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Nonlinear open mapping principles, with applications to the Jacobian equation and other scale-invariant PDEs

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ABSTRACT

For a nonlinear operator T satisfying certain structural assumptions, our main theorem states that the following claims are equivalent: i) T is surjective, ii) T is open at zero, and iii) T has a bounded right inverse. The theorem applies to numerous scale-invariant PDEs in regularity regimes where the equations are stable under weak* convergence. Two particular examples we explore are the Jacobian equation and the equations of incompressible fluid flow.
For the Jacobian, it is a long standing open problem to decide whether it is onto between the critical Sobolev space and the Hardy space. Towards a negative answer, we show that, if the Jacobian is onto, then it suffices to rule out the existence of surprisingly well-behaved solutions.
For the incompressible Euler equations, we show that, for any $p < \infty$, the set of initial data for which there are dissipative weak solutions in $L^p_t L^2_x$ is meagre in the space of solenoidal

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L^2 fields. Similar results hold for other equations of incompressible fluid dynamics.

Contents

1. Introduction

The open mapping theorem is one of the cornerstones of functional analysis. When X and Y are Banach spaces and $L\colon X \to Y$ is a bounded *linear* operator, it asserts the equivalence of the following two conditions:

(i) Qualitative solvability: for all $f \in Y$ there is $u \in X$ with $Lu = f$, i.e. $L(X) = Y$;
(ii) Quantitative solvability: for all $f \in Y$ there is $u \in X$ with $Lu = f$ and additionally $\|u\|_X \le C\|f\|_Y$.

In other words, the operator L is surjective if and only if it is open at the origin. More generally, following [45], we say that the *open mapping principle* holds for a surjective map T between Banach spaces if T is open at the origin.

In this paper, we obtain a quantitative version of the open mapping principle that applies to a large family of nonlinear translation and scale-invariant PDEs. Invariance under translations and scalings is an ubiquitous feature of physical processes and, therefore, of the associated equations. It is an example of the *relativity principle* that the solutions of a PDE representing a physical phenomenon should not have a form which depends on the location of the observer or the units that the observer is using to measure the system [15]. We refer the reader to [5] for the general role of scaling symmetries in physics and other sciences and to [66] for a systematic study of symmetries in PDEs. We also remark that, even from the purely functional analytic viewpoint, our result seems to be the first instance of an open mapping principle that is applicable to nonlinear PDEs. We refer the reader to Section 1.1 for further discussion.

It often happens that a PDE has not just one but *several scaling symmetries*: two important examples, which will be discussed at length below, are the Jacobian equation (see Section 1.2) and the incompressible Euler equations (see Section 1.3). An important theme in this paper is that, whenever a PDE has several scaling symmetries, these symmetries must be *compatible* in order for the equation to be solvable for all data.

Our main result encapsulates the two previous points: many scale-invariant PDEs satisfy a nonlinear open mapping principle and, for the equation to be solvable, the associated scalings need to be compatible.

Theorem A *(Rough version). Consider a constant-coefficient system of PDEs, posed over either $\mathbb{R}^n$ or $\mathbb{R}^n \times [0,\infty)$, which moreover is preserved under weak* convergence. Let T be the solution-to-datum operator associated with the PDE.*

Suppose that the equation $Tu = f$ is invariant under the scalings

$$u_\lambda(x,t) \equiv \frac{1}{\lambda^\alpha} u\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^\gamma}\right), \qquad f_\lambda(x,t) \equiv \frac{1}{\lambda^\delta} f\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^\gamma}\right), \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are fixed and the group parameter is $\lambda > 0$. Suppose further that the solutions and the data lie in homogeneous *dual function spaces X^* and Y^* which satisfy, for some $r, s \in \mathbb{R}$,*

$$\|u_\lambda\|_{X^*} \equiv \lambda^r \|u\|_{X^*}, \qquad \|f_\lambda\|_{Y^*} \equiv \lambda^s \|f\|_{Y^*}, \qquad \text{where } rs > 0.$$

The following statements are then equivalent:

(i) *For all $f \in Y^*$ there is $u \in X^*$ with $Tu = f$;*
(ii) *For all $f \in Y^*$ there is $u \in X^*$ with $Tu = f$ and $\|u\|_{X^*}^{s/r} \le C\|f\|_{Y^*}$.*

Moreover, suppose that T is invariant under another pair of scalings $u \mapsto \widetilde{u_\lambda}, \widetilde{f_\lambda}$, which satisfy

$$\|\widetilde{u_\lambda}\|_{X^*} \equiv \lambda^{\tilde r} \|u\|_{X^*}, \qquad \|\widetilde{f_\lambda}\|_{Y^*} \equiv \lambda^{\tilde s} \|f\|_{Y^*}, \qquad \text{where } \tilde r \tilde s > 0.$$

Then solvability of the equation $Tu = f$ requires compatibility of the scalings, i.e.

$$r/s \neq \tilde r/\tilde s \quad \Longrightarrow \quad T \text{ is non-surjective.} \tag{1.2}$$

For a precise and more general version of the theorem we refer the reader to Theorems 3.5 and 3.8, where some *inhomogeneous* spaces are also treated. Concerning the hypothesis of stability under weak* convergence, we note that it is typically satisfied by solutions above a certain regularity threshold: for instance, it holds for both the Navier–Stokes equations and the cubic wave equation in $\mathbb{R}^3 \times [0, +\infty)$ in the corresponding energy spaces. Moreover, by considering a relaxed version of the PDE, this assumption can sometimes be bypassed, as will be discussed in more detail in Section 1.3 below.

Theorem A gives justification for two ideas ubiquitous in the study of PDEs, cf. [53]. Firstly, given a PDE and a space of data, the scaling symmetries of the PDE should be exploited fully when one determines the solution space, and secondly, a priori estimates are often paramount to establishing existence of solutions. We describe the use of Theorem A more precisely below.

One can often rule out solvability in various function spaces simply by computing the scaling symmetries of the PDE and using (1.2): as an example, in Corollary 4.1 we recover the main result of [55] on the Jacobian equation. Sometimes, however, the PDE has very few scaling symmetries, and so the second part of Theorem A is not applicable. To circumvent this issue, it is often useful to relax the nonlinear PDE into a linear one, as in the Tartar framework [79,80]: doing so often enlarges the collection of available scaling symmetries, allowing Theorem A to be used. We illustrate this general technique on the Euler equations in §4 and on the Navier-Stokes equations in §5.

In function spaces that "scale correctly", the use of Theorem A is two-fold. In the direction of non-solvability, it is, in practice, much simpler to disprove an a priori estimate than to find a datum for which solvability fails; see §4 for an application to the Jacobian equation. In the direction of solvability, Theorem A justifies fully the method of a priori estimates [78, §1.7]. We give examples of evolutionary PDEs to demonstrate that the estimates obtained through the open mapping principle agree (up to a multiplicative constant) with well-known estimates such as the energy inequalities for Leray–Hopf solutions to the incompressible Navier-Stokes equations.

The rest of this introduction is structured as follows. In the next subsection we present a special version of Theorem A in a simple case, and then, in Section 1.2, we apply this version to the Jacobian equation. After proving our main theorem in Section 3, we expand on the applications of Theorem A to the equations of incompressible fluid dynamics in Section 1.3.

1.1. A simple abstract open mapping principle

Deciding whether nonlinear versions of the open mapping principle hold is a classical problem. In the bilinear setting, this question goes back to Rudin [71, page 67]:

Question 1.1. If X_1, X_2 and Y are Banach spaces and T is a continuous bilinear map of $X_1 \times X_2$ onto Y, does it follow that T is open at the origin?

The origin plays a special role in Question 1.1 since, if T is open at 0, then by scaling one obtains quantitative solvability: for all $f \in Y$ there exist $u_i \in X_i$ such that $T(u_1, u_2) = f$ and $\|u_1\|^2_{X_1} + \|u_2\|^2_{X_2} \leq C\|f\|_Y$. The simple example $T\colon \mathbb{R} \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $(t, f) \mapsto tf$ shows that in general, openness at 0 does not imply openness at all points.

It turns out that nonlinear open mapping principles *do not hold* even in the very simple setup of Question 1.1. A first counterexample was found by Cohen [20] and, a bit later, Horowitz [45] gave a *finite-dimensional* example by taking $T\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^4$ to be

$$T(x,y) \equiv (x_1y_1, x_1y_2, x_1y_3 + x_3y_1 + x_2y_2, x_3y_2 + x_2y_1). \tag{1.3}$$

We also refer the reader to [7,32] for further extensions and discussion.

To reconcile Theorem A with these counterexamples one needs to keep in mind that the operators arising in PDEs have additional structure. Here we state a precise, abstract (and simpler to prove) version of Theorem A. It gives conditions under which Rudin's question has a positive answer; we replace bilinearity by positive homogeneity to incorporate more examples.

Theorem B. *Let X and Y be separable or reflexive Banach spaces. We assume that:*

(A1) *$T\colon X^* \to Y^*$ is a positively homogeneous operator.*
(A2) *T is weak*-to-weak* sequentially continuous.*
(A3) *For $k \in \mathbb{N}$ there are isometric isomorphisms $\sigma_k^{X^*}\colon X^* \to X^*$, $\sigma_k^{Y^*}\colon Y^* \to Y^*$ such that*

$$T \circ \sigma_k^{X^*} = \sigma_k^{Y^*} \circ T \qquad \textit{for all } k \in \mathbb{N},$$
$$\sigma_k^{Y^*} f \overset{*}{\rightharpoonup} 0 \qquad \textit{for all } f \in Y^*.$$

Then T is onto if and only if T is open at the origin.

Condition (A3) should be thought of as *generalised translation-invariance* (while conditions (A1)–(A2) are self-explanatory). Indeed, when T is a constant-coefficient partial differential operator and X^* and Y^* are function spaces on $\mathbb{R}^n$, natural choices of $\sigma_k^{X^*}$ and $\sigma_k^{Y^*}$ include translations

$$\sigma_k^{X^*} u(x) \equiv u(x - ke) \text{ and } \sigma_k^{Y^*} f(x) \equiv f(x - ke), \qquad \text{where } e \in \mathbb{R}^n \setminus \{0\}.$$

Example (1.3) shows that assumption (A3) cannot be omitted, but it is unclear whether (A2) is needed. The roles of the conditions (A1)–(A3) are discussed further in Remark 2.2.

The assumptions of Theorem B arise naturally in compensated compactness [65,79], a theory dealing with operators which are sequentially weakly continuous over the space of solutions to a given linear, underdetermined PDE. Our motivation for Theorem B comes from an old problem of Coifman, Lions, Meyer and Semmes: they discovered in [21] that many compensated compactness operators have improved integrability, and hence their range is smaller than expected. In fact, in [42] it was shown that, under natural assumptions, *all* compensated compactness operators have improved integrability, see also [43] for a systematic study. These results lead to the following problem:

Question 1.2. What is the range of the operators arising in compensated compactness?

The most famous examples of operators arising in compensated compactness, all of which fall under the scope of Theorem B, are the Jacobian, the Hessian and the div-curl product. In the next subsection we will focus on the important case of the Jacobian.

1.2. Applications to the Jacobian equation

We consider the Jacobian equation

$$\mathrm{J}u \equiv \det \mathrm{D}u = f \qquad \text{in } \mathbb{R}^n, \tag{1.4}$$

where the solution is a map $u\colon \mathbb{R}^n \to \mathbb{R}^n$. This is a first-order, fully nonlinear, underdetermined equation, and it appears naturally in Differential Geometry [61] and Optimal Transport [12]. Equation (1.4) is also geometric, as formally the change of variables formula reads as

$$\int_E |\mathrm{J}u(x)|\,\mathrm{d}x = \int_{\mathbb{R}^n} \#\big(u^{-1}(y)\cap E\big)\,\mathrm{d}y, \qquad E\subset\mathbb{R}^n \text{ is measurable.} \tag{1.5}$$

Thus, for a smooth solution of (1.4), f measures the size of its image, counted with multiplicity.

Whenever f is positive and sufficiently regular there is a well-posedness theory for (1.4) which goes back to the works of Dacorogna and Moser [26,61], and which can alternatively be deduced from the regularity theory for the Monge–Ampère equation [17]. In fact, one may view the Monge–Ampère equation as the *elliptic, determined* analogue of (1.4). We also note that there is an existence theory for (1.4) for data which are regular but have arbitrary sign [25] and we refer the reader to the book [24] for a comprehensive bibliography on the subject.

We are interested in studying the existence and regularity of solutions to (1.4) for *low-regularity data*, which is to say that we take $f \in L^p$ for some finite p. There are essentially no existence results in this setting; moreover, non-existence results are also remarkably difficult to obtain, although see [16,60,69] for some endpoint statements as well as our recent works [39,40] for results in the general L^p case. One of the difficulties in establishing non-existence of regular solutions to (1.4) is that underdetermined equations often admit solutions with a surprising amount of regularity, particularly for rough data. This is the case for the divergence equation, which one may regard as the *linear* analogue of (1.4): as shown by Bourgain and Brezis in [10], see also [77,81], the divergence equation admits solutions with higher regularity than the ones obtained by solving the corresponding elliptic problem, i.e. the Poisson equation. In fact, the same is true for our nonlinear problem, as the solutions obtained through the Monge–Ampère equation are not always optimal, see [40] for further details and discussion.

Extending Question 1.2 in the case of the Jacobian, Iwaniec conjectured in [9,48] that solutions with optimal regularity exist and can even be selected with a continuous dependence on the data:

Conjecture 1.3. *There is a continuous* $E\colon \mathscr{H}^p(\mathbb{R}^n) \to \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ *with* $\mathrm{J} \circ E = \mathrm{Id}$.

In Conjecture 1.3 we take $p \in [1, \infty)$ and we note that $\mathscr{H}^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$, while $\mathscr{H}^1(\mathbb{R}^n)$ is the real Hardy space [75], which is the dual of the separable Banach space $\mathrm{VMO}(\mathbb{R}^n)$ [23]. We also emphasize that $\dot{W}^{1,np}$ denotes a *homogeneous* Sobolev space: the third author showed in [56] that the Jacobian is not even onto $\mathscr{H}^p(\mathbb{R}^n)$ if its domain is taken to be an inhomogeneous Sobolev space.

The appearance of the Hardy space at the endpoint $p = 1$ of Conjecture 1.3 has to do with the improved integrability of the Jacobian, first noticed by Müller [62]. In fact, Coifman, Lions, Meyer and Semmes proved in [21] that

$$u \in \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n) \quad \Longrightarrow \quad \mathrm{J}u \in \mathscr{H}^1(\mathbb{R}^n)$$

and that $\mathscr{H}^1(\mathbb{R}^n)$ is the smallest Banach space containing the range of the Jacobian, a statement which was recently generalized for $p > 1$ by Hytönen [46]. It is still an open question to determine whether the Jacobian is *surjective* into $\mathscr{H}^p(\mathbb{R}^n)$. Nonetheless, as an immediate consequence of Theorem B, we obtain:

Corollary C. *Fix* $1 \leq p < \infty$. *The following statements are equivalent:*

(i) $\mathrm{J}\colon \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \to \mathscr{H}^p(\mathbb{R}^n)$ *is surjective;*
(ii) *there is a* bounded *operator* $E\colon \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \to \mathscr{H}^p(\mathbb{R}^n)$ *such that* $\mathrm{J} \circ E = \mathrm{Id}$*;*
(iii) *for all* $f \in \mathscr{H}^p(\mathbb{R}^n)$ *there is* $u \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ *such that* $\mathrm{J}u = f$ *and*

$$\|\mathrm{D}u\|^n_{L^{np}(\mathbb{R}^n)} \lesssim \|f\|_{\mathscr{H}^p(\mathbb{R}^n)}. \tag{1.6}$$

Although Corollary C may appear purely abstract, this is not so, as we now explain.

