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On Choquet integrals and Poincaré-Sobolev inequalities

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A B S T R A C T

We consider integral inequalities in the sense of Choquet with respect to the Hausdorff content $\mathcal{H}^\delta_\infty$. In particular, if Ω is a bounded John domain in $\mathbb{R}^n$, $n \geq 2$, and $0 < \delta \leq n$, we prove that the corresponding $(\delta p/(\delta - p), p)$-Poincaré-Sobolev inequalities hold for all continuously differentiable functions defined on Ω whenever $\delta/n < p < \delta$. We prove also that the (p, p)-Poincaré inequality is valid for all $p > \delta/n$.

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1. Introduction

We are working on Euclidean n-space $\mathbb{R}^n$, $n \geq 2$. We recall the definition of Choquet integrals over sets E in $\mathbb{R}^n$ with respect to the Hausdorff content $\mathcal{H}^\delta_\infty$ and consider

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corresponding integral inequalities. In particular, we are interested in the Poincaré and Poincaré-Sobolev inequalities in this context.

Our main theorem, Theorem 3.7 gives the following corollary.

1.1 Corollary. *Let Ω be a bounded (α,β)-John domain in $\mathbb{R}^n$. If $0 < \delta \le n$ and $p \in (\delta/n, \delta)$, then there exists a constant c depending only on n, δ, p, and John constants α and β such that*

$$\inf_{b\in\mathbb{R}} \Big(\int_\Omega |u(x)-b|^{\frac{\delta p}{\delta - p}} d\mathcal{H}^\delta_\infty \Big)^{\frac{\delta-p}{\delta p}} \le c \Big(\int_\Omega |\nabla u(x)|^p d\mathcal{H}^\delta_\infty \Big)^{\frac{1}{p}} \tag{1.2}$$

for all $u \in C^1(\Omega)$.

We will show that the exponent $\delta p/(\delta - p)$ is the best possible exponent in this setting, Example 3.14. Theorem 3.7 states a version of the Poincaré-Sobolev inequality (3.8) where the dimension of the Hausdorff content is smaller on the left hand side than on the right hand side of the corresponding inequality.

We prove also the Poincaré inequality for any $p > \delta/n$, $\delta \in (0, n]$, that is, there is a constant c which depends only on δ, n, p, and John constants α and β such that the inequality

$$\inf_{b\in\mathbb{R}} \int_\Omega |u(x)-b|^p \, d\mathcal{H}^\delta_\infty \le c \int_\Omega |\nabla u(x)|^p \, d\mathcal{H}^\delta_\infty$$

holds for all $u \in C^1(\Omega)$ whenever Ω is a bounded (α,β)-John domain, Theorem 3.2.

We state and prove the corresponding Poincaré- and Poincaré-Sobolev -type inequalities for continuously differentiable functions with compact support defined on open, connected sets in Theorem 4.2.

If $\delta = n$, our results recover the earlier well-known results, [7]. Although there is a wealth literature on Poincaré- and Poincaré-Sobolev -type inequalities in various contexts, the authors of the present paper have not been able to find previous results where the integrals on the both sides of the corresponding inequalities are in the sense of Choquet with respect to the Hausdorff content $\mathcal{H}^\delta_\infty$, $0 < \delta < n$.

We point out that there are Poincaré-type inequalities for $C_0^\infty(\mathbb{R}^n)$ functions when only the left hand side is the Choquet integral with respect to Hausdorff content and the right hand side is the usual Lebesgue integral. We recall the following estimate which is a special case of the recent result [27, Theorem 1.7] and is called the inequality of D. R. Adams. There exists a constant $c(n)$ such that

$$\int_{\mathbb{R}^n} |u(x)| \, d\mathcal{H}^{n-1}_\infty \le c(n) ||\nabla u||_{L^1(\mathbb{R}^n)} \tag{1.3}$$

for every $u \in C_0^\infty(\mathbb{R}^n)$.

2. Hausdorff content and the Choquet integral

We recall the definition of Hausdorff content of a set E in $\mathbb{R}^n$, [15, 2.10.1, p. 169]. We refer to [3] and [4, Chapter 3], too. An open ball centered at x with radius $r > 0$ is written as $B(x, r)$.

2.1 Definition *(Hausdorff content).* Let E be a set in $\mathbb{R}^n$, $n \geq 2$. Suppose that $\delta \in (0, n]$. The Hausdorff content of E is defined by

$$\mathcal{H}^\delta_\infty(E) := \inf \left\{ \sum_{i=1}^\infty r_i^\delta : E \subset \bigcup_{i=1}^\infty B(x_i, r_i) \right\} \tag{2.2}$$

where the infimum is taken over all finite or countable ball coverings of E. The quantity (2.2) is called also the δ-Hausdorff content or δ-Hausdorff capacity or the Hausdorff content of E of dimension δ.

The Hausdorff content has the following properties:

(H1) $\mathcal{H}^\delta_\infty(\emptyset) = 0$;
(H2) if $A \subset B$ then $\mathcal{H}^\delta_\infty(A) \leq \mathcal{H}^\delta_\infty(B)$;
(H3) if $E \subset \mathbb{R}^n$ then

$$\mathcal{H}^\delta_\infty(E) = \inf_{E \subset U \text{ and } U \text{ is open}} \mathcal{H}^\delta_\infty(U);$$

(H4) if (K_i) is a decreasing sequence of compact sets then

$$\mathcal{H}^\delta_\infty\Big(\bigcap_{i=1}^\infty K_i\Big) = \lim_{i\to\infty} \mathcal{H}^\delta_\infty(K_i);$$

(H5) if (A_i) is any sequence of sets then

$$\mathcal{H}^\delta_\infty\Big(\bigcup_{i=1}^\infty A_i\Big) \leq \sum_{i=1}^\infty \mathcal{H}^\delta_\infty(A_i).$$

The proofs of properties (H1)–(H5) are straightforward. Properties (H1), (H2), (H3), and (H5) yield that $\mathcal{H}^\delta_\infty$ is an outer capacity in the sense of N. Meyers [25, p. 257]. By properties (H1), (H2) and (H5) the Hausdorff content is an outer measure.

