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ERA, 31(5): 2580–2594.
DOI: 10.3934/era.2023131
Received: 23 January 2023
Revised: 24 February 2023
Accepted: 27 February 2023
Published: 07 March 2023

Research article

On constrained minimizers for Kirchhoff type equations with Berestycki-Lions type mass subcritical conditions

Jing Hu and Jijiang Sun*

School of Mathematics and Computer Sciences, Nanchang University, Nanchang, Jiangxi 330031, China

* **Correspondence:** Email: sunjijiang2005@163.com.

Abstract: In this paper, for given mass $m > 0$, we focus on the existence and nonexistence of constrained minimizers of the energy functional

$$I(u) := \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\,dx\right)^2 - \int_{\mathbb{R}^3} F(u)dx$$

on $S_m := \left\{u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = m\right\}$,where $a, b > 0$ and F satisfies the almost optimal mass subcritical growth assumptions. We also establish the relationship between the normalized ground state solutions and the ground state to the action functional $I(u) - \frac{\lambda}{2}\|u\|_2^2$. Our results extend, nontrivially, the ones in Shibata (Manuscripta Math. 143 (2014) 221–237) and Jeanjean and Lu (Calc. Var. 61 (2022) 214) to the Kirchhoff type equations, and generalize and sharply improve the ones in Ye (Math. Methods. Appl. Sci. 38 (2015) 2603–2679) and Chen et al. (Appl. Math. Optim. 84 (2021) 773–806).

Keywords: Kirchhoff type equations; constrained minimizers; L^2-subcritical; Berestycki-Lions type conditions

1. Introduction and main results

In this paper, we are devoted to investigating the following Kirchhoff type problem:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u = \lambda u + f(u), \quad u \in H^1(\mathbb{R}^3), \tag{1.1}$$

with an L^2 constraint

$$\|u\|_{L^2(\mathbb{R}^3)}^2 = m,$$

where $f \in C(\mathbb{R}, \mathbb{R})$, a, b, m are positive constants and $\lambda \in \mathbb{R}$ is not a priori given, and will appear as a Lagrange multiplier.

Problems like (1.1) is related to the stationary analogue of the equation

$$u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u), \tag{1.2}$$

which was proposed by Kirchhoff in [1] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. In [2], Lions proposed an abstract framework for this problem and after that (1.2) began to receive more attention. Due to the strong physical meaning and the presence of the nonlocal term $\int_{\mathbb{R}^3} |\nabla u|^2 dx$, equations like (1.1) have been widely studied during the past decade. We mention that there are two totally different views to explore solutions for problem (1.1) in terms of the parameter $\lambda \in \mathbb{R}$. The first one is to fix the parameter λ. In this case, solutions without any L^2 constraint can be obtained as critical points of the associated functional. We refer the reader to [3–8] and the references therein. Nowadays, finding solutions with a prescribed L^2-norm for problem (1.1) has been the object of an intense activity. In this situation, the parameter λ is unknown and determined by the solution. For related works, one can see [9–19] and the references therein. Here, we would like to introduce some results for (1.1) with mass subcritical growth nonlinearities. In [14], Ye studied the existence and non-existence of normalized solutions for problem (1.1) with $f(u) = |u|^{p-2}u$ ($p \in (2, 6)$), and showed that $p = \frac{14}{3}$ is a L^2-critical exponent. Roughly speaking, for any given mass $m > 0$, when $p \in (2, \frac{14}{3})$, Ye proved that the functional I associated to (1.1) defined by

$$I(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right)^2 - \int_{\mathbb{R}^3} F(u) dx, \tag{1.3}$$

where $F(s) := \int_0^s f(t)dt$, is bounded from below on

$$S_m := \left\{u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = m\right\},$$

and when $p \in (\frac{14}{3}, 6)$, I is unbounded from below on S_m for any $m > 0$. Moreover, for any $p \in (2, \frac{14}{3})$, Ye established the sharp existence of global constraint minimizers for (1.1). Subsequently, for $p \in (2, \frac{14}{3})$, Zeng and Zhang [17] proved the existence and uniqueness of normalized solutions by using a different method. Recently, Li and Ye [11] considered the existence and concentration behavior of L^2-subcritical constraint minimizers for a class of Kirchhoff equations with potentials and the power-type nonlinearity. More recently, replacing $f(u)$ by $K(x)f(u)$ in (1.1), Chen et al. [20] considered the nonautonomous Kirchhoff type equations with mass sub- and super-critical case. More precisely, in the mass subcritical case, Chen et al. [20] obtained the global minimizers when K satisfies some suitable assumptions, and f satisfies

(T_1) $f \in C(\mathbb{R}, \mathbb{R})$, $f(t) = o(t)$ as $t \to 0$, and there exists constant $C > 0$ and $p \in (\frac{10}{3}, \frac{14}{3})$, such that $|f(t)| \le C(1 + |t|^{p-1})$;
(T_2) there exists $\mu_0 \in (2, \frac{14}{3})$, such that $f(t)t \ge \mu_0 F(t) > 0$ for all $t \in \mathbb{R}\backslash\{0\}$;
(T_3) there exists $q_0 \in (2, \frac{10}{3})$, such that $\lim_{|t|\to 0} \frac{F(t)}{|t|^{q_0}} > 0$ or $\lim_{|t|\to 0} \frac{F(t)}{|t|^{\frac{10}{3}}} = 0$.

