Wobbly moduli of chains, equivariant multiplicities and U(n0,n1)-Higgs bundles

Peón-Nieto, Ana

WOBBLY MODULI OF CHAINS, EQUIVARIANT MULTIPLICITIES AND $\mathrm{U}(n_0, n_1)$-HIGGS BUNDLES

ANA PEÓN-NIETO

ABSTRACT. We give a birational description of the reduced schemes underlying the irreducible components of the nilpotent cone and the $\mathbb{C}^\times$-fixed point locus of length two in the moduli space of Higgs bundles. By producing criteria for wobbliness, we are able to determine wobbly fixed point components of type (n_0, n_1) and prove that these are precisely $\mathrm{U}(n_0, n_1)$-wobbly components. We compute the virtual equivariant multiplicities of fixed points as defined by Hausel–Hitchin and find that they are polynomial for all partitions other than $(2, 1)$ and $(4, 3)$. In particular, this proves that they provide an obstruction to the existence of very stable fixed points only for very specific components.

CONTENTS

To Candela and Inés

1. INTRODUCTION

The moduli space $\mathrm{M}_X(n, d)$ of Higgs bundles of rank n and degree d on a Riemann surface X of genus at least two has been an object of intense study since their introduction by Hitchin 35 years ago [Hi1]. In spite of their already long history, they keep proving central through many applications both in geometry and theoretical physics. To cite a few examples, Higgs bundles appear in relation with integrable systems [Hi2], mirror symmetry in its many forms [HT, KW, GWZ, H], quantisation [G, GW], and the proof of geometric Langlands [DP, DP2].

In this article, we focus on some key objects in the geometry of moduli spaces of Higgs bundles. These are the fixed points of a $\mathbb{C}^\times$-action existing in the moduli space of Higgs bundles [S]. Such points determine the topology of $\mathrm{M}_X(n,d)$. Indeed, taking limits to zero retracts $\mathrm{M}_X(n,d)$ to the nilpotent cone, the fiber over zero of the Hitchin map. More precisely, it hits the fixed points under the $\mathbb{C}^\times$-action, which can also be described in terms of moduli spaces of chains [AG]. Through this action, it is possible to construct different interesting objects from fixed points. For instance, the components of the nilpotent cone can be constructed as upward flows to fixed point components (that is, taking limits at infinity to a given fixed point component). Another instance, meaningful in mirror symmetry, is the production of $\mathbb{C}^\times$-invariant branes such as downward flows to the so called very stable Higgs bundles [HH, H]. This notion, very stability, and its opposite, wobbliness, are at the core of many central problems, such as the determination of multiplicities of the components of the nilpotent cone [HH, Hi3], the definition of divisors inside $\mathrm{M}_X(n,d)$ [La, PPa, P], and the already cited mirror symmetry of Hitchin systems [HT, H] and geometric Langlands via abelianization of Higgs bundles [DP2, DP]. This explains the increasing interest in such objects. The present paper contributes by providing criteria for wobbliness of fixed points of length two, which allows to determine the components with very stable points, namely, those to which the existing techniques apply. Our work is motivated by the role played by wobbly bundles in several important strands of research.

Indeed, the study of very stable bundles can be traced back to the work of Laumon [La] on the nilpotent cone of the moduli space of Higgs bundles. Laumon, following Drinfeld, introduced very stable bundles, and proved that they form a dense open set in the moduli space of stable bundles. He announced the nowadays called Drinfeld conjecture. According to this, the complement of very stable bundles (subsequently named "wobbly" by Donagi–Pantev [DP]) is of pure codimension one. This result was proven by Pal–Pauly [PPa] in rank two, and by Pal [P] for arbitrary rank.

Wobbly bundles are also crucial in relation with geometric Langlands from abelianisation of Higgs bundles [DP]. According to Donagi–Pantev, the right setup towards this programme involves the study of parabolic Higgs bundles on the moduli space of vector bundles minus the "shaky" divisor. They conjectured the equality of the wobbly and shaky loci, a proof of which in the smooth moduli space was provided by the author [Pe]. A toy model of Donagi–Pantev's programme was produced by Hausel–Hitchin [HH] using fixed points of maximal nilpotent order rather than minimal (namely, the moduli space, involved in geometric Langlands).

More recently, criteria for wobbliness of fixed points in terms of properness of the Hitchin map [PPe1, Z, HH] has opened the way to the computation of multiplicities of the irreducible components of the nilpotent cone [HH], only known until then for the moduli space of bundles [BNR] and the Hitchin section [BR]. Indeed, downward flows to very stable fixed points are proper subvarieties of the moduli space, intersecting the nilpotent cone with generic multiplicity [HH]. Now, although very stability is generic, it can be empty [HH, PPe2]. Components with no very stable points are called wobbly, and very stable otherwise. The current knowledge of the geometry of very stable components is thus deeper than that of wobbly components. The determination of which are which is nonetheless unknown. The resolution of this basic problem is necessary in order to understand fundamental questions such as the structure of the nilpotent cone and the related dynamics of the $\mathbb{C}^\times$-flow, amongst others.

An obstruction to very stability of components is provided by Hausel and Hitchin [HH]. They define an invariant, called virtual equivariant multiplicity, which for very stable components recovers the actual multiplicity. These invariants are power series which are polynomial when the component contains a very stable point. Their failure to be such therefore implies wobbliness of the given component. Nonetheless, virtual equivariant multiplicities are known to be polynomial also for some concrete wobbly components (e.g., of type $(3,1)$ [HH]). Hence, they do not characterise wobbliness. In this paper we provide criteria for wobbliness for fixed points of nilpotent order two allowing to characterise wobbly components. We also compute their virtual equivariant multiplicities to gauge the accuracy of this obstruction. We find that they are almost always polynomial, except in ranks 3 and 7, and only in the former case they provide a perfect characterisation of wobbly components. This proves a conjecture of Hausel–Hitchin's [HH]. It also explains several observations in low rank showing different possibilities [HH, PPe2].

In his foundational work, Simpson [S] proved that fixed points are in fact Higgs bundles for real forms. This has been exploited in both directions: results about moduli spaces of real forms have been deduced from those for fixed points and viceversa. For example, the determination of irreducible components of the fixed point locus allowed to prove connectedness of $\mathrm{U}(p,q)$-Higgs bundle moduli spaces by Bradlow–García-Prada–Gothen–Heinloth [BGGH]. Very recently, real forms have been used to produce bounded invariants for moduli spaces of chains by Biquard–Collier–García-Prada–Toledo [BCGT]. This interplay between fixed points and real forms points to the basic question of determining to which extent phenomena such as wobbliness of fixed point components are controlled by the associated real form. In this paper we introduce the notion of $G_{\mathbb{R}}$-very stability and prove that very stability of length two components is equivalent to the a priori weaker notion of $\mathrm{U}(n_0,n_1)$-very stability.

The structure of the present paper is as follows. After a preliminary section (Section 2), we give in Section 3 a birational description of the reduced schemes underlying the fixed points and the associated components of the nilpotent cone with generic Higgs field of nilpotent order two (Theorem 3.6). These can be described in terms of moduli of chains of length two. The fixed points therein have underlying vector bundle of the form $F_0 \oplus F_1$, where $\mathrm{rk}(F_i) = n_i$ and Higgs field $\varphi \in H^0(F_1^* F_0 K)$. Our proof uses the Brill–Noether theoretic results by Russo–Teixidor i Bigas [RT]. In particular, we determine non emptiness of the fixed point components in terms of an invariant δ directly deducible from the Toledo invariant from the theory of $\mathrm{U}(n_0,n_1)$-Higgs bundles by Bradlow–García-Prada–Gothen [BGG] (albeit more natural for Brill-Noether theoretic reasons). This gives an alternative construction to the one in [BGG] in terms of moduli of triples. As an application, we recover minimal dimensionality of the space of sections $H^0(F_1^* F_0 K)$ in Corollary 3.10 (already known for $n_0 > n_1$ through Brill–Noether theory). From this, we give necessary conditions (conjecturally sufficient, see Remark 3.12) for those flowing down to full components of the wobbly divisor in the moduli space of bundles (Corollary 3.11). In Section 4, we give criteria for wobbliness of fixed points of nilpotent order two (Theorem 4.9), which we apply in Section 5 to completely classify wobbly fixed point components for fixed points of order two (Theorem 5.1).

This generalises the results obtained in rank three by the author and Pauly [PPe2], and explains the observations of Hausel–Hitchin for, e.g., partitions of type $(3,1)$ and $(2,1)$. After introducing the notion of $G_{\mathbb{R}}$-very stability (Definition 5.3), we prove the equivalence between wobbliness of length two components and the a priori stronger notion of $\mathrm{U}(n_0,n_1)$-wobbliness (Corollary 5.4). In Section 6 we move on to the computation of the virtual equivariant multiplicities of fixed points (Proposition 6.3), which has interesting consequences. Firstly, we compute the multiplicities of very stable fixed point components (Corollary 6.4). Secondly, we determine the range for which wobbliness of the components is captured by the invariants, i.e., the range for which the invariants are not polynomial (Theorem 6.5 and Corollary 6.6). It turns out that this obstruction occurs only in ranks 3 and 7 and only for partitions of the form (n_1+1,n_1). It moreover completely determines wobbliness only in rank 3 (Corollary 6.4).

Acknowledgements. Many thanks to P. Gothen, T. Hausel, and C. Pauly, for meaningful remarks and discussions. Thanks also to M. Teixidor i Bigas for discussing some aspects of Brill–Noether theory, and M. González and M. Mazzocco for their comments on a preliminary version of this manuscript.

2. Preliminaries and notation

2.1. **Higgs bundles, the Hitchin map, and fixed points.** Consider the moduli space $\mathrm{M}_X(n,d)$ of Higgs bundles of rank n and degree d on a Riemann surface X of genus at least two. Its closed points are S-equivalence classes of pairs (E,φ) where E is a vector bundle on X and $\varphi\in H^0(\mathrm{End}(E)\otimes K)$ where K is the canonical bundle of X. Recall the definition of semistability:

Definition 2.1. *The slope of a vector bundle (E,φ) of degree $\deg(E)$ and rank $rk(E)$ is the quotient*

$$\mu(E,\varphi)=\frac{\deg(E)}{rk(E)}.$$

A Higgs bundle (E,φ) is stable (resp. semistable) if for every subbundle $0\subsetneq F\subsetneq E$ such that $\varphi(F)\subseteq F\otimes K$, it holds that

$$\mu(F)<\mu(E)\qquad(\mathrm{resp.}\mu(F)\le\mu(E)).$$

Let

$$h:\mathrm{M}_X(n,d)\longrightarrow B_n=\bigoplus_{i=1}^n H^0(K^i)\qquad(E,\varphi)\mapsto\det(xId-\varphi)$$

be the Hitchin map. It is a proper map whose fibers are complex Lagrangians of pure dimension. The fiber over zero, the so called global nilpotent cone $h^{-1}(0)$, consists of Higgs bundles with nilpotent Higgs fields.

There is a natural $\mathbb{C}^\times$-action on the moduli space of Higgs bundles

$$t\cdot(E,\varphi)=(E,t\varphi).$$

The Hitchin map is $\mathbb{C}^\times$-equivariant for a suitable weighted action on B_n. This, together with properness of the Hitchin map, implies the existence of a limit

$$\lim_{t\longrightarrow 0}(E,\varphi)\in h^{-1}(0),$$

which is moreover a fixed point for the $\mathbb{C}^\times$-action. Similarly, equivariance of the Hitchin map and properness of the fibers imply that for any $(E,\varphi)\in h^{-1}(0)$

$$\lim_{t\longrightarrow\infty}(E,\varphi)\in h^{-1}(0)^{\mathbb{C}^\times},$$

where $h^{-1}(0)^{\mathbb{C}^\times}$ denotes the fixed point set of the $\mathbb{C}^\times$-action (necessarily nilpotent). These were classified by Simpson [S]. They are of the form

$$\mathcal{E}=(\bigoplus_{i=0}^{s}F_i,\bigoplus_{i=i}^{s}\phi_i)$$

where F_i is a rank n_i degree d_i vector bundle (where $\overline{n}=(n_0,\ldots,n_s)$ and $\overline{d}=(d_0,\ldots,d_s)$ are partitions of n and d respectively), and $\phi_i\in H^0(X,F_{i+1}^*F_iK)$.

Definition 2.2. *The type of the fixed point $\mathcal{E}=(\bigoplus_{i=0}^{s}F_i,\bigoplus_{i=i}^{s}\phi_i)$ is the pair $(\overline{n},\overline{d})$ of ranks and degrees of the graded terms. We will denote by $\mathbf{F}_{\overline{n},\overline{d}}$ an irreducible component of the fixed point set of given type $(\overline{n},\overline{d})$.*

Types determine the irreducible components of the fixed point set [B, Corollary 3.3]. Moreover, there is a correspondence between the irreducible components of the nilpotent cone and of the fixed point set. The relationship is given by the Bialynicki-Birula stratification: taking limits at ∞ defines a Zariski locally trivial fibration

$$\mathbf{F}^-\longrightarrow\mathbf{F}$$

where $\mathbf{F}$ denotes an irreducible component of the fixed point locus and

$$\mathbf{F}^-_{\overline{n},\overline{d}}:=\{(F,\psi)\in \mathrm{M}_X(n,d)\,:\,\lim_{t\to\infty}t\cdot(F,\psi)\in\mathbf{F}_{\overline{n},\overline{d}}\}.$$

Then, components of the nilpotent cone are precisely the closures $\mathbf{C}_{\overline{n},\overline{d}}:=\overline{\mathbf{F}^-}_{\overline{n},\overline{d}}\subset h^{-1}(0)$.

2.2. **Wobbly and very stable points and components.**

Definition 2.3. *Let $\mathcal{E}$ be a fixed point. Let*

$$\mathcal{E}^+=\{(F,\psi)\in\mathrm{M}_X(n,d)\,:\,\lim_{t\to0}(F,\psi)=\mathcal{E}\}.$$

A fixed point is **very stable** *if $\mathcal{E}^+\cap h^{-1}(0)=\{\mathcal{E}\}$. Otherwise it is called* **wobbly***.*

A fixed point component is called very stable if it contains a very stable point. Otherwise, it is called wobbly.

Given a $\mathbb{C}^\times$-module V, for $\lambda\in\mathbb{Z}$, let V_λ denote the λ weight space, and let $V^+=\bigoplus_{\lambda>0}V_\lambda$. Let $\chi_{\mathbb{C}^\times}$ denote the character of a $\mathbb{C}^\times$-module.For a fixed point $\mathcal{E}$, $T_{\mathcal{E}}\mathrm{M}_X(n,d)$ carries a $\mathbb{C}^\times$-action, as so does $\mathrm{Sym}(T^*_{\mathcal{E}})$. Let $T^+_{\mathcal{E}}$ denote the $\mathbb{C}^\times$ submodule of $T_{\mathcal{E}}\mathrm{M}_X(n,d)$ corresponding to positive weights.

