_0, r), x_0) \cong \boldsymbol{N}$, there is at least one n'-tuple $x = (x_0, \ldots)$ where φ_x^y is an isomorphism.

Next, we will prove that $D_{\boldsymbol{N}}$ is Borel. Remember that for some fixed x, the function φ_x^y is an isomorphism to $(B_{\boldsymbol{G}}(x_0, r), x_0)$ if it is a bijection that preserves the edge relation and the structure map.

Claim 3.4.1. The set of n'-tuples x where φ_x^y is a bijection to the set of $V(B_{\boldsymbol{G}}(x_0, r))$, that is,

(a) $|B_{\boldsymbol{G}}(x_0, r)| = n'$,

(b) $\forall i \in n' : x_i \in B_{\boldsymbol{G}}(x_0, r)$,

(c) $\forall i, j \in n' : i \neq j \implies x_i \neq x_j$,

is Borel.

Proof of claim. We will start with (b). The set of tuples that satisfy the property is

$$C_{\boldsymbol{N}} = \left\{ (x_0, \ldots, x_{n'-1}) \in V(\boldsymbol{G})^{n'} \;\middle|\; \forall i \in n' : x_i \in B_{\boldsymbol{G}}(x_0, r) \right\}.$$

Due to Lemma 3.3, the distance r neighbourhood graph $\boldsymbol{G}_r$, especially the edge relation $E(\boldsymbol{G}_r)$, is Borel. Thus for each i s.t. $1 \leq i < n'$, the set

$$C_{\boldsymbol{N},i} = \left\{ (x_0, \ldots, x_{n'-1}) \in V(\boldsymbol{G})^{n'} \;\middle|\; (x_0, x_i) \in E(\boldsymbol{G}_r) \right\}$$

is Borel. Now since $C_{\boldsymbol{N}} = \bigcap_{i \in n'} C_{\boldsymbol{N},i}$ is Borel.

At (c) we require that each x_i is unique. We will now show that the set

$$U_{\boldsymbol{G}} = \left\{ (x_0, \ldots, x_{n'-1}) \in V(\boldsymbol{G})^{n'} \;\middle|\; \forall i \in n', j < i \quad (x_i \neq x_j) \right\}.$$

is Borel. We can easily see that the set

$$U_{\boldsymbol{G},i,j} = \left\{ (x_0, \ldots, x_{n'-1}) \in V(\boldsymbol{G})^{n'} \;\middle|\; x_i \neq x_j \right\}$$

is Borel for each pair i, j s.t. $0 \leq j < i < n'$. Thus the set

$$U_{\boldsymbol{G}} = V(\boldsymbol{G})^{n'} \setminus \left(\bigcup_{i \in n'} \bigcup_{j < i} U_{\boldsymbol{G},i,j} \right)$$

is now Borel.

Coming back to (a), we will look for the vertices x_1 with $|B_{\boldsymbol{G}}(x_1, r)| = n'$. Let

$$E_0 = \left\{ (x, y) \in V(\boldsymbol{G})^2 \mid \mathrm{dist}_{\boldsymbol{G}}(x, y) \leq r \right\}.$$

As given by Lemma 3.3, E_0 is Borel. Also as $\boldsymbol{G}$ is locally finite, each fibre $P_x = \{y \in V(\boldsymbol{G}) \mid (x, y) \in E_0\}$ is finite. Therefore by Theorem 2.11 there exists some Borel uniformization function $f_0 \subseteq E_0$. Let us define for each $0 < i < n'$

$$E_i = E_{i-1} \setminus f_{i-1}$$
$$f_i \subseteq E_i, \;\; f_i \text{ is the Borel uniformization of } E_i.$$

Now each $\mathrm{dom}(f_i)$ contains the vertices $x \in V(\boldsymbol{G})$ that have $i + 1$ or more other vertices in their radius r neighbourhood. Since Theorem 2.11 also states that each $\mathrm{dom}(f_i)$ is Borel, we get the set

$$V_i = V(G) \setminus \left(\mathrm{dom}(f_i) \cup \bigcup_{j<i} V_j \right)$$

is Borel for each $i \in n'$. Now since $\mathrm{dom}(f_0)$ contained those vertices that have at least a single neighbour in the radius r neighbourhood graph, the set V_0 now contains those vertices that have exactly 0 neighbours. Using the same logic, each V_i contains the vertices that have exactly i other vertices in their radius r neighbourhoods. Thus we let

$$\begin{aligned} R_{\boldsymbol{N}} &= V_{n'} \times V(\boldsymbol{G})^{n'-1} \\ &= \{(x_0, \ldots, x_{n'-1}) \in V(\boldsymbol{G})^{n'} \;:\; |B_{\boldsymbol{G}}(x_0, r)| = n'\}. \end{aligned}$$

Now we have shown that the set of n'-tuples x where φ^y_x is a bijection to the set $V(B_{\boldsymbol{G}}(x_0, r))$ is the Borel set $C_{\boldsymbol{N}} \cap U_{\boldsymbol{N}} \cap R_{\boldsymbol{N}}$. ∎

Claim 3.4.2. The set of n'-tuples x where φ^y_x preserves the edge relation $E(\boldsymbol{G})$ and the root is Borel.

Proof of claim. Since y_0 is by definition the root of $\boldsymbol{N}$, the map always preserves the root. Thus we just need to show that the set of n'-tuples where φ^y_x preserves the edge relation, i.e. the set

$$E_{\boldsymbol{N}} = \left\{(x_1, \ldots x_{n'}) \in V(\boldsymbol{G})^{n'} \;\middle|\; \forall i, j \leq n' : (x_i, x_j) \in E(\boldsymbol{G}) \iff (y_i, y_j) \in E(\boldsymbol{N})\right\},$$

is Borel. Note that as $E(\boldsymbol{G})$ is Borel, for each pair $1 \leq i < j \leq n'$ the set

$$E_{\boldsymbol{N},i,j} = \left\{(x_1, \ldots, x_{n'}) \in V(\boldsymbol{G})^{n'} \;\middle|\; (x_i, x_j) \in E(\boldsymbol{G}) \iff (y_i, y_j) \in E(\boldsymbol{N})\right\}$$

is Borel. Now the set

$$E_{\boldsymbol{N}} = \bigcap_{i \leq n'} \bigcap_{j<i} E_{\boldsymbol{N},i,j},$$

is also Borel. The root is always preserved due to the definition of φ^y_x.

∎

Claim 3.4.3. The set of n'-tuples x where φ^y_x preserves the structure map σ is Borel.

Proof of claim. With a similar argument, we can show that the set of n'-tuples $S_{\boldsymbol{N}}$, where φ^y_x preserves the structure map σ is Borel. Let $l \in \omega$, and $z \in V(\boldsymbol{N})^l$. Let $p : l \to n'$ be the function where

$$z = (y_{p(0)}, y_{p(1)}, \ldots, y_{p(l-1)}).$$

We are basically just using it to indicate which elements y_i comprise z. Since $\sigma_{\boldsymbol{G}}$ is Borel, we can see that the set

$$S_{\boldsymbol{N},z} = \left\{(x_0, \ldots, x_{n'-1}) \in V(\boldsymbol{G})^{n'} \;\middle|\; \sigma_{\boldsymbol{G}}(x_{p(0)}, \ldots, x_{p(l-1)}) = \sigma_{\boldsymbol{N}}(y_{p(0)}, \ldots x_{p(l-1)})\right\}$$

is Borel. This is exactly the set of n'-tuples $(x_0, \ldots, x_{n'})$ where the structure given to $(x_{p(0)}, \ldots, x_{p(l-1)})$ in $\boldsymbol{G}$ is the same as given to z in $\boldsymbol{N}$. Thus the set of n'-tuples where φ^y_x preserves the structure map σ is the set

$$S_{\boldsymbol{N}} = \bigcap_{z \in V(\boldsymbol{N})^{<\omega}} S_{\boldsymbol{N},z},$$

which is also Borel. ∎

At this point, we have shown that the for each of the three requirements for isomorphism, the set of n'-tuples meeting that requirement is Borel. Thus the set where φ_x^y is an isomorphism $\varphi_x^y : \boldsymbol{N} \xrightarrow{\cong} (B_{\boldsymbol{G}}(x_0, r), x_0)$ is the Borel set

$$D_{\boldsymbol{N}} = C_{\boldsymbol{N}} \cap U_{\boldsymbol{N}} \cap R_{\boldsymbol{N}} \cap E_{\boldsymbol{N}} \cap S_{\boldsymbol{N}}.$$

Now since $\boldsymbol{G}$ is locally finite, each fibre

$$D_{\boldsymbol{N},x_1} = \left\{ (x_2, \ldots x_{n'}) \in V(\boldsymbol{G})^{n'-1} \;\middle|\; (x_1, \ldots x_{n'}) \in D_{\boldsymbol{N}} \right\}$$

is finite. Thus we can use Theorem 2.11 on the Borel set $D_{\boldsymbol{N}}$ to get the set

$$\begin{aligned} \exists^{2\ldots n'} D_{\boldsymbol{N}} &= \{x_1 \in V(\boldsymbol{G}) \mid (x_1, \ldots, x_{n'}) \in D_{\boldsymbol{N}}\} \\ &= \{x_1 \in V(\boldsymbol{G}) \mid (B_{\boldsymbol{G}}(x_1, r), x_1) \cong \boldsymbol{N}\} \\ &= f^{-1}([\boldsymbol{N}]), \end{aligned}$$

which is Borel. Therefore the function f is Borel. □

The following theorem is the proposition 4.6 by Kechris, Solceki, and Todercevic in [12].

Theorem 3.5 (Kechris-Solecki-Todorcevic, Proposition 4.6 [12])**.** *Let G be a Borel graph of maximum degree Δ. Then there exists a Borel proper $(\Delta + 1)$-colouring of G.*

This is our first main theorem. We use our previous two lemmas in conjunction with Theorem 3.5 to construct a locally unique Borel colouring, which we will then use to trick a fast LOCAL algorithm to solve the problem.

Theorem 3.6. *Let $\mathcal{G} \subseteq \mathcal{FSG}$ and let $\Pi = (t, P)$ be a local colouring problem. Fix $n \in \omega$ such that $\mathsf{Det}_{\Pi,\mathcal{G}}(n) \leq T < \infty$ and set $R = T + t$. If $\boldsymbol{G}$ is a Borel structured graph that is (R, n)-locally in $\mathcal{G}$ and such that $|B_{\boldsymbol{G}}(x, 2R)| \leq n$ for all $x \in V(\boldsymbol{G})$, then $\boldsymbol{G}$ has a Borel Π-coloring.*

Proof. Let A be a LOCAL algorithm witnessing the round complexity $\mathsf{Det}_{\Pi,\mathcal{G}}(n)$. Construct the distance $2R$ neighbourhood graph $\boldsymbol{G}'$, where $V(\boldsymbol{G}') = V(\boldsymbol{G})$ and

$$E(\boldsymbol{G}') = \left\{ (x, y) \in V(\boldsymbol{G})^2 \mid \mathrm{dist}_G(x, y) \leq 2R \right\}.$$

By Lemma 3.3, the graph $\boldsymbol{G}'$ is Borel with maximum degree $\Delta \leq |B_{\boldsymbol{G}}(x, 2R)| - 1 \leq n - 1$ for all $x \in V(\boldsymbol{G}')$. Thus by the Kechris-Solceki-Todorcevic Theorem 3.5 $\boldsymbol{G}'$ has a Borel proper n-coloring $c : V(\boldsymbol{G}) \to n$. Define the function $f : V(\boldsymbol{G}) \to \omega$ as

$$f(x) = \mathrm{A}(\boldsymbol{G}_c, r)(x) = \mathrm{A}([B_{\boldsymbol{G}_c}(x, r), x]).$$

This means that $f(x)$ is the output of the algorithm A on the isomorphism class of the structured rooted graph $(B_{\boldsymbol{G}_c}(x, r), x)$, where $\boldsymbol{G}_c$ is the graph $\boldsymbol{G}$ equipped with the additional structure from c. The algorithm then uses this locally unique n-colouring c as the identifier function. Since we constructed the function c to be unique in a large enough neighbourhood, the algorithm cannot see any repetitions.

Notice that A : $\mathcal{FSG}_\cdot \to \omega$ is trivially Borel. Now since the colouring c is Borel, the graph $\boldsymbol{G}_c$ remains a Borel structured graph. Thus due to Lemma 3.4, the function $h : V(\boldsymbol{G}_c) \to \mathcal{FSG}_\cdot$, where $h(x) = [B_{\boldsymbol{G}_c}(x, r), x]$, is Borel. Thus the function $f = \mathrm{A} \circ h$ is Borel. What remains to show is that f is a proper Π-colouring of $\boldsymbol{G}$.

Suppose that f is not a proper Π-colouring. Now there must be some $x \in V(\boldsymbol{G})$, such that the validation function P does not agree on the f-coloured radius t neighbourhood of x. As $\boldsymbol{G}$ is (R, n)-locally in $\mathcal{G}$, there must be some $[\boldsymbol{H}] \in \mathcal{G}$ such that $(B_{\boldsymbol{G}}(x, R), x) \cong (B_{\boldsymbol{H}}(y, R), y)$. Let φ be the isomorphism $(B_{\boldsymbol{G}}(x, R), x) \xrightarrow{\cong} \varphi : (B_{\boldsymbol{H}}(y, R), y)$. Now as all of the vertices in $(B_{\boldsymbol{G}}(x, R), x)$ were neighbours in $V(\boldsymbol{G}')$, the proper colouring c maps a unique colour to each of them. Thus we can extend the colouring $c \circ \varphi^{-1} : V(B_{\boldsymbol{H}}(y, R)) \to n$ into a bijection $\mathsf{Id} : V(\boldsymbol{H}) \to n$. Since $R = t + T$, for each vertex $x' \in B_{\boldsymbol{G}}(x, t)$ we now have

$$(B_{\boldsymbol{G}_c}(x', T), x') \cong (B_{\boldsymbol{H}_{\mathsf{Id}}}(\varphi(x'), T), \varphi(x')).$$

Now since the T-radius, c-coloured and Id-coloured neighbourhoods in $\boldsymbol{G}$ and $\boldsymbol{H}$ are isomorphic, the algorithm A has to output the same colour for both of them, i.e. $f(x') = \mathrm{A}(\varphi(x'))$ for each $x' \in B_{\boldsymbol{G}}(x, t)$. Thus the validation function of the problem P must also act the same for those neighbourhoods,

$$\begin{aligned} & (B_{\boldsymbol{G}_f}(x, t), x) \cong (B_{\boldsymbol{H}_{\mathrm{A}}}(y, t), y) \\ \implies \quad & P(B_{\boldsymbol{G}_f}(x, t), x) = P(B_{\boldsymbol{H}_{\mathrm{A}}}(y, t), y) \\ \implies \quad & P(B_{\boldsymbol{H}_{\mathrm{A}}}(y, t), y) = 0. \end{aligned}$$

This means that A does not produce a proper Π-coloring of $\boldsymbol{H} \in \mathcal{G}$ in T rounds, which contradicts the assumption $\mathsf{Det}_{\Pi,\mathcal{G}}(n) \leq T$. Therefore f must be a proper Borel Π-coloring of $\boldsymbol{G}$. □

3.B Continuous solutions

In this second section, we want to find the continuous analogue of Theorem 3.6. This time all of the functions need to be continuous, so we cannot use the Theorem 3.5 to construct the locally unique colouring. Instead, we work with topological graphs, where we can construct the continuous locally unique colouring ourselves and also provide continuous analogues to all the lemmas we used in Section 3.A. In this section, all of the results are by Bernshteyn from [2].

Definition 3.7 (ε-isomorphism)**.** Let $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ be finite structured graphs where $V(\boldsymbol{G}_1), V(\boldsymbol{G}_2) \subseteq X$. Let d be some metric on X. We say that $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are ε-isomorphic, if there is an isomorphism φ with $d(x, \varphi(x)) < \varepsilon$ for all $x \in V(\boldsymbol{G}_1)$.

Definition 3.8 (Topological structured graph)**.** Let $\boldsymbol{G}$ be a structured graph where $V(\boldsymbol{G})$ is a zero-dimensional Polish space. Let d be some metric inducing the topology on $V(\boldsymbol{G})$. $\boldsymbol{G}$ is a topologically structured graph if for each $x \in V(\boldsymbol{G})$, $\varepsilon > 0$ and $R > 0$ there is some $\delta > 0$ s.t. if $d(x, y) < \delta$, the rooted neighbourhoods $(B_{\boldsymbol{G}}(x, R), x)$ and $(B_{\boldsymbol{G}}(y, R), y)$ are ε-isomorphic.

Lemma 3.9. *If $\boldsymbol{G}$ is a locally finite topological graph, then the distance r neighbourhood graph $\boldsymbol{G}'$ is a topological graph for all $r \in \omega$.*

Proof. Let $\boldsymbol{G}$ be a topological graph, d some metric inducing the topology on $V(\boldsymbol{G})$, and $\boldsymbol{G}'$ the the distance r neighbourhood graph of $\boldsymbol{G}$, where $V(\boldsymbol{G}') = V(\boldsymbol{G})$, and

$$E(\boldsymbol{G}') = \{(x, y) \mid \mathrm{dist}_{\boldsymbol{G}}(x, y) \leq r\}.$$

Let $R \in \omega$, $\varepsilon > 0$, and $x \in V(\boldsymbol{G})$. Since $\boldsymbol{G}$ is a locally finite topological structured graph, there is some $\delta > 0$, s.t. for all $y \in V(\boldsymbol{G})$, if $d(x, y) < \delta$, then the rooted structured graphs $(B_{\boldsymbol{G}}(x, r \cdot R), x)$ and $(B_{\boldsymbol{G}}(y, r \cdot R), y)$ are ε-isomorphic. Now the rooted structured graphs $(B_{\boldsymbol{G}'}(x, R), x)$ and $(B_{\boldsymbol{G}'}(y, R), y)$ are clearly ε-isomorphic. □

Lemma 3.10. *Let $\boldsymbol{G}$ be a locally finite topological structured graph. Now the function*

$$f : V(\boldsymbol{G}) \times \omega \to \mathcal{FSG}_{\cdot}, \ f(x, R) = [B_{\boldsymbol{G}}(x, R), x]$$

is continuous.

Proof. Let $[\boldsymbol{N}] \in \mathcal{FSG}_{\cdot}$, and d be some metric inducing the topology on $V(\boldsymbol{G})$. As $\boldsymbol{G}$ is topological structured graph, for each $(x, R) \in f^{-1}([\boldsymbol{N}])$, there exists some $\delta > 0$ s.t. if $d(x, y) < \delta$, then $(y, R) \in f^{-1}([\boldsymbol{N}])$. Now $f^{-1}([\boldsymbol{N}])$ is a union of open sets. □

Lemma 3.11. *If $\boldsymbol{G}$ is a locally finite topological structured graph and $f : V(\boldsymbol{G}) \to \omega$ is a continuous function, then the graph $\boldsymbol{G}$ equipped with the additional structure from f, $\boldsymbol{G}_f$, is a locally finite topological structured graph.*

Proof. Let $x \in V(\boldsymbol{G})$, $R \in \omega$, $\varepsilon_0 > 0$, and d be some metric inducing the topology on $V(\boldsymbol{G})$. Now for each vertex $y \in B_{\boldsymbol{G}}(x, R)$, let ε_y be such that f is constant for the vertices within that distance of y. Since $\boldsymbol{G}$ is locally finite, there are only finitely many such y in $B_{\boldsymbol{G}}(x, R)$. Thus we can define

$$\varepsilon = \min(\varepsilon_0, \ \min(\{\varepsilon_y \mid y \in B_{\boldsymbol{G}}(x, R)\})).$$

Now ε is the distance s.t. for all vertices in $B_{\boldsymbol{G}}(x, R)$, the function f is constant within their ε-distance ball. As $\boldsymbol{G}$ is a topological structured graph, there exists some $\delta > 0$ s.t. for all $y \in V(\boldsymbol{G})$ with $d(x, y) < \delta$, $B_{\boldsymbol{G}}(x, R)$ and $B_{\boldsymbol{G}}(y, R)$ are ε-isomorphic.

