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Junnila, Heikki

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ON METRIC SPACES WITH NON-EQUIVALENT HEWITT AND SAMUEL REALCOMPACTIFICATIONS

HEIKKI JUNNILA, ANA S. MEROÑO

Abstract. Let D be a uniformly discrete space, let π be the product uniformity on the countable power $D^{\mathbb{N}}$ of D and let $e\pi$ be the uniformity on $D^{\mathbb{N}}$ induced by all the countable covers in π. Assume that the cardinality of D is Ulam measurable. Then $(D^{\mathbb{N}}, e\pi)$ has a Cauchy filterbase, consisting of closed sets, which is not countably centered. As a consequence, the Hewitt and Samuel realcompactifications of $(D^{\mathbb{N}}, \pi)$ are not equivalent.

1. Introduction

In the theory of uniform spaces, the well-known *Hewitt realcompactification* υX of a Tychonoff X space is defined as the completion of the uniform space $(X, wC(X))$ where $wC(X)$ is the weak uniformity [19] induced by the family $C(X)$ of all real-valued continuous functions on X. The space X is *realcompact* if $X = \upsilon X$, that is, if $(X, wC(X))$ is complete (see [10]). On the other hand, if we let $U_\mu(X)$ denote the set of all real-valued uniformly continuous functions on a uniform space (X, μ), then the *Samuel realcompactification* $H(U_\mu(X))$ of (X, μ) is defined as the completion of $(X, wU_\mu(X))$, where $wU_\mu(X)$ is the weak uniformity induced by $U_\mu(X)$ [9]. We say that the uniform space (X, μ) is *Samuel realcompact*, and we write $X = H(U_\mu(X))$, whenever $(X, wU_\mu(X))$ is complete.

These two realcompactifications are not equivalent in general. Given two realcompactifications $\alpha_1 X$ and $\alpha_2 X$ of a space X, we write $\alpha_1 X \leq \alpha_2 X$ if there exists a continuous map $\varphi : \alpha_2 X \to \alpha_1 X$ which keeps X pointwise fixed. The relation $\leq$ is a partial order in the family of all the realcompactifications of a Tychonoff space. Moreover, $\alpha_1 X$ and $\alpha_2 X$ are *equivalent*, written $\alpha_1 X = \alpha_2 X$, provided that $\alpha_1 X \leq \alpha_2 X$ and $\alpha_2 X \leq \alpha_2 X$. We have $\alpha_1 X = \alpha_2 X$ if, and only if, there exists an homeomorphism $\varphi : \alpha_1 X \to \alpha_2 X$ which keeps every point of X fixed. The following is a simple example of non-equivalence of υX and $H(U_\mu(X))$.

In this paper, we consider every metric space (X, d) also as a uniform space (X, μ_d), where μ_d is the uniformity on X induced by d.

Example 1. Consider the space of the rationals $(\mathbb{Q}, d)$, where d is the usual Euclidean metric. The space $\mathbb{Q}$ is realcompact because it is separable. However, $(\mathbb{Q}, \mu_d)$ is not Samuel realcompact. Indeed, it is not difficult to see that the weak uniformity $wU_{\mu_d}(\mathbb{Q})$ coincides with the metric uniformity μ_d on $\mathbb{Q}$. Therefore, $H(U_{\mu_d}(\mathbb{Q})) = \mathbb{R} \neq \mathbb{Q} = \upsilon\mathbb{Q}$.

The preceding example depends on non-completeness of the space $(\mathbb{Q}, \mu_d)$. However, it turns out that even for a complete metric space (X, d), we can have $H(U_{\mu_d}(X)) \neq \upsilon X$. To see this, we need to recall a characterization of Samuel realcompactness. In [9, Theorem 12], it is shown that a uniform space (X, μ) is Samuel realcompact if, and only if, (X, μ) is *Bourbaki-complete* and (X, μ) does not have a uniformly discrete subspace of measurable cardinality. Bourbaki-completeness is a uniform property, introduced in [7], which is stronger than usual

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completeness. Besides the above characterization of Samuel realcompactness, the only result on Bourbaki-completeness that we need in this paper is that every uniformly zero-dimensional complete space is Bourbaki-complete.

Example 2. Let $(J(\omega),\rho)$ be the hedgehog with ω-many spines, equipped with its usual metric ρ (see [6, Example 4.1.5]). The metric space $(J(\omega),\rho)$ is complete, but in [9, Example 20] it is shown that the space $(J(\omega),\rho)$ is not Bourbaki-complete and hence not Samuel realcompact. Therefore $H(U_{\mu_\rho}(J(\omega)))\neq J(\omega)$. On the other hand, the separable space $J(\omega)$ is realcompact and so $\upsilon J(\omega)=J(\omega)$. As a consequence, $H(U_{\mu_\rho}(J(\omega)))\neq \upsilon J(\omega)$.

