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Research article

Some identities of degenerate higher-order Daehee polynomials based on λ-umbral calculus

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Abstract: The degenerate versions of special polynomials and numbers, initiated by Carlitz, have regained the attention of some mathematicians by replacing the usual exponential function in the generating function of special polynomials with the degenerate exponential function. To study the relations between degenerate special polynomials, λ-umbral calculus, an analogue of umbral calculus, is intensively applied to obtain related formulas for expressing one λ-Sheffer polynomial in terms of other λ-Sheffer polynomials. In this paper, we study the connection between degenerate higher-order Daehee polynomials and other degenerate type of special polynomials. We present explicit formulas for representations of the polynomials using λ-umbral calculus and confirm the presented formulas between the degenerate higher-order Daehee polynomials and the degenerate Bernoulli polynomials, for example. Additionally, we investigate the pattern of the root distribution of these polynomials.

Keywords: generating function; λ-umbral calculus; λ-Sheffer polynomial; special polynomial; degenerate higher-order Daehee polynomial

1. Introduction

Special polynomials play a significantly important role in the development of several branches of mathematics, engineering, and physics by providing us with useful identities and properties. The study of special polynomials provides many useful identities, their relations, and representations associated with special numbers and polynomials. One of the powerful tools in this study is to investigate their generating functions [1, 2] and connections [3–6] using the umbral calculus [7]. Furthermore, to better understand generating functions in special polynomials, the degenerate type of special polynomials has been extensively studied in many areas such as probability theory, fuzzy theory, connection problems, and other combinatorial theories in recent years by many mathematicians [8–11]. Since the introduction of degenerate versions of special polynomials and numbers by Carlitz [12], many researchers have been interested in the relationships between them. In addition, the degenerate version of umbral calculus, called λ-umbral calculus, plays a very powerful role in studying the relationships between degenerate versions of special polynomials and numbers. Recently, the Daehee polynomials and numbers were originally introduced as a new type of special polynomials by Kim and Kim [13] and thereafter their related properties and relationships with other polynomials have been extensively studied.

In this study, we derive the formulas expressing degenerate higher-order Daehee polynomials in terms of the degenerate versions of other special polynomials by making use of λ-umbral calculus. These formulas provide the degenerate Daehee polynomials by taking $r = 1$ and the Daehee polynomials by letting $\lambda \to 0$. We first review the λ-analogue of umbral calculus: a class of λ-linear functionals on the polynomials, λ-differential operators based on the family of λ-linear functionals, and also λ-Sheffer sequences. See [14] and the references therein for more details on these contents.

The rest of this section briefly recalls some necessary notations and definitions that are needed throughout this paper. Throughout this paper, we assume that $\lambda \in \mathbb{R}\backslash\{0\}$ for simplicity.

The degenerate exponential function $e_\lambda^x(t)$ is defined by

$$e_\lambda^x(t) := (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \quad e_\lambda(t) := e_\lambda^1(t) = \sum_{n=0}^{\infty} (1)_{n,\lambda} \frac{t^n}{n!}, \quad \text{(see [10, 13, 15, 16])}, \tag{1.1}$$

where $(x)_{n,\lambda}$ is a λ-analogue of the falling factorial sequence which is given by

$$(x)_{n,\lambda} = x(x-\lambda)\cdots(x-(n-1)\lambda) \text{ for } n \geq 1 \text{ and } (x)_{0,\lambda} = 1, \quad \text{(see [14, 17])}. \tag{1.2}$$

Also, the degenerate logarithm function is given by $\log_\lambda(t) := \frac{1}{\lambda}(t^\lambda - 1)$, which is the compositional inverse of $e_\lambda(t)$, i.e.,

$$\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda(t)) = t.$$

In this study, we consider the degenerate higher-order Daehee polynomials $D_{n,\lambda}^{(r)}(x)$ which are given by the generating function to be

$$\left(\frac{\log_\lambda(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}, \quad r \in \mathbb{N}, \quad \text{(see [10, 15, 16])}. \tag{1.3}$$

Especially, we call $D_{n,\lambda}(x) := D_{n,\lambda}^{(1)}(x)$ the degenerate Daehee polynomials when $r = 1$ and $D_{n,\lambda} := D_{n,\lambda}(0)$ the degenerate Daehee numbers when $x = 0$.

The degenerate Stirling numbers of the first kind $S_{1,\lambda}(n,m)$ and the second kind $S_{2,\lambda}(n,m)$ are respectively given by

$$\frac{1}{m!}\left(\log_\lambda(1+t)\right)^m = \sum_{n=m}^{\infty} S_{1,\lambda}(n,m) \frac{t^n}{n!}, \quad (m \geq 0), \quad \text{(see [9, 18])} \tag{1.4}$$

and

$$\frac{1}{m!}(e_\lambda(t)-1)^m = \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!}, \quad (m \geq 0), \quad \text{(see [9, 18])}. \tag{1.5}$$

Note that the falling factorial sequence $(t)_n$ is given by

$$(t)_n = \begin{cases} t(t-1)(t-2)\cdots(t-(n-1)) & \text{for } n \geq 1, \\ (t)_0 = 1 & \text{when } n = 0, \end{cases} \qquad \text{(see [19])},$$

which provides the relation with the λ-analogue of the falling factorial sequence such as

$$(t)_{n,\lambda} = \sum_{m=0}^{n} S_{2,\lambda}(n,m)(t)_m, \quad (n \geq 0).$$

The main contribution of this paper is to provide various representations of the degenerate higher-order Daehee polynomials and numbers using λ-umbral calculus in terms of other well-known special polynomials and numbers. In more detail, we derive formulas for the n-th order of degenerate Daehee polynomials with the degenerate falling factorial polynomials, the degenerate type 2 Bernoulli polynomials, the degenerate Bernoulli polynomials, the degenerate Euler polynomials, the degenerate Mittag-Leffer polynomials, the degenerate Bell polynomials, and the degenerate Frobenius-Euler polynomials (see Theorems 2.1–2.7) as well as their inversion formulas. Therefore, we see that this technique enables us to represent various well-known polynomials in terms of degenerate higher-order Daehee polynomials and vice versa as a classical connection problem. In addition, to confirm the formulas, we present computational results between the degenerate higher-order Daehee polynomials and the degenerate Bernoulli polynomials for fixed variables. Moreover, we investigate the pattern of the root distribution of the polynomials.

2. Representations of degenerate higher-order Daehee polynomials

Now, we provide brief review of λ-umbral calculus: Let $\mathbb{P}$ be the algebra of polynomials in t over $\mathbb{C}$, i.e.,

$$\mathbb{P} = \mathbb{C}[t] = \left\{ \sum_{n=0}^{\infty} a_n t^n \,\middle|\, a_n \in \mathbb{C} \text{ with } a_n = 0 \text{ for all but finite number of } n \right\}.$$

and let $\mathcal{F}$ be the algebra of formal power series in t over the field $\mathbb{C}$ of complex numbers

$$\mathcal{F} = \left\{ f(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \,\middle|\, a_n \in \mathbb{C} \right\}.$$

Then, the λ-linear functional $\langle f(t)|\cdot\rangle_\lambda$ on $\mathbb{P}$ for $f(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \in \mathcal{F}$ is given by

$$\langle f(t)|(x)_{n,\lambda}\rangle_\lambda = a_n, \quad (n \geq 0), \quad \text{(see [14])}, \tag{2.1}$$

and it satisfies

$$\langle t^k|(x)_{n,\lambda}\rangle_\lambda = n!\delta_{n,k}, \quad \text{(see [14])}, \tag{2.2}$$

where $\delta_{n,k}$ is the Kronecker delta.

Note that the order of the formal power series for a nontrivial $f(t)$, $o(f(t))$, is represented by the smallest integer k for which a_k does not vanish. Especially, we call $f(t)$ a delta series when $o(f(t)) = 1$, and also we say $f(t)$ an invertible series when $o(f(t)) = 0$, (see [1, 7, 14] for details).

For a non-negative integer order k, the λ-differential operator $(t^k)_\lambda$ on $\mathbb{P}$ is defined by

$$(t^k)_\lambda(x)_{n,\lambda} = \begin{cases} (n)_k(x)_{n-k,\lambda} & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n, \end{cases} \quad \text{(see [14, 20])}. \tag{2.3}$$

In general, for $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$, the λ-differential operator $\left(f(t)\right)_\lambda$ is satisfied with

$$\left(f(t)\right)_\lambda(x)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k} a_k (x)_{n-k,\lambda}. \tag{2.4}$$

Or equivalently, one can express $\left(f(t)\right)_\lambda$ as

$$\left(f(t)\right)_\lambda = \sum_{k=0}^{\infty} \frac{a_k}{k!}(t^k)_\lambda.$$

For a delta series $f(t)$ and an invertible series $g(t)$, i.e., $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $p_{n,\lambda}(x)$ of polynomials $deg(p_{n,\lambda}(x)) = n$ satisfying the orthogonality condition

$$\left\langle g(t)(f(t))^k \middle| p_{n,\lambda}(x) \right\rangle_\lambda = n!\delta_{n,k}, \quad (n, k \ge 0). \tag{2.5}$$

Here, $p_{n,\lambda}(x)$ is called the λ-Sheffer sequence for $(g(t), f(t))$ denoted by $p_{n,\lambda}(x) \sim \left(g(t), f(t)\right)_\lambda$.

We recall that $p_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ if and only if

$$\frac{1}{g(\bar{f}(t))} e_\lambda^x(\bar{f}(t)) = \sum_{n=0}^{\infty} p_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [7, 20])}. \tag{2.6}$$

Here $\bar{f}(t)$ represents the compositional inverse of $f(t)$, i.e., $f\left(\bar{f}(t)\right) = \bar{f}\left(f(t)\right) = t$.

