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Modules with a small injectivity domain

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Abstract

An injective module is a module with the largest possible injectivity domain. A poor module is described as the opposite of an injective module, in the sense that a poor module is one whose injectivity domain is the smallest possible. A related concept to that of the poor module is that of a ring with no middle class. A ring has no right middle class if every right module is either poor or injective. Although, the concept we have the most interest on is that of the pauper module. A pauper module is a poor module with no proper poor direct summand. We will expose the importance of pauper modules regarding the characterization of poor modules over different rings. Furthermore, we shall characterize rings and their structures in function of their injectivity domains, in particular, regarding their poor and pauper modules. For any given ring, we have a particular interest in verifying certain conditions. The first condition being the existence of pauper modules. The other condition is that of ubiquity, for which we present two distinct cases. In the first one, every poor module contains a pauper direct summand. The second weaker one, is that every poor module contains a pauper module as a pure submodule.

Keywords— injective module, injectivity domain, poor module, no middle class, pauper module

Resumo

Esta dissertação tem como principal objetivo expor os conteúdos do artigo [3] de forma auto-contida. Neste é introduzido o estudo de módulos paupérrimos. Os conceitos principais que serão explorados são módulos pobres, anéis sem classe intermédia e módulos paupérrimos. Como veremos, as definições destes conceitos são derivadas da definição de módulos injetivos. Um módulo injetivo é um módulo cujo domínio de injetividade é máximo. Por outro lado, um módulo pobre é descrito como o oposto, isto é, um módulo diz-se pobre se o seu domínio de injetividade é mínimo. Notemos que esta dissertação não é um estudo completo em relação aos módulos pobres, nem sobre anéis sem classe intermédia. Um estudo mais abrangente é feito em [1, 2, 5, 8, 15, 27]. No esforço de manter esta dissertação auto-contida, o primeiro capítulo é dedicado a apresentar definições e resultados que variam entre resultados clássicos da teoria de módulos e anéis e resultados mais específicos e necessários relacionados com módulos injetivos.

O segundo capítulo é dedicado ao estudo de módulos pobres e também anéis sem classe intermédia. O estudo de módulos pobres foi iniciado em [1]. Nesse artigo começa-se por notar que, se um R-módulo N é semisimples, então N pertence ao domínio de injetividade de qualquer outro R-módulo (Proposição 1.1.12). Também temos, para um anel arbitrário R, que a interseção dos domínios de injetividade de todos os R-módulos, sobre a categoria dos R-módulos, é precisamente a classe dos módulos semisimples (Proposição 2.1.3). Por outras palavras, um módulo M é pobre se, para qualquer R-módulo N, quando M é N-injetivo, então N é semisimples.

A secção 2.1 é dedicada a introduzir conceitos essenciais, relacionados com módulos pobres, e alguns resultados mais ilustrativos, em relação à importância dos módulos pobres. Como por exemplo, o facto de qualquer anel admitir um módulo pobre (Teorema 2.1.2), e uma maneira explícita de obter módulos pobres (Proposição 2.1.7). Esta secção termina com a demonstração de que, $\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$, com $\mathbb{P}$ o conjunto dos números primos, é um $\mathbb{Z}$-módulo pobre.

Ainda em [1], define-se um anel R sem classe intermédia como um anel cujos R-módulos são todos injetivos ou são todos pobres. O estudo aqui apresentado acerca destes anéis parte maioritariamente de [15], mas também dos relevantes [5, 27]. Na secção 2.2, começamos por relacionar algumas classes de anéis com anéis sem classe intermédia. Por exemplo, um anel R é semisimples e Artiniano se e só se todos os R-módulos são pobres (Proposição 2.2.1). Também provamos que, se um anel R não tem classe intermédia, então qualquer anel quociente de R também não tem classe intermédia. Outro resultado relevante diz-nos que um anel sem classe intermédia à direita é semiartiniano à direita, ou Noetheriano à direita (Proposição 2.2.8). Além disso, podemos separar o primeiro caso em outros dois casos, R é Artiniano à direita, ou todos os R-módulos simples são injetivos (Proposição 2.2.11). O teorema mais importante desta secção oferece-nos uma caraterização da estrutura de um anel sem classe intermédia da seguinte forma: se R é um anel sem classe intermédia, então $R \cong S \times T$, onde S é um anel semisimples e Artiniano e $T = 0$ ou T pertence a uma das classes descrita em (a), (b), (c) do Teorema 2.2.14. Uma caracterização semelhante é dada na forma do Teorema 3.2.7. Terminamos esta secção com o Corolário 2.2.23, que garante que um anel comutativo sem classe intermédia é Artiniano.

O último capítulo é dedicada aos módulos paupérrimos. Um módulo diz-se paupérrimo se é pobre e não contém nenhuma parcela direta própria que seja pobre. O estudo de módulos paupérrimos é inspirado na necessidade de uma caracterização intrínseca de módulos pobres. Uma razão para esta definição ser necessária é o facto de o domínio de injetividade de uma soma direta entre dois módulos ser igual à interseção dos domínios dessas parcelas (Lema 2.1.4). Isto implica que um módulo ser pobre é uma espécie de propriedade absorvente em relação à soma direta, isto é, a soma direta de um R-módulo pobre com um outro R-módulo qualquer é também pobre. Isto implica, de forma geral, que não existe muito

interesse em algumas parcelas, daí querermos encontrar parcelas que sejam inerentemente pobres.

O estudo de módulos paupérrimos exposto nesta dissertação passa por verificar que diferentes tipos de anéis verificam duas propriedades. A primeira, e mais simples, consiste em verificar em que classes de módulos é que existem módulos paupérrimos (Existência que representaremos por (E)). A segunda propriedade passa por verificar que todos os módulos pobres numa dada classe de módulos admitem módulos paupérrimos como parcelas diretas (Ubiquidade que representaremos por (U)). No nosso contexto, uma classe de módulos que satisfaça (U) está totalmente caracterizada. No entanto, em geral, (U) não é fácil de verificar. Sendo assim, definimos uma condição de ubiquidade mais fraca (representada por (U')) da seguinte forma: todo o módulo pobre P contém um submódulo paupérrimo M tal que M é um submódulo puro de P. Para certos anéis, as condições (U) e (U') são equivalentes. Em particular, iremos verificar tal equivalência para anéis Noetherianos (Teorema 3.3.2). Em geral, a classe de módulos que consideramos é a categoria de R-módulos à direita. Neste caso, dizemos que R satisfaz (E), (U) ou (U').

Ao contrário dos módulos pobres, nem todos os anéis admitem módulos paupérrimos. Por exemplo, um anel semiartiniano à direita, que não seja semisimples, cujos R-módulos simples sejam injetivos, não admite módulos paupérrimos (Proposição 3.1.1). No terceiro capítulo, o nosso estudo de módulos paupérrimos inicia-se com alguns exemplos explícitos de módulos paupérrimos, como $\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$ e $\prod_{p\in\mathbb{P}}\mathbb{Z}_p$, com $\mathbb{P}$ o conjunto dos números primos (Exemplos 3.1.4(i) e (ii)). Daí continuaremos a dar outros exemplos mais abstratos, de anéis que admitem módulos paupérrimos, como anéis de dimensão uniforme finita (Proposição 3.1.5) e anéis semilocais (Proposition 3.1.6). Um anel R sem classe intermédia admite um módulo paupérrimo se e só se R é o produto direto de um anel semisimples S, com um anel Noetheriano T (Corolário 3.1.10).

Na secção 3.2, combinamos de forma natural a definição de módulo paupérrimo com a de anel sem classe intermédia. Num anel sem classe intermédia, um módulo é paupérrimo se e só se não é injetivo e é indecomponível. Assim sendo, faz sentido definirmos um anel sem classe intermédia indecomponível, à direita, se todos os R-módulos indecomponíveis, à direita, são pobres ou injetivos. Pela definição, torna-se claro que qualquer anel sem classe intermédia também é sem classe intermédia indecomponível. O recíproco é verdade para anéis Noetherianos comutativos (Corolário 3.2.6) e também para anéis Artinianos seriais (Teorema 3.2.8).

As secções 3.3 e 3.4 são focadas em anéis Noetherianos e semiartinianos, respetivamente. Alguns dos principais resultados permitem-nos concluir que, se um anel Noetheriano comutativo e hereditário satisfaz (U') (Teorema 3.3.10), então também satisfaz (U). A classe dos módulos cujo radical é zero, sobre um anel semiartiniano comutativo, satisfaz (U') (Proposição 3.4.3). Além disso, qualquer anel Artiniano serial satisfaz (U) (Proposição 3.4.8).

A última secção é dedicada a grupos abelianos, isto é, $\mathbb{Z}$-módulos. A classe de grupos abelianos de torção e a classe de grupos abelianos livres de torção de dimensão um satisfazem (U). Estas afirmações seguem, respetivamente, da Proposição 3.5.6 e do Corolário 3.5.12.

Remetemos para o apêndice conceitos necessários, mas que foram menos aprofundados pelo autor.

Palavras-Chave— módulo injetivo, domínio de injetividade, módulo pobre, anel sem classe intermédia, módulo paupérrimo

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Introduction

The main objective of this dissertation is to expose the contents of [3] in a self-contained manner. In the aforementioned article the concept of pauper modules is introduced, the main concepts to be explored alongside it are those of, poor modules and rings with no middle class. We will see how these concepts are all derived from the definition of injective module.

An injective module is a module with the largest possible injectivity domain. In [1] a poor module was defined as being the opposite of an injective module, in the sense that a poor module is one with the smallest injectivity domain. Still in article [1] the concept of ring with no middle class is also introduced. A ring R has no right middle class, if its right R-modules are all injective or poor. It is important to highlight that this dissertation is not a complete study of poor modules or rings with no middle class, a more thorough study is done in [1, 2, 5, 8, 15, 27].

In the effort of maintaining this dissertation self-contained, Chapter 1 is dedicated to introducing fundamental concepts and results regarding ring and module theory.