The above example depends on non-Bourbaki-completeness of the metric space $(J(\omega),\rho)$. This paper arose from attempts to see whether the failure of Bourbaki-completeness is essential for this type of examples. It turns out that it is at least consistent with ZFC that the Hewitt and Samuel realcompactifications agree for every Bourbaki-complete uniform space.

We denote by **MC** the assumption that there exists a measurable cardinal. If measurable cardinals exist, then the least measurable cardinal is strongly inaccessible. It follows that it is consistent with ZFC that there are no measurable cardinals, in other words, the assumption $\neg$**MC** is consistent with ZFC.

We refer the reader to [15] for basic properties of measurable cardinals. In this paper, we also use the weaker property of Ulam measurability. Recall than an *Ulam measure* on a set S is a non-trivial $\{0,1\}$-valued σ-additive measure defined for all subsets of S. A cardinal κ is *Ulam measurable* if there exists an Ulam measure on the set κ. Measurability and Ulam measurability are related in the following way: a cardinal κ is Ulam measurable if, and only if, κ is bigger than or equal to some measurable cardinal.

Proposition 1. *Let (X,μ) be a Bourbaki-complete uniform space such that the cardinal $|X|$ is not Ulam measurable. Then $H(U_\mu(X))=\upsilon X$.*

Proof. It follows from a result mentioned above that (X,μ) is Samuel realcompact. On the other hand, (X,μ) is complete, and it follows from the Katetov-Shirota Theorem that X is realcompact. As a consequence, $H(U_\mu(X))=X=\upsilon X$. □

Corollary 2. ($\neg$**MC**) *The equivalence $H(U_\mu(X))=\upsilon X$ holds for every Bourbaki-complete uniform space (X,μ).*

The above result and and the following example from [16] indicate that the equivalence of the Hewitt and Samuel realcompactifications of Bourbaki-complete uniform spaces is tightly connected with measurable cardinals.

Example 3. [16, Example 3.2.13] Let D be a uniformly discrete space of Ulam measurable cardinality and let π be the product uniformity on $D\times\beta D$. The uniform space $(D\times\beta D,\pi)$ is Bourbaki-complete but $H(U_\pi(D\times\beta D))\neq\upsilon(D\times\beta D)$.

The space $D\times\beta D$ above is non-metrizable. The main result of this paper (Theorem 7) shows that if measurable cardinals exist, then there also exist Bourbaki-complete metric spaces whose Hewitt and Samuel realcompactifications are non-equivalent. According to the corollary of the main theorem, if D is a discrete space of Ulam measurable cardinality and ρ is the first difference metric on $D^{\mathbb{N}}$, then the (Bourbaki-complete) metric space $(D^{\mathbb{N}},\rho)$ satisfies the non-equivalence $\upsilon D^{\mathbb{N}}\neq H(U_{\mu_\rho}(D^{\mathbb{N}}))$.

The results and examples above are related to the *Katetŏv-Shirota type Theorems.* The classical Katetŏv-Shirota Theorem states that a Tychonoff space X is realcompact if and only if X is topologically complete and X has no closed discrete subset of measurable cardinality (see [10] or [12]). For uniform spaces and the different realcompactifications that can be defined for them, there are also results similar to the Katetǒv-Shirota Theorem. For such "Katetǒv-Shirota type Theorems", see [5], [17], [8],[13] and [9].

For basic results and concepts, we refer the reader to [6] and [19] on topological and uniform spaces and to [10] and [4] on realcompactifications.

2. Preliminaries

We start by reviewing some known facts about z_u-sets and z_u-filters.

Definition 1. A subset Z of a uniform space (X, μ) is a *z_u-set* if there exists some (bounded) real-valued uniformly continuous function $f \in U_\mu(X)$ such that $f^{-1}(\{0\}) = Z$. We denote by $\mathcal{Z}_u(X)$ the family of all z_u-sets of (X, μ).

Note that the family $\mathcal{Z}_u(X)$ is closed under countable intersections.

For a uniform space (X, μ), we denote by $e\mu$ the uniformity on X induced by all countable covers from μ. Note that (X, μ) and $(X, e\mu)$ have the same z_u-sets. (see [2, Lemma 2.4]).

Clearly, every z_u-set of a uniform space (X, μ) is a zero-set of the topological space (X, τ_μ). On the other hand, every zero-set of a Tychonoff space X is a z_u-set when X is endowed with the fine uniformity. In metrizable uniform spaces the closed sets, the zero-sets and the z_u-sets all coincide.

Definition 2. A filter(base) $\mathcal{F}$ of a uniform space (X, μ) is a *z_u-filter(base)* if every member of $\mathcal{F}$ contains some set of the family $\mathcal{F} \cap \mathcal{Z}_u(X)$. A z_u-filter $\mathcal{F}$ is a *z_u-ultrafilter* if $\mathcal{F} \cap \mathcal{Z}_u(X)$ is maximal in $\mathcal{Z}_u(X)$, that is, $Z \in \mathcal{Z}_u(X)$ belongs to $\mathcal{F}$ if $Z \cap Z' \neq \emptyset$ for every $Z' \in \mathcal{F} \cap \mathcal{Z}_u(X)$.