Motivated by the above works and [21] which was concerned with global minimizers for the nonlinear scalar field equation with L^2 constraint (see also [22, 23]), in this paper, we aim to establish the existence of global L^2 constraint minimizers for problem (1.1) with Berestycki-Lions type conditions, which was first introduced by Berestycki and Lions [24], that we believe to be nearly optimal, and also discuss the relationship between the minimizers v of I on S_m and the ground state to equation (1.1) with $\lambda = \lambda(v)$, where $\lambda(v)$ denotes the Lagrange multiplier. To the best of our knowledge, so far, few results on this issue are known to the nonlocal problem. More precisely, we introduce the following assumptions:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$, $\lim_{t\to 0} \frac{f(t)}{t} = 0$ and $\limsup_{|t|\to\infty} \frac{|f(t)|}{|t|^5} < \infty$;
(f_2) $\limsup_{t\to\infty} \frac{F(t)}{|t|^{14/3}} \le 0$;
(f_3) There exists $\zeta \neq 0$, such that $F(\zeta) > 0$;
(f_4) $\liminf_{t\to 0} \frac{F(t)}{|t|^{10/3}} = +\infty$;
(f_4') $\limsup_{t\to 0} \frac{F(t)}{|t|^{10/3}} < +\infty$;
$(\widetilde{f_4'})$ $\limsup_{t\to 0} \frac{F(t)}{|t|^{10/3}} \le 0$.

Now, we state our first main result which reads as follows:

Theorem 1.1. *Assume that f satisfies $(f_1) - (f_3)$. Then, we have the following conclusions:*

(i) *If (f_4) holds, then for any $m > 0$, $E_m := \inf_{u\in S_m} I(u) < 0$ and is achieved for some $v \in S_m$ and, thus, I admits a constraint minimizer v on S_m.*
(ii) *If (f_4') holds, then there exists a number $m^* > 0$, such that $E_m = 0$ if $m \in (0, m^*]$ and $E_m < 0$ if $m > m^*$. Moreover, when $m > m^*$, E_m is achieved for some $v \in S_m$ and, thus, I admits a constraint minimizer v on S_m; and when $0 < m < m^*$, E_m is not achieved.*
(iii) *If we replace (f_4') by the stronger condition $(\widetilde{f_4'})$, then $E_{m^*} = 0$ is achieved for some $v \in S_{m^*}$ and, thus, I admits a constraint minimizer v on S_{m^*}.*
(iv) *The Lagrange multiplier $\lambda(v)$ corresponding to the minimizer $v \in S_m$ obtained above is negative.*
(v) *If (f_4') holds and we, in addition, assume that $f(t)t \le \frac{10}{3}F(t)$ for $t \in \mathbb{R}$, then E_{m^*} is not achieved.*

Remark 1.1. *It is clear that the nonlinearity $f(t) = |t|^{\frac{4}{3}}t$ fulfills the assumptions in Item (v). We would like to point out that, when $f(t) = |t|^{\frac{4}{3}}t$, Ye [14] derived the exact description of m^* and proved E_{m^*} is not achieved. The optimal achieved function for the well known Gagliardo-Nirenberg inequality plays a crucial role in [14]. However, the methods used in [14] are not available anymore for our general conditions case.*

Remark 1.2. *Due to the existence of nonlocal term, in contrast to the mass constrained nonlinear Schrödinger equations in [21,23], the behavior of f near 0 for Kirchhoff type equation depends heavily on the growth rate $\frac{10}{3}$, not on the mass critical exponent $\frac{14}{3}$. Moreover, from Item (v), the results for the case that $F(t)$ grows like $C|t|^{\frac{10}{3}}$ is totally different from those in [23, Theorem 1.4 (ii)] about the Schrödinger equations. In fact, in [23], the author showed that E_{m^*} is achieved when there exist positive constants C and δ, such that $F(t) = C|t|^{\frac{14}{3}}$ for $|t| \le \delta$. Therefore, our results extend, nontrivially, the ones in [21,23] to Kirchhoff type equations. However, for the Kirchhoff type equation, we do not know whether E_{m^*} is not achieved under the assumption that $F(t)$ grows locally like $C|t|^{\frac{10}{3}}$, i.e., $F(t) = C|t|^{\frac{10}{3}}$ for $|t| \le \delta$.*

Remark 1.3. *There are many functions satisfying our general assumptions and different to the pure power nonlinearity considered in [14], and not satisfying the Ambrosetti-Rabinowitz type conditions*

(T_2). *For example, the function*

$$f(t) = 2t\ln(1+|t|) + \frac{|t|\,t}{1+|t|},$$

satisfies $(f_1)-(f_3)$ *and* $(\widetilde{f_4'})$ *but it does not fulfill* (T_2). *The function*

$$f(t) = |t|^{p-2}t - |t|^{q-2}t, \quad 2 < p < q \le 6$$

satisfies $(f_1)-(f_3)$ *but does not satisfy* (T_2) *if* $q \ge \frac{14}{3}$. *Moreover, it satisfies* (f_4) *and* (f_4') *if* $p < \frac{10}{3}$ *and* $p \ge \frac{10}{3}$, *respectively. Therefore, Theorem 1.1 sharply improves and extends the results in [14, 20].*

Next, inspired by [21], we investigate the relationship between the global constrained minimizers v of I on S_m and the ground state of (1.1) with $\lambda = \lambda(v)$. Indeed, we have the following result.

Theorem 1.2. *Under the assumptions of Theorem 1.1, the following conclusions are held:*

(i) *The minimizer* v *of* I *on* S_m *is a ground state of* (1.1) *with* $\lambda = \lambda(v)$, *i.e.,* $J'_\lambda(v) = 0$ *and*

$$E_m - \frac{\lambda}{2}m = c_\lambda := \inf\{J_\lambda(u) | u \in H^1(\mathbb{R}^3)\backslash\{0\}, J'_\lambda(u) = 0\},$$

where the C^1 *action functional* $J_\lambda : H^1(\mathbb{R}^3) \to \mathbb{R}$ *defined by*

$$J_\lambda(u) = I(u) - \frac{\lambda}{2}\int_{\mathbb{R}^3} |u|^2\,dx. \tag{1.4}$$

In particular, the minimizer v *has constant sign and is radially symmetric up to translation (i.e.,* $v(x) = v(r)$, *where* $r = |x|$) *and monotone with respect to* r.