For given a pair of λ-Sheffer sequences $p_{n,\lambda}(x) \sim \left(g(t), f(t)\right)_\lambda$ and $q_{n,\lambda}(x) \sim \left(h(t), \ell(t)\right)_\lambda$, we have the relation:

$$p_{n,\lambda}(x) = \sum_{k=0}^{n} \mu_{n,k} q_{k,\lambda}(x), \tag{2.7}$$

where $\mu_{n,k}$ is obtained by

$$\mu_{n,k} = \frac{1}{k!} \left\langle \frac{h\left(\bar{f}(t)\right)}{g\left(\bar{f}(t)\right)} \left(\ell(\bar{f}(t))\right)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda.$$

Likewise, if $q_{n,\lambda}(x)$ is expressed in terms of $p_{n,\lambda}(x)$ as

$$q_{n,\lambda}(x) = \sum_{k=0}^{n} \nu_{n,k} p_{k,\lambda}(x), \tag{2.8}$$

then $\nu_{n,k}$ can be obtained by

$$\nu_{n,k} = \frac{1}{k!} \left\langle \frac{g\left(\bar{\ell}(t)\right)}{h\left(\bar{\ell}(t)\right)} \left(f(\bar{\ell}(t))\right)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda.$$

It is easily shown that for $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$,

$$\langle f(t)g(t)|p(x)\rangle_\lambda = \langle g(t)|\,(f(t))_\lambda\, p(x)\rangle_\lambda = \langle f(t)|\,(g(t))_\lambda\, p(x)\rangle_\lambda, \quad \text{(see [14]).}$$

We also note that from $(x)_{n,\lambda} \sim (1,t)_\lambda$, any λ-Sheffer sequence $p_{n,\lambda}(x) \sim (g(t), f(t))_\lambda$ is represented by

$$p_{n,\lambda}(x) = \sum_{k=0}^{n} \frac{1}{k!} \left\langle \frac{1}{g(\bar{f}(t))} (\bar{f}(t))^k \middle| (x)_{n,\lambda} \right\rangle_\lambda (x)_{k,\lambda}. \tag{2.9}$$

Now, we want to present representations of the degenerate higher-order Daehee polynomials $D_{n,\lambda}^{(r)}(x)$ by using the algebraic properties of λ-Sheffer sequences.

From (1.3), we have that $\sum_{n=0}^{\infty} D_{n,\lambda}^{(r)}(x)\frac{t^n}{n!} = \left(\frac{\log_\lambda(1+t)}{t}\right)^r e_\lambda^x(\log_\lambda(1+t))$, so that we consider $f(t) = e_\lambda(t)-1$, $\bar{f}(t) = \log_\lambda(1+t)$, and $g(t) = \frac{e_\lambda(t)-1}{t}$ in the view of (2.6) to obtain

$$D_{n,\lambda}^{(r)}(x) \sim \left(\left(\frac{e_\lambda(t)-1}{t}\right)^r, e_\lambda(t)-1 \right)_\lambda. \tag{2.10}$$

If we let $p_{n,\lambda}(x) = \sum_{\ell=0}^{n} \mu_\ell D_{\ell,\lambda}^{(r)}(x)$, then, by using (2.5) we have

$$\begin{aligned}
\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^r (e_\lambda(t)-1)^k \middle| p_{n,\lambda}(x) \right\rangle_\lambda &= \sum_{\ell=0}^{n} \mu_\ell \left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^r (e_\lambda(t)-1)^k \middle| D_{\ell,\lambda}^{(r)}(x) \right\rangle_\lambda \\
&= \sum_{\ell=0}^{n} \mu_\ell \ell! \delta_{k,\ell} \\
&= k!\mu_k,
\end{aligned}$$

which implies

$$\mu_k = \frac{1}{k!} \left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^r (e_\lambda(t)-1)^k \middle| p_{n,\lambda}(x) \right\rangle_\lambda.$$

Thus, for $p_{n,\lambda}(x) \in \mathbb{P}$ we have

$$p_{n,\lambda}(x) = \sum_{k=0}^{n} \frac{1}{k!} \left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^r (e_\lambda(t)-1)^k \middle| p_{n,\lambda}(x) \right\rangle_\lambda D_{k,\lambda}^{(r)}(x).$$

Then, the formula between $D_{n,\lambda}^{(r)}(x)$ and $(x)_{n,\lambda}$ is obtained.

Theorem 2.1. *For $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, we have*

$$D_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \left(\sum_{\ell=k}^{n} \binom{n}{\ell} S_{1,\lambda}(\ell,k) D_{n-\ell,\lambda}^{(r)} \right) (x)_{k,\lambda}.$$

Reversely, we have the inversion formula given by

$$(x)_{n,\lambda} = \sum_{k=0}^{n} \left(\sum_{\ell=k}^{n} \sum_{j=0}^{r} \binom{n}{\ell} \binom{r}{j} (-1)^{r-j} S_{2,\lambda}(\ell,k) \frac{(j)_{n+r-\ell,\lambda}}{(n-\ell+r)_r} \right) D_{k,\lambda}^{(r)}(x).$$

Proof. Let $D^{(r)}_{n,\lambda}(x) = \sum_{k=0}^{n} \mu_{n,k}(x)_{k,\lambda}$. Then, by (1.4), (2.3), and (2.9), we can obtain

$$\begin{aligned}
\mu_{n,k} =& \frac{1}{k!}\left\langle \left(\frac{\log_\lambda(1+t)}{t}\right)^r (\log_\lambda(1+t))^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
=& \left\langle \left(\frac{\log_\lambda(1+t)}{t}\right)^r \middle| \left(\frac{1}{k!}(\log_\lambda(1+t))^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=& \frac{1}{\ell!}\sum_{\ell=k}^{\infty} S_{1,\lambda}(\ell,k)\left\langle \left(\frac{\log_\lambda(1+t)}{t}\right)^r \middle| \left(t^\ell\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=& \sum_{\ell=k}^{n}\binom{n}{\ell} S_{1,\lambda}(\ell,k)\left\langle \left(\frac{\log_\lambda(1+t)}{t}\right)^r \middle| (x)_{n-\ell,\lambda}\right\rangle_\lambda \\
=& \sum_{\ell=k}^{n}\binom{n}{\ell} S_{1,\lambda}(\ell,k) D^{(r)}_{n-\ell,\lambda},
\end{aligned}$$

which shows the first formula.

For the inversion formula, we first note that

$$\left(\frac{e_\lambda(t)-1}{t}\right)^r = \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\sum_{m=0}^{\infty}(j)_{m,\lambda}\frac{t^{m-r}}{m!},$$

which implies

$$\begin{aligned}
\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^r \middle| (x)_{n-\ell,\lambda}\right\rangle_\lambda &= \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}(j)_{n+r-\ell,\lambda}\frac{(n-\ell)!}{(n-\ell+r)!} \\
&= \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}(j)_{n+r-\ell,\lambda}\frac{1}{(n-\ell+r)_r}.
\end{aligned} \tag{2.11}$$

Now, let $(x)_{n,\lambda} = \sum_{k=0}^{\infty} \nu_{n,k} D^{(r)}_{k,\lambda}(x)$. Then, from (1.5), (2.7), and (2.11), ν_k satisfies

$$\begin{aligned}
\nu_{n,k} =& \frac{1}{k!}\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^r (e_\lambda(t)-1)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\
=& \left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^r \middle| \left(\frac{1}{k!}(e_\lambda(t)-1)^k\right)_\lambda (x)_{n,\lambda}\right\rangle_\lambda \\
=& \sum_{\ell=k}^{n}\binom{n}{\ell} S_{2,\lambda}(\ell,k)\left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^r \middle| (x)_{n-\ell,\lambda}\right\rangle_\lambda \\
=& \sum_{\ell=k}^{n}\binom{n}{\ell}\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j} S_{2,\lambda}(\ell,k)\frac{(j)_{n+r-\ell,\lambda}}{(n-\ell+r)_r},
\end{aligned}$$

which shows the second result.

Next, we consider the degenerate Bernoulli polynomials $\beta_{n,\lambda}(x)$, which is defined by the generating function to be

$$\frac{t}{e_\lambda(t)-1}e^x_\lambda(t) = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}, \qquad \text{(see [21]).}$$

Then, the connection formulas between $D^{(r)}_{n,\lambda}(x)$ and $\beta_{n,\lambda}(x)$ are as follows.

Theorem 2.2. *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$D_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \frac{(k+r-1)_{r-1}}{(n+r-1)_{r-1}} S_{1,\lambda}(n+r-1, k+r-1)\beta_{k,\lambda} \text{ for } r \in \mathbb{N}.$$

As the inversion formula, we have

$$\beta_{n,\lambda}(x) = \sum_{k=0}^{n} \left(\sum_{\ell=k}^{n} \sum_{j=0}^{r-1} \binom{n}{\ell} \binom{r-1}{j} \frac{(-1)^{r-1-j}(j)_{n-\ell+r-1,\lambda}}{(n-\ell+r-1)_{r-1}} S_{2,\lambda}(\ell, k) \right) D_{k,\lambda}^{(r)}(x) \text{ for } r > 1,$$

and

$$\beta_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n,k) D_{k,\lambda}(x) \text{ for } r = 1.$$

Proof. First, note that $\beta_{n,\lambda}(x)$ is the λ-Sheffer sequence for

$$\beta_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)-1}{t}, t \right)_\lambda. \tag{2.12}$$

Let us consider $D_{\ell,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \mu_{n,k}\beta_{k,\lambda}(x)$. By (1.4), (2.9), (2.10) and (2.12), we obtain

$$\begin{aligned}
\mu_{n,k} =& \frac{1}{k!} \left\langle \frac{\frac{t}{\log_\lambda(1+t)}}{\left(\frac{t}{\log_\lambda(1+t)}\right)^r} (\log_\lambda(1+t))^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
=& \frac{1}{k!} \left\langle t^{1-r} (\log_\lambda(1+t))^{k+r-1} \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
=& \frac{\langle k+1 \rangle_{r-1}}{(k+r-1)!} \left\langle t^{1-r} (\log_\lambda(1+t))^{k+r-1} \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
=& \frac{(k+r-1)_{r-1}}{(n+r-1)_{r-1}} S_{1,\lambda}(n+r-1, k+r-1),
\end{aligned}$$

which implies the first formula.