Let $y \in V(\boldsymbol{G})$ be s.t. $d(x, y) < \delta$, and let φ witness the isomorphism between $B_{\boldsymbol{G}}(x, R)$ and $B_{\boldsymbol{G}}(y, R)$. For each $z \in B_{\boldsymbol{G}}(x, R)$, we have that $d(z, \varphi(z)) < \varepsilon$. Now by the definition of ε, we have that $f(z) = f(\varphi(z))$ for all $z \in B_{\boldsymbol{G}}(x, R)$. Since $\varepsilon \leq \varepsilon_0$, the graphs $B_{\boldsymbol{G}_f}(x, R)$ and $B_{\boldsymbol{G}_f}(y, R)$ are ε_0-isomorphic. Therefore $\boldsymbol{G}_f$ is a topological structured graph. □

Lemma 3.12. *If $\boldsymbol{G}$ is a locally finite topological structured graph, then there is a continuous proper colouring $c : V(\boldsymbol{G}) \to \omega$.*

Proof. As $V(\boldsymbol{G})$ is a zero-dimensional Polish space, it has a countable base $(U_i)_{i\in\omega}$ consisting of clopen sets. For each $i \in \omega$, let us define

$$V_i = \{x \in V(\boldsymbol{G}) \mid x \in U_i \text{ and } N_{\boldsymbol{G}}(x) \cap U_i = \emptyset\}.$$

Let $x \in V(\boldsymbol{G})$. For each $y \in N_{\boldsymbol{G}}(x)$, let A_y be some open neighbourhood of x s.t. $y \notin A_y$. Since $\boldsymbol{G}$ is locally finite, $\bigcap \{A_y \mid y \in N_{\boldsymbol{G}}(x)\}$ is a open set containing x and none of its neighbours. Since each $x \in V(\boldsymbol{G})$ has an open neighbourhood that does not contain any of its neighbours in $\boldsymbol{G}$, each x must belong to at least one V_i.

Let $1_i : V(\boldsymbol{G}) \to \{0,1\}$ be the indicator function for the set U_i. Since U_i is clopen by definition, the indicator function is continuous. Now let $\boldsymbol{G}_i$ denote the graph $(\boldsymbol{G}, 1_i)$. By Lemma 3.11, the graph $\boldsymbol{G}_i$, which is $\boldsymbol{G}$ with the added structure 1_i, is a locally finite topological structured graph. At this point, notice that whether $x \in V(\boldsymbol{G})$ belongs to V_i is completely defined by the isomorphism class of radius 1 neighbourhood of x in $\boldsymbol{G}_i$. Since $\boldsymbol{G}_i$ is locally finite, each of these isomorphism classes is in the set $\mathcal{FSG}_\cdot$. The Lemma 3.10 states that the function $f_i : V(\boldsymbol{G}_i) \to \mathcal{FSG}_\cdot$ mapping x to the isomorphism class of its radius 1 neighbourhood in $\boldsymbol{G}_i$ is continuous.

Now let us define $\mathcal{N} \subset \mathcal{FSG}_\cdot$ as

$$\mathcal{N} = \left\{ [\boldsymbol{N}] \in \mathcal{FSG}_\cdot \;\middle|\; \begin{array}{l} \boldsymbol{N} \text{ is a radius 1 neighbourhood of the root, where} \\ \text{the root has colour 1 and all other vertices have colour 0} \end{array} \right\}.$$

Note that the vertices might already have one or more colours in $\boldsymbol{G}$. In this definition, we are talking about the colours given by 1_i, so we just need to take into account the encoding we used to combine 1_i to the structure map of $\boldsymbol{G}$ while constructing $\boldsymbol{G}_i$.

Now x belongs to V_i iff $f_i(x) \in \mathcal{N}$. Since each f_i is continuous, each set V_i is clopen, as

$$V_i = \bigcup_{\boldsymbol{N} \in \mathcal{N}} f_i^{-1}([\boldsymbol{N}])$$

Finally for each $i \in \omega$, we let

$$W_i = V_i \setminus \left(\bigcup_{j<i} V_j \right).$$

Now the clopen sets W_i form a partition of $V(\boldsymbol{G})$. Also each set W_i is independent in $\boldsymbol{G}$, since $x \in W_i \implies N_{\boldsymbol{G}}(x) \cap W_i = \emptyset$. Thus we can define the continuous proper coloring $c : V(\boldsymbol{G}) \to \omega$,

$$c(x) = i \iff x \in W_i.$$

□

Lemma 3.13. *If $\boldsymbol{G}$ is a locally finite topological graph with finite maximum degree Δ, then there is a continuous proper colouring $c : V(\boldsymbol{G}) \to (\Delta + 1)$.*

Proof. Let $p : V(\boldsymbol{G}) \to \omega$ be a continuous proper colouring of $V(\boldsymbol{G})$ from Lemma 3.12. We will first describe a process which recolours the graph $\boldsymbol{G}$ with $\Delta + 1$ colours, and then we show that this process can be simulated with a continuous function.

The recolouring process takes ω rounds to complete. In round i, each vertex in $p^{-1}(i)$ picks the least colour from $\Delta + 1$ that is not already chosen by one of its neighbours. Since adjacent elements cannot have the same colour in p, they will not have to make the decision at the same time. Thus this is a well-defined process that produces a proper $\Delta + 1$-colouring of $\boldsymbol{G}$.

To see that this colouring will also be continuous, we notice that the decision made by x during round $p(x)$ can depend only on the decision made by its neighbours with lower colours in p. This means that, given $x \in V(\boldsymbol{G})$, the furthest vertex that x:s decision can depend on must be within its radius $p(x)$ neighbourhood. In other words, if we know the radius $p(x)$ neighbourhood of x in the graph $(\boldsymbol{G}, p)$, then we can determine the colour that x chooses. Thus there is some function $h : \mathcal{FSG}. \to (\Delta + 1)$ that simulates this process. Since $\mathcal{FSG}.$ is a discrete space, h is trivially continuous. Now as we let $f : V(\boldsymbol{G}) \times \omega \to \mathcal{FSG}.$ be the continuous function provided by Lemma 3.10, the continuous function

$$c(x) = h(f(x, p(x))).$$

simulates the process that we defined. Thus it is a continuous proper $\Delta+1$-colouring of $\boldsymbol{G}$. $\square$

At this point, we are ready to prove the continuous analogue of Theorem 3.6. The proof is very much the same, except this time all functions have to be continuous.

Theorem 3.14. *Let $\mathcal{G} \subseteq \mathcal{FSG}$ and let $\Pi = (t, P)$ be a local colouring problem. Fix $n \in \omega$ such that $\mathsf{Det}_{\Pi,\mathcal{G}}(n) \leq T < \infty$ and set $R = T + t$. If $\boldsymbol{G}$ is a topological structured graph that is (R, n)-locally in $\mathcal{G}$ and such that $|B_{\boldsymbol{G}}(x, 2R)| \leq n$ for all $x \in V(\boldsymbol{G})$, then $\boldsymbol{G}$ has a continuous Π-coloring.*

Proof. Let A be a LOCAL algorithm witnessing the round complexity $\mathsf{Det}_{\Pi,\mathcal{G}}(n)$. Construct the distance $2R$ neighbourhood graph $\boldsymbol{G}'$, where $V(\boldsymbol{G}') = V(\boldsymbol{G})$, and

$$E(\boldsymbol{G}') = \left\{(x, y) \in V(\boldsymbol{G})^2 \mid \operatorname{dist}_{\boldsymbol{G}}(x, y) \leq 2R\right\}.$$

By Lemma 3.9, $\boldsymbol{G}'$ is a topological structured graph with $\Delta(\boldsymbol{G}') \leq |B_{\boldsymbol{G}}(x, 2R)| \leq n-1$ for all $x \in V(\boldsymbol{G}')$. Thus by Lemma 3.13, $\boldsymbol{G}'$ has a continuous proper n-coloring $c : V(\boldsymbol{G}) \to n$.

Define the function $f : V(\boldsymbol{G}) \to \omega$, $f(x) = \mathrm{A}([B_{(\boldsymbol{G},c)}(x, R)])$. Notice that as A is trivially continuous, f is continuous by Lemma 3.10. Now using the same deduction as we did in Theorem 3.6, we can show that f is a proper Π colouring. $\square$

Chapter 4

Results from descriptive combinatorics

In this chapter, we focus more on descriptive combinatorics. We will also provide a useful continuous version of the very powerful Lovász Local Lemma, LLL. It belongs to the category of probabilistic methods, which are techniques used to prove the existence of a certain graph by showing a positive probability for choosing it at random. Later on, we use this continuous version to produce interesting descriptive results in continuous combinatorics. We amended some proofs with added detail, but all of the results are by Bernshteyn in [3] unless stated otherwise.

4.A Basic definitions

From now on, we will always use Γ to denote a countably infinite group with discrete topology.

Definition 4.1 (Group action space)**.** Let Γ be a group with the identity element $\mathbf{1}_\Gamma$ and X a set. The left **group action** of Γ on X is a function $\alpha : \Gamma \times X \to X$, where the following axioms hold:

$$\begin{aligned} &\text{Identity:} && \alpha(\mathbf{1}_\Gamma, x) = x \\ &\text{Compatibility:} && \alpha(\gamma, \alpha(\delta, x)) = \alpha(\gamma\delta, x). \end{aligned}$$

We will use the notation $\alpha(\gamma, x) = \gamma \cdot x$ on most occasions. If the action $\alpha : \Gamma \times X \to X$ exists, we say that Γ **acts** on X and denote it by $\Gamma \curvearrowright X$. A subset $Y \subseteq X$ is said to be Γ**-invariant** if it is closed under the group action α. If Γ acts both on X and Y and f is a function $f : X \to Y$ such that $\forall \gamma \in \Gamma,\ x \in X \quad f(\gamma \cdot x) = \gamma \cdot f(x)$, we say that f is Γ**-equivariant**. If $X = \Gamma$, the group is said to **act on itself** by left multiplication.

The group Γ acts **freely** on X if

$$\forall \gamma \in \Gamma, \forall x \in X \quad \gamma \cdot x = x \implies \gamma = \mathbf{1}_\Gamma.$$

$\mathrm{Free}(X)$ is the largest Γ-invariant subspace of X where Γ acts freely.

If X is also a topological space, we say that X is a Γ-**space** if the group action $\alpha : \Gamma \times X \to X$ is continuous.

Definition 4.2 (Shift)**.** We denote the set of k-colourings of Γ as ${}^{\Gamma}k$. The base of ${}^{\Gamma}k$ is formed by the sets

$$\{x : \Gamma \to k \mid \forall \gamma \in D \quad (x(\gamma) = y(\gamma)\} ,$$

where $D \subset \Gamma$ is a finite set and $y \in {}^{\Gamma}k$. This makes the set a compact, zero-dimensional Polish space. It is also a Γ-space, as we equip it with the action $\Gamma \curvearrowright {}^{\Gamma}k$,

$$\alpha(\gamma, f)(\delta) = f(\delta\gamma) \quad \text{for all } f \in {}^{\Gamma}k,\ \gamma, \delta \in \Gamma.$$

The Γ-spaces of the form ${}^{\Gamma}k$ are called a **Bernoulli shifts**, or just **shifts**. A closed Γ-invariant subset of a shift is called a **subshift**. A subshift $X \subseteq {}^{\Gamma}k$ is free if Γ acts freely on it.

Pay attention to how we have defined the group action on the shift ${}^{\Gamma}k$. The standard would be to define $\gamma \cdot f(\delta) = f(\gamma^{-1}\delta)$, but we digress from this since our definition becomes very useful later on, and the structures they produce are isomorphic.

4.B Continuous graphs and the continuous LLL

In this section, we focus on proving a continuous version of the LLL. Our version is more stringent on the problems that can be solved with it, but in return, it yields continuous solutions. Before expressing those requirements, however, we will first prove some descriptive results on certain kinds of graphs. These will be later used to prove the continuous version of the LLL.

Definition 4.3 (Continuous graphs)**.** Let G be a graph, where $V(G)$ is a zero-dimensional Polish space. We say that G is **continuous** if for every clopen $U \subseteq V(G)$, the neighbourhood of U, $N_G(U) = \bigcup_{x \in U} N_G(x)$, is clopen.

The definition of continuous graphs might look restricting, but later on, we will show that many of the graphs we discuss are actually continuous.

Lemma 4.4. *Every locally finite continuous graph G admits a partition $\{I_i \mid i \in \omega\}$ into countably many clopen independent sets.*

Proof. Let $\{U_i \mid i \in \omega\}$ be a countable base of clopen sets for $V(G)$. For each $i \in \omega$, let $V_i = U_n \setminus N_G(U_i)$. Now as G is continuous, each V_i is independent and clopen. Since G is locally finite and $V(G)$ is metrizable, for each $x \in V(G)$ there is some open neighbourhood disjoint from $N_G(x)$. Thus $\bigcup_{i \in \omega} V_i = V(G)$. Now we get the partition by setting $I_i = V_i \setminus \left(\bigcup_{j<i} V_j\right)$. □

Lemma 4.5. *Every locally finite continuous graph G has a clopen maximal independent set $I \subseteq V(G)$.*

Proof. Let $\{I_i \mid i \in \omega\}$ be a partition of $V(G)$ into clopen independent sets. Let us define I'_i for each $i \in \omega$ recursively so that $I'_0 = I_0$, and $I'_{n+1} = I_{n+1} \setminus \left(\bigcup_{j<i} N_G(I'_j)\right)$. As G is continuous, each I'_i is clopen, and thus the set

$$I = \bigcup_{i \in \omega} I'_i$$

is open. Note that $V(G) \setminus I = \bigcup_{i \in \omega} I_i \setminus I_i'$, so therefore I is also closed. Thus I is clopen. □

Lemma 4.6. *If G is a continuous graph with finite maximum degree Δ, then $\chi_c(G) \leq \Delta + 1$.*

Proof. We will construct a partition of $V(G)$ into $\Delta + 1$ clopen independent sets. Let $G_0 = G$, and apply Lemma 4.5 to get I_0, a clopen maximal independent set of G_0. Now let $G_{n+1} = G_n[V(G_n) \setminus I_n]$, and I_{n+1} the clopen maximal independent set of G_{n+1}. After $\Delta + 1$ iterations there are no vertices left, as such a vertex would need to have a neighbour in each $I_0, \dots, I_\Delta$, which is impossible on a graph with maximal degree Δ. Now the function $f : V(G) \to \Delta + 1$, $f(x) = i \iff x \in I_i$ is a continuous proper colouring of G. □

Later on we especially use Lemmas 4.5 and 4.6. But for now, let us define the problems on which the LLL is defined on.

Definition 4.7 (Constraint Satisfaction Problem)**.** Let X be a set. For finite $D \subseteq X$, a subset $B \subseteq {}^D k$ is called an **(X,k)-constraint**. By $\mathrm{dom}(B)$ we denote the set D. A k-colouring $f : X \to k$ **violates** B iff $f \restriction_{\mathrm{dom}(B)} \in B$. If f does not violate B, it **satisfies** it. A **constraint satisfaction problem**, **CSP**, on X with range k is a set of (X,k)-constraints. It is denoted as $\mathcal{B} : X \to^? k$. A solution to CSP $\mathcal{B} : X \to^? k$ is a function $f : X \to k$ that satisfies each $B \in \mathcal{B}$.

The neighbourhood of B, $N(B)$, is the set

$$N(B) = \{B' \in \mathcal{B} \mid B' \neq B \text{ and } \mathrm{dom}(B') \cap \mathrm{dom}(B) \neq \emptyset\}.$$

Let us now introduce the classical version of Lovász Local Lemma. The first version was developed by Lovász and Erdős in 1975 [7]. Nowadays the most used version is the following symmetric LLL by Lovász, published by Spencer in [16].

Theorem 4.8 (Classical Lovász Local Lemma)**.** *Let $\mathcal{B}$ be a CSP, $p(\mathcal{B}) = \sup_{B \in \mathcal{B}} \mathbb{P}[B]$ and $d(\mathcal{B}) = \sup_{B \in \mathcal{B}} |N(B)|$. If*

$$e \cdot p(\mathcal{B}) \cdot (d(\mathcal{B}) + 1) < 1,$$

then there is a solution to $\mathcal{B}$.

This version is not useful for us, however, since we are interested in topologically well-behaving solutions. Thus we will prove a version of the lemma that produces continuous solutions. Before that, however, we still need some more definitions and lemmas.

Definition 4.9 (Definitions on CSP:s)**.** Let $\mathcal{B} : X \to^? k$ be a CSP on X. For each $B \in \mathcal{B}$, let the **probability** of B be defined as

$$\mathbb{P}[B] = \frac{|B|}{k^{|\mathrm{dom}(B)|}}.$$

You can see that this is the same as the probability of f violating B, for $f \in {}^X k$ chosen from a uniform distribution.

We define the **maximum vertex degree** $\text{vdeg}(\mathcal{B})$ and the **order** $\text{ord}(\mathcal{B})$ of $\mathcal{B}$ as

$$\text{vdeg}(\mathcal{B}) = \sup_{x \in X} |\{B \in \mathcal{B} \ : \ x \in \text{dom}(B)\}|,$$
$$\text{ord}(\mathcal{B}) = \sup_{B \in \mathcal{B}} |\text{dom}(B)|.$$

If both $\text{vdeg}(\mathcal{B})$ and $\text{ord}(\mathcal{B})$ are finite, we call $\mathcal{B}$ **bounded**.

Definition 4.10 (Continuous CSP:s)**.** Let X be a zero-dimensional Polish space and $\mathcal{B} : X \to^? k$. For $B^* \subseteq {}^n k$, let $B^*(x_1, \dots, x_n)$ for distinct $x_i \in X$ be an (X,k)-constraint given by

$$B^*(x_1, \dots, x_n) = \{\varphi \circ \iota \mid \varphi \in B^*\}, \quad \text{where } \iota(x_i) = i - 1.$$

We say that the CSP $\mathcal{B}$ is **continuous** if for any $n \in \omega$, $B^* \subseteq {}^n k$, and clopen sets $U_2, \dots, U_n$, the set

$$\{x_1 \in X \mid \exists x_2 \in U_2, \dots, x_n \in U_n, \text{ s.t. each } x_i \text{ is distinct and } B^*(x_1, \dots, x_n) \in \mathcal{B}\},$$

is clopen.

Definition 4.11 (CSP to Graph)**.** Given a CSP $\mathcal{B} : X \to^? k$, let us define the graph $G_{\mathcal{B}}$ as $V(G_{\mathcal{B}}) = X$, $(x, y) \in E(G_{\mathcal{B}}) \iff \{x, y\} \subseteq \text{dom}(B)$ for some $B \in \mathcal{B}$.

Lemma 4.12. *If $\mathcal{B} : X \to^? k$ is a bounded continuous CSP on a zero-dimensional Polish space X, then the graph $G_{\mathcal{B}}$ is continuous.*

Proof. Let $G = G_{\mathcal{B}}$ and $U \subseteq X$ be a clopen set. Note that $x_1 \in N_G(U)$ iff we find some constraint B with $\{x_1, y\} \subseteq \text{dom}(B)$ for some $y \in U$. That is, for some $2 \leq i \leq n \leq \text{ord}(\mathcal{B})$ and some $B^* \in {}^n k$, x_1 is in the following clopen set

$$\{x_1 \in X \mid \exists x_2 \in X, \dots, x_i \in U, \dots, x_n \in X \text{ s.t. each } x_i \text{ is distinct and } B^*(x_1, \dots, x_n) \in \mathcal{B}\}.$$

Now we can just let $N_G(U)$ be the finite union over such sets for all $2 \leq i \leq n \leq \text{ord}(\mathcal{B})$ and $B^* \subseteq {}^n k$, and thus $N_G(U)$ is clopen. □

At this point, we have been able to connect the bounded CSPs to the continuous graphs that we talked about before. The next few lemmas will show that these kind of well behaving CSPs can be continuously solved piece by piece, as long as each piece is a clopen independent set.