(ii) *For any given* $\lambda \in \{\lambda(v) : v \in S_m$ *is a minimizer for* I *on* $S_m\}$, *any ground state* $w \in H^1(\mathbb{R}^3)$ *of* (1.1) *is a minimizer of* I *on* S_m, *i.e.,* $w \in S_m$ *and* $I(w) = E_m$.

The remainder of this paper is organized as follows: In Section 2, we give some preliminary lemmas that will be frequently used in the proofs of our main theorems. Section 3 is devoted to dealing with the proof of Theorems 1.1 and 1.2.

Throughout this paper, we use the standard notations. We denote by $C, c_i, C_i, i = 1, 2, \cdots$ for various positive constants whose exact value may change from lines to lines but are not essential to the analysis of the problem. $\|\cdot\|_q$ denotes the usual norm of $L^q(\mathbb{R}^3)$ for $q \ge 2$. We use "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function space, respectively. We will write $o(1)$ to denote quantity that tends to 0 as $n \to \infty$.

2. Preliminaries

In this section, we collect some known results and prove some lemmas, which will be used frequently in what follows. We start with recalling the well-known
Gagliardo-Nirenberg inequality: for $p \in (2, 6)$, there exists a constant $C_p > 0$, such that

$$\|u\|_p^p \le C_p \|\nabla u\|_2^{p\gamma_p} \|u\|_2^{p(1-\gamma_p)}, \quad \forall\, u \in H^1(\mathbb{R}^3), \tag{2.1}$$

where $\gamma_p = \frac{3(p-2)}{2p}$.

The following well-known Brezis-Lieb type splitting result (see [25, Lemma 3.2]) will be useful to study our problem.

Lemma 2.1. *Assume that f satisfies (f_1) and $\{u_n\} \subset H^1(\mathbb{R}^3)$ is bounded and $u_n \rightharpoonup u$ a.e. in $\mathbb{R}^3$ for some $u \in H^1(\mathbb{R}^3)$, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |F(u_n) - F(u_n - u) - F(u)|\, dx = 0. \tag{2.2}$$

Now we summarize some properties of I on S_m which play an important role in our proof.

Lemma 2.2. *Assume that (f_1)–(f_3) are satisfied. Then, the following conclusions hold:*

(i) *For any $m > 0$, $E_m = \inf_{u\in S_m} I(u)$ is well defined and $E_m \le 0$.*
(ii) *There exists $m_0 > 0$, such that $E_m < 0$ for any $m > m_0$.*
(iii) *If (f_4) holds, then one has $E_m < 0$ for any $m > 0$.*
(iv) *If (f_4') holds, then one has $E_m = 0$ for $m > 0$ small enough.*
(v) *The function $m \to E_m$ is continuous and nonincreasing.*

Proof. (i) Note that (f_1) and (f_2) imply that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that

$$F(t) \le C_\varepsilon |t|^2 + \varepsilon |t|^{14/3}, \quad \text{for all } t \in \mathbb{R}. \tag{2.3}$$

Then, for any $u \in H^1(\mathbb{R}^3)$, from (2.3) and (2.1), we deduce that

$$\int_{\mathbb{R}^3} F(u)dx \le C_\varepsilon \int_{\mathbb{R}^3} |u|^2\, dx + \varepsilon \int_{\mathbb{R}^3} |u|^{\frac{14}{3}}\, dx \le C_\varepsilon \|u\|_2^2 + \varepsilon C_{\frac{14}{3}} \|\nabla u\|_2^4 \|u\|_2^{\frac{2}{3}}. \tag{2.4}$$

Thus, by (1.3) and (2.4), choosing $\varepsilon = \frac{b}{8C_{\frac{14}{3}} m^{\frac{1}{3}}}$, for $u \in S_m$, we have

$$I(u) \ge \frac{a}{2} \|\nabla u\|_2^2 + \frac{b}{8} \|\nabla u\|_2^4 - C_\varepsilon m, \tag{2.5}$$

which implies I is coercive and bounded from below on S_m, and, thus, E_m is well-defined.

For any $u \in H^1(\mathbb{R}^3)$ and $s \in \mathbb{R}$, we define $(s * u)(x) := e^{3s/2} u(e^s x)$ for a.e. $x \in \mathbb{R}^3$. Fixed $u \in S_m \cap L^\infty(\mathbb{R}^3)$, it is clear that $s * u \in S_m$ and

$$\|\nabla(s * u)\|_2 \to 0 \quad \text{and} \quad \|s * u\|_\infty \to 0, \quad \text{as } s \to -\infty.$$

Then, by (f_1) and (1.3), we have

$$\lim_{s\to-\infty} I(s * u) = \lim_{s\to-\infty} \left(\frac{a}{2} \|\nabla(s * u)\|_2^2 + \frac{b}{4} \|\nabla(s * u)\|_2^4 - \int_{\mathbb{R}^3} F(s * u)dx \right) = 0.$$

Thus, $E_m \le 0$ for any $m > 0$.

(ii) In view of (f_3) and arguing as in the proof of Theorem 2 in [24], we can find a function $u \in H^1(\mathbb{R}^3)$, such that $\int_{\mathbb{R}^3} F(u)dx > 0$. For any $m > 0$, we set $u_m(x) := u\left(\left(\frac{\|u\|_2^2}{m}\right)^{\frac{1}{3}} x\right)$. Clearly, $u_m \in S_m$. Then, it follows from (1.3) that

$$I(u_m) = \frac{a m^{\frac{1}{3}}}{2\|u\|_2^{\frac{2}{3}}} \|\nabla u\|_2^2 + \frac{b m^{\frac{2}{3}}}{4\|u\|_2^{\frac{4}{3}}} \|\nabla u\|_2^4 - \frac{m}{\|u\|_2^2} \int_{\mathbb{R}^3} F(u)dx,$$

which implies that $E_m \le I(u_m) < 0$ for $m > 0$ large enough.