To find the inversion formula, we first note that from (1.1) for $r > 1$

$$\begin{aligned}
\left(\frac{e_\lambda(t)-1}{t} \right)^{r-1} =& t^{1-r} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} e_\lambda^j(t) \\
=& \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} \sum_{m=0}^{\infty} (j)_{m,\lambda} \frac{t^{m+1-r}}{m!}.
\end{aligned} \tag{2.13}$$

Thus, by (2.2) and (2.13)

$$\begin{aligned}
\left\langle \left(\frac{e_\lambda(t)-1}{t} \right)^{r-1} \middle| (x)_{n-\ell,\lambda} \right\rangle_\lambda =& \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} (j)_{n-\ell+r-1,\lambda} \frac{(n-\ell)!}{(n-\ell+r-1)!} \\
=& \begin{cases} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} \frac{(j)_{n-\ell+r-1,\lambda}}{(n-\ell+r-1)_{r-1}} & \text{if } r > 1, \\ \delta_{n,\ell} & \text{if } r = 1. \end{cases}
\end{aligned} \tag{2.14}$$

Now, if we consider $\beta_{n,\lambda}(x) = \sum_{k=0}^{\infty} \nu_{n,k} D_{k,\lambda}^{(r)}(x)$, then by (1.5), (2.7), and (2.14), $\nu_{n,k}$ satisfies

$$\begin{aligned}
\nu_{n,k} &= \frac{1}{k!} \left\langle \frac{\left(\frac{e_\lambda(t)-1}{t}\right)^r}{\frac{e_\lambda(t)-1}{t}} (e_\lambda(t)-1)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
&= \sum_{\ell=k}^{n} \binom{n}{\ell} S_{2,\lambda}(\ell,k) \left\langle \left(\frac{e_\lambda(t)-1}{t}\right)^{r-1} \middle| (x)_{n-\ell,\lambda} \right\rangle_\lambda \\
&= \begin{cases} \sum_{\ell=k}^{n} \binom{n}{\ell} S_{2,\lambda}(\ell,k) \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-1-j} \frac{(j)_{n-\ell+r-1,\lambda}}{(n-\ell+r-1)_{r-1}} & \text{if } r > 1, \\ S_{2,\lambda}(n,k) & \text{if } r = 1, \end{cases}
\end{aligned}$$

which provides the formula.

Next, we consider the degenerate type 2 Bernoulli polynomials $b_{n,\lambda}(x)$, which are defined by the generating functions to be

$$\frac{t}{e_\lambda(t) - e_\lambda^{-1}(t)} e_\lambda^x(t) = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{(see [22]).}$$

Note that $b_{n,\lambda}(x)$ satisfies

$$b_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t) - e_\lambda^{-1}(t)}{t}, t \right)_\lambda . \tag{2.15}$$

Then, we can have the following relation between $D_{n,\lambda}^{(r)}(x)$ and $b_{n,\lambda}(x)$.

Theorem 2.3. *For $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, we have*

$$D_{n,\lambda}^{(r)}(x) = \begin{cases} \sum_{k=0}^{n} \sum_{m=k}^{n} \binom{n}{m} S_{1,\lambda}(m,k) \left(D_{n-m,\lambda}^{(r-1)} + \sum_{\ell=0}^{n-m} (-1)^\ell (n-m)_\ell D_{n-m-\ell,\lambda}^{(r-1)} \right) b_{k,\lambda} & \text{for } r > 1. \\ S_{1,\lambda}(n,k) + \sum_{m=k}^{n} \binom{n}{m} S_{1,\lambda}(m,k) (-1)^{n-m} (n-m)_{n-m} & \text{for } r = 1. \end{cases}$$

For the inversion formula, we have

$$\begin{aligned}
b_{n,\lambda}(x) = \frac{1}{2} \sum_{k=0}^{n} &\left(\sum_{\ell=k}^{n} \sum_{j=0}^{r-1} \sum_{m=0}^{n+r-1} \binom{n}{\ell} \binom{r-1}{j} \binom{n+r-1}{m} S_{2,\lambda}(\ell,k) \frac{(-1)^{r-1-j} (j)_{m,\lambda}}{(n+r-1)_{r-1}} \right. \\
&\left. \times \mathcal{E}_{n+r-1-m,\lambda}\left(\frac{1}{2}\right) \right) D_{k,\lambda}^{(r)}(x) \text{ for } r > 1
\end{aligned}$$

and

$$b_{n,\lambda}(x) = \frac{1}{2} \sum_{k=0}^{n} \left(\sum_{m=k}^{n} \binom{n}{m} S_{2,\lambda}(m,k) \mathcal{E}_{n-m,\lambda}\left(\frac{1}{2}\right) \right) D_{k,\lambda}(x) \text{ for } r = 1.$$

Proof. Let us consider $D_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}\mu_{n,k}b_{k,\lambda}(x)$. By (1.4), (2.9), (2.10), and (2.15), we get

$$\begin{aligned}\mu_{n,k}=&\frac{1}{k!}\left\langle \frac{\frac{1+t-\frac{1}{1+t}}{\log_\lambda(1+t)}}{\left(\frac{t}{\log_\lambda(1+t)}\right)^r}(\log_\lambda(1+t))^k\middle|(x)_{n,\lambda}\right\rangle_\lambda\\
=&\left\langle\left(\frac{t(t+2)}{t+1}\right)\left(\frac{\log_\lambda(1+t)}{t}\right)^r\frac{1}{k!}(\log_\lambda(1+t))^{k-1}\middle|(x)_{n,\lambda}\right\rangle_\lambda\\
=&\left\langle\left(\frac{t+2}{t+1}\right)\left(\frac{\log_\lambda(1+t)}{t}\right)^{r-1}\frac{1}{k!}(\log_\lambda(1+t))^{k}\middle|(x)_{n,\lambda}\right\rangle_\lambda\\
=&\sum_{m=k}^{n}\binom{n}{m}S_{1,\lambda}(m,k)\left\langle\left(\frac{\log_\lambda(1+t)}{t}\right)^{r-1}\middle|\left(1+\sum_{\ell=0}^{\infty}(-t)^\ell\right)_\lambda(x)_{n-m,\lambda}\right\rangle_\lambda.\end{aligned}\tag{2.16}$$

From (2.3), it is noted that

$$\left(1+\sum_{\ell=0}^{\infty}(-t)^\ell\right)_\lambda(x)_{n-m,\lambda}=(x)_{n-m,\lambda}+\sum_{\ell=0}^{n-m}(-1)^\ell(n-m)_\ell(x)_{n-m-\ell,\lambda}.\tag{2.17}$$

By applying the note (2.17) in (2.16), we have

$$\mu_{n,k}=\begin{cases}\sum_{m=k}^{n}\binom{n}{m}S_{1,\lambda}(m,k)\left(D_{n-m,\lambda}^{(r-1)}+\sum_{\ell=0}^{n-m}(-1)^\ell(n-m)_\ell D_{n-m-\ell,\lambda}^{(r-1)}\right) & \text{for } r>1,\\ S_{1,\lambda}(n,k)+\sum_{m=k}^{n}\binom{n}{m}S_{1,\lambda}(m,k)(-1)^{n-m}(n-m)_{n-m} & \text{for } r=1.\end{cases}$$

To find the inversion formula, let us consider $b_{n,\lambda}(x)=\sum_{k=0}^{\infty}\nu_{n,k}D_{k,\lambda}^{(r)}(x)$. From (1.5), (2.7), and (2.15), $\nu_{n,k}$ satisfies

$$\begin{aligned}\nu_{n,k}=&\frac{1}{k!}\left\langle\frac{\left(\frac{e_\lambda(t)-1}{t}\right)^r}{\frac{e_\lambda(t)-e_\lambda^{-1}(t)}{t}}(\log_\lambda(t+1))^k\middle|(x)_{n,\lambda}\right\rangle_\lambda\\
=&\sum_{\ell=k}^{n}\binom{n}{\ell}S_{2,\lambda}(\ell,k)\left\langle\frac{e_\lambda(t)}{e_\lambda(t)+1}\left(\frac{e_\lambda(t)-1}{t}\right)^{r-1}\middle|(x)_{n-\ell,\lambda}\right\rangle_\lambda\\
=&\frac{1}{2}\sum_{\ell=k}^{n}\binom{n}{\ell}S_{2,\lambda}(\ell,k)\left\langle\frac{2e_\lambda^{\frac{1}{2}}(t)}{e_\lambda^{\frac{1}{2}}(t)+e_\lambda^{-\frac{1}{2}}(t)}\left(\frac{e_\lambda(t)-1}{t}\right)^{r-1}\middle|(x)_{n-\ell,\lambda}\right\rangle_\lambda.\end{aligned}\tag{2.18}$$

Since $\frac{e_\lambda^x(t)-1}{t}=\sum_{n=0}^{\infty}\frac{(x)_{n+1,\lambda}}{n+1}\frac{t^n}{n!}$, we have that for $r>1$

$$\begin{aligned}\left(\frac{e_\lambda(t)-1}{t}\right)^{r-1}=&t^{1-r}\sum_{j=0}^{r-1}\binom{r-1}{j}e_\lambda^j(t)(-1)^{r-1-j}\\
=&t^{1-r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\sum_{m=0}^{\infty}(j)_{m,\lambda}\frac{t^m}{m!}.\end{aligned}\tag{2.19}$$