It appears plausible that Conjecture 1.3 is *false*: our motivation for considering this scenario comes in part from our related works [39,40], see also Section 4.4. For instance, in [40] we showed that the analogue of Conjecture 1.3 concerning the Dirichlet problem for (1.4) over a bounded domain Ω fails in a very strong sense. Roughly speaking, for $f \in L^p$ one cannot expect solutions of $\mathrm{J}u = f$ in any space higher than $\mathrm{id} + W^{1,p}_0(\Omega, \mathbb{R}^n)$. This is proved through a geometric argument, relying on (1.5) and on the condition $u = \mathrm{id}$ on the boundary $\partial\Omega$.

Let us say that a continuous solution of (1.4) which satisfies (1.5) is *admissible*; here our choice of terminology is inspired by the fluid dynamics literature. Any solution in $\dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$, for $p > 1$, is admissible, but there are continuous solutions in $\dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ which are not admissible, see e.g. [58]. Thus, when $p = 1$, it is possible for the PDE (1.4) to hold a.e. in $\mathbb{R}^n$, and hence also in the sense of distributions, and yet for its geometric information to be completely lost! We also note that geometric information on solutions is essential to study (1.4) beyond disproving Conjecture 1.3: for instance, in [39] we used a parametric version of the isoperimetric inequality to identify the energy-minimal admissible solutions of (1.4) in the simple case where f is spherically symmetric.

In fact, studying admissible solutions is not necessarily restrictive: Using Corollary C we can show that the existence of rough solutions implies the existence of admissible solutions. This observation is made precise in the following result:

Theorem D. *Let $\Omega \subset \mathbb{R}^n$ be an open set and take $f \in \mathscr{H}^1(\mathbb{R}^n)$ such that $f \geq 0$ in Ω. Assume that $\mathrm{J}\colon \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n) \to \mathscr{H}^1(\mathbb{R}^n)$ is onto. Then there is a solution $u \in \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ of (1.4) such that:*

(i) *u is continuous in Ω;*
(ii) *u has the Lusin (N) property in Ω;*
(iii) *$\int_{\mathbb{R}^n} |\mathrm{D}u|^n \,\mathrm{d}x \leq C\|f\|_{\mathscr{H}^1}$ with $C > 0$ independent of f.*

In particular, u is admissible over Ω, as it satisfies (1.5) there.

Moreover, if $n = 2$ and there is an open set $\Omega' \Subset \Omega$ with $f = 0$ a.e. in Ω', then:

(iv) *for any set $E \subseteq \Omega'$, we have $u(\partial E) = u(\overline{E})$;*
(v) *for $y \in u(\Omega')$, if C denotes a connected component of $u^{-1}(y) \cap \Omega'$ then C intersects $\partial\Omega'$.*

Theorem D provides a vital practical tool towards a negative answer to Conjecture 1.3. It is proved through a regularisation argument: due to Corollary C, powerful tools from Geometric Function Theory become available. In the supercritical regime $p > 1$, the first part of Theorem D holds automatically, although one can still use the a priori estimate (1.6) to get solutions with additional structure, see Section 4 for further details. The second part of Theorem D also holds in any dimension if p is taken to be sufficiently large.

1.3. Applications to the equations of incompressible fluid flow

In order to give a representative application of Theorem A to evolutionary PDEs we consider the incompressible Euler equations

$$\partial_t u + u \cdot \nabla u - \nabla P = 0, \tag{1.7}$$

$$\operatorname{div} u = 0, \tag{1.8}$$

$$u(\cdot, 0) = u^0 \tag{1.9}$$

in $\mathbb{R}^n \times [0, \infty)$, for $n \geq 2$. Note that (1.7)–(1.9) are invariant under scalings of the form

$$u_\lambda(x,t) \equiv \frac{1}{\lambda^\alpha} u\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^{\alpha+\beta}}\right), \; u^0_\lambda(x,t) \equiv \frac{1}{\lambda^\alpha} u^0\left(\frac{x}{\lambda^\beta}\right), \; P_\lambda(x,t) \equiv \frac{1}{\lambda^{2\alpha}} P\left(\frac{x}{\lambda^\beta}, \frac{t}{\lambda^{\alpha+\beta}}\right)$$

for any $\alpha, \beta > 0$. Recall also that smooth solutions of (1.7)–(1.9), with strong enough decay properties at infinity, conserve the kinetic energy $\int_{\mathbb{R}^n} |u(x,t)|^2 \, dx$ in time.

Weak solutions of (1.7)-(1.9) can, nevertheless, violate energy conservation and exhibit various other kinds of wild behaviour. Scheffer constructed in [72] solutions of the Euler equations which are compactly supported and square integrable in space-time and a simpler construction on the torus was given by Shnirelman in [73]. A systematic study of energy-dissipating solutions via convex integration was initiated by De Lellis and Székelyhidi in the groundbreaking works [27,28], culminating in the solution of the Onsager Conjecture in [27,47]. For more information on Onsager's conjecture see the recent reviews [14,30] and the references contained therein.

In view of the highly underdetermined nature of the Euler equations (in particular, the ability of fluids to come to rest in *finite time*), it is a priori not completely clear at which rates energy decay can occur if one starts from a generic square integrable initial datum. On the one hand, on the flat torus $\mathbb{T}^n$, Wiedemann has shown in [82] the existence of weak solutions of the Euler equations for all solenoidal, square integrable initial data by applying the methodology developed in [27,28]. By closely examining the proof, the kinetic energy of Wiedemann's solutions can be chosen to decay exponentially in time. On the other hand, in the case of Leray-Hopf solutions of Navier-Stokes equations, it is a well-known fact that the exponential energy decay does not carry over from the torus to the whole space (see §5.2).

Theorem A rather immediately rules out L^q-type energy decay, $q < \infty$, for a Baire-generic square integrable initial datum. The precise statement is given in Corollary E below; we use the customary notation $L^2_\sigma \equiv \{v \in L^2 : \operatorname{div} v = 0\}$. In §5.1 we also make a separate smallness statement about the set of initial data for which a solution can come to rest within a prescribed time interval.

Corollary E. *Let $n \geq 2$, $2 < p < \infty$ and $M > 0$. The set of initial data for which* (1.7)–(1.9) *have a solution u with $\|u\|_{L^p_t L^2_{\sigma,x}} \leq M$ is nowhere dense in L^2_σ.*

In particular, for a residual set of initial data in L^2_σ, the Cauchy problem (1.7)–(1.9) *has no solution in $L^\infty_t L^2_x \cap [\bigcup_{2<p<\infty} L^p_t L^2_x]$.*

To deduce Corollary E from Theorem A we consider a *linear* relaxation of the equations (1.7)–(1.9); this is an idea in the spirit of Tartar's framework for studying oscillations and concentrations in conservation laws [79,80]. Such a relaxation is used here in order to render the associated solution-to-datum operator weak*-to-weak* continuous and to introduce extra scaling symmetries into the problem. Corollary E is proved in §5.1.

The proof of Corollary E also applies to many other models in fluid dynamics. For instance, concerning the Navier–Stokes equations, we prove in an elementary fashion upper bounds for the generic energy dissipation rate of distributional solutions. Another example is given by the equations of ideal magnetohydrodynamics, for which the analogue of Corollary E holds true. In that context bounded solutions with compact support in space-time were constructed in [34]. On the torus $\mathbb{T}^3$, solutions in $L^\infty_t H^\beta_x$, for a small $\beta > 0$, violating magnetic helicity conservation were constructed in [6].

2. A nonlinear open mapping principle for positively homogeneous operators

The main goal of this section is to prove Theorem B. A related nonlinear uniform boundedness principle is proved in Proposition 2.3 and a precise statement concerning atomic decompositions in terms of T is proved in Proposition 2.5.

We already motivate Theorem B in the setting of Corollary C. By adapting the usual proof of the standard Open Mapping Theorem to $\mathrm{J}\colon \dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n) \to \mathscr{H}^p(\mathbb{R}^n)$ one obtains the following statement: if $\mathrm{J}(\dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n)) = \mathscr{H}^p(\mathbb{R}^n)$, then for every $f \in \mathscr{H}^p(\mathbb{R}^n)$ there exist $u, v \in \dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n)$ with

$$\mathrm{J}u + \mathrm{J}v = f \qquad \text{and} \qquad \int_{\mathbb{R}^n} (|\mathrm{D}u|^{np} + |\mathrm{D}v|^{np})\,\mathrm{d}x \leq C\|f\|^p_{\mathscr{H}^p}. \tag{2.1}$$

Thus, quantitative control is gained at the expense of introducing an extra term $\mathrm{J}v$.

One could attempt to show the non-surjectivity of J by disproving the a priori estimate in (2.1). However, the extra term $\mathrm{J}v$ makes this a formidable task since the equation $\mathrm{J}u + \mathrm{J}v = f$ admits much more pathological solutions than $\mathrm{J}u = f$. As a prototypical example, there exist Lipschitz maps $u, v\colon \mathbb{R}^2 \to \mathbb{R}^2$ vanishing in the lower half-plane and satisfying $\mathrm{J}u + \mathrm{J}v = 1$ in the upper half-plane [49, Lemma 5]. In Theorem B and Corollary C, the extra Jacobian $\mathrm{J}v$ is removed, which leads to a dramatically less daunting task than disproving (2.1). We explore this further in §4.

2.1. The proof of Theorem B

Here we give a slightly more precise version of Theorem B:

Theorem 2.1. *Let X and Y be Banach spaces such that $\mathbb{B}_{X^*}$ is sequentially weak* compact. We make the following assumptions:*

- (A1) *$T\colon X^* \to Y^*$ is a weak*-to-weak* sequentially continuous operator.*
- (A2) *$T(au) = a^s T(u)$ for all $a > 0$ and $u \in X^*$, where $s > 0$.*
- (A3) *For $k \in \mathbb{N}$ there are isometric isomorphisms $\sigma_k^{X^*}: X^* \to X^*$, $\sigma_k^{Y^*}: Y^* \to Y^*$ such that*

$$T \circ \sigma_k^{X^*} = \sigma_k^{Y^*} \circ T \qquad \text{for all } k \in \mathbb{N}, \qquad \sigma_k^{Y^*} f \overset{*}{\rightharpoonup} 0 \qquad \text{for all } f \in Y^*.$$

Then the following conditions are equivalent:

(i) *$T(X^*)$ is non-meagre in Y^*.*

(ii) $T(X^*) = Y^*$.
(iii) *T is open at the origin.*
(iv) *For every $f \in Y^*$ there exists $u \in X^*$ such that*

$$Tu = f, \qquad \|u\|_{X^*}^s \leq C\|f\|_{Y^*}. \tag{2.2}$$

A sufficient condition for $\mathbb{B}_{X^*}$ to be sequentially weak* compact is that X is a *weak Asplund space* [74, Theorem 3.5]. For instance, reflexive or separable spaces are weak Asplund [31].

Proof of Theorem B. We have (iv) ⇒ (iii) ⇒ (ii) ⇒ (i) and so we just prove (i) ⇒ (iv).

Assume that (i) holds. We may write $T(X^*)$ as a union $\cup_{\ell=1}^{\infty} K_\ell$, where

$$K_\ell \equiv \left\{f \in Y^* : \text{there exists } u \in X^* \text{ with } Tu = f \text{ and } \|u\|_{X^*}^s \leq \ell\right\}.$$

Since balls in X^* are sequentially weak* compact, by (A1), the sets K_ℓ are sequentially weak* closed, in particular norm-closed. Now, by the Baire Category Theorem, some K_ℓ contains a closed ball $\bar{B}_r(f_0)$.

Our next aim is to show that $\bar{B}_r(0) \subset K_\ell$. Suppose, therefore, that $\|f\|_{Y^*} \leq r$. By (A3), $f + \sigma_k^{Y^*} f_0 \overset{*}{\rightharpoonup} f$, and thus it suffices to show that $f + \sigma_k^{Y^*} f_0 \in K_\ell$ for all $k \in \mathbb{N}$. Given $k \in \mathbb{N}$, we note that $f_0 + (\sigma_k^{Y^*})^{-1} f \in \bar{B}_r(f_0)$. Hence, we may choose $u_k \in X^*$ such that $Tu_k = f_0 + (\sigma_k^{Y^*})^{-1} f$ and $\|u_k\|_{X^*}^s \leq \ell$. By (A3),

$$\begin{aligned} &f + \sigma_k^{Y^*} f_0 = \sigma_k^{Y^*}(f_0 + (\sigma_k^{Y^*})^{-1} f) = \sigma_k^{Y^*} Tu_k = T(\sigma_k^{X^*} u_k), \\ &\|\sigma_k^{X^*} u_k\|_{X^*}^s = \|u_k\|_{X^*}^s \leq \ell, \end{aligned}$$

which yields $f + \sigma_k^{Y^*} f_0 \in K_\ell$, and so $\bar{B}_r(0) \subset K_\ell$.

Assumption (A2) now yields $\bar{B}_R(0) \subset K_{\ell R/r}$ for all $R > 0$, and so (iv) holds with $C = \ell/r$. □

Remark 2.2. We make some comments on the roles of each assumption of Theorem B. Note that we only used (A2) at the very end of the proof to move from the local statement $\bar{B}_r(0) \subset K_\ell$ to the global quantitative solvability statement (iv). This motivates us to replace positive homogeneity by more general scaling symmetries, and this is done in §3. It is an interesting problem whether (ii) and (iii) continue to be equivalent if one simply discards assumption (A2).

Recall that in view of Horowitz's example (1.3), assumption (A3) cannot be omitted. We also note that (A3) never holds if Y is finite-dimensional and that moreover, when the target is two-dimensional, it is not needed: Downey has shown that, in this case, the answer to Question 1.1 is positive [33, Theorem 12]. The assumption (A1), in turn, is not always necessary, but it holds automatically in finite dimensional examples. An infinite dimensional case where it is not needed is the following multiplication operator

$(f,g) \mapsto fg\colon L^p \times L^q \to L^r$, where $1/p + 1/q = 1/r$: this operator does not satisfy (A1), although it verifies the open mapping principle [3,4].

2.2. Examples

The theory of Compensated Compactness provides many nonlinear operators to which Theorem B applies. Here we give a general formulation in the spirit of [41], see also [65,79], which we then illustrate with more concrete examples.