We point out that the Hausdorff content $\mathcal{H}^\delta_\infty$ does not have the following property: if (E_i) is an increasing sequence of sets then

$$\mathcal{H}^\delta_\infty\big(\bigcup_{i=1}^\infty E_i\big) = \lim_{i\to\infty} \mathcal{H}^\delta_\infty(E_i), \tag{2.3}$$

we refer to [10], and also [11,30]. Thus the Hausdorff content $\mathcal{H}^\delta_\infty$ is not a capacity in the sense of Choquet [8].

Let us recall the dyadic counterpart of $\mathcal{H}^\delta_\infty$, that is

$$\tilde{\mathcal{H}}^\delta_\infty(E) := \inf\left\{\sum_{i=1}^\infty \ell(Q_i)^\delta : E \subset \bigcup_{i=1}^\infty Q_i\right\} \tag{2.4}$$

where the infimum is taken over all dyadic cube coverings of E. Here $\ell(Q)$ is the side length of a cube Q. It is known that $\mathcal{H}^\delta_\infty(E)$ and $\tilde{\mathcal{H}}^\delta_\infty(E)$ are comparable to each other for all sets E in $\mathbb{R}^n$, that is there are finite positive constants $c_1(n)$ and $c_2(n)$ such that

$$c_1(n)\mathcal{H}^\delta_\infty(E) \le \tilde{\mathcal{H}}^\delta_\infty(E) \le c_2(n)\mathcal{H}^\delta_\infty(E)\,,$$

we refer to [2] and [28, Chapter 2, Section 7]. By [31, Proposition 2.1 and Proposition 2.2] the dyadic Hausdorff content $\tilde{\mathcal{H}}^\delta_\infty(E)$ is a capacity in the sense of Choquet only when $n-1 \le \delta \le n$. D. Yang and W. Yuan overcame this obstacle by defining a new dyadic Hausdorff content $\tilde{\tilde{\mathcal{H}}}^\delta_\infty(E)$ by requiring in (2.4) that E is a subset of the interior of the set $\bigcup_i Q_j$, [31, Definition 2.1]. Now this new dyadic Hausdorff content $\tilde{\tilde{\mathcal{H}}}^\delta_\infty(E)$ is a capacity in the sense of Choquet for all $0 < \delta \le n$. By [31, Proposition 2.3] this new Hausdorff content $\tilde{\tilde{\mathcal{H}}}^\delta_\infty$ is comparable to the Hausdorff content $\mathcal{H}^\delta_\infty$, and constants depend only on n. By [31, Theorem 2.1 and Proposition 2.4] $\tilde{\tilde{\mathcal{H}}}^\delta_\infty$ is a strongly subadditive Choquet capacity for all $0 < \delta \le n$. For the strongly subadditivity we refer to [3].

Let $0 < \delta \le n$. We recall the definition of the δ-dimensional Hausdorff measure for $E \subset \mathbb{R}^n$,

$$\mathcal{H}^\delta(E) := \lim_{\rho\to 0^+} \inf\left\{\sum_{i=1}^\infty r_i^\delta : E \subset \bigcup_{i=1}^\infty B(x_i, r_i) \text{ and } r_i \le \rho \text{ for all } i\right\},$$

where the infimum is taken over all such finite or countable ball coverings of E that the radius of a ball is at most ρ. Thus there are finite positive constants $c_1(n)$ and $c_2(n)$ such that $c_1(n)\mathcal{H}^n(E) \le |E| \le c_2(n)\mathcal{H}^n(E)$ for all Lebesgue measurable sets E in $\mathbb{R}^n$. For the properties of the Hausdorff measure we refer to [13, Chapter 2] and [21, pp. 54–58].

The authors would like to thank Tuomas Orponen for clarifying the relationship between the Hausdorff measure and Hausdorff content.

2.5 Proposition. *There exists a constant $c(n) > 0$ such that for all $E \subset \mathbb{R}^n$ hold $\mathcal{H}^n_\infty(E) \le \mathcal{H}^n(E) \le c(n)\mathcal{H}^n_\infty(E)$.*

Proof. By definitions we have $\mathcal{H}^n_\infty(E) \le \mathcal{H}^n(E)$.

For an open ball we have $\mathcal{H}^n(B(x,r)) \le c(n)r^n$. Fix $\varepsilon > 0$. Let us take an open ball $B(x_i, r_i)$ covering of E such that $\sum_i r_i^n \le \mathcal{H}^n_\infty(E) + \varepsilon$. Since $\mathcal{H}^n$ is an outer measure, we have by monotonicity and subadditivity that

$$\mathcal{H}^n(E) \le \sum_i \mathcal{H}^n(B(x_i, r_i)) \le \sum_i c(n) r_i^n \le c(n)(\mathcal{H}^n_\infty(E) + \varepsilon).$$

Since this holds for all $\varepsilon > 0$, the claim follows. □

2.6 Remark. We use ball coverings for the definition of the δ-dimensional Hausdorff measure. In the definitions of different Hausdorff contents we use also ball coverings and dyadic cube coverings as in [2], [3], [9], [26], [31]. If one wishes to take coverings with arbitrary sets we refer to the following result. The proof of [13, Theorem 2.5] gives for all measurable $E \subset \mathbb{R}^n$ that

$$\inf \left\{ \sum_{i=1}^\infty \omega(n) \Big(\frac{\operatorname{diam} C_i}{2} \Big)^n : E \subset \bigcup_{i=1}^\infty C_i \right\} = |E|,$$

where the infimum of the left-hand side is taken over all covering of E, and $\omega(n) := \frac{\pi^{\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2}+1)}$.