As is well known, every z_u-filterbase is contained in a z_u-ultrafilter (see [10, 2.5]).

Note that every (ultra)filter of a set S can be considered as a z_u-(ultra)filter when we give S the discrete topology and the fine uniformity.

Definition 3. A filter(base) $\mathcal{F}$ of a uniform space (X, μ) is a *Cauchy filter(base)* if for every uniform cover $\mathcal{U} \in \mu$, some member of $\mathcal{U}$ contains a member of $\mathcal{F}$.

Recall that a uniform space is *complete* if every Cauchy z_u-filter converges and that Cauchy z_u-ultrafilters are used in the completion of a uniform space. Indeed, the points in the completion $(\widetilde{X}, \widetilde{\mu})$ of a uniform space (X, μ) are exactly the Cauchy z_u-ultrafilters of (X, μ). The uniformity $\widetilde{\mu}$ has a base consisting of all covers $\widetilde{\mathcal{U}} = \{\widetilde{U} : U \in \mathcal{U}\}$, where $\mathcal{U} \in \mu$ and $\widetilde{U} = \{\mathcal{F} : \mathcal{F}$ is a Cauchy z_u-ultrafilter and $F \subset U$ for some $F \in \mathcal{F}\}$ for each $U \in \mathcal{U}$ (see [1, Chapter II, Section 3.7] and [10, Sections 15.7, 15.8, 15.9]).

It follows from the foregoing that the points of the Samuel realcompactification $H(U_\mu(X))$ of (X, μ) are the Cauchy z_u-ultrafilters of the uniform space $(X, wU_\mu(X))$.

We still need names for two properties of filters.

Definition 4. A filterbase $\mathcal{F}$ is *countably centered* (*countably complete*) if $\bigcap \mathcal{E} \neq \emptyset$ ($\bigcap \mathcal{E} \in \mathcal{F}$) for every countable $\mathcal{E} \subset \mathcal{F}$.

Note that every countably centered (z_u-)ultrafilter is countably complete.

There is a close connection between countably complete ultrafilters and Ulam measures. If m is an Ulam measure on a set S, then the family $\{E \subset S : m(E) = 1\}$ is a countably complete free ultrafilter on S. If $\mathcal{F}$ is a countably complete free ultrafilter on S, then we obtain an Ulam measure m on S by setting $m(E) = 1$ for $E \in \mathcal{F}$ and $m(E) = 0$ for $E \notin \mathcal{F}$.

In our study of the equivalence of υX and $H(U_\mu(X))$, it is useful to consider yet another realcompactification $\upsilon_u X$, the *Wallman realcompactification* of (X,μ) (see [18] or [2] for the definitions). The following lemma, contained in [2, Definition 2.10, Lemma 2.11 and Proposition 2.12], tells everything we need to know here about the realcompactification $\upsilon_u X$. Additional basic facts on various realcompactifications of uniform spaces can be found in [9], [13] and [2].

Lemma 3. *The realcompactification $\upsilon_u X$ of a uniform space (X,μ) is the subspace of the Samuel realcompactification $H(U_\mu(X))$ consisting of all Cauchy z_u-ultrafilters of $(X, wU_\mu(X))$ which are countably centered.*

For a uniform space (X,μ), we have $\upsilon_u X \leq \upsilon X$. In general, these two realcompactifications are not equivalent.

Example 4. Let D be an uncountable discrete space. We assume that the cardinality of D is not Ulam measurable. Then D is realcompact, that is, $\upsilon D = D$ (see [10, 12.1-12.6]). Let $D \cup \{\infty\}$ be the one-point compactification of D and let ν be the uniformity on D inherited from this compactification. We are going to show that $\upsilon_u D = D \cup \{\infty\}$. From this it follows that $\upsilon D \neq \upsilon_u D$ for the uniform space (D,ν).

Note that complements of finite subsets of D are z_u-sets of (D,ν). Let $\mathcal{F}$ be the Fréchet filter $\{D \setminus A : A \text{ is a finite subset of } D\}$ of D. Then $\mathcal{F}$ is a z_u-filter of (D,ν) which converges to the point ∞ in $D \cup \{\infty\}$. As a consequence, $\mathcal{F}$ is a Cauchy z_u-filter of (D,ν). Since D is uncountable, $\mathcal{F}$ is countably centered. Let $\mathcal{U}$ be a z_u-ultrafilter of (D,ν) containing $\mathcal{F}$. By [13, Corollary 1.3], $\mathcal{U}$ is countably centered. Moreover, $\mathcal{U}$ converges to ∞. Since ν is a precompact uniformity, $wU_\nu(D) = \nu$. By the foregoing, we can write $\upsilon_u D = D \cup \{\infty\}$.

Our next result characterizes the equivalence of the realcompactifications $\upsilon_u X$ and $H(U_\mu(X))$ of a uniform space (X,μ).