Then, (2.19) implies that for $r > 1$

$$\begin{aligned}\frac{2e_\lambda^{\frac{1}{2}}(t)}{e_\lambda^{\frac{1}{2}}(t)+e_\lambda^{-\frac{1}{2}}(t)}\left(\frac{e_\lambda(t)-1}{t}\right)^{r-1} &= t^{1-r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\left(\sum_{m=0}^{\infty}(j)_{m,\lambda}\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\mathcal{E}_{k,\lambda}\left(\frac{1}{2}\right)\frac{t^k}{k!}\right)\\ &= t^{1-r}\sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}(j)_{k,\lambda}\mathcal{E}_{m-k,\lambda}\left(\frac{1}{2}\right)\frac{t^m}{m!}\\ &= \sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}(j)_{k,\lambda}\mathcal{E}_{m-k,\lambda}\left(\frac{1}{2}\right)\frac{t^{m+1-r}}{m!},\end{aligned} \tag{2.20}$$

where $\mathcal{E}_{n,\lambda}(x)$ are the type 2 degenerate Euler polynomials defined by the following generating function

$$\frac{2}{e_\lambda^{\frac{1}{2}}(t)+e_\lambda^{-\frac{1}{2}}(t)}e_\lambda^x(t) = \sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [23]).} \tag{2.21}$$

Here we call $\mathcal{E}_{n,\lambda} := \mathcal{E}_{n,\lambda}(0)$ the type 2 degenerate Euler numbers if $x = 0$. Thus, for $m = n - \ell + r - 1$ in (2.20) for $r > 1$

$$\begin{aligned}\left\langle \frac{2e_\lambda^{\frac{1}{2}}(t)}{e_\lambda^{\frac{1}{2}}(t)+e_\lambda^{-\frac{1}{2}}(t)}\left(\frac{e_\lambda(t)-1}{t}\right)^{r-1}\middle|(x)_{n-\ell,\lambda}\right\rangle_\lambda &= \sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\sum_{k=0}^{n-\ell+r-1}\binom{n-\ell+r-1}{k}(j)_{k,\lambda}\\ &\quad\times\mathcal{E}_{n-\ell+r-1-k,\lambda}\left(\frac{1}{2}\right)\frac{n!}{(n+r-1)!}\\ &= \sum_{j=0}^{r-1}\binom{r-1}{j}(-1)^{r-1-j}\sum_{k=0}^{n-\ell+r-1}\binom{n-\ell+r-1}{k}(j)_{k,\lambda}\\ &\quad\times\mathcal{E}_{n-\ell+r-1-k,\lambda}\left(\frac{1}{2}\right)\frac{1}{(n-\ell+r-1)_{r-1}},\end{aligned}$$

and

$$\left\langle \frac{2e_\lambda^{\frac{1}{2}}(t)}{e_\lambda^{\frac{1}{2}}(t)+e_\lambda^{-\frac{1}{2}}(t)}\middle|(x)_{n-\ell,\lambda}\right\rangle_\lambda = \mathcal{E}_{n-\ell,\lambda}\left(\frac{1}{2}\right) \quad \text{for } r = 1,$$

which provides the inversion formula with (2.18).

We consider the degenerate Euler polynomials $\mathcal{E}_{k,\lambda}$ that is defined by the generating function to be

$$\left(\frac{2}{e_\lambda(t)+1}\right)e_\lambda^x(t) = \sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [12,21]),}$$

which satisfies that

$$\mathcal{E}_{n,\lambda}(x) \sim \left(\frac{e_\lambda(t)+1}{2}, t\right)_\lambda. \tag{2.22}$$

Then, the representation formula between $D_{n,\lambda}^{(r)}(x)$ and $\mathcal{E}_{n,\lambda}(x)$ holds true.

Theorem 2.4. *For $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, we have*

$$D_{n,\lambda}^{(r)}(x) = \frac{1}{2} \sum_{k=0}^{n} \left(\sum_{\ell=k}^{n} S_{1,\lambda}(\ell,k) \binom{n}{\ell} \left((n-\ell) D_{n-\ell-1,\lambda}^{(r)} + 2 D_{n-\ell,\lambda}^{(r)} \right) \right) \mathcal{E}_{k,\lambda}(x).$$

As the inversion formula, we have

$$\mathcal{E}_{n,\lambda}(x) = \sum_{k=0}^{n} \left(\sum_{\ell=k}^{n} \sum_{j=0}^{r} \sum_{l=0}^{n+r} \binom{n}{\ell} \binom{r}{j} \binom{n+r}{l} S_{2,\lambda}(\ell,k) (-1)^{r-j} \frac{(j)_{l,\lambda}}{(n+r)_r} \mathcal{E}_{n+r-l,\lambda} \right) D_{k,\lambda}(x).$$

Proof. Let $D_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \mu_{n,k} \mathcal{E}_{k,\lambda}$. Then, By (1.4), (2.9), (2.10), and (2.22), we can obtain

$$\begin{aligned} \mu_{n,k} =& \frac{1}{k!} \left\langle \frac{\frac{(1+t)+1}{2}}{\left(\frac{t}{\log_\lambda(1+t)}\right)^r} (\log_\lambda(1+t))^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ =& \left\langle \frac{t+2}{2} \left(\frac{\log_\lambda(1+t)}{t} \right)^r \frac{1}{k!} (\log_\lambda(1+t))^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ =& \frac{1}{2} \sum_{\ell=k}^{\infty} \binom{n}{\ell} S_{1,\lambda}(\ell,k) \left\langle (t+2) \left(\frac{\log_\lambda(1+t)}{t} \right)^r \middle| (x)_{n-\ell,\lambda} \right\rangle_\lambda, \end{aligned} \tag{2.23}$$

where

$$\left\langle t \left(\frac{\log_\lambda(1+t)}{t} \right)^r \middle| (x)_{n-\ell,\lambda} \right\rangle_\lambda = \left\langle \sum_{k=0}^{n} D_{k,\lambda}^{(r)} \frac{t^{k+1}}{k!} \middle| (x)_{n-\ell,\lambda} \right\rangle_\lambda = D_{n-\ell-1,\lambda}^{(r)}(n-\ell) \tag{2.24}$$

and

$$\left\langle \sum_{k=0}^{n} D_{k,\lambda}^{(r)} \frac{t^k}{k!} \middle| (x)_{n-\ell,\lambda} \right\rangle_\lambda = D_{n-\ell,\lambda}^{(r)}. \tag{2.25}$$

Therefore, combining (2.24) and (2.25) to (2.23), we have

$$\mu_{n,k} = \frac{1}{2} \sum_{\ell=k}^{n} \binom{n}{\ell} S_{1,\lambda}(\ell,k) \left((n-\ell) D_{n-\ell-1,\lambda}^{(r)} + 2 D_{n-\ell,\lambda}^{(r)} \right).$$

To find the inversion formula, let $\mathcal{E}_{n,\lambda}(x) = \sum_{k=0}^{\infty} \nu_{n,k} D_{k,\lambda}^{(r)}(x)$, where $\nu_{n,k}$ satisfies

$$\begin{aligned} \nu_{n,k} =& \frac{1}{k!} \left\langle \frac{\left(\frac{e_\lambda(t)-1}{t}\right)^r}{\frac{e_\lambda(t)+1}{2}} (e_\lambda(t)-1)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\ =& \sum_{\ell=k}^{n} \binom{n}{\ell} S_{2,\lambda}(\ell,k) \left\langle \left(\frac{e_\lambda(t)-1}{t} \right)^r \frac{2}{e_\lambda(t)+1} \middle| (x)_{n,\lambda} \right\rangle_\lambda. \end{aligned} \tag{2.26}$$

We note that

$$\left(\frac{e_\lambda(t)-1}{t} \right)^r = t^{-r} \sum_{j=0}^{r} \binom{r}{j} e_\lambda^j(t) (-1)^{r-j}$$

$$=t^{-r}\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\sum_{m=0}^{\infty}(j)_{m,\lambda}\frac{t^m}{m!}$$
$$=\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\sum_{m=0}^{\infty}(j)_{m,\lambda}\frac{t^{m-r}}{m!}$$

and

$$\left(\frac{e_\lambda(t)-1}{t}\right)^r\frac{2}{e_\lambda(t)+1}=\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\binom{m}{l}(j)_{l,\lambda}\mathcal{E}_{m-l,\lambda}\right)\frac{t^{m-r}}{m!}.$$

Thus,

$$\left\langle\left(\frac{e_\lambda(t)-1}{t}\right)^r\frac{2}{e_\lambda(t)+1}\middle|(x)_{n,\lambda}\right\rangle_\lambda=\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\sum_{l=0}^{n+r}\binom{n+r}{l}\frac{(j)_{l,\lambda}}{(n+r)_r}\mathcal{E}_{n+r-l,\lambda}. \tag{2.27}$$

Combining (2.27) to (2.26) gives

$$\nu_{n,k}=\sum_{\ell=k}^{n}\sum_{j=0}^{r}\sum_{l=0}^{n+r}\binom{n}{\ell}\binom{r}{j}\binom{n+r}{l}S_{2,\lambda}(\ell,k)(-1)^{r-j}\frac{(j)_{l,\lambda}}{(n+r)_r}\mathcal{E}_{n+r-l,\lambda}.$$

The degenerate Mittag-Leffer polynomials $M_{n,\lambda}(x)$ are given by the generating function to be

$$e_\lambda^x\left(\log_\lambda\left(\frac{1+t}{1-t}\right)\right)=\sum_{n=0}^{\infty}M_{n,\lambda}(x)\frac{t^n}{n!},\quad \text{(see [24])}.$$

It is noted that

$$M_{n,\lambda}(x)\sim\left(1,\frac{e_\lambda(t)-1}{e_\lambda(t)+1}\right)_\lambda. \tag{2.28}$$

Then, we have the representation formulas between $D_{n,\lambda}^{(r)}(x)$ and $M_{n,\lambda}(x)$.