(iii) For any $m > 0$, we choose $u \in S_m \cap L^\infty(\mathbb{R}^3)$. By (f_4), for $M := \frac{a\|\nabla u\|_2^2}{\|u\|_{\frac{10}{3}}^{\frac{10}{3}}} > 0$, there exists $\delta > 0$, such that $F(t) \ge M|t|^{\frac{10}{3}}$ for any $|t| \le \delta$. Then, for any $s < 0$ small enough, such that $\|s * u\|_\infty \le \delta$ and $e^{2s}\|\nabla u\|_2^2 < \frac{2a}{b}$, by (1.3), we have

$$\begin{aligned} E_m \le I(s*u) &\le \frac{ae^{2s}}{2}\|\nabla u\|_2^2 + \frac{be^{4s}}{4}\|\nabla u\|_2^4 - Me^{2s}\int_{\mathbb{R}^3}|u|^{\frac{10}{3}}dx \\ &= \frac{be^{4s}}{4}\|\nabla u\|_2^4 - \frac{ae^{2s}}{2}\|\nabla u\|_2^2 \\ &< 0. \end{aligned}$$

(iv) Fixed $p \in (\frac{10}{3}, \frac{14}{3})$. By (f_2) and (f_4'), there exists $C > 0$, such that

$$F(t) \le C\left(|t|^{\frac{10}{3}} + |t|^{\frac{14}{3}} + |t|^p\right), \quad \text{for all } t \in \mathbb{R}.$$

For any $u \in H^1(\mathbb{R}^3)$, from (2.1), we have

$$\begin{aligned} \int_{\mathbb{R}^3} F(u)dx &\le C\int_{\mathbb{R}^3}\left(|u|^{\frac{10}{3}} + |u|^{\frac{14}{3}} + |u|^p\right)dx \\ &\le C\left(C_{\frac{10}{3}}\|\nabla u\|_2^2\|u\|_2^{\frac{4}{3}} + C_{\frac{14}{3}}\|\nabla u\|_2^4\|u\|_2^{\frac{2}{3}} + C_p\|\nabla u\|_2^{\frac{3(p-2)}{2}}\|u\|_2^{\frac{6-p}{2}}\right). \end{aligned} \tag{2.6}$$

Taking m small enough, such that

$$CC_{\frac{10}{3}}m^{\frac{2}{3}} \le \frac{a}{4} \text{ and } CC_{\frac{14}{3}}m^{\frac{1}{3}} \le \frac{b}{8}, \tag{2.7}$$

for any $u \in S_m$, by (1.3) and (2.6), we conclude that

$$\begin{aligned} I(u) &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \int_{\mathbb{R}^3}F(u)dx \\ &\ge \|\nabla u\|_2^2\left(\frac{a}{2} + \frac{b}{4}\|\nabla u\|_2^2 - C\left(C_{\frac{10}{3}}m^{\frac{2}{3}} + C_{\frac{14}{3}}m^{\frac{1}{3}}\|\nabla u\|_2^2 + C_p m^{\frac{6-p}{4}}\|\nabla u\|_2^{\frac{3p-10}{2}}\right)\right) \\ &\ge \|\nabla u\|_2^2\left(\frac{a}{4} + \frac{b}{8}\|\nabla u\|_2^2 - CC_p m^{\frac{6-p}{4}}\|\nabla u\|_2^{\frac{3p-10}{2}}\right). \end{aligned} \tag{2.8}$$

By Young's inequality and (2.8), one has

$$\begin{aligned} CC_p m^{\frac{6-p}{4}}\|\nabla u\|_2^{\frac{3p-10}{2}} &= \left[\frac{b}{2(3p-10)}\right]^{\frac{3p-10}{4}}\|\nabla u\|_2^{\frac{3p-10}{2}}\left[\frac{2(3p-10)}{b}\right]^{\frac{3p-10}{4}}CC_p m^{\frac{6-p}{4}} \\ &\le \frac{b}{8}\|\nabla u\|_2^2 + \frac{14-3p}{4}(CC_p)^{\frac{4}{14-3p}}\left[\frac{2(3p-10)}{b}\right]^{\frac{3p-10}{14-3p}}m^{\frac{6-p}{14-3p}} \\ &\le \frac{b}{8}\|\nabla u\|_2^2 + \frac{a}{4}, \end{aligned} \tag{2.9}$$

if we choose $m > 0$ satisfies

$$m^{\frac{6-p}{14-3p}} \le (CC_p)^{\frac{4}{3p-14}} \frac{a}{14-3p}\left[\frac{b}{2(3p-10)}\right]^{\frac{3p-10}{14-3p}}. \tag{2.10}$$

Therefore, from (2.8) and (2.9), we deduce $I(u) \ge 0$ for any $u \in S_m$ if we choose $m > 0$ small enough, such that (2.7) and (2.10) hold. Therefore, from (i), we infer that $E_m = 0$ for $m > 0$ small enough.

(v) To show the continuity, it is equivalent to prove that for a given $m > 0$, and any positive sequence m_k, such that $m_k \to m$ as $k \to \infty$, one has

$$\lim_{k\to\infty} E_{m_k} = E_m. \tag{2.11}$$

In view of the definition of E_{m_k}, for every $k \in \mathbb{N}$, let $u_k \in S_{m_k}$, such that

$$I(u_k) \le E_{m_k} + \frac{1}{k} \le \frac{1}{k}. \tag{2.12}$$

From (2.5), it follows that $\{u_k\}$ is bounded in $H^1(\mathbb{R}^3)$. By (f_1), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$, such that

$$|f(t)| \le \varepsilon|t| + C_\varepsilon|t|^5 \quad \text{and} \quad |F(t)| \le \varepsilon|t|^2 + C_\varepsilon|t|^6, \quad \text{for all } t \in \mathbb{R}. \tag{2.13}$$

Then, noting that $\sqrt{\frac{m}{m_k}}u_k \in S_m$, from $m_k \to m$ as $k \to \infty$, (2.13) and (2.12), similar to the proof of [23, Lemma 2.4], we obtain that

$$E_m \le I\Big(\sqrt{\frac{m}{m_k}}u_k\Big) = I(u_k) + o(1) \le E_{m_k} + o(1). \tag{2.14}$$

On the other hand, choosing a minimization sequence $\{v_n\} \in S_m$ for I, we can follow the same line as in (2.14) to obtain that $E_{m_k} \le E_m + o(1)$. Therefore, we obtain (2.11).