Definition 2.4 ([HH])**.** *The virtual equivariant multiplicity of $\mathcal{E}$ is the fraction*

$$m_{\mathcal{E}}(t)=\frac{\chi_{\mathbb{C}^\times}(\mathrm{Sym}(T^{*+}_{\mathcal{E}}))}{\chi_{\mathbb{C}^\times}(\mathrm{Sym}(B^*_n))}$$

where B_n is the Hitchin base.

Remark 2.5. When $\mathcal{E}$ is very stable, this is a polynomial whose value $m_{\mathcal{E}}(1)$ at one matches the multiplicity of the irreducible component containing the point [HH, Theorem 5.2]. When the component has no very stable points, there is no reason why $m_{\mathcal{E}}(t)$ should be a polynomial.

2.3. $\mathrm{U}(n_0, n_1)$**-Higgs bundles.** Let $G_{\mathbb{R}} < \mathrm{GL}(n, \mathbb{C})$ be a real form. Let $H_{\mathbb{R}} < G_{\mathbb{R}}$ be a maximal compact subgroup. Denote by $\mathfrak{g}_{\mathbb{R}} := Lie(G_{\mathbb{R}})$, $\mathfrak{h}_{\mathbb{R}} := Lie(H_{\mathbb{R}})$. Let

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}} \oplus \mathfrak{m}_{\mathbb{R}}$$

be the Cartan decomposition. Let $H := H_{\mathbb{R}}^{\mathbb{C}}$, $\mathfrak{h} := \mathfrak{h}_{\mathbb{R}}^{\mathbb{C}}$, $\mathfrak{m} := \mathfrak{m}_{\mathbb{R}}^{\mathbb{C}}$ be the complexifications.

Definition 2.6. *A $G_{\mathbb{R}}$-Higgs bundle is a Higgs bundle (E, φ) where $E = E_H(\mathbb{C}^n)$ is the associated bundle (for the standard representation) for a principal H-bundle E_H and $\varphi \in H^0(E_H(\mathfrak{m}) \otimes K)$, where H acts on $\mathfrak{m}$ via the isotropy representation and we identify $E_H(\mathfrak{m}) \subset \mathrm{End}(E_H(\mathbb{C}^n))$ via $\mathfrak{m} \subset \mathfrak{gl}_n(\mathbb{C}) \cong \mathrm{End}(\mathbb{C}^\times)$.*

A stability condition can be defined yielding a moduli space $\mathrm{M}_X(G_{\mathbb{R}})$ admitting a natural map

$$\mathrm{M}_X(G_{\mathbb{R}}) \longrightarrow \mathrm{M}_X(n, d). \tag{2.1}$$

By abuse of notation, we will identify the image of this map with $\mathrm{M}_X(G_{\mathbb{R}})$, althought the fibers of the map (2.1) are non trivial [GP].

Definition 2.7. *A $\mathrm{U}(n_0, n_1)$-Higgs bundle is a Higgs bundle (E, φ) where $E = V \oplus W$ where $rk(V) = n_0$, $rk(W) = n_1$ and $\varphi = (\beta, \gamma)$ with $\beta : V \longrightarrow WK$, $\gamma : W \longrightarrow VK$.*

The ranks (n_0, n_1) and degrees (d_0, d_1) of V and W combine in the so called Toledo invariant, defined by

$$\tau = 2\frac{n_1 d_0 - n_0 d_1}{n_0 + n_1} \tag{2.2}$$

This invariant is bounded by

$$0 \leq |\tau| \leq 2\min\{n_0, n_1\}(g-1). \tag{2.3}$$

The irreducible components of $\mathrm{M}_X(\mathrm{U}(n_0, n_1))$ are classified by the degrees (d_0, d_1) except in the maximal Toledo case when $n_0 > n_1$, for which the Higgs bundle is generically semistable, with S-equivalence class $((F_1K^* \oplus S) \oplus F_1, 1 : F_1 \longrightarrow F_1K^*K)$.

3. Fixed points of length two and $\mathrm{U}(n_0, n_1)$-Higgs bundles

Let $\overline{n} = (n_0, n_1)$, $\overline{d} = (d_0, d_1)$. In this section, we will consider irreducible components of fixed points $\mathbf{F}_{\overline{n},\overline{d}} \subset \mathrm{M}_X(n, d)^{\mathbb{C}^\times}$. Namely, $\mathcal{E} \in \mathbf{F}_{\overline{n},\overline{d}}$ is of the form

$$\mathcal{E} = (F_0 \oplus F_1, \varphi), \qquad \varphi \in H^0(F_1^* F_0 K).$$

Observe that the map

$$t : \mathrm{M}_X(n, d) \longrightarrow \mathrm{M}_X(n, -d) \qquad (E, \varphi) \mapsto (E^*, {}^t\varphi)$$

maps $\mathbf{F}_{\overline{n},\overline{d}}$ to $\mathbf{F}_{(n_1,n_0),-(d_1,d_0)}$. So it is enough to study the fixed points for which $n_0 \geq n_1$. These have the advantage of generically having $\mathrm{Ker}(\varphi) = F_0$ (see Theorem 3.6).

Let us begin by giving a birational description of the reduced scheme underlying the irreducible components. By [La, Prop. 3.8], any component $\mathbf{F}_{\overline{n},\overline{d}}$ satisfies that F_0 and F_1 are generically semistable. Consider

$$Z'_{\overline{n},\overline{d}} = \left\{(F_0, F_1) \in \mathrm{N}_X(n_0, d_0) \times \mathrm{N}_X(n_1, d_1) \,:\, h^0(F_0 F_1^* K) > 0\right\}.$$

When $n_0 > n_1$, this is a family of twisted Brill-Noether loci [RT] over $\mathrm{N}_X(n_0, d_0)$. Define the invariant

$$\delta := d_0 n_1 - d_1 n_0 + 2n_0 n_1(g-1). \tag{3.1}$$

Lemma 3.1. *Let $n_0 = n_1$. There exists a rational projective bundle*

$$\mathbb{P}(\tilde{\mathcal{H}}_\delta) \dashrightarrow \mathrm{Sym}^{\frac{\delta}{n_0}} X \times \mathrm{N}_X(n_0, d_0)$$

with fiber $\mathbb{P}(H^0(F_0^* \otimes \mathcal{O}_D))$.

Proof. Consider the universal bundle $\mathcal{F}_0 \longrightarrow X \times \mathrm{N}_X(n_0, d_0)$ (only defined up to tensorization by a line bundle). Consider the sheaf $\mathcal{S}_0 := \mathbb{P}(\mathcal{O}_{\mathrm{Sym}} \boxtimes \mathcal{F}_0^*)$ on

$$S_0 := \mathrm{Sym}^{\frac{\delta}{n_0}} X \times (X \times \mathrm{N}_X(n_0, d_0)).$$

We note that the sheaf $\mathcal{S}_0$ is independent of the choice of a universal bundle $\mathcal{F}_0$. Now, let

$$S := \mathrm{Sym}^{\frac{\delta}{n_0}} X \times (X \times \mathrm{N}_X(n_0, d_0)) \times H^0(S_0, \mathcal{S}_0)$$
$$\overline{S} := \mathrm{Sym}^{\frac{\delta}{n_0}} X \times (X \times \mathrm{N}_X(n_0, d_0)) \times \mathbb{P}(H^0(S_0, \mathcal{S}_0) \oplus \mathbb{C})$$

and let $S \overset{i}{\hookrightarrow} \overline{S}$ be the open immersion. Let p_1, p_2, p_3 denote the projections to $\mathrm{Sym}^{\frac{\delta}{n_0}} X$, $X \times \mathrm{N}_X(n_0, d_0)$ and $\mathbb{P}(H^0(S_0, \mathcal{S}_0) \oplus \mathbb{C})$ respectively. We denote by $\mathcal{S} := (p_1 \times p_2)^* \mathcal{S}_0 = \mathbb{P}(p_1^* \mathcal{O}_{\mathrm{Sym}} \otimes p_2^* \mathcal{F}_0^*)$.

Consider the sheaf $\mathcal{O}'_{H^0} := i_*(p_3 \circ i)^* \mathcal{O}_{H^0}$, whose global sections are precisely $H^0(\mathcal{S})$. Then

$$\mathcal{O}'_{H^0} \longrightarrow \overline{S}$$

satisfies that the restriction to $X \times \{F_0, D, f\}$ is precisely the line of sections

$$\mathbb{C}^\times \cdot f : F_0 \longrightarrow \mathcal{O}_D.$$

Now, let $\pi : \overline{S} \longrightarrow \mathrm{Sym}^{\frac{\delta}{n_0}} X \times \mathrm{N}_X(n_0, d_0)$ denote the projection. It is a proper flat map, and so by [EGA, Theorem III.3.2.1], the sheaf $\mathcal{H}_\delta := R^0 \pi_* \mathcal{O}'_{H^0}$ is coherent, with fiber over (F_0, D) equal to $\mathbb{P}(H^0(F_0^* \otimes \mathcal{O}_D))$, so it is a vector bundle over a dense open subset. □

Corollary 3.2. *For $n_0 = n_1$, $Z'_{\overline{n},\overline{d}}$ contains an irreducible component $Z_{\overline{n},\overline{d}}$ given by the closure of the image of $\mathbb{P}(\mathcal{H}_\delta)$ under the rational map $r : (F_0, D, \mathbb{C}^\times f) \mapsto (\mathrm{Ker}(\mathbb{C}^\times f) K, F_0)$.*

Lemma 3.3. *The scheme $Z'_{\overline{n},\overline{d}}$ contains an irreducible component $Z_{\overline{n},\overline{d}}$ containing pairs (F_0, F_1) with $F_1 K^* \subset F_0$ of maximal rank. The dimension is*

$$\dim Z_{\overline{n},\overline{d}} = \begin{cases} (n_0^2 + n_1^2)(g-1) + 2 & \text{if } \delta > n_0 n_1 (g-1) \\ (n_0^2 + n_1^2 - n_0 n_1)(g-1) + \delta + 1 & \text{if } \delta \leq n_0 n_1 (g-1). \end{cases} \tag{3.2}$$

If $\delta > n_0 n_1(g-1)$, then $Z' = Z = \mathrm{N}_X(n_0, d_0) \times \mathrm{N}_X(n_1, d_1)$.

Proof. If $\delta > n_0 n_1 (g-1)$, by Riemann–Roch, $Z'_{\overline{n},\overline{d}} = \mathrm{N}_X(n_1, d_1) \times \mathrm{N}_X(n_0, d_0)$, whence the statement.

If $\delta \leq n_0 n_1 (g-1)$, $n_0 > n_1$, then for generic F_0, when $d_0 n_1 - (d_1 - 2(g-1) n_1) n_0 > (n_0 - n_1) n_1 (g-1)$ (namely, for non maximal Toledo), there always exists a subbundle of the form $F_1 K^*$ [RT, Prop. 1.5]. By [RT, Theorem 0.3], for generic F_0, the space of bundles with a section $F_1 K^* \longrightarrow F_0$ is a $\delta - n_1(n_0 - n_1)(g-1)$ dimensional variety. Putting all this together one obtains the result.

When δ is minimal, the result is still valid as within the moduli space, $F_0 = F_1K^* \oplus S$ where $\mu(S) = \mu(F_1K^*)$, so the result holds too by stability of F_1 and S [RT].

The case $n_0 = n_1$ is covered in the proof of [RT, Theorem 0.3]. □

Lemma 3.4. *There exist rational projective bundles*

$$\mathbb{P}(\mathcal{H}_\delta) \dashrightarrow Z_{\overline{n},\overline{d}}\ , \qquad \mathbb{P}(\mathcal{E}_\delta) \dashrightarrow Z_{\overline{n},\overline{d}}\ , \qquad \mathbb{P}(\mathcal{H}_\delta \oplus \mathcal{E}_\delta) \dashrightarrow Z_{\overline{n},\overline{d}}$$

with fibers

$$\mathbb{P}(\mathcal{H}_\delta)_{(F_0,F_1)} = \mathbb{P}(H^0(F_1^*F_0K)), \qquad \mathbb{P}(\mathcal{E}_\delta)_{(F_0,F_1)} = \mathbb{P}(H^1(F_1^*F_0)),$$
$$\mathbb{P}(\mathcal{H}_\delta \oplus \mathcal{E}_\delta)_{(F_0,F_1)} = \mathbb{P}(H^0(F_1^*F_0K) \oplus H^1(F_1^*F_0K)).$$

Proof. Existence of $\mathbb{P}(\mathcal{H}_\delta)$ follows like in Lemma 3.1 by considering

$$\mathcal{S}_0 := \mathcal{F}_1^* \boxtimes \mathcal{F}_0 \boxtimes K \longrightarrow X \times \mathrm{N}_X(n_0, d_0) \times \mathrm{N}_X(n_1, d_1)$$

where $\mathcal{F}_i \longrightarrow X \times \mathrm{N}_X(n_i, d_i)$ are the universal bundles.

Existence of $\mathbb{P}(\mathcal{E}_\delta)$ is proven in [L].

The existence of $\mathbb{P}(\mathcal{H}_\delta \oplus \mathcal{E}_\delta)$ follows in a similar fashion. □

The following is well known, we sketch a proof for the reader's convenience.

Lemma 3.5. *If the component $\mathbf{C}_{\overline{n},\overline{d}}$ contains a point with underlyings stable bundle, then it is birational to an affine bundle over $\mathbf{F}_{\overline{n},\overline{d}}$ with fiber $H^1(F_1^*F_0K)$ over a stable $(F_0 \oplus F_1, \varphi) \in \mathbf{F}_{\overline{n},\overline{d}}$.*

Proof. By [HH, Prop. 3.11], $\lim_{t\to\infty} t \cdot (E, \phi) = (F_0 \oplus F_1, \varphi)$ if and only if E underlies an extension $e \in H^1(F_1^*F_0)$ and ϕ induces φ on the graded object. Since by assumption E is generically stable, then the only ambiguity is given by scalar multiplication on the Higgs field ϕ. This action can be seen on the level of extensions and fixed points as follows: identify e with a pair (i, π) where $i : F_0 \hookrightarrow E$ is the inclusion and $\pi : E \twoheadrightarrow F_1$ the projection. Then (e, φ) is identified with $(E, i \circ \varphi \circ \pi)$, so that $t \cdot (e, \varphi) := (te, \varphi)$ is identified with $t \cdot (E, i \circ \varphi \circ \pi) = (E, t \cdot i \circ \varphi \circ \pi)$. Thus, the fiber is the whole affine space. □

Theorem 3.6. *Let $n_0 \geq n_1$ with $n_0 + n_1 = n$, and let $d_0 + d_1 = d$. Let δ be as in* (3.1).