Definition 4.13. Let X be a set and $X' \subseteq X$. Given a function $g : X' \to k$ and an (X,k-constraint) B with domain D, we define B/g as

$$B/g = \{\varphi : D \setminus X' \to k \mid \varphi \cup g \restriction_{D \cap X'} \in B\}.$$

We can think of the set of those possible "extensions" of g to the domain D that violate the constraint B. Note that it is possible that D is a subset of X', i.e. g is already defined in the domain D. In this case B/g is either $\{\emptyset\}$ or $\emptyset$, depending on if g violates B or not. $\mathcal{B}/g$ is defined as expected,

$$\mathcal{B}/g = \{B/g \mid B \in \mathcal{B}\}.$$

We call a CSP $\mathcal{B}$ **good** if it is bounded, and for all $B \in \mathcal{B}$

$$\mathbb{P}[B] \cdot \mathrm{vdeg}(\mathcal{B})^{|\mathrm{dom}(B)|} < 1$$

Lemma 4.14. *Let $\mathcal{B} : X \to^? k$ be a bounded continuous CSP on a zero-dimensional Polish space X. If $X' \subseteq X$ is a clopen set and $g : X' \to k$ is continuous, then the CSP $\mathcal{B}/g : X \setminus X' \to^? k$ is also continuous.*

Proof. Given $B^* \subseteq {}^n k$, and clopen subsets $U_2, \dots, U_n \subseteq X \setminus X'$, we will need to show that the set

$$\{x_1 \in X \setminus X' \mid \exists x_2 \in U_2, \dots, x_n \in U_n,\ x_i \text{ are distinct and } B^*(x_1, \dots, x_n) \in \mathcal{B}/g\}$$

is clopen. Now for each $B^*_e \in {}^m k$ such that $B^* = \{\varphi \restriction_n \mid \varphi \in B^*_e\}$, and each sequence of colours $\alpha_{n+1}, \dots, \alpha_m \in k$, we can create the following clopen set

$$(X \setminus X') \cap \left\{ x_1 \in X \;\middle|\; \begin{array}{l} \exists x_2 \in U_2, \dots, x_n \in U_n, x_{n+1} \in g^{-1}(\alpha_{n+1}), \dots, x_m \in g^{-1}(\alpha_m), \\ \text{all } x_i \text{ are distinct, and } B^*_e(x_1, \dots, x_m) \in \mathcal{B} \end{array} \right\}.$$

The wanted set is given by union over these finitely many clopen sets, and thus it is clopen. ☐

Lemma 4.15. *Let $\mathcal{B} : X \to^? k$ be a good continuous CSP on a zero-dimensional Polish space X, and let $I \subseteq X$ be a clopen independent set in $G_{\mathcal{B}}$. There exists a continuous colouring $g : I \to k$ such that $\mathcal{B}/g$ is good.*

Proof. Note that as $\mathrm{vdeg}(\mathcal{B}/g) \le \mathrm{vdeg}(\mathcal{B})$, we can instead just show that

$$\mathbb{P}[B/g] \cdot \mathrm{vdeg}(\mathcal{B})^{|\mathrm{dom}(B/g)|} < 1 \tag{4.15.1}$$

holds for all $B/g \in \mathcal{B}/g$. For each $x \in I$, let $\mathcal{B}_x \subseteq \mathcal{B}$ be the set of constraints that have x in their domain. We see that $|\mathcal{B}_x| \le \mathrm{vdeg}(\mathcal{B})$ for all x. As I is independent in the graph $G_{\mathcal{B}}$, the sets $\mathcal{B}_x$ are disjoint. This means that the value of $g(x)$ fully determines the values of $\mathbb{P}[B/g]$ for all $B \in \mathcal{B}_x$. Thus for each $B \in \mathcal{B}_x$ and colour $\alpha \in k$, we define

$$\begin{aligned} \mathbb{P}[B/g] &= \mathbb{P}[B \mid x \mapsto \alpha] \\ &= \frac{|\{\varphi \in B \mid \varphi(x) = \alpha\}|}{k^{|\mathrm{dom}(B)|-1}} \end{aligned}$$

For colour $\alpha \in k$, and vertex $x \in I$, and constraint $B \in \mathcal{B}_x$ we say that α is **good** for x and B if,

$$\mathbb{P}[B \mid x \mapsto \alpha] \le \mathbb{P}[B] \cdot \mathrm{vdeg}(\mathcal{B}). \tag{4.15.2}$$

In fact, for each $x \in I$, there is a colour that is good for all $B \in \mathcal{B}_x$. Let $x \in I$ and $B \in \mathcal{B}_x$. We notice that

$$\begin{aligned} \mathbb{P}[B] &= \sum_{\alpha \in k} \mathbb{P}[B \mid x \mapsto \alpha] \cdot \mathbb{P}[g(x) = \alpha] \\ &= \frac{1}{k} \cdot \sum_{\alpha \in k} \mathbb{P}[B \mid x \mapsto \alpha] \end{aligned}$$

Claim 4.15.3. The number of colours that are not good for B and x must be less than $k/\mathrm{vdeg}(\mathcal{B})$.

Proof of claim. Suppose to the contrary that there are at least $k/\mathrm{vdeg}(\mathcal{B})$ bad colours for B and x. Now by using Eq. (4.15.2), we get that

$$\begin{aligned}
\mathbb{P}[B] &\geq \frac{1}{k} \cdot \sum_{\substack{\alpha \in k \\ \alpha \text{ is bad for } B,\, x}} \mathbb{P}[B \mid x \mapsto \alpha] \\
&> \frac{1}{k} \cdot \sum_{\substack{\alpha \in k \\ \alpha \text{ is bad for } B,\, x}} \mathbb{P}[B] \cdot \mathrm{vdeg}(\mathcal{B}) \\
&= \frac{1}{k} \cdot \frac{k}{\mathrm{vdeg}(\mathcal{B})} \cdot \mathbb{P}[B] \cdot \mathrm{vdeg}(\mathcal{B}) \\
&= \mathbb{P}[B].
\end{aligned}$$

■

Therefore since there are at most $\mathrm{vdeg}(\mathcal{B})$ constraints in $\mathcal{B}_x$ and there are k colours to choose from, there must be a colour $\alpha \in k$ that is good for each $B \in \mathcal{B}_x$ and x.

We now define $g : I \to k$ to take the least colour that is good for x and all $B \in \mathcal{B}_x$.

Claim 4.15.4. The function g is continuous.

Proof of claim. Let $B^* \subseteq {}^n k$, and $\alpha \in k$. Notice that just by knowing B^*, we can determine whether α will be good for the constraint $B^*(x_1, x_2, \ldots, x_n)$ and the vertex x_1, regardless of our choice of the non-repeating sequence of x_i:s. We say that such α is good for B^*. Now we can form the following finite sets for all $\alpha \in k$,

$$\mathcal{B}^*_\alpha = \{B^* \subseteq {}^n k \mid n \leq \mathrm{ord}(\mathcal{B}),\ \alpha \text{ is not good for } B^*\}.$$

Now the set $\mathcal{B}^*_\alpha$ holds all those B^*, where α will be bad for $B^*(x_1, x \ldots, x_n)$ and x_1, regardless of how we choose the elements x_i. Since $\mathcal{B}$ is continuous, we can use these sets to create the clopen sets

$$F_\alpha = I \cap \left(\bigcup_{B^* \in \mathcal{B}^*_\alpha} \{x_1 \in X \mid \exists x_2, \ldots, x_n \in X,\ \text{all } x_i \text{ are distinct, and } B^*(x_1, \ldots, x_m) \in \mathcal{B}\} \right).$$

Now the set F_α contains exactly those vertices $x \in I$ that have some constraint $B \in \mathcal{B}_x$ s.t. α is bad for x and B. Since g always chooses the smallest good colour for each vertex, we can create the clopen colour sets for each $\alpha \in k$,

$$g^{-1}(\alpha) = (I \setminus F_\alpha) \setminus \left(\bigcup_{\beta < \alpha} g^{-1}(\beta) \right).$$

■

Finally we will show that each $B/g \in \mathcal{B}/g$ satisfies

$$\mathbb{P}[B/g] \cdot \mathrm{vdeg}(\mathcal{B})^{|\mathrm{dom}(B)|} < 1.$$

As we noticed at the start, we can just show that Eq. (4.15.1) holds instead. Now if $\mathrm{dom}(B) \cap I = \emptyset$, then $B/g = B$ and B satisfies Eq. (4.15.1) as $\mathcal{B}$ is good. Since I is independent in the graph $G_{\mathcal{B}}$, the other case is that the intersection has one element. We let x be that element, and now $B \in \mathcal{B}_x$. We see that since g chooses a good colour for B and x,

$$\begin{aligned}
\mathbb{P}[B/g] \cdot \mathrm{vdeg}(\mathcal{B})^{|\mathrm{dom}(B/g)|} &= \mathbb{P}[B \mid x \mapsto g(x)] \cdot \mathrm{vdeg}(\mathcal{B})^{|\mathrm{dom}(B)|-1} \\
&\leq \mathbb{P}[B] \cdot \mathrm{vdeg}(\mathcal{B}) \cdot \mathrm{vdeg}(\mathcal{B})^{|\mathrm{dom}(B)|-1} \\
&= \mathbb{P}[B] \cdot \mathrm{vdeg}(\mathcal{B})^{|\mathrm{dom}(B)|} \\
&< 1.
\end{aligned}$$

Thus the CSP $\mathcal{B}/g$ is good. □

Now we are ready to prove our continuous version of the Lovász Local Lemma.

Theorem 4.16 (The continuous version of Lovász Local Lemma)**.** *Let $\mathcal{B} : X \to^? k$ be a good continuous CSP on a zero-dimensional Polish space X. There exists a continuous solution $g : X \to k$ to $\mathcal{B}$.*

Proof. As $\mathcal{B}$ is a good continuous CSP, it is bounded. Thus the graph $G_{\mathcal{B}}$ is continuous and its maximum degree Δ is finite. Now Lemma 4.6 gives us the continuous proper $\Delta + 1$-colouring of $G_{\mathcal{B}}$, $f : V(G_{\mathcal{B}}) \to \Delta + 1$. Notably, the colour classes $I_i = f^{-1}(i)$ form a partition of $V(G_{\mathcal{B}}) = X$, where the sets are clopen and independent in the graph. We will now create a sequence of functions $g_i : I_i \to k$. For each $i \in \Delta + 1$, let g_i be the continuous colouring obtained by applying Lemma 4.15 to $\mathcal{B} \setminus (g_0 \cup \ldots \cup g_{i-1})$ and I_i. Now we argue that $g = \bigcup_{i \in \Delta+1} g_i$, $g : X \to k$ is a continuous solution to $\mathcal{B}$. Suppose to the contrary that g violates some $B \in \mathcal{B}$. Now $B/g = \{\emptyset\}$, but this means that

$$\begin{aligned}
\mathbb{P}[B/g] &= \frac{|B|}{k^{\mathrm{dom}(B)}} \\
&= 1,
\end{aligned}$$

which is in contradiction, since $\mathcal{B}/g$ is good due to Lemma 4.15. □

4.C Continuous pattern-avoiding solutions

In this section, we extend our theory into graphs and sets that are defined using Γ, which we defined to be any countably infinite discrete group. We will also use Γ to define the pattern-avoiding colouring problems on these graphs and sets. This work culminates into an equivalence result, where we show that a problem of this kind is solvable continuously on all free zero-dimensional Polish Γ-spaces, i.e. spaces where the action is continuous, if and only if it is solvable continuously on the free part of the shift ${}^{\Gamma}2$. Later in Section 4.D, we extend this equivalence to other statements about a certain family of graphs, and also about LOCAL algorithms.

Definition 4.17 (Schreier graph)**.** Let $\Gamma \curvearrowright X$, and $S \subset \Gamma$ be a finite subset. The **Schreier graph** of S, $G(X,S)$, has the vertex set X. The edges of $G(X,S)$ are $(x, \sigma \cdot x)$ for each $\sigma \in (S \cup S^{-1}) \setminus \{\mathbf{1}_\Gamma\}$. If X is a free zero-dimensional Polish Γ-space, then the Schreier graph is continuous. The neighbourhood of a clopen set $U \subseteq X$ is open, as

$$\begin{aligned} N_{G(X,S)}(U) &= \left(\left(S \cup S^{-1}\right) \setminus \{\mathbf{1}_\Gamma\}\right) \cdot U \\ &= \pi_1\left(\left(\left(\left(S \cup S^{-1}\right) \setminus \{\mathbf{1}_\Gamma\}\right) \times X\right) \cap \alpha^{-1}(U)\right), \end{aligned}$$

where α is the group action $\alpha : \Gamma \times X \to X$, and π_1 is the open projection function $\pi_1 : \Gamma \times X \to X$.

We also equip the graph with the bidirectional edge labeling $\lambda : X^2 \rightharpoonup \Gamma$, where $\lambda(x, \sigma \cdot x) = \sigma$ iff $\sigma \in (S \cup S^{-1}) \setminus \{\mathbf{1}_\Gamma\}$. This implies that for an edge (x,y), $\lambda(x,y) = (\lambda(y,x))^{-1}$, meaning that the vertices x and y "see" the edge coloured differently. We do not make use of this labeling until Section 4.D.

A keen reader might have already noticed the similarity of Schreier graphs to the more well-known Cayley graphs. Indeed, Cayley graphs can be considered as Schreier graphs where we let the group Γ act on itself.

Next, we will define two ways to bound the size of "gaps" in some subset of Γ.

Definition 4.18 (S-syndeticity and S-separateness)**.** Let $\Gamma \curvearrowright X$, and $S \subset \Gamma$ be a finite subset. We say that $A \subseteq X$ is **S-syndetic** if $S^{-1} \cdot A = X$, and **S-separated** if $\forall x, y \in A$, $x \neq y \implies x \notin S \cdot y$. Notice that set A is S-syndetic iff it is independent in the Schreier graph $G(X,S)$.

The syndeticity bounds the "gaps" of a subset by not allowing too large parts to be missing, while separateness requires some "gaps" to exist between each element of the subset. The next definition will be used in our main lemma, which will eventually be used to produce both Borel and continuous colourings.

Definition 4.19 (S-similarity)**.** Let $\Gamma \curvearrowright X$, and $f : X \rightharpoonup k$ be a partial colouring. For a finite subset $S \subset \Gamma$, we say that x and y are S-similar in f, $x \equiv_f^S y$ if

$$\forall \sigma \in S, \{\sigma \cdot x, \sigma \cdot y\} \subseteq \mathrm{dom}(f) \implies f(\sigma \cdot x) = f(\sigma \cdot y).$$

The following lemma will be our main tool in this section. We will use it along with suitable partitions to construct both Borel and continuous solutions piece by piece.

Lemma 4.20 (Main lemma)**.** *For any finite subset $F \subset \Gamma$, there exists a finite subset $S \subset \Gamma$ such that if X is a free zero-dimensional Polish Γ-space that is partitioned into three clopen sets, $X = C_0 \cup C \cup U$, where C is F-syndetic and U is S-separated, then given any element $\mathbf{1}_\Gamma \neq \gamma \in \Gamma$, every continuous 2-colouring $f_0 : C_0 \to 2$ can be extended to a continuous 2-colouring $g : C_0 \cup C \to 2$ s.t.*

$$\forall x \in X,\ x \not\equiv_f^S \gamma \cdot x. \tag{4.20.1}$$

Proof. Let $F \subset \Gamma$ be a finite set. Without loss of generality, we can suppose that F is symmetric and $\mathbf{1}_\Gamma \in F$. Let $m \in \omega$ be such that $2^m > (2m|F|)^{600}$. This value will become useful only at the very end when we use Theorem 4.16. Let M be any finite symmetric subset of Γ with $\mathbf{1}_\Gamma \in M$ s.t. $|M| = m|F|$. Now let

$$N = FM \cup MF,$$

and

$$S = N^5 F.$$

We will now prove that the lemma holds with S.

Let X be a free, zero-dimensional Polish Γ-space that is partitioned into three clopen sets $X = C_0 \cup C \cup U$, where C is F-syndetic and U is S-separated. Fix any element $\mathbf{1}_\Gamma \neq \gamma \in \Gamma$, and let

$$\Delta = N^4 F \gamma F N^4 \setminus \{\mathbf{1}_\Gamma\}.$$

Now let Z be a clopen maximal independent set in the Schreier graph $G(C, N^4)$, which makes it a maximal N^4-separated clopen set. We make the following claims about Z.

Claim 4.20.2. $C \subseteq N^4 Z$

Proof of claim. Suppose $x \in C$, but $x \notin N^4 Z$. Now since N is symmetric and $\mathbf{1}_\Gamma \in N$, we have that $x \notin Z$ and $\sigma \cdot x \notin Z$ for all $\sigma \in N^4$. Now $Z \cup \{x\} \supsetneq Z$ is a N^4-separated set, which contradicts the maximality of Z. ■

Claim 4.20.3. Z is N^4F-syndetic

Proof of claim. Let $x \in X$. Since C is F-syndetic, there is $\sigma \in F$ and $y \in C$ such that $\sigma^{-1} \cdot y = x$. Also since $C \subseteq N^4 Z$, there are $\delta \in N^4$ and $z \in Z$ such that $y = \delta \cdot z$, i.e. $x = \sigma^{-1} \delta \cdot z$. Since F and N^4 are symmetric, $\delta^{-1} \sigma \in N^4 Z$, and thus Z is N^4F-syndetic. ■

Let us now restate the goal of this proof. We want to extend a continuous 2-colouring $f_0 : C_0 \to 2$ to the domain $C_0 \cup C$, so that the extended function abides by Eq. (4.20.1). Let $g : C_0 \cup (C \setminus (N \cdot Z)) \to 2$ be an *arbitrary* continuous extension of f_0. We show that g can always be extended to satisfy Eq. (4.20.1).

Claim 4.20.4. Any function $f : C_0 \cup C$ extending g satisfies Eq. (4.20.1), if it satisfies

$$\forall z \in Z \; \forall \delta \in \Delta, \; z \not\equiv_f^N \delta \cdot z. \tag{4.20.5}$$

Proof of claim. Since Z is N^4F-syndetic, for any $x \in X$, there exists $\beta \in N^4F$ such that $\beta \cdot x \in Z$. Now apply Eq. (4.20.5) with $z = \beta \cdot x$ and $\delta = \beta\gamma\beta^{-1}$. Now $\beta \cdot x \not\equiv_f^N \beta\gamma \cdot x$, i.e. there exists some $\sigma \in N$ s.t. $\{\sigma\beta \cdot x, \sigma\beta\gamma \cdot x\} \subseteq \mathrm{dom}(f)$ and $f(\sigma\beta \cdot x) \neq f(\sigma\beta\gamma \cdot x)$. Since $S = N^5 F$, we have that $\sigma\beta \in S$, which implies that $x \not\equiv_f^S \gamma \cdot x$, i.e. Eq. (4.20.1) holds. ■

This means that since our $f : C_0 \cup C \to 2$ will extend g, whether it satisfies Eq. (4.20.5) and thus Eq. (4.20.1) will be determined by the values f gets in $C \setminus (N \cdot Z)$. Using this, we will create, with proper encoding, a CSP corresponding to the extensions of g that satisfy our equations.