Theorem 2.5. *For $n\in\mathbb{N}\cup\{0\}$ and $r\in\mathbb{N}$, we have*

$$D_{n,\lambda}^{(r)}(x)=\sum_{k=0}^{n}\left(\frac{1}{k!}\sum_{m=0}^{n}D_{m,\lambda}^{(r)}\binom{n}{m}\binom{n-m-1}{n-m-k}\frac{(-1)^{n-m-k}}{2^{n-m}}(n-m)!\right)M_{k,\lambda}.$$

As the inversion formula, we have

$$M_{n,\lambda}(x)=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{n}\frac{n!}{k!m!}K_m(\lambda)2^{m+k+r-1}\right)D_{k,\lambda}^{(r)}(x),$$

where $K_n(x|\lambda)$ are the Korobov polynomials of the first kind given by the generating function

$$\frac{\lambda t}{(1+t)^\lambda-1}(1+t)^x=\sum_{n=0}^{\infty}K_n(x|\lambda)\frac{t^n}{n!},\quad \text{(see [25])}.$$

In particular, when $x=0$ $K_n(\lambda):=K_n(0|\lambda)$ are called Korobov numbers of the first kind, that is,

$$\frac{t}{\log_\lambda(1+t)}=\sum_{n=0}^{\infty}K_n(\lambda)\frac{t^n}{n!}.$$

Proof. Let $D_{\ell,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \mu_{n,k} M_{k,\lambda}$. Then, by (1.4), (2.9), (2.10), and (2.28), we can obtain

$$\begin{aligned}\mu_{n,k} &= \frac{1}{k!}\left\langle \frac{1}{\left(\frac{(1+t)-1}{\log_\lambda(1+t)}\right)^r}\left(\frac{1+t-1}{1+t+1}\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \frac{1}{k!}\left\langle \left(\frac{\log_\lambda(1+t)}{t}\right)^r \left(\frac{t}{t+2}\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \frac{1}{k!}\sum_{m=0}^{n} D_{m,\lambda}^{(r)}\binom{n}{m}\left\langle \left(\frac{t}{t+2}\right)^k \middle| (x)_{n-m,\lambda}\right\rangle_\lambda,\end{aligned}$$

where

$$\begin{aligned}\left\langle \left(\frac{t}{t+2}\right)^k \middle| (x)_{n-m,\lambda}\right\rangle_\lambda &= \left\langle \sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2^{k+\ell}}\binom{k+\ell-1}{\ell} t^{k+\ell} \middle| (x)_{n-m,\lambda}\right\rangle_\lambda \\ &= \frac{(-1)^{n-m-k}}{2^{n-m}}\binom{n-m-1}{n-m-k}(n-m)!.\end{aligned} \tag{2.29}$$

Thus,

$$\mu_{n,k} = \frac{1}{k!}\sum_{m=0}^{n} D_{m,\lambda}^{(r)}\binom{n}{m}\binom{n-m-1}{n-m-k}\frac{(-1)^{n-m-k}}{2^{n-m}}(n-m)!.$$

To find the inversion formula, let $M_{n,\lambda}(x) = \sum_{k=0}^{\infty} \nu_{n,k} D_{k,\lambda}^{(r)}(x)$, where

$$\begin{aligned}\nu_{n,k} &= \frac{1}{k!}\left\langle \frac{\left(\frac{1+t}{1-t}-1\right)^r}{\log_\lambda\left(\frac{1+t}{1-t}\right)}\left(\frac{1+t}{1-t}-1\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \frac{1}{k!}\left\langle \frac{\left(\frac{1+t-(1-t)}{1-t}\right)^r}{\log_\lambda\left(\frac{1+t}{1-t}\right)}\left(\frac{1+t-1+t}{1-t}\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \frac{1}{k!}\left\langle \frac{\left(\frac{2t}{1-t}\right)^r}{\log_\lambda\left(\frac{1+t}{1-t}\right)}\left(\frac{2t}{1-t}\right)^k \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= \frac{1}{k!}\sum_{m=0}^{n}\frac{1}{m!}K_m(\lambda)\left\langle \left(\frac{2t}{1-t}\right)^{m+k+r-1} \middle| (x)_{n,\lambda}\right\rangle_\lambda,\end{aligned}$$

where

$$\begin{aligned}\left\langle \left(\frac{2t}{1-t}\right)^{m+k+r-1} \middle| (x)_{n,\lambda}\right\rangle_\lambda &= 2^{m+k+r-1}\left\langle \sum_{\ell=0}^{\infty} t^{\ell+m+k+r-1} \middle| (x)_{n,\lambda}\right\rangle_\lambda \\ &= 2^{m+k+r-1} n!.\end{aligned}$$

Thus,

$$\nu_{n,k} = \frac{1}{k!}\sum_{m=0}^{n}\frac{n!}{m!}K_m(\lambda)2^{m+k+r-1}.$$

Next, let us consider the degenerate Bell polynomials $Bel_{n,\lambda}(x)$, which are defined by the generating function to be

$$e_{\lambda}^{x(e_{\lambda}(t)-1)} = \sum_{n=0}^{\infty} Bel_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [26, 27])}.$$

Note that $Bel_{n,\lambda}(x)$ are the λ-Sheffer sequences of

$$Bel_{n,\lambda}(x) \sim \left(1, \log_{\lambda}(1+t)\right)_{\lambda}, \tag{2.30}$$

which satisfies that

$$Bel_{n,\lambda}(x) = \sum_{k=0}^{n} S_{2,\lambda}(n,k)(x)_{k,\lambda}, \quad \text{(see [26])}.$$

Then, we have the representation formulas between $D_{n,\lambda}^{(r)}(x)$ and $Bel_{n,\lambda}(x)$.

Theorem 2.6. *For $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, it holds:*

$$D_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n}\left(\sum_{m=k}^{n}\sum_{\ell=m}^{n}\binom{n}{\ell}S_{1,\lambda}(m,k)S_{1,\lambda}(\ell,m)D_{n-\ell,\lambda}^{(r)}\right)Bel_{k,\lambda}(x).$$

Also the inversion formula are established

$$Bel_{n,\lambda}(x) = \sum_{k=0}^{n}\left(\sum_{m=k}^{n}\sum_{\ell=0}^{n}\sum_{j=0}^{r}\binom{r}{j}S_{2,\lambda}(m,k)S_{2,\lambda}(n,\ell)\frac{(-1)^{r-j}}{m!}(j)_{\ell+r-m,\lambda}\right)D_{k,\lambda}^{(r)}(x).$$

Proof. Let $D_{\ell,\lambda}^{(r)}(x) = \sum_{k=0}^{n}\mu_{n,k}Bel_{k,\lambda}(x)$. Then, by (1.4), (2.9), (2.10), and (2.30), we can obtain

$$\begin{aligned}
\mu_{n,k} =& \frac{1}{k!}\left\langle \frac{1}{\left(\frac{(1+t)-1}{\log_{\lambda}(1+t)}\right)^r}\left(\log_{\lambda}(1+\log_{\lambda}(1+t))\right)^k \middle| (x)_{n,\lambda}\right\rangle_{\lambda} \\
=& \frac{1}{k!}\left\langle \left(\frac{\log_{\lambda}(1+t)}{t}\right)^r \left(\log_{\lambda}(1+\log_{\lambda}(1+t))\right)^k \middle| (x)_{n,\lambda}\right\rangle_{\lambda} \\
=& \left\langle \left(\frac{\log_{\lambda}(1+t)}{t}\right)^r \middle| \left(\left(\frac{1}{k!}\log_{\lambda}(1+\log_{\lambda}(1+t))\right)^k\right)_{\lambda} (x)_{n,\lambda}\right\rangle_{\lambda} \\
=& \sum_{m=k}^{n} S_{1,\lambda}(m,k)\sum_{\ell=m}^{n}\binom{n}{\ell}S_{1,\lambda}(\ell,m)\left\langle \left(\frac{\log_{\lambda}(1+t)}{t}\right)^r \middle| (x)_{n-\ell,\lambda}\right\rangle_{\lambda} \\
=& \sum_{m=k}^{n}\sum_{\ell=m}^{n}\binom{n}{\ell}S_{1,\lambda}(m,k)S_{1,\lambda}(\ell,m)D_{n-\ell,\lambda}^{(r)}.
\end{aligned}$$

To find the inversion formula, let $Bel_{n,\lambda}(x) = \sum_{k=0}^{n}\nu_{n,k}D_{k,\lambda}^{(r)}(x)$, then $\nu_{n,k}$ satisfies

$$\begin{aligned}
\nu_{n,k} =& \frac{1}{k!}\left\langle \left(\frac{e_{\lambda}(t)-1}{t}\right)^r (e_{\lambda}(t)-1)^k \middle| Bel_{n,\lambda}(x)\right\rangle_{\lambda} \\
=& \sum_{m=k}^{\infty} S_{2,\lambda}(m,k)\left\langle \left(\frac{e_{\lambda}(t)-1}{t}\right)^r \frac{t^m}{m!} \middle| \sum_{\ell=0}^{n} S_{2,\lambda}(n,\ell)(x)_{\ell,\lambda}\right\rangle_{\lambda}
\end{aligned}$$

$$= \sum_{m=k}^{n} S_{2,\lambda}(m,k) \sum_{\ell=0}^{n} S_{2,\lambda}(n,\ell) \left\langle \left(\frac{e_\lambda(t)-1}{t} \right)^r \frac{t^m}{m!} \middle| (x)_{\ell,\lambda} \right\rangle_\lambda ,$$

where

$$\begin{aligned} \left\langle \left(\frac{e_\lambda(t)-1}{t} \right)^r \frac{t^m}{m!} \middle| (x)_{\ell,\lambda} \right\rangle_\lambda &= \left\langle \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \sum_{l=0}^{\infty} (j)_{l,\lambda} \frac{t^{l+m-r}}{l!m!} \middle| (x)_{\ell,\lambda} \right\rangle_\lambda \\ &= \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} (j)_{\ell+r-m,\lambda} \frac{1}{m!}. \end{aligned}$$

Therefore, we have

$$\nu_{n,k} = \sum_{m=k}^{n} \sum_{\ell=0}^{n} \sum_{j=0}^{r} \binom{r}{j} S_{2,\lambda}(m,k) S_{2,\lambda}(n,\ell) \frac{(-1)^{r-j}}{m!} (j)_{\ell+r-m,\lambda}.$$

The degenerate Frobenius-Euler polynomials $h_{n,\lambda}^{(\alpha)}(x|u)$ of order α are defined by the generating function as

$$\left(\frac{1-u}{e_\lambda(t)-u} \right)^\alpha e_\lambda^x(t) = \sum_{n=0}^{\infty} h_{n,\lambda}^{(\alpha)}(x|u) \frac{t^n}{n!}, \quad u(\neq 1) \in \mathbb{C}.$$

When $x=0$, $h_{n,\lambda}^{(\alpha)}(u) := h_{n,\lambda}^{(\alpha)}(0|u)$ are called the degenerate Frobenius-Euler numbers.
We note that $h_{n,\lambda}^{(\alpha)}(x|u)$ satisfy

$$h_{n,\lambda}^{(\alpha)}(x|u) \sim \left(\left(\frac{e_\lambda(t)-u}{1-u} \right)^\alpha, t \right)_\lambda . \tag{2.31}$$

Then, we have the representation formulas between $D_{n,\lambda}^{(r)}(x)$ and $h_{n,\lambda}^{(\alpha)}(x|u)$.