Then the fixed point components $\mathbf{F}_{\overline{n},\overline{d}} \subset \mathrm{M}_X(n, d)$ are labelled by (d_1, δ), and are non-empty if and only if

$$n_1(n_0 - n_1)(g - 1) \leq \delta < 2n_0n_1(g - 1). \tag{3.3}$$

Moreover:

1. *The corresponding reduced component of the nilpotent cone is birational to the rational bundle*
$$\mathbb{P}(\mathcal{E}_\delta \oplus \mathcal{H}_\delta) \dashrightarrow Z_{\overline{n},\overline{d}}$$
where all sheaves are defined in Lemma 3.4 and $Z_{\overline{n},\overline{d}}$ is an irreducible component as in Lemma 3.3.
2. *The scheme $\mathbf{F}^{red}_{\overline{n},\overline{d}}$ is birational to $\mathbb{P}(\mathcal{H}_\delta)$. In particular*

(3.4)
$$\dim \mathbf{F}_{\overline{n},\overline{d}} = \begin{cases} (n_0^2 + n_1^2 - n_0n_1)(g-1) + \delta + 1 & \text{if } \delta > n_1(n_0 - n_1)(g-1) \\ (n_0^2 + n_1^2 - n_0n_1)(g-1) + \delta + 2 & \text{if } \delta = n_1(n_0 - n_1)(g-1) \end{cases}$$

(3) If $n_0 > n_1$, the component $\mathbf{F}^{red}_{\overline{n},\overline{d}}$ is birational to the rational projective bundle

$$\mathbb{P}(\tilde{\mathcal{E}}_\delta) \dashrightarrow \tilde{Z}_{\overline{n},\overline{d}}$$

where

$$\tilde{Z}_{\overline{n},\overline{d}} = \mathrm{N}_X(n_1, d_1) \times \mathrm{N}_X(n_0 - n_1, d_0 - d_1 + 2n_1(g-1))$$

and $\mathbb{P}(\tilde{\mathcal{E}}_\delta) \longrightarrow \tilde{Z}_{\overline{n},\overline{d}}$ has fiber

$$\mathbb{P}(\tilde{\mathcal{E}}_\delta)|_{(F_1,S)} = \begin{cases} \mathbb{P}(H^1(S^*F_1K^*)) & \text{if } \delta > n_1(n_0-n_1)(g-1) \\ 0 & \text{if } \delta = n_1(n_0-n_1)(g-1) \end{cases}$$

(4) If $n_0 = n_1$, then $\mathbf{F}^{red}_{\overline{n},\overline{d}}$ is birational to the rational bundle

$$\mathbb{P}(\tilde{\mathcal{H}}_\delta)/\mathbb{P}(\mathrm{Aut}(\mathcal{O}_{\mathrm{Sym}})) \dashrightarrow \mathrm{Sym}^{\frac{\delta}{n_0}} X \times \mathrm{N}_X(n_0, d_0)$$

where $\tilde{\mathcal{H}}_\delta$ is defined in Lemma 3.1.

Proof. Necessity of the bounds follows from the impossibility to have semistable Higgs bundles otherwise. Indeed, assume there exists a maximal rank Higgs bundle (we will see this is always the case). Then, the lower bound follows from the fact that the slope of $F \oplus F_1K^*$ is at most equal than that of $F_0 \oplus F_1$. Similarly, for $\delta = 2n_0n_1(g-1)$, $\mu(F_0) = \mu(F_1)$, so no fixed points happen for this invariant. Likewise for $\delta > 2n_0n_1(g-1)$, $\mu(F_0) > \mu(F_1)$ so no semistable Higgs bundle can exists with those invariants (these arguments can be found in [BGG,]).

Sufficiency of the bounds, and uniqueness of the components for the given invariants follow from the construction below.

Let us prove (1). By [RT, Theorem 0.1], when $n_0n_1(g-1) \leq \delta < 2n_0n_1(g-1)$, there exist stable extensions in $H^1(F_1^*F_0)$ for general F_0, F_1. Thus, there exists a rational map

$$\mathbb{P}(\mathcal{E}_\delta \oplus \mathcal{H}_\delta) \dashrightarrow \mathbf{C}_{\overline{n},\overline{d}} \tag{3.5}$$

$$(F_0 \overset{i}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} F_1, \varphi) \longmapsto (E, \overbrace{i \circ \varphi \circ \pi}^{\phi})$$

for some irreducible component $\mathbf{C}_{\overline{n},\overline{d}}$ of the nilpotent cone with the given invariants. The map is injective up to the action of an automorphism of (E, ϕ). Since E is generically stable, this is just a scalar action. In particular

$$\dim(\mathbb{P}(\mathcal{E}_\delta \oplus \mathcal{H}_\delta) \leq n^2(g-1)+1.$$

Now, we have

$$\begin{aligned} \dim(\mathbb{P}(\mathcal{E}_\delta \oplus \mathcal{H}_\delta)) &\geq -\chi(F_1^*F_0) + \chi(F_1^*F_0K) - 1 + \dim(Z_{\overline{n},\overline{d}}) = \\ &\overset{Lm.3.3}{=} 3n_0n_1(g-1) - \delta + \delta - n_0n_1(g-1) + (n_0^2+n_1^2)(g-1) + 2 - 1 = \\ &= n^2(g-1)+1, \end{aligned} \tag{3.6}$$

where we use $\mu(F_0) < \mu(F_1)$ when $\delta \neq 2n_0n_1(g-1)$. Note that for any $\mathcal{E} = (F_0 \oplus F_1, \varphi) \in \mathbf{F}_{\overline{n},\overline{d}}$ and any extension $e \in H^1(F_1^*F_0)$, the underlying bundle E admits a nilpotent Higgs field

$$\phi : E \twoheadrightarrow F_1 \xrightarrow{\varphi} F_0K \hookrightarrow EK$$

that makes it semistable.

A similar argument proves that for $n_1(n_0-n_1)(g-1)\leq\delta\leq n_0n_1(g-1)$,

$$\dim(\mathbb{P}(\mathcal{E}_\delta\oplus\mathcal{H}_\delta)\geq-\chi(F_1^*F_0)+1+\dim(Z_{\overline{n},\overline{d}})-1 \tag{3.7}$$

$$\stackrel{Lm.3.3}{=}3n_0n_1(g-1)-\delta+1+\delta+1+(g-1)((n_0-n_1)^2+n_0n_1)-1=$$
$$=n^2(g-1)+1.$$

To prove (3.5) is well defined, we need to take a closer look at $Z_{\overline{n},\overline{d}}$. By [RT, Theorem 0.3], the generic point $(F_1,F_0)\in Z_{\overline{n},\overline{d}}$ satisfies that the general section $\varphi:F_1\longrightarrow F_0K$ is injective and saturated (if $n_0>n_1$) or with maximal rank at the support of F_0K/F_1 (if $n_0=n_1$). The condition on δ then makes sure that $(F_0\oplus F_1,\varphi)$ is stable as a Higgs bundle. Indeed, by the upper bound on δ, $\mu(F_0)\leq\mu(F_1)-(g-1)$, with equality only if $\delta=n_1(n_0-n_1)(g-1)$. On the other hand, the lower bound of δ implies that $\mu(F_1\oplus F_1K^*)=\mu(F_1)-g+1\leq\mu(E)$, with equality only if $\delta=n_1(n_0-n_1)(g-1)$. Assuming there exists a Higgs field with maximal rank, any φ-invariant subbundle is of the form $N_0\oplus N_1$ with $N_i\subset F_i$, $N_1K^*\subset N_0$. Now, by semistability of F_i, we have, letting $r_i=\mathrm{rk}(N_i)$,

$$\mu(N_0\oplus N_1)=\frac{\mu(N_0)r_1+\mu(N_1)r_0}{r_0+r_1}\leq\frac{\mu(F_0)r_1+\mu(F_1)r_0}{r_0+r_1}$$
$$\leq\mu(F_1)-\frac{n_0n_1(g-1)r_1}{r_0+r_1}\leq\mu(F_1)-g+1\leq\mu(E).$$

Thus, for a general $e\in H^1(F_1^*F_0)$, the image of (e,φ) under (3.5) must also be semistable, and the remaining arguments go through.

Item (2) is now clear, provided that the rational map be well defined therein. This is guaranteed by the value of δ. Thus, to prove (3.4), we subtract

$$\dim\mathbf{F}_{\overline{n},\overline{d}}=n^2(g-1)+1-h^1(F_1^*F_0)=n^2(g-1)+1-h^1(F_1^*F_0)=$$
$$=\begin{cases}(n_0^2+n_1^2-n_0n_1)(g-1)+\delta+1 & \text{if } \delta>n_1(n_0-n_1)(g-1),\\ (n_0^2+n_1^2-n_0n_1)(g-1)+\delta+2 & \text{if } \delta=n_1(n_0-n_1)(g-1),\end{cases}$$

where we have used that $h^0(F_1^*F_0)=1$ generically if $\delta=n_1(n_0-n_1)(g-1)$, and vanishes otherwise. Thus, (2) follows.

This proves that every component with maximal rank Higgs fields flowing up to a component $\mathbf{F}_{\overline{n},\overline{d}}$ is birational to a sheaf as above. We will see that these are all after proving (4) and (3).

To see (3), we note that the existence of $\tilde{\mathcal{E}}_\delta$ follows from [L]. By irreducibility of all components involved, it is enough to find an injective rational map from one to the other and prove that dimensions match. Now, consider the map

$$\mathbb{P}(\tilde{\mathcal{E}}_\delta)\dashrightarrow\mathbf{F}^{red}_{\overline{n},\overline{d}} \tag{3.8}$$

$$(F_1,S,f_0)\longmapsto\left(F_0\oplus F_1,\varphi:F_1K^*\stackrel{i}{\hookrightarrow}F_0\right).$$

In the above f_0 denotes an extension class $F_1K^*\stackrel{i}{\hookrightarrow}F_0\stackrel{\pi}{\twoheadrightarrow}S$ and F_0 the underlying vector bundle. It is well defined as by [RT, Theorem 0.3] the twisted

Brill–Noether locus $BN^0_{n_1,d_1-2n_1(g-1)}(F_0)$ (see (3.13)) is non empty and the general $F_1K^* \in BN^0(F_0)$ is of maximal rank. Moreover, the map (3.8) is injective, as automorphisms are scalar on both sides by stability of the Higgs bundle for $\delta > n_1(n_0-n_1)(g-1)$ and for $\delta = n_1(n_0-n_1)(g-1)$, the map is simply $(F_1,S) \mapsto (F_1K^* \oplus S) \oplus F_1, 1)$ where $1 : F_1 \cong F_1K^*K$ is the unique isomorphism up to automorphisms of F_1. To check the equality of dimensions, at a general point

$$\dim_{(F_1,S,f_0)} \mathbb{P}(\tilde{\mathcal{E}}_\delta) =$$

$$= \begin{cases} h^1(S^*F_1K^*) + (n_1^2 + (n_0-n_1)^2)(g-1) + 2 - 1 & \text{if } \delta > n_1(n_0-n_1)(g-1), \\ (n_1^2+(n_0-n_1)^2)(g-1)+2 & \text{if } \delta = n_1(n_0-n_1)(g-1). \end{cases}$$

Since $\mu(S) \geq \mu(F_1K^*)$ and both bundles are generally stable (so that $H^0(S^*F_1K^*) = 0$ for general pairs), by Riemann–Roch

(3.9)

$$h^1(S^*F_1K^*) = -\left(\deg(S^*F_1K^*) - n_1(n_0-n_1)(g-1)\right) = \delta + n_1(n_0-n_1)(g-1),$$

and so

$$\dim_{(F_1,S,f_0)} \overset{(3.4)}{=} \dim \mathbf{F}_{\overline{n},\overline{d}}.$$

This proves that there exists at most one component with a Higgs field of maximal rank flowing to a fixed point with semistable graded pieces. Indeed, (3) is a map to $\mathrm{M}_X(n,d)$ with dense open image inside any such component, so by irreducibility, the statement follows.

For item (4), when $n_0 = n_1$ $F_1K^* \hookrightarrow F_0$, has torsion cokernel $\mathcal{T}$. There is a map

$$\begin{array}{ccc} \mathbb{P}(\tilde{\mathcal{H}}_\delta) & \dashrightarrow & \mathbf{F}^{red}_{\overline{n},\overline{d}} \\ & & \\ (D, F_0, \mathbb{C}^\times \cdot f) & \longmapsto & (F_0 \oplus \ker(f)K, \varphi : \mathrm{Ker}(\mathbb{C}^\times \cdot f) \hookrightarrow F_0) \end{array} \tag{3.10}$$

which is well defined over points for which f is surjective and hits a component with maximal rank Higgs fields. Note that this map is injective up to the action of a torus $\mathbb{T} := \mathrm{Aut}(\mathcal{O}_D)/\mathbb{C}^\times \cong (\mathbb{C}^\times)^{\frac{\delta}{n_0}-1}$, as the kernel of $f \in H^0(F_0^*\mathcal{O}_D)$ is well defined up to the action of $\mathrm{Aut}(\mathcal{O}_D)$. Thus, there exists a unique component with maximal rank Higgs fields.

Now, we claim that every component $\mathbf{C}_{\overline{n},\overline{d}}$ has a point with a maximal rank Higgs field. This proves that all components are determined by the invariants $(\overline{n},\overline{d})$, or equivalently, $(\overline{n}, d_1, \delta)$. There are three (non exclusive) options for the rank not to be maximal: either $\mathrm{rk}(\mathrm{Ker}(\phi)) > n_0$, or F_1 is not saturated in F_0 (when $n_0 > n_1$), or $D = \mathrm{supp}(F_0/F_1K^*)$ is never reduced (when $n_0 = n_1$).