Theorem 4. *The following conditions are equivalent for a uniform space (X,μ):*

(1) $\upsilon_u X = H(U_\mu(X))$;

(2) *Every Cauchy z_u-ultrafilter of $(X, wU_\mu(X))$ is countably centered;*

(3) *Every Cauchy z_u- filterbase of $(X, wU_\mu(X))$ is countably centered.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\upsilon_u X = H(U_\mu(X))$. Then there exists a homeomorphism $\varphi : \upsilon_u X \to H(U_\mu(X))$ which keeps the points of X fixed. By Lemma 3, we have $\upsilon_u X \subset H(U_\mu(X))$. Let $i : \upsilon_u X \to H(U_\mu(X))$ be the inclusion map. Since the continuous mappings φ and i agree on the dense subset X of $\upsilon_u X$, we have $i = \varphi$. It follows, since φ is an onto mapping, that $\upsilon_u X = H(U_\mu(X))$. By Lemma 3, every Cauchy z_u-ultrafilter of $(X, wU_\mu(X))$ is countably centered.

(2) $\Rightarrow$ (3) Assume that (2) holds, and let $\mathcal{F}$ be a Cauchy z_u-filterbase of $(X, wU_\mu(X))$. By the Kuratowski-Zorn Lemma, $\mathcal{F}$ can be extended to a z_u-ultrafilter $\mathcal{G}$. Note that $\mathcal{G}$ is a Cauchy filter. By (2), $\mathcal{G}$ is countably centered. Hence also $\mathcal{F}$ has this property.

(3) $\Rightarrow$ (1) This follows from Lemma 3. □

For a metric space (X,d), the Hewitt realcompactification υX can be considered as a topological subspace of the Samuel realcompactification (see, for instance, [8, Proposition 40]). This follows from Lemma 3 because υX is equivalent with the realcompactification $\upsilon_u X$. It is shown in [11, Definition 5.1 and Theorem 5.3] and [2, Definition 2.10 and Theorem 2.12] that both $\upsilon_u X$ and υX are Wallman realcompactifications : $\upsilon_u X$ for the Wallman base $\mathcal{Z}_u(X)$

of all z_u-sets of (X, μ_d) and υX for the Wallman base $\mathcal{Z}(X)$ of all zero-sets of X. Since $\mathcal{Z}(X) = \mathcal{Z}_u(X)$ for a metric space (X, d), the equivalence of $\upsilon_u X$ and υX is immediate.

We will use the following consequence of Theorem 4 and the preceding considerations in the proof of our main result, Theorem 7.

Corollary 5. *Let (X, d) be a metric space such that $\upsilon X = H(U_{\mu_d}(X))$. Then every Cauchy z_u-filterbase of $(X, e\mu_d)$ is countably centered.*

Proof. Assume on the contrary that $(X, e\mu_d)$ has a Cauchy z_u-filterbase $\mathcal{N}$ which is not countably centered. Since the uniformity $e\mu_d$ is finer that the uniformity $wU_{\mu_d}(X)$, the filterbase $\mathcal{N}$ is also Cauchy in $(X, wU_{\mu_d}(X))$. It follows from Theorem 4 that $\upsilon_u X \neq H(U_\mu(X))$. This, however, leads to a contradiction, because we have $\upsilon_u X = \upsilon X$ for the metric space (X, d). □

The following example shows that the necessary condition given in Corollary 5 for the equivalence $\upsilon X = H(U_{\mu_d}(X))$ is not always sufficient, even for a complete metric space (X, d).

Example 5. In Example 2, we saw that the separable hedgehog-space $(J(\omega), \rho)$ satisfies the non-equivalence $H(U_{\mu_\rho}(J(\omega))) \neq \upsilon J(\omega)$. The space $J(\omega)$ is Lindelöf and it follows that the uniformity μ_ρ has a base consisting of countable covers. As a consequence, we have $\mu_\rho = e\mu_\rho$ and thus the space $(J(\omega), e\mu_\rho)$ is complete. It follows that every Cauchy z_u-filter of $(J(\omega), e\mu_\rho)$ is fixed and therefore countably centered.

Let D be a discrete space. We denote by $\mathtt{u}$ the metric uniformity induced by the 0-1-metric δ on D, and we note that $\mathtt{u}$ is the fine uniformity on D. We say that $(D, \mathtt{u})$ is a *uniformly discrete space*. Note that the uniformity $\mathtt{u}$ is induced by the collection of all partitions of D. A base for the countable modification $e\mathtt{u}$ of $\mathtt{u}$ is given by the collection of all countable partitions of D. Note that every countable cover of D is in $e\mathtt{u}$.

The content of the following lemma is a part of the folklore in the theory of uniform realcompactifications.