Theorem 2.7. *For $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$, the representation holds:*

$$D_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \left(\sum_{m=k}^{n} \sum_{\ell=0}^{\alpha} \binom{n}{m} \binom{\alpha}{\ell} \frac{(n-m)_\ell}{(1-u)^\ell} S_{1,\lambda}(m,k) D_{n-m-\ell,\lambda}^{(r)} \right) h_{k,\lambda}^{(\alpha)}(x|u).$$

As the inversion formula, we have

$$h_{n,\lambda}^{(\alpha)}(x|u) = \sum_{k=0}^{n} \left(\sum_{\ell=0}^{n} \binom{n}{k+\ell} \frac{(r+k)!}{k!(\ell+r+k)!} S_{2,\lambda}(r+k+\ell, r+k) h_{n-k-\ell,\lambda}^{(\alpha)}(u) \right) D_{k,\lambda}^{(r)}(x).$$

Proof. Let $D^{(r)}_{n,\lambda}(x) = \sum_{k=0}^{n} \mu_{n,k} h^{(\alpha)}_{k,\lambda}(x|u)$. By (1.4), (2.9), (2.10), and (2.31), we have

$$\begin{aligned}
\mu_{n,k} =& \frac{1}{k!}\left\langle \frac{\left(\frac{1+t-u}{1-u}\right)^{\alpha}}{\left(\frac{t}{\log_\lambda(1+t)}\right)^{r}} (\log_\lambda(1+t))^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
=& \left\langle \left(\frac{\log_\lambda(1+t)}{t}\right)^r \left(1+\frac{t}{1-u}\right)^{\alpha} \middle| \left(\frac{1}{k!}(\log_\lambda(1+t))^k\right)_\lambda (x)_{n,\lambda} \right\rangle_\lambda \\
=& \sum_{m=k}^{n} \binom{n}{m} S_{1,\lambda}(m,k) \left\langle \left(\frac{\log_\lambda(1+t)}{t}\right)^r \left(1+\frac{t}{1-u}\right)^{\alpha} \middle| (x)_{n-m,\lambda} \right\rangle_\lambda \\
=& \sum_{m=k}^{n} \binom{n}{m} S_{1,\lambda}(m,k) \left\langle \left(\frac{\log_\lambda(1+t)}{t}\right)^r \middle| \left(\sum_{\ell=0}^{\alpha} \binom{\alpha}{\ell}\left(\frac{t}{1-u}\right)^{\ell}\right)_\lambda (x)_{n-m,\lambda} \right\rangle_\lambda \\
=& \sum_{m=k}^{n} \binom{n}{m} S_{1,\lambda}(m,k) \sum_{\ell=0}^{\alpha} \binom{\alpha}{\ell} \frac{(n-m)_\ell}{(1-u)^\ell} \left\langle \left(\frac{\log_\lambda(1+t)}{t}\right)^r \middle| (x)_{n-m-\ell,\lambda} \right\rangle_\lambda \\
=& \sum_{m=k}^{n} \binom{n}{m} S_{1,\lambda}(m,k) \sum_{\ell=0}^{\alpha} \binom{\alpha}{\ell} \frac{(n-m)_\ell}{(1-u)^\ell} D^{(r)}_{n-m-\ell,\lambda},
\end{aligned}$$

which implies the first formula.

Conversely, we assume that $h^{(\alpha)}_{n,\lambda}(x|u) = \sum_{k=0}^{n} nu_{n,k} D^{(r)}_{k,\lambda}(x)$. Then, $\nu_{n,k}$ satisfies

$$\begin{aligned}
\nu_{n,k} =& \frac{1}{k!}\left\langle \frac{\left(\frac{e_\lambda(t)-1}{t}\right)^{r}}{\left(\frac{e_\lambda(t)-u}{1-u}\right)^{\alpha}} (e_\lambda(t)-1)^k \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
=& \frac{1}{k!}\left\langle \left(\frac{1-u}{e_\lambda(t)-u}\right)^{\alpha} \frac{(e_\lambda(t)-1)^{r+k}}{t^r} \middle| (x)_{n,\lambda} \right\rangle_\lambda \\
=& \sum_{\ell=0}^{n} \binom{n}{k+\ell} \frac{(r+k)!}{k!(\ell+r+k)!} S_{2,\lambda}(r+k+\ell, r+k) \left\langle \left(\frac{1-u}{e_\lambda(t)-u}\right)^{\alpha} \middle| (x)_{n-k-\ell,\lambda} \right\rangle_\lambda \\
=& \sum_{\ell=0}^{n} \binom{n}{k+\ell} \frac{(r+k)!}{k!(\ell+r+k)!} S_{2,\lambda}(r+k+\ell, r+k) h^{(\alpha)}_{n-k-\ell,\lambda}(u),
\end{aligned}$$

which shows the second assertion.

3. Illustrations of formulas

3.1. Distribution of roots of the polynomials

In this subsection, we present the pattern of the zeros of the polynomials. The understanding of patterns of zeros of degenerate polynomials can provide useful information about the original polynomials which can be obtained as limit of λ approaches zero. For example, the first three consecutive degenerate higher-order Daehee polynomials of degree r are given by

$$\begin{aligned}
D_{1,\lambda}^{(r)}(x) &= x + \frac{r(\lambda-1)}{2},\\
D_{2,\lambda}^{(r)}(x) &= x^2 + (r(\lambda-1)-1)x + \frac{1}{12}r(\lambda-1)(3r(\lambda-1)+\lambda-5),\\
D_{3,\lambda}^{(r)}(x) &= x^3 + \frac{1}{2}(3r(\lambda-1)-9)x^2 + \frac{1}{4}(8+r(\lambda-1)(3r(\lambda-1)+\lambda-11))x\\
&\quad + \frac{1}{8}r(\lambda-1)(r(\lambda-1)-2)(r(\lambda-1)+\lambda-3),
\end{aligned}$$

which approach to the higher-order Daehee polynomials of degree r as $\lambda \to 0$. We observe the patterns of roots by the changing parameters λ and r on the polynomials. In order to do this, we fix the degree of the polynomials as $n = 40$, and compute the roots of $D_{40,\lambda}^{(r)}(x)$ with fixed $r = 3$ and six different parameters $\lambda = \pm 1, \pm 10, \pm 100$ with the help of the Mathematica tool. The results are displayed in Figure 1. Next, we increase the degree of the polynomials and investigate the distribution of the roots of the polynomials.

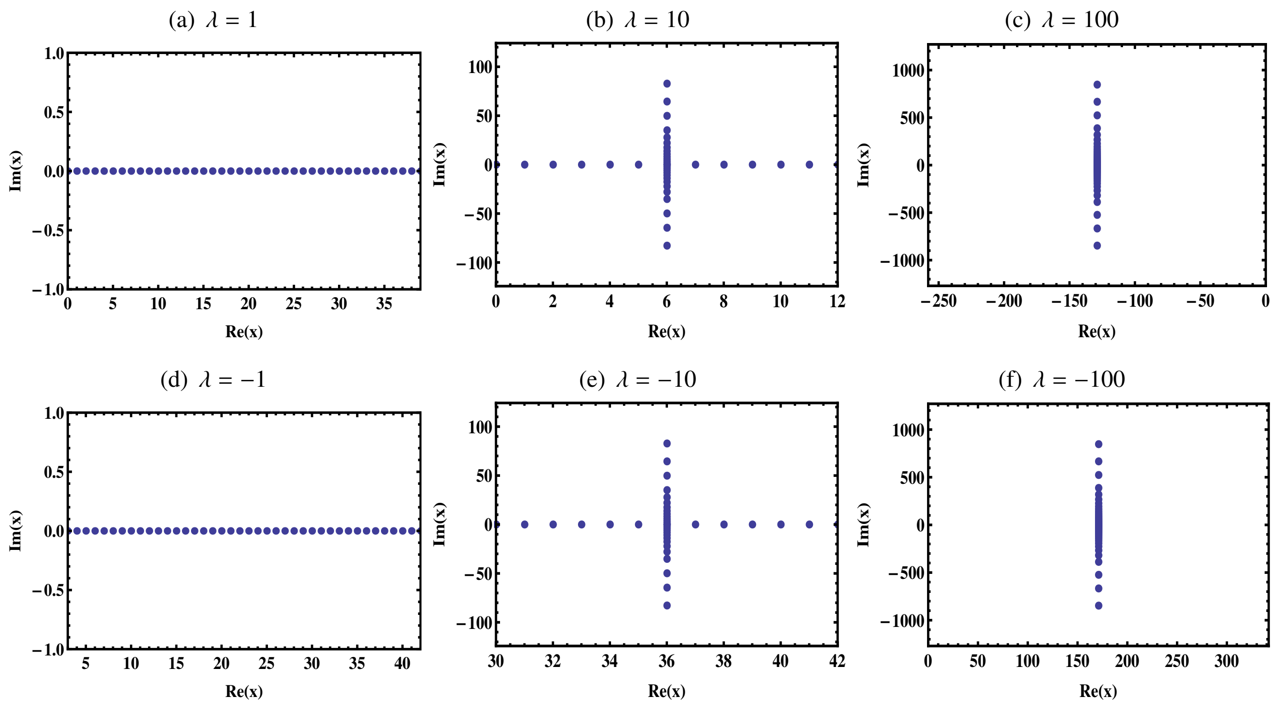

Figure 1. The computed roots of $D_{40,\lambda}^{(3)}(x)$ with variable λ.