Chapter 2 is divided in two sections. The first section is dedicated to introducing poor modules and some of its fundamental results such as proving the existence of a poor module for any ring (Theorem 2.1.2), as well as an explicit way to construct poor modules, Proposition 2.1.7. We end Section 2.1 with a result that will follow us throughout, for p primes, $\oplus_p \mathbb{Z}_p$ is a poor $\mathbb{Z}$-module.

The second section of Chapter 2 is dedicated to rings with no middle class, and most of it follows from [15] and less preeminent, but also relevant [5, 27]. In this section we start by seeing the relation between classes of rings and rings with no middle class. For example, a ring R is semisimple Artinian if and only if, all of its R-modules are poor, Proposition 2.2.1. As another example, we also prove that if a ring has no middle class, then a factor rings has no middle class either. Another important result states that, a ring with no right middle class is right semiartinian or right Noetherian, Proposition 2.2.8. Furthermore, in the first case that the ring is semiartnian we can unfold the result in two other cases, the ring is right Artinian, or a V-ring, Proposition 2.2.11. The main theorem of Section 2.2 offers as a characterization of the structure of a ring R with no middle class as follows, $R \cong S \times T$, where S is a semisimple Artinian ring and $T = 0$, or is described as in (a), (b), (c) of Theorem 2.2.14. We finish this section with a Corollary that states that, a commutative ring with no middle class is Artinian.

Chapter 3 is dedicated to pauper modules. The study of pauper modules is born of the necessity of an intrinsic characterization of poor modules. We will prove that the injectivity domain of a direct sum is the intersection of the injectivity domains, Lemma 2.1.4, and as a result it becomes clear that a poor module acts as a sort of an absorption property regarding direct sums, which justifies the definition of pauper module as follows. A module is pauper if, it is poor and no proper direct summand of it is poor.

The study of pauper modules done in this dissertation can be summed up to which rings satisfy the following two properties. The first one is existence, which we will represent by (E), meaning, which classes of modules admit pauper modules. The second one is Ubiquity, represented by (U), which states that for every poor module P contained in a class of modules $\mathcal{A}$ there exists a pauper module M in $\mathcal{A}$ such that M is a direct summand of P. In our context, this means that $\mathcal{A}$ is completely characterized. However proving (U) is no trivial matter, so we define a weaker version of ubiquity, represented by (U') as follows, if for every poor module P in $\mathcal{A}$ there exists a pauper module M such that M is a pure submodule of P in $\mathcal{A}$. In general the class of modules we will be considering is the category of right R-modules, meaning we omit $\mathcal{A}$ and simply state that a ring R satisfies (E), (U) or (U'). For certain rings we will see that (U) and (U') coincide, we will see this is true for Noetherian rings, Theorem 3.3.2.

Unlike poor modules, not every ring admits pauper modules. For example, Proposition 3.1.1, shows us that right semiartinian right V-rings that are not semisimple do not admit pauper modules. In Section 3.1 we give explicit examples of pauper modules such as, $\oplus_p \mathbb{Z}_p$ e $\prod_p \mathbb{Z}_p$, for primes p (Examples 3.1.4(i) e (ii)). Finite uniform dimensions modules also admit pauper summands (Proposition 3.1.5), and so do semilocal rings (Proposition 3.1.6).

In Section 3.2 we combine in a natural way the definitions of pauper module and ring with no middle class. In a ring with no middle class, a module is pauper if and only if it is not injective and it is indecomposable. Therefore, it makes sense to define a ring with no right indecomposable middle class by saying that, every indecomposable right R-module is poor or injective. By definition it is clear that a ring with no middle class is in particular, a ring with no indecomposable middle class. The other way around is also true for commutative Noetherian rings and serial Artinian rings, by Corollary 3.2.6 and Theorem 3.2.8 respectively.

The remaining sections are fairly self-explanatory by their name. Some of the most important results are that a commutative hereditary Noetherian ring satisfies (U') (Theorem 3.3.10), then it also satisfies (U). Any Artinian serial ring satisfies (U), by Proposition 3.4.8. Furthermore, both the class of torsion abelian groups satisfies, and the class of torsion-free rank one groups satisfy (U), by Proposition 3.5.6 and Corollary 3.5.12 respectively.

The Appendix is dedicated to necessary concepts that were not as fully developed by the author.

Chapter 1

Basic Definitions, notations and results

This chapter is dedicated to establishing notations, definitions and necessary results in order to study poor and subsequently pauper modules. Most of the concepts and results presented here are well-known in the theory of modules over rings, and can be found in [4, 17, 18, 22, 23, 24, 25, 30]. When appropriate we present some group related concepts as well. The title of each section is fairly self-explanatory, with the last section being dedicated to the remaining needed classes of modules/rings that did not fit elsewhere in a natural way.

We represent the set of all prime numbers by $\mathbb{P}$.

By an abelian group G we mean a commutative group under addition with ”0” (zero) representing its identity.

We shall always consider a ring R to be unitary and unless otherwise stated we will assume every module to be a right R-module, which we denote by $M \in$ Mod-R (or M_R), where Mod-R represents the category of all right modules over R.

We write $N \leq M$ to say N is a submodule of the R-module M.

The set $Hom_R(M, N)$ corresponds to all the R-homomorphisms from M to N. In particular, $End_R(M)$ represents all the R-homomorphisms from M to itself.

Given a family of modules $(M_i)_{i \in I}$, we define the direct product of modules as follows

$$\prod_{i \in I} M_i = \{(x_i)_{i \in I} : x_i \in M_i,\ i \in I\}.$$

Furthermore we represent the direct sum by

$$\bigoplus_{i \in I} M_i = \{(x_i)_{i \in I} \in \prod_{i \in I} M_i \mid x_i = 0, \text{ for all but finite } i \in I\}.$$

Given a module M, the set $M^{(I)} = \sum_{i \in I} M$ represents the direct sum of I copies of M, while M^I represents the direct product of I copies of M.

1.1 Semisimple and Injective Modules

These are the core building blocks of what is to come, so we dedicate this section to exploring these two classes of modules and presenting important correlations between them.

The fundamental idea of a semisimple module is that it can be uniquely decomposed (up to isomorphism) as a direct sum of its simple submodules. Recall that a module is simple if it is non-zero and it contains no proper submodules.

Definition. We say that a module M is *semisimple,* if for any submodule N of M there exists a submodule K of M such that $M = N \oplus K$, that is N is a direct summand of M.

A ring R is said to be a *right (resp. left) semisimple ring,* if the right module R_R (resp. ${}_RR$) is semisimple.

SSMod-R represents the class of all semisimple modules over the ring R.

The following characterization is interchangeable with the definition.

Proposition 1.1.1. [25, Theorem 2.4] *For a right R-module M, the following properties are equivalent:*

(a) *M is semisimple.*

(b) *M is the direct sum of a family of simple submodules.*

(c) *M is the sum of a family of simple modules.*

Proof. (a) $\Rightarrow$ (c) Let M_1 be the sum of all simple submodules of M. As M is semisimple, for a suitable submodule M_2 of M, we have $M = M_1 \oplus M_2$. If $M_2 \neq 0$, then M_2 contains a simple submodule, because every non-zero semisimple module contains a simple module. But then such submodule is in M_1, a contradiction. Thus, $M = M_1$.

(c) $\Rightarrow$ (a) Take $M = \sum_{i\in I} M_i$, where each M_i is a simple submodule of M. For notation purposes, we write $M_J = \sum_{j\in J} M_j$, for every $J \subseteq I$. Take N an arbitrary submodule of M. We want to prove that N is a direct summand of M. Consider Ω the set of all subsets $J \subseteq I$ with the following properties: M_J is a direct sum, and $N \cap M_J = 0$. We can apply Zorn's Lemma to Ω (note that the empty set is module of Ω) so we can choose a maximal subset $J \in \Omega$. Thus, for such J, let $M' = N + M_J = N \oplus (\oplus_{j\in J} M_j)$. We are left to show that $M = M'$. It is enough to check $M_i \leq M'$, for all $i \in I$. If there is some $M_i \not\leq M'$, then $M' \cap M_i = 0$, because M_i is simple. Now

$$M' + M_i = N \oplus (\oplus_{j\in J} M_j) \oplus M_i$$

which contradicts the maximality of J.

(c) $\Rightarrow$ (b) Take $N = 0$, in the previous step.

(b) $\Rightarrow$ (c) Tautology. □

Remark. By the implication (c) $\Rightarrow$ (a) we have that, if $M = \sum_{i\in I} M_i$, for a family of simple submodules of M, then, for any $N \leq M$, there exists a subset $J \subseteq I$ such that $M = N \oplus (\oplus_{j\in J} M_j)$.

The following proposition describes the structure of semisimple rings.

Proposition 1.1.2. *A ring R is right semisimple, if and only if it is a finite direct sum of some of its right minimal ideals $\mathfrak{R}_1, \cdots, \mathfrak{R}_t$ i.e., $R = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_t$.*

Proof. Let R be a right semisimple ring, that is R_R is semisimple. Therefore we can write $R_R = \bigoplus_{i\in I} \mathfrak{R}_i$, where $\{\mathfrak{R}_i : i \in I\}$ are simple submodules of R_R. Note that these are minimal right ideals over the ring. So it is enough to prove that the set I is finite. Since $R = \sum_{i\in I} \mathfrak{R}_i$, there exists a finite subset $J \subseteq I$ such that $1 = \sum_{j\in J} a_j$, where $a_j \in \mathfrak{R}_j$ for $j \in J$. Let us assume I is infinite and take $i \in I \setminus J$.