Lemma 6. *Let $(D, \mathtt{u})$ be a uniformly discrete space. The following conditions are equivalent for a filter $\mathcal{F}$ of D.*

(1) *$\mathcal{F}$ is a Cauchy filter of $(D, wU_{\mathtt{u}}(D))$*

(2) *$\mathcal{F}$ is a Cauchy filter of $(D, e\mathtt{u})$;*

(3) *$\mathcal{F}$ is a countably complete ultrafilter;*

Moreover, $\upsilon D = \upsilon_u D = H(U_{\mathtt{u}}(D))$. The uniform space $(D, \mathtt{u})$ is Samuel realcompact if, and only if, it is realcompact, that is, exactly when the cardinal $|D|$ is not Ulam measurable.

Proof. (1) ⇔ (2) This equivalence follows from the general fact that, for a uniformly 0-dimensional space (X, μ), the uniformities $wU_\mu(X)$ and $e\mu$ are the same as $e\mu$ is induced by all the countable uniform partitions of (X, μ) (see [14, 1.7]).

(2) ⇒ (3) Assume that $\mathcal{F}$ is a Cauchy filter of $(D, e\mathtt{u})$. For every $E \subset D$, the partition $\{E, D \backslash E\}$ of D belongs to the uniformity $e\mu$ and it follows that either $E \in \mathcal{F}$ or $D \backslash E \in \mathcal{F}$. Hence, $\mathcal{F}$ is an ultrafilter. To show that $\mathcal{F}$ is countably complete, let $\{F_n : n \in \mathbb{N}\} \subset \mathcal{F}$. The countable cover $\mathcal{U} = \{D \backslash F_n : n \in \mathbb{N}\} \cup \{\bigcap_{n \in \mathbb{N}} F_n\}$ of D is a member of the uniformity $e\mathtt{u}$. Since $\mathcal{F}$ is a Cauchy filter and $D \backslash F_k \notin \mathcal{F}$ for every $k \in \mathbb{N}$, we must have $\bigcap_{n \in \mathbb{N}} F_n \in \mathcal{F}$.

(3) ⇒ (2). Assume that (3) holds. To verify (2), let $\{U_n : n \in \mathbb{N}\} \in e\mathtt{u}$. Then $\{U_n : n \in \mathbb{N}\}$ covers D and thus $\bigcap_{n \in \mathbb{N}} (D \setminus U_n) = \emptyset \notin \mathcal{F}$. It follows, since $\mathcal{F}$ is countably complete, that there exists $k \in \mathbb{N}$ such that $D \setminus U_k \notin \mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter, we have $U_k \in \mathcal{F}$.

Since D is metrizable, we have $\upsilon D = \upsilon_u D$. By the foregoing and Lemma 3, it is clear that $\upsilon_u D = H(U_{\mathtt{u}}(D))$. Finally, a well-known result (see [10, Theorem 12.2]) shows that $\upsilon D = D$ if, and only if, the cardinal of D is not Ulam measurable. □

3. The space $\mathfrak{m}^{\mathbb{N}}$

Corollary 2 shows that we need measurable cardinals if we want to find Bourbaki-complete metric spaces with non-equivalent Hewitt and Samuel realcompactifications, and Lemma 6 shows that we must go beyond uniformly discrete spaces. In this section, we show that if measurable cardinals exist, then it is enough to consider countable powers of uniformly discrete spaces.

Let D be an infinite discrete space. We equip the countable power $D^{\mathbb{N}}$ with the product uniformity π where each factor D is endowed with the uniformity induced by the 0-1 metric. The uniformity π can be determined by the usual product metric d or by the "first difference metric" ρ (see [6, Example 4.2.12]). Note that ρ is an ultrametric and hence the uniform space $(D^{\mathbb{N}}, \pi)$ is uniformly zero-dimensional. It follows, since the metric ρ is complete, that $(D^{\mathbb{N}}, \pi)$ is Bourbaki-complete. The space $(D^{\mathbb{N}}, \rho)$ is known as a *Baire metric space of weigth* $|D|$.

Since a cardinal κ is a set of cardinality κ, we can replace the uniformly discrete space D with the uniformly discrete space λ, where $\lambda = |D|$; this will be useful in the following proof. Hence we will consider cardinals also as uniformly discrete spaces. For every $n \in \mathbb{N}$, we denote by p_n the projection from $\lambda^{\mathbb{N}}$ onto the uniformly discrete space $(\lambda^n, \mathtt{u})$, i.e., the mapping $\langle \alpha_1, \alpha_2, \ldots \rangle \mapsto \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$. The projection p_n is uniformly continuous and sets of the form $p_n^{-1}(A)$, where $A \subset \lambda^n$, are clopen and hence z_u-sets in $\lambda^{\mathbb{N}}$. Note that the uniformity $e\pi$ of $\lambda^{\mathbb{N}}$ is induced by all partitions of $\lambda^{\mathbb{N}}$ of the form $\{p_n^{-1}(A) : A \in \mathcal{A}\}$, where $n \in \mathbb{N}$ and $\mathcal{A}$ is a countable partition of the uniformly discrete space λ^n.