For further investigation, we computed the roots of the polynomials by increasing the degree n of polynomials from 1 to 40 in Figure 2.

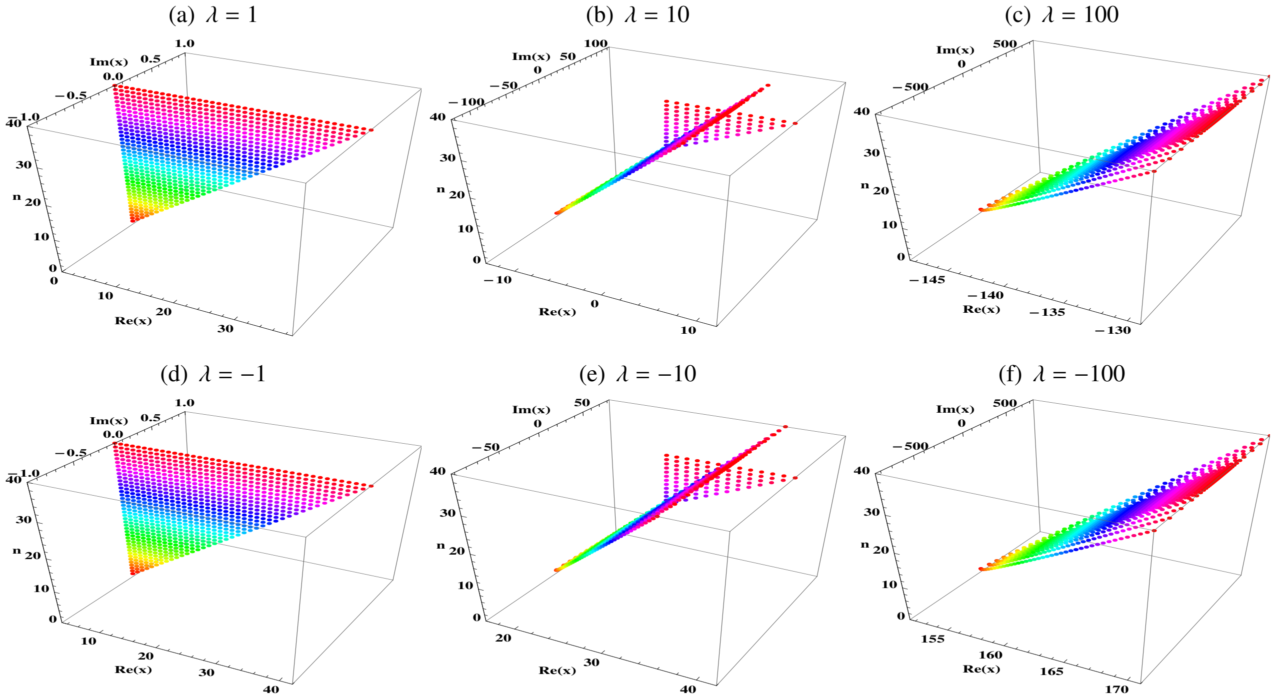

Figure 2. The roots of $D^{(3)}_{n,\lambda}(x)$ with variable λ and different degree $n = 1, 2, \cdots, 40$.

Finally, we investigated the distribution of the roots of $D^{(r)}_{40,\lambda}(x)$ with a fixed $\lambda = 10$ and three different parameters $r = 3, 4, 5$ and the results were displayed in Figure 3.

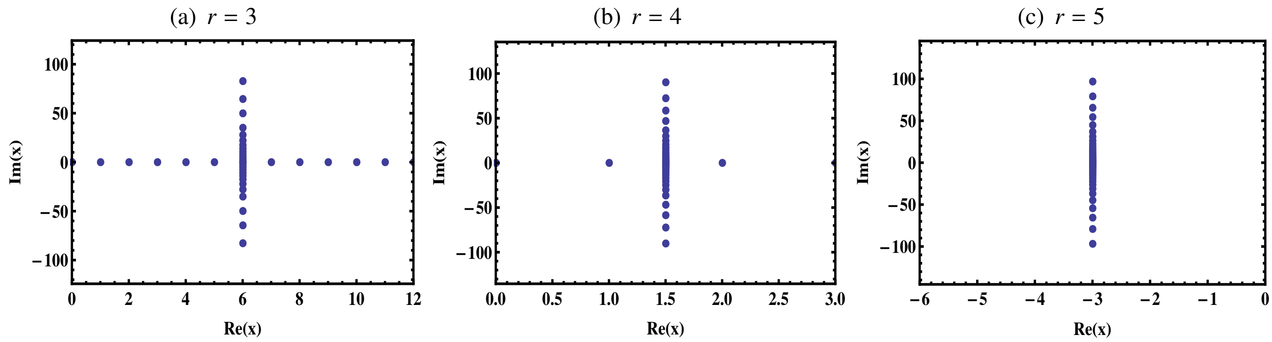

Figure 3. The roots of $D^{(r)}_{40,10}(x)$ when $r = 3, 4$, and 5.

3.2. *Examples of formulas*

In this subsection, we provide the explicit formulas presented in Theorem 2.2 that show the representations of the degenerate higher-order Daehee polynomials in terms of the degenerate Bernoulli polynomials and vice versa. To better understand, we present the graphs of $D^{(r)}_{n,\lambda}(x)$ with $\lambda = 0.1$ and $r = 3$ for $n = 0, 1, \cdots, 5$ and of $D^{(r)}_{n,\lambda}(x)$ with $\lambda = 0.1$ for various orders $r = 1, 2, \cdots, 5$ in Figure 4.

Next, we compute the combinatorial results of $\mu_{n,k}$ and $\nu_{n,k}$ presented in the proof of Theorem 2.2 to confirm the connection formulas presented. To do this, we compute $D_{n,\lambda}^{(r)}(x)$ and $\beta_{n,\lambda}(x)$ for $r = 3$, $\lambda = 0.1$, and $n = 0, 1, \cdots, 5$ and expand them using the coefficients $\mu_{n,k}$ and $\nu_{n,k}$ which are computed with two decimal place accuracy. The expressions presented confirm the results of Theorem 2.2.

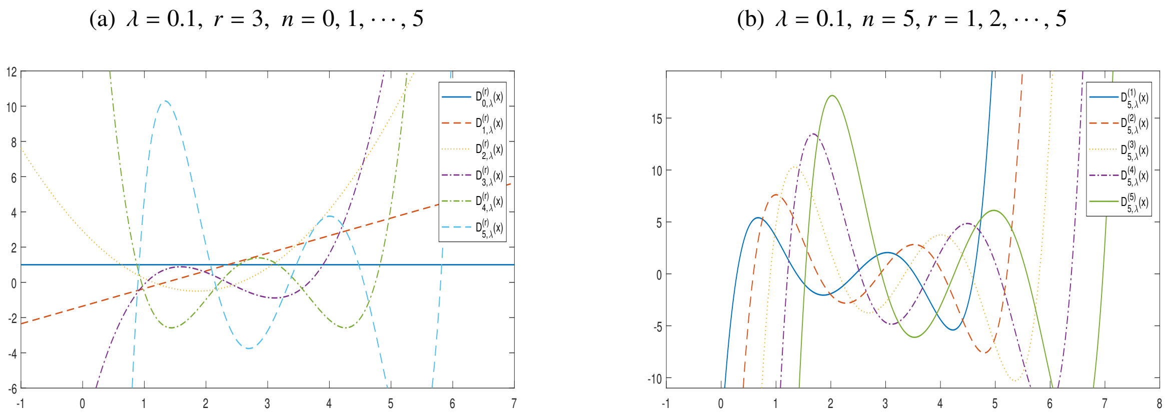

Figure 4. Graph of degenerate higher-order Daehee polynomials $D_{n,\lambda}^{(r)}(x)$.