Case 1: $n_0 = n_1$ and $D = \mathrm{supp}(F_0/F_1K^*)$ is never a reduced divisor. Then, there is a rational map

$$\begin{array}{ccc} \mathbf{F}^{red}_{\overline{n},\overline{d}} & \dashrightarrow & \mathbb{P}(\tilde{\mathcal{H}}_\delta) \\ & & \\ (F_0 \oplus F_1, \varphi) & \longmapsto & (\mathrm{supp}\,(F_0/F_1K^*)\,, F_0, f : F_0 \twoheadrightarrow F_0/F_1K^*) \end{array} \tag{3.11}$$

where $\mathbb{P}(\tilde{\mathcal{H}}_\delta)$ is as in Lemma 3.1, whose image only hits the sheaf over the non reduced divisors. Note that $\mathrm{Aut}(\mathcal{T})$ contains as a strict subgroup $(\mathbb{C}^\times)^{\delta/n_0}$. Indeed, since $\deg(\mathcal{T}) = \delta/n_0$, as an $\mathcal{O}_D$ module, $\mathcal{T} := F_0/F_1K^*$ is still of rank one. Equivalently, letting D^r denote the reduced divisor underlying D, the ranks at the points of D^r equal the multiplicities in D. Then $h^0(F_0^* \otimes_{\mathcal{O}_D} \mathcal{T}) = h^0(F_0^* \otimes_{\mathcal{O}_{D^r}} \mathcal{T}) = \delta$. Thus

$$\dim \mathbf{F}_{\overline{n},\overline{d}} \leq \mathbb{P}(\tilde{\mathcal{H}}_\delta)/\mathbb{P}(\mathrm{Aut}(\mathcal{T})) < n_0^2(g-1) + 1 + s + h^0(F_0^* \otimes \mathcal{T}) - 1 - (\frac{\delta}{n_0} - 1) =$$

$$= n_0^2(g-1) + \delta - (\frac{\delta}{n_0} - 1 - s),$$

where s is the generic degree of the reduced divisor D^r. The above is strictly smaller than (3.4). Since the dimension $h^1(F_1^*F_0)$ does not depend on the rank of the Higgs field, it follows from Lemma 3.5 that

$$\dim \mathbf{C}_{\overline{n},\overline{d}} < n_0^2(g-1) + \delta - (\frac{\delta}{n_0} - 1 - s) + 3n_0^2 - \delta - 1 < 4n_0^2(g-1) + 1.$$

Case 2: $n_0 > n_1$ and generically $\mathrm{rk}(\mathrm{Ker}(\phi)) = r > n_0$. Then, for fixed points it must also be $\mathrm{rk}(\mathrm{Ker}(\phi)) > n_0$. Then, there exists rational map

$$\mathbf{F}_{\overline{n},\overline{d}} \dashrightarrow \mathbb{P}(\tilde{\mathcal{E}}_0) \oplus \mathbb{P}(\tilde{\mathcal{E}}_1)$$

where $\mathbb{P}(\tilde{\mathcal{E}}_i) \dashrightarrow \mathrm{N}_X(n_i - r + n_0, d_i - d) \times \mathrm{N}_X(r - n_0, d)$ has fiber over (A, B) equal to $\mathbb{P}(H^1(B^*A))$ if $i = 0$ and $\mathbb{P}(H^1(A^*B))$ if $i = 1$. The image of this map hits the restriction to the diagonal $\mathrm{N}_X(n_0 - r + n_0, d_0 - d) \times \mathrm{N}_X(r - n_0, d) \times \mathrm{N}_X(n_1 - r + n_0, d_1 - d) \longrightarrow \mathrm{N}_X(n_0 - r + n_0, d_0 - d) \times \mathrm{N}_X(r - n_0, d) \times \mathrm{N}_X(r - n_0, d) \times \mathrm{N}_X(n_1 - r + n_0, d_1 - d)$. A dimensional computations shows that the dimension of these fixed points is strictly less than (3.4), and so by Lemma 3.5 the total dimension of the component is strictly smaller than half the dimension, a contradiction.

Case 3: $F_1K^* \subset F_0$ never saturated. Then, by assumption, for all points in this component satisfy that $\varphi \in H^0(\overline{F}_1^* F_0 K)$ where $\overline{F}_1$ denotes the saturation of F_1 in F_0K. Then, let $\mathbb{P}(\tilde{\mathcal{E}}'_\delta) \dashrightarrow \mathrm{N}_X(n_1, \overline{d}_1) \times \mathrm{N}_X(n_0 - n_1, d_0 - \overline{d}_1 + 2g - 2)$ be as in Lemma 3.4, but with $\overline{d}_1$ the generic degree of the saturation of $\mathrm{Im}(\varphi)K^* \subset F_0$. Then, there is a rational map

$$\begin{aligned} \mathbf{F}_{\overline{n},\overline{d}} &\dashrightarrow \mathbb{P}(\tilde{\mathcal{E}}'_\delta) \\ (F_0 \oplus F_1, \varphi) &\longmapsto (\overline{F}_1, F_0/\overline{F}_1K^*, f_0) \end{aligned}$$

which is well defined and injective as $\varphi \in H^0(\overline{F}_1^* F_0 K)$. But $h^1(\overline{F}_1 K^* S^*$ is structly smaller than (3.9) (by generic stability of $\overline{F}_1$ and $S := F_0/\overline{F}_1K^*$, so again we have fixed points of dimension strictly smaller than necessary. □

Remark 3.7. Condition (3.3) is equivalent to (2.3), as

$$\delta = -\frac{n}{2}|\tau| + 2n_0n_1(g-1). \tag{3.12}$$

The top bound for δ corresponds to minimal Toledo invariant, while the low bound corresponds to maximal Toledo.

Remark 3.8. For maximal Toledo, the Higgs field splits as $F_1 \oplus F_1K^* \oplus S$, with $\mu(S) = \mu(F_1 \oplus F_1K^*)$ and the Higgs field the constant section of $1 : F_1 \longrightarrow F_1K^*K$. An easy computation shows that F_0 is generically semistable, so the fixed point is indeed of type (n_0, n_1).

Lemma 3.9. *The rational projections* $\mathbf{F}_{\overline{n},\overline{d}} \dashrightarrow \mathrm{N}_X(n_i, d_i)$ *are dominant for* $\delta > n_1(n_0-n_1)(g-1)$. *For* $\delta = n_1(n_0-n_1)(g-1)$, *only the projection to* $\mathrm{N}_X(n_1, d_1)$ *is.*

If $n_0 = n_1$, *there are dominant maps onto both factors separately.*

Proof. If $n_0 > n_1$, dominance of $\mathbf{F}_{\overline{n},\overline{d}}$ over $\mathrm{N}_X(n_0, d_0)$ for non maximal Toledo follows from [RT, Prop. 1.11]. Dominance over $\mathrm{N}_X(n_1, d_1)$ is a consequence of the description in terms of extensions in Theorem 3.6. Similarly, when $n_0 = n_1$, dominance of $\mathbf{F}_{\overline{n},\overline{d}}$ over $\mathrm{N}_X(n_0, d_0)$ is clear by construction. Dominance over $\mathrm{N}_X(n_1, d_1)$ follows from the fact that elementary transformations define a birational morphism between moduli spaces of different degrees. □

The following well known result in Brill–Noether theory follows from Theorem 3.6 (1).

Corollary 3.10. *Let* $(F_0, F_1) \in Z_{\overline{n},\overline{d}}$ *be general. Then*

$$\dim H^0(F_1^*F_0K) = \begin{cases} \delta - n_0n_1(g-1) & \textit{if } \delta > n_0n_1(g-1) \\ 1 & \textit{if } \delta \leq n_0n_1(g-1). \end{cases}$$

In particular, let $BN^0(n_0n_1, \delta) \subset \mathrm{N}_X(n_0n_1, \delta)$ *denote the Brill–Noether locus, and let*

$$BN^0_{n_1,d_1-2n_1(g-1)}(F_0) := \{R \in \mathrm{N}_X(n_1, d_1 - 2n_1(g-1)) \,:\, h^0(R^*F_0) \neq 0\} \tag{3.13}$$

denote the twisted Brill–Noether locus. Then, for a general F_0, *tensorisation defines a map* $BN^0_{n_1,d_1-2n_0(g-1)}(F_0) \longrightarrow BN^0(n_0n_1, \delta)$ $R \mapsto F_1^*F_0K$ *intersecting the open stratum of* $BN^0(n_0n_1, \delta)$ *consisting of bundles with a minumum number of sections at a non empty subset.*

Proof. For $\delta > n_0n_1(g-1)$, then

$$h^1(F_1^*F_0) + h^0(F_0F_1^*K) - 1 \overset{Thm. 3.6(1)}{=} \dim \mathbb{P}(\mathcal{E}_\delta \oplus \mathcal{H}_\delta) - \dim Z_{\overline{n},\overline{d}} \overset{(3.6)}{=}$$
$$= -\chi(F_1^*F_0) + \chi(F_0F_1^*K) - 1.$$

Since $h^1(F_1^*F_0) = -\chi(F_1^*F_0)$ if $\delta \neq n_1(n_0-n_1)(g-1)$, then it must be $h^0(F_1^*F_0K) - 1 = \chi(F_1^*F_0K) - 1$. This means proves the result for $\delta > n_0n_1(g-1)$.

Similarly, for $\delta \leq n_0n_1(g-1)$, then

$$h^1(F_1^*F_0) + h^0(F_0F_1^*K) - 1 \overset{Thm. 3.6(1)}{=} \dim \mathbb{P}(\mathcal{E}_\delta \oplus \mathcal{H}_\delta) - \dim Z_{\overline{n},\overline{d}} \overset{(3.7)}{=} -\chi(F_1^*F_0) + 1 - 1.$$

Thus $h^0(F_1^*F_0K) = 1$ if $\delta \neq n_1(n_0-n_1)(g-1)$, as $h^1(F_1^*F_0) = -\chi(F_1^*F_0)$. For if $\delta = n_1(n_0-n_1)(g-1)$, $h^1(F_1^*F_0) = h^0(F_1^*F_1K^*) \oplus h^0(F_1^*SK^*)$, which is generally zero by Theorem 3.6(4).

□

As a consequence of Corollary 3.10, we can establish necessary conditions on the invariant δ to determine wobbly divisors in the moduli space $\mathrm{N}_X(n, d)$ (see [PPa, PPe2] for the rank two and three cases).

Corollary 3.11. *Let $\mathbf{C}^{\delta}_{\overline{n},\overline{d}}$ be the irreducible component of the nilpotent cone with fixed points with associated invariant δ. Then $\mathbf{C}^{\delta}_{\overline{n},\overline{d}}$ flows down to a wobbly divisor in $\mathrm{N}_X(n,d)$ only if*

$$n_1(n_0-n_1)(g-1)\leq\delta\leq n_0n_1(g-1)+1. \tag{3.14}$$

Proof. In what follows we consider underlying reduced schemes.

By [P], the wobbly locus is a divisor. So the intersection $\mathbf{C}^{\delta}_{\overline{n},\overline{d}}\cap\mathrm{N}_X(n,d)$ of the irreducible component determined by the fixed points and the moduli space is either a divisor or embedded in one.

Let $E\in\mathbf{W}$ be wobbly. Then, by smoothness of (E,φ), we have

$$T_{(E,\varphi)}\mathrm{M}\cong T_E\mathrm{N}\oplus H^0(\mathrm{End}(E)K),$$

where we have used that locally around (E,φ), $\mathrm{M}\cong T^*N$. Then, if $\omega: T^*\mathrm{N}_X\longrightarrow\mathrm{N}_X$ is the forgetful map, it follows that $d\omega|_{T_E\mathrm{N}}\equiv id$. Now, $\varphi^{\perp}\subset T_E\mathrm{N}$ is a divisor. Since the components of the nilpotent cone are Lagrangian, it follows that for any nilpotent $\psi\in H^0(\mathrm{End}(E)K)$ with $(E,\psi)\in\mathbf{C}^{\delta,smooth}_{\overline{n},\overline{d}}$, then $\psi^{\perp}\subset T_E\mathrm{N}_X$. Since the downward flow is a Zariski locally trivial Lagrangian fibration that matches ω, we have that $d\omega(\psi^{\perp})\cong\psi^{\perp}\subset T_{(E,0)}\mathrm{M}_X$.

This, in addition to Corollary 3.10 proves that uniqueness of the Higgs field on a given component of the nilpotent cone for $E\in\mathbf{W}^{smooth}$ is a necessary condition, which is equivalent to (3.14). $\square$

Remark 3.12. Sufficiency of the condition (3.14) should also hold (as is the case in rank two [PPa] and three [PPe2]). However, it requires to prove that every component with $\delta\leq n_0n_1(g-1)$ intersects the moduli space of bundles, together with a careful analysis of the forgetful map $d\omega|_{\mathbf{C}}$ to determine local injectivity.

4. Criteria for wobbliness

In this section we give criteria to decide when a fixed point is wobbly. We start by defining types of wobbly Higgs bundles, dependant of the upward flow from them.

Definition 4.1. *Let $\overline{m}$ be a partition of n. A wobbly fixed point $\mathcal{E}$ is called of wobbly type $\overline{m}$ if $\mathcal{E}=\lim_{t\to0}(E,\psi)$ and $\mathcal{E}\neq\mathcal{E}'=\lim_{t\to\infty}(E,\psi)$ is of type $\overline{m}$.*

Remark 4.2. The wobbly type is not uniquely determined (see Corollary 4.11).

Let us recall a result from [HH] that will appear many times in this section.

Lemma 4.3. [HH, Prop 3.4] *A Higgs bundle (E,ψ) satisfies $\lim_{t\to0}t(E,\psi)=(F_0\oplus F_1,\varphi)$ if and only if $E\in H^1(F_0^*F_1)$ and the Higgs field ψ satisfies*

$$\varphi: F_1\hookrightarrow E\xrightarrow{\psi}EK\twoheadrightarrow\tilde{F}_0K.$$

Let us start by a preliminary lemma.

Lemma 4.4. *Let a fixed point $\mathcal{E}$ be of type $\overline{m}$. Then $\mathcal{E}=\lim_{t\to0}(E,t\psi)$ with $rk(\mathrm{Ker}(\psi^{i+1})\geq\sum_{k=0}^{i}m_k$. Moreover, if $m_i\leq m_{i-i}$ and $\mathcal{E}'=\lim_{t\to\infty}(E,t\psi)$ is general in its fixed point component, then equality is general.*

Proof. Let $\mathcal{E} = \lim_{t\to 0}(E,\psi)$ where the iterated kernel filtration of ψ is of type $\overline{m}$. By [HH, Prop. 3.11], E admits a filtration $E_0 \subsetneq \cdots \subsetneq E_r = E$ with $E_i/E_{i-1} = F_i$ where $\mathcal{E}' = (\bigoplus F_i, \bigoplus \phi_i)$ and $\psi(E_i) \subset E_{i-1}K$ inducing $\phi_i : F_i \longrightarrow F_{i-1}K$. This means that $\psi^{i+1}(E_i) = 0$, i.e., $E_i \subseteq \ker(\psi^{i+1})$.

Now, let $\mathcal{E}'$ be general inside its component. Then the statement follows from [La, Prop 1.9]. $\square$

Lemma 4.5. *Let $e \in H^1(F_0^*F_1)$ be an extension class, and let E be the underlying vector bundle. Let*

$$Res : H^0(\mathrm{End}(E) \otimes K) \longrightarrow H^0(F_0F_1^*K)$$

be the natural map. Let $\phi \in H^0(F_0F_1^*K)$. Then the following are equivalent:

(1) There is an equality $\phi = Res(\psi')$.
(2) The pairing given by Serre duality satisifies $\langle \phi, e\rangle = 0$.
(3) The field ϕ lifts to $H^0(F_1^*EK)$ and $H^0(F_0(E)^*K)$.
(4) The field ϕ lifts to $H^0(F_1^*EK)$ or $H^0(F_0(E)^*K)$.