Let $\mathcal{A}$ be an l-th order homogeneous linear operator, which for simplicity we assume to have constant coefficients; that is, for $v \in C^\infty(\mathbb{R}^n, \mathbb{V})$,

$$\mathcal{A}v = \sum_{|\alpha|=l} A_\alpha \partial^\alpha v, \qquad A_\alpha \in \operatorname{Lin}(\mathbb{V}, \mathbb{W}),$$

where $\mathbb{V}, \mathbb{W}$ are finite-dimensional vector spaces. For $p \in [1, +\infty)$ and $s \in \mathbb{N}$, $s \geq 2$, take

$$X^* = L^{ps}_{\mathcal{A}}(\mathbb{R}^n, \mathbb{V}), \qquad Y^* = \mathscr{H}^p(\mathbb{R}^n).$$

Here $L^{ps}_{\mathcal{A}}(\mathbb{R}^n, \mathbb{V})$ is the space of those $v \in L^{ps}(\mathbb{R}^n, \mathbb{V})$ such that $\mathcal{A}v = 0$ in the sense of distributions. We will further need the following standard non-degeneracy assumption:

$$\text{the symbol of } \mathcal{A}, \text{ seen as a matrix-valued polynomial, has constant rank.} \tag{2.3}$$

Whenever (2.3) holds, we say that $\mathcal{A}$ has *constant rank*. We will not discuss this assumption here but it holds in all of the examples below; the reader may find other characterizations of constant rank operators in [42,67].

Let $T\colon X^* \to Y^*$ be a homogeneous sequentially weakly continuous operator. Under the assumption (2.3), such operators were completely characterised in [41], and they are often called *Compensated Compactness quantities*. They can be realised as certain constant-coefficient partial differential operators and so they necessarily satisfy (A3) if one takes the isometries $\sigma_k^{X^*}, \sigma_k^{Y^*}$ to be translations. The following are standard examples of such operators:

(i) $\mathcal{A} = \operatorname{curl}$ and $T = \mathrm{J}$. For this example, take $\mathbb{V} = \mathbb{R}^{n\times n}$ and choose $\mathcal{A}$ in such a way that $\mathcal{A}v = 0$ if and only if $v = \mathrm{D}u$, for some $u\colon \mathbb{R}^n \to \mathbb{R}^n$. For instance, we may take $(\operatorname{curl} v)_{ijk} = \partial_k v_{ij} - \partial_j v_{ik}$. We also choose $s = n$ and so $X^* = \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$. The only positively n-homogeneous sequentially weakly continuous operator $X^* \to Y^*$ is the Jacobian, and in particular we recover Corollary C.

(ii) $\mathcal{A} = \operatorname{curl}^2$ and $T = \mathrm{H}$. Here $\mathcal{A}$ is chosen similarly to the previous example, but now $\mathcal{A}v = 0$ if and only if $v = \mathrm{D}^2 u$, for some $u\colon \mathbb{R}^n \to \mathbb{R}$. Again we take $s = n$ and so $X^* = \dot{W}^{2,np}(\mathbb{R}^n, \mathbb{R}^n)$. We may take $T = \mathrm{H}\colon X^* \to Y^*$ to be the Hessian, and Theorem B shows that it satisfies the open mapping principle.

The two previous examples admit a straightforward generalisation, where one considers s-th order minors (instead of the determinant) and a j-th order curl (instead of $j = 1, 2$).

(iii) $\mathcal{A} = (\text{div}, \text{curl})$ and $T = \langle \cdot, \cdot \rangle$. In this example, $s = 2$ and T is the standard inner product acting on a pair $v \equiv (B, E)\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$; here, B is thought of as a "magnetic field" and E as an "electric field". As before, Theorem B shows that T satisfies the open mapping principle.

We conclude this subsection by comparing the above example with [21]. There, the authors address the problem of deciding whether Compensated Compactness quantities are surjective, particularly when $p = 1$. Thus Theorem B can be read as saying that openness at zero is a necessary condition for a positive answer to this problem.

2.3. A nonlinear uniform boundedness principle

We also present a nonlinear version of the Uniform Boundedness Principle in the spirit of Theorem B; under certain structural conditions, a family of operators which is pointwise bounded in a neighbourhood of the origin is uniformly bounded in a (possibly smaller) neighbourhood of the origin.

Proposition 2.3. *Let X and Z be Banach spaces and let I be an index set. Suppose the following conditions hold:*

(i) *For every $i \in I$, the mapping $T_i\colon X \to Z$ is such that $u \mapsto \|T_i u\|_Z \colon X \to \mathbb{R}$ is weakly sequentially lower semicontinuous.*
(ii) *There is $\varepsilon > 0$ such that $\sup_{i\in I} \|T_i(u)\|_Z < \infty$ whenever $\|u\|_X \le \varepsilon$.*
(iii) *For $j \in \mathbb{N}$ there are isometric isomorphisms $\sigma_k^X \colon X \to X$ and $\sigma_k^Z \colon Z \to Z$ such that*

$$T_i \circ \sigma_k^X = \sigma_k^Z \circ T_i \qquad \text{for all } i \in I \text{ and } k \in \mathbb{N},$$
$$\sigma_k^X u \rightharpoonup 0 \qquad \text{for all } u \in X.$$

Then there exists $\delta > 0$ such that

$$\sup_{\|u\|_X \le \delta} \sup_{i\in I} \|T_i u\|_Z < \infty.$$

Proof. By (ii), we may write $\varepsilon\mathbb{B}_X = \cup_{\ell=1}^\infty C_\ell$, where $C_\ell \equiv \{u \in \varepsilon\mathbb{B}_X \colon \sup_{i\in I} \|T_i u\|_Z \le \ell\}$ and (i) shows that each C_ℓ is norm closed. Thus, by the Baire Category Theorem, some C_ℓ contains a closed ball $\bar{B}_\delta(u_0)$.

Let now $\|u\|_X \le \delta$ and $i \in I$. By (iii), we have $u + \sigma_k^X u_0 = \sigma_k^X [u_0 + (\sigma_k^X)^{-1} u] \in \bar{B}(u_0, \delta)$ and moreover $u + \sigma_k^X u_0 \rightharpoonup u$. So by (i) and again (iii), we have

$$\|T_i u\|_Z \leq \liminf_{k\to\infty} \|T_i \sigma_k^X [u_0 + (\sigma_k^X)^{-1} u]\|_Z = \liminf_{k\to\infty} \|\sigma_k^Z T[u_0 + (\sigma_k^X)^{-1} u]\|_Z \leq \ell.$$

The proof is complete. □

We note that, in the linear case, it is possible to prove the Banach–Steinhaus Uniform Boundedness Principle without using Baire's Category Theorem: the proof relies, instead, on the so-called "gliding hump method". For an extension of the classical Uniform Boundedness Principle using this method, we refer the reader to [37].

2.4. Atomic decompositions in terms of T

The main motivation behind this subsection is Theorem 2.4. It establishes an analogue of the atomic decomposition of $\mathscr{H}^1(\mathbb{R}^n)$, giving a weak factorization on $\mathscr{H}^p(\mathbb{R}^n)$ in the spirit of the classical work of Coifman, Rochberg and Weiss [22]:

Theorem 2.4. *Let $p \in [1,\infty)$. For every $f \in \mathscr{H}^p(\mathbb{R}^n)$ there are functions $u_i \in \dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n)$ and real numbers c_i such that*

$$f = \sum_{i=1}^{\infty} c_i \mathrm{J} u_i, \qquad \|u_i\|_{\dot{W}^{1,np}(\mathbb{R}^n)} \leq 1, \qquad \sum_{i=1}^{\infty} |c_i| \lesssim \|f\|_{\mathscr{H}^p(\mathbb{R}^n)}. \tag{2.4}$$

In particular, $\mathscr{H}^p(\mathbb{R}^n)$ is the smallest Banach space containing the range $\mathrm{J}(\dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n))$.

Theorem 2.4 was proved in [21] for $p = 1$, while the case $p > 1$ is much harder and was established only recently by Hytönen in [46]. It is conceivable that the operator $\mathrm{J}\colon \dot{W}^{1,np}(\mathbb{R}^n,\mathbb{R}^n) \to \mathscr{H}^p(\mathbb{R}^n)$ is not surjective but (2.4) improves to a *finitary* decomposition of $\mathscr{H}^p(\mathbb{R}^n)$ in terms of Jacobians. In Proposition 2.5, we formulate a rather precise classification of infinitary and finitary decompositions in the setting of Theorem B.

Take $\omega \in \overline{\mathbb{N}} \equiv \mathbb{N} \cup \{\infty\}$. Given T as in Theorem B, if every $f \in Y^*$ can be written as

$$f = \sum_{j=1}^{\omega} c_j \, T u_j, \qquad c_j \in \mathbb{R},\ u_j \in \mathbb{B}_{X^*}, \tag{2.5}$$

then, following [32], T is said to be $1/\omega$-*surjective*. If, furthermore,

$$\sum_{j=1}^{\omega} |c_j| \lesssim \|f\|_{Y^*} \tag{2.6}$$

for all $f \in Y^*$, then T is said to be $1/\omega$-*open*. Dixon [32] generalised Horowitz's example by constructing, for every $m \in \mathbb{N}$, a continuous $1/m$-surjective bilinear map between Banach spaces which is not $1/m$-open. In fact, in Dixon's notation, the constants

c_j are subsumed by the elements u_j. The formalism (2.5)–(2.6) is, however, more standard in the context of atomic decompositions.

In Proposition 2.5 we show that, for $\omega \in \overline{\mathbb{N}}$, and under the assumptions of Theorem B, $1/\omega$-surjectivity implies $1/\omega$-openness.

Proposition 2.5. *Suppose X, Y and T satisfy the assumptions of Theorem B. Let us define, for $\omega \in \overline{\mathbb{N}}$, the sets*

$$\Lambda_\omega \equiv \left\{ \sum_{j=1}^{\omega} c_j T u_j : u_j \in \mathbb{B}_{X^*},\, c_j \in \mathbb{R} \text{ and } \sum_{j=1}^{\omega} |c_j| < \infty \right\}.$$

If Λ_∞ is not meagre in Y^*, there is $\omega \in \overline{\mathbb{N}}$ such that $\Lambda_\omega = Y^*$ and $\bigcup_{m<\omega} \Lambda_m$ is meagre in Y^*; moreover, T is $1/\omega$-open.

Proof. We show that if $\bigcup_{m<\infty} \Lambda_m$ is not meagre in Y^*, then there is $m \in \mathbb{N}$ such that $\Lambda_m = Y^*$ and Λ_{m-1} is meagre in Y^*. Note that, for each $m \in \mathbb{N}$, the set Λ_m is closed; it follows from the Baire Category Theorem that one of the sets Λ_m contains a ball. By using the s-homogeneity of T, we write $\Lambda_m = \{\sum_{j=1}^m d_j T v_j \colon d_j \in \mathbb{R},\, v_j \in X^*\}$. By applying Theorem B to the $(s+1)$-homogeneous operator

$$\tilde{T} \colon \mathbb{R}^m \times (X^*)^m \to Y^*, \qquad \tilde{T}\big(\{d_j\}_{j=1}^m, \{v_j\}_{j=1}^m\big) \equiv \sum_{j=1}^m d_j T v_j\,,$$

we find that for each $f \in Y^*$ there are $d_j \in \mathbb{R}$ and $v_j \in X^*$ such that

$$\sum_{j=1}^m d_j T v_j = f, \qquad \sum_{j=1}^m (|d_j|^{s+1} + \|v_j\|_{X^*}^{s+1}) \lesssim \|f\|_{Y^*}. \tag{2.7}$$

We set $c_j = d_j \|v_j\|_{X^*}^s$ and denote $u_j = v_j/\|v_j\|_{X^*}$ if $v_j \neq 0$ and $u_j = 0$ if $v_j = 0$. Thus $c_j\, T u_j = d_j\, T v_j$ for $j = 1, \ldots, m$. Consequently, through Young's inequality, (2.7) yields

$$\sum_{j=1}^m c_j\, T u_j = f, \qquad \sum_{j=1}^m |c_j| \lesssim \|f\|_{Y^*}, \qquad u_j \in \mathbb{B}_{X^*}. \tag{2.8}$$

It now suffices choose the smallest $m \in \mathbb{N}$ such that $T \colon X^* \to Y^*$ is $1/m$-surjective; the $1/m$-openness of T is given by (2.8).

We finally show that if $\bigcup_{m<\infty} \Lambda_m$ is meagre but Λ_∞ is non-meagre, then in fact $\Lambda_\infty = Y^*$ and T is $1/\infty$-open. We denote $V \equiv \{\varepsilon\, T u \colon \varepsilon = \pm 1,\, u \in \mathbb{B}_{X^*}\} \subset Y^*$. Now V is bounded and symmetric and, by assumption, $\{\sum_{j=1}^\infty c_j f_j : f_j \in V \text{ for all } j \text{ and } \sum_{j=1}^\infty |c_j| < \infty\}$ is non-meagre in Y^*. By [56, Lemma 3.1], $\{\sum_{j=1}^\infty c_j T u_j \colon \sum_{j=1}^\infty |c_j| = 1,\, u_j \in \mathbb{B}_{X^*}\} \subset Y^*$ contains a ball centred at the origin. It immediately follows that given $f \in Y^*$, conditions (2.5)–(2.6) can be satisfied with $\omega = \infty$. □

Remark 2.6. It is tempting to try and prove the last part of Proposition 2.5 by defining an auxiliary operator $\tilde{T}\colon \ell^{s+1}(\mathbb{N}) \times \ell^{s+1}(\mathbb{N}; X^*) \to Y^*$ via $T(\{d_j\}_{j=1}^\infty, \{v_j\}_{j=1}^\infty) \equiv \sum_{j=1}^\infty d_j T v_j$ and using Theorem B on $\tilde{T}$, in analogy to the case $\omega < \infty$. However, such an operator is never weak*-to-weak* continuous unless $T \equiv 0$. Indeed, suppose $Tu \neq 0$ and set $d_{jk} = \delta_{jk}$ and $v_{jk} = \delta_{jk} u$. Now $\tilde{T}(\{d_{jk}\}_{j=1}^\infty, \{v_{jk}\}_{j=1}^\infty) = Tu$ for all $k \in \mathbb{N}$ but $(\{d_{jk}\}_{j=1}^\infty, \{v_{jk}\}_{j=1}^\infty) \overset{*}{\rightharpoonup} 0$.

Example 2.7. Let us denote by $\mathcal{H}$ the Hilbert transform and by $T\colon L^2(\mathbb{R},\mathbb{R}) \to \mathscr{H}^1(\mathbb{R})$ the operator $T(\chi,\eta) \equiv \mathcal{H}\chi\,\mathcal{H}\eta - \chi\,\eta$. The strong factorization $\mathscr{H}^1(\mathbb{C}_+) = \mathscr{H}^2(\mathbb{C}_+) \cdot \mathscr{H}^2(\mathbb{C}_+)$ of analytical Hardy spaces yields the surjectivity result

$$\mathscr{H}^1(\mathbb{R}) = \left\{T(\chi,\eta) : \chi,\eta \in L^2(\mathbb{R})\right\}, \tag{2.9}$$

see e.g. [21, page 258]. Thus, in this case, $\Lambda_1 = \mathscr{H}^1(\mathbb{R})$.