Let $h : Z \to 2^{|N|}$ encode our extension to the set $C \cap (Z \cdot N)$. For this section, it is useful to think of $2^{|N|}$ as a binary number. Let $(v_i)_{i\in|N|}$ be some enumeration of N, and let $h_i : Z \to 2$ indicate the function

$$\forall z \in Z,\ h_i(z) = \text{the } i\text{:th bit of } h(z)$$

for each $i \in |N|$. This encoding may have redundant bits, as it may be that $N \cdot Z \not\subseteq C$. This does not however affect our proof in any way. We also make the following claim about the encoding.

Claim 4.20.6. Each $x \in C \cap (N \cdot Z)$ has a unique encoding $\sigma \cdot z \in N \cdot Z$

Proof of claim. Suppose $\sigma_0 \cdot z_0 = \sigma_1 \cdot z_1$, and suppose that $z_0 \neq z_1$. Immediately $\sigma_1^{-1}\sigma_0 \cdot z_0 = z_1$ violates the N^4-separateness of Z. Now since $z_0 = z_1$, let us suppose that $\sigma_0 \neq \sigma_1$. Again we immediately get $\sigma_1^{-1}\sigma_0 \cdot z_0 = z_1$, which now violates the freeness of X. ■

To decode h, let $f^h : C_0 \cup C \to 2$ be the function defined as

$$f^h(x) = \begin{cases} g(x), & \text{if } x \in C_0 \cup (C \setminus (N \cdot Z)) \\ h_i(z), & \text{if } x \in C \cap (N \cdot Z),\ \text{where } x = v_i \cdot z,\ v_i \in N,\ z \in Z. \end{cases}$$

Notice that the encoding is constructed in such a way that choosing the colours for f^h on the restriction $C \cap (N \times z)$ uniformly can be done by choosing the colour $h(z)$ uniformly from $2^{|N|}$.

Claim 4.20.7. If h is continuous, then so is f^h.

Proof of claim. By the definition of f^h,

$$(f^h)^{-1}(b) = g^{-1}(b) \cup \{x \in C \cap (N \cdot Z) \mid \exists v_i \in N,\ z \in Z,\ x = v_i \cdot z, h_i(z) = b\}.$$

As g is continuous, $g^{-1}(b)$ is open. Thus all that remains is to show that the second set is open. Let $v_i \in N$, and let $\mathcal{F}_{i,b}$ be the set of N-length bit sequences (or numbers less than $2^{|N|}$), where the i:th bit is b. Let $c \in \mathcal{F}_{i,b}$. Now the set

$$\alpha^{-1}(h^{-1}(c)) = \{(\gamma, x) \in \Gamma \times X \mid \exists z \in Z,\ \gamma \cdot x = z,\ h(z) = c\}$$

is open. Therefore the set

$$\begin{aligned}(\{v_i^{-1}\} \times C) \cap \alpha^{-1}(h^{-1}(c)) &= \left\{(v_i^{-1}, x) \in \{v_i^{-1}\} \times C \;\middle|\; \exists z \in Z,\ v_i^{-1} \cdot x = z,\ h(z) = c\right\} \\ &= \left\{(v_i^{-1}, x) \in \{v_i^{-1}\} \times C \;\middle|\; \exists z \in Z,\ x = v_i \cdot z,\ h(z) = c\right\}\end{aligned}$$

is open. Notably, this means that the set

$$\pi_1\left((\{v_i^{-1}\} \times C) \cap \alpha^{-1}(h^{-1}(c))\right) = \{x \in C \cap (\{v_i\} \cdot Z) \mid \exists z \in Z,\ x = v_i \cdot z,\ h(z) = c\}$$

is open, where π_1 is the canonical open projection $\pi_1 : \Gamma \times X \to X$. Therefore the union

$$\begin{aligned}&\bigcup_{i\in|N|} \bigcup_{c\in\mathcal{F}_{i,b}} \pi_1\left((\{v_i^{-1}\} \times C) \cap \alpha^{-1}(h^{-1}(c))\right) \\ &= \bigcup_{i\in|N|} \{x \in C \cap (\{v_i\} \cdot Z) \mid \exists z \in Z, x = v_i \cdot z, h_i(z) = b\} \\ &= \{x \in C \cap (N \cdot Z) \mid \exists v_i \in N,\ z \in Z,\ x = v_i \cdot z, h_i(z) = b\}\end{aligned}$$

is open, and thus f^h is continuous. ■

Now that we have our encoding, we will define a CSP $\mathcal{B} : Z \to^? 2^{|N|}$ such that $h : Z \to 2^{|N|}$ is a solution to $\mathcal{B}$ iff f^h satisfies Eq. (4.20.5), i.e.

$$\forall z \in Z \; \forall \delta \in \Delta, \; z \not\equiv^N_{f^h} \delta \cdot z.$$

As at this point we are extending the function $g : C_0 \cup (C \setminus (N \cdot Z)) \to 2$, the truth value of the statement $z \not\equiv^N_{f^h} \delta \cdot z$ for fixed $z \in Z$ and $\delta \in \Delta$ depends only on the values of f^h on $(N \cdot z) \cup (N\delta \cdot z)$. This fact yields us a natural way to construct our CSP: For each pair $(z, \delta) \in Z \times \Delta$, let $B_{z,\delta}$ be the constraint with domain

$$\begin{aligned} \mathrm{dom}(B_{z,\delta}) &= \{x \in Z \mid (N \cdot x) \cap ((N \cdot z) \cup (N\delta \cdot z)) \neq \emptyset\} \\ &= Z \cap ((N^2 \cup N^2\delta) \cdot z), \end{aligned}$$

where $h : Z \to 2^{|N|}$ satisfies the constraint iff $z \not\equiv^N_{f^h} \delta \cdot z$. Notice that the domain is not simply $(N \cup N\delta) \cdot z$, since for example $f^h(v_i \cdot z) = f^h(v_i\delta \cdot z)$ could depend on some $h(x)$, where $x \in Z$, $x \neq z$ and $v_i\delta \cdot z = v_j \cdot x$ for some $v_j \in N$.

We let $\mathcal{B} : Z \to^? 2^{|N|}$ be the CSP consisting of all these constraints $B_{z,\delta}$ and then, we finally show that $\mathcal{B}$ satisfies the requirements for the continuous version of LLL, Theorem 4.16.

- $\mathrm{ord}(\mathcal{B}) \leq 2$: Take any $B_{z,\delta} \in \mathcal{B}$. Since Z is N^4-separated, the domain $Z \cap ((N^2 \cup N^2\delta) \cdot z)$ has at most 2 elements, so $\mathrm{ord}(\mathcal{B}) \leq 2$.

- $\mathrm{vdeg}(\mathcal{B}) \leq 2^{11}m^{10}|F|^{22}$: Take any $x \in Z$. We need to bound the number of constraints that take values on x, i.e. the pairs $(z, \delta) \in Z \times \Delta$ s.t. $x \in \mathrm{dom}(B_{z,\delta})$. Since $N = FM \cup MF$ and $|M| = m|F|$, we have that $|N| \leq 2m|F|^2$. Therefore $|\Delta| \leq |N^4F\gamma FN^4| \leq 2^8m^8|F|^{18}$. Now fix a $\delta \in \Delta$. For any z s.t. $x \in (N^2 \cup N^2\delta) \cdot z$, we must have $z \in (N^2 \cup \delta^{-1}N^2) \cdot x$. Therefore there are at most $2^3m^2|F|^4$ choices for z s.t. $x \in \mathrm{dom}(B_{z,\delta})$, for any given $\delta \in \Delta$. Thus overall there can be at most $2^8m^8|F|^{18} \cdot 2^3m^2|F|^4 = 2^{11}m^{10}|F|^{22}$ such constraints, i.e. $\mathrm{vdeg}(\mathcal{B}) \leq 2^{11}m^{10}|F|^{22}$.

- $\forall B \in \mathcal{B} \quad \mathbb{P}(B) \leq 2^{-m/6}$: Take any $z \in Z$, $\delta \in \Delta$. We want to show that $P[B_{z,\delta}] \leq 2^{-m/6}$, i.e. a random extension f^h of g violates the constraint with probability at most $2^{-m/6}$. By definition, this happens only if z is N-similar to $\delta \cdot z$ in f^h. Let $E \subseteq \{v \in N \mid v \cdot z \in C, v\delta \cdot z \in C_0 \cup C\}$. We call E the set of **eligible** elements. Note that if v is eligible, $v \cdot z$ will be coloured by our choice for $h(z)$, but $v\delta \cdot z$ might already be coloured in g. Regardless of the case, as $f^h(v \cdot z)$ is chosen at random, the probability that $f^h(v \cdot z) = f^h(v\delta \cdot z)$ is $1/2$.

  Remember that $N \supseteq FM$, as therefore $N \cdot z \supseteq FM \cdot z$. Now since X is free, $|M \cdot z| = |M|$. From the fact that C is F-syndetic, we get that $|C \cap (N \cdot z)| \geq |C \cap (FM \cdot z)| \geq |M|/|F| = m$. Also since the third set U in our partition $U = X \setminus (C_0 \cup C)$ is S-separated and $S \supseteq N^2$, we have that $|(N \cdot y) \cap U| \leq 1$. Therefore there are at least $|E| \geq m - 1 \geq m/2$ eligible elements. Let G be a graph with $V(G) = (N \cdot z) \cup (N\delta \cdot z)$, and $E(G) = \{(v \cdot z, v\delta \cdot z) \mid v \in E\}$. Let $x \in V(G)$. Since X is free, x can have at most two edges: $(x, v_1\delta \cdot z)$ and $(v_2 \cdot z, x)$. Therefore there exists an independent edge set $I \subseteq E(G)$ with

$|I| \geq |E(G)|/3 = |E|/3$. Let $E' = \{v \in E \mid (v \cdot z, v\delta \cdot z) \in I\}$. Since the I was independent, the events $f^h(v \cdot z) = f^h(v\delta \cdot z)$ are mutually independent for all $v \in E'$. Therefore the probability that each equality is true when h is chosen at random $2^{-|E'|} \leq 2^{-m/6}$.

Now since we set m s.t. $2^m > (2m|F|)^{600}$, we have that

$$\mathbb{P}[B] \cdot \text{vdeg}(\mathcal{B})^{|\text{dom}(B)|} \leq 2^{-m/6} \cdot \left(2^{11}m^{10}|F|^{22}\right)^2 < 1$$

for all $B \in \mathcal{B}$, and therefore $\mathcal{B}$ is good. The CSP $\mathcal{B}$ is also continuous, the proof is provided in appendix Appendix A. Thus by Theorem 4.16, $\mathcal{B}$ has a continuous solution $h : Z \to 2^{|N|}$, and therefore the function $f^h : C_0 \cup C \to 2$ is continuous and it satisfies Eq. (4.20.5).

□

The Lemma 4.20 will be iterated in the proofs of the next few theorems. In the first theorem, we iterate it over the set Γ, which gives us a Borel function. In the theorem after that, with some compactness arguments, we can work with only finitely many iterations, which gives us a continuous solution.

The following is a result of Seward and Tucker-Drob in [15]. This proof is a simpler version by Bernshteyn in [3].

Theorem 4.21. *If $\Gamma \curvearrowright X$ is a free Borel action of Γ on a standard Borel space X, then there is a Γ-equivariant Borel map $\pi : X \to Y$, where $Y \subset {}^{\Gamma}2$ is a free subshift.*

Proof. We will begin this proof by constructing the set Y. First, let us define the finite sets H_i, F_i, $S_i \subset \Gamma$ for all $i \in \omega$. Let H_0 be an arbitrary finite subset of Γ. Now we define the other sets recursively.

- For every H_i, let $\delta_i \in \Gamma$ be any element s.t. $H_i \cap (H_i\delta_i) = \emptyset$. (There are at most $|H_i|^2$ unsuitable elements.)
- Let $F_i = H_i \cup (H_i\delta_i)$.
- Let S_i' be the set given by Lemma 4.20 when applied with F_i.
- Let $S_i \supseteq S_i'$ be symmetric and $S_i \supseteq F_i^{-1}F_i$.
- Let $H_{i+1} = S_iH_i$.

Next, let $(\gamma_i)_{i\in\omega}$ be an arbitrary enumeration of $\Gamma \setminus \{\mathbf{1}_\Gamma\}$. For each $i \in \omega$, let us define the shift

$$Y_i = \{y : 2 \to \Gamma \mid \exists \sigma \in S_i \;\; y(\sigma) \neq y(\sigma\gamma_i)\}.$$

Since each S_i is finite, we can see that each Y_i is clopen. Also if $y \in Y_i$, we have that $\gamma_i \cdot y \neq y$. Therefore Γ acts freely on the set $Y' = \bigcap_{i\in\omega} Y_i$. Finally, we can define $Y = \bigcap_{i\in\omega} (\gamma_i \cdot Y')$, and now we have Y that is a closed, free, and Γ-invariant shift, i.e. a free subshift.

Remember that X is a standard Polish space where Γ is a free Borel action. By standard descriptive set theory results, we can equip X with a topology that has the same Borel sets and where X is zero-dimensional Γ-space, see Kechris [11]. Next, we will make an important claim about the sets H_i, F_i, and S_i.

Claim 4.21.1. If $W \subseteq X$ is an H_i-syndetic clopen set, then there is a partition $W = C \cup U$ into clopen sets s.t. C is F_i-syndetic and U is S_i-separated and H_{i+1}-syndetic.

Proof of claim. Let us apply Lemma 4.5 on the graph $G(W, S_i)$ to get us the clopen S_i-separated set U. Since $\mathbf{1}_\Gamma \in S_i$, S_i is symmetric, and U is maximal, we get that $S_i \cdot U \supseteq W$. Now as W is H_i-syndetic and $H_{i+1} = S_i H_i$, we get that $H_i^{-1} S_i^{-1} U = X$, i.e. U is H_{i+1}-syndetic. Now to see that C is F_i-syndetic, we need to show that $(F_i \cdot x) \cap C \neq \emptyset$ for any $x \in X$. Remember that $F_i = H_i \cup (H_i \delta_i)$, where H_i and $H_i \delta_i$ are disjoint. Since W is H_i-syndetic, the sets $H_i \cdot x$ and $H_i \delta \cdot x$ each contain at least one point in W. Since U is $F_i^{-1} F_i$-separated, only one of those points can be in U, and therefore one must be in C, i.e. $(F_i \cdot x) \cap C \neq \emptyset$. ■

Next, we will utilize Claim 4.21.1 to construct the sets U_i, $C_i \subseteq X$ for all $i \in \omega$. Let us start with X, which is trivially H_0-syndetic and clopen. We use the claim to partition X into C_0 and U_0, where C_0 is F_0-syndetic and U_0 is S_0-separated and H_1-syndetic. Now we recursively define the rest: Once U_{i-1} is defined, partition it into the clopen F_i-syndetic set C_i and the clopen, S_i-separated, and H_{i+1}-syndetic set U_i. Now let us apply Lemma 4.20 on each pair C_i, U_i, with $\gamma = \gamma_i$. This yields us an increasing sequence of functions $f_0 \subseteq f_1 \subseteq \dots$ for all $i \in \omega$, where $f_i : \bigcup_{j \leq i} C_i \to 2$ is continuous 2-colouring that satisfies

$$\forall j \leq i,\ x \in X, \quad x \not\equiv_{f_i}^{S_j} \gamma_j \cdot x.$$

Now let $f : X \to 2$ be an arbitrary Borel extension of $\bigcup_{i \in \omega} f_i$. Let $\pi_f : X \to {}^\Gamma 2$ be the Borel map, where

$$\forall x \in X, \gamma \in \Gamma \quad \pi_f(x)(\gamma) = f(\gamma \cdot x).$$

Since $\pi_f(x) \in {}^\Gamma 2$, we have that

$$(\delta \cdot \pi_f(x))(\gamma) = \pi_f(x)(\gamma\delta) = f(\gamma\delta \cdot x) = \pi_f(\delta \cdot x)(\gamma),$$

so π_f is also Γ-invariant.

We claim that π_f is the Γ-equivariant Borel map that we are looking for, i.e. $\pi_f(x) \in Y$ for all $x \in X$. Since $Y = \bigcap_{i \in \omega} \gamma_i \cdot Y'$, we need to show that $\pi_f(x) \in \gamma_i \cdot Y'$ for all $i \in \omega$. Notice that as π_f is Γ-equivariant,

$$\begin{aligned} \forall x \in X,\ i \in \omega \quad \pi_f(x) \in \gamma_i \cdot Y' &\iff \forall x \in X,\ i \in \omega \quad \pi_f(\gamma_i^{-1} \cdot x) \in Y' \\ &\iff \forall x \in X \quad \pi_f(x) \in Y' \end{aligned}$$

All that remains is to show that $\pi_f(x) \in Y' = \bigcap_{i \in \omega} Y_i$ holds for all $x \in X$, i.e.

$$\forall i \in \omega,\ x \in X, \exists \sigma \in S_i \quad \pi_f(x)(\sigma) \neq \pi_f(x)(\sigma\gamma_i).$$

As $\pi_f(x)(\sigma) \neq \pi_f(x)(\sigma\gamma_i) \iff f(\sigma \cdot x) \neq f(\sigma\gamma_i \cdot x)$, this is the same as

$$\forall i \in \omega,\ x \in X \quad x \not\equiv_f^{S_i} \gamma_i \cdot x.$$

Now since f extends each f_i that we constructed using Lemma 4.20, the equation holds for all $i \in \omega$ and $x \in X$. □

We will present our last way to define problems. This definition is somewhat similar to the local colouring problem, and in fact, we will simulate certain local colouring problems later in Section 4.D.

Definition 4.22 (k-pattern)**.** For $k \geq 1$, a k-pattern is a partial function $p : \Gamma \rightharpoonup k$ with a finite domain. Given an action $\Gamma \curvearrowright X$ and a k-colouring $f : X \to k$, we say that p **occurs** in f if for some $x \in X$,

$$\forall \gamma \in \operatorname{dom}(p) \quad p(\gamma) = f(\gamma \cdot x).$$

For a finite subset $F \subset \Gamma$, let $\mathcal{P}_F(X, f) \subseteq {}^F k$ denote the set of all k-patterns that occur in f.

Definition 4.23 ($\mathcal{P}$-avoiding colouring)**.** Let X be a Γ-space and $\mathcal{P}$ a set of k-patterns. A colouring $f : X \to k$ is $\mathcal{P}$-avoiding, if no pattern $p \in \mathcal{P}$ occurs in f.

Definition 4.24 (Weak containment)**.** Let X and Y be zero-dimensional Polish Γ-spaces. We say that X is **weakly contained** in Y, $X \preceq Y$, if for all $k \geq 1$, finite $F \subset \Gamma$, and continuous k-colouring $f : X \to k$, there exists a continuous k-colouring $g : Y \to k$ s.t. $\mathcal{P}_F(Y, g) = \mathcal{P}_F(X, f)$. If $X \preceq Y$ and $Y \preceq X$, we say that X and Y are **weakly equivalent**, $X \simeq Y$.