Proof. Consider the exact diagram

$$\begin{tikzcd}
F_1F_0^*K \arrow[r, hook] \arrow[d, hook] & F_1E^*K \arrow[r, two heads] \arrow[d, hook] & \mathrm{End}(F_0)K \arrow[d, hook] \\
EF_0^*K \arrow[r, hook] \arrow[d, two heads] & \mathrm{End}(E)K \arrow[r, two heads] \arrow[d, two heads] & EF_1^*K \arrow[d, two heads] \\
\mathrm{End}(F_1)K \arrow[r, hook] & F_0E^*K \arrow[r, two heads] & F_0F_1^*K.
\end{tikzcd}$$

The associated long-exact diagram in cohomology reads
(4.1)

$$\begin{tikzcd}
H^0(F_1F_0^*K) \arrow[r, hook] \arrow[d, hook] & H^0(F_1E^*K) \arrow[r] \arrow[d, hook] & H^0(\mathrm{End}(F_1)K) \arrow[r, "-\cup e"] \arrow[d, hook] & H^1(F_1F_0^*K) \arrow[r] \arrow[d, hook] & \dots \\
H^0(EF_0^*K) \arrow[r, hook] \arrow[d] & H^0(\mathrm{End}(E)K) \arrow[r] \arrow[d] & H^0(EF_1^*K) \arrow[r, "\cup e"] \arrow[d] & H^1(EF_0^*K) \arrow[r] \arrow[d] & \dots \\
H^0(\mathrm{End}(F_0)K) \arrow[r, hook] \arrow[d, "\cup e"'] & H^0(F_0E^*K) \arrow[r] \arrow[d, "\cup e"'] & H^0(F_0F_1^*K) \arrow[r, "-\cup e"] \arrow[d, "\cup e"'] & H^1(\mathrm{End}(F_1)K) \arrow[r] \arrow[d] & \dots \\
H^1(F_1F_0^*K) \arrow[r] & H^1(F_1E^*K) \arrow[r] & H^1(\mathrm{End}(F_1)K) \arrow[r] & 0
\end{tikzcd}$$

Let $\phi \in H^0(F_0F_1^*K)$.

It is clear that (1) $\Rightarrow$ (3)$\Rightarrow$ (4).

Let us prove that (4) $\Leftrightarrow$ (2) $\Rightarrow$ (1), which finishes the proof.

Let ϕ lift to $\phi_1 \in H^0(EF_1^*K)$. Since the map $H^0(F_0F_1^*K) \longrightarrow H^1(\mathrm{End}(F_1)K)$ is given by $\cup e$ for the given extension, sections lifting to $H^0(EF_1^*K)$ are contained in the hyperplane $e^\perp \subset H^0(F_0F_1^*K)$, so (2) holds. The same argument proves that if ϕ lifts to $\phi_2 \in H^0(F_0E^*K)$, then it must be $\phi \perp e$. Now, assume $e \perp \phi$. Note that $H^0(EF_1^*K)$ surjects onto $e^\perp \subset H^0(F_1^*F_0K)$, as the cokernel of $H^0(EF_1^*K) \longrightarrow H^0(F_0F_1^*K)$ is either 0 or $H^1(\mathrm{End}(F_1)K) \cong \mathbb{C}$ (by stability of F_1). Hence $e \perp$

ϕ implies that (4) and it implies (3). Now, continuing the argument $\phi_1 \cup e \in H^1(EF_0K)$ maps to zero under $H^1(EF_0K) \longrightarrow H^1(\mathrm{End}(F_0)K)$. If it weren't $\phi_1 \cup e = 0$, then it would lift to $H^1(F_1F_0^*K)$, which in turn must lift to $H^0(\mathrm{End}(F_0)K)$. If the latter is not zero, then ϕ must be zero. So it must be $\phi_1 \cup e = 0$, namely, ϕ lifts to $H^0(\mathrm{End}(E) \otimes K)$. $\square$

Proposition 4.6. *Let $n_1 \leq n_0$. A fixed point $\mathcal{E} = (F_0 \oplus F_1, \varphi) \in \mathbf{F}_{n_0,n_1,d_0,\delta}$ with φ injective is wobbly of type (n_0, n_1) if and only if there is a torsion subsheaf $\mathcal{T} \subset F_0/F_1K^*$ such that φ factors as*

$$F_1 \longrightarrow \overline{F_1} \hookrightarrow \tilde{F}_0K \hookrightarrow F_0K$$

where $\overline{F_1}$ is the saturation of F_1 inside F_0K along $\mathcal{T}$ and $\tilde{F}_0 = \mathrm{Ker}(F_0 \longrightarrow \mathcal{T})$.

Proof. Assume $F_0/F_1K^* = M \oplus \mathcal{T}'$ where M is locally free and $\mathcal{T}'$ is torsion, then we have that φ factors as

$$\varphi : F_1 \to \overline{F}_1' \xrightarrow{\overline{\varphi}} F_0K.$$

where $\overline{F}_1'$ is the saturation of F_1 inside F_0K along $\mathcal{T}'$.

Assume that for some $\mathcal{T} \subset \mathcal{T}'$, $\overline{\varphi}(\overline{F}_1) \subset \tilde{F}_0K$, where $\tilde{F}_0 = \mathrm{Ker}(F_0 \longrightarrow \mathcal{T})$ and $\overline{F}_1 \subset \overline{F}_1'$ is the saturation of F_1 inside F_0K along $\mathcal{T}$. Namely

$$\varphi : F_1 \longrightarrow \overline{F}_1 \xrightarrow{\overline{\varphi}} \tilde{F}_0K \hookrightarrow F_0K.$$

By Lemma 4.3, we want to find an extension $e \in H^1(F_1F_0^*)$ whose underlying bundle E fits in a commutative diagram

$$\begin{tikzcd}
 & F_1 \arrow[r, equal] \arrow[d, hook] & F_1 \arrow[d, hook] \\
\tilde{F}_0 \arrow[r, hook] \arrow[d, equal] & E \arrow[r, two heads] \arrow[d, two heads] & \overline{F}_1 \arrow[d, two heads] \\
\tilde{F}_0 \arrow[r, hook] & E_0 \arrow[r, two heads] & \mathcal{T},
\end{tikzcd}$$

Indeed, then, the Higgs field

$$\psi : E \longrightarrow \overline{F}_1 \xrightarrow{\overline{\varphi}} \tilde{F}_0K \hookrightarrow EK$$

satisfies the conditions of (4.3), so $\lim_{t\to 0}(E, \psi) = (F_0 \oplus F_1, \varphi)$. Moreover, by construction, ψ is of type (n_0, n_1).

Since $H^0(F_1^*\tilde{F}_0K) \subseteq H^0(F_1^*F_0K)$, then $H^0(F_1^*\tilde{F}_0K)^\perp \subseteq H^1(F_1F_0^*)$. Let $\tilde{e} \in H^0(F_1^*\tilde{F}_0K)^\perp$ be general. Since $H^0(F_1^*\tilde{F}_0K)^\perp \subset \mathrm{Im}(H^0(F_1 \otimes \mathcal{T}) \longrightarrow H^1(F_1F_0^*))$, then the image of $\tilde{e}$ under $H^1(F_1F_0^*) \longrightarrow H^1(F_1\tilde{F}_0^*)$ is trivial. Namely, if $\tilde{E}$ is the underlying bundle, then there is an exact diagram

$$\begin{tikzcd}
F_1 \arrow[r, equal] \arrow[d, hook] & F_1 \arrow[d, hook] & \\
\tilde{E} \arrow[r] \arrow[d, two heads] & E \arrow[r] \arrow[d, two heads] & \mathcal{T} \arrow[d, equal] \\
\tilde{F}_0 \arrow[r] & F_0 \arrow[r] & \mathcal{T}
\end{tikzcd}$$

and $\tilde{E} \cong F_1 \oplus \tilde{F}_0$.

Now, since $\tilde{e}$ is general, in particular it is not zero unless $H^1(F_1F_0^*) \cong H^1(F_1\tilde{F_0}^*)$. Assume first $H^1(F_1F_0^*) \ncong H^1(F_1\tilde{F_0}^*)$. Then, E can be taken to be non trivial as an extension, and so if $E/\tilde{F}_0$ had torsion, say $\tilde{\mathcal{T}}$, then $\tilde{\mathcal{T}} \not\subset \mathcal{T}$. But then there would be an exact diagram

$$\begin{tikzcd}
\tilde{F}_0 \arrow[r, equal] \arrow[d, hook] & \tilde{F}_0 \arrow[d, hook] & \\
\tilde{E} \arrow[r] \arrow[d, two heads] & E \arrow[r] \arrow[d, two heads] & \mathcal{T} \arrow[d, hook] \\
F_1 \arrow[r] & E/\tilde{F}_0 \arrow[r] & \mathcal{T}'
\end{tikzcd}$$

which is not possible. Thus E is an extension of $\tilde{F}_0$ by $\overline{F}_1$ and we can conclude.

If $H^1(F_1F_0^*) \cong H^1(F_1\tilde{F_0}^*)$, then there exists a non zero $u \in H^0(F_1\tilde{F}_0^*) \setminus H^0(F_1E_0^*)$.

Then $(u, i) : \tilde{F}_0 \longrightarrow F_0 \oplus F_1$ is a subbundle with torsion free quotient for general u. Then

$$\tilde{F}_0 \hookrightarrow F_0 \oplus F_1 \twoheadrightarrow \overline{F}_1$$

is exact and we may define

$$\psi : F_0 \oplus F_1 \twoheadrightarrow \overline{F}_1 \xrightarrow{\overline{\varphi}} \tilde{F}_0K \hookrightarrow F_0K \oplus F_1K.$$

Conversely, if $\mathcal{E} = (F_0 \oplus F_1, \varphi) \in \mathbf{F}_{\overline{n},\overline{d}}$ is wobbly of type (n_0, n_1), let $\lim_{t\to 0}(E, \psi) = (F_0 \oplus F_1, \varphi)$ for some ψ with iterated kernel filtration $E_0 \subset E_1 = E$. Then, by Lemma 4.3, there is a short exact sequence $F_1 \hookrightarrow E \twoheadrightarrow F_0$, and $\psi|_{F_1} \equiv \varphi$. That is:

$$\varphi : F_1 \hookrightarrow E \xrightarrow{\psi} EK \twoheadrightarrow F_0K.$$

Now, since $\ker(\psi) = E_0$, it follows that the composition $F_1 \longrightarrow E \hookrightarrow E/E_0$ is either zero or an embedding, as by assumption φ is an embedding (so $F_1 \cap E_0 \subset \mathrm{Ker}\varphi = 0$) and $\mathrm{rk}(E/E_0) \leq n_1$ by Lemma 4.4. It cannot be zero as otherwise $F_1 \subset E_0$ contradicting the fact that $\varphi \neq 0$. Thus $F_1 \longrightarrow E_1$ and it must be torsion by Lemma 4.4. This torsion quotient $\mathcal{T}$ cannot be zero as otherwise $(E, \psi) = \mathcal{E}$. Moreover, $\mathcal{T}$ must be the same as the quotient F_1/E_0, and so $\varphi = F_1 \longrightarrow E_1 = \overline{F}_1K^* \xrightarrow{\psi} E_0K = \tilde{F}_0K$ where the saturation $\overline{F}_1$ and elementary modification $\tilde{F}_0$ are taken with respect to $\mathcal{T}$. □

Lemma 4.7. *Assume $\varphi : F_1 \longrightarrow F_0K$ is injective and F_1 is stable. Let $S = F_0/F_1K^*$ be torsion free. Then $\varphi^\perp = \mathrm{Im}(H^1(S^*F_1) \longrightarrow H^1(F_0^*F_1))$.*

Proof. By injectivity of $\varphi|_{F_1}$, $F_1 \cap F_1K^* = 0$. Now, since S is torsion free, there is a long exact sequence in cohomology

$$\cdots \longrightarrow H^1(S^*F_1) \longrightarrow H^1(F_0^*F_1) \twoheadrightarrow \underbrace{H^1(\mathrm{End}(F_1)K)}_{\cong\mathbb{C}} \tag{4.2}$$

where the last isomorphism follows from stability of F_1. Then, $\varphi^\perp$ projects to either zero or the whole $H^1(\mathrm{End}(F_1)K)$. Given that φ is injective, then $H^1(\mathrm{End}(F_1)K) = \varphi^*$, so the latter is not possible. Thus $\varphi^\perp \subset \mathrm{Im}(H^1(S^*F_1) \longrightarrow H^1(F_0^*F_1))$. Equality follows from hyperplanarity of $\varphi^\perp$. □

Proposition 4.8. *Let $n_0 > n_1$. Let $\mathcal{E} = (F_0 \oplus F_1, \varphi) \in \mathbf{F}_{n_0,n_1,d_0,\delta}$ be general. In particular, we assume φ to be injective, S torsion free and F_1 stable. Suppose $\mathcal{E}$ is wobbly, and let $\lim_{t\longrightarrow 0}(E,\psi) = \mathcal{E}$. Then, $E = F_1 \oplus F_0$.*

Proof. By Lemma 4.5, $E \in \varphi^\perp \subset H^1(F_0^* F_1)$, which equals $H^1(S^* F_1)$ by Lemma 4.7. In particular, $F_1 K^* \hookrightarrow E$, as the extension class of E maps to zero inside $H^1(F_1^* K F_1)$. This implies that $F_1 \oplus F_1 K^* \subset E$ is a vector subbundle. Consider the exact diagram derived from the short exact sequences $F_1 K^* \subset F_0 \twoheadrightarrow S$ and $F_1 \subset E \twoheadrightarrow F_0$

$$\begin{tikzcd}
H^0(F_0^* F_1 K^* K) \arrow[r] \arrow[d] & H^0(\mathrm{End}(F_0)K) \arrow[r] \arrow[d] & H^0(F_0^* S K) \arrow[d] \\
H^0(E^* F_1 K^* K) \arrow[r] \arrow[d] & H^0(E^* F_0 K) \arrow[r] \arrow[d] & H^0(E^* S K) \arrow[d] \\
H^0(F_1^* F_1 K^* K) \arrow[r] \arrow[d, "\cup - e"'] & H^0(F_1^* F_0 K) \arrow[r] \arrow[d, "\cup - e"'] & H^0(F_1^* S K) \arrow[d, "\cup - e"'] \\
H^1(F_0^* F_1 K^* K) \arrow[r] & H^1(\mathrm{End}(F_0)K) \arrow[r] & H^1(F_0^* S K).
\end{tikzcd}$$

Then, by assumption φ lifts to $H^0(\mathrm{End}(E)K)$, which by Lemma 4.5 is equivalent to it lifting to $\alpha \in H^0(E^* F_0 K)$. On the other hand, we know that $\varphi \in H^0(F_1^* F_1 K^* K)$, and the former must also lift to $H^0(E^* F_1 K^* K)$ as $e \in \varphi^\perp$. Now, there are two options:

1. If $\varphi^\perp = 0$, then we are done. This happens generically when $\delta \le n_0 n_1 (g-1) + 1$, as by Corollary 3.10, in this case $h^1(F_0^* F_1) = h^0(F_1^* F_0 K) = 1 = h^1(\mathrm{End}(F_1)K)$, so that $\mathrm{Im}(H^1(S^* F_1) \longrightarrow H^1(F_0^* F_1)) = 0$.
2. If $\varphi^\perp \neq 0$, then $\delta > n_0 n_1 (g-1) + 1$. By Corollary 3.10, generically $h^1(F_0 F_1^* K) = 0 = h^0(F_0^* F_1)$. Hence, since S is torsion free, a lift of φ to $H^0(E^* F_1)$ provides a splitting of $F_1 \longrightarrow E$. □

Theorem 4.9. *Let $n_0 \ge n_1$. A general fixed point $(F_0 \oplus F_1, \varphi) \in \mathbf{F}_{\overline{n},\overline{d}}$ is wobbly if and only if*

(1) $F_0/F_1 K^*$ has a torsion subsheaf $\mathcal{T}$ such that φ factors as

$$F_1 \longrightarrow \overline{F_1} \hookrightarrow \tilde{F}_0 K \hookrightarrow F_0 K$$

where $\overline{F_1}$ is the saturation of $\mathrm{Im}(\varphi)$ inside $F_0 K$ and $\tilde{F}_0 = \mathrm{Ker}(F_0 \longrightarrow \mathcal{T})$.