Another example is obtained by considering the operator $\mathrm{J}\colon W^{1,np}(\mathbb{R}^n,\mathbb{R}^n) \to \mathscr{H}^p(\mathbb{R}^n)$, where $n \geq 2$ and $p \in [1,\infty)$; we emphasise that the Sobolev space is *inhomogeneous*. In this case, Λ_∞ is meagre in $\mathscr{H}^p(\mathbb{R}^n)$, see [55] and Corollary 4.1. However, if we instead consider the Jacobian as defined on $\dot{W}^{1,np}$, then $\Lambda_\infty = \mathscr{H}^p(\mathbb{R}^n)$ by the results of [46], although it is unclear whether this is optimal. We note that for $\mathrm{J}\colon \dot{W}^{1,2p}(\mathbb{R}^2,\mathbb{R}^2) \to \mathscr{H}^p(\mathbb{R}^2)$, the statement $\Lambda_1 = \mathscr{H}^p(\mathbb{R}^2)$ is equivalent to

$$\mathscr{H}^p(\mathbb{R}^2) = \left\{|\mathcal{S}\omega|^2 - |\omega|^2 \colon \omega \in L^{2p}(\mathbb{R}^2,\mathbb{R}^2)\right\},$$

compare with (2.9). Here $\mathcal{S}$ is the Beurling–Ahlfors transform, which one may think of as the square of a complex Hilbert transform [50].

We are not aware of operators satisfying the assumptions of Theorem B and for which there is $1 < m \in \mathbb{N}$ such that $\Lambda_m = Y^*$ but $\bigcup_{m'<m} \Lambda_{m'}$ is meagre in Y^*.

3. A general nonlinear open mapping principle for scale-invariant problems

The main aim of this section is to formalise Theorem A and generalise Theorem B to a wider class of translation-invariant, scaling-invariant PDEs. We divide the rigorous version of Theorem A into Theorems 3.5 and 3.8.

Our motivation for generalising Theorem B is two-fold. On the one hand, from an abstract perspective, obtaining more general nonlinear versions of the open mapping principle is of interest in its own right. In particular, we point out that, in Theorem B, multi-linearity is only used in order to move from small data to arbitrary data by a simple scaling argument. Here we wish to allow more general scaling symmetries. On the other hand, solutions to partial differential equations typically satisfy a priori estimates. Having at hand a version of Theorem B that ensures the existence of a priori estimates can give a heuristic justification for the methods used in the construction of solutions to these equations.

3.1. Two model examples

The formulation of Theorem 3.5 is rather technical, and so we start by motivating it via two familiar examples, the Navier-Stokes equations and the cubic wave equation.

Example 3.1. Consider the inhomogeneous, incompressible Navier-Stokes equations in $\mathbb{R}^3 \times [0, \infty)$:

$$\partial_t u + u \cdot \nabla u - \nu \Delta u - \nabla P = 0, \tag{3.1}$$

$$\operatorname{div} u = 0, \tag{3.2}$$

$$u(\cdot, 0) = u^0, \tag{3.3}$$

where u is the velocity field, P is the pressure, $\nu > 0$ is the viscosity and u^0 is the initial datum. The equations are invariant under the scalings $u \to u_\lambda$, $P \to P_\lambda$ and $u^0 \to u^0_\lambda$,

$$u_\lambda(x,t) \equiv \frac{1}{\lambda} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right), \qquad P_\lambda(x,t) \equiv \frac{1}{\lambda^2} P\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right), \qquad u^0_\lambda(x) = \frac{1}{\lambda} u^0\left(\frac{x}{\lambda}\right).$$

For simplicity, we concentrate on the familiar solution and datum spaces

$$X^* = L^\infty_t L^2_{\sigma,x} \cap L^2_t \dot{W}^{1,2}_x, \qquad Y^* = L^2_\sigma.$$

Note that X^* and Y^* are homogeneous: $\|u_\lambda\|_{X^*} = \lambda^{1/2}\|u_\lambda\|_{X^*}$ and $\|u^0_\lambda\|_{Y^*} = \lambda^{1/2}\|u^0_\lambda\|_{Y^*}$ for all $\lambda > 0$, $u \in X^*$ and $u^0 \in Y^*$. Recall that $u \in X^*$ is called a *weak solution* of (3.1)–(3.3) if u satisfies

$$\int_0^\tau \langle u, \partial_t \varphi\rangle \,\mathrm{d}t + \int_0^\tau \langle u \otimes u, \mathrm{D}\varphi\rangle \,\mathrm{d}t - \nu \int_0^\tau \langle \mathrm{D}u, \mathrm{D}\varphi\rangle \,\mathrm{d}t + \langle u^0, \varphi(0)\rangle - \langle u(\tau), \varphi(\tau)\rangle = 0 \tag{3.4}$$

for all $\varphi \in C^\infty_c(\mathbb{R}^3 \times [0,\infty), \mathbb{R}^3)$ with $\operatorname{div} \varphi = 0$ and almost every $\tau > 0$. In (3.4), $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2_x.

Our aim is to express the solvability of (3.1)–(3.3) equivalently as surjectivity of a suitable nonlinear map T from (a subset of) X^* onto Y^*. Openness of T at 0 will then be equivalent to an a priori estimate as in Theorem 2.1. Up to a multiplicative constant, the a priori estimate coincides with the familiar energy inequality.

Formally, we choose T to be the *solution-to-initial datum map* $T(u) = u(\cdot, 0)$. The rigorous formulation of Theorem A is, however, complicated by the fact that $T(u)$ is not well-defined for all $u \in X^*$. We overcome this issue by restricting the domain of definition of T and setting

$$D \equiv \{u \in L^\infty_t L^2_{\sigma,x} \cap L^2_t \dot{W}^{1,2}_x : u \text{ is a weak solution of (3.1)–(3.3) for some } u^0 \in L^2_\sigma\},$$

$$T\colon D \to Y^*, \quad T(u) \equiv u^0 \text{ if (3.4) holds.}$$

The set D is not a vector space, but fortunately, the proof of Theorem B does not require the additive structure of the domain space; in fact, the proof adapts readily to more general scaling invariant problems as long as the sets $D_\ell \equiv \{u \in D\colon \|u\|_{X^*} \le \ell,\ \|Tu\|_{Y^*} \le \ell\}$ are sequentially weak* compact. This latter condition appears as assumption $(\widehat{\text{A4}})$ in Theorem 3.5.

Theorem 3.5 now says that solvability of (3.1)–(3.3) for all $u^0 \in L^2_\sigma$ is equivalent to solvability with the a priori estimate

$$\|u\|_{L^\infty_t L^2_x} + \|u\|_{L^2_t \dot{W}^{1,2}_x} \le C\|u(\cdot,0)\|_{L^2}.$$

Such an estimate is well-known to be satisfied by Leray–Hopf solutions [70].

In Example 3.1, we deliberately chose homogeneous domain and target spaces X^* and Y^* for simplicity. However, Theorem 3.5 also incorporates inhomogeneous function spaces as well as product spaces. In Example 3.4 we motivate this via the cubic wave equation, but first we recall some notions from interpolation theory.

Definition 3.2. Suppose that X_1 and X_2 are Banach spaces embed into a topological vector space Z. We set

$$\|u\|_{X_1\cap X_2} \equiv \max\{\|u\|_{X_1}, \|u\|_{X_2}\},$$
$$\|u\|_{X_1+X_2} \equiv \inf\{\|u_1\|_{X_1} + \|u_2\|_{X_2}\colon u = u_1 + u_2,\ u_1 \in X_1,\ u_2 \in X_2\}.$$

If $X_1 \cap X_2$ is dense in both X_1 and X_2, then (X_1, X_2) is called a *conjugate couple.*

The duals of spaces of the form $X_1 \cap X_2$ are well-known, cf. [8, Theorem 2.7.1]:

Theorem 3.3. *Let (X_1, X_2) be a conjugate couple. Then, up to isometric isomorphism, it holds that $(X_1 \cap X_2)^* = X_1^* + X_2^*$ and $(X_1 + X_2)^* = X_1^* \cap X_2^*$.*

Example 3.4. Consider the cubic wave equation in $(1 + 3)$-dimensions

$$\partial_{tt}u - \Delta u + u^3 = 0 \text{ in } [0, +\infty) \times \mathbb{R}^3 \tag{3.5}$$
$$(u(\cdot,0), \partial_t u(\cdot,0)) = (u^0, u^1). \tag{3.6}$$

We are interested in initial data in the energy space $Y^* = [\dot{H}^1(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)] \times L^2(\mathbb{R}^3)$ and we look for solutions in the space

$$X^* = L^\infty_t \dot{H}^1_x \cap L^\infty_t L^4_x \cap L^\rho_t L^\sigma_x([0,\infty) \times \mathbb{R}^3), \qquad \text{where } \frac{1}{\rho} + \frac{3}{\sigma} = \frac{1}{2}.$$

Thus we have $X^*_{1,1} = L^\infty_t \dot{H}^1_x$, $X^*_{1,2} = L^\infty_t L^4_x$, $X^*_{1,3} = L^\rho_t L^\sigma_x$ and $Y^*_{1,1} = \dot{H}^1$, $Y^*_{1,2} = L^4$ and $Y^*_{2,1} = L^2$. Recall that $u \in X^*$ is a weak solution of (3.5)–(3.6) if, for every test function $\varphi \in C^\infty_c(\mathbb{R} \times \mathbb{R}^3, \mathbb{R}^3)$,

$$\int_0^\tau u\,\partial_{tt}\varphi + \langle \mathrm{D}u, \mathrm{D}\varphi\rangle\,\mathrm{d}t + \langle u^3, \varphi\rangle\,\mathrm{d}t = \langle u^0, \varphi(0,\cdot)\rangle - \langle u^1, \varphi_t(0,\cdot)\rangle \\ - \langle u(\tau,\cdot), \varphi(\tau,\cdot)\rangle + \langle u(\tau,\cdot), \varphi_t(\tau,\cdot)\rangle \tag{3.7}$$

for almost every $\tau > 0$. The equation is invariant under translations as well as the scalings $u \to u_\lambda$, $u^0 \to u^0_\lambda$ and $u^1 \to u^1_\lambda$ where

$$u_\lambda(t,x) \equiv \lambda u(\lambda t, \lambda x) \qquad u^0_\lambda(x) \equiv \lambda u^0_\lambda(\lambda x) \qquad u^1_\lambda(x) \equiv \lambda^2 u^1_\lambda(\lambda^2 x).$$

We define

$$D = \{u \in X^*\colon u \text{ is a weak solution of (3.5)–(3.6) for some } (u^0, u^1) \in Y^*\},$$
$$T\colon D \to Y^*, Tu = (u^0, u^1) \text{ if (3.7) holds.}$$

It is easy to compute

$$\|u^0_\lambda\|_{L^4} = \lambda^{\frac14}\|u^0\|_{L^4}, \qquad \|(u^0_\lambda, u^1_\lambda)\|_{\dot H^1\times L^2} = \lambda^{\frac12}\|(u^0,u^1)\|_{\dot H^1\times L^2},$$
$$\|u_\lambda\|_{L^\infty_t L^4_x} = \lambda^{\frac14}\|u\|_{L^\infty_t L^4_x}, \qquad \|u_\lambda\|_{L^\infty_t \dot H^1_x \cap L^\rho_t L^\sigma_x} = \lambda^{\frac12}\|u\|_{L^\infty_t \dot H^1_x\cap L^\rho_t L^\sigma_x}.$$

Thus $r_{11} = r_{13} = \frac12$, $r_{12} = \frac14$ and $s_{11} = s_{21} = \frac12$, $s_{12} = \frac14$ in the notation of Theorem 3.5.

Theorem 3.5 now says that solvability of (3.5)–(3.6) for all $(u^0, u^1) \in [\dot H^1 \cap L^4] \times L^2$ is equivalent to solvability with the a priori estimate

$$\begin{cases} \|u\|^{\frac12}_{L^\infty_t \dot H^1_x\cap L^\rho_t L^\sigma_x} + \|u\|_{L^\infty_t L^4_x} \le C\|(u^0,u^1)\|_{\dot H^1\cap L^4\times L^2}, & \text{if } \|(u^0,u^1)\|_{\dot H^1\cap L^4\times L^2} \le 1, \\ \|u\|_{L^\infty_t \dot H^1\cap L^\rho_t L^\sigma_x} + \|u\|^2_{L^\infty_t L^4_x} \le C\|(u^0,u^1)\|_{\dot H^1\cap L^4\times L^2}, & \text{if } \|(u^0,u^1)\|_{\dot H^1\cap L^4\times L^2} > 1. \end{cases}$$

Taking powers and estimating the right-hand sides, we conclude in particular that solvability of the equation implies the more familiar-looking estimate

$$\|u\|^2_{L^\infty_t \dot H^1_x\cap L^\rho_t L^\sigma_x} + \|u\|^4_{L^\infty_t L^4_x} \le C\left(\|u^0\|^2_{\dot H^1} + \|u^0\|^4_{L^4} + \|u^1\|^2_{L^2}\right).$$

The estimate in the Strichartz space $L^\rho_t L^\sigma_x$ is known from [38], and the reader may also find the stronger estimate

$$\frac12\|u\|^2_{L^\infty_t \dot H^1_x} + \frac14\|u\|^4_{L^\infty_t L^4_x} \le \frac12\|u^0\|^2_{\dot H^1} + \frac14\|u^0\|^4_{L^4}$$

in [2, Theorem 8.41].

3.2. A more general nonlinear open mapping principle

We can now formulate the main result of this section. On a first reading, it is advisable to consider $I = J_i = M = N_\mu = 1$ as in Example 3.1. Recall that when a direct sum of Banach spaces $X = \oplus_{i=1}^I X_i$ is endowed with the norm $\|w\|_X \equiv \sum_{i=1}^I \|w_i\|_{X_i}$, the dual norm of $X^* = \oplus_{i=1}^I X_i^*$ is of the form $\|u\|_{X^*} = \max_{1\le i\le I} \|u_i\|_{X_i^*}$.

Theorem 3.5. *For $i = 1, \dots I$, $j = 1, \dots J_i$ and $\mu = 1, \dots M$, $\nu = 1, \dots N_\mu$ let $X_{i,j}$ and $Y_{\mu,\nu}$ be Banach spaces. Consider X^*, Y^* of the form*

$$X^* = \bigoplus_{i=1}^{I} \Big(\bigcap_{j=1}^{J_i} X_{i,j}^* \Big), \qquad Y^* = \bigoplus_{\mu=1}^{M} \Big(\bigcap_{\nu=1}^{N_\mu} Y_{\mu,\nu}^* \Big)$$

for some $I, M, J_i, N_\mu \in \mathbb{N}$. Suppose $0 \in D \subset X^*$.

We make the following assumptions:

($\widehat{\text{A1}}$) *If $u_j \overset{*}{\rightharpoonup} u$ in D and $Tu_j \overset{*}{\rightharpoonup} f$ in Y^*, then $Tu = f$.*

($\widehat{\text{A2}}$) *For $\lambda > 0$, there exist bijections $u \mapsto u_\lambda : D \to D$ and $f \mapsto f_\lambda : Y^* \to Y^*$ such that*

$$\begin{aligned} &T(u_\lambda) = (Tu)_\lambda && \text{for all } u \in D, \lambda > 0,\\ &\|(u_\lambda)_i\|_{X_{i,j}^*} = \lambda^{r_{i,j}} \|u_i\|_{X_{i,j}^*} && \text{for all } \lambda > 0,\ i = 1,\dots,I,\ j = 1,\dots,J_i,\ u \in D,\\ &\|(f_\lambda)_\mu\|_{Y_{\mu,\nu}^*} = \lambda^{s_{\mu,\nu}} \|f_\mu\|_{Y_{\mu,\nu}^*} && \text{for all } \lambda > 0,\ \mu = 1,\dots,M,\ \nu = 1,\dots,N_\mu,\ f \in Y^*, \end{aligned}$$

where $0 < r_{i,j}$ and $0 < s_1 \le s_{\mu,\nu} \le s_2$.