We recall the definition of the Choquet integral. In the present paper Ω is always assumed to be a domain in $\mathbb{R}^n$, $n \ge 2$, that is, an open, connected set. For a function $f : \Omega \to [0, \infty]$ the integral in the sense of Choquet with respect to Hausdorff content is defined by

$$\int_\Omega f(x)\, d\mathcal{H}^\delta_\infty := \int_0^\infty \mathcal{H}^\delta_\infty \big(\{x \in \Omega : f(x) > t\} \big)\, dt. \tag{2.7}$$

Note that $\mathcal{H}^\delta_\infty$ is monotone. Hence, for every function $f : \Omega \to [0, \infty]$ the corresponding distribution function $t \mapsto \mathcal{H}^\delta_\infty \big(\{x \in \Omega : f(x) > t\} \big)$ is decreasing with respect to t. By decreasing property we know that the distribution function $t \mapsto \mathcal{H}^\delta_\infty \big(\{x \in \Omega : f(x) > t\} \big)$ is measurable with respect to Lebesgue measure. Thus, $\int_0^\infty \mathcal{H}^\delta_\infty \big(\{x \in \Omega : f(x) > t\} \big)\, dt$ is well-defined as a Lebesgue integral. The right hand side of (2.7) can be understood also as an improper Riemann integral. Although the Choquet integral is well-defined for non-measurable functions we study here only measurable functions. We recall that the Choquet integral is a nonlinear integral and used in non-additive measure theory.

The Choquet integral with respect to Hausdorff content has the following properties:

(C1) $\displaystyle\int_\Omega a f(x)\, d\mathcal{H}^\delta_\infty = a \int_\Omega f(x)\, d\mathcal{H}^\delta_\infty$ for every $a \ge 0$;

(C2) $\displaystyle\int_\Omega f(x)\, d\mathcal{H}^\delta_\infty = 0$ if and only if $f(x) = 0$ for $\mathcal{H}^\delta_\infty$-almost every $x \in \Omega$;

(C3) $\displaystyle\int_\Omega \chi_E(x)\, d\mathcal{H}^\delta_\infty = \mathcal{H}^\delta_\infty(\Omega \cap E)$;

(C4) if $A \subset B$, then $\int_A f(x)\, d\mathcal{H}^\delta_\infty \le \int_B f(x)\, d\mathcal{H}^\delta_\infty$;

(C5) if $0 \le f \le g$, then $\int_\Omega f(x)\,d\mathcal{H}^\delta_\infty \le \int_\Omega g(x)\,d\mathcal{H}^\delta_\infty$;

(C6) $\int_\Omega f(x)+g(x)\,d\mathcal{H}^\delta_\infty \le 2\Big(\int_\Omega f(x)\,d\mathcal{H}^\delta_\infty + \int_\Omega g(x)\,d\mathcal{H}^\delta_\infty\Big)$;

(C7) $\int_\Omega f(x)g(x)\,d\mathcal{H}^\delta_\infty \le 2\Big(\int_\Omega f(x)^p\,d\mathcal{H}^\delta_\infty\Big)^{1/p}\Big(\int_\Omega g(x)^q\,d\mathcal{H}^\delta_\infty\Big)^{1/q}$ when $p,q>1$ are Hölder conjugates, that is $\frac{1}{p}+\frac{1}{q}=1$.

For the proofs of these properties we refer to [3] and [4, Chapter 4].

Finally, we note that for a function $f:\Omega\to[0,\infty]$

$$\int_0^\infty \mathcal{H}^\delta_\infty\big(\{x\in\Omega : f(x)^p>t\}\big)\,dt = \int_0^\infty pt^{p-1}\mathcal{H}^\delta_\infty\big(\{x\in\Omega : f(x)>t\}\big)\,dt.$$

Namely, by changing of the variables, $t^{1/p}=\lambda$ we obtain

$$\begin{aligned}\int_0^\infty \mathcal{H}^\delta_\infty\big(\{x\in\Omega : f(x)^p>t\}\big)\,dt &= \int_0^\infty \mathcal{H}^\delta_\infty\big(\{x\in\Omega : f(x)>t^{1/p}\}\big)\,dt\\ &= \int_0^\infty p\lambda^{p-1}\mathcal{H}^\delta_\infty\big(\{x\in\Omega : f(x)>\lambda\}\big)\,d\lambda.\end{aligned}$$

From now on we study functions with values in $[-\infty,\infty]$, and the Choquet integral is taken of the absolute value of the function. We need the following lemma.

2.8 Lemma. *Let Ω be an open subset of $\mathbb{R}^n$ and let $0<\delta\le n$. Then there exist constants $c_1(n)$ and $c_2(n)$ such that*

$$\frac{1}{c_1(n)}\int_\Omega |f(x)|\,d\mathcal{H}^n_\infty \le \int_\Omega |f(x)|\,dx \le c_1(n)\int_\Omega |f(x)|\,d\mathcal{H}^n_\infty \tag{2.9}$$

and

$$\int_\Omega |f(x)|\,dx \le \frac{c_2(n)}{\delta}\Big(\int_\Omega |f(x)|^{\frac{\delta}{n}}\,d\mathcal{H}^\delta_\infty\Big)^{\frac{n}{\delta}} \tag{2.10}$$

for all measurable functions $f:\Omega\to[-\infty,\infty]$.

Proof. By Cavalier's principle we have

$$\int_\Omega |f(x)|\,dx = \int_0^\infty |\{x : |f(x)|>t\}|\,dt$$

and hence the inequalities (2.9) follow by Proposition 2.5.

For the inequality (2.10) we need to show that

$$\int_\Omega |f(x)|\,d\mathcal{H}^n_\infty \le \frac{c(n)}{\delta}\Big(\int_\Omega |f(x)|^{\frac{\delta}{n}}\,d\mathcal{H}^\delta_\infty\Big)^{\frac{n}{\delta}}.$$

Let us estimate the integrand on the right hand side. The rest of the proof follows by the proof of [26, Lemma 3]. Since the mapping $t \mapsto t^{\delta/n}$ is concave on $[0,\infty)$, we have the inequality $\big(\sum_{i=1}^m r_i^n\big)^{\frac{\delta}{n}} \le \sum_{i=1}^m (r_i^n)^{\frac{\delta}{n}}$, where $r_i > 0$. Thus $(\mathcal{H}^n_\infty(E))^{\frac{1}{n}} \le (\mathcal{H}^\delta_\infty(E))^{\frac{1}{\delta}}$. We obtain by changing the variables that

$$\begin{aligned}\int_0^\infty \mathcal{H}^n_\infty\Big(\{x : |f(x)| > t\}\Big)\,dt &= \frac{n}{\delta}\int_0^\infty \mathcal{H}^n_\infty\Big(\{x : |f(x)| > t^{n/\delta}\}\Big)t^{\frac{n}{\delta}-1}\,dt\\ &= \frac{n}{\delta}\int_0^\infty \mathcal{H}^\delta_\infty\Big(\{x : |f(x)|^{\delta/n} > t\}\Big)^{\frac{n}{\delta}} t^{\frac{n}{\delta}-1}\,dt.\end{aligned}$$