(2) F_0 is wobbly.

(3) F_1 is wobbly.

(4) $h^0(\tilde{S}^* F_1 K) > 0$ where

$$\overline{\mathrm{Im}(\varphi)K^*} \hookrightarrow F_0 \twoheadrightarrow \tilde{S}$$

is exact, and $\overline{\mathrm{Im}(\varphi)K^*}$ denotes the saturation of the image of $\mathrm{Im}(\varphi)K^*$ in F_0.

Proof. Since we assume the point to be general, we will suppose that φ is injective, and F_1 stable.

It is clear after Proposition 4.6 that under conditions (1)–(4) $(F_0\oplus F_1,\varphi)$ admits a nilpotent Higgs bundle making it wobbly. Indeed, Proposition 4.6 proves the sufficiency of (1). Let $\gamma_{(i)}$ be the section whose existence follows from the assumptions in item (i) (for example, $\gamma_{(2)}\in H^0(\mathrm{End}(F_0)K)$ nilpotent). Then $(F_0\oplus F_1,\varphi+\gamma_{(i)})$ is nilpotent and flows down to $\mathcal{E}$.

Let us next see the necessity of (1)–(4). Assume there exists $(E,\psi)\neq\mathcal{E}$ such that $\lim_{t\longrightarrow 0} t(E,\psi)=\mathcal{E}$. Assume first that ψ has order two. Let $F_0'=\mathrm{Ker}(\psi)$, and consider the exact diagram

$$\begin{tikzcd}
 & F_0' \arrow[r, equal] \arrow[d, hook] & F_0' \arrow[d, hook] \\
F_1 \arrow[r, hook] \arrow[d, equal] & E \arrow[r, two heads] \arrow[d, two heads] & F_0 \arrow[d, two heads] \\
F_1 \arrow[r, hook] & F_1' \arrow[r, two heads] & M.
\end{tikzcd}$$

If M has torsion, note that since $S:=F_0/\mathrm{Im}(\varphi)K^*\subset M$, then $\overline{F}_1\subset F_1'$, $F_0'\subset\tilde{F}_0$ and φ factors as stipulated in (1) by Lemma 4.3.

Hence, we may assume that M is torsion free. In this case, we may apply Proposition 4.8 to conclude that $E=F_0\oplus F_1$. Then, there is a section $\psi|_{F_0}:F_0\longrightarrow F_0K\oplus F_1K$ which factors through $F_0/\mathrm{Ker}(\psi)=M$. Since $\mathrm{Im}(\psi)\subset F_0'K$, it follows that either $\psi|_{F_0}\equiv 0$ or F_0 is wobbly. Since $\psi|_{F_0}\equiv 0$ would imply $F_0=F_0'$, which contradicts $\mathcal{E}\neq(E,\psi)$, we are done.

More generally, for higher order Higgs fields ψ, if S has torsion, the existence of (E,ψ) implies, as in the case of order two, that (1) must hold. Indeed, we have $F_1\cap F_1K^*$, so that we have an exact diagram

$$\begin{tikzcd}
 & F_1K^* \arrow[r, equal] \arrow[d, hook] & F_1K^* \arrow[d, hook] \\
F_1 \arrow[r, hook] \arrow[d, equal] & E \arrow[r, two heads] \arrow[d, two heads] & F_0 \arrow[d, two heads] \\
F_1 \arrow[r, hook] & E/F_1K^* \arrow[r, two heads] & S.
\end{tikzcd}$$

Then, $\overline{F}_1\subset E/F_1K^*$, $\overline{F}_1K^*\subset\tilde{F}_0$ and we can conclude that φ factors as in (1).

So we may assume that S is torsion free. Thus, by Proposition 4.8, it must be $E=F_1\oplus F_0$ and

$$\psi=\begin{pmatrix} a & b\\ \varphi & c\end{pmatrix},\qquad a\in H^0(\mathrm{End}(F_1)K),\ c\in H^0(\mathrm{End}(F_0)K), b\in H^0(F_0^*F_1K).$$

Now, consider the iterated kernel filtration $M_k=\mathrm{Ker}(\psi^{k+1})$, and let $M_{s-1}\subsetneq M_s=E$. Then

$$M_k=N_k\oplus N_k',\qquad N_k=M_k\cap F_1,\ N_k'=M_k\cap F_0.$$

Let $R_k=M_k/M_{k-1}$ be the graded terms, which are locally free as Ψ^k induces an embedding $R_k\hookrightarrow M_kK$. Note that $R_k=L_k\oplus L_k'$ with $L_k=N_k/N_{k-1}$ $L_k'=N_k'/N_{k-1}'$. Then, $L_0'=R_0=M_0\subset F_0$ by injectivity of φ. Moreover, since $\varphi:F_1\cong F_1K^*K\subset F_0K$, it follows that $L_1\hookrightarrow L_0'K$ and $L_0'':=L_0'/L_1K^*\cong\mathrm{Im}(L_0'\longrightarrow S)$

under the composition $L_0' \hookrightarrow F_0 \twoheadrightarrow S$. Similarly, one sees that L_i' is an extension of the form $L_{i-1}K^* \hookrightarrow L_i' \twoheadrightarrow L_i''$. On the other hand, we have that $L_s \longrightarrow R_s \twoheadrightarrow L_s''$, as $\mathrm{Im}(\psi) \longrightarrow R_1$ is equivalently 0. Thus, F_1K^* must be a successive extension of the L_iK^*. Now, locally writing ψ in block form with respect to the graded terms in F_1, F_1K^*, S, we have that a and c are upper triangular and b maps the blocks as shown in the following table:

$$\begin{array}{cccccccccc} & L_0' & L_1' & L_2' & \dots & L_{s-1}' & L_0'' & \dots & L_s'' \\ L_1 & 0 & 0 & b_{13} & \dots & b_{1s} & 0 & \dots & b_{1,2s} \\ L_2 & 0 & 0 & \ddots & \ddots & & & & \\ \vdots & \vdots & \vdots & & \ddots & & & & \\ L_s & 0 & 0 & \dots & & & & & 0 \end{array}$$

where we have used that $\psi : M_k \longrightarrow M_{k-1}K$. Then, either a or c are nilpotent, so we would be in cases (3) and (2) respectively, or they are zero. In this latter case, one would have that $\mathrm{Ker}(\psi^2) \cap F_1 = N_2$, which by definition is $\mathrm{Ker}(\psi^3) \cap F_1$. Now, since $\mathrm{Ker}(\psi^2) \cap F_1 \subseteq \mathrm{Ker}(\psi^3) \cap F_1$ with equality only if $\psi^2 = 0$, it must be $\psi^2|_{F_1} = b \circ \varphi \equiv 0$. Namely, b must factor as $F_0 \longrightarrow S \longrightarrow F_1K$. □

Remark 4.10. Theorem 4.9 generalises a criterion found by Pauly and the author in rank three [PPe2].

As a corollary of the proof of Theorem 4.9 it is possible to specify a bit more precisely the type of the wobbly bundle. This is useful towards the understanding of the dynamics defiend by the $\mathbb{C}^\times$-flow within the nilpotent cone.

Corollary 4.11. *Let $\mathcal{E} \in \mathbf{F}_{n_0,n_1,d_0,\delta}$ be wobbly. Then*

(1) If S has torsion, then $\mathcal{E}$ is also of wobbly type (n_0, n_1).
(2) If S is torsion free and F_1 and F_0 are very stable, then $H^0(S^*F_1K) \neq 0$ and $\mathcal{E}$ is of wobbly type $(n_1, n_1, n_0 - n_1)$.

5. Wobbliness of length two components

5.1. **Wobbly and very stable components.** Theorem 4.9 allows us to classify all fixed point components which are wobbly.

Theorem 5.1. *Fixed point components of length two are wobbly if and only if $n_0 > n_1$ and*

$$\delta < 3n_1(n_0 - n_1)(g-1) \tag{5.1}$$

In particular, if $n_0 \geq 3n_1$, all fixed point components of length two are wobbly.

Proof. The last statement is clear from (5.1) and (3.3), as the latter range is a subrange of the former. Indeed,

$$3n_1(n_0 - n_1)(g-1) \geq 2n_0n_1(g-1) \iff 3(n_0 - n_1) \geq 2n_0 \iff n_0 \geq 3n_1.$$

Now, to prove the first statement, since very stability is generic, it is enough to prove that the general point is wobbly for the given range.

First note that for $\delta < 3n_1(n_0 - n_1)(g-1)$, it follows that there exists

$$\beta : F_0 \longrightarrow F_1K \text{ s.t. } \varphi \circ \beta = 0.$$

Indeed, let $\tilde{S}$ be the torsion free part of F_0/F_1K^*. Then

$$\deg(\tilde{S}) \leq d_0 - d_1 + 2n_1(g-1)$$

so that

$$\deg(\tilde{S}^*F_1K) \geq (n_0-n_1)d_1 + 2(n_0-n_1)n_1(g-1) - n_1(d_0-d_1+2n_1(g-1))$$

$$= n_0d_1 - n_1d_0 + 2n_1(n_0-2n_1)(g-1) =$$

$$= -\delta + 4n_1(n_0-n_1)(g-1).$$

So by Riemann–Roch, if (5.1) holds, then

$$-\delta + 4n_1(n_0-n_1)(g-1) > (n_0-n_1)n_1(g-1) \Rightarrow h^0(S^*F_1K) > 0$$

for all pairs (S, F_1) whenever $\delta < 3n_1(n_0-n_1)(g-1)$.

This proves sufficiency of the condition. To check necessity, assume first $n_0 = n_1$. Then, depending on the invariants, either generically $F_1K^* \longrightarrow F_0$ is an isomorphism or it has torsion cokernel. In the first case, wobbliness of the fixed point is equivalent to wobbliness of F_1 by Theorem 4.9. This is non general by Lemma 3.9.

If F_0/F_1K^* is torsion, then all conditions are non generic, as the only potentially generic condition Theorem 4.9(1), which is impossible for degree reasons.

Let us now check that for $n_0 > n_1$ generic bundles cannot be wobbly if (5.1) does not hold. By Lemma 3.9, since by assumption $\delta > 3n_1(n_0-n_1)(g-1)$ and $3n_1(n_0-n_1)(g-1) > n_1(n_0-n_1)(g-1)$, $\mathbf{F}_{\overline{n},\overline{d}}$ dominates $\mathrm{N}_X(n_i,d_i)$. So wobbliness of F_0 and F_1 is not generic. Similarly, the existence of torsion inside F_0/F_1K^* is not generic unless $n_1 = n_0$. Finally, within the given range, the existence of a section $S \longrightarrow F_1K$ has positive codimension. □

Remark 5.2. Theorem 5.1 explains some phenomena observed for particular cases. For example, it is known [PPe2] that in rank three, components of type $(2,1)$ and $(1,2)$ are wobbly if and only if $\delta < 3g-3$. This proves Hausel–Hitchin's conjecture [HH, Remark 5.12], formulated after the computation of the virtual equivariant multiplicites (see Remark 6.7). Similarly, in rank four, all components of type $(3,1)$ were known to be wobbly [HH, Remark 5.13], which is the first fully wobbly case covered by the criterion in Theorem 5.1, corresponding to $n_1 = 1$.

5.2. $\mathrm{U}(n_0,n_1)$**-very stability and wobbly components.** This section gathers some observations resulting in Corollary 5.4.

Definition 5.3. *Let $G_{\mathbb{R}} < \mathrm{GL}(n,\mathbb{C})$ be a real form. A fixed point $\mathcal{E} \in \mathrm{M}_X(G_{\mathbb{R}})$ is $G_{\mathbb{R}}$-very stable if there exists no nilpotent $G_{\mathbb{R}}$-Higgs bundle (E,ψ) such that $\lim_{t\to 0}(E,\psi) = \mathcal{E}$.*

Similarly, we call a fixed point component $\mathbf{F}_{\overline{n},\overline{d}}$ $G_{\mathbb{R}}$-wobbly if it has no $G_{\mathbb{R}}$-very stable points.

Clearly, if a component is very stable, then it is $G_{\mathbb{R}}$-very stable. As a corollary to Theorem 5.1 we find that for length two fixed points this is an equivalence.

Corollary 5.4. *The fixed point component $\mathbf{F}_{n_0,n_1,d_0,\delta}$ is wobbly if and only if it is $\mathrm{U}(n_0,n_1)$-wobbly.*

Proof. By Theorem 5.1, a component is wobbly if and only if it is flown down into by flows of the form $(F_0 \oplus F_1, \psi)$ where

$$\psi = \left(\begin{pmatrix} 0 & \beta \\ \varphi & 0 \end{pmatrix}\right).$$

This is by definition an $U(n_0, n_1)$-Higgs bundle. $\square$

6. Virtual equivariant multiplicities of wobbly components

In what follows, we analyse the failure of polynomiality of equivariant multiplicities.