($\widehat{\text{A3}}$) *There exist sequences of isometric bijections $\sigma_k^D : D \to D$ with $\sigma_k^D(0) = 0$ and isometric isomorphisms $\sigma_k^{Y^*} : Y^* \to Y^*$ such that*

$$T \circ \sigma_k^D = \sigma_k^{Y^*} \circ T \qquad \text{for all } k \in \mathbb{N},$$
$$\sigma_k^{Y^*} f \overset{*}{\rightharpoonup} 0 \qquad \text{for all } f \in Y^*.$$

($\widehat{\text{A4}}$) *For $\ell \in \mathbb{N}$, the sets $D_\ell \equiv \{u \in D : \|u\|_{X^*} \le \ell, \|Tu\|_{Y^*} \le \ell\}$ are weakly* sequentially compact in X^*.*

The following conditions are then equivalent:

(i) *$T(D)$ is non-meagre in Y^*.*
(ii) *$T(D) = Y^*$.*
(iii) *T is open at the origin.*
(iv) *For every $f \in Y^*$ there exists $u \in D$ such that*

$$Tu = f, \qquad \begin{cases} \sum_{i=1}^I \sum_{j=1}^{J_i} \|u_i\|_{X_{i,j}^*}^{s_2/r_{i,j}} \le C\|f\|_{Y^*}, & \|f\|_{Y^*} \le 1,\\ \sum_{i=1}^I \sum_{j=1}^{J_i} \|u_i\|_{X_{i,j}^*}^{s_1/r_{i,j}} \le C\|f\|_{Y^*}, & \|f\|_{Y^*} > 1. \end{cases} \tag{3.8}$$

Proof. We first show (i) ⇒ (iii), so assume (i) holds. Write $D = \cup_{\ell=1}^{\infty} D_\ell$ and note that we have $T(D) = \cup_{\ell=1}^{\infty} T(D_\ell)$. Since each set D_ℓ is weak* sequentially compact and $T\colon D \to Y^*$ has weak*-to-weak* sequentially closed graph, the sets $T(D_\ell)$ are closed in Y^* and, therefore, complete. By the Baire Category Theorem, one of the sets $T(D_\ell)$ contains a ball $\bar{B}_\eta(f_0)$. Clearly $\eta \le \ell$. We first show that

$$T(\bar{B}_D(0,\ell)) \supset \bar{B}_{Y^*}(0,\eta). \tag{3.9}$$

Here $\bar{B}_D = \bar{B}_{X^*} \cap D$.

Suppose $f \in Y^*$ with $\|f\|_{Y^*} \le \eta$. We show the stronger statement $f \in T(D_\ell)$. By $(\widehat{A1})$ and $(\widehat{A4})$, the set $T(D_\ell)$ is weakly* sequentially closed, and so, by $(\widehat{A3})$, it suffices to show that $f + \sigma_k^{Y^*} f_0 \in T(D_\ell)$ for all $k \in \mathbb{N}$. Given $k \in \mathbb{N}$ we write $f + \sigma_k^{Y^*} f_0 = \sigma_k^{Y^*}(f_0 + (\sigma_k^{Y^*})^{-1} f)$ and note that $f_0 + (\sigma_k^{Y^*})^{-1} f \in B_{Y^*}(f_0,\eta) \subset T(D_\ell)$. Since σ_k^D and $\sigma_k^{Y^*}$ are isometries and $T \circ \sigma_k^D = \sigma_k^{Y^*} \circ T$, we get $\sigma_k^{Y^*}(T(D_\ell)) \subset T(D_\ell)$, and so $f + \sigma_k^{Y^*} f_0 \in T(D_\ell)$, as claimed.

We are ready to show openness of T at zero. Let $\varepsilon > 0$; our aim is to find $\delta > 0$ such that $T(\bar{B}_D(0,\varepsilon)) \supset \bar{B}_{X^*}(0,\delta)$. Denoting $\tau_\lambda^D \equiv (u \mapsto u_\lambda)$ and $\tau_\lambda^{Y^*} \equiv (f \mapsto f_\lambda)$, we first note that for each $\lambda > 0$ we have

$$\tau_\lambda^D(\bar{B}_D(0,\ell)) = \{u \in D\colon \|u_i\|_{X^*_{i,j}} \le \lambda^{r_{i,j}}\ell \text{ for } i = 1,\dots,I, j = 1,\dots,J_i\}.$$

By choosing $\lambda = \min_{1\le i\le I, 1\le j\le J_i} (\varepsilon/\ell)^{1/r_{i,j}}$ we get $\max_{1\le i\le I,1\le j\le J_i} \lambda^{r_{i,j}}\ell \le \varepsilon$ so that

$$T(\bar{B}_D(0,\varepsilon)) \supset T(\tau_\lambda^D \bar{B}_D(0,\ell)) = \tau_\lambda^{Y^*} T(\bar{B}_D(0,\ell)).$$

By using (3.9) and selecting $\delta = \min_{1\le i\le I,\, 1\le i\le J_i} \min_{1\le\mu\le M,\, 1\le\nu\le N_\mu} \eta(\varepsilon/\ell)^{s_{\mu,\nu}/r_{i,j}}$ we get

$$\begin{aligned}\tau_\lambda^{Y^*} T(\bar{B}_D(0,\ell)) &\supset \tau_\lambda^{Y^*}(\bar{B}_{Y^*}(0,\eta))\\ &= \bar{B}_{\cap_{\nu=1}^{N_1} Y^*_{1,\nu}}(0,\tilde{\lambda}_1\eta) \times \cdots \times \bar{B}_{\cap_{\nu=1}^{N_M} Y^*_{M,\nu}}(0,\tilde{\lambda}_M\eta) \supset \bar{B}_{Y^*}(0,\delta),\end{aligned}$$

where for $1 \le i \le I$, $\tilde{\lambda}_i \equiv \max_{1\le j\le J_i} \lambda^{s_{i,j}}$.

We now prove (iii) ⇒ (iv), so as above take some $\varepsilon > 0$ and get $\delta > 0$ in such a way that $\bar{B}_{Y^*}(0,\delta) \subset T(\bar{B}_D(0,\varepsilon))$. Assume, without loss of generality, that $\delta \le 1$. Let $f \in Y^*$ and define $\lambda > 0$ via

$$\|f\|_{Y^*} \equiv \mu = \min_{1\le\mu\le M,\, 1\le\nu\le N_\mu} \lambda^{s_{\mu,\nu}}\delta = \begin{cases}\lambda^{s_2}\delta, & \mu \le \delta,\\ \lambda^{s_1}\delta, & \mu > \delta.\end{cases}$$

In either case, let j_0 be such that $\mu = \lambda^{s_{j_0}}\delta$. Then

$$\begin{aligned}f \in \tau_\lambda^{Y^*}(\bar{B}_{Y^*}(0,\delta)) &\subset \tau_\lambda^{Y^*} T(\bar{B}_D(0,\varepsilon)) = T\tau_\lambda^D(\bar{B}_D(0,\varepsilon))\\ &= T\{u \in D\colon \|u_i\|_{X^*_{i,j}} \le \lambda^{r_{i,j}}\varepsilon \text{ for } i = 1,\dots,I,\, j = 1,\dots,J_i\}.\end{aligned}$$

Suppose now $u \in D$ satisfies $\|u_i\|_{X^*_{i,j}} \le \lambda^{r_{i,j}}\varepsilon$ for $i = 1, \dots, I$, $j, \dots, I_j$. Then, for all such i, j,

$$\|u_i\|_{X^*_{i,j}}^{s_{j0}/r_{i,j}} \le \lambda^{s_{j0}} \varepsilon^{s_{j0}/r_{i,j}} \le \frac{\varepsilon^{s_{j0}/r_{i,j}}}{\delta}\mu.$$

We conclude that

$$f \in T\left\{u \in D\colon \|u_i\|_{X_i^*} \le \lambda^{r_i}\varepsilon \text{ for all } i\right\} \subset T\left\{u \in D\colon \sum_{i=1}^{I}\sum_{j=1}^{J_i} \|u_i\|_{X^*_{i,j}}^{s_{j0}/r_{i,j}} \le C\mu\right\},$$

where

$$C = \sum_{i=1}^{I}\sum_{j=1}^{J_i} \frac{\varepsilon^{s_{j0}/r_{i,j}}}{\delta},$$

which yields (3.8) in the cases $\|f\|_{Y^*} \le \delta$ and $\|f\|_{Y^*} > 1$. If $\|f\|_{Y^*} \in (\delta, 1]$, one obviously has $\lambda^{s_1} \approx_\delta \lambda^{s_2}$ so that (3.8) holds for all f.

We conclude the proof of the theorem by noting that (iv) ⇒ (ii) ⇒ (i). □

Remark 3.6. Inspection of the proof of Theorem 3.5 shows that, in the statement of the theorem, one may replace all occurrences of Y^* with K, where $K \subset Y^*$ is a closed convex cone. Recall that K is said to be a *cone* if $af \in K$ whenever $a > 0$ and $f \in K$. Such a generalisation is occasionally useful, since it may be interesting to consider smaller data sets. For instance, in the case of Conjecture 1.3, it is natural to look at the sets of radially symmetric data $K = \{f \in \mathscr{H}^p(\mathbb{R}^n) : f(x) \equiv f(|x|)\}$ as well as, for $p > 1$, the set of non-negative data $K = \{f \in L^p(\mathbb{R}^n) : f \ge 0\}$.

3.3. A simple linear version

In Theorem 3.5, surjectivity can only hold if all the scaling symmetries are compatible; Theorem 3.8 below makes this precise. For our applications, the full *nonlinear* strength of Theorems 3.5 and 3.8 is not always needed, as often we can relax nonlinear PDEs into linear ones. We therefore formulate a simple linear variant of Theorem 3.5 in Proposition 3.7. The formulation of Proposition 3.7 aims at compatibility with Theorem 3.5 instead of maximal generality.

Proposition 3.7. *Let $X^* = \oplus_{i=1}^{I}(\cap_{j=1}^{J_i} X^*_{i,j})$ and $Y^* = \oplus_{\mu=1}^{M}(\cap_{\nu=1}^{N_\mu} Y^*_{\mu,\nu})$ be dual Banach spaces. Suppose D is a vector subspace of X^* and the following conditions hold:*

(A1) *$T\colon D \to Y^*$ is linear.*
(A2) *$D_\ell \equiv \{u \in D\colon \|u\|_{X^*} \le \ell, \|Tu\|_{Y^*} \le \ell\}$ is weakly[*] sequentially compact for all $\ell \in \mathbb{N}$.*

(A3) *If* $u^j \overset{*}{\rightharpoonup} u$ *and* $Tu^j \overset{*}{\rightharpoonup} f$, *then* $Tu = f$.

The following conditions are then equivalent:

(i) $T(D)$ *is non-meagre.*
(ii) *For all* $f \in Y^*$ *there exists* $u \in D$ *such that* $Tu = f$ *and* $\|u\|_{X^*} \le C\|f\|_{Y^*}$.

Proof. Suppose $T(D)$ is non-meagre. Write $T(D) = \cup_{\ell\in\mathbb{N}} T(D_\ell)$. By assumption, each D_ℓ is weakly* sequentially compact. With assumption (A3), this implies that each $T(D_\ell)$ is norm closed. By the Baire category theorem, some $T(D_\ell)$ contains a closed ball $\bar{B}_{Y^*}(f_0, r)$. By linearity, $\bar{B}_{Y^*}(0, 2r) = \bar{B}_{Y^*}(f_0, r) - \bar{B}_{Y^*}(f_0, r) \subset T(D_\ell) - T(D_\ell) = T(D_{2\ell})$. The claim follows by scaling. □

3.4. A general non-solvability result

We are now ready to formalise the part of Theorem A which says that incompatibility of two scalings leads to non-surjectivity.

Theorem 3.8. *Consider the setup and assumptions of either Theorem 3.5 or Proposition 3.7. Suppose, additionally, that there exist other bijections* $u \mapsto \widetilde{u_\lambda}$, $f \mapsto \widetilde{f_\lambda}$ *satisfying* ($\widehat{\text{A2}}$):

$$
\begin{aligned}
&T\widetilde{u_\lambda} = \widetilde{(Tu)_\lambda} && \text{for all } u \in D, \lambda > 0,\\
&\|(\widetilde{u_\lambda})_i\|_{X^*_{i,j}} = \lambda^{\tilde{r}_{i,j}}\|u_i\|_{X^*_{i,j}} && \text{for all } \lambda > 0,\ i = 1, \dots, I,\ j = 1, \dots, J_i,\ u \in D,\\
&\|(\widetilde{f_\lambda})_\mu\|_{Y^*_{\mu,\nu}} = \lambda^{\tilde{s}_{\mu,\nu}}\|f_\mu\|_{Y^*_{\mu,\nu}} && \text{for all } \lambda > 0,\ i = 1, \dots, M,\ \nu = 1, \dots, N_\mu,\ f \in Y^*,
\end{aligned}
$$

where $0 < \tilde{r}_{ij}$ *and* $0 < \tilde{s}_1 \le \tilde{s}_{\mu,\nu} \le \tilde{s}_2$. *If*

$$\frac{\tilde{s}_1}{\tilde{r}_{i,j}} > \frac{s_1}{r_{i,j}} \text{ for all } i, j \quad \text{or} \quad \frac{\tilde{s}_2}{\tilde{r}_{i,j}} < \frac{s_2}{r_{i,j}} \text{ for all } i, j,$$

then $T(\bar{B}_D(0, R))$ *is nowhere dense in* Y^* *for every* $R > 0$.

Moreover, the conclusion of Theorem 3.8 also follows if $\tilde{s}_1/\tilde{r}_{i,j} \ge s_1/r_{i,j}$ *for all* i, j *but in addition, for some* $i_0 \in \{1, \dots, I\}$ *and* $j_0 \in \{1, \dots, J_{i_0}\}$ *such that*

$$T(u_1, \dots, u_{i_0-1}, 0, u_{i_0+1}, \dots, u_I) \equiv 0,$$

we have $\tilde{s}_1/\tilde{r}_{i_0,j_0} > s_1/r_{i_0,j_0}$. *An analogous statement holds if* $\tilde{s}_2/\tilde{r}_{i,j} \le s_2/r_{i,j}$ *for all* i, j.