In the following proof, we employ the product $\mathcal{F} \cdot \mathcal{G}$ (sometimes called *tensor product* or *Fubini product*) of two filters $\mathcal{F}$ and $\mathcal{G}$: if $\mathcal{F}$ is a filter of a set X and $\mathcal{G}$ a filter of a set Y, then $\mathcal{F} \cdot \mathcal{G}$ is the family $\Big\{H \subset X \times Y : \big\{x \in X : \{y \in Y : (x,y) \in H\} \in \mathcal{G}\big\} \in \mathcal{F}\Big\}$ of subsets of $X \times Y$. The family $\mathcal{F} \cdot \mathcal{G}$ is a filter, and if both $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters, then $\mathcal{F} \cdot \mathcal{G}$ is an ultrafilter. We refer the reader to [3, p. 157-159] for basic facts about product filters.

Note that the filter $\mathcal{F} \cdot \mathcal{G}$ has a base consisting of all sets of the form $\bigcup \big\{\{a\} \times B_a : a \in A\big\}$, where $A \in \mathcal{F}$ and $B_a \in \mathcal{G}$ for every $a \in A$. Using this base, we can easily see that if $\mathcal{F}$ and $\mathcal{G}$ are countably complete filters, then so is $\mathcal{F} \cdot \mathcal{G}$: if $A^1, A^2, \ldots \in \mathcal{F}$ and $B_a^n \in \mathcal{G}$ for all $n \in \mathbb{N}$ and $a \in A^n$, then $\bigcap_{n \in \mathbb{N}} \bigcup \big\{\{a\} \times B_a^n : a \in A^n\big\} = \bigcup \big\{\{a\} \times \bigcap_{n \in \mathbb{N}} B_a^n : a \in \bigcap_{n \in \mathbb{N}} A^n\big\} \in \mathcal{F} \cdot \mathcal{G}$.

Theorem 7. (**MC**) *Let $\mathfrak{m}$ be the least measurable cardinal. There exists a closed discrete subspace X of $\mathfrak{m}^{\mathbb{N}}$ such that $\upsilon X \neq H(U_\pi(X))$.*

Proof. There exists a countably complete free ultrafilter $\mathcal{F}$ on the measurable cardinal $\mathfrak{m}$. Since $\mathfrak{m}$ is the least measurable cardinal, the filter $\mathcal{F}$ is uniform, i.e., $|F| = \mathfrak{m}$ for each $F \in \mathcal{F}$.

We identify each 1-element sequence $\langle \alpha \rangle$ with the element α. Thus we have $\mathfrak{m}^1 = \mathfrak{m}$ and $\mathcal{F}$ is an ultrafilter on $\mathfrak{m}^1$. We set $\mathcal{F}_1 = \mathcal{F}$ and we define families $\mathcal{F}_2$, $\mathcal{F}_3$, ... recursively by the formula $\mathcal{F}_{n+1} = \mathcal{F}_n \cdot \mathcal{F}_1$. Since $\mathcal{F}_1$ is a countably complete ultrafilter on $\mathfrak{m}^1$, induction shows that each $\mathcal{F}_n$ is a countably complete ultrafilter on $\mathfrak{m}^n$.

For the rest of the proof, we are going to work on the subspace

$$X = \{\langle \alpha_1, \alpha_2, \ldots \rangle \in \mathfrak{m}^{\mathbb{N}} : \text{ for every } n \in \mathbb{N},\ \alpha_n > \alpha_{n+1} \text{ or } \alpha_{n+1} = 0\}.$$

of $\mathfrak{m}^{\mathbb{N}}$. Note that the subspace X is closed and discrete. We will denote the uniformity inherited by X from $(\mathfrak{m}^{\mathbb{N}}, \pi)$ with the same symbol π.

For every $n \in \mathbb{N}$, we denote by $\mathtt{p}_n$ the projection $X \to \mathfrak{m}^n$, in other words, the mapping $\langle \alpha_1, \alpha_2, \ldots, \alpha_k, \ldots \rangle \mapsto \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$, where $\langle \alpha_1, \alpha_2, \ldots, \alpha_k, \ldots \rangle \in X$.

For every $n \in \mathbb{N}$, let $X_n = \mathtt{p}_n(X)$. For all $n \in \mathbb{N}$ and $\gamma < \mathfrak{m}$, let $X_{n,\gamma} = \{\langle \alpha_1, \ldots, \alpha_n \rangle \in X_n : \alpha_n > \gamma\}$. For each $n \in \mathbb{N}$, let $F_n = \mathtt{p}_n^{-1}(X_{n,0})$. Note that the sets F_n are non-empty and that $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$ as there is no infinite strictly decreasing sequence of ordinals.