For any $0 \neq a \in \mathfrak{R}_i$, we have

$$a = 1.a = (\sum_{j \in J} a_j)a = \sum_{j \in J} a_j a \in \sum_{j \in J} \mathfrak{R}_j$$

therefore

$$a \in \mathfrak{R}_i \cap (\sum_{j \in J} \mathfrak{R}_j) = 0$$

and we have arrived at a contradiction. Hence I is finite. $\square$

Lemma 1.1.3. (Schur) [25, Lemma 3.6] *Let V be a simple R-module. Then $End_R(V)$ is a division ring.*

Proof. Let $0 \neq \phi \in End_R(V)$. Then $\operatorname{im}\phi \neq 0$ and $\ker\phi \neq V$. Since V is simple and $\ker\phi$ and $\operatorname{im}\phi$ are submodules of V, we infer that $\operatorname{im}\phi = V$ and $\ker\phi = 0$. This means that any non-zero R-endomorphism admits an inverse, thus $End_R(M)$ is a division ring. $\square$

The following will be used without mention. It gives us a fairly useful and straightforward characterization of the relation between simple modules.

Proposition 1.1.4. *For any simple R-modules V_1, V_2 we have $Hom_R(V_1, V_2) \neq 0$ if and only if V_1 and V_2 are isomorphic.*

Proof. ($\Leftarrow$) If $V_1 \cong_R V_2$ then clearly there is a non-zero homomorphism.

($\Rightarrow$) Let $0 \neq \phi : V_1 \to V_2$. By a similar argument done for Schur's Lemma and by the fact that both modules are simple we are able to infer that $\operatorname{im}\phi = V_2$ and $\ker\phi = 0$. Now by the First Isomorphism Theorem we conclude that $V_1/0 \cong V_2$. $\square$

The Wedderburn-Artin Theorem is fundamental in the study of semisimple rings/modules, since it allows us to determine the class of (right or left) semisimple rings. We skip this proof since it is not particularly important in this work.

Theorem 1.1.5. (Wedderburn-Artin) [25, Theorem 3.5] *Let R be any right semisimple ring. Then we have a ring isomorphism $R \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_t}(D_t)$, for suitable division rings $D_1, \ldots, D_t$ and positive integers $n_1, \ldots, n_t$. The number t is uniquely determined, as are the pairs $(n_1, D_1), \ldots, (n_t, D_t)$ (up to permutation). There are exactly t mutually non-isomorphic right simple modules over R.*

Remark. A consequence of the Wedderburn-Artin Theorem is that the condition of a ring being right semisimple is equivalent to it being left semisimple. This is true because $\mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_t}(D_t)$ is both right and left semisimple, so we often omit the "right/left" condition and simply state that a ring is semisimple.

Now we must introduce some notions regarding R-homomorphisms.

Definition. A finite (or infinite) sequence of R-modules and R-homomorphism

$$\ldots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \longrightarrow \ldots$$

is said *exact* if for every pair of successive R-homomorphisms (f_i, f_{i+1}) we have that $\operatorname{im} f_n = \ker f_{n+1}$.

An exact sequence of the form

$$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is called a *short exact sequence*.

It is well-understood that a sequence of this form is exact if and only if, $\ker g = \operatorname{im} f$, g is surjective and f is injective.

Lemma 1.1.6. [4, Lemma 5.1] *Let $f : M \to N$ and $f' : N \to M$ be R-homomorphisms such that $ff' = 1_N$. Then f is an epimorphism, f' is a monomorphism and $M = \ker f \oplus \operatorname{im} f'$.*

Proof. For any $n \in N$ we have $ff'(n) = n$, so $ff'(N) = N$, hence f must be an epimorphism. Now if we take $n \in N$ such that $f'(n) = 0$, then $n = ff'(n) = f(0) = 0$, therefore f' is injective. To prove the remainder, take $m \in M$ and it follows that $f(m - f'f(m)) = f(m) - f(m) = 0$ and $m = (m - f'f(m) + f'f(m)) \in \ker f + \operatorname{im} f'$. Now if $m = f'(n) \in \ker f \cap \operatorname{im} f'$, then $0 = f(m) = ff'(n) = n$ and $m = f'(n) = f'(0) = 0$. Thus $\ker f \cap \operatorname{im} f' = 0$. $\square$

If $f : M \to N$ and $f' : N \to M$ are R-homomorphisms with $ff' = 1_N$, we say that f is a *split epimorphism* and that f' is a *split monomorphism.*

Definition. A short exact sequence

$$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is *split* or *split exact* in case f is a split monomorphism and g is a split epimorphism.

Lemma 1.1.7. [4, Proposition 5.2] *The following statements about a short exact sequence*

$$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$$

in Mod-R are equivalent:

(a) *The short exact sequence splits.*

(b) *The monomorphism $f : K \to M$ splits.*

(c) *The epimorphism $g : M \to N$ splits.*

(d) *$\operatorname{im} f = \ker g$ is a direct summand of M.*

(e) *Every homomorphism $h : K \to X$ factors through f (i.e., there exists a homomorphism $t : M \to X$ such that $h = tf$.)*

(f) *Every homomorphism $h : X \to N$ factors through g (i.e., there exists a homomorphism $v : X \to M$ such that $h = gv$.)*

Proof. It is clear that (a) implies (b) and (c) by definition. Furthermore, by the previous lemma, (b) and (c) imply (d). So it is enough to prove (d) $\Rightarrow$ (e) $\Rightarrow$ (b) and (d) $\Rightarrow$ (f) $\Rightarrow$ (c).

(d) $\Rightarrow$ (e) Assume $M = \operatorname{im} f \oplus L$, where $L \leq M$, and let $h : K \to X$ be a homomorphism. Since f is a monomorphism, then for each $m \in M$ there exists a unique $k \in K$ and $l \in L$ such that $m = f(k)+l$. Now define $t : M \to X, m = f(k) + l \mapsto h(k)$. It is clear that t is a homomorphism and that $h = tf$.

(e) $\Rightarrow$ (b) Since (e) is true for every homomorphism, then in particular take $h = 1_K$ and $X = K$, so f splits.

(d) $\Rightarrow$ (f) Suppose $M = \ker g \oplus L$, where $L \leq M$, and let $h : X \to N$ be a homomorphism. By definition of direct sum it follows that $L \cap \ker g = 0$ and $g(M) = g(L)$, hence $g_{|L} : L \to N$ is an isomorphism and let $g' : N \to L$ be its inverse. Therefore $v = g'h : X \to M$ is a homomorphism such that $h = gv$.

(f) $\Rightarrow$ (c) Take $h = 1_N$ and $X = N$, thus g splits. $\square$

The following proposition tells us that every submodule and every factor module of a semisimple module is semisimple.

Proposition 1.1.8. [4, Proposition 9.4] *Let M be a semisimple module with semisimple decomposition $M = \oplus_A T_\alpha$. If*

$$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is an exact sequence, then this sequence splits and both K and N are semisimple.

Furthermore, there is a subset $B \subseteq A$ and isomorphisms

$$N \cong \oplus_B T_\beta \text{ and } K \cong \oplus_{A\setminus B} T_\alpha.$$

Proof. Since im f is a submodule of M, then by the proof of Proposition 1.1.1, there is a subset $B \subseteq A$ such that $M = (\text{im } f) \oplus (\oplus_B T_\beta)$, which means that the sequence as defined above splits and $N \cong M/\text{im } f \cong \oplus_B T_\beta$. We also have $M = (\oplus_{A\setminus B} T_\alpha) \oplus (\oplus_B T_\beta)$, hence $K \cong \text{im } f \cong \oplus_{A\setminus B} T_\alpha$. $\square$

Before we shift our attention towards injective modules, we introduce the fundamental concept of essential submodule.

Definition. A module M is said to be an *essential extension* of a submodule N or N is said to be an *essential submodule* of M, if for every submodule H of M such that $H \cap N = 0$, then $H = 0$. We denote this by $N \leq_e M$.

The following two lemmas explore the properties of essential submodules and will be utilized often.

Lemma 1.1.9. [4, Lemma 5.19] *A submodule $N \leq M$ is essential in M if and only if, for each $x \in M\setminus 0$, there exists an $r \in R$ such that $xr \in N\setminus 0$.*

Proof. ($\Rightarrow$) If $N \leq_e M$ and $x \in M\setminus 0$, then $xR \cap N \neq 0$.

($\Leftarrow$) Take $x \in L\setminus 0$, with $L \leq M$. Now there is an $r \in R$ such that $0 \neq xr \in N$, which implies that $0 \neq xr \in N \cap L$. $\square$

Proposition 1.1.10. [23, Proposition 1.1] *Let $M, N, K, K', N' \in$ Mod-R. Then we have the following:*

1. *If $K \leq N \leq M$, then $K \leq_e M$ if and only if $K \leq_e N \leq_e M$;*

2. *If $K \leq_e N \leq M$ and $K' \leq_e N' \leq M$, then $K \cap K' \leq_e N \cap N'$;*

3. *If $f : N \to M$ is an R-homomorphism and $K \leq_e M$, then $f^{-1}(K) \leq_e N$;*

4. *If $\{K_i\}_{i\in I}$ is an independent family of submodules of M, and if $K_i \leq_e N_i \leq M$, for each $i \in I$, then $\{N_i\}_{i\in I}$ is an independent family and $\oplus_{i\in I} K_i \leq_e \oplus_{i\in I} N_i$.*

Proof. 1. First assume $K \leq_e N \leq_e M$ and consider $0 \neq M_0 \leq M$. Since $N \leq_e M$ we have $M_0 \cap N \neq 0$. As $K \leq_e N$, so $(M_0 \cap N) \cap K \neq 0$, thus $M_0 \cap K \neq 0$. Therefore $K \leq_e M$. Conversely, assume that $K \leq_e M$. Then any non-zero submodule of M has a non-trivial intersection with K. In particular, we have $N \leq M$, so any non-zero submodule of N has non-zero intersection with K, hence $K \leq_e N$. For

$0 \neq M_0 \leq M$, again by hypothesis, $M_0 \cap K \neq 0$ and $K \leq N$ so we infer that $M_0 \cap N \neq 0$. Therefore $N \leq_e M$.

2. Consider B a non-zero submodule of $N \cap N'$. Since by hypothesis $K \leq_e N$, then $B \cap K \neq 0$. Furthermore, since $K' \leq_e N'$, then $(B \cap K) \cap K' \neq 0$. Hence $K \cap K' \leq_e N \cap N'$.