Proof. We prove the case where $\tilde{s}_1/\tilde{r}_{i,j} > s_1/r_{i,j}$ for all i, j. Seeking a contradiction, assume $T(\bar{B}_D(0, R))$ is not nowhere dense in some ball $\bar{B}_{Y^*}(0, \ell)$, where $\ell \ge R$. By weak* sequential compactness of $D_\ell = \{u \in \bar{B}_D(0, \ell) : Tu \in \bar{B}_{Y^*}(0, \ell)\}$ and since T has a weak*-to-weak* closed graph, the set $T(\bar{B}_D(0,R))$ is closed, and thus $T(\bar{B}_D(0,R))$ contains a ball $\bar{B}_{Y^*}(f_0,r)$. By Theorem 3.5, whenever $\|f\|_{Y^*}>1$, there exists $u\in X^*$ such that $Tu=f$ and $\sum_{i=1}^I\sum_{j=1}^{J_i}\|u_i\|_{X^*_{i,j}}^{s_1/r_{i,j}}\le C\|f\|_{Y^*}$ and $\bar{u}\in X^*$ such that $T\bar{u}=f$ and $\sum_{i=1}^I\sum_{j=1}^{J_i}\|\bar{u}_i\|_{X^*_{i,j}}^{\tilde{s}_1/\tilde{r}_{i,j}}\le C\|f\|_{Y^*}$. Below, we denote the inverses of $u\mapsto u_\lambda$ and $f\mapsto f_\lambda$ by $u\mapsto u_{-\lambda}$ and $f\mapsto f_{-\lambda}$.

Fix $f\in Y^*$ with $\|f\|_{Y^*}=2$; our aim is to show that $T0=f$ and derive a contradiction. First note that $\|f_\lambda\|_{Y^*}=2\lambda^{s_1}$. Choose $\bar{u}\in X^*$ with $T\bar{u}=f_\lambda$ and $\sum_{i=1}^I\sum_{j=1}^{J_i}\|\tilde{u}_i\|_{X^*_{i,j}}^{\tilde{s}_1/\tilde{r}_{i,j}}\le C\lambda^{s_1}$. Choose $u\in D$ such that $\bar{u}=u_\lambda$, and note that $u=(u_\lambda)_{-\lambda}=\bar{u}_{-\lambda}$. Now

$$\begin{aligned} &Tu=T\bar{u}_{-\lambda}=(T\bar{u})_{-\lambda}=f,\\ &\sum_{i=1}^I\sum_{j=1}^{J_i}\lambda^{r_{i,j}\tilde{s}_1/\tilde{r}_{i,j}}\|u_i\|_{X^*_{i_j}}^{\tilde{s}_1/\tilde{r}_{i,j}}=\sum_{i=1}^I\sum_{j=1}^{J_i}\|\bar{u}_i\|_{X^*_{i,j}}^{\tilde{s}_1/\tilde{r}_{i,j}}\le C\lambda^{s_1}. \end{aligned}$$

We conclude that $\sum_{i=1}^I\sum_{j=1}^{J_i}\lambda^{r_{i,j}\tilde{s}_1/\tilde{r}_{i,j}-s_i}\|u_i\|_{X^*_{i,j}}^{\tilde{s}_1/\tilde{r}_{i,j}}\le C$. Thus, by letting $\lambda\to\infty$ we find a sequence of solutions u^ℓ of $Tu^\ell=f$ with $\sum_{i=1}^I\sum_{j=1}^{J_i}\|u_i^\ell\|_{X^*_{i,j}}^{\tilde{s}_1/\tilde{r}_{i,j}}\to 0$. Now $\|u^\ell\|_{X^*}\to 0$ so that $u^\ell\overset{*}{\rightharpoonup}0$, which yields $T0=f$. Thus $1=\|T0\|_{Y^*}=\|T(0_\lambda)\|_{Y^*}=\|(T0)_\lambda\|_{Y^*}=\lambda^{s_1}$ for all $\lambda>0$. We have reached a contradiction. The case $\tilde{s}_2/\tilde{r}_{i,j}<s_2/r_{i,j}$ has an analogous proof where one sets $\|f\|_{Y^*}\le 1$ and lets $\lambda\to 0$, and the proof of the second claim of the theorem only requires obvious modifications. □

4. Applications to the Jacobian equation

This section expands on the relation between open mapping principles and Question 1.3.

4.1. Scaling analysis for the Jacobian equation

We begin with the following question: given $p\ge 1$ and $q,r\in(1,\infty]$, is there a solution $u\in L^q(\mathbb{R}^n,\mathbb{R}^n)\cap\dot{W}^{1,r}(\mathbb{R}^n,\mathbb{R}^n)$, or a solution $u\in\dot{W}^{1,r}(\mathbb{R}^n,\mathbb{R}^n)$, of

$$\mathrm{J}u=f\qquad\text{a.e. in }\mathbb{R}^n\tag{4.1}$$

for every $f\in\mathscr{H}^p(\mathbb{R}^n)$? One of the motivations for considering this question is that, for positive radial data, a radial solution is only in $W^{1,p}_{\mathrm{loc}}$; moreover, in the case of the Dirichlet problem, one must have $r\le p$. We also wish to put the main result of [56], namely the non-surjectivity of $J\colon W^{1,np}(\mathbb{R}^n,\mathbb{R}^n)\to\mathscr{H}^p(\mathbb{R}^n)$, into the general framework of this paper. The result is, indeed, deduced from Theorem 3.8 by simple arithmetic:

Corollary 4.1. *Suppose $n\ge 2$, and $p\in[1,\infty)$, $q\in(1,\infty]$ and $r\in[n,\infty)$. The following statements hold:*

(i) $\mathrm{J}(\dot{W}^{1,r}(\mathbb{R}^n,\mathbb{R}^n)) \cap \mathscr{H}^p(\mathbb{R}^n)$ *is meagre in* $\mathscr{H}^p(\mathbb{R}^n)$ *unless* $r = np$.
(ii) $\mathrm{J}(L^q(\mathbb{R}^n,\mathbb{R}^n) \cap \dot{W}^{1,r}(\mathbb{R}^n,\mathbb{R}^n)) \cap \mathscr{H}^p(\mathbb{R}^n)$ *is meagre in* $\mathscr{H}^p(\mathbb{R}^n)$ *unless* $q = \infty$, $r = n$ *and* $p = 1$.

In particular, $\mathrm{J}(W^{1,np}(\mathbb{R}^n,\mathbb{R}^n))$ *is meagre in* $\mathscr{H}^p(\mathbb{R}^n)$.

Proof. We prove (ii), claim (i) has a similar proof. Fix p, q, r. The assumptions of Theorems 3.5 and 3.8 are satisfied once we set $D \equiv \{u \in L^q(\mathbb{R}^n,\mathbb{R}^n) \cap \dot{W}^{1,r}(\mathbb{R}^n,\mathbb{R}^n) \colon \mathrm{J}u \in \mathscr{H}^p(\mathbb{R}^n)\}$. The scaling $u_\lambda = \lambda u(\cdot/\lambda)$, $f_\lambda = f(\cdot/\lambda)$, under which J is invariant, gives

$$\|u_\lambda\|_{L^q} = \lambda^{1+n/q}\|u\|_{L^q}, \quad \|u_\lambda\|_{\dot{W}^{1,r}} = \lambda^{n/r}\|u\|_{\dot{W}^{1,r}}, \quad \|f\|_{\mathscr{H}^p} = \lambda^{n/p}\|f\|_{\mathscr{H}^p},$$

so that $s/r_1 = (n/p)/(1 + n/q) = qn/(p(q+n)$ and $s/r_2 = r/p$, whereas the scalings $u \mapsto \widetilde{u_\lambda} \equiv \lambda u$ and $f \mapsto \widetilde{f_\lambda} \equiv \lambda^n f$ give $\tilde{s}/\tilde{r}_1 = \tilde{s}/\tilde{r}_2 = n$. The claim follows from Theorem 3.8 since if $\|u\|_{L^q} = 0$ or $\|u\|_{\dot{W}^{1,np}} = 0$, we immediately get $\mathrm{J}u = 0$. $\square$

Interestingly, we note that if $p = 1$ the algebra $L^\infty(\mathbb{R}^n,\mathbb{R}^n) \cap \dot{W}^{1,n}(\mathbb{R}^n,\mathbb{R}^n)$ is not ruled out as a solution space. Surjectivity of $\mathrm{J} \colon L^\infty(\mathbb{R}^n,\mathbb{R}^n) \cap \dot{W}^{1,n}(\mathbb{R}^n,\mathbb{R}^n) \to \mathscr{H}^1(\mathbb{R}^n)$ would be the natural analogue of [10, Theorem 1] for the Jacobian equation on $\mathbb{R}^n$.

4.2. Tools from Geometric Function Theory

Before proceeding further we collect, for the convenience of the reader, useful results about Sobolev maps and mappings of finite distortion. The following notions are relevant in relation to the change of variables formula:

Definition 4.2. Let $u \colon \Omega \to \mathbb{R}^n$ be a continuous map which is differentiable a.e. in Ω. Then:

(i) u has the *Lusin (N) property* if $|u(E)| = 0$ for any $E \subset \Omega$ such that $|E| = 0$;
(ii) u has the *(SA) property* if $|u(E)| = 0$ for any open set $E \subset \Omega$ with $\mathrm{J}u = 0$ a.e. in E.

In the one-dimensional case, the Lusin (N) property is well understood: for instance, on an interval, a continuous function of bounded variation has the Lusin (N) property if and only if it is absolutely continuous. However, in higher dimensions, the situation is much more complicated, although we have the following characterisation, proved in [59]:

Proposition 4.3. *Let* $u \in W^{1,n}(\Omega,\mathbb{R}^n)$ *be a continuous map with* $\mathrm{J}u \geq 0$ *in* Ω*. Then* u *has the Lusin (N) property if and only if it has the (SA) property.*

We remark that Proposition 4.3 is in general false if $\mathrm{J}u \not\geq 0$, see [68] for a counterexample. The following result, see [58], is also useful for our purposes:

Proposition 4.4. *Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a continuous map such that, for some $K \geq 1$,*

$$\operatorname{diam}(u(B_r(x)) \leq K \operatorname{diam}(u(\partial B_r(x))) \qquad \text{for all } B_r(x) \Subset \Omega. \tag{4.2}$$

Then u has the Lusin (N) property.

The change of variables formula is closely related to the Jacobian determinant:

Theorem 4.5. *Let $u \in C^0(\Omega, \mathbb{R}^n) \cap W^{1,n}(\Omega, \mathbb{R}^n)$ be a map with the Lusin (N) property. Then*

$$\int_E |\mathrm{J}u| \,\mathrm{d}x = \int_{\mathbb{R}^n} \mathcal{N}(y, u, E) \,\mathrm{d}y \qquad \text{for all measurable sets } E \subset \Omega, \tag{4.3}$$

where $\mathcal{N}$ is the multiplicity function*, defined as $\mathcal{N}(y, u, E) \equiv \#\{x \in E : u(x) = y\}$.*

The reader may find the proof of Theorem 4.5, together with a wealth of information on geometric properties of Sobolev maps, in [36].

We now recall some useful facts about mappings of finite distortion and, for simplicity, we focus on the planar case $n = 2$, see [1]. The reader can also find these and higher-dimensional results in [44,50].

Definition 4.6. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ be such that $0 \leq \mathrm{J}u \in L^1_{\mathrm{loc}}(\Omega)$. We say that u is a *map of finite distortion* if there is a function $K \colon \Omega \to [1, \infty]$ such that $K < \infty$ a.e. in Ω and

$$|\mathrm{D}u(x)|^2 \leq K(x) \,\mathrm{J}u(x) \quad \text{ for a.e. } x \text{ in } \Omega.$$

If u has finite distortion, we can set $\mathrm{K}u(x) = \frac{|\mathrm{D}u|^2}{\mathrm{J}u(x)}$ if $\mathrm{J}u(x) \neq 0$ and $\mathrm{K}u(x) = 1$ otherwise; this function is the (optimal) *distortion* of u.

In Definition 4.6, $|\cdot|$ denotes the operator norm of a matrix. We summarise some of the key analytic and topological properties of mappings of finite distortion in the plane:

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^2$ and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ be a map of finite distortion. Then:*

(i) *u has a continuous representative and, whenever $r < R$ and $B_R(x_0) \subset \Omega$,*

$$(\operatorname{diam} u(B_r(x_0)))^2 \leq \frac{C}{\log(R/r)} \int_{B_R(x_0)} |\mathrm{D}u|^2 \,\mathrm{d}x;$$

(ii) *u has the Lusin (N) property;*
(iii) *u is differentiable a.e. in Ω;*

(iv) *if* $\mathrm{K}u \in L^1(\Omega)$ *then* u *is open and discrete;*
(v) *if* $\mathrm{K}u \in L^1(\Omega)$ *then for each* $\Omega' \Subset \Omega$ *there is* $m = m(\Omega')$ *such that*

$$\mathcal{N}(y, u, \Omega') \leq m \qquad \text{for all } y \in u(\Omega').$$

Whenever u is a map of finite distortion we always implicitly assume that u denotes the continuous representative of the equivalence class in $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$. If u is such that $\mathrm{K}u \in L^1(\Omega)$, we say that u has *integrable distortion*; the theory of such maps was pioneered in [51].

We remark that the first three properties of Theorem 4.7 are a consequence of the fact that mappings of finite distortion are *monotone in the sense of Lebesgue*:

Proposition 4.8. *Let* $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ *be a map of finite distortion; then* (4.2) *holds. In fact, if we measure the diameter in* $\mathbb{R}^2$ *with respect to the* ℓ^∞ *norm, we can take* $K = 1$.

4.3. Existence of admissible solutions

In this subsection we focus on the case $n = 2$ for simplicity and we assume throughout that $\mathrm{J}\colon \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2) \to \mathscr{H}^p(\mathbb{R}^2)$ is surjective. We are particularly interested in the case $p = 1$. Our goal is to illustrate the way in which Theorem B yields the following principle:

the existence of rough solutions implies the existence of well-behaved solutions.

The following is an example a rough solution, and something that we would like to avoid:

Example 4.9 *([58]).* There is a map $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ such that

$$\mathrm{J}u = 0 \text{ a.e. in } \mathbb{R}^2 \qquad \text{and} \qquad u([0,1] \times \{0\}) = [0,1]^2.$$

In particular, u does not have the Lusin (N) property and (4.3) does not hold.

The main result of this subsection is the following theorem, which shows that in some sense it suffices to deal with non-pathological solutions.

Theorem 4.10. *Let* $\Omega \subset \mathbb{R}^2$ *be an open set and take* $f \in \mathscr{H}^1(\mathbb{R}^2)$ *such that* $f \geq 0$ *in* Ω. *Assume that* $\mathrm{J}\colon \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2) \to \mathscr{H}^1(\mathbb{R}^2)$ *is onto. Then there is a solution* $u \in \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ *of* (1.4) *such that:*

(i) u *is continuous in* Ω*;*
(ii) u *has the Lusin (N) property in* Ω.
(iii) $\int_{\mathbb{R}^2} |\mathrm{D}u|^2 \,\mathrm{d}x \leq C\|f\|_{\mathscr{H}^1}$ *with* $C > 0$ *independent of* f.

In particular, u satisfies the change of variables formula (4.3). Moreover, let $\Omega' \Subset \Omega$ be an open set such that $f = 0$ a.e. in Ω'. Then:

(iv) *for any bounded set $E \subseteq \Omega'$, we have $u(\partial E) = u(\overline{E})$;*
(v) *for $y \in u(\Omega')$, if C denotes a connected component of $u^{-1}(y) \cap \Omega'$ then C intersects $\partial\Omega'$.*

Before proceeding with the proof, we note that (iv) is a type of degenerate monotonicity which had already appeared in the study of the hyperbolic Monge–Ampère equation [19,52].