By induction on n, we show that $X_{n,\beta} \in \mathcal{F}_n$ for each $\beta < \mathfrak{m}$. For $n = 1$, this holds because $\mathcal{F}_1$ is a uniform ultrafilter on $\mathfrak{m}$ and $X_{1,\beta} = \{\alpha < \mathfrak{m} : \alpha > \beta\}$ for each $\beta < \mathfrak{m}$. Assume that the claim has been established for $n = k$. To prove it for $n = k + 1$, let $\beta < \mathfrak{m}$. Set $A = \big\{\alpha < \mathfrak{m} : \{\langle \alpha_1, \ldots, \alpha_k \rangle \in \mathfrak{m}^k : \langle \alpha_1, \ldots, \alpha_k, \alpha \rangle \in X_{k+1,\beta}\} \in \mathcal{F}_k\big\}$. Note that, for all $\alpha, \alpha_1, \ldots \alpha_k \in \mathfrak{m}$, we have $\langle \alpha_1, \ldots, \alpha_k, \alpha \rangle \in X_{k+1,\beta}$ if, and only if, $\langle \alpha_1, \ldots, \alpha_k \rangle \in X_k$ and $\alpha_k > \alpha > \beta$. It follows, since $X_{1,\gamma} \in \mathcal{F}_1$ and $X_{k,\gamma} \in \mathcal{F}_k$ for each $\gamma < \mathfrak{m}$, that

$$\begin{aligned} A &= \big\{\alpha : \{\langle \alpha_1, \ldots, \alpha_k \rangle \in X_k : \alpha_k > \alpha > \beta\} \in \mathcal{F}_k\big\} \\ &= \big\{\alpha > \beta : \{\langle \alpha_1, \ldots, \alpha_k \rangle \in X_k : \alpha_k > \alpha\} \in \mathcal{F}_k\big\} \\ &= \{\alpha > \beta : X_{k,\alpha} \in \mathcal{F}_k\} = X_{1,\beta} \in \mathcal{F}_1 \end{aligned}$$

The foregoing shows that $X_{k+1,\beta} \in \mathcal{F}_k$, and this completes the induction.

The result of the preceding paragraph shows, in particular, that for each $n \in \mathbb{N}$, the set X_n belongs to the countably complete ultrafilter $\mathcal{F}_n$. As a consequence, for each $n \in \mathbb{N}$, the family $\mathcal{G}_n = \{G \in \mathcal{F}_n : G \subset X_n\}$ is a countably complete ultrafilter of X_n.

Let $n \in \mathbb{N}$ and $G \in \mathcal{G}_n$. We show that $\mathtt{p}_{n+1}(\mathtt{p}_n^{-1}(G)) \in \mathcal{G}_{n+1}$. Denote by $\hat{G}$ the set $\mathtt{p}_{n+1}(\mathtt{p}_n^{-1}(G))$ and note that $\hat{G} = \{\langle \alpha_1, .., \alpha_{n+1} \rangle \in X_{n+1} : \langle \alpha_1, .., \alpha_n \rangle \in G\}$. To prove that $\hat{G} \in \mathcal{F}_{n+1}$, it suffices to show that the set $A = \big\{\alpha : \{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \alpha_1, \ldots, \alpha_n, \alpha \rangle \in \hat{G}\} \in \mathcal{F}_n\big\}$ belongs to the family $\mathcal{F}_1$. We have

$$\begin{aligned} A &= \big\{\alpha : \{\langle \alpha_1, \ldots, \alpha_n \rangle : \langle \alpha_1, \ldots, \alpha_n, \alpha \rangle \in X_{n+1} \text{ and } \langle \alpha_1, \ldots, \alpha_n \rangle \in G\} \in \mathcal{F}_n\big\} \\ &= \big\{\alpha : \{\langle \alpha_1, \ldots, \alpha_n \rangle \in G : \alpha_n > \alpha > 0\} \in \mathcal{F}_n\big\} \\ &= \big\{\alpha > 0 : \{\langle \alpha_1, \ldots, \alpha_n \rangle \in G : \alpha_n > \alpha\} \in \mathcal{F}_n\big\} \\ &= \{\alpha > 0 : G \cap X_{n,\alpha} \in \mathcal{F}_n\} = \{\alpha : \alpha > 0\} = X_{1,0} \end{aligned}$$

Since $X_{1,0} \in \mathcal{F}_1$, we have shown that $\hat{G} \in \mathcal{F}_{n+1}$. Moreover, we have $\hat{G} \subset \mathtt{p}_{n+1}(X) = X_{n+1}$. As a consequence, $\hat{G} \in \mathcal{G}_{n+1}$, in other words, $\mathtt{p}_{n+1}(\mathtt{p}_n^{-1}(G)) \in \mathcal{G}_{n+1}$.