If we write $h(t) := \mathcal{H}^\delta_\infty(\{x : |f(x)|^{\delta/n} > t\}$, the function h is decreasing. Hence, we obtain

$$th(t) \le \int_0^t h(t)\,ds \le \int_0^t h(s)\,ds \le \int_0^\infty h(s)\,ds.$$

Thus, combining the estimates gives

$$\begin{aligned}\int_0^\infty \mathcal{H}^n_\infty\Big(\{x : |f(x)| > t\}\Big)\,dt &\le \frac{n}{\delta}\Big(\int_0^\infty h(s)\,ds\Big)^{\frac{n}{\delta}-1}\int_0^\infty h(t)\,dt\\ &\le \frac{n}{\delta}\Big(\int_0^\infty h(s)\,ds\Big)^{\frac{n}{\delta}}\\ &= \frac{n}{\delta}\Big(\int_\Omega |f(x)|^{\frac{\delta}{n}}\,d\mathcal{H}^\delta_\infty\Big)^{\frac{n}{\delta}}. \quad \square\end{aligned}$$

Let $\kappa \in [0,n)$. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the centered fractional Hardy-Littlewood maximal function of f is written as

$$M_\kappa f(x) := \sup_{r>0} r^{\kappa-n}\int_{B(x,r)} |f(y)|\,dy.$$

The non-fractional centered maximal function $M_0 f$ is written as Mf. If f is a defined only on Ω in $\mathbb{R}^n$, then f is defined to be zero on $\mathbb{R}^n \setminus \Omega$ in the definition of M_κ.

D. R. Adams in 1986 [2] and J. Orobitg and J. Verdera in 1998 [26] proved boundedness of the maximal operator in the sense of Choquet with respect to Hausdorff content for $p = 1$ and $p > \delta/n$, respectively. These papers as well as [3] seem to assume that the dyadic Hausdorff content is always a Choquet capacity and they used this to conclude that the Choquet integral is sublinear. However, Yang and Yan [31] showed that the dyadic Hausdorff content is a Choquet capacity if and only if the dimension δ satisfies $n - 1 < \delta \le n$. The modified dyadic Hausdorff content $\tilde{\tilde{\mathcal{H}}}^{\delta}_{\infty}$ is a strongly subadditive Choquet capacity for all $0 < \delta \le n$, and thus [12, Theorem 6.3, p. 75], see also [8, 54.2] and [6, pp. 248–249], yield

$$\int_{\Omega} \sum_{i=1}^{\infty} f_i(x) d\tilde{\tilde{\mathcal{H}}}^{\delta}_{\infty} \le \sum_{i=1}^{\infty} \int_{\Omega} f_i(x) d\tilde{\tilde{\mathcal{H}}}^{\delta}_{\infty}.$$

Since $\tilde{\tilde{\mathcal{H}}}^{\delta}_{\infty}$ is comparable with $\mathcal{H}^{\delta}_{\infty}$ we obtain

$$\int_{\Omega} \sum_{i=1}^{\infty} f_i(x) d\mathcal{H}^{\delta}_{\infty} \le c(n) \sum_{i=1}^{\infty} \int_{\Omega} f_i(x) d\mathcal{H}^{\delta}_{\infty},$$

as it is pointed out in [31, Remark 2.4]. Hence the following theorem holds.

2.11 Theorem *(Adams–Orobitg–Verdera). Let $\delta \in (0, n)$. Then there exists a constant c depending only on n, δ, and p such that for every $p > \delta/n$ and for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} (Mf(x))^p \, d\mathcal{H}^{\delta}_{\infty} \le c \int_{\mathbb{R}^n} |f(x)|^p \, d\mathcal{H}^{\delta}_{\infty}.$$

Note that in Theorem 2.11 the exponent p can be smaller than 1. We need also the next result by Adams that covers the previous theorem. It shows that the fractional maximal operator is bounded when the Choquet integrals are taken with respect to the Hausdorff content. We point out that the dimension of the Hausdorff content is smaller on the left hand side in the following inequality than on the right hand side.

2.12 Theorem *(Theorem 7(a) of [3]). Suppose that $\delta \in (0, n]$ and $\kappa \in [0, n)$. If $p \in (\delta/n, \delta/\kappa)$, then there exists a constant c depending only on n, δ, κ, and p such that for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} (M_{\kappa} f(x))^p \, d\mathcal{H}^{\delta-\kappa p}_{\infty} \le c \int_{\mathbb{R}^n} |f(x)|^p \, d\mathcal{H}^{\delta}_{\infty}.$$

3. Inequalities for C^1-functions

We recall the definition of John domains. The notion was introduced by F. John in [20] where it was called an inner radius and outer radius property. Later, domains with this property were named as John domains.

3.1 Definition. Suppose that Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$. The domain Ω is an (α, β)-John domain if there exist constants $0 < \alpha \leq \beta < \infty$ and a point $x_0 \in \Omega$ such that each point $x \in \Omega$ can be joined to x_0 by a rectifiable curve $\gamma_x : [0, \ell(\gamma_x)] \to \Omega$, parametrized by its arc length, such that $\gamma_x(0) = x$, $\gamma_x(\ell(\gamma_x)) = x_0$, $\ell(\gamma_x) \leq \beta$, and

$$\operatorname{dist}\big(\gamma_x(t), \partial\Omega\big) \geq \frac{\alpha}{\beta} t \quad \text{for all} \quad t \in [0, \ell(\gamma_x)].$$

The point x_0 is called a John center of Ω.

Examples of John domains are convex domains and domains with Lipschitz boundary, but also domains with fractal boundaries such as the von Koch snow flake. Outward spires are not allowed.

We show that the Poincaré inequality in the sense of Choquet with respect to Hausdorff content is valid in John domains. From now on we denote the integral average of a function u over a ball B by u_B where the integrals are taken with respect to the Lebesgue measure.