The degenerate higher-order Daehee polynomials $D_{n,\lambda}^{(r)}(x)$ with $\lambda = 0.1$, $r = 3$ for $n = 1, 2, \cdots, 5$ are expressed in terms of $\beta_{n,\lambda}(x)$ as follows:

$$\begin{aligned}
D_{5,\lambda}^{(3)}(x) =& x^5 - \frac{67}{4}x^4 + \frac{419}{4}x^3 - \frac{30083}{100}x^2 + \frac{3868143}{10000}x - \frac{68088951}{400000} \\
=& \beta_{5,\lambda}(x) - \frac{27}{2}\beta_{4,\lambda}(x) + 69\beta_{3,\lambda}(x) - \frac{819}{5}\beta_{2,\lambda}(x) + \frac{1749843}{10000}\beta_{1,\lambda}(x) - \frac{6312807}{100000}\beta_{0,\lambda}(x), \\
D_{4,\lambda}^{(3)}(x) =& x^4 - \frac{57}{5}x^3 + \frac{179}{4}x^2 - \frac{34941}{500}x + \frac{1749843}{50000} \\
=& \beta_{4,\lambda}(x) - 9\beta_{3,\lambda}(x) + \frac{1413}{50}\beta_{2,\lambda}(x) - \frac{8883}{250}\beta_{1,\lambda}(x) + \frac{707427}{50000}\beta_{0,\lambda}(x), \\
D_{3,\lambda}^{(3)}(x) =& x^3 - \frac{141}{20}x^2 + \frac{593}{40}x - \frac{8883}{1000} \\
=& \beta_{3,\lambda}(x) - \frac{27}{5}\beta_{2,\lambda}(x) + \frac{351}{40}\beta_{1,\lambda}(x) - \frac{8037}{2000}\beta_{0,\lambda}(x), \\
D_{2,\lambda}^{(3)}(x) =& x^2 - \frac{37}{10}x + \frac{117}{40} \\
=& \beta_{2,\lambda}(x) - \frac{27}{10}\beta_{1,\lambda}(x) + \frac{309}{200}\beta_{0,\lambda}(x), \\
D_{1,\lambda}^{(3)}(x) =& x - \frac{27}{20} \\
=& \beta_{1,\lambda}(x) - \frac{9}{10}\beta_{0,\lambda}(x).
\end{aligned}$$

Conversely, the degenerate Bernoulli polynomials $\beta_{n,\lambda}(x)$ with $\lambda = 0.1$, $r = 3$ for $n = 1, 2, \cdots, 5$ are represented in terms of $D_{n,\lambda}^{(r)}(x)$:

$$\begin{aligned}
\beta_{5,\lambda}(x) =& x^5 - \frac{13}{4}x^4 + \frac{67}{20}x^3 - \frac{26}{25}x^2 - \frac{3}{50}x + \frac{9009}{400000} \\
=& D_{5,\lambda}^{(3)}(x) + \frac{27}{2}D_{4,\lambda}^{(3)}(x) + \frac{105}{2}D_{3,\lambda}^{(3)}(x) + \frac{6579}{100}D_{2,\lambda}^{(3)}(x) + \frac{13527}{625}D_{1,\lambda}^{(3)}(x) + \frac{2898}{3125}D_{0,\lambda}^{(3)}(x), \\
\beta_{4,\lambda}(x) =& x^4 - \frac{12}{5}x^3 + \frac{41}{25}x^2 - \frac{6}{25}x - \frac{2673}{100000}, \\
=& D_{4,\lambda}^{(3)}(x) + 9D_{3,\lambda}^{(3)}(x) + \frac{1017}{50}D_{2,\lambda}^{(3)}(x) + \frac{459}{40}D_{1,\lambda}^{(3)}(x) + \frac{5751}{6250}D_{0,\lambda}^{(3)}(x), \\
\beta_{3,\lambda}(x) =& x^3 - \frac{33}{20}x^2 + \frac{13}{20}x - \frac{99}{4000}, \\
=& D_{3,\lambda}^{(3)}(x) + \frac{27}{5}D_{2,\lambda}^{(3)}(x) + \frac{1161}{200}D_{1,\lambda}^{(3)}(x) + \frac{9}{10}D_{0,\lambda}^{(3)}(x), \\
\beta_{2,\lambda}(x) =& x^2 - x + \frac{33}{200} \\
=& D_{2,\lambda}^{(3)}(x) + \frac{27}{10}D_{1,\lambda}^{(3)}(x) + \frac{177}{200}D_{0,\lambda}^{(3)}(x), \\
\beta_{1,\lambda}(x) =& x - \frac{9}{20} \\
=& D_{1,\lambda}^{(3)}(x) + \frac{9}{10}D_{0,\lambda}^{(3)}(x).
\end{aligned}$$

4. Conclusions

The study of special polynomials provides useful tools in differential equations, fuzzy theory, probability, orthogonal polynomials, and special functions and numbers. These researches are conducted using various tools, including generating functions, p-adic analysis, combinatorial methods, and umbral calculus. Recently, degenerate versions of special polynomials and numbers have been investigated using λ-analogues of these methods, and their arithmetical and combinatorial properties and relations have been studied by several mathematicians. These degenerate versions of special polynomials and numbers have been applied in differential equations and probability theories, providing new applications. In this paper, we explore the connection problems between the degenerate higher-order Daehee polynomials and other degenerate types of special polynomials. We present explicit formulas for representations with the help of umbral calculus and vice versa. In addition, we illustrate the results with some explicit examples. In order to better understanding the polynomials, the distribution of roots are presented.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. R. Dere, Y. Simsek, H. Srivastava, A unified presentation of three families of generalized Apostol type polynomials based upon the theory of the umbral calculus and the umbral algebra, *J. Number Theory*, **133** (2013), 3245–3263. https://doi.org/10.1016/j.jnt.2013.03.004
2. E. Doha, Y. Youssri, On the connection coefficients and recurrence relations arising from expansions in series of modified generalized Laguerre polynomials: Applications on a semi-infinite domain, *Nonlinear Eng.*, **8** (2019), 318–327. https://doi.org/10.1515/nleng-2018-0073
3. W. Abd-Elhameed, Y. Youssri, Solutions of the connection problems between Fermat and generalized Fibonacci polynomials, *JP J. Algebra, Number Theory Appl.*, **38** (2016), 349–362. https://doi.org/10.17654/NT038040349
4. W. Abd-Elhameed, Y. Youssri, New connection formulae between Chebyshev and Lucas polynomials: New expressions involving Lucas numbers via hypergeometric functions, *Adv. Stud. Contemp. Math.*, **28** (2018), 357–367.
5. W. Abd-Elhameed, Y. Youssri, Connection formulae between generalized Lucas polynomials and some Jacobi polynomials: Application to certain types of fourth-order BVPs, *Int. J. Appl. Comput. Math.*, **6** (2020), 45. https://doi.org/10.1007/s40819-020-0799-4
6. W. Abd-Elhameed, Y. Youssri, N. El-Sissi, M. Sadek, New hypergeometric connection formulae between Fibonacci and Chebyshev polynomials, *Ramanujan J.*, **42** (2017), 347–361. https://doi.org/10.1007/s11139-015-9712-x
7. S. Roman, *The Umbral Calculus*, Academic Press, London, UK, 1984.
8. T. Kim, D. S. Kim, Some identities on truncated polynomials associated with degenerate Bell polynomials, *Russ. J. Math. Phys.*, **28** (2021), 342–355. https://doi.org/10.1134/S1061920821030079
9. T. K. Kim, D. S. Kim, H. I. Kwon, A note on degenerate stirling numbers and their applications, *Proc. Jangjeon Math. Soc.*, **21** (2018), 195–203.
10. T. Kim, D. Kim, H. Lee, J. Kwon, Representations by degenerate Daehee polynomials, *Open Math.*, **20** (2022), 179–194. https://doi.org/10.1515/math-2022-0013
11. J. Kwon, P. Wongsason, Y. Kim, D. Kim, Representations of modified type 2 degenerate poly-Bernoulli polynomials, *AIMS Math.*, **7** (2022), 11443–11463. https://doi.org/10.3934/math.2022638
12. L. Carlitz, A degenerate Staudt-Clausen theorem, *Arch. Math.*, **7** (1956), 28–33. https://doi.org/10.1007/BF01900520
13. D. S. Kim, T. Kim, Daehee numbers and polynomials, *Appl. Math. Sci.*, **7** (2013), 5969–5976. https://doi.org/10.12988/ams.2013.39535
14. D. S. Kim, T. Kim, Degenerate Sheffer sequences and λ-Sheffer sequences, *J. Math. Anal. Appl.*, **493** (2021), 124521. https://doi.org/10.1016/j.jmaa.2020.124521
15. T. Kim, D. S. Kim, H. Kim, J. Kwon, Some results on degenerate Daehee and Bernoulli numbers and polynomials, *Adv. Differ. Equations*, **2020** (2020), 1–13. https://doi.org/10.1186/s13662-020-02778-8

16. J. W. Park, J. Kwon, A note on the degenerate high order Daehee polynomials, *Appl. Math. Sci.*, **9** (2015), 4635–4642. http://doi.org/10.12988/ams.2015.56416
17. D. S. Kim, T. Kim, A note on a new type of degenerate Bernoulli numbers, *Russ. J. Math. Phys.*, **27** (2020), 227–235. https://doi.org/10.1134/S1061920820020090
18. T. Kim, A note on degenerate stirling polynomials of the second kind, *arXiv preprint*, (2017), arXiv:1704.02290. https://doi.org/10.48550/arXiv.1704.02290
19. J. Riordan, *An Introduction to Combinatorial Analysis*, Princeton University Press, 1958.
20. T. Kim, D. S. Kim, H. K. Kim, H. Lee, Some properties on degenerate Fubini polynomials, *Appl. Math. Sci. Eng.*, **30** (2022), 235–248. https://doi.org/10.1080/27690911.2022.2056169
21. L. Carlitz, Degenerate Stirling, Bernoulli and Eulerian numbers, *Util. Math.*, **15** (1979), 51–88.
22. T. Kim, L. C. Jang, D. S. Kim, H. Y. Kim, Some identities on type 2 degenerate Bernoulli polynomials of the second kind, *Symmetry*, **12** (2020), 510. https://doi.org/10.3390/sym12040510
23. G. W. Jang, T. Kim, A note on type 2 degenerate Euler and Bernoulli polynomials, *Adv. Stud. Contemp. Math.*, **29** (2019), 147–159.
24. D. S. Kim, T. Kim, T. Mansour, J. J. Seo, Degenerate Mittag-Leffler polynomials, *Appl. Math. Comput.*, **274** (2016), 258–266. https://doi.org/10.1016/j.amc.2015.11.014
25. N. Korobov, Special polynomials and their applications, *Math. Notes*, **2** (1996), 77–89.
26. T. Kim, D. S. Kim, Degenerate polyexponential functions and degenerate Bell polynomials, *J. Math. Anal. Appl.*, **487** (2020), 124017. https://doi.org/10.1016/j.jmaa.2020.124017
27. T. Kim, D. S. Kim, H. Y. Kim, J. Kwon, Some identities of degenerate Bell polynomials, *Mathematics*, **8** (2020), 40. https://doi.org/10.3390/math8010040