3. By contradiction, assume that $f^{-1}(K)$ is not an essential submodule of N. Then N admits a submodule $A \neq 0$ such that $A \cap f^{-1}(K) = 0$. In particular $A \cap \ker f = 0$, so $A \cong f(A)$ and $0 \neq f(A) \leq M$. However $A \cap f^{-1}(K) = 0$, hence $f(A) \cap K = 0$, a contradiction.

4. First consider the case where the index set is finite. Assume $I = \{1, 2\}$. By 2., $K_1 \cap K_2 = 0 \leq_e N \cap N_2$, hence $N_1 \cap N_2 = 0$, therefore $\{N_1, N_2\}$ is independent. Now consider the projections $\pi_1 : N_1 \oplus N_2 \to N_1$ and $\pi_2 : N_1 \oplus N_2 \to N_2$. By 3., $K_1 \oplus N_2 \leq_e N_1 \oplus N_2$ and $N_1 \oplus K_2 \leq_e N_1 \oplus N_2$. Again by 2., $K_1 \oplus K_2 \leq_e N_1 \oplus N_2$. Now by induction, consider $\#I = n$ and assume $K_1 \oplus \cdots \oplus K_{n-1} \leq_e N_1 \oplus \cdots N_{n-1}$. By the same argument done above, $(N_1 \oplus \cdots \oplus N_{n-1}) \cap N_n = 0$. Therefore $\{M_1, \ldots, M_n\}$ is independent, and $(K_1 \oplus \cdots \oplus K_{n-1}) \oplus K_n \leq_e (N_1 \oplus \cdots \oplus N_{n-1}) \oplus N_n$. We are left to prove the case where I is an arbitrary index set. In general, a family $\{N_i\}_{i \in I}$ is independent if every finite subfamily is independent. However, that is what we have just shown. So take $0 \neq n \in \oplus_{i \in I} N_i$. Then $n \in \oplus_{i \in J} N_i$, for some $J \subseteq I$ finite. By the first part of the proof, it follows that $\oplus_{i \in J} K_i \leq_e \oplus_{i \in J} N_i$, and $0 \neq nR \cap (\oplus_{i \in J} K_i) \subseteq nR \cap (\oplus_{i \in I} K_i)$. This shows that the intersection of a non-zero submodule of $\oplus_{i \in I} N_i$ and $\oplus_{i \in I} K_i$ is different from zero. Hence $\oplus_{i \in I} K_i \leq_e \oplus_{i \in I} N_i$. $\square$

Definition. We say that $K \leq M$ is a *closed submodule* of M, if K has no proper essential extension in M. This means that given L a submodule of M such that $K \leq_e L$, then $K = L$.

We are now ready to define injective modules.

Definition. Given right R-modules M and N, we say that M is N*-injective* if, for each monomorphism $f : K \to N$ of right R-modules (equivalently, for each submodule K of N with f the inclusion), and each R-homomorphism $h : K \to M$, there exists an R-homomorphism $h' : N \to M$ such that $h = h'f$, i.e. the follow diagram commutes.

$$\begin{tikzcd}
 & M & \\
0 \arrow[r] & K \arrow[u, "h"] \arrow[r, "f"] & N \arrow[ul, dotted, "h'"']
\end{tikzcd}$$

When this is the case h' is said to be an *extension* of h or that h can be *extended* to h'.

Definition. We define the *injectivity domain of M* as follows

$$\mathfrak{I}n^{-1}(M) = \{N \in \text{Mod-R} : M \text{ is } N\text{-injective}\}.$$

We say M is an *injective module* (over R) if its injective domain is the class of all right R-modules. We say that R is *injective on the right (resp. left)* if R_R (resp. ${}_R R$) is an injective module.

The following is an important structural equivalence. It allows us to reduce the study of a semisimple module to that of cyclic submodules of factor modules. Its proof is beyond the scope of this thesis.

Proposition 1.1.11. [11, Corollary 7.14] *A module M is semisimple if and only if every cyclic submodule of a factor of M is M-injective.*

The following shows that SSMod-R $\subseteq \mathfrak{I}n^{-1}(M)$, for every module M.

Proposition 1.1.12. *Let M be an arbitrary module, and N an arbitrary semisimple module. Then M is N-injective.*

Proof. Let K be a submodule of N and let $h : K \to M$ be a homomorphism. Since N is semisimple, we have $N = K \oplus N_1$, for some submodule N_1 of N. Define an extension $h' : N \to M$ as $h'(x) = h(x)$, if $x \in K$ and $h'(x) = 0$, if $x \in N_1$. Therefore M is N-injective. $\square$

Let us present some fundamental properties of injective modules. The next result, also known as Baer's Test, tells us that in order to verify if a module M is injective, it is enough to take $N = R_R$. We skip this proof, as we intend to prove a generalization of this result in the form of Proposition 1.1.16.

Theorem 1.1.13. (Baer's Criterion) [24, Theorem 3.7] *A right R-module M is injective if and only if, for any right ideal I of R, any R-homomorphism $h : I \to M$ can be extended to $h' : R \to M$.*

Proposition 1.1.14. [30, Lemma 1.2] *Let M and N be arbitrary modules. If M is N-injective, then any monomorphism $M \to N$ splits.*

Proof. Suppose M is N-injective and take an arbitrary monomorphism $g : M \to N$. Since M is N-injective. Take $h = id_M : M \to M$, then there exists $h' : N \to N$ such that $id_M = h'g$. Thus g splits. $\square$

Proposition 1.1.15. [30, Proposition 1.3] *Let M be N-injective and $N_1 \leq N$. Then M is N_1-injective and (N/N_1)-injective*

Proof. It is clear that M is N_1-injective. Now take X/N_1 a submodule of N/N_1, and a homomorphism $\phi : X/N_1 \to M$. Furthermore we take the canonical epimorphisms $\pi : N \to N/N_1$ and $\pi_1 = \pi_{|X} : X \to X/N_1$. Since M is N-injective, there is a homomorphism $\theta : N \to M$ that extends $\phi\pi_1$, because $X \leq N$. Now we have $\theta(N_1) = \phi\pi_1(N_1) = \phi(0) = 0$, meaning $N_1 = \ker \pi \leq \ker \theta$. Therefore, there exists $\psi : N/N_1 \to M$ such that $\psi\pi = \theta$. For any $x \in X$, $\psi(x + N_1) = \psi\pi(x) = \theta(x) = \phi\pi_1(x) = \phi(x + N_1)$. To better illustrate this we define the diagram.

$$\begin{tikzcd}
M & \\
X/N_1 \arrow[u, "\phi"] \arrow[r] & N/N_1 \arrow[ul, dotted, "\psi"'] \\
X \arrow[u, "\pi_1"] \arrow[r] & N \arrow[u, "\pi"'] \arrow[uul, dotted, "\theta"]
\end{tikzcd}$$

Therefore ψ extends ϕ, hence M is (N/N_1)-injective. $\square$

Proposition 1.1.16. [30, Proposition 1.4] *A module M is N-injective if and only if M is nR-injective, for every $n \in N$.*

Proof. $(\Rightarrow)$ Follows from Proposition 1.1.15.

$(\Leftarrow)$ Now assume M is nR-injective for every $n \in N$. Take $N_0 \leq N$ and $\phi : N_0 \to M$ an R-homomorphism. By Zorn's Lemma we can find a maximal submodule N_1 and a homomorphism ψ such that $N_0 \leq N_1 \leq N$ and $\psi : N_1 \to M$ extends ϕ, so we have $N_1 \leq_e N$. Assume that $N_1 \neq N$ and consider an element $n \in N \backslash N_1$ and define $K = \{r \in R : nr \in N_1\}$. By Lemma 1.1.9, $nK \neq 0$. We consider $\mu = \psi_{|nK} : nK \to M$, which by our assumption can be extended to a homomorphism $\sigma : nR \to M$. Let $\epsilon : N_1 + nR \to M$, $\epsilon(n_1 + nr) = \phi(n_1) + \sigma(nr)$. If $n_1 + nr = 0$, then we have $r \in K$ and $\phi(n_1) + \sigma(nr) = \psi(n_1) + \mu(nr) = \psi(n_1) + \psi(nr) = \psi(n_1 + nr) = 0$. So ϵ is well-defined. However the pair $(N_1 + nR, \epsilon)$ contradicts the maximality of (N_1, ψ). Therefore $N_1 = N$ and $\psi : N \to M$ extends ϕ. Thus M is N-injective. $\square$

Proposition 1.1.17. [30, Proposition 1.5] *A module M is $(\oplus_{i\in I} N_i)$-injective if and only if M is N_i-injective for every $i \in I$.*

Proof. $(\Rightarrow)$ Follows from Proposition 1.1.15.

$(\Leftarrow)$ Assume that M is $(\oplus_{i\in I} N_i)$-injective for all $i \in I$. Let $N = \oplus_{i\in I} N_i$ and take a submodule X of N, also consider a homomorphism $\phi : X \to M$. Now by Zorn's Lemma we may assume that ϕ cannot be extendend to any homomorphism $Y \to M$, where $Y \leq N, Y \neq N$. Therefore $X \leq_e N$. Now we want to show that $X = N$. By contradiction assume there exists $j \in I$ and $n \in N_j$ such that $n \notin X$. Since M is N_j-injective, M is also nR-injective, again by Proposition 1.1.15. By a similar argument done in the previous proposition, we can extend ϕ to a homomorphism $\psi : X + nR \to M$, which contradicts the maximality of ϕ. Therefore $X = N$, which means M is N-injective. $\square$

We skip the proof of the following results.

Proposition 1.1.18. [30, Proposition 1.6] *Let N and $\{M_i : i \in I\}$ be modules. Then $\prod_{i\in I} M_i$ is N-injective if and only if M_i is N-injective, for every $i \in I$.*

Proposition 1.1.19. [4, Proposition 18.6] *Every right R-module can be embedded in an injective right R-module.*

The previous proposition leads us to the definition of the injective envelope of a module. It is a "minimal" embedding of M in an injective module.