3.2 Theorem. *Suppose that Ω is a bounded (α, β)-John domain in $\mathbb{R}^n$. If $\delta \in (0, n]$ and $p \in (\delta/n, \infty)$, then there exists a constant c depending only on n, δ, p, and John constants α and β such that*

$$\inf_{b \in \mathbb{R}} \int_\Omega |u(x) - b|^p \, d\mathcal{H}^\delta_\infty \leq c(n, p, \delta) \beta^p \left(\frac{\beta}{\alpha}\right)^{2np} \int_\Omega |\nabla u(x)|^p \, d\mathcal{H}^\delta_\infty \tag{3.3}$$

for all $u \in C^1(\Omega)$.

Proof. Suppose that $0 < \delta < n$. Let $u \in C^1(\Omega)$. We may assume that the right hand side of the above inequality is finite. By Lemma 2.8 we have

$$\int_\Omega |\nabla u(x)|^{\frac{pn}{\delta}} \, dx \leq c(n, \delta) \Big(\int_\Omega |\nabla u(x)|^p d\mathcal{H}^\delta_\infty \Big)^{\frac{n}{\delta}}.$$

By the assumption $p > \frac{\delta}{n}$. Hence we have $\frac{pn}{\delta} > 1$. This together with the boundedness of Ω yields that $|\nabla u| \in L^1(\Omega)$. Thus the Riesz potential and the maximal function of $|\nabla u|$ are well-defined.

Since Ω is a John domain we obtain by [29, Theorem], [7], [23], and [18] the pointwise estimate

$$|u(x) - u_B| \le c(n, \alpha, \beta) \int_\Omega \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy \tag{3.4}$$

for every $x \in \Omega$. Here, $B = B(x_0, c(n)\alpha^2/\beta)$ and $c(n, \alpha, \beta) = c(n)(\beta/\alpha)^{2n}$ by [18]. The Riesz potential can be estimated by the Hardy-Littlewood maximal operator. Thus by [32, Lemma 2.8.3] we have

$$|u(x) - u_B| \le c(n,p) \left(\frac{\beta}{\alpha}\right)^{2n} \operatorname{diam}(\Omega) M|\nabla u|(x)$$

for every $x \in \Omega$. Hence, by properties (C5) and (C1) of the Choquet integral we obtain

$$\int_\Omega |u(x) - u_B|^p d\mathcal{H}^\delta_\infty \le c(n,p) \left(\frac{\beta}{\alpha}\right)^{2np} \operatorname{diam}(\Omega)^p \int_\Omega (M|\nabla u|(x))^p d\mathcal{H}^\delta_\infty.$$

Since the maximal operator is bounded in the sense of Choquet with respect to Hausdorff content by Theorem 2.11, we obtain

$$\int_\Omega |u(x) - u_B|^p d\mathcal{H}^\delta_\infty \le c(n,\delta,p) \left(\frac{\beta}{\alpha}\right)^{2np} \operatorname{diam}(\Omega)^p \int_\Omega |\nabla u(x)|^p d\mathcal{H}^\delta_\infty.$$

Since Ω is a bounded John domain, we have $\operatorname{diam}(\Omega) \le 2\beta$.

If $\delta = n$, then by [7], [23], [18] we have

$$\inf_{b\in\mathbb{R}} \int_\Omega |u(x) - b|^p dx \le c(n,p)\beta^p \left(\frac{\beta}{\alpha}\right)^{2np} \int_\Omega |\nabla u(x)|^p\, dx$$

for all $u \in C^1(\Omega)$ with $|\nabla u| \in L^p(\Omega)$. Now the claim follows by Lemma 2.8. □

3.5 Remark. If Ω is a bounded convex domain then the same proof yields

$$\inf_{b\in\mathbb{R}} \int_\Omega |u(x) - b|^p\, d\mathcal{H}^\delta_\infty \le c(n,\delta,p) \frac{\operatorname{diam}(\Omega)^{np+p}}{|\Omega|^p} \int_\Omega |\nabla u(x)|^p\, d\mathcal{H}^\delta_\infty$$

for all $u \in C^1(\Omega)$. Here we have used [14, Lemma 7.16] instead of the previous estimate for functions in a John domain.

Next we estimate the Riesz potential by the Hedberg-type pointwise estimate where we have the Choquet integral. The classical version of this pointwise estimate goes back

to [17]. We use the fractional maximal function by following the idea of [1]. We also refer to [22, Lemma 3.4].

3.6 Lemma. *Let $\kappa \in [0,1)$, $\delta \in (0,n]$, and $p \in (\delta/n, \delta)$. Then there exists a constant c depending only on n, δ, κ, and p such that*

$$\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}}\,dy \le c\Big(M_\kappa f(x)\Big)^{\frac{\delta-p}{\delta-\kappa p}} \Big(\int_{\mathbb{R}^n} |f(y)|^p\,d\mathcal{H}^\delta_\infty\Big)^{\frac{1-\kappa}{\delta-\kappa p}}$$

for all $x \in \mathbb{R}^n$ and all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $A_k = \{y \in \mathbb{R}^n : 2^{-k}r \le |x-y| < 2^{-k+1}r\}$. We estimate

$$\begin{aligned}\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-1}}\,dy &= \sum_{k=1}^\infty \int_{A_k} \frac{|f(y)|}{|x-y|^{n-1}}\,dy \\ &\le \sum_{k=1}^\infty (2^{-k}r)^{1-n} \int_{B(x,2^{-k+1}r)} |f(y)|\,dy \\ &\le \frac{2^{\kappa-1+n}}{1-2^{\kappa-1}} r^{1-\kappa} M_\kappa f(x),\end{aligned}$$

where in the last step the sum of a geometric series is used.