Theorem 4.25. *If X is a nonempty free zero-dimensional Polish Γ-space, then* $\operatorname{Free}({}^\Gamma 2) \preceq X$*.*

Proof. Let us start by fixing $k \geq 1$, finite $F \subset \Gamma$, and a continuous colouring $f : \operatorname{Free}({}^\Gamma 2) \to k$. We need to show that there is a continuous colouring $g : X \to k$ s.t. each k-pattern $p : F \to k$ that occurs in f occurs in g as well, and vice versa. That is, for all $p : F \to k$,

$$\exists y \in \operatorname{Free}({}^\Gamma 2), \forall \gamma \in F \quad p(\gamma) = f(\gamma \cdot y) \iff \exists x \in X, \forall \gamma \in F \quad p(\gamma) = g(\gamma \cdot x).$$

We say that a finite set $D \subset \Gamma$ f**-determines** a point $x \in \operatorname{Free}({}^\Gamma 2)$ iff for all $z \in {}^\Gamma 2$,

$$z \restriction_D = x \restriction_D \implies f(z) = f(x).$$

Remember that the base of ${}^\Gamma 2$ consists of clopen sets $\{y \in {}^\Gamma 2 \mid y \restriction_{\operatorname{dom}(x)} = x\}$ for all partial functions $x : \Gamma \rightharpoonup 2$ with finite domain. This means that a point x is f-determined by some set iff it has a basic clopen neighbourhood where f is constant. Thus it is clear that f is continuous iff each for each $x \in {}^\Gamma 2$ there is a finite set $D \subset \Gamma$ that f-determines it.

Since 2-patterns are partial functions from Γ to 2 with finite domain, the functions that extend the pattern also form a basic clopen set. With this fact in mind, we make the following statement:

Claim 4.25.1. For each k-pattern $p \in \mathcal{P}_F(\operatorname{Free}({}^\Gamma 2, f)$, there is a 2-pattern s_p such that each $z \in {}^\Gamma 2$ that extends s_p witnesses the occurrence of p in f,

$$\forall \gamma \in F \quad f(\gamma \cdot z) = p(\gamma).$$

Proof of claim. Let p occur in f. Now there is some $x \in X$ s.t. $f(\gamma \cdot x) = p(\gamma)$ for all $\gamma \in F$. Since f is continuous, for each $\gamma \in F$ there must be some finite set D_γ that f-determines $\gamma \cdot x$. Now as we let $D = \bigcup_{\gamma \in F} D_\gamma$, we get that D f-determines $\gamma \cdot x$ for each $\gamma \in F$. Thus for all $z \in {}^\Gamma 2$ and $\gamma \in F$,

$$z \restriction_D = x \restriction_D \implies f(\gamma \cdot z) = f(\gamma \cdot x) = p(\gamma).$$

Now if we let $s_p = x \restriction_D$, we get that for any $z \in {}^\Gamma 2$ that extends s_p,

$$\forall \gamma \in F \quad f(\gamma \cdot z) = p(\gamma).$$

■

Let

$$D = \left\{ x \in \mathrm{dom}(s_p) \mid p \in \mathcal{P}_F(\mathrm{Free}({}^\Gamma 2), f) \right\},$$

where s_p is the 2-pattern given by the previous claim.

From this point on, the proof proceeds similarly to the proof of Theorem 4.21. Let H_0 be an arbitrary symmetric finite subset of Γ with size $|H_0| > |D|$. Now we define the sets H_i, F_i, $S_i \subset \Gamma$ for all $i \in \omega$.

- For every H_i, let $\delta_i \in \Gamma$ be any element s.t. $H_i \cap (H_i \delta_i) = \emptyset$. (There are at most $|H_i|^2$ unsuitable elements.)
- Let $F_i = H_i \cup (H_i \delta_i)$.
- Let S'_i be the set given by Lemma 4.20 when applied with F_i.
- Let $S_i \supseteq S'_i$ be symmetric and $S_i \supseteq F_i^{-1} F_i$.
- Let $H_{i+1} = S_i H_i$.

Now fix an arbitrary enumeration $(\gamma_i)_{i \in \omega}$ for the elements of $\Gamma \setminus \{\mathbf{1}_\Gamma\}$, and define the clopen set

$$Y_i = \{ y : \Gamma \to 2 \mid \exists \sigma \in S_i \, (y(\sigma) \neq y(\sigma \gamma_i)) \}$$

for each $i \in \omega$. Now we define

$$Y_{<N} = \bigcap_{j \in \omega} \bigcap_{i \in N} (\gamma_j \cdot Y_i)$$

for each $N \in \omega$. We have previously shown that $Y = \bigcap_{N \in \omega} Y_{<N}$ is a nonempty free subshift, i.e. it is closed Γ-invariant subset of $\mathrm{Free}({}^\Gamma 2)$. Thus each $Y_{<N}$ is a nonempty subshift, but not necessarily free. Even though our function $f : \mathrm{Free}({}^\Gamma 2) \to k$ might not be defined on the whole set $Y_{<N}$, with large enough N we can create a continuous function $f^* : Y_{<N} \to k$ that approximates f in some meaningful way.

Claim 4.25.2. There exists $N \in \omega$ and a continuous colouring $f^* : Y_{<N} \to k$ such that for each $z \in Y_{<N}$ there exists $y \in Y$ so that:

1. For all $\delta \in D$, $z(\delta) = y(\delta)$.
2. For all $\gamma \in F$, $f^*(\gamma \cdot z) = f(\gamma \cdot y)$.

Proof of claim. First, we will show that there is a finite set $L \subset \Gamma$ that f-determines each $y \in Y$. Let $L \subset \Gamma$ be a finite subset, and let $V_L \subseteq Y$ be the set of points f-determined by L. Remember that $y \in {}^\Gamma 2$ is L-determined by L iff f takes constant values in the neighbourhood defined by $y \restriction_D$. Therefore V_L is the union of such basic clopen sets $\{y \in Y \mid y \restriction_D = p\}$ where f takes constant values, for all $p \in {}^D 2$. Thus V_L is relatively open in Y. Now let $(L_i)_{i\in\omega}$ be an arbitrary enumeration of the finite subsets of Γ. Since f is continuous, each $y \in Y$ is f-determined by some L_i, and thus the set of V_{L_i}:s cover Y. As Y is a closed subset of a compact space, it is compact and thus there is some finite subcover $Y = V_{L_1} \cup \ldots \cup V_{L_r}$. Therefore each $y \in Y$ is f-determined by the finite set $L = L_1 \cup \ldots \cup L_r$.

Next we will find and fix such an $N \in \omega$ that for each $z \in Y_{<N}$,

$$\exists y \in Y \ \forall \delta \in (D \cup L \cup LF) \quad z(\delta) = y(\delta), \tag{4.25.3}$$

i.e. we will always find some $y \in Y$ that agrees with $z \in Y_{<N}$ when restricted to $D \cup L \cup LF$. Let $Q \subseteq {}^\Gamma 2$ be the set of such $z \in {}^\Gamma 2$ where Eq. (4.25.3) fails, i.e. the union of basic clopen sets

$$\{z : \Gamma \to 2 \mid \forall \delta \in (D \cup L \cup LF) \quad z(\delta) = p(\delta)\}$$

over those 2-patterns $p : (D \cup L \cup LF) \to 2$ that no $y \in Y$ extends. Due to this construction, Q is clopen. Clearly as the sequence $Q \cap Y_{<0} \supseteq Q \cap Y_{<1} \supseteq \ldots$ is a decreasing family of closed sets in the compact space ${}^\Gamma 2$ and $\bigcap_{i\in\omega}(Q \cap Y_{<i}) = Q \cap Y = \emptyset$, there must be some $N \in \omega$ s.t. $Q \cap Y_{<N} = \emptyset$, i.e. where Eq. (4.25.3) holds for all $z \in Y_{<N}$

Finally, we can define the approximating function $f^* : Y_{<N} \to k$. Let $f^*(z) = c$ iff there is some $y \in Y$ such that $z \restriction_L = y \restriction_L$ and $f(y) = c$. In Eq. (4.25.3) we have shown that such $y \in Y$, and since each y is f-determined by L, the function is well defined. Now by definition each $z \in Y_{<N}$ is f^*-defined by L, and therefore f^* is continuous.

Next, we will show that Claim 4.25.2 holds for f^*. Due to Eq. (4.25.3), there must be such a $y \in Y$ that agrees with z when restricted to $D \cup L \cup LF$, and that $f^*(z) = f(y)$. Therefore

$$\begin{aligned} \exists \gamma \in F \quad f^*(\gamma \cdot z) \neq f(\gamma \cdot y) &\implies \exists \gamma \in F \quad (\gamma \cdot z) \restriction_L \neq (\gamma \cdot y) \restriction_L \\ &\implies \exists \gamma \in F, \ \delta \in L \quad z(\delta\gamma) \neq y(\delta\gamma). \end{aligned}$$

Thus the statement Claim 4.25.2 holds for f^*. ■

Now let X be a nonempty free zero-dimensional Polish Γ-space. Let $N \in \omega$ and $f^* : Y_{<N} \to k$ be the ones given by Claim 4.25.2. We will finally start to construct the continuous k-colouring g for which $g : X \to k$ $\mathcal{P}_F(X, g) = \mathcal{P}_F(\mathrm{Free}({}^\Gamma 2), f)$. To achieve this, we will first construct a continuous Γ-equivariant map $\pi : X \to Y_{<N}$ and let $g = f^* \circ \pi$.

Let $W \subseteq X$ be a clopen maximal $D^{-1}H_0^2D$-separated subset, given by Lemma 4.5. Since Γ acts freely on the nonempty set X, the set W must be infinite. Now since the set $\mathcal{P}_F(\mathrm{Free}({}^\Gamma 2), f) \subseteq {}^F k$ is finite, we can partition W *arbitrarily* into clopen nonempty sets W_p that are indexed by $p \in \mathcal{P}_F(\mathrm{Free}({}^\Gamma 2), f)$. For a given $p \in \mathcal{P}_F(\mathrm{Free}({}^\Gamma 2))$, let s_p be the 2-pattern given by Claim 4.25.1 and define $B_p =$

$\mathrm{dom}(s_p) \cdot W_p$. Now let B be the union of sets B_p over all k-patterns p that occur in f. The set B is partitioned by B_p:s, since W is $D^{-1}D$-separated.

At this point, we want to create the conditions required to iterate Claim 4.21.1 and Lemma 4.20 in the same way as we did in the proof of Theorem 4.21. This time, however, instead of starting the iteration from an empty function, we start by defining a colouring on B whose properties will be useful later on. Let $b : B \to 2$ be a continuous 2-colouring, defined as

$$\forall p \in \mathcal{P}_F(\mathrm{Free}(^\Gamma 2), f),\ \delta \in B_p,\ w \in W_p \quad b(\delta \cdot w) = s_p(\delta). \tag{4.25.4}$$

Now since we start the iteration with $b : B \to 2$, we need to prove the following fact about $X \setminus B$:

4.25.5. The set $X \setminus B$ is H_0-syndetic.

Let $x \in X$. As W is $D^{-1}H_0^2D$-separated, there can be at most one $w \in W$ s.t. $w \in D^{-1}H_0 \cdot x$. Thus $|(D \cdot W) \cap (H_0 \cdot x)| \leq D < H_0$. Since $B \subseteq D \cdot W$, we have that $(H_0 \cdot x) \setminus B \neq \emptyset$, i.e. $\exists \delta \in H_0 \quad \delta \cdot x \in X \setminus B$.

Now using Claim 4.21.1, let us partition $X \setminus B = C_0 \cup U_0$ into clopen sets, such that C_0 is F_0-syndetic and U_0 is both S_0-separated and H_1-syndetic. Now we can iteratively construct the subsets U_i, C_i, for all $i \in N$, s.t.

- $U_{i-1} = C_i \cup U_i$
- The set C_i is F_i-syndetic and clopen, and U_i is S_i-separated, H_{i+1}-syndetic, and clopen.

Notice that here we use the same enumeration of $\Gamma \setminus \{\mathbf{1}_\Gamma\}$, $(\gamma_i)_{i\in\omega}$, that we used while constructing the sets Y_i. Let us use Lemma 4.20, where $X = B \cup C_0 \cup U_0$ is the required partition, $b : B \to 2$ the continuous colouring to be extended, and γ_0 the required non-identity element. This will yield us the continuous colouring $h_0 : B \cup C_0 \to 2$ that satisfies $\forall x \in X \quad x \not\equiv^{S_0}_{h_0} \gamma_0 \cdot x$. Now we repeatedly apply the lemma $N-1$ more times to acquire the continuous functions $b \subseteq h_0 \subseteq \ldots \subseteq h_{N-1}$, such that

$$\forall i \in N,\ x \in X \quad x \not\equiv^{S_i}_{h_{N-1}} \gamma_i \cdot x$$

Let $h : X \to 2$ be an arbitrary continuous extension to h_{N-1}, and define $\pi_h : X \to Y_{<N}$ to be the Γ-equivariant continuous map given by

$$\forall x \in X, \gamma \in \Gamma \quad \pi_h(x)(\gamma) = h(\gamma \cdot x).$$

To see that π_h is Γ-equivariant, refer to the similar definition in the proof of Theorem 4.21. To see that $\pi_h(x) \in Y_{<N}$ for each $x \in X$, remember that since h extends each h_i,

$$\begin{aligned} \forall x \in X \quad x \not\equiv^{S_i}_{h} \gamma_i \cdot x &\iff \forall x \in X, \exists \sigma \in S_i \quad h(\sigma \cdot x) \neq h(\sigma\gamma_i \cdot x) \\ &\iff \forall x \in X, \exists \sigma \in S_i \quad \pi_h(x)(\sigma) \neq \pi(x)(\sigma\gamma_i). \end{aligned}$$

Thus we let $g : X \to k$ be the continuous k-colouring given by $f^* \circ \pi_h$. It remains to argue that $\mathcal{P}_F(\mathrm{Free}(^\Gamma 2), f) = \mathcal{P}_F(X, g)$, i.e. every k-pattern p that occurs in f occurs in g, and vice versa.

4.25.6. $\mathcal{P}_F(\mathrm{Free}(^\Gamma 2), f) \subseteq \mathcal{P}_F(X, g)$.

Let $p \in \mathcal{P}_F(\text{Free}({}^\Gamma 2), f)$. Let $w \in W_p$ be an arbitrary point, and let $z = \pi_h(w) \in Y_{<N}$. Now using the statement Claim 4.25.2 with z promises us a $y \in Y$ s.t.

1. For all $\delta \in D$, $z(\delta) = y(\delta)$.
2. For all $\gamma \in F$, $f^*(\gamma \cdot z) = f(\gamma \cdot y)$.

Since π_h is Γ-equivariant, we have that

$$\forall \gamma \in F \quad g(\gamma \cdot w) = f^*(\pi_h(\gamma \cdot w)) = f^*(\gamma \cdot \pi_h(w)) = f^*(\gamma \cdot z).$$

Now from point 2., we get that

$$\forall \gamma \in F \quad g(\gamma \cdot w) = f^*(\gamma \cdot z) = f(\gamma \cdot y).$$

From the definition of b, we have that $z(\delta) = s_p(\delta)$ for all $\delta \in \text{dom}(s_p)$, so therefore by 1., y extends s_p. Remember that since s_p was defined in Claim 4.25.1 so that for any $x \in \text{Free}({}^\Gamma 2)$ extending it, we have that

$$\forall \gamma \in F \quad f(\gamma \cdot x) = p(\gamma).$$

Thus

$$\forall \gamma \in F \quad g(\gamma \cdot w) = f^*(\gamma \cdot z) = f(\gamma \cdot y) = p(\gamma),$$

and therefore p occurs in g.

4.25.7. $\mathcal{P}_F(\text{Free}({}^\Gamma 2), f) \supseteq \mathcal{P}_F(X, g)$.

Let p occur in g, and let $x \in X$ be the witness for that occurrence, $g(\gamma \cdot x) = p(\gamma)$ for all $\gamma \in F$. Let $z = \pi_h(x) \in Y_{<N}$. Now Claim 4.25.2 yields us such a $y \in Y$ that for all $\gamma \in F$, $f^*(\gamma \cdot z) = f(\gamma \cdot y)$, and therefore

$$f(\gamma \cdot y) = f^*(\gamma \cdot z) = f^*(\gamma \cdot \pi_h(x)) = f^*(\pi_h(\gamma \cdot x)) = g(\gamma \cdot x) = p(\gamma)$$

for all $\gamma \in F$. Therefore p occurs in f as well. ☐

We can now make a central statement about continuously solving problems that can be expressed with k-patterns.

Theorem 4.26. *Let $\mathcal{P}$ be a finite set of k-patterns. The following statements are equivalent:*

(I) There is a continuous $\mathcal{P}$-avoiding k-colouring of $\text{Free}({}^\Gamma 2)$.

(II) Every free zero-dimensional Polish Γ-space admits a continuous $\mathcal{P}$-avoiding k-colouring.

Proof. The direction (II) $\implies$ (I) is immediate. The other direction is given by Theorem 4.25. ☐

This theorem basically says that when continuously solving problems on free zero-dimensional Polish Γ-spaces, the free part of ${}^\Gamma 2$ will always be the hardest case.

4.D Local colourings

Thus far, we have been able to construct continuous solutions to problems presented as sets of k-patterns. For a subshift $Y \subseteq \mathrm{Free}({}^{\Gamma}2)$, the existence of a continuous solution $f : Y \to k$ was equivalent to there being some finite $D \subset \Gamma$ so that knowing $x(\delta)$ for each $\delta \in D$, we could determine $f(x)$. In this section, we show that for a given set of k-patterns $\mathcal{P}$, we have some control over the set D. This eventually leads us to make the connection to LOCAL algorithms.

Many results in this chapter require graphs to have edges that are bidirectionally labeled. This means that our edges are still undirected, we just label each edge with two labels, one for each direction. Our definition for Schreier graphs already includes this kind of labeling, we just have not made use of it until now.

Definition 4.27 (Local colouring of a subshift)**.** Let $X \subseteq {}^{\Gamma}n$ be a subshift, $F \subset \Gamma$ a finite subset, and $k \geq 1$. A k-colouring $f : X \to k$ is called **F-local** if for all $x \in X$, the colour $f(x)$ is determined by $x \restriction_F$. Using the terminology from the proof of Theorem 4.25, we would say that each $x \in X$ is f-determined by F. Using compactness arguments, we can show that f is F-local for some finite $F \subset \Gamma$ iff it is continuous. It is also equivalent to there being a mapping $\rho : {}^{F}n \to k$ such that for all $x \in X$, $f(x) = \rho(\{(\sigma, y) \in F \times n \mid x(\sigma) = y\})$.

Definition 4.28 (The subshift $X_{D,n}$)**.** Let $D \subset \Gamma$ be a finite subset, and $n \geq 1$. Define the shift $X_{D,n} \subseteq {}^{\Gamma}n$

$$X_{D,n} = \left\{x \in {}^{\Gamma}n \mid \forall \gamma \in \Gamma, \forall\, \sigma \in D \setminus \{\mathbf{1}_\Gamma\} \quad (x(\gamma) \neq x(\sigma\gamma))\right\}.$$

The shift can be constructed as the intersection of clopen sets,

$$X_{D,n} = \bigcap_{\gamma \in \Gamma} \bigcap_{\sigma \in D} \left\{x \in {}^{\Gamma}n \mid x(\gamma) \neq x(\sigma\gamma)\right\},$$

and is thus closed. We can also see that $X_{D,n}$ is Γ-invariant, and thus it is a subshift.

Definition 4.29 (Graph homomorphism)**.** Let $\varphi : V(G) \to V(G')$ be a function mapping vertices of the graph G to the vertices of graph G'. The function is a **graph homomorphism** between G and G' if the endpoints of each edge in G are mapped to be adjacent in G',

$$\forall (x, y) \in E(G) \quad (\varphi(x), \varphi(y)) \in E(G').$$

If (G, λ) and (G', λ') are edge-labeled graphs, we also require the homomorphism to map the labels accordingly,

$$\forall (x, y) \in E(G) \quad \lambda(x, y) = \lambda'(\varphi(x), \varphi(y)).$$

The following theorem is another fundamental result from distributed computing by Cole and Vishkin [5], and Goldberg, Plotking and Shannon [8]. Here the theorem is translated to use the notation of descriptive combinatorics.