To show that E_m is nonincreasing in $m > 0$, we first claim that for any $m > 0$,

$$E_{tm} \le tE_m, \quad \text{for any } t > 1. \tag{2.15}$$

Indeed, for any $u \in S_m$ and $t > 1$, set $v(x) := u(t^{-\frac{1}{3}}x)$. Then, $v \in S_{tm}$ and we deduce that

$$\begin{aligned} E_{tm} \le I(v) &= \frac{at^{\frac{1}{3}}}{2}\|\nabla u\|_2^2 + \frac{bt^{\frac{2}{3}}}{4}\|\nabla u\|_2^4 - t\int_{\mathbb{R}^3} F(u)dx \\ &= tI(u) + \frac{at^{\frac{1}{3}}(1-t^{\frac{2}{3}})}{2}\|\nabla u\|_2^2 + \frac{bt^{\frac{2}{3}}(1-t^{\frac{1}{3}})}{4}\|\nabla u\|_2^4 \\ &< tI(u). \end{aligned} \tag{2.16}$$

Since $u \in S_m$ is arbitrary, we obtain the inequality (2.15). As a consequence, from (i) and (2.15), it follows that E_m is nonincreasing.

In view of Lemma 2.1, $m^* := \inf\{m \in (0, +\infty), E_m < 0\}$ is well-defined and it is easy to obtain the following property of m^*.

Lemma 2.3. *Assume that* (f_1)*–*(f_3) *are satisfied. Then, the following statements are true:*

(i) *If* (f_4) *holds, then* $m^* = 0$.
(ii) *If* (f_4') *holds, then* $m^* > 0$*; in addition,* $E_m = 0$ *for* $m \in (0, m^*]$ *and* $E_m < 0$ *for* $m \in (m^*, +\infty)$.

The following subadditivity property is crucial in the proof of Theorem 1.1.

Lemma 2.4. *Assume that* (f_1)*–*(f_3) *are satisfied and either* (f_4) *or* (f_4') *holds. Then, for any* $m > m^*$*, we have* $E_m < E_k + E_{m-k}$ *for all* $k \in (0, m)$.

Proof. For any $m > m^*$, let $\{u_n\} \subset S_m$, such that $I(u_n) \to E_m$. We claim that there exists $\delta > 0$, such that

$$\liminf_{n\to\infty} \|\nabla u_n\|_2^2 \geq \delta. \tag{2.17}$$

Indeed, if (2.17) is not true, then passing to a subsequence, $\|\nabla u_n\|_2^2 \to 0$. Thus, by (2.13) and Sobolev's inequality, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(u_n)dx = 0.$$

Then, recalling $m > m^*$, by Lemma 2.3 and (1.3), we deduce that

$$0 > E_m = \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} \left(\frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \int_{\mathbb{R}^3} F(u_n)dx \right) = 0,$$

a contradiction. Therefore, it follows from (2.17) that

$$\begin{aligned} E_{tm} &\leq tI(u_n) + \frac{at^{\frac{1}{3}}(1 - t^{\frac{2}{3}})}{2} \|\nabla u_n\|_2^2 + \frac{bt^{\frac{2}{3}}(1 - t^{\frac{1}{3}})}{4} \|\nabla u_n\|_2^4 \\ &\leq tE_m + \frac{at^{\frac{1}{3}}(1 - t^{\frac{2}{3}})\delta}{2} + \frac{bt^{\frac{2}{3}}(1 - t^{\frac{1}{3}})\delta^2}{4} + o(1), \end{aligned}$$

which implies that for any $t > 1$ and $m > m^*$,

$$E_{tm} < tE_m. \tag{2.18}$$

For $k \in (0, m)$, if $k > m^*$ and $m - k > m^*$, using (2.18), we have

$$E_m < E_k + E_{m-k}. \tag{2.19}$$

On the other hand, if $k \leq m^*$ or $m - k \leq m^*$, from Lemma 2.3, we deduce that $E_k = 0$ or $E_{m-k} = 0$. Then, using (2.18), we also show that (2.19) holds.

Remark 2.1. *It is worth mentioning that the strict inequality in Lemma 2.4 is obtained without the priori assumption "*E_m *is achieved for any* $m > m^*$*", and so our result settles an open question proposed by Jeanjean and Lu in [21, Remark 2.3] in the general conditions framework.*

As in [21], we give a mountain pass type characterization of the nontrivial solutions of (1.1) with $\lambda \in \mathbb{R}$, as below.

Lemma 2.5. *Assume that f satisfies (f_1). If $J'_\lambda(\omega) = 0$ for some $\omega \in H^1(\mathbb{R}^3)\backslash\{0\}$, where the functional J_λ is defined by* (1.4)*, then for any $\delta > 0$ and any $L > 0$, there exist a constant $T = T(\omega, L) > 0$ and a continuous path $\gamma : [0, T] \to H^1(\mathbb{R}^3)$, such that*
(i) $\gamma(0) = 0$, $J_\lambda(\gamma(T)) < -1$, $\max_{t\in[0,T]} J_\lambda(\gamma(t)) = J_\lambda(\omega)$;
(ii) $\gamma(\tau) = \omega$ for some $\tau \in (0, T)$, $J_\lambda(\gamma(t)) < J_\lambda(\omega)$ for any $t \in [0, T]$ such that $\|\gamma(t) - \omega\| \geq \delta$;
(iii) $m(t) = \|\gamma(t)\|_2^2$ is a strictly increasing continuous function with $m(T) > L$.