Outside the ball $B(x,r)$ we use Hölder's inequality and Lemma 2.8 to obtain

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus B(x,r)} \frac{|f(y)|}{|x-y|^{n-1}}\,dy \le \Big(\int_{\mathbb{R}^n\setminus B(x,r)} |f(y)|^{\frac{np}{\delta}}\,dy\Big)^{\frac{\delta}{np}} \Big(\int_{\mathbb{R}^n\setminus B(x,r)} |x-y|^{\frac{np(1-n)}{np-\delta}}\,dy\Big)^{\frac{np-\delta}{np}} \\ &\le c(n,\delta,p)\Big(\int_{\mathbb{R}^n\setminus B(x,r)} |f(y)|^p\,d\mathcal{H}^\delta_\infty\Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}^n\setminus B(x,r)} |x-y|^{\frac{np(1-n)}{np-\delta}}\,dy\Big)^{\frac{np-\delta}{np}}.\end{aligned}$$

The last term on the right hand side is

$$\int_{\mathbb{R}^n\setminus B(x,r)} |x-y|^{\frac{np(1-n)}{np-\delta}}\,dy = \frac{\omega_{n-1}}{(n-1)\frac{np(n-1)}{np-\delta} - n} r^{n-\frac{np(n-1)}{np-\delta}},$$

where ω_{n-1} is the $n-1$-dimensional Hausdorff measure of the sphere, [17, Lemma]. Note that $n - \frac{np(n-1)}{np-\delta} < 0$, since $p \in (\delta/n, \delta)$. Thus we have

$$\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}}\,dy \le c\big(r^{1-\kappa} M_\kappa f(x) + \|f\| r^{1-\frac{\delta}{p}}\big),$$

where $\|f\| := \Big(\int_{\mathbb{R}^n\setminus B(x,r)} |f(y)|^p \, d\mathcal{H}^\delta_\infty\Big)^{\frac{1}{p}}$. By choosing

$$r = \Big(\frac{M_\kappa f(x)}{\|f\|}\Big)^{-\frac{p}{\delta-\kappa p}}$$

we obtain

$$\int\limits_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}} \, dy \le c(M_\kappa f(x))^{1-\frac{p(1-\kappa)}{\delta-\kappa p}} \|f\|^{\frac{p(1-\kappa)}{\delta-\kappa p}}$$

for all $x \in \mathbb{R}^n$. This inequality yields the claim. □

The previous lemma gives our main result. Note that if $\kappa > 0$ then the dimension of the Hausdorff content is lower on the left and side than on the right hand side.

3.7 Theorem. *Let Ω be a bounded (α,β)-John domain in $\mathbb{R}^n$. Suppose that $\delta \in (0,n]$, $\kappa \in [0,1)$, and $p \in (\delta/n, \delta)$. Then there exists a constant c depending only on n, δ, κ, p, and John constants α and β such that*

$$\inf_{b\in\mathbb{R}} \Big(\int\limits_\Omega |u(x)-b|^{\frac{p(\delta-\kappa p)}{\delta-p}} \, d\mathcal{H}^{\delta-\kappa p}_\infty\Big)^{\frac{\delta-p}{p(\delta-\kappa p)}} \le c\Big(\int\limits_\Omega |\nabla u(x)|^p \, d\mathcal{H}^\delta_\infty\Big)^{\frac{1}{p}} \tag{3.8}$$

for all $u \in C^1(\Omega)$.

Proof. Suppose that $0 < \delta < n$ and $u \in C^1(\Omega)$. We may assume that the right hand side of inequality (3.8) is finite. As in the proof of Theorem 3.2 we have $|\nabla u| \in L^1(\Omega)$. By [29, Theorem], [7], [23], and [18] for an (α,β)-John domain the pointwise estimate

$$|u(x) - u_B| \le c(n,\alpha,\beta) \int\limits_\Omega \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy$$

holds for every $x \in \Omega$. Here $B = B(x_0, c(n)\alpha^2/\beta)$ and $c(n,\alpha,\beta) = c(n)(\alpha/\beta)^{2n}$ by [18]. Next we apply Lemma 3.6 with the understanding that $|\nabla u|$ is zero outside Ω. We obtain

$$\begin{aligned} |u(x) - u_B|^{\frac{p(\delta-\kappa p)}{\delta-p}} &\le c\Big(\int\limits_\Omega \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy\Big)^{\frac{p(\delta-\kappa p)}{\delta-p}} \\ &\le c\Big(\int\limits_\Omega |\nabla u(y)|^p \, d\mathcal{H}^\delta_\infty\Big)^{\frac{p(1-\kappa)}{\delta-p}} (M_\kappa |\nabla u|(x))^p \end{aligned}$$

for every $x \in \Omega$. Here the constants depend on n, δ, κ, p, α, and β. By integrating with respect to $\mathcal{H}^{\delta-\kappa p}_\infty$ and using the properties (C5) and (C1) of the Choquet integral we obtain

$$\int_\Omega |u(x)-u_B|^{\frac{p(\delta-\kappa p)}{\delta-p}}\,d\mathcal{H}_\infty^{\delta-\kappa p} \le c\Big(\int_\Omega |\nabla u(y)|^p\,d\mathcal{H}_\infty^\delta\Big)^{\frac{p(1-\kappa)}{\delta-p}} \int_\Omega (M_\kappa|\nabla u|)^p d\mathcal{H}_\infty^{\delta-\kappa p}.$$

Adams's result for boundedness of the fractional Hardy-Littlewood maximal operator, Theorem 2.12 implies

$$\begin{aligned}\int_\Omega |u(x)-u_B|^{\frac{p(\delta-\kappa p)}{\delta-p}}\,d\mathcal{H}_\infty^{d} &\le c\Big(\int_\Omega |\nabla u(y)|^p\,d\mathcal{H}_\infty^\delta\Big)^{\frac{p(1-\kappa)}{\delta-p}} \int_\Omega |\nabla u(x)|^p d\mathcal{H}_\infty^\delta\\ &= c\Big(\int_\Omega |\nabla u(x)|^p\,d\mathcal{H}_\infty^\delta\Big)^{\frac{\delta-p\kappa}{\delta-p}}.\end{aligned}$$

Hence the claim follows by raising both sides of the previous inequality to the power $\frac{\delta-p}{p(\delta-\kappa p)}$. □

The $(\delta p/(\delta-p),p)$-Poincaré-Sobolev inequality

$$\inf_{b\in\mathbb{R}}\Big(\int_\Omega |u(x)-b|^{\frac{\delta p}{\delta-p}}\,d\mathcal{H}_\infty^\delta\Big)^{\frac{\delta-p}{\delta p}} \le c\Big(\int_\Omega |\nabla u(x)|^p d\mathcal{H}_\infty^\delta\Big)^{\frac{1}{p}}$$

in Corollary 1.1 follows now from Theorem 3.7 when we choose $\kappa=0$. When $\delta=n$ we recover the classical Sobolev inequality.