Lemma 6.1. *Let $\mathcal{E} = (F_0 \oplus F_1, \varphi)$ be a smooth fixed point. Then the $\mathbb{C}^\times$-weight subsheaves of on $\mathrm{End}(E)$ and $\mathrm{End}(E)K$ are as described below:*

Component	$F_1^*F_0$	$F_0^*F_1$	$\mathrm{End}(F_0)$	$\mathrm{End}(F_1)$
Weight	*1*	*-1*	*0*	*0*

Proof. The weight w_i of the action on F_i can be computed by imposing $n_0w_0 + n_1w_1 = 0$, and $w_0 - w_1 = 1$. This implies that the gauge transformation

$$Ad\begin{pmatrix} t^{\frac{n_1}{n}} Id_{F_0} & \\ & t^{\frac{-n_0}{n}} Id_{F_1} \end{pmatrix}$$

takes $\mathcal{E}$ to $t\mathcal{E}$. Thus, the weight on F_0 is $w_0 = \frac{n_1}{n}$ and the weight on F_1 is $w_1 = \frac{-n_0}{n}$. One easily computes from that the weights on the subsheaves $F_iF_j^*$. $\square$

Proposition 6.2. *Let $\mathcal{E} = (F_0 \oplus F_1, \varphi) =: (E, \varphi)$ be general; in particular, assume it is smooth and φ is injective. Then, the $\mathbb{C}^\times$-module $T^+_{\mathcal{E}}$ consists of the following weight spaces:*

$$(T^+_{\mathcal{E}})_1 \cong \frac{H^0(F_1^*F_0K)}{H^0(\mathrm{End}(F_1)) \oplus \mathrm{End}(F_0))} \oplus \left(\frac{H^0(\mathrm{End}(F_1)K) \oplus \mathrm{End}(F_0)K)}{H^0(F_0^*F_1)}\right)^*$$

$$(T^+_{\mathcal{E}})_2 \cong H^1(F_0F_1^*)$$

where $S := F_0/F_1K^*$ is locally free if $n_0 > n_1$ and $\mathcal{T} := F_0/F_1K^*$ is torsion otherwise. In particular

$$\dim(T^+_{\mathcal{E}})_1 = \begin{cases} \delta + 1 + n_0^2(g-1) & n_0 = n_1; \\ (n_0^2 + n_1^2 - n_0n_1)(g-1) + 1 + \delta & n_0 > n_1. \end{cases}$$

$$\dim(T^+_{\mathcal{E}})_2 = -\delta + 3n_0n_1(g-1).$$

Proof. Recall that $T_{\mathcal{E}} = \mathbb{H}^1(C_\bullet)$ for the complex

$$C_\bullet : \mathrm{End}(F_0 \oplus F_1) \xrightarrow{\mathrm{ad}(\varphi)} \mathrm{End}(F_0 \oplus F_1)K.$$

Hence, using the long exact sequence in hypercohomology defining $\mathbb{H}^1$, we have a short exact sequence

$$\begin{array}{c} \operatorname{Coker}\left(H^0(\operatorname{End}(E)) \overset{h^0(\operatorname{ad}(\varphi))}{\to} H^0(\operatorname{End}(E)\otimes K)\right) \\ \Big\downarrow \\ T_{\mathcal{E}} \\ \Big\downarrow \\ \operatorname{Ker}\left(H^1(\operatorname{End}(E)) \overset{h^1(\operatorname{ad}(\varphi))}{\to} H^1(\operatorname{End}(E)\otimes K)\right). \end{array} \tag{6.1}$$

Now, we have the following exact sequences:

$$F_0F_1^* \hookrightarrow F_0F_1^* \overset{ad[\varphi]}{\twoheadrightarrow} 0 \tag{6.2}$$

$$F_0^*F_1 \overset{ad[\varphi]}{\hookrightarrow} \operatorname{End}(F_0)K \oplus \operatorname{End}(F_1)K \twoheadrightarrow \begin{cases} \operatorname{End}(F_0)K \oplus F_1K\otimes\mathcal{T} & n_0 = n_1 \\ F_0^*SK \oplus \operatorname{End}(F_1)K & n_0 > n_1. \end{cases}$$

$$\left.\begin{array}{ll} 0 & n_0 = n_1 \\ S^*F_0 & n_0 > n_1. \end{array}\right\} \hookrightarrow \operatorname{End}(F_0) \overset{ad[\varphi]}{\longrightarrow} F_0F_1^*K \twoheadrightarrow \begin{cases} F_0K\otimes\mathcal{T} & n_0 = n_1 \\ 0 & n_0 > n_1. \end{cases}$$

$$\operatorname{End}(F_1) \overset{ad[\varphi]}{\hookrightarrow} F_1^*F_0K \twoheadrightarrow \begin{cases} F_0K\otimes\mathcal{T} & n_0 = n_1 \\ F_1^*\otimes S\otimes K & n_0 > n_1. \end{cases}$$

Hence, from (6.1), (6.2), and [G, Prop. 4.1] one sees that the positive weight contributions to $\mathbb{H}^1(C_\bullet)$ come from the weight one space

$$\frac{H^0(F_1^*F_0K)}{H^0(\operatorname{End}(F_1))\oplus \operatorname{End}(F_0))} \oplus \left(\frac{H^0(\operatorname{End}(F_1)K)\oplus \operatorname{End}(F_0)K)}{H^0(F_0^*F_1)}\right)^*$$

and the weight two space $H^1(F_0F_1^*)$. The latter has dimension

$$\dim(T_{\mathcal{E}}^+)_2 = h^1(F_0F_1^*) = -\delta + 3n_0n_1(g-1)$$

by stability of $\mathcal{E}$ and unstability of $F_0\oplus F_1$.

To compute the exact dimensions of the weight one spaces, by Corollary 3.10 we have that generically

$$h^0(F_0^*F_1)_{gen} = \begin{cases} 0 & \text{if } \delta > n_0n_1(g-1) \\ 1 + n_0n_1(g-1) - \delta & \text{if } \delta \le n_0n_1(g-1) \end{cases}$$

so

$$\dim\left(\frac{H^0(\operatorname{End}(F_1)K)\oplus \operatorname{End}(F_0)K)}{H^0(F_0^*F_1)}\right)^* =$$

$$= \begin{cases} 2 + (n_0^2+n_1^2)(g-1) & \delta > n_0n_1(g-1) \\ 2 + (n_0^2+n_1^2-n_0n_1)(g-1)+\delta & (n_0-n_1)n_1(g-1) = \delta, \\ 1 + (n_0^2+n_1^2-n_0n_1)(g-1)+\delta & \text{otherwise.} \end{cases}$$

In the above we have used $H^1(\operatorname{End}(F_i)K) = 1$ (by stability of F_i when $\delta \neq n_1(n_0 - n_1)(g-1)$ and for the general pair (F_0, F_1) otherwise by Theorem 3.6).

For $n_0 > n_1$ we have that $\operatorname{Im}(\operatorname{End}(F_1)\oplus \operatorname{End}(F_0) \longrightarrow F_1^*F_0K) = \operatorname{Im}(\operatorname{End}(F_0) \longrightarrow F_1^*F_0K)$ by (6.2). For $n_0 = n_1$, we have that $\operatorname{Im}(H^0(\operatorname{End}(F_1) \oplus \operatorname{End}(F_0) \longrightarrow$

$H^0(F_1^* F_0 K)) = \mathbb{C}$ by stability of F_1 and S. So

$$\dim \frac{H^0(F_1^* F_0 K)}{H^0(\mathrm{End}(F_0) \oplus \mathrm{End}(F_1))} = \begin{cases} \delta - n_0 n_1(g-1) - 1 & \text{if } \delta > n_0 n_1(g-1); \\ -1 & \text{if } \delta = n_1(n_0 - n_1)(g-1) \\ 0 & \text{otherwise .} \end{cases}$$

Thus, we have

$$\dim(T^+_{\mathcal{E}})_1 = 1 + (n_0^2 + n_1^2 - n_0 n_1)(g-1) + \delta$$

which finishes the proof. □

Proposition 6.3. *Let $\mathcal{E}$ be a fixed point in a component with invariant δ. Then*

$$m_{\mathcal{E}}(t) = (1+t)^e p(t)$$

where

$$e = e(\delta, n_0, n_1) = (g-1)\left(-3n_0 n_1 + \sum_{k=1}^{\lfloor \log_2(n) \rfloor} 2^k \left\lfloor \frac{n}{2^k} \right\rfloor^2 + (2^k - 1)\left\lfloor \frac{n}{2^k} \right\rfloor\right) + \delta,$$

and

$$p(t) = \prod_{2 \not| m \leq n} (\sum_{j=0}^{m-1} t^j)^{(2m-1)(g-1)} \prod_{1 \leq k \leq \lfloor \log_2 n \rfloor} \prod_{2 \not| \frac{m}{2^k} = \left[\frac{m}{2^k}\right] \leq n} \left(\frac{\sum_{j=0}^{m-1} t^j}{(1+t)^k} \right)^{(2m-1)(g-1)}$$

is a product of cyclotomic polynomials coprime with $(1-t)(1+t)$.

Proof. By Proposition 6.2, and the fact that $(1-t^2) = (1+t)(1-t)$, we have

$$m_{\mathcal{E}}(t) = \frac{(1-t)^g \prod_{i=2}^n (1-t^i)^{(2i-1)(g-1)}}{(1-t)^{\dim(T^+_{\mathcal{E}})_1 - \delta + 3n_0 n_1(g-1)} (1+t)^{-\delta + 3n_0 n_1(g-1)}}.$$

Now, by the properties of cyclotomic polynomials, we have that $1 - t^i = (1-t)(1 + \cdots + t^{i-1})$ and

$$(1+t)^k | (1 - t^{2^k m}) \qquad , \left((1+t), \frac{(1 - t^{2^k m})}{(1+t)^k} \right) = 1 \text{ if } (m, 2) = 1.$$

Thus, we obtain that

$$m_{\mathcal{E}}(t) = (1-t)^{e_1} (1+t)^{e_2} \prod_{2 \not| m \leq n} (\sum_{j=0}^{m-1} t^j)^{(2m-1)(g-1)} \prod_{1 \leq k \leq \lfloor \log_2 n \rfloor} \prod_{2 \not| \frac{m}{2^k} = \left[\frac{m}{2^k}\right] \leq n} \left(\frac{\sum_{j=0}^{m-1} t^j}{(1+t)^k} \right)^{(2m-1)(g-1)},$$

where

$$\begin{aligned} e_1 &= g + \sum_{i=2}^n (2i-1)(g-1) - \dim(T^+_{\mathcal{E}})_1 - \dim(T^+_{\mathcal{E}})_2 \\ &= 1 + n^2(g-1) - \dim(T^+_{\mathcal{E}}) = 0 \end{aligned}$$

where we have used that downward flows hav Lagrangian fibers. On the other hand

$$e_2 = \sum_{2 \leq 2i \leq n} (4i-1)(g-1) + \sum_{4 \leq 4i \leq n} (8i-1)(g-1) + \cdots + (2^{\lfloor \log_2(n) \rfloor + 1} - 1)(g-1) + \delta - 3n_0 n_1 (g-1).$$

Now, fixing k,

$$\sum_{2^k \leq 2^k i \leq n} (2^{k+1} i - 1) = \sum_{i=1}^{\left\lfloor \frac{n}{2^k} \right\rfloor} (2^{k+1} i - 1) =$$

$$= 2^k\left(\left\lfloor \frac{n}{2^k}\right\rfloor^2 + \left\lfloor \frac{n}{2^k}\right\rfloor\right) - \left\lfloor \frac{n}{2^k}\right\rfloor = 2^k\left\lfloor \frac{n}{2^k}\right\rfloor^2 + (2^k-1)\left\lfloor \frac{n}{2^k}\right\rfloor.$$

Thus

$$e_2 = (g-1)\left(-3n_0n_1 + \sum_{k=1}^{\lfloor \log_2(n)\rfloor} 2^k\left\lfloor \frac{n}{2^k}\right\rfloor^2 + (2^k-1)\left\lfloor \frac{n}{2^k}\right\rfloor\right) + \delta.$$

□

A direct consequence of Theorem 6.3 and [HH, Theorem 5.2] is the following.

Corollary 6.4. *The multiplicities of very stable components of type* (n_0, n_1) *and invariant* δ *are*

$$m_{\mathcal{E}}(\mathbf{F}_{\overline{n},\overline{d}}) = 2^{e_2(\delta,\overline{n})} \prod_{2\not| m\leq n} m^{(2m-1)(g-1)} \prod_{1\leq k\leq \lfloor \log_2 n\rfloor}\ \prod_{2\not|\frac{m}{2^k}=\left[\frac{m}{2^k}\right]\leq n} \left(\frac{m}{2^k}\right)^{(2m-1)(g-1)}.$$

Next we compute polynomiality of equivariant multiplicities.

Theorem 6.5. *Let* $\mathcal{E} \in \mathbf{F}_{\overline{n},\overline{d}}$ *be a smooth fixed point. Then, the virtual equivariant multiplicity* $m_{\mathcal{E}}(t)$ *is a polynomial if and only if*

(1) the partition $(n_0, n_1) \neq (2,1), (4,3)$*, or*
(2) the partition $(n_0, n_1) = (2,1)$ *and* $3(g-1) \leq \delta$*, or*
(3) the partition $(n_0, n_1) = (4,3)$ *and* $8(g-1) < \delta$*.*

Proof. Note that by Proposition 6.3, the statement is equivalent to showing that $e_2 \geq 0$ outside of the specified partitions.

Consider the function of n_1

$$f_n(n_1) = \overbrace{3(n-n_1)n_1}^{r_n(n_1)} - \underbrace{\sum_{k=1}^{\lfloor \log_2(n)\rfloor} 2^k\left\lfloor \frac{n}{2^k}\right\rfloor^2 + (2^k-1)\left\lfloor \frac{n}{2^k}\right\rfloor}_{h(n)}.$$

Then $e_2(n_0, n_1, \delta) = \delta - f_{n_0+n_1}(n_1)$.

Now, a local minimum computation shows that

$$r_n(n_1) \leq r_n\left(\left\lfloor \frac{n}{2}\right\rfloor\right). \tag{6.3}$$

Also, by definition

$$h(2m) = h(2m+1). \tag{6.4}$$

Since $e_2(n_0, n_1, \delta) = \delta - f_n(n_1)(g-1) \geq 0 \iff \delta \geq f_n(n_1)(g-1)$, it is enough to prove that $f(n_1)$ is small enough for maximising partitions. So if the upper bound of $f_n(n_1)$ is smaller than the lower bound δ_{min} for δ, we are done.

If $n = 2m$, then $\delta_{min} = 0$ corresponds to the maximal Toledo of the corresponding $\mathrm{U}(n-n_1, n_1)$-Higgs bundle moduli space. By Theorem 5.1, the component $\mathbf{F}_{(m,m)}$ is very stable, so $e_2(m, m, 0) \geq 0$ by [HH, Corollary 5.4]. Thus by (6.3) and (6.4)

$$\delta(n_0, n_1) \geq 0 = \delta_{min}(m,m) \geq f_{2m}(m)(g-1) \geq f_{2m}(n_1)(g-1) \qquad \forall n_0 + n_1 = 2m.$$

Similarly, for $n = 2m+1$, note that $r_{2m+1}(n_1) = r_{2m}(n_1) + 3n_1(g-1)$. Thus, by (6.4)

$$f_{2m+1}(n_1)(g-1) = f_{2m}(n_1)(g-1) + 3n_1(g-1).$$

Since by (6.4)

$$f_{2m}(m) - f_{2m}(n_1) = r_{2m}(m) - r_{2m}(n_1) = 3(m^2 - n_1(2m - n_1)),$$

in order to prove that $f_{2m+1}(n_1)(g-1) \leq \delta$, it is enough to prove that

$$f_{2m}(m) - 3(m^2 - n_1(2m - n_1)) + 3n_1 \leq n_1(n_0 - n_1) = n_1(2m + 1 - 2n_1).$$

Since $f_{2m}(m) \leq 0$, for the above to hold for all values of δ, it is enough to prove that

$$-3(m^2 - n_1(2m - n_1)) + 3n_1 \leq n_1(2m + 1 - 2n_1),$$

which is equivalent to

$$n_1^2 - n_1(4m + 2) + 3m^2 \geq 0. \tag{6.5}$$

Now, the parabola in n_1 defined by the lower bound of (6.5) has two real roots, only one of which is smaller than $m + 1 > n_1$. This root $\lambda_1(m)$ satisfies

$$m - 1 < \lambda_1(m) = 1 + 2m - \sqrt{1 + m^2 + 4m} < m.$$

Note that necessarily $n_1 \leq m$ when $n = 2m + 1$, as by assumption $n_0 > n_1$. Since for $m = n_1$ equation (6.5) does not hold, and there are no roots other than $\lambda_1(m)$ below m, it follows that for $n_1 < m$, $n_1^2 - n_1(4m + 1) + 3m^2 > 0$. This shows that for $n_0 \neq n_1 + 1$, the virtual equivariant multiplicity is a polynomial. Now, by Theorem 5.1 and [HH, Cor. 5.4], this assertion is also true for $n_0 = n_1 + 1$ not in the range (5.1). We next prove that for $(n_1 + 1, n_1) \neq (4, 3), (2, 1)$, $m_{\mathcal{E}}(t)$ is always a polynomial.