Definition. A module E is called an *injective hull* or *injective envelope of* a module M, if E is an essential extension of M (i.e. $M \leq_e E$) and E is injective.

We denote the injective hull of M by $E(M)$.

Theorem 1.1.20. [4, Theorem 18.10] *Every module has an injective envelope. It is unique to within isomorphism.*

The following result will be fairly useful.

Lemma 1.1.21. [4, Proposition 18.12] *In the category Mod-R we have the following statements:*

1. *M is injective if and only if $M = E(M)$;*
2. *If $M \leq_e N$, then $E(M) = E(N)$;*
3. *If $M \leq Q$, with Q an injective module, then $Q = E(M) \oplus E'$;*
4. *If $\bigoplus\limits_{\alpha\in A} E(M_\alpha)$ is injective, then $E(\bigoplus\limits_{\alpha\in A} M_\alpha) = \bigoplus\limits_{\alpha\in A} E(M_\alpha)$.*

Proof. 1. This is immediate from the definition of injective hull.

2. By definition $N \leq_e E(N)$ and by hypothesis $M \leq_e N$. Then $M \leq_e E(N)$, by Proposition 1.1.10(1). Now by definition $E(N)$ is injective, thus $E(N)$ is an injective envelope of M.

3. Since Q is injective, there is a morphism $g : E(M) \to Q$ making the following commute

$$\begin{tikzcd}
Q & \\
M \arrow[u, "f"] \arrow[r, "i"] & E(M) \arrow[ul, dotted, "g"']
\end{tikzcd}$$

where f is the inclusion. As $M \leq_e E(M)$, g is also a monomorphism. Since $E(M)$ is injective, then g splits, by Proposition 1.1.14. Therefore $Q = E(M) \oplus E'$, where E' is some submodule of Q.

4. Assume that $\bigoplus_{\alpha \in A} E(M_\alpha)$ is injective. Take the injective envelopes $E(M_\alpha)$, for each $\alpha \in A$. By Proposition 1.1.10(4), $\oplus_A M_\alpha \leq \oplus_A E(M_\alpha)$. Thus $\oplus_A E(M_\alpha) = E(\oplus_A M_\alpha)$. $\square$

The following is a useful characterization of injectivity in terms of its essential extensions.

Proposition 1.1.22. [24, Lemma 3.28] *A module M is injective if and only if it has no proper essential extensions.*

Proof. Assume M is injective and consider a proper extension, $M \lneq E$. By Proposition 1.1.14, the inclusion $M \to E$ splits, hence $E = M \oplus N$, for some $0 \neq N \leq E$. Then $N \cap M = 0$ and M is not essential in E.

Conversely, assume M has no proper essential extensions, and embed M in an injective module J, by Proposition 1.1.19. Take $\phi : M \to J$ to be such an embedding. By Zorn's Lemma there exists a maximal submodule $S \subseteq J$ such that $S \cap \phi(M) = 0$. Note that for any non-zero quotient submodule S'/S of J/S we have $(S'/S) \cap ((\phi(M) \oplus S)/S) \neq 0$. Then $(\phi(M) \oplus S)/S \leq_e J/S$. However by assumption of no proper essential extension, this only holds if $(\phi(M) \oplus S)/S = J/S$. Therefore $J = \phi(M) \oplus S$ and since J is injective, by a particular case of Proposition 1.1.18 we conclude that $\phi(M)$ is injective, and so is M. $\square$

The definition of injectivity is given in function of homomorphisms between modules, while the definition of semisimple modules is given in terms of direct summands of submodules. The following proposition shows us that short exact sequences and splitting homomorphisms offer a much needed correlation regarding a semisimple ring.

Proposition 1.1.23. *Let $K, M, N \in$ Mod-R. The following are equivalent:*

(a) *R is semisimple.*

(b) [4, Proposition 13.9] *Every short exact sequence*

$$0 \to K \to M \to N \to 0$$

of right R-modules splits.

(c) [4, Proposition 13.9] *Every right R-module is semisimple.*

(d) [4, Corollary 13.10] *Every monomorphism in Mod-R splits.*

(e) [4, Corollary 13.10] *Every epimorphism in Mod-R splits.*

Proof. Since R_R is a right module (c) $\Rightarrow$ (a) is clear.

(a) $\Rightarrow$ (c) A finite (or infinite) direct sum of semisimple modules is itself semisimple. It will be shown in Proposition 1.3.1, that every module is the epimorphic image of a direct sum of copies of R_R. So by hypothesis this sum is semisimple.

(c) $\Rightarrow$ (b) By Proposition 1.1.8.

(b) $\Rightarrow$ (c) Let M be a module and let $K \leq M$. Consider the short exact sequence, $0 \xrightarrow{i} M \xrightarrow{\pi} M/K \to 0$, where i is the inclusion and π is the canonical epimorphism. By hypothesis, this sequence splits and so $K = \operatorname{im} i = \ker \pi$ is a direct summand of M.

(b) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) Follows from Lemma 1.1.7. $\square$

Corollary 1.1.24. [4, Corollary 18.8] *A ring R is semisimple if and only if every right module is injective.*

Proof. ($\Leftarrow$) Assuming every right R-module is injective, by Proposition 1.1.14, every monomorphism in Mod-R splits. Then, by Proposition 1.1.23, R is semisimple.

($\Rightarrow$) By the previous proposition we know that $M \hookrightarrow E(M)$ splits, for every R-module M. So M is a direct summand of $E(M)$. Since $E(M)$ is injective. Then by Proposition 1.1.18, M is injective. $\square$

Much like the name indicates the following definition is a weaker form of injectivity. We will see a fair amount of use for it later on, especially in Chapter 2.

Definition. A module M is said to be *quasi-injective* if M is M-injective.

A ring R is said to be a *quasi-injective ring (QI-ring)* if all its quasi-injective modules are injective.

Definition. Let M be a module and $N \leq M$. We say that N is a *fully invariant submodule of M* if $f(N) \leq N$, for every $f \in End_R(M)$, i.e., $End_R(M)N \leq N$.

The following proposition gives us a fairly useful way of looking at quasi-injective modules regarding the injective hull of a module.

Proposition 1.1.25. [23, Proposition 2.13] *A module M is quasi-injective if and only if M is a fully invariant submodule of $E(M)$.*

Proof. Let $A = End_R(E(M))$.

($\Leftarrow$) Assume that $AM \leq M$ and let N be a submodule of M. Any R-homomorphism $f : N \to M$ can be extendend to some endomorphism g in A. Note that $g_{|M}$ is an endomorphism of M that extends f. Thus by definition we conclude that M is quasi-injective.

($\Rightarrow$) Now assume that M is quasi-injective, and let $g \in A$. Now we restrict the domain of g to be $M \cap g^{-1}M$, meaning we get a map from $M \cap g^{-1}M$ to M, which can be extended to a map $h \in End_R(M)$, by quasi-injectivity. Thus h can be extended to $f \in A$ such that $f(M) \leq M$ and $(f-g)(M \cap g^{-1}M) = 0$. Since $f(M) \leq M$, this implies that

$$M \cap (f-g)^{-1}M \leq M \cap g^{-1}M \leq \ker(f-g)$$

and from this we infer that $(f-g)M \cap M = 0$. Since $M \leq_e E(M)$, then it follows by definition of essential extension that $(f-g)M = 0$. Finally $gM = fM \leq M$, thus $AM \leq M$. $\square$

The following definition is fundamental in the theory of modules over rings.

Definition. We define the *right annihilator of a module* M (over R) as follows

$$ann_R(M) = \{a \in R : xa = 0, \, x \in M\}.$$

If $ann_R(M) = 0$, M is said to be a *faithful module.*

When there is no ambiguity about the ring we simply write $ann(M)$. When we consider an element $a \in R$ (resp. a subset X of R) it may not be clear if $ann(a)$ (resp. $ann(X)$) represents the right annihilator or the left annihilator. So we denote the right annihilator of a over R (resp. the right annihilator of I over R), by $ann_r(a)$ (resp. $ann_r(X)$), and the left annihilator of a over R (resp. the left annihilator of X over R), by $ann_l(a)$ (resp. $ann_l(X)$).

Now we introduce the last major concept of this section that directly relates to injectivity, the notion of divisibility.

Definition. Let M be a right R-module, if $x \in M$ and $a \in R$, we say that x is *divisible by* a if $x \in Ma$, i.e., there exists an element $y \in M$ such that $x = ya$.

The module M is *divisible* if, for any $x \in M$ and $a \in R$, x is divisible by a.

Remark. As described in the definition above, for such an element $y \in M$ to exist we have the necessary following condition: For $b \in R$, $ab = 0$ implies that $xb = 0$. We represent this in terms of the annihilators by $ann_r(a) \subseteq ann(x)$. So x is divisible by a, only if $ann_r(a) \subseteq ann(x)$.

Proposition 1.1.26. [24, Proposition 3.17] *For any module M the following statements are equivalent:*

(a) *M is a divisible module.*

(b) *For any $a \in R$, any homomorphism $f : aR \to M$ extends to a homomorphism from R_R to M.*

Proof. (a) $\Rightarrow$ (b) Let $a \in R$, $f \in Hom_R(aR, M)$, and $x = f(a) \in M$. Then $u \in ann_r(a)$ implies that $au = 0$ and $xu = f(a)u = f(au) = f(0) = 0$. Therefore $u \in ann(x)$. Now by definition of divisibility, $x = ya$ for some $y \in M$. Hence f extends to $R_R \to M$ by $1 \mapsto y$.

(b) $\Rightarrow$ (a) Let $x \in M$ and $a \in R$ be such that $ann_r(a) \subseteq ann(x)$. The morphism $f : aR \to M$, $f(as) = xs$, for all $s \in R$, is a well-defined homomorphism. By hypothesis, f extends to a homomorphism $g : R \to M$. Let $y = g(1) \in M$. Clearly, $x = f(a) = g(a) = g(1.a) = g(1).a = ya$. $\square$

Corollary 1.1.27. [24, Corollary 3.17'] *If M is an injective module, then it is divisible. The converse holds if R is a principal right ideal ring.*

Proof. If a module M is injective, the condition (b) in Proposition 1.1.26 is obviously true and the first part becomes clear. In case R is a principal right ideal ring, the converse follows by Baer's Criterion. $\square$

Remark. It is well known that the structure of abelian groups coincides with that of $\mathbb{Z}$-modules. So naturally let us define divisible abelian group.