Proof. For any $\omega \in H^1(\mathbb{R}^3)\backslash\{0\}$ with $J'_\lambda(\omega) = 0$, we define a continuous function

$$\gamma(t) := \begin{cases} \omega(\frac{\cdot}{t}) & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases}$$

Then, it is clear that $m(t) := \|\gamma(t)\|_2^2 = t^3 \|\omega\|_2^2$ is strictly increasing with respect to t and $m(t) \to \infty$ as $t \to \infty$. Since ω is a critical point of J_λ, it follows from (1.4) and the Pohozaev identity (see [3])

$$P(\omega) := \frac{a}{2} \|\nabla\omega\|_2^2 + \frac{b}{2} \|\nabla\omega\|_2^4 - \frac{3}{2}\lambda\|\omega\|_2^2 - 3 \int_{\mathbb{R}^3} F(\omega)dx = 0 \tag{2.20}$$

that

$$\begin{aligned} J_\lambda(\gamma(t)) &= \frac{a}{2} \|\nabla\gamma(t)\|_2^2 + \frac{b}{4} \|\nabla\gamma(t)\|_2^4 - \int_{\mathbb{R}^3} F(\gamma(t))dx - \frac{\lambda}{2} \|\gamma(t)\|_2^2 \\ &= \frac{a}{2}t \|\nabla\omega\|_2^2 + \frac{b}{4}t^2 \|\nabla\omega\|_2^4 - t^3 \int_{\mathbb{R}^3} F(\omega)dx - \frac{\lambda}{2}t^3 \|\omega\|_2^2 \\ &= \frac{a}{2}t \|\nabla\omega\|_2^2 + \frac{b}{4}t^2 \|\nabla\omega\|_2^4 - \frac{t^3}{6} \left(a \|\nabla\omega\|_2^2 + b \|\nabla\omega\|_2^4\right) \\ &= \left(\frac{t}{2} - \frac{t^3}{6}\right) a \|\nabla\omega\|_2^2 + \left(\frac{t^2}{4} - \frac{t^3}{6}\right) b \|\nabla\omega\|_2^4 . \end{aligned}$$

Thus, by a simple computation, $J_\lambda(\gamma(t))$ has a unique maximum at $t = 1$ and $J_\lambda(\gamma(t)) \to -\infty$ as $t \to \infty$. Consequently, from the above argument, for any $L > 0$, there exists a large enough constant $T = T(\omega, L) > 0$, such that $J_\lambda(\gamma(T)) < -1$ and $m(T) > L$ and the continuous path $\gamma(t) : [0, T] \to H^1(\mathbb{R}^3)$ is desired.

3. Proof of main theorems

In this section, we devote to proving our main theorems. We first give the proof of Theorem 1.1.

Proof. [Proof of Theorem 1.1] (i) Fixed $m > 0$, from Lemma 2.2 (iii), one has $E_m < 0$. Let $\{u_n\} \subset S_m$ be a minimization sequence, such that $I(u_n) \to E_m$. By (2.5), $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Up to subsequence, there exists $u \in H^1(\mathbb{R}^3)$, such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, $u_n \to u$ in $L^s_{loc}(\mathbb{R}^3)$ for $s \in [2, 6)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$. Denote

$$\rho := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n|^2 \, dx.$$

Suppose $\rho = 0$. In view of Lions' Lemma [26, Lemma 1.21], one has $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $s \in (2, 6)$. Note that by (f_1) and (f_2), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$, such that

$$F(t) \leq \varepsilon|t|^2 + C_\varepsilon|t|^{\frac{14}{3}}, \quad \text{for all } t \in \mathbb{R}. \tag{3.1}$$

Then, using (3.1) and (2.1), we obtain

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3} F(u_n)dx \le 0.$$

Consequently, in view of Lemma 2.2 (iii), we deduce that

$$0 > E_m = \lim_{n\to\infty} I(u_n) \ge -\limsup_{n\to\infty}\int_{\mathbb{R}^3} F(u_n)dx \ge 0,$$

a contradiction. Thus, $\{u_n\}$ is non-vanishing, i.e., $\rho > 0$. Passing to a subsequence if necessary, there exists $\{y_n\} \subset \mathbb{R}^3$ and $v \in H^1(\mathbb{R}^3)\backslash\{0\}$, such that $u_n(x + y_n) =: \tilde{u}_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, $\tilde{u}_n \to v$ in $L^p_{loc}(\mathbb{R}^3)$ for $p \in [2, 6)$ and $\tilde{u}_n(x) \to v(x)$ a.e. in $\mathbb{R}^3$. Clearly, $\|\widetilde{u_n}\|_2^2 = m$, $I(\tilde{u}_n) \to E_m$ and $\|v\|_2^2 \le m$. Then, from Lemma 2.1, we infer that

$$\begin{aligned} E_m &= \lim_{n\to\infty} I(\tilde{u}_n) \\ &= I(v) + \lim_{n\to\infty}[I(\tilde{u}_n - v) + \frac{b}{2}\|\nabla v\|_2^2\,\|\nabla(\tilde{u}_n - v)\|_2^2] \\ &\ge E_{\|v\|_2^2} + E_{m-\|v\|_2^2}. \end{aligned} \tag{3.2}$$

If $\|v\|_2^2 < m$, it follows from Lemma 2.3 (i), Lemma 2.4 and (3.2) that

$$E_m \ge E_{\|v\|_2^2} + E_{m-\|v\|_2^2} > E_m,$$

a contradiction. Therefore, $\|v\|_2^2 = m$ and so it follows from (3.2) that $\tilde{u}_n \to v$ and $I(v) = E_m$. Hence, $E_m < 0$ is achieved at $v \in S_m$.