When we apply the above result for $n+1 \in \mathbb{N}$ and for the set $\mathtt{p}_{n+1}(\mathtt{p}_n^{-1}(G)) \in \mathcal{G}_{n+1}$, we get that $\mathtt{p}_{n+2}(\mathtt{p}_{n+1}^{-1}(\mathtt{p}_{n+1}(\mathtt{p}_n^{-1}(G)))) \in \mathcal{G}_{n+2}$. Since $\mathtt{p}_{n+1}^{-1}(\mathtt{p}_{n+1}(\mathtt{p}_n^{-1}(G))) = \mathtt{p}_n^{-1}(G)$, it follows that $\mathtt{p}_{n+2}(\mathtt{p}_n^{-1}(G)) \in \mathcal{G}_{n+2}$. Induction and a repetition of the preceding argument establish the more general result that $\mathtt{p}_k(\mathtt{p}_n^{-1}(G)) \in \mathcal{G}_k$ whenever $k > n$ and $G \in \mathcal{G}_n$.

For each $n \in \mathbb{N}$, let $\mathcal{N}_n = \{\mathtt{p}_n^{-1}(G) : G \in \mathcal{G}_n\}$. Set $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$. We show that the family $\mathcal{N}$ of subsets of X is a filter-base. Let $M, N \in \mathcal{N}$. There exist $m, n \in \mathbb{N}$, $G \in \mathcal{G}_m$ and $H \in \mathcal{G}_n$ such that $M = \mathtt{p}_m^{-1}(G)$ and $N = \mathtt{p}_n^{-1}(H)$. Without loss of generality, we may assume that $n \le m$. If $n = m$, then $G \cap H \in \mathcal{G}_n$ and $M \cap N = \mathtt{p}_n^{-1}(G \cap H) \in \mathcal{N}$. Assume that $n < m$. By the result in the preceding paragraph, we have $\mathtt{p}_m(N) \in \mathcal{G}_m$. Since $\mathcal{G}_m$ is a filter, the set $K = \mathtt{p}_m(N) \cap G$ belongs to $\mathcal{G}_m$. Now it is easy to see that the member $p_m^{-1}(K)$ of $\mathcal{N}$ is contained in $M \cap N$. We have shown that the family $\mathcal{N}$ is a filter-base. Note that the family $\mathcal{N}$ is not countably centered, since we have $F_n \in \mathcal{N}_n \subset \mathcal{N}$ for every n.

To complete the proof, we show that $\mathcal{N}$ is a Cauchy filterbase of $(X, e\pi)$. For every $n \in \mathbb{N}$, the projection $\mathtt{p}_n(\mathcal{N})$ is a filterbase of X_n containing the ultrafilter $\mathcal{G}_n$ of X_n; as a consequence, $\mathtt{p}_n(\mathcal{N}) = \mathcal{G}_n$. Since $\mathcal{G}_n$ is countably complete, Lemma 6 shows that $\mathcal{G}_n$ is a Cauchy filter of $(X_n, e\mathtt{u})$. When we recall the definition of the base of the uniformity $e\pi$ of $\lambda^{\mathbb{N}}$ from the beginning of this section, we see that the clopen filterbase $\mathcal{N}$ is Cauchy in $(X, e\pi)$. Corollary 5 now shows that $\upsilon X \neq H(U_\pi(X))$. □

Note that even though the space X above has Ulam measurable cardinality, it is "uniformly locally small" in the sense that no member of the uniform partition $\{B_\rho(x,1) : x \in X\}$ of X has Ulam measurable cardinality.

We close this paper with the following consequence of Theorem 7.

Corollary 8. *For a uniformly discrete space D, we have $\upsilon D^{\mathbb{N}} = H(U_\pi(D^{\mathbb{N}}))$ if, and only if, the cardinal $|D|$ is not Ulam measurable.*

Proof. Necessity. Assume that $|D|$ is Ulam measurable. Then $\mathfrak{m} \leq |D|$ and it follows that $(\mathfrak{m}^{\mathbb{N}}, \rho)$ embeds isometrically into $(D^{\mathbb{N}}, \rho)$. The embedding transforms the $e\pi$-Cauchy filterbase $\mathcal{N}$ of the closed discrete subset X of $\mathfrak{m}^{\mathbb{N}}$ from the above proof to a closed $e\pi$-Cauchy filterbase $\mathcal{M}$ of $D^{\mathbb{N}}$. Like $\mathcal{N}$, the filterbase $\mathcal{M}$ fails to be countably centered. Corollary 5 gives the conclusion that $\upsilon D^{\mathbb{N}} \neq H(U_\pi(D^{\mathbb{N}}))$.

Sufficiency. Assume that $|D|$ is not Ulam measurable. Then $|D^{\mathbb{N}}|$ is not Ulam measurable. Since $(D^{\mathbb{N}}, \pi)$ is Bourbaki-complete, we have $\upsilon D^{\mathbb{N}} = H(U_\pi(D^{\mathbb{N}}))$ by Proposition 1. □

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Heikki Junnila
Department of Mathematics and Statistics
University of Helsinki
00140 Helsinki, Finland

heikki.junnila@helsinki.fi

Ana S. Meroño
Departamento de Matemática Aplicada a la Ingeniería Industrial
E.T.S. Ingenieros Industriales
Universidad Politécnica de Madrid
Madrid, Spain

anasoledad.merono@upm.es