Definition. An abelian group G is *divisible* if for all $x \in G$ and for all $n \in \mathbb{Z}\backslash 0$, there exists $y \in G$ such that $ny = x$. We represent this by $n \mid x$. Thus G is divisible if $G = nG$, $\forall n \in \mathbb{Z}\backslash 0$.

For a prime p, we have that G is *p-divisible* if $G = pG$.

Remark. From the definition above, note that:

1. An abelian group is divisible if and only if it is p-divisible for every p prime. This is clear since every positive integer n can be factored as the product of primes i.e., $n = p_1^{k_1} p_2^{k_2} ... p_n^{k_n}$, with primes p_i and natural numbers k_i.

1.2 Noetherian and Artinian Modules

The concepts of Noetherian and Artinian modules are essential and unavoidable in module and ring theory. We shall present some of the most well-known results, as well as proving some meaningful relations to injectivity and semisimplicity.

Let us start by presenting the rigorous definitions of generated and cogenerated.

Definition. Let $\mathcal{A}$ be a class of modules. A module M *is (finitely) generated by* $\mathcal{A}$ or $\mathcal{A}$ *(finitely) generates* M*, in case there is a (finite) indexed set* $(U_i)_{i\in I}$ *in* $\mathcal{A}$ *and an epimorphis*m

$$\bigoplus_{i\in I} U_i \to M \to 0.$$

A module M *is (finitely) cogenerated by* $\mathcal{A}$ or $\mathcal{A}$ *(finitely) cogenerates* M*, in case there is a (finite) indexed set* $(U_i)_{i\in I}$ *in* $\mathcal{A}$ *and a monomorphism*

$$0 \to M \to \prod_{i\in I} U_i.$$

Definition. Let M be a module, we denote the lattice of all submodules of M by $(Sub(M), \leq)$. Now we say that

• M is *Noetherian* if $Sub(M)$ satisfies the *ascending chain condition* (ACC) i.e., an ascending chain $M_0 \leq M_1 \leq \cdots \leq M_p \leq \cdots$ of submodules is *stable* (there is a $p \in \mathbb{N}$ such that for all $n \geq p$, $M_p = M_n$).

• M is *Artinian* if $Sub(M)$ satisfies the *descending chain condition* (DCC) i.e., a descending chain $M_0 \geq M_1 \geq \cdots \geq M_p \geq \cdots$ of submodules is *stable* (there is a $p \in \mathbb{N}$ such that for all $n \geq p$, $M_p = M_n$).

A ring R is said to be *right Artinian (resp. Noetherian)* if the module R_R is Artinian (resp. Noetherian).

The upcoming Propositions 1.2.1 and 1.2.2 will be used as interchangeable with the definitions of Noetherian and Artinian modules, respectively.

Proposition 1.2.1. [4, Proposition 10.9] *Let M be a module. The following are equivalent:*

(a) *M is Noetherian.*

(b) *Every non-empty subset of $Sub(M)$ has a maximal element.*

(c) *All submodules of M are finitely generated.*

Proof. The first three implications are done by negation.

(b) $\Rightarrow$ (a) Assume that M is non-Noetherian, meaning there is a chain of submodules $M_0 \leq M_1 \leq \cdots \leq M_n \leq \cdots$ that is not stable. Then it is clear that the set $\{M_n : n \in \mathbb{N}\}$ has no maximal element.

(a) $\Rightarrow$ (b) Assume there is a non-empty set S of submodules of M without maximal element. Take $M_0 \in S$. As M_0 is not maximal in S, there exists $M_1 \in S$ such that $M_0 < M_1$. Since S has no maximal element we can choose a submodule $M_2 \in S$ such that $M_0 < M_1 < M_2$. By recursion we obtain an infinite chain

$$M_0 < M_1 < \cdots < M_n < \cdots$$

which does not stablize. Therefore M is not Noetherian.

(a) $\Rightarrow$ (c) Assume that M has a submodule N that is not finitely generated. By recursion we define a sequence $x_1, \dots, x_n, \dots$ of elements in N has follows: first take $x_1 \in N$, secondly for each $n \geq 2$, choose $x_n \in N$ such that x_n does not belong to the submodule of M generated by $\{x_1, x_2, \dots, x_{n-1}\}$. Note that the choice of such x_n is possible since N is not finitely generated. Now for each $n \geq 1$, we define M_n to be generated by $\{x_1, \dots, x_n\}$. By construction we infer that the chain $M_1 < \cdots M_n < \cdots$ does not stabilize. Hence M is non-Noetherian.

(c) $\Rightarrow$ (a) Assume every submodule of M is finitely generated and let $M_0 \leq M_1 \leq \cdots M_n \leq \cdots$ be a chain of submodules. Define $H = \bigcup_{n \in \mathbb{N}} M_n$ and suppose H is generated by $\{x_1, x_2, \dots, x_r\}$. Now for each $i \in \{1, \dots, r\}$, let k_i be a positive integer such that $x_i \in M_{k_i}$. Take $k = max\{k_1, \dots, k_r\}$, so that $\{x_1, \dots, x_r\} \subseteq M_k$. Therefore for all $n \geq k$, we have $H \leq M_k \leq M_n \leq H$, so M is Noetherian. □

The proof of the following proposition is done by analogous arguments.

Proposition 1.2.2. [4, Proposition 10.10] *Let M be a module. The following statements are equivalent:*

(a) *M is Artinian.*

(b) *Every non-empty subset of $Sub(M)$ has a minimal element.*

(c) *Every factor module of M is finitely cogenerated.*

Proposition 1.2.3. *Let $M \in$ Mod-R. The following hold:*

1. If we have an R-homomorphism $f : M \to M'$ and M is Noetherian (resp. Artinian), then $f(M)$ is Noetherian (resp. Artinian);

2. [25, Result 1.20] Let $N \leq M$. Then M is Noetherian (resp. Artinian) if and only if N and M/N are Noetherian (resp. Artinian);

3. [25, Result 1.20] If M_1, M_2 are Noetherian (resp. Artinian) submodules of M, then $M_1 \oplus M_2$ is Noetherian (resp. Artinian).

Proof. We will only prove the Noetherian case.

1. Assume M is Noetherian, let $f \in Hom_R(M, M')$ and take $M'_1 \leq \cdots \leq M'_n \leq \cdots$ a chain of submodules of $f(M)$. Then
$$f^{-1}(M'_1) \leq \cdots \leq f^{-1}(M'_n) \leq \cdots$$
is a chain of submodules in M. Since M is Noetherian there exists a $p \in \mathbb{N}$ such that for all $n \geq p$, we have $f^{-1}(M'_p) = f^{-1}(M'_n)$. Now for each $n \in \mathbb{N}$ we have $M'_n \leq f(M)$, so we infer that $ff^{-1}(M'_n) = M'_n$. We have just shown that for each $n \geq p$, $M'_p = M'_n$, therefore $f(M)$ is Noetherian.

2. ($\Rightarrow$) Any chain in N is in particular a chain of M. Since M is Noetherian any ascending chain in N also stabilizes. For the second part, define the canonical epimorphism $\pi : M \to M/N$, so $\pi(M) = M/N$ is Noetherian by 1.

($\Leftarrow$) Conversely, let $M_1 \leq \cdots \leq M_n \leq \cdots$ be a chain of submodules of M, then

$$M_1 \cap N \leq \cdots \leq M_n \cap N \leq \cdots \text{ and } (M_1 + N)/N \leq \cdots \leq (M_n + N)/N \leq \cdots$$

are chains of N and M/N respectively. By hypothesis these chains stabilize for some $p \in \mathbb{N}$ so for every $n \geq p$, it follows that

$$M_p \cap N = M_n \cap N \text{ and } (M_p + N)/N = (M_n + N)/N.$$

Take $x \in M_n$, then $x + N \in (M_n + N)/N = (M_p + N)/N$, so there exist $y \in M_p$ and $n \in N$ such that $x + N = (y + n) + N = y + (n + N) = y + N$, so $x - y \in M_n \cap N = M_p \cap N$. Now $x \in M_p$, therefore $M_n = M_p$. Hence M is Noetherian.

3. Assume M_1 and M_2 are Noetherian. We have the homomorphism $q : M_1 \to (M_1 \oplus M_2)/M_2$, $x \mapsto x+M_2$, so by 1. $q(M_1) = (M_1 \oplus M_2)/M_2$ is Noetherian and by 2. $M_1 \oplus M_2$ is Noetherian. □

Proposition 1.2.4. [30, Theorem 1.7] *Given a module M and a family of modules $\{N_i : i \in I\}$ the following statements are equivalent:*

(a) $\oplus_{i \in I} N_i$ *is M-injective.*

(b) $\oplus_{i \in J} N_i$ *is M-injective for every countable subset $J \subseteq I$.*

(c) *N_i is M-injective for every $i \in I$, and for every choice of $x_n \in N_{i_n} (n \in \mathbb{N})$ for distinct $i_n \in I$ such that $ann(a) \subseteq \cap_{n=1}^{\infty} ann(x_n)$, for some $a \in M$, the ascending chain*

$$\bigcap_{n=1}^{\infty} ann(x_n) \subseteq \bigcap_{n=2}^{\infty} ann(x_n) \subseteq \cdots \subseteq \bigcap_{n=k}^{\infty} ann(x_n) \subseteq \cdots$$

becomes stationary.

Proof. (a) ⇒ (b) Immediate from Proposition 1.1.18.