Choosing $\delta=n$ and $\kappa=1/p$ in Theorem 3.7 gives the following corollary.

3.9 Corollary. *Let Ω be a bounded (α,β)-John domain in $\mathbb{R}^n$. Suppose that $p\in(1,n)$. Then there exists a constant c depending only on n, p, and John constants α and β such that*

$$\inf_{b\in\mathbb{R}}\Big(\int_\Omega |u(x)-b|^{\frac{p(n-1)}{n-p}}\,d\mathcal{H}_\infty^{n-1}\Big)^{\frac{n-p}{p(n-1)}} \le c\Big(\int_\Omega |\nabla u(x)|^p\,dx\Big)^{\frac{1}{p}} \tag{3.10}$$

for all $u\in C^1(\Omega)$.

3.11 Remark. We point out that the proofs of Theorem 3.2 and Theorem 3.7 give stronger inequalities than (3.3) and (3.8), respectively. If Ω is a bounded (α,β)-John domain in $\mathbb{R}^n$, $\delta\in(0,n]$, $\kappa\in[0,1)$, and $p\in(\delta/n,\delta)$, then there exist constants $c_1=c_1(\alpha,\beta,\delta,n,p)$ and $c_2=c_2(\alpha,\beta,\delta,\kappa,n,p)$ such that the inequalities

$$\int_\Omega |u(x)-u_B|^p\,d\mathcal{H}_\infty^\delta \le c(n,p,\delta)\beta^p\Big(\frac{\beta}{\alpha}\Big)^{2np}\int_\Omega |\nabla u(x)|^p\,d\mathcal{H}_\infty^\delta$$

and

$$\Big(\int_\Omega |u(x)-u_B|^{\frac{p(\delta-\kappa p)}{\delta-p}}\,d\mathcal{H}_\infty^{\delta-\kappa p}\Big)^{\frac{\delta-p}{p(\delta-\kappa p)}} \le c\Big(\int_\Omega |\nabla u(x)|^p\,d\mathcal{H}_\infty^{\delta}\Big)^{\frac{1}{p}}$$

are valid for all $u \in C^1(\Omega)$. We recall that $B = B(x_0, c(n)\alpha^2/\beta)$ and the integral average has been calculated with respect to the Lebesgue measure.

3.12 Remark. Let Ω be a bounded (α,β)-John domain in $\mathbb{R}^n$. Suppose that $p \in (1,n)$. Choosing $|f(x)| = |u(x)-u_B|^{\frac{np}{n-p}}$ in Lemma 2.8 gives that there exists a constant c_1 such that

$$\Big(\int_\Omega |u(x)-u_B|^{\frac{np}{n-p}}\,dx\Big)^{\frac{n-p}{pn}} \le c_1\Big(\int_\Omega |u(x)-u_B|^{\frac{p(n-1)}{n-p}}\,d\mathcal{H}_\infty^{n-1}\Big)^{\frac{n-p}{p(n-1)}}.$$

Corollary 3.9 gives that there exists a constant c_2 such that we have

$$\begin{aligned}&\Big(\int_\Omega |u(x)-u_B|^{\frac{np}{n-p}}\,dx\Big)^{\frac{n-p}{pn}}\\ &\le c_1\Big(\int_\Omega |u(x)-u_B|^{\frac{p(n-1)}{n-p}}\,d\mathcal{H}_\infty^{n-1}\Big)^{\frac{n-p}{p(n-1)}}\\ &\le c_2\Big(\int_\Omega |\nabla u(x)|^p\,dx\Big)^{\frac{1}{p}}.\end{aligned}$$

This shows some of the benefits which come from using Choquet integrals in Poincaré-Sobolev inequalities.

3.13 Remark. Note that by Lemma 3.6 and Theorem 2.12 the Riesz potential $I_1 f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}}\,dy$ is bounded with respect to Hausdorff content. If $0 < \delta \le n$, $\kappa \in [0,1)$, and $p \in (\delta/n, \delta)$, then

$$\Big(\int_{\mathbb{R}^n} (I_1(f(x)))^{\frac{p(\delta-\kappa p)}{\delta-p}}\,d\mathcal{H}_\infty^{\delta-\kappa p}\Big)^{\frac{\delta-p}{p(\delta-\kappa p)}} \le c(n,\delta,\kappa)\Big(\int_{\mathbb{R}^n} |f(x)|^p\,d\mathcal{H}_\infty^{\delta}\Big)^{\frac{1}{p}}$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Next we show that the exponent $\frac{p(\delta-\kappa p)}{\delta-p}$ in Theorem 3.7 is the best possible exponent in this setting. This example is based on the example, [5, Example 4.41, p. 109].

3.14 Example. Let $\Omega := B^n(0,1) \setminus \{0\}$, $0 < \delta \le n$, $\kappa \in [0,1)$, and $p \in (\delta/n, \delta)$. Let us define $v(x) := |x|^\mu$, where $\mu < 0$ is chosen later. Then $v \in C^\infty(\Omega)$. We show that, if $q > \frac{p(\delta-\kappa p)}{\delta-p}$, then there exists μ such that $\int_\Omega |v(x)-a|^q d\mathcal{H}_\infty^{\delta-\kappa p} = \infty$ for any $a \in \mathbb{R}$ and at the same time $\int_\Omega |\nabla v(x)|^p d\mathcal{H}_\infty^{\delta} < \infty$.

Let $a \in \mathbb{R}$. For the function v itself we use Lemma 2.8 to obtain

$$\begin{aligned} c(n,\delta,\kappa,p)\Big(\int_\Omega |v(x)-a|^q\,d\mathcal{H}_\infty^{\delta-\kappa p}\Big)^{\frac{n}{\delta-\kappa p}} &\ge \int_\Omega |v(x)-a|^{\frac{qn}{\delta-\kappa p}}\,dx \\ &\ge \int_{B(0,r)} |\tfrac{1}{2}v(x)|^{\frac{qn}{\delta-\kappa p}}\,dx \\ &= c(n,\kappa,p,\delta,q)\int_0^r \rho^{\frac{\mu q n}{\delta-\kappa p}+n-1}\,d\rho, \end{aligned}$$

for some $r > 0$. The last integral is infinite whenever $\frac{\mu q n}{\delta-\kappa p} + n - 1 \le -1$, that is if $\mu \le -\frac{\delta-\kappa p}{q}$.