Proof. The point of the proof is to perturb f appropriately; then the solution u is obtained as a limit of mappings of integrable distortion.

Take an exhaustion of Ω by bounded open sets Ω_j: that is, $\Omega_j \subset \Omega_{j+1}$ and $\bigcup_{j=1}^\infty \Omega_j = \Omega$. Fix j and let B_j^+ be a ball containing Ω_j and let B_j^- be another ball, disjoint from Ω_j, and with the same volume as B_j^+. Consider the perturbations

$$f_{\varepsilon,j} \equiv f + \varepsilon a_j, \qquad a_j \equiv \chi_{B_j^+} - \chi_{B_j^-},$$

which satisfy $f_{\varepsilon,j} \geq \varepsilon$ a.e. in Ω_j. Clearly $a_j \in \mathscr{H}^1(\mathbb{R}^2)$, being bounded, compactly supported and with zero mean. Hence, as $\varepsilon \to 0$, $f_{\varepsilon,j} \to f$ in $\mathscr{H}^1(\mathbb{R}^2)$ and, from Corollary C, we see that we can choose solutions $u_{\varepsilon,j}$ of $\mathrm{J}u_{\varepsilon,j} = f_{\varepsilon,j}$ such that, for all $\varepsilon > 0$,

$$\int_{\mathbb{R}^2} |\mathrm{D}u_{\varepsilon,j}|^2 \leq C\|f_{\varepsilon,j}\|_{\mathscr{H}^1} \leq C(j).$$

Since the maps $u_{\varepsilon,j}$ have integrable distortion in Ω_j, we can apply Theorem 4.7(i) to conclude that the family $(u_{\varepsilon,j})_\varepsilon$ is equicontinuous; we also normalise the maps so that $u_{\varepsilon,j}(x_0) = 0$ for some fixed $x_0 \in \Omega_1$. Therefore, by taking a diagonal subsequence of $(u_{\varepsilon,j})$, we get a sequence (u_k) which converges both locally uniformly in Ω and weakly in $\dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ to a limit u. This already proves (i) and (iii).

To prove (ii) it suffices to show that $|u(E)| = 0$ whenever $E \subset \Omega$ is a *bounded* null set; we may therefore assume that $E \subset \Omega_j$ for some large j. We note that, for each $\varepsilon > 0$, the map $u_{\varepsilon,j}$ satisfies (4.2), cf. Proposition 4.8. In Ω_j, u is the uniform limit of a subsequence of $(u_{\varepsilon,j})_\varepsilon$, and hence u also satisfies (4.2). Thus, by Proposition 4.4, u has the Lusin (N) property in Ω_j, which implies (ii).

For (iv) we may again suppose that $E \subset \Omega_j \cap \Omega'$ for some large j. Since j is fixed we write for simplicity $u_\varepsilon \equiv u_{\varepsilon,j}$ in the rest of the proof. As u_ε has integrable distortion in Ω_j, it is open in Ω_j and hence $\partial u_\varepsilon(E) \subseteq u_\varepsilon(\partial E)$. Suppose that there is $y \in u(\overline{E}) \backslash u(\partial E)$. On the one hand, there is some $\delta > 0$ such that, for all ε small enough,

$$B_\delta(y) \cap \partial u_\varepsilon(\operatorname{int} E) \subset B_\delta(y) \cap u_\varepsilon(\partial E) = \emptyset;$$

on the other hand, since $y \in u(\operatorname{int} E)$, for all ε small enough,

$$B_\delta(y) \cap u_\varepsilon(\operatorname{int} E) \neq \emptyset.$$

It follows that $B_\delta(y) \subseteq u_\varepsilon(\operatorname{int} E)$. We also have that $|u_\varepsilon(\operatorname{int} E)| \to 0$ as $\varepsilon \to 0$: by the change of variables formula,

$$|u_\varepsilon(\operatorname{int} E)| \leq \int\limits_{u_\varepsilon(\operatorname{int} E)} \mathcal{N}(y, u_\varepsilon, \operatorname{int} E) \, \mathrm{d}y = \int\limits_E \mathrm{J} u_\varepsilon = \varepsilon|E| \to 0.$$

Thus, since $|B_\delta(y)| \leq |u_\varepsilon(E)|$, by sending $\varepsilon \to 0$ we see that no such y can exist. Hence the sets $u(\overline{E})$ and $u(\partial E)$ are the same.

Finally, (v) follows from (iv), as shown for instance in [52, Lemma 2.10]. □

In view of the change of variables formula, it is useful to control the multiplicity function: this seems crucial, for instance, if one intends to disprove the surjectivity of the Jacobian through a geometric argument involving perturbations of an appropriate datum. In that direction we have the following proposition, in which we again assume that the Jacobian is surjective.

Proposition 4.11. *Let $\Omega \subset \mathbb{R}^2$ be an open set and let $Y \equiv \{f \in \mathscr{H}^p(\mathbb{R}^2) : f \geq c$ a.e. in $\Omega\}$, where $c > 0$. Suppose that $f_j \in Y$ is a sequence converging weakly to f in $\mathscr{H}^p(\mathbb{R}^2)$. For any maps $u_j \in \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$ satisfying $\mathrm{J}u_j = f_j$ and the a priori estimate (1.6), we have that*

$$\sup_j \sup_{y \in u_j(\Omega')} \mathcal{N}(y, u_j, \Omega') < \infty, \qquad \text{whenever } \Omega' \Subset \Omega.$$

Proof. We claim that the sequence u_j is equicontinuous and converges to $u \in \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$, a solution of $\mathrm{J}u = f$, uniformly in Ω'. Once the claim is proved, the conclusion follows: u has integrable distortion in Ω and so by Theorem 4.7(v) it is at most m-to-one in Ω', for some $m \in \mathbb{N}$. Thus, for all j sufficiently large, u_j is also at most m-to-one in Ω': if not, there are arbitrarily large j and points $x_1^{(j)}, \dots, x_{m+1}^{(j)} \in \Omega'$ such that $u_j(x_i^{(j)}) = y$ for some $y \in \mathbb{R}^n$ and all $i \in \{1, \dots, m+1\}$. By compactness, we can further assume that $x_i^{(j)} \to x_i$ for $i = 1, \dots, m+1$. However, there are at least two different points $y_1 \neq y_2$ such that

$$\{y_1, y_2\} \subset u(\{x_1, \dots, x_{m+1}\});$$

for the sake of definiteness, say $u(x_1) = y_1, u(x_2) = y_2$. Let $\varepsilon < |y_1 - y_2|$ and take j sufficiently large so that, for $i = 1, 2$,

$$|u_j(x_i^{(j)}) - u_j(x_i)| < \frac{\varepsilon}{4}, \qquad |u_j(x_i) - u(x_i)| < \frac{\varepsilon}{4};$$

this is possible from equicontinuity of the sequence u_j and the fact that it converges to u uniformly. The triangle inequality gives $|y_1 - y_2| = |u(x_1) - u(x_2)| < \varepsilon$, a contradiction.

To prove the claim, we assume that the Jacobian is surjective and we use Corollary C. If $p > 1$ we appeal to Morrey's inequality,

$$[u_j]_{C^{0,1-2/p}(\mathbb{R}^2)} \lesssim_p \|\mathrm{D}u_j\|_{L^{2p}(\mathbb{R}^2)} \le C,$$

while for $p = 1$ we use Theorem 4.7(i) instead. Either way, after normalizing the maps so that $u_j(x_0) = 0$, where $x_0 \in \Omega$, we see that the sequence (u_j) is precompact in the local uniform topology over Ω'. Hence we may assume that u_j converges to some map $u \in \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$ uniformly in Ω' and also weakly in $\dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$. □

4.4. A case study

We illustrate the use of Corollary C and Theorem D via an example, and as above we confine ourselves to the case $n = 2$. Consider piecewise constant data supported on thinning annuli: with $\mathbb{A}(r, R) \equiv \{x \in \mathbb{R}^2 : r < |x| < R\}$, set

$$f_j \equiv \chi_{\mathbb{A}(1,\sqrt{1+1/j})} - \chi_{\mathbb{A}(\sqrt{1+1/j},\sqrt{1+2/j})} \in \mathscr{H}^1(\mathbb{R}^2). \tag{4.4}$$

We remark that we used similar data to construct counterexamples to the existence of solutions with low regularity data in the case of the Dirichlet problem in [40]. For data of the type (4.4) it is not entirely clear if solutions $u_j \in \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ exist; moreover, even if we assume that u_1 exists, there seems to be no obvious argument to construct u_j from u_1.

Using Corollary C, in order to conclude non-surjectivity of $\mathrm{J}\colon \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2) \to \mathscr{H}^1(\mathbb{R}^2)$ it suffices to assume the existence of solutions and to prove that they must satisfy

$$\lim_{j\to\infty} \frac{\int_{\mathbb{R}^2} |\mathrm{D}u_j|^2 \,\mathrm{d}x}{\|f_j\|_{\mathscr{H}^1}} = \infty. \tag{4.5}$$

More specifically, we propose to study the question of whether the solutions u_j constructed in Theorem D satisfy $\inf_{j\in\mathbb{N}} \int_{B(0,1)} |\mathrm{D}u_j|^2 \,\mathrm{d}x > 0$. Note that despite the fact that $f_j \equiv 0$ in $B(0,1)$, one cannot have $u_j \equiv 0$ in $B(0,1)$, as then u_j would have integrable distortion in $B(0, \sqrt{1 + 1/j})$, in contradiction with a theorem of Iwaniec and Šverák from [51]. Theorem D also puts further conditions on solutions u_j which prohibit various kinds of pathological behaviour within $B(0,1)$. It is also natural to study $\inf_{j\in\mathbb{N}} \int_{B(0,1)} |\mathrm{D}v_j|^2 \,\mathrm{d}x$ for *energy-minimal solutions*, that is, solutions of $\mathrm{J}v_j = f_j$ satisfying $\int_{\mathbb{R}^2} |\mathrm{D}v_j|^2 \,\mathrm{d}x = \min\{\int_{\mathbb{R}^2} |\mathrm{D}u|^2 \,\mathrm{d}x : \mathrm{J}u = f_j\}$. Energy-minimal solutions are studied at length in [39,55].

We indicate some of the difficulties one would run into without Corollary C and Theorem D, *even if* (4.5) *were to be proved.* Assuming (4.5), it is by no means clear what kind of datum $f \in \mathscr{H}^1(\mathbb{R}^2)$ would be outside $\mathrm{J}(\dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2))$. By scaling, one could assume that $\|\mathrm{D}v_j\|_{L^2} = 1$ and pass to a weakly convergent subsequence, but it seems difficult to say anything definite about the weak limit.

A plausible way to get a concrete datum $f \in \mathscr{H}^1(\mathbb{R}^2) \setminus \mathrm{J}(\dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2))$ would be to consider an infinite sum $f \equiv \sum_{j=1}^{\infty} c_j f_j(\lambda_j x - x_j)$ of scaled and translated copies of f_j with mutually disjoint supports. If the energy-minimal solutions for f_j were compactly supported, we could conclude the non-surjectivity of $\mathrm{J}\colon \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2) \to \mathscr{H}^1(\mathbb{R}^2)$ from (4.5) by patching together maps with mutually disjoint supports. However, in general, and for such f_j in particular, energy-minimal solutions must be supported on the whole plane [55, Proposition 8.4], which rules out such a patching strategy. It also seems rather non-trivial to control the supports of the solutions given by Theorem D.

As another candidate collection of data we propose $f_j \equiv \chi_{(-1,0)\times(0,1/j)} - \chi_{(0,1)\times(0,1/j)}$ —in this case, a compactly supported Lipschitz solution u_1 for f_1 was constructed by the third author at [55, p. 59]. Now the simple scaling $u_j(x_1, x_2) \equiv u_1(x_1, jx_2)/\sqrt{j}$ gives a solution for $\mathrm{J}u_j = f_j$ for each $j \in \mathbb{N}$, but such solutions satisfy (4.5). It is, again, natural to study whether (4.5) holds for the solutions of Theorem D and energy-minimal solutions.

5. Applications to incompressible fluid mechanics

This section is dedicated to illustrating the practical use of the nonlinear open mapping principles in evolutionary problems. As examples we consider the incompressible Euler and Navier–Stokes equations, proving in particular Corollary E. In combination with scaling analysis, Theorem 3.8 is used to rule out incorrectly scaling solution spaces. Besides being useful to prove non-solvability, this strategy also gives an elementary way of proving upper bounds on the energy dissipation rates for Baire-generic initial data.

5.1. The incompressible Euler equations and the proof of Corollary E

Our next aim is to prove Corollary E on the incompressible Euler equations in $\mathbb{R}^n \times [0, \infty)$, $n \geq 2$. Recall that given $u^0 \in L^2_\sigma$, a mapping $u \in L^p_t L^2_{\sigma,x}$, $2 \leq p \leq \infty$, is a weak solution of the Cauchy problem (1.7)–(1.9) if

$$\int_0^\infty \int_{\mathbb{R}^n} (u \cdot \partial_t \varphi + u \otimes u : \mathrm{D}\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^n} u^0 \cdot \varphi(\cdot, 0)\,\mathrm{d}x = 0 \quad \forall \varphi \in C^\infty_{c,\sigma}(\mathbb{R}^n \times [0,\infty), \mathbb{R}^n). \tag{5.1}$$

We cannot deduce Corollary E directly via Theorem 3.5. Indeed, the integral condition (5.1) leads to a well defined mapping T from a weak solution $u \in L^p_t L^2_{\sigma,x}$ of (1.7)–(1.9) to the initial data $u^0 \in L^2_\sigma$ but does not easily lend itself to a domain of definition $D \subset L^p_t L^2_{\sigma,x}$ satisfying condition $\widehat{\text{(A4)}}$ of Theorem 3.5. We therefore consider a *relaxed problem*, where $u \otimes u \in L^{p/2}_t L^1_x$ is replaced by a general matrix-valued mapping S.

In order to apply Theorem 3.8 we embed $L^1(\mathbb{R}^n, \mathbb{R}^{n\times n})$ into the space of signed Radon measures $\mathbf{M}(\mathbb{R}^n, \mathbb{R}^{n\times n})$ which is the dual of the separable Banach space $C_0(\mathbb{R}^n, \mathbb{R}^{n\times n})$.

We endow $\mathbf{M}(\mathbb{R}^n, \mathbb{R}^{n\times n})$ with the dual norm. In the relaxed problem we require $u \in L^p_t L^2_{\sigma,x}$ and $S \in L^{p/2}_t \mathbf{M}_x$ to satisfy

$$\int_0^\infty \int_{\mathbb{R}^n} (u \cdot \partial_t \varphi + S : \mathrm{D}\varphi) \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^n} u^0 \cdot \varphi(\cdot, 0)\,\mathrm{d}x = 0 \quad \forall \varphi \in C^\infty_{c,\sigma}(\mathbb{R}^n \times [0,\infty), \mathbb{R}^n). \tag{5.2}$$

Unlike (5.1), *due to linearity, condition* (5.2) *is stable under weak* convergence*. Furthermore, the linear Proposition 3.7 is applicable to this relaxed problem.