In order to do this it is enough check it for wobbly components of type (n_1+1, n_1), that is, $n_1(g-1) \leq \delta \leq 3n_1(g-1)$. Thus, to check if

$$3n_1(n_1 + 1)(g - 1) - h(2n_1 + 1)(g - 1) \leq \delta$$

holds, it is enough to prove that $3n_1(n_1+1)(g-1) - h(2n_1+1)(g-1) \leq n_1(g-1)$, or, equivalently, that $3n_1^2 + 2n_1 \leq h(2n_1+1)$ for all partitions other than the prescribed ones.

In order to do this, we first prove it for $n = 2^k + 1$. In this case

$$h(2^k + 1) = 2^{2k} + 2^k(k - 2) + 1 > 3n_1^2 + 2n_1 = \frac{3}{4}2^{2k} + 2^{k-1} \quad \forall k > 1.$$

This proves the case of all ranges $n = 2^k + 1$ except $n = 3$, which we will take care of separately. Now, all odd numbers between $2^k + 1$ and $2^{k+1} - 1$ are obtained by adding successive powers 2^j to $2^k + 1$ for $j < k$. So it is enough to check that if $h(2n_1 + 1) > 3n_1^2 + 2n_1$, then also $h(2n_1 + 1 + 2^j) > 3(n_1 + 2^{j-1})^2 + 2(n_1 + 2^{j-1})$ when $j < k := \lfloor \log_2(2n_1 + 1) \rfloor$.

Now

$$h(2n_1 + 1 + 2^j) \geq h(2n_1 + 1) + \sum_{1 \leq s \leq j} 2^s \left(2^{j-s+1} \left\lfloor \frac{2n_1 + 1}{2^s} \right\rfloor + 2^{2(j-s)} \right) =$$

$$= h(2n_1 + 1) + 2^{2j}(1 - 2^{-j}) + 2^{j+1} \sum_{1 \leq s \leq j} \left\lfloor \frac{2n_1 + 1}{2^s} \right\rfloor. \qquad (**)$$

Now, we may further bound

$$(**) \geq \begin{cases} h(2n_1 + 1) + 2^{2j}(1 - 2^{-j}) + 2^{j+1}\left(n_1 + \frac{n_1}{2}\right) & \text{if } n_1 \text{ even} \\ h(2n_1 + 1) + 2^{2j}(1 - 2^{-j}) + 2^{j+1}\left(n_1 + \frac{n_1 - 1}{2} + 1\right) & \text{if } n_1 \geq 5 \text{ odd} \end{cases}$$

Note that for $n_1 \geq 5$ it must be $k = \log_2(2n_1+1) \geq 3$. On the other hand, we have

$$3(n_1+2^{j-1})^2 + 2(n_1+2^{j-1}) = 3n_1^2 + 2n_1 + 3\cdot 2^j n_1 + 3\cdot 2^{2j-2} + 2^j.$$

So to prove the statement for n_1 even or $n_1 \geq 5$ it is enough to check that

$$\begin{cases} 2^{2j}(1-2^{-j}) + 2^{j+1}\left(n_1 + \frac{n_1}{2}\right) \geq 3\cdot 2^j n_1 + 3\cdot 2^{2j-2} + 2^j & \text{if } n_1 \text{ even} \\ 2^{2j}(1-2^{-j}) + 2^{j+1}\left(n_1 + \frac{n_1-1}{2} + 1\right) \geq 3\cdot 2^j n_1 + 3\cdot 2^{2j-2} + 2^j & \text{if } n_1 \geq 5 \text{ odd} \end{cases}$$

and that $h(5) > 3\cdot 2^2 + 2\cdot 2$. This is an easy computation.

Hence, the only partitions to be checked are $(2,1)$ and $(4,3)$, which are computed in Table 1.

□

Corollary 6.6. *Let $\mathcal{E}$ be a smooth point in a wobbly fixed point component. Then $m_{\mathcal{E}}(t)$ is not a polynomial if and only if $\mathcal{E} \in \mathbf{F}_{2,1,\delta}$ where $(g-1) \leq \delta \leq 3(g-1)$ or $\mathcal{E} \in \mathbf{F}_{4,3,\delta}$ where $3(g-1) \leq \delta \leq 8(g-1)$. In particular, wobbly components $\mathbf{F}_{4,3,\delta}$ where $8(g-1) < \delta < 9(g-1)$ have polynomial virtual equivariant multiplicities.*

Proof. This is a rephrasing of Theorem 6.5 once noticing that, when non empty, the detection range for wobbly components is

$$n_1(g-1) \leq \delta < (3n_1(n_1+1) - h(2n_1+1))(g-1). \tag{6.6}$$

The exact range is thus determined by the top bound in (6.6). In particular, the range (6.6) is non empty if and only if $h(2n_1+1) \leq (3n_1^2 + 2n_1)$, and includes all wobbly components if and only if $h(2n_1+1) = 3n_1^2$. Table 1 shows the values for $(2,1)$ and $(4,3)$, thus finishing the proof. □

n_1	$h(2n_1+1)$	$3n_1(n_1+1)$	Non-polynomial $m_{\mathcal{E}}(t)$	Polynomial $m_{\mathcal{E}}(t)$
1	3	6	$(g-1) \leq \delta \leq 3(g-1)$	$\emptyset$
3	28	36	$3(g-1) \leq \delta \leq 8(g-1)$	$8(g-1) < \delta < 9(g-1)$

TABLE 1. (Non-)polynomiality of wobbly virtual multiplicities

Remark 6.7. In rank 3, the only fixed point components which are not of type $(1,1,1)$ are of type $(2,1)$. Hence, a combination of [HH, Theorem 1.2], Theorem 5.1 (or its rank three version from [PPe2]) and Theorem 6.5 shows that, in this case, wobbliness is totally determined by non polynomiality of the virtual equivariant multiplicities. This was conjectured in [HH, Remark 5.12]. Similarly, Theorem 6.5 explains why in rank 4, components of type $(3,1)$ have always polynomial equivariant multiplicities (as noticed in [HH, Remark 5.13]).

REFERENCES

[AG] L. ÁAlvarez-Cóonsul, O. Garcíıa-Prada, and A. Schmitt, *On the geometry of moduli spaces of holomorphic chains over compact Riemann surfaces*, Int. Math. Res. Papers (2006), 1–82.

[A] Y. André, *On nef and semistable hermitian lattices, and their behaviour under tensor product*, Tohoku Math. J. (2) 63(4): 629–649 (2011). DOI: 10.2748/tmj/1325886284.

[BNR] A. Beauville, M.S. Narasimhan, and S. Ramanan, *Spectral curves and the generalised theta divisor.* Journal für die reine und angewandte Mathematik 398 (1989): 169–179. http://eudml.org/doc/153148

[BCGT] O. Biquard, B. Collier, O. García-Prada, D. Toledo, *Arakelov–Milnor inequalities and maximal variations of Hodge structure.* Preprint, arXiv:2101.02759 [math.AG]

[BR] I. Biswas, S. Ramanan, *An Infinitesimal Study of the Moduli of Hitchin Pairs*, J. London Math. Soc.- 2nd Series, vol. 49,219–231, (1994).

[BLP] I. Biswas, M. Logares, Ana Peón-Nieto. *Symplectic Geometry of a Moduli Space of Framed Higgs Bundles*, International Mathematics Research Notices, Volume 2021, Issue 8, April 2021, 5623–5650, https://doi.org/10.1093/imrn/rnz016.

[B] T. Bozec, *Irreducible components of the global nilpotent cone*. International Mathematics Research Notices, Volume 2022, Issue 23, December 2022, Pages 19054–19077,https://doi.org/10.1093/imrn/rnab035

[BGG] S. Bradlow, O. García-Prada, P.B. Gothen, *Moduli spaces of holomorphic triples over compact Riemann surfaces*. Math. Ann. 328, 299–351 (2004). https://doi.org/10.1007/s00208-003-0484-z

[BGGH] S.B. Bradlow, O. García-Prada, P.B. Gothen and J. Heinloth, *Irreducibility of Moduli of Semi-Stable Chains and Applications to* $\mathrm{U}(p,q)$*-Higgs Bundles*, in Andrew Dancer, Jorgen Ellegaard Andersen, and Oscar García-Prada (eds), Geometry and Physics: Volume II: A Festschrift in honour of Nigel Hitchin (Oxford,2018);doi.org/10.1093/oso/9780198802020.003.0018.

[BGG] S.B. Bradlow, O. García-Prada, and P.B. Gothen *Surface group representations and* $\mathrm{U}(p,q)$*-Higgs bundles*, J. Differential Geometry 64 (2003) 111–170.

[BGGH] S.B. Bradlow, O. García-Prada, P.B. Gothen, and J. Heinloth, *Irreducibility of the moduli of chains and* $\mathrm{U}(p,q)$*-Higgs bundles*, Geometry and Physics: Volume II: A Festschrift in honour of Nigel Hitchin (Oxford, 2018);https://doi.org/10.1093/oso/9780198802020.003.0018.

[DP] R. Donagi and T. Pantev, *Geometric Langlands and non-abelian Hodge theory*, Surveys in differential geometry, Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry, 85–116, International Press, Somerville, MA, 2009.

[DP2] R. Donagi and T. Pantev,*Parabolic Hecke eigensheaves*. Preprint. https://doi.org/10.48550/arXiv.1910.02357

[EGA] A. Grothendieck, J. Dieudonné, *Élements de Géométrie Algébrique, Chapter III*, Publ. Math. IHES, 11, 1961.

[F] G. Faltings, *Stable G-bundles and projective connections*, J. Algebraic Geom. 2 (1993), 507–568.

[FGOP] E. Franco, P. B. Gothen, A. G. Oliveira, A. Peón-Nieto, *Unramified covers and branes on the Hitchin system*, Advances in Mathematics 377, (2020). https://doi.org/10.1016/j.aim.2020.107493.

[GP] O. García-Prada and A. Peón-Nieto, *Abelianization of Higgs bundles for quasi-split real groups*. Transformation Groups (2021). doi.org/10.1007/s00031-021-09658-9

[G] P.B. Gothen, *The Betti numbers of the moduli space of rank 3 Higgs bundles*, Internat. J. Math. 5 (1994) 861–875.

[G] S. Gukov, *Quantization via mirror symmetry*. Jpn. J. Math. 6, 65–119 (2011). doi.org/10.1007/s11537-011-1033-2

[GWZ] M. Groechening, D. Wyss and P. Ziegler, *Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration*. Inventiones Mathematicae, 2020, vol. 221, no 2, p. 505–596.

[GW] S. Gukov and E. Witten, *Branes and quantization*, Adv. Theor. Math. Phys. 13(5): 1445–1518 (2009).

[H] T. Hausel, *Enhanced mirror symmetry for Langlands dual Hitchin systems*, arXiv:2112.09455,(to appear in Proceedings of ICM 2022).

[HH] T. Hausel and N. Hitchin, *Very stable Higgs bundles, virtual equivariant multiplicity and mirror symmetry*,Invent. math. 228, 893–989 (2022). https://doi.org/10.1007/s00222-021-01093-7

[HT] T. Hausel and M. Thaddeus, *Mirror symmetry, Langlands duality, and the Hitchin system*. Invent. math. 153, 197–229 (2003). https://doi.org/10.1007/s00222-003-0286-7

[Hi1] N. Hitchin. *The self-duality equations on a Riemann surface*. Proc. London Math. Soc. (3), 55 (1):59–126, (1987).

[Hi2] N. Hitchin. *Stable bundles and integrable systems*. Duke Math. J. 54(1): 91–114 (1987). DOI: 10.1215/S0012-7094-87-05408-1

[Hi3] N. Hitchin, *Multiplicity algebras for rank 2 bundles on curves of small genus*, Preprint, `https://doi.org/10.48550/arXiv.2203.03424`.

[KW] A. Kapustin, E. Witten, *Electric-magnetic duality and the geometric Langlands program*, Comm. Number Theory Phys. **1** (2007), 1–236.

[L] H. Lange, *Universal families of extensions*, J. Algebra, 83, 101–112 (1983).

[La] G. Laumon, *Un analogue global du cône nilpotent*, Duke Math. Jour. 57 (1988), 647–671.

[NR] M.S. Narasimhan and S. Ramanan, Moduli of vector bundles on a compact Riemann surface, Ann. Math., 89 (1969), 14–51.

[P] S. Pal, *On Drinfeld's conjecture*. Preprint `https://doi.org/10.48550/arXiv.2202.11874`

[PPa] S. Pal, C. Pauly, *The wobbly divisors of the moduli space of rank-2 vector bundles*, Advances in Geometry 21 (4), 473–482.

[PPe1] C. Pauly and A. Peón-Nieto, *Very stable bundles and properness of the Hitchin map*, A. Geom Dedicata (2018). `https://doi.org/10.1007/s10711-018-0333-6`.

[PPe2] C. Pauly and A. Peón-Nieto, The nilpotent cone in rank 3, in preparation.

[Pe] A. Peón-Nieto, *Wobbly and shaky bundles and resolutions of rational maps*, Preprint `https://doi.org/10.48550/arXiv.2007.1344`.

[RT] B. Russo, M. Teixidor-i-Bigas, *On a conjecture of Lange*, J. Alg. Geom. 8(3) (1997).

[S] C.T. Simpson, *Higgs bundles and local systems*, Pub. Math. IHES, Tome 75 (1992), 5–95.

[Z] H. Zelaci, *On very stablity of principal G-bundles*. Geom Dedicata 204, 165–173 (2020). `https://doi.org/10.1007/s10711-019-00447-z`.

Ana Peón-Nieto, School of Mathematics, University of Birmingham, Watson Building, Edgebaston, Birmingham B15 2TT, UK

Email address: `a.peon-nieto@bham.ac.uk`