(b) ⇒ (c) Again Proposition 1.1.18 implies that N_i is M-injective for every $i \in I$. Consider $x = (x_n) \in \prod_{n=1}^{\infty} N_{i_n}$ and $\phi : aR \to \prod_{n=1}^{\infty} N_{i_n}$, $ar \mapsto xr$. Take $J = \cup_{n=1}^{\infty}(\cap_{j \geq n} ann(x_j))$ and let $\bar{\phi} = \phi_{|aJ}$. Therefore $\bar{\phi}$ is a homomorphism from aJ into $\bigoplus_{j=1}^{\infty} N_{i_j}$. Since $\bigoplus_{j=1}^{\infty} N_{i_j}$ is M-injective, then it is also aR-injective, meaning $\bar{\phi}$ extends to some $\psi : aR \to \bigoplus_{j=1}^{\infty} N_{i_j}$. Thus $xJ = \bar{\phi}(aJ) = \psi(aJ) \leq \psi(aR) = \psi(a)R \leq \bigoplus_{f \in F} N_{i_f}$, where F is a finite subset of $\mathbb{N}$. Let $F = \{1, 2, \ldots, k-1\}$, so that $x_j J = 0$, for $j \geq k$ and hence $J = \cap_{j \geq k} ann(x_j)$, meaning the sequence $\cap_{j \geq n} ann(x_j) (n \in \mathbb{N})$ becomes stationary.

(c) ⇒ (a) By contradiction, assume that $\bigoplus_{i \in I} N_i$ is not M-injective. Then by Proposition 1.1.16 we infer that $\bigoplus_{i \in I} N_i$ is not aR-injective, for some $a \in M$. Therefore there exists a right ideal K of R and a homomorphism $f : aK \to \bigoplus_{i \in I} N_i$ that cannot be extended to aR. Since $\bigoplus_{i \in F} N_i$ is M-injective, for all finite subsets $F \subseteq I$, by Proposition 1.1.18, then $f(aK) \nleq \bigoplus_{i \in F} N_i$, for any finite subset $F \subseteq I$. However f can be extended to $g : aR \to \prod_{i \in I} N_i$, because $\prod_{i \in I} N_i$ is M-injective. Now let $x = g(a)$, then $ann(a) \leq ann(x) = \cap_{i \in I} ann(x_i)$, where x_i is the i-component of $x \in \prod_{i \in I} N_i$. Define $S_k = \{i \in I : x_i k \neq 0\}$, $k \in K$. Then for every $k \in K$, S_k is a finite subset of I. However $J = \cup_{k \in K} S_k$ is not finite, since $xK = f(aK) \nleq \bigoplus_{i \in F} N_i$, for any finite subset $F \subseteq I$. Now by induction take $k_n \in K\,(n \in \mathbb{N})$ and indices $i_l \in I$ such that $i_l \in S_{k_l}$ and $i_l \notin \cup_{n=1}^{l-1} S_{k_n}$. We denote the i_n-component of x by x_n. Therefore $ann(m) \subseteq \cap_{n=1}^{\infty} ann(x_n)$ and the sequence $\cap_{i \geq n} ann(x_n)$ is strictly increasing, which contradicts our assumption. So we conclude $\bigoplus_{i \in I} N_i$ is M-injective. □

The following theorem tells us that any direct sum of injective R-modules is injective if and only if R is Noetherian.

Proposition 1.2.5. [30, Theorem 1.11] *The direct sum of any family of N-injective right R-modules is N-injective if and only if every cyclic (or finitely generated) submodule of N is right Noetherian. In particular, the direct sum of every family of injective right R-modules is injective if and only if R is right Noetherian.*

Proof. ($\Leftarrow$) Assume that nR is Noetherian for every $n \in N$, and consider a direct sum $\oplus_{\alpha \in A} M_\alpha$ of N-injective modules M_α. Let $N_1 \leq nR$ and $\phi : N_1 \to \oplus_{\alpha \in A} M_\alpha$ be a homomorphism. Since N_1 is finitely generated, then $\phi(N_1) \leq \oplus_{\alpha \in F} M_\alpha$ for $F \subseteq A$ finite. By Proposition 1.1.18, $\oplus_{\alpha \in F} M_\alpha$ is N-injective. Therefore ϕ can be extended to $\psi : nR \to \oplus_{\alpha \in F} M_\alpha$. Hence $\oplus_{\alpha \in A} M_\alpha$ is N-injective, by Proposition 1.1.16.

($\Rightarrow$) Assume that the direct sum of any family of N-injective modules is N-injective. Take an arbitrary $n \in N$. We will prove that nR is right Noetherian, this means we will prove that an ascending sequence $ann(n) = N_0 \leq N_1 \leq N_2 \leq \cdots$ of right ideals of R is stationary. Let $M_i = E(R/N_i), i \in \mathbb{N}$. Since each M_i is N-injective, then by hypothesis $\oplus_{i=1}^{\infty} M_i$ is N-injective. Now consider the set $\{m_i = 1 + N_i : i \in \mathbb{N}\}$. By Proposition 1.2.4, we infer that the ascending sequence $\cap_{i \geq k} ann(m_i)(k \in \mathbb{N})$ becomes stationary. As $ann(m_i) = N_i$ for every $i \in \mathbb{N}$, $N_k = ann(m_k) = \cap_{i \geq k} ann(m_i)$. Therefore the sequence $N_1 \leq N_2 \leq \cdots$ becomes stationary. Hence nR is Noetherian. $\square$

Definition. Let M be a non-zero module. A finite chain of $n+1$ submodules of M

$$\{0\} = M_0 \lneq M_1 \lneq \cdots \lneq M_n = M$$

is called a *composition series of length* n for M, if each quotient M_i/M_{i-1} is simple, this is to say each M_{i-1} is maximal in M_i.

Definition. If a module M admits two composition series of the form $0 = M_0 \lneq M_1 \lneq \cdots M_n = M$ and $0 = N_0 \lneq N_1 \lneq \cdots \lneq N_p = M$, then these composition series are said to be *equivalent* if $n = p$ and there is a permutation σ of $\{1, 2, \ldots, n\}$ such that

$$M_i/M_{i+1} \cong N_{\sigma(i)}/N_{\sigma(i)+1}, \text{ for } i = 1, 2, \ldots, n.$$

Theorem 1.2.6. (Jordan-Hölder) [4, Theorem 11.3] Given a ring R, if an R-module M has a composition series, then every pair of composition series for M are equivalent.

Remark. By the Jordan-Hölder Theorem if a module has a composition series, then all composition series are equivalent and, in particular have the same length. Therefore we can define the composition length of a module as the length of its composition series (when they exist).

We denote the composition length of M by $cl(M)$. If $M = 0$, then $cl(M) = 0$. If M has a composition series of length n, we write $cl(M) = n$. If M does not have a composition series, then $cl(M) = \infty$.

The following proposition tells us that the composition series condition is equivalent to the descending and ascending chain conditions combined.

Proposition 1.2.7. [4, Proposition 11.1] *A non-zero module M has a composition series if and only if M is Noetherian and Artinian.*

Proof. ($\Leftarrow$) Suppose M is Noetherian and Artinian. Recursively choose an ascending chain $(M_n)_{n \in \mathbb{N}_0}$ of submodules of M as follows. Take $M_0 = 0$. If $n \geq 1$ and $M_{n-1} = M$, take $M_n = M$. Otherwise, if $M_{n-1} \neq M$, let M_n be a minimal element in the set of all submodules of M that contain M_{n-1} properly.

Note that such minimal M_n exists by Artinian assumption. Now by Noetherian hypothesis, there is $n \in \mathbb{N}$ such that $M_n = M$. Let k be the lowest index such that $M_k = M$. Finally by our construction of the submodules M_n we have, $0 = M_0 \lneq M_1 \lneq \cdots \lneq M_k = M$.

($\Rightarrow$) Now assume that M has a composition series. The proof is done by induction on the length of the composition series. If $n = 1$, then M is simple and the series is trivial. Take a composition series of length $n \geq 2$

$$0 = M_0 \lneq M_1 \lneq \cdots \lneq M_n = M,$$

hence M_{n-1} has a composition series of length $n-1$ and M/M_{n-1} is simple. Now by Theorem 1.2.3(2) we conclude the proof. $\square$

Corollary 1.2.8. *Any semisimple ring is right and left Noetherian and Artinian.*

Proof. Let R be a semisimple ring, then by Proposition 1.1.2 there exist right minimal ideals $\mathfrak{R}_1, \ldots, \mathfrak{R}_t$ of R such that

$$R = \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_t.$$

Note that the right minimal ideals of R are simple submodules of R_R. Therefore the following chain of submodules is a composition series of R_R

$$0 \lneq \mathfrak{R}_1 \lneq \mathfrak{R}_1 \oplus \mathfrak{R}_2 \lneq \cdots \lneq \mathfrak{R}_1 \oplus \cdots \oplus \mathfrak{R}_t = R_R$$

and by the previous proposition we get the desired result. $\square$

1.3 Free, Projective and Uniform Modules

The concept of free module is the one that most closely resembles the structure of vector spaces, that is not to say they coincide or that they behave in the same way, but they do admit a basis.

Definition. A module M with a subset $X = \{e_i : i \in I\}$ is called *free,* with X as its basis, if and only if the following "linear extension property" holds: for any family of elements $\{n_i : i \in I\}$ in a module N there exists a unique R-homomorphism $f : M \to N$ such that $f(e_i) = n_i$, for all $i \in I$.

An equivalent way to define a free R-module is to say that M is isomorphic to a direct sum of copies of R_R i.e., $M \cong \bigoplus_{i \in I} R_R$.

Proposition 1.3.1. [4, Theorem 8.1] *If a right R-module M has a spanning set $X \subseteq M$, then there is an epimorphism $R^{(X)} \to M$. Moreover, R finitely generates M if and only if M has a finite spanning set.*

Proof. Let X be the spanning set for M. For each $x \in M$, the left multiplication $\rho_x : r \mapsto xr$ is a right R-homomorphism $R_R \to M$. Let $\rho = \oplus_X \rho_x$ be the direct sum of these homomorphisms. Then $\rho : R^{(X)} \to M$ and im $\rho = \sum_X$ im $\rho_x = \sum_X xR = M$. Thus ρ is an epimorphism. The last statement is clear. $\square$

Now we introduce the dual concept to the injective module.