(ii) By Lemma 2.3 (ii), when $m > m^*$ one has $E_m < 0$ and when $0 < m \le m^*$ one has $E_m = 0$. For $m > m^*$, one can follow the same line in the proof of Item (i) to obtain that $E_m < 0$ is achieved at some $v \in S_m$. Now we show that if $0 < m < m^*$ then $E_m = 0$ is not achieved. Indeed, arguing indirectly, we assume that there exists $m \in (0, m^*)$, such that $E_m = 0$ is achieved at some $v \in S_m$. Then, from Lemma 2.3 (ii) and (2.16), it follows that

$$0 = E_{m^*} < \frac{m^*}{m} I(v) = \frac{m^*}{m} E_m = 0,$$

a contradiction.

(iii) Let $m_n = m^* + \frac{1}{n}$. Then, from Lemma 2.3 (ii), $E_{m_n} < 0$ for all $n \in \mathbb{N}^+$. Similar to the proof of Item (i), there exists $\{u_n\} \subset S_{m_n}$, such that

$$I(u_n) = E_{m_n} < 0, \text{ for all } n \in \mathbb{N}^+. \tag{3.3}$$

Since by Lemmas 2.2 (v) and 2.3 (ii),

$$I(u_n) = E_{m_n} \to E_{m^*} = 0, \tag{3.4}$$

it follows from (2.5) that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Set

$$\rho := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)} |u_n|^2\, dx.$$

Assume $\rho = 0$. From Lions' Lemma [26, Lemma 1.21], $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $s \in (2, 6)$. By (f_2) and $(\widetilde{f_4'})$, for any $\varepsilon > 0$, there exist C_ε, such that

$$F(t) \le \varepsilon |t|^{\frac{10}{3}} + C_\varepsilon |t|^{\frac{14}{3}}, \quad \text{for all } t \in \mathbb{R}. \tag{3.5}$$

Then,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(u_n)dx \le 0.$$

Thus, by (3.4), one has

$$0 = E_{m^*} = \lim_{n\to\infty} I(u_n) \ge \lim_{n\to\infty} \left(\frac{a}{2} \|\nabla u_n\|_2^2 + \frac{b}{4} \|\nabla u_n\|_2^4 \right),$$

which implies $\|\nabla u_n\|_2 \to 0$. Then, it follows from (1.3), (3.5) and (2.1) that

$$I(u_n) \ge \frac{1}{4} \|\nabla u_n\|_2^2 \left(2a + b \|\nabla u_n\|_2^2 - 4\varepsilon \|u_n\|_2^{\frac{4}{3}} - 4C_\varepsilon \|u_n\|_2^{\frac{2}{3}} \|\nabla u_n\|_2^2 \right).$$

Therefore, if we choose $\varepsilon > 0$ small enough, $I(u_n) \ge 0$ for large $n \in \mathbb{N}^+$. This contradicts (3.3). Thus, $\rho > 0$. Up to subsequence, there exists $\{y_n\} \subset \mathbb{R}^3$ and $v \in H^1(\mathbb{R}^3)\backslash\{0\}$, such that $u_n(x+y_n) =: \overline{u}_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, $\overline{u}_n \to v$ in $L^p_{loc}(\mathbb{R}^3)$ for $p \in [2, 6)$ and $\overline{u}_n(x) \to v(x)$ a.e. in $\mathbb{R}^3$. Then, $\|\overline{u}_n\|_2^2 = \|u_n\|_2^2 \to m^*$, $I(\overline{u}_n) \to E_{m^*}$ and $\|v\|_2^2 \le m^*$. As a consequence, by (3.4), Lemma 2.1, Lemma 2.2 (v) and Lemma 2.3 (ii), we obtain

$$\begin{aligned} 0 = E_{m^*} &= \lim_{n\to\infty} I(\overline{u}_n) \\ &= I(v) + \lim_{n\to\infty} [I(\overline{u}_n - v) + \frac{b}{2} \|\nabla \overline{u}\|_2^2 \|\nabla(\overline{u}_n - v)\|_2^2] \\ &\ge E_{\|v\|_2^2} + E_{m^* - \|v\|_2^2} = 0, \end{aligned} \tag{3.6}$$

which implies $\|\nabla(\overline{u}_n - v)\|_2^2 \to 0$. Then, using (3.5), (2.1) and (1.3), one can show that

$$\lim_{n\to\infty} I(\overline{u}_n - v) \ge 0.$$

Therefore, from (3.6), it follows that $I(v) = \lim_{n\to\infty} I(\overline{u}_n) = E_{m^*} = 0$. Noting that by Item (ii), E_m is not achieved for any $m \in (0, m^*)$, we conclude that $\|v\|_2^2 = m^*$. Hence, $E_{m^*} = 0$ is achieved at $v \in S_{m^*}$.

(iv) For any minimizer $v \in S_m$ of I, from the Pohozaev identity associated to (1.1) (see (2.20)) and the fact that $I(v) = E_m \le 0$, we deduce that

$$0 \ge I(v) = I(v) - \frac{1}{3}P(v) = \frac{a}{3}\|v\|_2^2 + \frac{b}{12}\|v\|_2^4 + \frac{1}{2}\lambda(v)m$$

and, therefore, $\lambda(v) < 0$.