Theorem 4.30. *Let γ be a non-identity element of Γ, and let $D \subset \Gamma$ be a finite set containing γ. Let $n \geq 2$ and $F^* = \{\mathbf{1}_\Gamma, \gamma\}^{\log^* n + 2}$. Now the Schreier graph $G(X_{D,n}, \{\gamma\})$ admits an F^*-local proper 6-colouring.*

It will be used in the proof of the next rather technical lemma, which shows that for some subshifts $X_{D,n}$, the problem encoded by $\mathcal{P}$ can be solved locally.

Lemma 4.31 (Local colourings of $X_{D,n}$)**.** *Let $\mathcal{P}$ be a finite set of k-patterns such that every free zero-dimensional Polish Γ-space admits a continuous $\mathcal{P}$-avoiding colouring. Now there is a finite set $F \subset \Gamma$ such that given $n \geq 2$, a finite subset D with $F \subseteq D \subset \Gamma$, and $F^* = F^{\log^* n}$, then the subshift $X_{D,n}$ admits an F^*-local $\mathcal{P}$-avoiding k-colouring.*

Proof. Let $(\gamma_i)_{i\in\omega}$ be an enumeration of the non-identity elements of Γ. Let us define $X_i = X_{\{\gamma_i\},6} \subset {}^{\Gamma}6$ and equip it with the induced topology for each $i \in \omega$. Now X_i is the set of 6-colourings of Γ where $x(\sigma) \neq x(\gamma_i\sigma)$ for all $\sigma \in \Gamma$. We define $X = \prod_{i\in\omega} X_i$, and let Γ be the diagonal action on X,

$$\forall \sigma \in \Gamma,\ (x_0, x_1, \ldots) \in X \quad \sigma \cdot (x_0, x_1, \ldots) = (\sigma \cdot x_0, \sigma \cdot x_1, \ldots).$$

We equip the space X with the product topology, which makes it a compact zero-dimensional Polish Γ-space. Since $\sigma \cdot x \neq x$ for all $x \in X$ and $\sigma \in \Gamma$, it is also free.

Let $\mathcal{P}$ be a finite set of k-patterns such that every free zero-dimensional Polish Γ-space admits a continuous $\mathcal{P}$-avoiding colouring. By this assumption, let $f : X \to k$ be a $\mathcal{P}$-avoiding k-colouring.

By the definition of X, the base of X is formed by sets of the form

$$\{x = (x_0, x_1, \ldots) \in X \mid \forall \sigma \in R,\ \forall i \in N\, (x_i(\sigma) = y_i(\sigma))\} \tag{4.31.1}$$

where $R \subset \Gamma$ is a finite set, $N \in \omega$, and $y_0, \ldots, y_{N-1}$ are 6-patterns with domain R. Since f is continuous, each $x \in X$ has some parameters N and R so that f is constant in its neighbourhood of the form Eq. (4.31.1). As these neighbourhoods cover the space X, using a compactness argument we can find some finite N and R s.t. f is constant on all the sets

$$\{x = (x_0, x_1, \ldots) \in X \mid \forall \sigma \in R,\ \forall i \in N\, (x_i(\sigma) = y_i(\sigma))\},$$

for each sequence of 6-patterns $y_0 : R \to 6, \ldots, y_N : R \to 6$. In other words, for any $x \in X$, if we know the value of $x_i(\sigma)$ for each $i \in N$ and $\sigma \in R$, we can determine the value $f(x)$. Thus there is some function $\rho : {}^{N\times R}6 \to k$ such that

$$\forall (x_0, x_1, \ldots) \in X \quad f((x_0, x_1, \ldots)) = \rho(\{((i, \sigma), m) \in {}^{N\times R}6 \mid x_i(\sigma) = m\}).$$

Now using ρ, let us define the continuous function $f^* : \prod_{i\in N} X_i \to k$ s.t.

$$\forall (x_0, x_1, \ldots) \in \prod_{i\in N} X_i \quad f^*((x_0, \ldots, x_{N-1})) = \rho(\{((i, \sigma), m) \in {}^{N\times R}6 \mid x_i(\sigma) = m\}). \tag{4.31.2}$$

Notably this means that for any $(x_0, x_1, \ldots) \in X$, $f((x_0, x_1, \ldots)) = f^*(x_0, \ldots, x_{N-1})$.

We now claim that the lemma holds for

$$F = (\{\mathbf{1}_\Gamma, \gamma_0, \ldots, \gamma_{N-1}\} \cup R)^4.$$

Let $n \geq 2$, and $D \supseteq F$ be a finite set. Let us also define $F_i^* = \{\mathbf{1}_\Gamma, \gamma_i\}^{\log^* n+2}$ for each $i \in N$. Now we use the result from distributed computing, Theorem 4.30, to construct the F_i^*-local proper 6-colourings $f_i : X_{D,n} \to 6$ of the Schreier graphs $G(X_{D,n}, \{\gamma_i\})$. For each $i \in N$, let $\pi_i : X_{D,n} \to X_i$ be a map, where

$$\pi_i(x)(\gamma) = f_i(\gamma \cdot x)$$

for all $x \in X_{D,n}$ and $\gamma \in \Gamma$. To clarify, here π_i is a function between subshifts, so x and $\pi_i(x)$ are functions from Γ to n and 6 respectively. To see that indeed $\mathrm{ran}(\pi_i) = X_i$ for each $i \in N$, remember that the functions f_i are proper 6-colourings of the graphs $G(X_{D,n}, \{\gamma_i\})$. Therefore for any $i \in N$ and $x \in X_{D,n}$, we have that

$$\begin{aligned} & \forall \sigma \in \Gamma \quad f_i(\sigma \cdot x) \neq f_i(\gamma_i \sigma \cdot x) \\ \Longrightarrow \quad & \forall \sigma \in \Gamma \quad \pi_i(x)(\sigma) \neq \pi_i(x)(\gamma_i \sigma) \\ \Longrightarrow \quad & \pi_i(x) \in X_i. \end{aligned}$$

From the definitions, we can also see that each π_i is Γ-equivariant:

$$\delta \cdot (\pi_i(x))(\gamma) = \pi_i(x)(\gamma\delta) = f_i(\gamma\delta \cdot x) = \pi_i(\delta \cdot x)(\gamma).$$

Now we can define the Γ-equivariant map $\pi : X_{D,n} \to \prod_{i \in N} X_i$ as $\pi(x) = (\pi_0(x), \ldots, \pi_{N-1}(x))$.

Claim 4.31.3. The function $f^* \circ \pi : X_{D,n} \to k$ is a $\mathcal{P}$-avoiding k-colouring of $X_{D,n}$.

Proof of claim. Suppose otherwise and let $p \in \mathcal{P}$ occur in $f^* \circ \pi$. Now there is $x \in X_{D,n}$ s.t.

$$\forall \sigma \in \mathrm{dom}(p) \quad f^*(\pi(\sigma \cdot x)) = p(\sigma).$$

Let $y = \pi(x) \in \prod_{i \in N} X_i$. Since π is Γ-equivariant, we have that

$$\forall \sigma \in \mathrm{dom}(p) \quad f^*(\sigma \cdot y) = p(\sigma).$$

Due to the definition of f^* at Eq. (4.31.2), all such $z \in X$ that $(z_0, \ldots, z_{N-1}) = (y_0, \ldots, y_{N-1})$ witness the occurrence of p in f, which is a contradiction. ■

Finally we will show that $f^* \circ \pi$ is $F^{\log^* n}$-local. Remember that by Eq. (4.31.2), in order to determine $(f^* \circ \pi)(x)$ we only need to know the value of $\pi(x)(\delta)$ for each $\delta \in R$. By the definition of π, this is the same as knowing $f_i(\delta \cdot x)$ for each $i \in N$ and $\delta \in R$. Since each f_i is F_i^*-local, the value of $f_i(\delta \cdot x)$ is determined by the values $(\delta \cdot x)(\gamma) = x(\gamma\delta)$ for each $\gamma \in F_i^*$. Thus if we know the value of x in $F_i^* R$, we can determine the value of $f_i(\delta \cdot x)$ for each $\delta \in R$. Therefore $f^* \circ \pi$ is $(F_0^* \cup \ldots \cup F_{N-1}^*)R$-local, and since $F^* = F^{\log^* n} \supseteq (F_0^* \cup \ldots \cup F_{N-1}^*)R$, we are done. □

Now we shall introduce the *finite* graphs $H_{D,n}$. Afterwards, we will show that with suitable parameters, we can find $\mathcal{P}$-avoiding colourings for these graphs.

Definition 4.32 (The S-labeled graph $H_{D,n}$)**.** Let $S \subset \Gamma$ and $D \supseteq S \cup S^{-1} \cup \{\mathbf{1}_\Gamma\}$ be finite subsets of Γ. Let $\mathrm{Inj}(A, B) \subset {}^A B$ be the set of injective functions from A to B. For $n \in \omega$, we define the graph $H_{D,n}$ so that $V(H_{D,n}) = \mathrm{Inj}(D, n)$ and for all $q, q' \in V(H_{D,n})$ and $\sigma \in S$, there is a bidirectionally labeled edge from q to q' iff

$$\forall \delta, \delta' \in D \quad (\delta = \delta'\sigma \implies q(\delta) = q'(\delta')).$$

If the above holds for the ordered pair (q, q') and σ, we say that q and q' are **σ-compatible** and label the edge $\lambda(q, q') = \sigma$. Notice that again $\lambda(q, q') = \sigma \iff \lambda(q', q) = \sigma^{-1}$. Also if q and q' are σ-compatible, we have $q'(\mathbf{1}_\Gamma) = q(\sigma)$ and since they are injective functions, they cannot also be τ-compatible for any $\tau \neq \sigma$. Thus the edge labeling is properly defined.

Next, we would like to construct $\mathcal{P}$-avoiding colourings of graphs $H_{D,n}$. However, we cannot use the standard Definition 4.22 of p occurring in a colouring of $H_{D,n}$. It requires that $q(\sigma\delta) = p(\sigma)$ for all $\sigma \in \text{dom}(p)$, but it may very well be that $\text{dom}(p)\delta \nsubseteq D$ for any $\delta \in D$. We will instead use homomorphisms to define p occurring in $H_{D,n}$.

Definition 4.33 ($\mathcal{P}$-avoiding colouring of S-labeled graphs)**.** Let $S \subset \Gamma$ be a finite set. We say that a k-pattern p is S-connected, if the Schreier graph $G(\text{dom}(p), S)$ is a connected graph. Remember that since $\text{dom}(p)$ is itself a subset of Γ, in $G(\text{dom}(p), S)$ the group is acting on itself via left multiplication. If p is an S-connected k-pattern and G a graph with bidirectionally S-labeled edges, we say that p occurs in the k-colouring $f : V(G) \to k$, if there is a graph homomorphism $\varphi : V(G(\text{dom}(p), S)) \to V(G)$ such that $f(\varphi(\sigma)) = p(\sigma)$.

We can see that for the Schreier graph $G(X, S)$, the two definitions for p:s occurrence coincide: Since $G(\text{dom}(p), S)$ is connected, for any $x \in X$ the function $\varphi(\sigma) = \sigma \cdot x$ is a graph homomorphism. Now $f(\varphi(\sigma)) = f(\sigma \cdot x)$, so clearly φ witnesses the occurrence of p according to the new definition iff x does according to the old.

Lemma 4.34. *Let D be a finite set with $S \cup S^{-1} \cup \{\mathbf{1}_\Gamma\} \subseteq D \subset \Gamma$, and let n be an integer with $n \geq |D|^2$. Now for every free zero-dimensional Polish Γ-space X, there exists a continuous graph homomorphism from $G(X, S)$ to $H_{D,n}$.*

Proof. Since the Schreier graph $G(X, DD^{-1})$ has maximum degree $|D|^2 - 1 < n$, so therefore by Lemma 4.6 there exists a continuous proper n-colouring $f : X \to n$ that colours the graph. Now we define the continuous function $h : X \to \text{Inj}(D, n)$ as

$$\forall \delta \in D \quad h(x)(\delta) = f(\delta \cdot x).$$

To see that h is continuous, we use the construction

$$\forall y \in \text{Inj}(D, n) \quad h^{-1}(y) = \bigcap_{\sigma \in D} \sigma^{-1} \cdot f^{-1}(y(\sigma)).$$

Now notice that if $g(x)(\delta) = g(x)(\delta')$ for $\delta \neq \delta'$, then $f(\delta \cdot x) = f(\delta'\delta^{-1} \cdot (\delta \cdot x))$. Since f is a proper colouring of $G(X, DD^{-1})$, this is a contradiction and therefore $\text{dom}(h) = \text{Inj}(D, n)$.

Finally we notice that x and $\sigma{\cdot}x$ are σ-compatible for each $x \in X$ and $\sigma \in S \cup S^{-1}$, since

$$\forall \delta, \delta' \in D \quad (\delta = \delta'\sigma \implies h(x)(\delta) = h(\sigma \cdot x)(\delta')) \, .$$

Thus the map $h : V(G(X, S)) \to V(H_{D,n})$ correctly maps the edge labels, and thus is a continuous graph homomorphism. □

The following lemma will finally let us connect our previous results to the realm of finite graphs.

Lemma 4.35. *Let $\mathcal{P}$ be a finite set of S-connected k-patterns such that every free zero-dimensional Polish Γ-space admits a continuous $\mathcal{P}$-avoiding k-colouring. Then there is a finite set F with $S \cup S^{-1} \cup \{\mathbf{1}_\Gamma\} \subseteq F \subset \Gamma$ so that:*

Given an integer $n \geq 2$ and set $F = F^{\log^* n}$. Given a finite set $F^* \subseteq D \subset \Gamma$, if $n \geq 2|D|$, then $H_{D,n}$ admits a $\mathcal{P}$-avoiding k-colouring.

Proof. Without loss of generality, let $\mathbf{1}_\Gamma \in p$ for each $p \in \mathcal{P}$. For each $p \in \mathcal{P}$, let d_p be the **diameter** of the graph $G(\mathrm{dom}(p), S)$, i.e. $\max(\{\mathrm{dist}_G(x, y) \mid x, y \in \mathrm{dom}(p)\}$. Since each p is S-connected and has a finite domain, each graph diameter is finite. Also since $\mathcal{P}$ is a finite set, we can define $d = \max(\{d_p \mid p \in \mathcal{P}\})$.

Now we apply Lemma 4.31 with $\mathcal{P}$ to get F_0. We claim that this lemma holds for

$$F = (F_0 \cup \{\mathbf{1}_\Gamma\})(S \cup S^{-1} \cup \{\mathbf{1}_\Gamma\})^d.$$

Let $n \geq 2$, $F^* = F^{\log^* n}$, D be a finite set with $F^* \subseteq D \subset \Gamma$ and let $F_0^* = F_0^{\log^* n}$. Now by Lemma 4.31 the subshift $X_{D,n}$ admits an F_0^*-local $\mathcal{P}$-avoiding k-colouring $f : X_{D,n} \to k$. Thus there is a map $\rho : {}^{F_0^*}n \to k$ such that

$$\forall x \in X_{D,n} \quad f(x) = \rho\left(\{(\delta, m) \in F_0^* \times n \mid x(\delta) = m\}\right). \tag{4.35.1}$$

Let us now construct the $\mathcal{P}$-avoiding k-colouring of $H_{D,n}$. Since $F_0^* \subseteq D$, we can define the function $g : \mathrm{Inj}(D, n) \to k$ as

$$\forall y \in \mathrm{Inj}(D, n) \quad g(y) = \rho\left(\{(\delta, m) \in F_0^* \times n \mid y(\delta) = m\}\right). \tag{4.35.2}$$

First, we will show that for each $y \in \mathrm{Inj}(D, n)$ there exists $x \in X_{D,n}$ such that $y(\delta) = x(\delta)$ for all $\delta \in D$. Remember that the elements of $X_{D,n} \subset {}^\Gamma n$ are the proper n-colourings of the Cayley graph $G(\Gamma, D)$. Since the maximum degree of $G(\Gamma, D)$ is $|D \cup D^{-1}| - 1 < 2|D| \leq n$, each $y \in \mathrm{Inj}(D, n)$ can be greedily extended into a proper n-colouring $x \in X_{D,n}$.

Suppose now that some $p \in \mathcal{P}$ occurs in g, i.e. there is a homomorphism $\varphi : V(G(\mathrm{dom}(p), S) \to V(H_{D,n})$ such that $g(\varphi(\gamma)) = p(\gamma)$. We have just shown that for $\varphi(\mathbf{1}_\Gamma) \in \mathrm{Inj}(D, n)$, there exists some $x \in X_{D,n}$ s.t. $y(\delta) = x(\delta)$ for all $\delta \in D$. We will now show that x witnesses the occurrence of p in f, which is a contradiction.

Let $\gamma \in \mathrm{dom}(p)$. Since p is S-connected and the graph $G(\mathrm{dom}(p), S)$ has diameter at most d, we have that $\gamma = \sigma_0 \ldots \sigma_{d-1} \in (S \cup S^{-1} \cup \{\mathbf{1}_\Gamma\})^d$. Let us start by showing that $f(\sigma_0 \cdot x) = g(\varphi(\sigma_0)) = p(\sigma_0)$.

Since the edge $(\mathbf{1}_\Gamma, \sigma_0)$ has label σ_0 in $G(\mathrm{dom}(p), S)$ and φ is a homomorphism, the functions $\varphi(\mathbf{1}_\Gamma)$ and $\varphi(\sigma_0)$ must be σ_0-compatible. Thus we have that

$$\begin{aligned} & \forall \delta, \delta' \in D \quad (\delta = \delta'\sigma_0 \implies \varphi(\mathbf{1}_\Gamma)(\delta) = \varphi(\sigma_0)(\delta') \\ \iff \quad & \forall \delta' \in D\sigma_0^{-1} \cap D \quad \varphi(\mathbf{1}_\Gamma)(\delta'\sigma_0) = \varphi(\sigma_0)(\delta'). \end{aligned}$$

Now by the definition of D and F_0^*, we have that $F_0^* \subseteq D\sigma_0^{-1} \cap D$. Therefore

$$\forall \delta \in F_0^* \quad (\sigma_0 \cdot x(\delta) = x(\delta\sigma_0) = \varphi(\mathbf{1}_\Gamma)(\delta\sigma_0) = \varphi(\sigma_0)(\delta)).$$

By using Eq. (4.35.1) and Eq. (4.35.2), we can see that $f(\sigma_0 \cdot x) = g(\varphi(\sigma_0)) = p(\sigma_0)$.

Now let $t < d-1$ and suppose that $f(\sigma_0\sigma_1 \dots \sigma_t \cdot x) = g(\varphi(\sigma_0\sigma_1 \dots \sigma_t)) = p(\sigma_0\sigma_1 \dots \sigma_t)$. Since $F_0^* \subseteq D(S \cup S^{-1} \cup \{\mathbf{1}_\Gamma\})^d \cap D$, we have that $F_0^* \subseteq D\sigma_0^{-1} \dots \sigma_{t+1}^{-1} \cap D$. Thus by the same reasoning as above, we can show that

$$f(\sigma_0\sigma_1 \dots \sigma_{t+1} \cdot x) = g(\varphi(\sigma_0\sigma_1 \dots \sigma_{t+1})) = p(\sigma_0\sigma_1 \dots \sigma_{t+1}).$$

Now by induction along the path to γ, we can show that $f(\gamma \cdot x) = p(\gamma)$ for all $\gamma \in \operatorname{dom}(p)$. Thus x witnesses the occurrence of p in f, which is a contradiction. □

Definition 4.36 (The graph family $\mathcal{H}_S$)**.** Let $S \subset \Gamma$ be a finite subset. Now let $(F_i)_{i \in \omega}$ be a sequence of arbitrary finite subsets of Γ so that

$$(S \cup S^{-1} \cup \{\mathbf{1}_\Gamma\}) \subseteq F_0 \subset F_1 \subset F_2 \subset \dots,$$

and also $\bigcup_{i \in \omega} F_i = \Gamma$. Now for each $i \in \omega$, let $n_i \geq 2$ be an integer such that $n_i \geq |F_i|^{2\log^* n_i}$, and set $D_i = F_i^{\log^* n_i}$. We now define the family of graphs $\mathcal{H}_S$ as

$$\mathcal{H}_S = \{H_{D_i,n_i} \mid i \in \omega\}.$$

Theorem 4.37. *Let $S \subset \Gamma$ be a finite subset, and let $\mathcal{P}$ be a finite set of S-connected k-patterns. The following statements are equivalent:*

(II) Every free zero-dimensional Polish Γ-space admits a continuous $\mathcal{P}$-avoiding k-colouring.