Relaxations such as (5.2) are studied in *Tartar's framework*, where a system of nonlinear PDEs is decoupled into a set of linear PDEs (conservation laws) and pointwise constraints (constitutive laws) [79,80]. Tartar's framework has been very useful in convex integration both in the Calculus of Variations [63,64], as well as in fluid dynamics [27,28]. Specific constitutive laws do not play a role in the proof of Corollary E, and in fact, an analogous result holds for many other incompressible models of fluid mechanics. The result also trivially extends to *subsolutions*, that is solutions of the linear equations which take values in the so-called Λ-convex hull. Subsolutions can be interpreted as coarse-grained averages, see e.g. [18,29].

Corollary E follows immediately from the next lemma:

Lemma 5.1. *Let $n \geq 2$, $M > 0$ and $p \in (2, \infty)$. It is only for a nowhere dense set of data $u^0 \in L^2_\sigma$ that there exists a solution $(u, S) \in L^p_t L^2_{\sigma,x} \times L^{p/2}_t \mathbf{M}_x$ of* (5.2) *with $\|u\|_{L^p_t L^2_{\sigma,x}} \leq M$.*

Proof. Denote $D = \{(u, S) \in L^p_t L^2_{\sigma,x} \times L^{p/2}_t \mathbf{M}_x \colon$ (5.2) holds for some $u^0 \in L^2_\sigma\}$ and define $T \colon D \to L^2_\sigma$ by $T(u, S) \equiv u^0$. The linear map T clearly satisfies the assumptions of Proposition 3.7. Our intention is to verify the assumptions of Theorem 3.8.

Let $(u, S) \in D$. Given $\lambda > 0$ we set

$$u_\lambda(x,t) \equiv u\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right), \quad S_\lambda(x,t) \equiv S\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right), \quad u^0_\lambda(x,t) \equiv u^0\left(\frac{x}{\lambda}\right). \tag{5.3}$$

Now (5.2)–(5.3) imply that $(u_\lambda, S_\lambda) \in D$ and $T(u_\lambda, S_\lambda) = u^0_\lambda$. We compute

$$\|u_\lambda\|_{L^p_t L^2_x} = \lambda^{\frac{n}{2}+\frac{1}{p}} \|u\|_{L^p_t L^2_x}, \quad \|S_\lambda\|_{L^{p/2}_t \mathbf{M}_x} = \lambda^{n+\frac{2}{p}} \|S\|_{L^{p/2}_t \mathbf{M}_x}, \quad \|u^0_\lambda\|_{L^2} = \lambda^{\frac{n}{2}} \|u^0\|_{L^2}.$$

We set $\widetilde{u_\lambda} \equiv \lambda u$ and $\widetilde{f_\lambda} \equiv \lambda f$. In the notation of Theorem 3.8, we have

$$s_1 = \frac{n}{2} \quad \tilde{s}_1 = 1$$
$$r_{1,1} = \frac{n}{2} + \frac{1}{p} \quad \tilde{r}_{1,1} = 1 \quad r_{2,1} = n + \frac{2}{p} \quad \tilde{r}_{2,1} = 1.$$

Thus the claim follows from Theorem 3.8. □

We conclude this subsection by briefly comparing Corollary E with the existing literature and we focus on the case $n = 2$, where the picture is more complete. Following [28], we say that an initial datum u^0 is *wild* if (1.7)–(1.9) admits infinitely many admissible weak solutions. Combining the results of [76] with [57, Theorem 4.2], we arrive at the following:

Theorem 5.2. *When $n = 2$, the set of wild initial data is a dense, meagre subset of L^2_σ.*

We also note that some wild initial data admit compactly supported solutions [28], while Corollary E shows that such solutions exist only for a meagre set of initial data. In fact from Corollary E we deduce immediately the following:

Corollary 5.3. *Take $\tau > 0$ and $M > 0$. A solution $u \in L^\infty_t L^2_{\sigma,x}$ with $\operatorname{supp}(u) \subset \mathbb{R}^n \times [0,\tau]$ and $\|u\|_{L^\infty_t L^2_{\sigma,x}} \le M\|u^0\|_{L^2_\sigma}$ exists only for a nowhere dense set of data $u^0 \in L^2_\sigma$.*

5.2. Energy decay rate in the Navier–Stokes equations

We also illustrate the use of Theorem 3.8 in the presence of viscosity; we use the Navier–Stokes equations in $\mathbb{R}^n \times [0,\infty)$, $n \ge 2$, as an example. Given an initial datum $u^0 \in L^2_\sigma$, recall that weak solutions of (3.1)–(3.3) were defined in $L^\infty_t L^2_{\sigma,x} \cap L^2_t \dot{H}^1_x$ in §3.1. Furthermore, a weak solution is called a *Leray–Hopf solution* if it satisfies the energy inequality

$$\frac{1}{2}\int_{\mathbb{R}^3} |u(x,t)|^2 \, \mathrm{d}x + \nu \int_s^t \int_{\mathbb{R}^3} |\mathrm{D}u(x,\tau)|^2 \, \mathrm{d}x \, \mathrm{d}\tau \le \frac{1}{2}\int_{\mathbb{R}^3} |u(x,s)|^2 \, \mathrm{d}x \qquad \text{for all } t > s$$

for a.e. $s \in [0,\infty)$, including $s = 0$. Leray showed in his milestone paper [54] that for every initial datum $u^0 \in L^2_\sigma$ there exists a Leray–Hopf solution $u \in L^\infty_t L^2_{\sigma,x} \cap L^2_t \dot{H}^1_x$ with $u(\cdot,0) = u^0$. We briefly recall some of the pertinent results on energy decay of Leray–Hopf solutions and refer to the recent review [11] for more details and references.

Leray asked in [54] whether $\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}^3} |u(x,t)|^2 \, \mathrm{d}x \to 0$ as $t \to \infty$ for all Leray–Hopf solutions. An affirmative answer was given by theorems of Kato and Masuda, see [11, Theorem 2–3]. Schonbeck has shown that there is no uniform energy decay rate for general data $u^0 \in L^2_\sigma$; more precisely, for every $\beta, \varepsilon, T > 0$ there exists $u^0 \in \beta \mathbb{B}_{L^2_\sigma}$ such that a Leray–Hopf solution satisfies $\mathcal{E}(T) \ge (1-\varepsilon)\mathcal{E}(0)$. Furthermore, whenever $u^0 \in L^2_\sigma \setminus \cup_{1 \le p < 2} L^p$, the energy $\mathcal{E}(t)$ does not undergo polynomial decay. Several precise statements on the decay rate of $\mathcal{E}(t)$ under extra integrability assumptions on $u^0 \in L^2_\sigma$ are given in [11].

In Corollary 5.5 below, we recover the lack of polynomial decay for a Baire-generic datum. The result applies to all *distributional solutions* of (3.1)–(3.3), which we define as mappings $u \in L^2_{\mathrm{loc},t} L^2_{\sigma,x}(\mathbb{R}^3 \times [0,\infty), \mathbb{R}^3)$ such that

$$\int_0^\infty \langle u, \partial_t \varphi\rangle \,\mathrm{d}t + \int_0^\infty \langle u\otimes u, \mathrm{D}\varphi\rangle \,\mathrm{d}t + \nu \int_0^\infty \langle u, \Delta\varphi\rangle \,\mathrm{d}t + \langle u^0, \varphi(0)\rangle = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R}^3\times[0,\infty),\mathbb{R}^3)$ with $\operatorname{div}\varphi = 0$.

Proposition 5.4. *Let $p\in(2,\infty)$ and $M>0$. It is only for a nowhere dense set of initial data that (3.1)–(3.3) admits a distributional solution with $\|u\|_{L^p_t L^2_x}\le M$.*

Proof. We consider the relaxed problem where we require $u\in L^p_t L^2_{\sigma,x}$, $S^1\in L^{p/2}_t \mathbf{M}_x$ and $S^2\in L^p_t \dot H^{-1}_x$ to satisfy

$$\int_0^\infty \langle u, \partial_t \varphi\rangle \,\mathrm{d}t + \int_0^\infty \langle S^1, \mathrm{D}\varphi\rangle \,\mathrm{d}t + \nu \int_0^\infty \langle S^2, \mathrm{D}\varphi\rangle \,\mathrm{d}t + \langle u^0, \varphi(0)\rangle = 0 \tag{5.4}$$

for all $\varphi\in C_c^\infty([0,\infty),\mathbb{R}^3)$ with $\operatorname{div}\varphi=0$. As before, denote by

$$D\subset L^p_t L^2_{\sigma,x}\oplus L^{p/2}_t\mathbf{M}_x\oplus L^p_t\dot H^{-1}_x \equiv X^*$$

the set of triples (u,S^1,S^2) such that (5.4) holds. Again, the assumptions of Proposition 3.7 are satisfied. We set $u_\lambda(x,t)=u(x/\lambda,t/\lambda)$ and $S^i(x,t)=S^i(x/\lambda,t/\lambda)$. A simple computation gives

$$\|u_\lambda\|_{L^p_t L^2_x} = \lambda^{n/2+1/p}\|u\|_{L^p_t L^2_x}, \qquad \|S^1_\lambda\|_{L^{p/2}_t\mathbf{M}_x} = \lambda^{n+2/p}\|S^1\|_{L^{p/2}_t\mathbf{M}_x},$$
$$\|u^0_\lambda\|_{L^2} = \lambda^{n/2}\|u\|_{L^2}, \qquad \|S^2_\lambda\|_{L^p_t\dot H^{-1}_x} = \lambda^{n/2+1+2/p}\|S^2\|_{L^p_t\dot H^{-1}_x}.$$

As before, we set $\widetilde{u_\lambda}\equiv\lambda u$ and $\widetilde{f_\lambda}\equiv\lambda f$. The claim now follows from Theorem 3.8. □

Corollary 5.5. *Let $\varepsilon>0$. For a Baire-generic initial datum $u^0\in L^2_\sigma$, solutions in $L^\infty_t L^2_{\sigma,x}$ satisfy $\|t^\varepsilon\mathcal{E}\|_{L^\infty(\tau,\infty)}=\infty$ for every $\tau>0$.*

Proof. Given $\varepsilon>0$ choose $p>2/\varepsilon$. By Proposition 5.4, for a Baire generic initial datum $u_0\in L^2_\sigma$ there is no solution $u\in(L^p\cap L^\infty)_t L^2_{\sigma,x}$. Given such a datum u_0, suppose, by way of contradiction, that a solution $u\in L^\infty_t L^2_{\sigma,x}$ satisfies $\|t^\varepsilon\mathcal{E}\|_{L^\infty(\tau,\infty)}<\infty$. Thus there exists $M>0$ such that $\mathcal{E}(t)\le Mt^{-\varepsilon}$ for a.e. $t\ge\tau$. Now $\int_\tau^\infty\|u(t)\|^p_{L^2}\,\mathrm{d}t\le M^{p/2}\int_\tau^\infty t^{-p\varepsilon/2}\,\mathrm{d}t<\infty$ so that $u\in L^p_t L^2_{\sigma,x}$. We have reached the sought contradiction. □

Remark 5.6. In view of Wiedemann's results in [82], *the analogue of Corollary E is false on the torus $\mathbb{T}^n$*. An analogous remark applies to the Navier-Stokes equations on $\mathbb{R}^2$ and $\mathbb{R}^3$. Indeed, on the torus and on smooth, bounded domains, all Leray-Hopf solutions have an exponentially decaying kinetic energy from some time point on. More detailed information on the decay rate of energy and enstrophy in bounded domains and the torus can be found in [35]. Moreover, recently, Buckmaster and Vicol have constructed $C_tH_x^\beta$-regular, $\beta > 0$, mild solutions in $\mathbb{T}^3 \times [0,T]$ with a prescribed smooth, non-negative kinetic energy profile [13]. In particular, mild solutions are non-unique and may come to rest in finite time.

6. Concluding discussion

In this section, we discuss the advantages of the nonlinear open mapping principles proved in this paper, when compared to the classical Banach–Schauder theorem. We also point out some of the limitations of our results, as well as directions for future work.

We begin by recalling the standard proof of the Banach–Schauder open mapping theorem in a special case. If a bounded *linear* map $L\colon X \to Y$ between Banach spaces is surjective and X is reflexive, the Baire category theorem then yields a constant $C > 0$ and a ball $B_Y(f_0, r)$ such that $L(B_X(0,C)) \supseteq B_Y(f_0,r)$, and the proof is completed as follows. First, by linearity, $L(-B_X(0,C)) = -L(B_X(0,C)) \supseteq -B_Y(f_0,r)$, so that, by linearity again,

$$L(B_X(0,2C)) = L(B_X(0,C)) - L(B_X(0,C)) \supset B_Y(f_0,r) - B_Y(f_0,r) = B_Y(0,2r).$$

We notice that this proof uses in a fundamental way three properties:

(i) the linearity of the operator L;
(ii) the vector space structure of the domain of definition of L;
(iii) the symmetry of the range of L.

Concerning (i), we note that if one attempts to generalise the above proof to nonlinear operators, then surjectivity only leads to "1/2-openness" and, more generally, $1/n$-surjectivity leads to $1/2n$-openness. To our knowledge, Theorem B and Proposition 2.5 give the first abstract results on Rudin's problem, cf. Question 1.1, which yield $1/n$-openness from $1/n$-surjectivity.

With respect to (ii), another key novelty of our work is that the domain of definition D of the operator T need not be a vector space. This is crucial when applying open mapping theorems to typical Cauchy problems in nonlinear evolutionary PDEs as is done in §3–5.

Finally, we note that (iii) is not needed for our results either. In fact, Theorems B and 3.5 apply when the target space is a closed convex cone such as $\{f \in L^p(\mathbb{R}^n)\colon f \geq 0 \text{ a.e.}\}$, for $1 < p < \infty$, cf. Remark 3.6, and also when the symmetry of the range is non-trivial to check, as is the case for the Hessian operator $\mathrm{H}\colon \dot{W}^{1,2}(\mathbb{R}^2) \to \mathscr{H}^1(\mathbb{R}^2)$.

We now discuss some of the limitations of our work. From a PDE perspective, the main weak point of Theorem 3.5 is that assumption $\widehat{\text{(A3)}}$ seems difficult to adapt to function spaces defined over the flat torus $\mathbb{T}^n$ or bounded domains. For instance, on

$\mathbb{T}^n$, translations $\sigma_k^{Y^*} f(x) = f(x - ke)$ typically fail the condition $\sigma_k^{Y^*} f \overset{*}{\rightharpoonup} 0$. On $\mathbb{R}^n$, translations can often be replaced by scalings such as $\sigma_k^{X^*} f(x) = k^\alpha f(kx)$, but such operators are of course not invertible on function spaces over the torus or bounded domains.

Despite the fact that Theorem 3.5 applies to many different equations, it would be interesting to look for generalisations, in order to account for other physical PDEs. Note, for instance, that even if one does not assume that T is positively homogeneous, as in Theorem B, the proof of this theorem still provides $\delta, M_\delta > 0$ such that $T(B_{X^*}(0, M_\delta)) \supseteq B_{Y^*}(0, \delta)$. It thus seems natural to ask whether one can achieve openness at the origin, i.e., whether one gets $\lim_{\delta \searrow 0} M_\delta = 0$. Another interesting problem is to decide whether the weak*-to-weak* closed graph assumption on the operators is an artifact of our proofs or a fundamental requirement for the validity of a nonlinear open mapping principle. We hope to address these questions in future work.

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