For the gradient we obtain $|\nabla v(x)| = |\mu||x|^{\mu-1}$. Thus, $|\nabla v(x)|^p > t$ provided that $|x| < ct^{\frac{1}{p(\mu-1)}}$. By using the inequality $\mathcal{H}_\infty^\delta(B(0,r)) \le r^\delta$, we obtain

$$\begin{aligned} \int_\Omega |\nabla v(x)|^p\,d\mathcal{H}_\infty^\delta &= \int_0^\infty \mathcal{H}_\infty^\delta(\{|\nabla v(x)|^p > t\})\,dt \\ &\le \mathcal{H}_\infty^\delta(B(0,1)) + \int_1^\infty \mathcal{H}_\infty^\delta(B(0,ct^{\frac{1}{p(\mu-1)}})\,dt \\ &\le \mathcal{H}_\infty^\delta(B(0,1)) + c\int_1^\infty t^{\frac{\delta}{p(\mu-1)}}\,dt. \end{aligned}$$

The last integral is finite provided that $\frac{\delta}{p(\mu-1)} < -1$ i.e. if $\mu > 1-\frac{\delta}{p}$. Since $q > \frac{p(\delta-\kappa p)}{\delta-p}$, we have $1-\frac{\delta}{p} < -\frac{\delta-\kappa p}{q}$. Thus we may choose the parameter μ such that $1-\frac{\delta}{p} < \mu \le -\frac{\delta-\kappa p}{q}$.

3.15 Remark. If Ω is an unbounded domain such that $\Omega = \cup_{i=0}^\infty \Omega_i$ where $\Omega_i \subset \Omega_{i+1}$ and Ω_i is an (α_i,β_i)-John domain for some $0 < \alpha_i \le \beta_i < \infty$, $i = 0,1,\dots$. If $\beta_i/\alpha_i \le c$ for all i, then the $(np/n-p),p)$-Poincaré-Sobolev inequality holds for all functions $u \in L_p^1(\Omega)$, [19, Theorem 4.1]. This result corresponds to the case $\delta = n$.

4. Inequalities for C_0^1-functions

The Poincaré inequality and Poincaré-Sobolev inequality for C_0^1-functions follow in a similar fashion as for C^1-functions, respectively. The main difference is to use for functions $u \in C_0^1(\Omega)$ the estimate

$$|u(x)| \le c(n)\int_\Omega \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy \quad \text{for all } x \in \mathbb{R}^n, \tag{4.1}$$

[24, 1.1.10 Theorem 2], instead of the corresponding inequality (3.4) for $C^1(\Omega)$-functions defined on a John domain. Inequality (4.1) yields the following theorem, where the part (a) holds only in a bounded domain while the part (b) can also be applied for unbounded domains. In fact, if the domain is bounded in the part (b), then Hölder's inequality implies the part (a) too.

4.2 Theorem. *Let $\delta \in (0, n]$.*

(a) *If Ω is a bounded domain in $\mathbb{R}^n$ and $p \in (\delta/n, \infty)$, then there exists a constant c depending only on n, δ, and p such that*

$$\int_\Omega |u(x)|^p \, d\mathcal{H}^\delta_\infty \le c \operatorname{diam}(\Omega)^p \int_\Omega |\nabla u(x)|^p \, d\mathcal{H}^\delta_\infty$$

for all $u \in C^1_0(\Omega)$.

(b) *If Ω is a domain in $\mathbb{R}^n$, $\kappa \in [0, 1)$, and $p \in (\delta/n, \delta)$, then there exists a constant c depending only on n, δ, κ, and p such that*

$$\Big(\int_\Omega |u(x)|^{\frac{p(\delta-\kappa p)}{\delta-p}} \, d\mathcal{H}^{\delta-\kappa p}_\infty \Big)^{\frac{\delta-p}{p(\delta-\kappa p)}} \le c \Big(\int_\Omega |\nabla u(x)|^p \, d\mathcal{H}^\delta_\infty \Big)^{\frac{1}{p}}$$

for all $u \in C^1_0(\Omega)$.

4.3 Remark. Let $\kappa = 0$ and $\delta = n - 1$. Both limit cases $p = \frac{\delta}{n}$ and $p = \delta$ are excluded from Theorem 4.2(b).

- However, by combining the inequality of Adams (1.3) and Lemma 2.8 we obtain the inequality

$$\int_{\mathbb{R}^n} |u(x)| \, d\mathcal{H}^{n-1}_\infty \le c(n, \delta) \Big(\int_{\mathbb{R}^n} |\nabla u(x)|^{\frac{\delta}{n}} \, d\mathcal{H}^\delta_\infty \Big)^{\frac{n}{\delta}}$$

  for every $\delta \in (0, n]$ whenever $u \in \mathcal{C}^\infty_0(\mathbb{R}^n)$. Note that if $p = \frac{\delta}{n}$ and $\delta = n - 1$, then $\frac{\delta p}{\delta - p} = 1$. Hence, the above inequality can be seen as a limit case if $p = \frac{\delta}{n}$ with $\delta = n - 1$ for Theorem 4.2(b) where $\kappa = 0$.
- Corresponding to the upper limiting case $p = \delta = n - 1$, the authors of the present paper showed in [16, Corollary 1.3]: If Ω is a bounded (α, β)-John domain in $\mathbb{R}^n$, then there exist positive constants a and b such that

$$\int_\Omega \exp\left(a |u(x) - u_B|^{\frac{n}{n-1}} \right) d\mathcal{H}^{n-1}_\infty \le b$$

for all $u \in L^1_n(\Omega) \cap C^1(\Omega)$ with $||\nabla u||_{L^n(\Omega)} \le 1$. Here $B = B(x_0, c(n)\alpha^2/\beta)$. Moreover, $||\nabla u||_{L^n(\Omega)} \le c(n)\big(\int_\Omega |\nabla u|^{n-1}\, d\mathcal{H}^{n-1}_\infty\big)^{1/(n-1)}$ by Lemma 2.8. We also refer to [22, Corollary 1.4].

Data availability

No data was used for the research described in the article.

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