(III) There is a graph in $\mathcal{H}_S$ that admits a $\mathcal{P}$-avoiding k-colouring.

(IV) All but finitely many graphs in $\mathcal{H}_S$ admit a $\mathcal{P}$-avoiding k-colouring.

Proof. The implication (IV) $\implies$ (III) is immediate. Now assume (II), and let $F \subset \Gamma$ be the finite subset given by applying Lemma 4.35 on $\mathcal{P}$. Now (IV) holds since $F_i \subseteq F$ fails for only finitely many i.

Now suppose (III) holds and let $H_{D_i,n_i} \in \mathcal{H}_S$ admit a $\mathcal{P}$-avoiding k-colouring $f : V(H_{D_i,n_i}) \to k$. Since $\operatorname{Inj}(D_i, n_i)$ is finite, we let it have discrete topology, which makes f trivially continuous. Let X be a free zero-dimensional Polish Γ-space, and since $n_i \geq |D_i|^2$, we can use Lemma 4.34 to construct the continuous graph homomorphism $h : V(G(X,S)) \to V(H_{D_i,n_i})$. Now we need to prove that the continuous function $f \circ h$ is $\mathcal{P}$-avoiding.

Suppose to the contrary that $x \in X$ witnesses the occurrence of p in $f \circ h$. Let $\varphi : V(G(\operatorname{dom}(p), S)) \to V(G(X,S))$ be the graph homomorphism given by $\varphi(\sigma) = \sigma \cdot x$ for all $\sigma \in S$. Now the function $h \circ \varphi : V(G(\operatorname{dom}(p), S)) \to V(H_{D_i,n_i})$ is a graph homomorphism. Since for all $\sigma \in \operatorname{dom}(p)$,

$$f((h \circ \varphi)(\sigma)) = f(h(\sigma \cdot x)) = p(\sigma),$$

φ witnesses the occurrence of p in f, which is a contradiction. Therefore (II) holds. □

The following is another classical theorem in distributed computing by Goldberg, Plotkin, and Shannon in [8].

Theorem 4.38. *There is a deterministic* LOCAL *algorithm that computes a proper $(d+1)$-colouring of an n-vertex graph G with a maximum degree d in $\log^* n + O(d^2)$ rounds.*

We will use this theorem to finally join our previous work with the LOCAL model. Before anything else, let us remind ourselves of Theorem 3.14.

Theorem 3.14. *Let $\mathcal{G} \subseteq \mathcal{FSG}$ and let $\Pi = (t, P)$ be a local colouring problem. Fix $n \in \omega$ such that $\mathsf{Det}_{\Pi,\mathcal{G}}(n) \leq T < \infty$ and set $R = T + t$. If $\boldsymbol{G}$ is a topological structured graph that is (R, n)-locally in $\mathcal{G}$ and such that $|B_{\boldsymbol{G}}(x, 2R)| \leq n$ for all $x \in V(\boldsymbol{G})$, then $\boldsymbol{G}$ has a continuous Π-coloring.*

Translating this theorem to our new notation requires some work, but it provides us with one direction in our final theorem.

Theorem 4.39. *Let $S \subset \Gamma$ be a finite set and let $\mathcal{P}$ be a finite set of S-connected k-patterns. The following statements are equivalent:*

(II) Every free zero-dimensional Polish Γ-space admits a continuous $\mathcal{P}$-avoiding k-colouring.

(V) There is a deterministic distributed algorithm in the LOCAL *model that, given an n-vertex S-labeled subgraph G of $G(\Gamma, S)$, outputs a $\mathcal{P}$-avoiding k-colouring of G in $O(\log^* n)$ rounds.*

Proof. **(V)** $\implies$ **(II)**. Let us suppose that (V) holds, and let X be a free zero-dimensional Polish Γ-space. First, we show that Schreier graphs of free Γ-spaces are topological structured graphs.

Claim 4.39.1. The Schreier graph $\boldsymbol{G} = G(X, S)$ is a topological structured graph.

Proof of claim. Let d be any metric inducing the topology on X, let $\varepsilon > 0$, and let $R > 0$ be an integer. Since the group action $\alpha : \Gamma \times X \to X$ is continuous, we can for each $\gamma \in (S \cup S^{-1})^R$ determine some $\delta > 0$ s.t. if $d(x, y) < \delta$, then $d(\gamma \cdot x, \gamma \cdot y) < \epsilon$. We take δ to be the least of these, and now since X is free, if for any $x, y \in X$ we have $d(x, y) < \delta$, then the neighbourhoods $B_{\boldsymbol{G}}(x, R)$ and $B_{\boldsymbol{G}}(y, R)$ are ϵ-isomorphic. ■

Next, we show that the problem of producing $\mathcal{P}$-avoiding colourings can be expressed as a local colouring problem.

Claim 4.39.2. $\mathcal{P}$ can be encoded by a local colouring problem.

Proof of claim. Since $\mathcal{P}$ is S-connected and finite, let t be the maximum distance between $\sigma, \sigma' \in \mathrm{dom}(p)$ in the graph $G(\mathrm{dom}(p), S)$, for any $p \in \mathcal{P}$. Now let $P : \mathcal{FSG}. \to \{0, 1\}$ be the function

$$P([B_G(x,r),x]) = \begin{cases} 0, & \text{if } r \neq t \text{ or there is a graph homomorphism from} \\ & \mathrm{dom}(p) \text{ to any } \boldsymbol{H} \in [B_G(x,r),x]. \\ 1, & \text{otherwise.} \end{cases}$$

Now a k-colouring is validated by P iff it is a $\mathcal{P}$-avoiding. Let $\Pi = (t, P)$ be the local colouring problem given by $\mathcal{P}$. ■

Finally we need to create the set $\mathcal{G}$ where $\boldsymbol{G}$ is (R, n)-locally for some $n \in \omega$ and $R = T + t$, where $\mathsf{Det}_{\Pi,\mathcal{G}}(n) \leq T$.

Claim 4.39.3. There are $n, R, T \in \omega$, such that $\mathsf{Det}_{\Pi,\mathcal{G}}(n) \leq T$, $R = T + t$, $\boldsymbol{G}$ is (R, n)-locally in $\mathcal{G}$, and $|B_{\boldsymbol{G}}(x, 2R)| \leq n$ for all $x \in X$.

Proof of claim. Since we have a LOCAL algorithm that produces a solution to Π in $O(\log^* n)$ rounds when run on an n-vertex subgraph of $G(X, S)$, there is some $N \in \omega$ s.t. when run on an N-vertex subgraph, the algorithm takes less than T rounds where

$$(2|S|)^{2T+2t} \leq N.$$

Now let $n = N$, and set $R = T + t$. We let $\mathcal{G}$ be the set of rooted isomorphism classes of n-vertex subgraphs of $\boldsymbol{G}$. Now we have that $\mathsf{Det}_{\Pi,\mathcal{G}}(n) \leq T$, and each $[B_{\boldsymbol{G}}(x, R), x]$ is an n-vertex rooted subgraph of $\boldsymbol{G}$, so $\boldsymbol{G}$ is (R, n)-locally in $\mathcal{G}$. Also we defined n so that $|B_{\boldsymbol{G}}(x, 2R)| \leq n$ for all $x \in X$. ■

Now we can simply apply Theorem 3.14 to produce a continuous $\mathcal{P}$-avoiding k-colouring of X.

(II) $\implies$ **(V)**. Suppose (II) holds, and let $F \subset \Gamma$ be the finite set given by applying Lemma 4.35 on $\mathcal{P}$. Now let m be an integer with $m > |F|^{3 \log^* m}$, and let $D = F^{\log^* m}$. Now due to Lemma 4.35 there is a $\mathcal{P}$-avoiding function from $V(H_{D,m})$ to k. Thus all the LOCAL algorithm needs to do is compute a graph homomorphism from $V(G)$ to $H_{D,m}$. This means that for each vertex $x \in V(G)$, the algorithm has to compute an injective function $q_x : D \to m$ so that if x and y are adjacent in X, then q_x and q_y are $\lambda(x, y)$-compatible. We could use a similar tactic as in the proof of Lemma 4.34, where we compute a locally injective (proper) m-colouring of G and use that to define q_x for each $x \in X$. However, there can be elements in $D \cdot x$ that are not in $V(G)$ at all. Also even if $\delta \cdot x$ is in $V(G)$ it can be far away from x in G, which causes problems in the LOCAL model. Therefore we will compute the function q_x for each x directly.

We say that the pairs $(x, \delta), (y, \delta') \in V(G) \times D$ are **one-step equivalent**, $(x, \delta) \sim_1 (y, \delta')$, if x and y are adjacent in G and $\delta = \delta' \lambda(x, y)$. The relation $\sim_1$ is symmetric and transitive, and since $(S \cup S^{-1} \cup \{\mathbf{1}_\Gamma\}) \subseteq F \subseteq D$, it is also reflexive. Now let $\sim$ be the equivalence relation on $V(G) \times D$ generated by $\sim_1$, i.e. $(x, \delta) \sim (y, \delta')$ iff $(x, \delta) = (y, \delta')$ or for some $t \in \omega$,

$$\exists (z_0, \gamma_0), \ldots (z_t, \gamma_t) \in V(G) \times D \quad (x, \delta) \sim_1 (z_0, \gamma_0) \sim_1 \ldots \sim_1 (z_t, \gamma_t) \sim_1 (y, \delta'). \tag{4.39.4}$$

If $(x, \delta) \sim (y, \delta')$, we say that they are **equivalent**.

Claim 4.39.5. The function mapping x to q_x is a graph homomorphism if and only if $q_x(\delta) = q_y(\delta')$, whenever $(x, \delta) \sim (y, \delta')$.

Proof of claim. Suppose that the mapping $x \mapsto q_x$ is a graph homomorphism, and that $(x, \delta) \sim (y, \delta')$. If $(x, \delta) = (y, \delta')$, we are done. Otherwise, there is some finite path of the form Eq. (4.39.4). Let (z_i, γ_i) and (z_{i+1}, γ_{i+1}) be any two consecutive pairs on the path. Since z_i and z_{i+1} are neighbours in G, q_{z_i} and $q_{z_{i+1}}$ are $\lambda(z_i, z_{i+1})$-compatible. Now since $\gamma_i = \gamma_{i+1} \lambda(z_i, z_{i+1})$, we have that $q_{z_i}(\gamma_i) = q_{z_{i+1}}(\gamma_{i+1})$. Since this is true for all consecutive pairs on the path, we can see that $q_x(\delta) = q_y(\delta')$.

Now suppose that x and y are neighbours in G, and that $(x, \delta) \sim (y, \delta')$ implies that $q_x(\delta) = q_y(\delta')$. Therefore if there are such $\delta, \delta' \in D$ s.t. $\delta = \delta'\lambda(x, y)$, we have that $q_x(\delta) = q_y(\delta')$. Thus q_x and q_y are be $\lambda(x, y)$-compatible, which makes the map $x \mapsto q_x$ a graph homomorphism. ■

Note that if (x, δ) and (y, δ') are equivalent, then $\delta x = \delta' y$. Using this fact, we make the following claim:

Claim 4.39.6. For given $x \in V(G)$ and $\delta, \delta' \in D$, there is at most one $y \in V(G)$ s.t. $(x, \delta) \sim (y, \delta')$. Also for given $x, y \in V(G)$ and $\delta \in D$, there is at most one $\delta' \in D$ s.t. $(x, \delta) \sim (y, \delta')$.

Proof of claim. Since G is a subgraph of $G(\Gamma, S)$, x and y are also elements of Γ. Thus we can see that $\delta' = \delta x y^{-1}$ and $y = (\delta')^{-1}\delta x$. ■

Now let us define $[x]$ as

$$[x] = \{y \in V(G) \mid \exists \delta, \delta' \in D \quad (x, \delta) \sim (y, \delta')\}.$$

The relation "$y \in [x]$" is both reflexive and symmetric, but not necessarily transitive.

Claim 4.39.7. For every $x \in V(G)$ and $y \in [x]$, we have that $\mathrm{dist}_G(x, y) < |D|$.

Proof of claim. If $x = y$, we are done. Otherwise, there is some finite path of the form Eq. (4.39.4). By minimizing the length of the path, we may assume that the pairs on the path are unique. Now by Claim 4.39.6, the second element of each pair $\delta, \gamma_0, \ldots, \gamma_t, \delta'$ must also be unique in the path. Thus there are at most $|D|$ pairs on the path between (x, δ) and (y, δ'), and therefore $\mathrm{dist}_G(x, y) < D$. ■

Let G' be the graph with vertex set $V(G') = V(G)$, and where x and y are connected iff there is $z \in V(G)$ s.t. $z \in [x]$ and $y \in [z]$. By Claim 4.39.6, $|[x]| < D^2$ for all $x \in V(G)$, and therefore the maximum degree $\Delta(G')$ is less than $|D|^4$. Now by the Claim 4.39.7, we can simulate one round of communication in the graph G' by $|D| = O(1)$ rounds of communication in G. Thus we can use Theorem 4.38 to compute the proper $N = |D|^4$-colouring of the graph G', $\varphi : V(G) \to N$, in $O(\log^* n)$ rounds. For each $i \in N$, let X_i denote the set $\varphi^{-1}(i)$, which is independent in G'.

Now the LOCAL algorithm shall compute the injective function $q_x : D \to m$ in N steps, each taking $|D|^2 = O(1)$ rounds. Thus far it has taken $O(\log^* n)$ rounds to compute the G'-independent sets.

During step number i, each vertex $x \in X_i$ computes the values $q_x(\delta)$ for each $\delta \in D$. By doing so, it also determines each $q_y(\delta')$ for all $y \in X_j$, where $(x, \delta) \sim (y, \delta')$ and $j > i$. This way we make sure that the mapping $x \mapsto q_x$ will be homomorphic, due to Claim 4.39.5. To be precise, in the i:th step, the algorithm running on $x \in X_i$ repeats the following substeps for each $\delta \in D$:

1. If $q_x(\delta)$ is already fixed, move on to the next δ.

2. Otherwise, compute the set $C = \{q_y(\varepsilon) \mid y \in [x], \varepsilon \in D \text{ and } q_y(\epsilon) \text{ is defined}\}$ in $|D|$ rounds. The set C contains each colour that is already in use by some q_y for $y \in [x]$. Since $|[x]| < |D|^2$, we have that $|C| < |D|^3 < m$, and therefore there must be a free colour $\alpha \in m \setminus C$. This ensures that every function will remain injective in the next substep.

3. Let $q_x(\delta) = \alpha$, and send this information to each vertex in $[x]$ in $|D|$ rounds. Each vertex $y \in [x]$ now checks if $(x, \delta) \sim (y, \delta')$ for some $\delta' \in D$. If such δ' is found, it fixes $q_y(\delta') = \alpha$. This construction ensures that $(x, \delta) \sim (q, \delta') \implies q_x(\delta) = q_y(\delta')$.

Since each set X_i is independent in G', all the elements $x \in X_i$ can run these steps in parallel without affecting each other.

The LOCAL algorithm computes a homomorphism from G to $H_{D,m}$. This takes $O(\log^* n) + O(|D|^6) = O(\log^* n)$ rounds in total. Since the $\mathcal{P}$-avoiding k-colouring of $H_{D,m}$ produced by Lemma 4.35 does not depend on G in any way, the algorithm can compute the colouring locally in zero rounds. Thus the algorithm computes the $\mathcal{P}$-avoiding k-colouring of G in $O(\log^* n)$ rounds.

□

Appendix A

Proof of the continuity of $\mathcal{B}$ in main lemma

As a reminder, given $B^* \subseteq {}^n2^{|N|}$ and clopen sets $U_2, \dots U_n \subseteq Z$, we want to show that the set

$$\{z \in Z \mid \exists x_2 \in U_2 \dots x_n \in U_n \text{ s.t. } x_i \text{ are distinct and } B^*(z, x_2, \dots, x_n) \in \mathcal{B}\}$$

is clopen. Here we may utilize the fact that $\operatorname{ord}(\mathcal{B}) \le 2$, as then the set is empty for any $n > 2$, and therefore we need to consider at most one clopen set U at a time.

First lets consider the case where $B^* \subseteq {}^1(2^{|N|})$. We want to show that

$$\{z \in Z \mid \exists z' \in Z, \delta \in \Delta \,(B^*(z) = B_{z',\delta})\}$$

is clopen. Notice that for any pair $z' \in Z$, $\delta \in \Delta$, the truth value of $z' \equiv^N_{f^h} \delta \cdot z'$ depends only on $f^h(v \cdot z')$ and $f^h(v\delta \cdot z')$ for all $v \in N$, whenever $\{v \cdot z', v\delta \cdot z'\} \subseteq C_0 \cup C$. We previously showed that $|C \cap (N \cdot z')| \ge m$ and $|(N\delta \cdot z') \cap U| \le 1$, so therefore there must be some $v_i \in N$ s.t. $\{v_i \cdot z', v_i\delta \cdot z'\} \subseteq C_0 \cup C$ and $v_i \cdot z' \in C \cap (N \cdot z')$. Therefore the N-equivalence depends on $f^h(v_i \cdot z')$ and moreover on $h_i(z')$. Thus if there is some constraint $B^*(z) \in \mathcal{B}$ with domain size 1, it must be $B_{z,\delta}$ for some $\delta \in \Delta$. Now our set looks like the following

$$\{z \in Z \mid \exists \delta \in \Delta \,(B^*(z) = B_{z,\delta})\}.$$

Before using the definition of our constraints, for a given $\delta \in \Delta$, let us define the sets $Z^i_\delta = \{z \in Z \mid |\operatorname{dom}(B_{z,\delta})| = i\}$. Notice that $z \in Z^0_\delta$ iff $z \not\equiv^N_g \delta \cdot z$, so therefore we can define

$$\begin{aligned} Z^0_\delta &= \left\{z \in Z \mid z \not\equiv^N_g \delta \cdot z\right\} \\ &= \bigcup_{v \in N} \Big(\big(v^{-1} \cdot C_0\big) \cap \big(v^{-1}\delta \cdot C_0\big) \cap \\ &\qquad\qquad \bigcup_{b \in 2} \big((v^{-1} \cdot g^{-1}(b)) \cap (\delta^{-1} v^{-1} \cdot g^{-1}(b)) \big) \Big). \end{aligned}$$

We also know that if for $z \in Z^1_\delta$, we must have $\operatorname{dom}(B_{z,\delta}) = \{z\}$. Notably $h(\delta \cdot z)$ does not have any effect on the statement $z \equiv^N_{f^h} \delta \cdot z$, i.e. $\forall v \in N$ either $\{v \cdot z, v\delta \cdot z\} \not\subseteq$

$\mathrm{dom}(f^h)$, or $v\delta \cdot z \in \mathrm{dom}(g)$. Thus

$$Z^1_\delta = \bigcap_{v \in N} \big((v^{-1} \cdot U) \cup (\delta^{-1} v^{-1} \cdot U) \cup (\delta^{-1} v^{-1} \cdot (g^{-1}(0) \cup g^{-1}(1)))\big) \setminus Z^0_\delta.$$

As the sets Z^i_δ for $i \in 3$ form a partition of Z, we have $Z^2_\delta = Z \setminus (Z^0_\delta \cup Z^1_\delta)$.