(v) From Item (ii), $m^* > 0$. Arguing indirectly, we suppose that there exists $v \in S_{m^*}$ such that $I(v) = E_{m^*} = 0$. Then,

$$\frac{a}{2}\|\nabla v\|_2^2 + \frac{b}{4}\|\nabla v\|_2^4 = \int_{\mathbb{R}^3} F(v)dx, \tag{3.7}$$

and there exists $\lambda(v) \in \mathbb{R}$, such that v is a solution of (1.1) with $\lambda = \lambda(v)$. As in Item (iv), $\lambda < 0$. Moreover, v lies in the corresponding Nehari manifold, i.e.,

$$a\|\nabla v\|_2^2 + b\|\nabla v\|_2^4 = \lambda\|v\|_2^2 + \int_{\mathbb{R}^3} f(v)v dx, \tag{3.8}$$

and satisfies the folowing Pohozaev identity

$$\frac{a}{6}\|\nabla v\|_2^2 + \frac{b}{6}\|\nabla v\|_2^4 = \frac{\lambda}{2}\|v\|_2^2 + \int_{\mathbb{R}^3} F(v)dx, \tag{3.9}$$

Noting that $f(t)t \leq \frac{10}{3}F(t)$, combining (3.7) and (3.8), we conclude that

$$\frac{a}{5}\|\nabla v\|_2^2 - \frac{b}{20}\|\nabla v\|_2^4 \geq -\frac{3\lambda}{10}\|v\|_2^2. \tag{3.10}$$

In view of (3.7) and (3.9), we then obtain that

$$\frac{2a}{3}\|\nabla v\|_2^2 + \frac{b}{6}\|\nabla v\|_2^4 = -\lambda\|v\|_2^2,$$

which, jointly with (3.10), implies $\|\nabla v\|_2^4 = 0$. Hence, $v = 0$, contrary to $v \in S_{m^*}$. The proof is complete.

Now we present the proof of Theorem 1.2.

Proof. [Proof of Theorem 1.2] (i) In order to show that the minimizer $v \in S_m$ of I is a ground state of (1.1) with $\lambda = \lambda(v)$, it is equivalent to prove that for any $\omega \in H^1(\mathbb{R}^3)\backslash\{0\}$, such that $J'_\lambda(\omega) = 0$,

$$J_\lambda(\omega) \geq J_\lambda(v) = E_m - \frac{1}{2}\lambda m.$$

In view of Lemma 2.5, for $L := m > 0$, there exists a continuous path $\gamma : [0, T] \to H^1(\mathbb{R}^3)$ satisfying $J_\lambda(\omega) = \max_{t\in[0,T]} J_\lambda(\gamma(t))$ and there exists $t_0 \in (0, T)$, such that $\|\gamma(t_0)\|_2^2 = m$. As a consequence,

$$J_\lambda(\omega) = \max_{t\in[0,T]} J_\lambda(\gamma(t)) \geq J_\lambda(\gamma(t_0)) = I(\gamma(t_0)) - \frac{1}{2}\lambda m \geq E_m - \frac{1}{2}\lambda m,$$

as required.

Now, we prove that any minimizer v of I on S_m has constant sign. Indeed, for any given minimizer $v \in S_m$ of I, using the notations $v^+ := \max\{0, v\}$ and $v^- := \min\{0, v\}$, if $m^\pm := \|v^\pm\|_2^2 \neq 0$, then $m = m^+ + m^-$ and, thus, by (2.15), we have

$$\begin{aligned} E_m = I(v) &= I(v^+) + I(v^-) + \frac{b}{2}\int_{\mathbb{R}^3} |\nabla v^+|^2 dx \int_{\mathbb{R}^3} |\nabla v^-|^2 dx \\ &\geq I(v^+) + I(v^-) \geq E_{m^+} + E_{m^-} \geq \frac{m^+}{m}E_m + \frac{m^-}{m}E_m = E_m, \end{aligned}$$

which implies

$$\int_{\mathbb{R}^3} |\nabla v^+|^2 dx \int_{\mathbb{R}^3} |\nabla v^-|^2 dx = 0.$$

Therefore, we obtain $v^+ = 0$ or $v^- = 0$, a contradiction. Hence, v has constant sign. Without loss of generality, we may assume $v \geq 0$. Noting that by regularity, any nonnegative ground state of (1.1) with $\lambda = \lambda(v)$ is of class C^1, we also deduce from [27, Theorem 2] that v is radially symmetric with respect to the origin up to translation in $\mathbb{R}^3$ (i.e., $v(x) = v(r)$, where $r = |x|$). Moreover, in view of [28, Lemma 3.2], we can follow the same line of the proof of [21, Theorem 1.4] to prove that v is nonincreasing with respect to the radial variable. Therefore, we obtain that the minimizer v is radially symmetric up to translation and monotone with respect to r. We omit the details and leave them to the reader.

(ii) Obviously, from (i), we infer that any ground state $\omega \in H^1(\mathbb{R}^3)$ of (1.1) satisfies

$$J_\lambda(\omega) = E_m - \frac{1}{2}\lambda m. \tag{3.11}$$

Arguing indirectly, we assume that $\|\omega\|_2^2 \neq m$. For given $\delta := \left|\sqrt{m} - \|\omega\|_2\right| > 0$ and $L := m > 0$, from Lemma 2.5, there exists a continuous path $\gamma : [0, T] \to H^1(\mathbb{R}^3)$ and there exists $t_0 \in (0, T)$, such that $\|\gamma(t_0)\|_2^2 = m$ and $\|\gamma(t_0) - \omega\|_2 \geq \delta$. Then, from Lemma 2.5 (ii), we have

$$J_\lambda(\omega) > J_\lambda(\gamma(t_0)) = I(\gamma(t_0)) - \frac{1}{2}\lambda m \geq E_m - \frac{1}{2}\lambda m,$$

which contradicts with (3.11). It follows that $\|\omega\|_2^2 = m$ and $I(\omega) = E_m$. This completes the proof.

Acknowledgments

We would like to thank the referee for his/her valuable comments and helpful suggestions, which have led to an improvement of the presentation of this paper. This work is supported by NSFC (No. 11861046) and Natural Science Foundation of Jiangxi Provincial (No. 20212BAB201026).

Conflict of interest

We declare no conflicts of interest in this paper.

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