Let us now use the definition of our constraints. The set we want to show to be clopen is

$$\begin{aligned}
&\{z \in Z \mid \exists \delta \in \Delta \, (B^*(z) = B_{z,\delta})\} \\
={}& \bigcup_{\delta \in \Delta} \Big(\Big\{ z \in Z^1_\delta \mid \forall \varphi \in {}^{1}(2^{|N|}) \, \big(h(z) = \varphi(0) \iff z \equiv^N_{f^h} \delta \cdot z \big) \Big\} \Big) \\
={}& \bigcup_{\delta \in \Delta} \left(\left(\bigcap_{\varphi \in B^*} \big\{ z \in Z^1_\delta \mid h(z) = \varphi(0) \text{ and } z \equiv^N_{f^h} \delta \cdot z \big\} \right) \cap \right. \\
& \left. \left(\bigcap_{\varphi \in {}^{1}(2^{|N|}) \setminus B^*} \big\{ z \in Z^1_\delta \mid h(z) = \varphi(0) \text{ and } z \not\equiv^N_{f^h} \delta \cdot z \big\} \right) \right).
\end{aligned} \tag{A.0.1}$$

As we have ensured that the N-similarity depends only on $h(z)$, the sets are well defined without fixing h any further. Now let us take a look at the set

$$\big\{ z \in Z^1_\delta \mid h(z) = \varphi(0) \text{ and } z \not\equiv^N_{f^h} \delta \cdot z \big\}$$

above, for given $\delta \in \Delta$ and $\varphi \in {}^{1}(2^{|N|})$. As the set contains only elements from Z^1_δ, for any element $v_i \in N$ that witnesses $f^h(v_i \cdot z) \neq f^h(v_i\delta \cdot z)$, at least $f^h(v_i \cdot z)$ or $f^h(v_i\delta \cdot z)$ must depend on $h(z)$. Thus if we also fix v_i, there are three possible cases

1. $v_i \cdot z \in C \cap (N \cdot z)$ and $v_i\delta \cdot z \in \mathrm{dom}(g)$,
2. $v_i \cdot z \in \mathrm{dom}(g)$ and $v_i\delta \cdot z \in C \cap (N \cdot z)$, so there is $v_j \in N$ s.t. $v_i\delta = v_j$,
3. $v_i \cdot z \in C \cap (N \cdot z)$ and $v_i\delta \cdot z \in C \cap (N \cdot z)$, so there is $v_j \in N$ s.t. $v_i\delta = v_j$.

We show that each of these is clopen. Let $\overline{b_i}$ notate the *inverse* of the i:th bit of $\varphi(0)$. Now the first set is simply

$$Z^1_\delta \cap (v_i^{-1} \cdot (C \cap (N \cdot z))) \cap (\delta^{-1} v_i^{-1} \cdot g^{-1}(\overline{b_i})).$$

If $v_i\delta = v_j$ for some $v_j \in N$, the second case is also possible. That set is

$$Z^1_\delta \cap (v_i^{-1} \cdot g^{-1}(\overline{b_j})) \cap Z^1_\delta \cap (\delta^{-1} v_i^{-1} \cdot (C \cap (N \cdot z))).$$

If in addition to $v_i\delta = v_j$, we have that $\overline{b_i} \neq \overline{b_j}$, then the third case is also possible. Then that set is

$$Z^1_\delta \cap (v_i^{-1} \cdot (C \cap (N \cdot z))) \cap (\delta^{-1} v_i^{-1} \cdot (C \cap (N \cdot z))).$$

Now for fixed $\delta \in \Delta$ and $\varphi \in {}^{1}(2^{|N|})$, we get the set

$$\big\{ z \in Z^1_\delta \mid h(z) = \varphi(0) \text{ and } z \not\equiv^N_{f^h} \delta \cdot z \big\}$$

if we take the union of the first kind of sets over N, the second kind of sets over $\{v_i \in N \mid \exists v_j \in N\ v_i\delta = v_j\}$, and the third kind over $\{v_i \in N \mid \exists v_j \in N\ v_i\delta = v_j,$ and $\overline{b_i} \neq \overline{b_j}\}$. As all of the three sets were clopen and every set that we iterate over is finite, their union is also clopen. Thus its complement, the set

$$\left\{z \in Z^1_\delta \mid h(z) = \varphi(0) \text{ and } z \equiv^N_{f^h} \delta \cdot z\right\}$$

is also clopen. Therefore by Eq. (A.0.1), if $B^* \subseteq {}^1(2^{|N|})$, then

$$\{z \in Z \mid B^*(z) \in \mathcal{B}\}$$

is clopen.

Now let $B^* \subseteq {}^2(2^{|N|})$ and $U_2 \subseteq Z$ a clopen set. We will show that

$$\{z \in Z \mid \exists x \in U_2 \text{ s.t. } z \neq x, B^*(z,x) \in \mathcal{B}\}$$

is clopen. Let $z, z' \in Z$, $x \in U_2$, $\delta \in \Delta$ be s.t. $z \neq x$ and $B^*(z,x) = B_{z',\delta}$. Remember that h violates $B_{z',\delta}$ iff $z' \equiv^N_{f^h} \delta \cdot z'$, which depends on $f^h(v \cdot z')$ and $f^h(v\delta \cdot z')$ for all $v \in N$, whenever they are defined. Previously we showed that this N-similarity must depend on $h(z')$, so $z' \in \mathrm{dom}(B_{z',\delta})$. Now as $\mathrm{dom}(B_{z',\delta}) = \{z, x\}$, we know that either $z = z'$ or $x = z'$. Therefore we need to show that the set

$$\bigcup_{\delta \in \Delta} \{z \in Z \mid \exists x \in U_2\, (z \neq x,\ B^*(z,x) = B_{z,\delta} \text{ or } B^*(z,x) = B_{x,\delta})\}$$

is clopen. Let us fix $\delta \in \Delta$, and first take a look at the case $z = z'$. We want to show that

$$\left\{z \in Z^2_\delta \mid \exists x \in U_2\, (z \neq x, B^*(z,x) = B_{z,\delta})\right\}$$

is clopen. Remember that Z^2_δ consists of such $z \in Z$ that $|\mathrm{dom}(B_{z,\delta})| = 2$. Next, we would need to find a single $x \in U_2$ s.t. the equivalence holds. But as δ is fixed and $|\mathrm{dom}(B_{z,\delta})| = 2$, we do not actually need to enforce the uniqueness of x, as long as we find one.

Let $\varphi \in {}^2(2^{|N|})$ and let $h : Z \to 2^{|N|}$ be a function s.t. $h(z) = \varphi(0)$ and $h(x) = \varphi(1)$ *for all* $x \in U_2$. Now suppose $\varphi \notin B^*$. If $z \in \{z \in Z^2_\delta \mid \exists x \in U_2\, (z \neq x, B^*(z,x) = B_{z,\delta})\}$, it must be that $z \not\equiv^N_{f^h} \delta \cdot z$, i.e. there must be $v_i, v_j N$ s.t. one of the following cases is true

1. $v_i \cdot z \in C$, $v_i\delta \cdot z \in \mathrm{dom}(g)$, and $f^h(v_i \cdot z) \neq g(v_i\delta \cdot z)$,

2. $v_j = v_i\delta$, $v_i \cdot z \in \mathrm{dom}(g)$, $v_j \cdot z \in C$, and $g(v_i \cdot z) \neq f^h(v_j \cdot z)$,

3. $v_j = v_i\delta$, $v_i \cdot z \in C$, $v_j \cdot z \in C$, and $f^h(v_i \cdot z) \neq f^h(v_j \cdot z)$,

4. $v_j \neq v_i\delta$, $v_j \cdot x = v_i\delta \cdot z$, $v_i \cdot z \in \mathrm{dom}(g)$, $v_j \cdot x \in C$, and $g(v_i \cdot z) \neq f^h(v_j \cdot x)$,

5. $v_j \neq v_i\delta$, $v_j \cdot x = v_i\delta \cdot z$, $v_i \cdot z \in C$, $v_j \cdot x \in C$, and $f^h(v_i \cdot z) \neq f^h(v_j \cdot x)$,

Thus we can take the union of sets matching these cases over $v_i, v_j \in N$. Let $v_i, v_j \in N$ and b^0_n, b^1_n the n:th bits of $\varphi(0)$ and $\varphi(1)$. The sets matching to the cases above are

1.

$$\left(v_i^{-1}\cdot C\right)\cap\left(\delta^{-1}v_i^{-1}\cdot g^{-1}(b_i^0)\right)$$

2.

$$\text{if } v_j=v_i\delta,\ \left(v_i^{-1}\cdot g^{-1}(b_j^0)\right)\cap\left(v_j^{-1}\cdot C\right),\ \text{otherwise } \emptyset$$

3.

$$\text{if } v_j=v_i\delta \text{ and } b_i^0\neq b_j^0,\ \left(v_i^{-1}\cdot C\right)\cap\left(v_j^{-1}\cdot C\right),\ \text{otherwise } \emptyset$$

4.

$$\text{if } v_j\neq v_i\delta,\ \left(\delta^{-1}v_i^{-1}\cdot C\right)\cap\left(v_i^{-1}\cdot g^{-1}(b_j^1)\right),\ \text{otherwise } \emptyset$$

5.

$$\text{if } v_j\neq v_i\delta \text{ and } b_i^0\neq b_j^1,\ \left(v_i^{-1}\cdot C\right)\cap\left(\delta^{-1}v_i^{-1}\cdot C\right),\ \text{otherwise } \emptyset$$

Each of these sets is clopen, and therefore their union over all $v_i, v_j \in N$ is clopen. That set is

$$\left\{z\in Z_\delta^2 \mid \exists x\in U_2\left(z\neq x,\ x\in\operatorname{dom}(B_{z,\delta}),\ z\not\equiv_{fh}^N \delta\cdot z\right)\right\},$$

where δ, φ and therefore h are fixed. To form the set

$$\left\{z\in Z_\delta^2 \mid \exists x\in U_2\left(z\neq x,\ x\in\operatorname{dom}(B_{z,\delta}),\ z\equiv_{fh}^N \delta\cdot z\right)\right\},$$

we can just find their union

$$\left\{z\in Z_\delta^2 \mid \exists x\in U_2\left(z\neq x,\ x\in\operatorname{dom}(B_{z,\delta})\right)\right\}.$$

We can see that this is the set

$$Z_\delta^2\cap\left(\bigcup_{v_i\in N}\ \bigcup_{v_j\in N\setminus\{v_i\delta\}}\left((\delta^{-1}v_i^{-1}v_j^{-1}\cdot U_2)\cap(v_i^{-1}\cdot(C_0\cup C))\right)\right),$$

which is clopen, and therefore the set

$$\left\{z\in Z_\delta^2 \mid \exists x\in U_2\left(z\neq x,\ x\in\operatorname{dom}(B_{z,\delta}),\ z\equiv_{fh}^N \delta\cdot z\right)\right\},$$

is also clopen.

Thus we have shown that

$$\begin{aligned}&\left\{z\in Z_\delta^2 \mid \exists x\in U_2\left(z\neq x, B^*(z,x)=B_{z,\delta}\right)\right\}\\ =&\bigcap_{\varphi\in B^*}\left(\left\{z\in Z_\delta^2 \mid \exists x\in U_2\left(z\neq x,\ x\in\operatorname{dom}(B_{z,\delta}),\ z\equiv_{fh}^N \delta\cdot z\right)\right\}\right)\cap\\ &\bigcap_{\varphi\in {}^2(2^{|N|})\setminus B^*}\left(\left\{z\in Z_\delta^2 \mid \exists x\in U_2\left(z\neq x,\ x\in\operatorname{dom}(B_{z,\delta}),\ z\not\equiv_{fh}^N \delta\cdot z\right)\right\}\right)\end{aligned}$$

is clopen.

Last thing we need to do is show that

$$\left\{z\in Z \mid \exists x\in U_2\left(z\neq x, B^*(z,x)=B_{x,\delta}\right)\right\}$$

is clopen, for fixed $\delta \in \Delta$. This is the same set as

$$\begin{aligned} &\{z \in Z \mid \exists v_i, v_j \in N\,(v_i\delta \cdot x = v_j \cdot z,\ v_i\delta \neq v_j,\ x \in U_2,\ B^*(z,x) = B_{x,\delta})\} \\ = &\bigcup_{v_i \in N} \bigcup_{v_j \in N\setminus\{v_i\delta\}} \left(\left\{z \in Z \mid B^*(z,x) = B_{x,\delta},\ \text{where } x = \delta^{-1}v_i^{-1}v_j \cdot z \in U_2\right\}\right). \end{aligned}$$

Let us fix $v_i \in N$ and $v_j \in N \setminus \{v_i\delta\}$, let $v_x = \delta^{-1}v_i^{-1}v_j$, and for any $\varphi \in {}^2(2^{|N|})$ let $h : Z \to 2^{|N|}$ be s.t. . Now we have that

$$\begin{aligned} &\{z \in Z \mid B^*(z, v_x \cdot z) = B_{v_x \cdot z,\delta}\} \\ = &\bigcap_{\varphi \in B^*} \left(\left\{z \in Z \mid v_x \cdot z \equiv^N_{f^h} \delta v_x \cdot z,\ \text{where } h(z) = \varphi(0),\ h(v_x \cdot z) = \varphi(1)\right\}\right) \cap \\ &\bigcap_{\varphi \in {}^2(2^{|N|})\setminus B^*} \left(\left\{z \in Z \mid v_x \cdot z \not\equiv^N_{f^h} \delta v_x \cdot z,\ \text{where } h(z) = \varphi(0),\ h(v_x \cdot z) = \varphi(1)\right\}\right) \end{aligned}$$

Let us fix $\varphi \in {}^2(2^{|N|})$ and again take a look at one of the sets,

$$\left\{z \in Z \mid v_x \cdot z \not\equiv^N_{f^h} \delta v_x \cdot z,\ \text{where } h(z) = \varphi(0),\ h(v_x \cdot z) = \varphi(1)\right\}.$$

For any z in that set, we must have that $v_x \cdot z \not\equiv^N_{f^h} \delta v_x \cdot z$, i.e. $v_k, v_l \in N$ s.t. one of the following cases is true

1. $v_k v_x \cdot z \in C$, $v_k \delta v_x \cdot z \in \mathrm{dom}(g)$, and $f^h(v_k v_x \cdot z) \neq g(v_k \delta v_x \cdot z)$,
2. $v_l = v_k\delta$, $v_k v_x \cdot z \in \mathrm{dom}(g)$, $v_l v_x \cdot z \in C$, and $g(v_k v_x \cdot z) \neq f^h(v_l v_x \cdot z)$,
3. $v_l = v_k\delta$, $v_k v_x \cdot z \in C$, $v_l v_x \cdot z \in C$, and $f^h(v_k v_x \cdot z) \neq f^h(v_l v_x \cdot z)$,
4. $v_l \neq v_k\delta$, $v_l v_x \cdot x = v_k v_x \delta \cdot z$, $v_k v_x \cdot z \in \mathrm{dom}(g)$, $v_l v_x \cdot x \in C$, and $g(v_k v_x \cdot z) \neq f^h(v_l v_x \cdot x)$,
5. $v_l \neq v_k\delta$, $v_l v_x \cdot x = v_k v_x \delta \cdot z$, $v_k v_x \cdot z \in C$, $v_l v_x \cdot x \in C$, and $f^h(v_k v_x \cdot z) \neq f^h(v_l v_x \cdot x)$,

From this on the proof is equivalent to what we did in the case $z = z$. Thus we may conclude that for any $B^* \subseteq {}^2(2)^{|N|}$ and clopen $U_2 \subseteq Z$, the set

$$\{z \in Z \mid \exists x \in U_2\,(z \neq x,\ B^*(z,x) \in \mathcal{B})\}$$

is clopen.

Bibliography

[1] Leonid Barenboim and Michael Elkin. "Distributed Graph Coloring: Fundamentals and Recent Developments". In: *Synthesis Lectures on Distributed Computing Theory* 4 (July 2013), pp. 28, 33. DOI: 10.2200/S00520ED1V01Y201307DCT011.

[2] Anton Bernshteyn. *Distributed Algorithms, the Lovász Local Lemma, and Descriptive Combinatorics*. 2020. DOI: 10.48550/ARXIV.2004.04905. URL: https://arxiv.org/abs/2004.04905.

[3] Anton Bernshteyn. *Probabilistic constructions in continuous combinatorics and a bridge to distributed algorithms*. 2021. DOI: 10.48550/ARXIV.2102.08797. URL: https://arxiv.org/abs/2102.08797.

[4] Sebastian Brandt et al. *On Homomorphism Graphs*. 2021. DOI: 10.48550/ARXIV.2111.03683. URL: https://arxiv.org/abs/2111.03683.

[5] Richard Cole and Uzi Vishkin. "Deterministic coin tossing with applications to optimal parallel list ranking". In: *Information and Control* 70.1 (1986), pp. 32–53. ISSN: 0019-9958. DOI: https://doi.org/10.1016/S0019-9958(86)80023-7.

[6] R. Diestel. *Graph Theory*. Graduate Texts in Mathematics. Springer Berlin Heidelberg, 2018. ISBN: 9783662575604.

[7] Paul Erdős and László Lovász. "Problems and results on 3-chromatic Hypergraphs and some related questions". In: *Coll Math Soc J Bolyai* 10 (Jan. 1974).

[8] A. Goldberg, S. Plotkin, and G. Shannon. "Parallel Symmetry-Breaking in Sparse Graphs". In: *Proceedings of the Nineteenth Annual ACM Symposium on Theory of Computing*. STOC '87. New York, New York, USA: Association for Computing Machinery, 1987, 315â€"324. ISBN: 0897912217. DOI: 10.1145/28395.28429.

[9] Jan Grebík and Václav Rozhoň. *Classification of Local Problems on Paths from the Perspective of Descriptive Combinatorics*. 2021. DOI: 10.48550/ARXIV.2103.14112. URL: https://arxiv.org/abs/2103.14112.

[10] Jan Grebík and Václav Rozhoň. *Local Problems on Grids from the Perspective of Distributed Algorithms, Finitary Factors, and Descriptive Combinatorics*. 2021. DOI: 10.48550/ARXIV.2103.08394. URL: https://arxiv.org/abs/2103.08394.

[11] Alexander S. Kechris. *Classical descriptive set theory*. 1995, p. 83.

[12] A.S Kechris, S Solecki, and S Todorcevic. "Borel Chromatic Numbers". In: *Advances in Mathematics* 141.1 (1999), p. 13. ISSN: 0001-8708. DOI: https://doi.org/10.1006/aima.1998.1771.

[13] K. Kunen. *Set Theory: An Introduction to Independence Proofs*. North-Holland Publishing Company, 1980. ISBN: 9780444854018.

[14] Moni Naor and Larry Stockmeyer. "What Can Be Computed Locally?" In: *Proceedings of the Twenty-Fifth Annual ACM Symposium on Theory of Computing*. STOC '93. San Diego, California, USA: Association for Computing Machinery, 1993, pp. 184–193. ISBN: 0897915917. DOI: 10.1145/167088.167149.

[15] Brandon Seward and Robin D. Tucker-Drob. *Borel structurability on the 2-shift of a countable group*. 2014. DOI: 10.48550/ARXIV.1402.4184. URL: https://arxiv.org/abs/1402.4184.

[16] Joel Spencer. "Asymptotic lower bounds for Ramsey functions". In: *Discrete Mathematics* 20 (1977), pp. 69–76. ISSN: 0012-365X. DOI: https://doi.org/10.1016/0012-365X(77)90044-9.