Definition. Given right R-modules M and N, we say that M is N-*projective,* if for each epimorphism $\pi : N \to K$ and each homomorphism $h : M \to K$ there exists a homomorphism $h' : M \to N$ such that

$h = \pi h'$ i.e., the follow diagram commutes

$$\begin{tikzcd} & M \arrow[dl, dotted, "h'"'] \arrow[d, "h"] & \\ N \arrow[r, "\pi"] & K \arrow[r] & 0 \end{tikzcd}.$$

Definition. We define the *projectivity domain* of M as follows

$$\mathcal{P}r^{-1}(M) = \{N \in \text{Mod-R} : M \text{ is N-projective}\}.$$

We say M is a *projective module* (over R) if its projectivity domain is the class of all right R-modules. The ring R is *self-projective on the right (resp. left)* if R_R (resp. ${}_RR$) is projective.

Proposition 1.3.2. *Every free module is projective.*

Proof. Let F be a free module with basis B. Take $f : F \to N$ and $\pi : M \to N$ to be a homomorphism and an epimorphism respectively. Now for each $b \in B$ we choose $m_b \in M$ such that $\pi(m_b) = f(b)$. By linear extension, there exists a uniquely determined homomorphism $g : F \to M$, such that $\forall b \in B$, $g(b) = m_b$. Thus $\pi g(b) = \pi(m_b) = f(b)$ and so by linear extension $\pi g = f$. Therefore F is projective. $\square$

Proposition 1.3.3. [4, Proposition 17.2] *Let R be a ring and P a module. Then the following are equivalent:*

(a) *P is projective.*

(b) *Every R-epimorphism $M \to P \to 0$ splits.*

(c) *P is isomorphic to a direct summand of a free right R-module.*

Proof. (a) $\Rightarrow$ (b) Take an epimorphism $f : M \to P$. Since P is projective, there exists a homomorphism $g : P \to M$ such that $fg = id_p$. Hence the epimorphism splits.

(b) $\Rightarrow$ (c) Let X be a spanning set for P. By Proposition 1.3.1, we have an epimorphism f from the free module $F = \oplus_X R_R$ onto P. Then by hypothesis $F \xrightarrow{f} P \to 0$ is a splitting epimorphism and $\ker f$ is a direct summand of F. Therefore, P is isomorphic to a direct summand of F.

(c) $\Rightarrow$ (a) By Proposition 1.3.2 and the fact that a direct summand of a projective module is also projective. $\square$

Now we introduce a notion of dimension for modules, but of course this is not as straightforward as it is for vector spaces, where we characterize dimension in function of the cardinality of a basis. With modules, we often lack a basis to begin with, so we will define uniform dimension.

Definition. Let M be a module. We define M to be an *uniform module*, if for any two non-zero submodules $N, N' \leq M$ we have $N \cap N' \neq 0$. Equivalently, M is uniform if every non-zero submodule of M is an essential submodule of M. A ring R is a *right (resp. left) uniform ring* if R_R (resp. ${}_RR$) is uniform.

Lemma 1.3.4. [24, Theorem 6.1] *Let $U = U_1 \oplus \cdots \oplus U_m$ and $V = V_1 \oplus \cdots \oplus V_n$ be essential submodules of a right R-module M, where the $U_i's$ and $V_j's$ are uniform modules. Then $m = n$.*

Proof. Assume that $n \geq m$. We claim that $\bar{U} = U_2 \oplus \cdots \oplus U_m$ intersects some V_j trivially. Otherwise, $\bar{U} \cap V_j \leq_e V_j \, (1 \leq j \leq n)$, because V_j is uniform. Furthermore, by Proposition 1.1.10(4) it would follow that

$$(\bar{U} \cap V_1) \oplus \cdots \oplus (\bar{U} \cap V_n) \leq_e V_1 \oplus \cdots \oplus V_n = V,$$

and also $\bar{U} \cap V \leq_e V \leq_e M$. Again, by Proposition 1.1.10(1) $\bar{U} \leq_e M$, a contradiction. Therefore without loss of generality we can assume that $\bar{U} \cap V_1 = 0$. Let $U' = \bar{U} \oplus V_1$. We have $U' \cap U_1 \neq 0$, otherwise $U_1 + U_2 + \cdots + U_m + V_1$ would be a direct sum, but this contradicts $U \leq_e M$. Now we have

$$(U' \cap U_1) \oplus U_2 \oplus \cdots \oplus U_m \leq_e U_1 \oplus \cdots \oplus U_m \leq_e M.$$

Since $(U' \cap U_1) \oplus U_2 \oplus \cdots \oplus U_m \leq U'$, then $U' \leq_e M$. We have "replaced" the summand U_1 by V_1. Repeating this process for V_2 we can go from U' to $U'' = V_1 \oplus V_2 \oplus U_3 \oplus \cdots \oplus U_m \leq_e M$. After m steps, $U^{(m)} = V_1 \oplus \cdots \oplus V_m \leq_e M$. However we also have $V = V_1 \oplus \cdots \oplus V_n \leq_e M$, so $m = n$. □

Definition. We say that a module has *uniform dimension n* or *Goldie dimension n*, denoted by $u.dim(M) = n$, if there is an essential submodule $V \leq_e M$ that is the direct sum of n uniform submodules, i.e., if for a set of uniform submodules $\{U_i\}_{i=1}^n$ we have that $V = \oplus_{i=1}^n U_i$ is an essential submodule of M. If no such positive integer n exists, we write $u.dim(M) = \infty$. A ring has right uniform dimension n, if $u.dim(R_R) = n$.

The lemma above justifies that the uniform dimension is well-defined.

Lemma 1.3.5. [24, Proposition 6.3] *Given a right R-mdoule M, suppose $u.dim(M) = n < \infty$. Then any direct sum of non-zero submodules $N = N_1 \oplus \cdots \oplus N_k \leq M$ has $k \leq n$ summands.*

Proof. If $n = 1$ the result is clear. Assume $n \geq 2$ and let $V \leq_e M$ be such that it is the direct sum of n uniform submodules. Now $N_i' = N_i \cap V \neq 0$ and $N_i' \oplus \cdots \oplus N_k' \leq V$. We can assume $M = V$, say $M = U_1 \oplus \cdots \oplus U_n$, where the $U_i's$ are uniform. Let $\bar{N} = N_2 \oplus \cdots \oplus N_k$. Analogous to the proof of Lemma 1.3.4, we may assume $\bar{N} \cap V_1 = 0$. Now project M onto $V_2 \oplus \cdots \oplus V_n$. Therefore we have an embedding of $\bar{N}$ into $V_2 \oplus \cdots \oplus V_n$. Now by induction in n, we have $k - 1 \leq n - 1$, hence $k \leq n$. □

Proposition 1.3.6. [24, Proposition 6.4] *We have $u.dim(M) = \infty$ if and only if M contains an infinite direct sum of non-zero submodules.*

Proof. The "if" part follows from Lemma 1.3.5.

For the "only if" part assume that M does not contain an infinite direct sum of non-zero submodules. We claim that any non–zero submodule $N \leq M$ contains a uniform submodule. If this is not true, then N cannot be uniform, meaning it contains some $A_1 \oplus B_1$ with $A_1 \neq 0 \neq B_1$. Then B_1 is not uniform either. Hence it contains some $A_2 \oplus B_2$, with $A_2 \neq 0 \neq B_2$. Repeating this process, we arrive at an infinite direct sum $A_1 \oplus A_2 \oplus \cdots \leq M$, a contradiction. Now take $U_1 \leq M$ uniform. If U_1 is not essential in M, then $U_1 \oplus U_2 \leq M$, for some $U_2 \neq 0$ that we can assume to be uniform. If $U_1 \oplus U_2$ is not essential in M, again we can take a non-zero uniform submodule U_3 such that $U_1 \oplus U_2 \oplus U_3 \leq M$. However, by assumption, we can only repeat this process a finite number of times. Eventually we have $U_1 \oplus \cdots \oplus U_n \leq_e M$, with all $U_i's$ uniform submodules of M. Therefore, by definition, $u.dim(M) = n$. □

Corollary 1.3.7. [24, Corollary 6.6] *For a right R-module M, the uniform dimension of M is the supremum of the set:*

$$\{k : M \textit{ contains a direct sum of } k \textit{ non-zero submodules}\}.$$

Proof. Let $k \leq \infty$ be such supremum. If $k = \infty$, then by Lemma 1.3.5, $u.dim(M) = \infty$. If $k < \infty$, then by the previous proposition, $u.dim(M)$ must be finite, and again by Lemma 1.3.5 we conclude that $u.dim(M) = k$. □

Let us relate the notion of uniform dimension with Noetherian and Artinian modules.

Proposition 1.3.8. [24, Corollary 6.7] *Let M be a right R-module.*

(a) *If M is a Noetherian or an Artinian module, then $u.dim(M) < \infty$.*

(b) *If M has a composition series of length n, then $u.dim(M) \leq n$. This is an equality if and only if M is semisimple.*

Proof. (a) If the module is Noetherian or Artinian, then either of the chain conditions rules out the existence of an infinite direct sum of non-zero submodules in M.

(b) Assume we have a composition series of length n. If M contains $N_1 \oplus ... \oplus N_k$, where every $N_i \neq 0$, then

$$k \leq \sum_{i=1}^{k} length(N_i) \leq length(M) = n.$$

This implies that $u.dim(M) \leq n$. If M is semisimple then $u.dim(M) = n$.

Conversely, if $u.dim(M) = n$, then $N_1 \oplus ... \oplus N_n \subseteq M$, for some $N_i \neq 0$, and by the inequality above for $k = n$, it follows that $M = N_1 \oplus ... \oplus N_n$, with $length(N_i) = 1$, for every i. Thus M is semisimple. □

The following lemma tells us that the uniform dimension of M is equal to the uniform dimension of its injective hull. This fact holds for both the finite and infinite cases.

Lemma 1.3.9. [24, Corollary 6.10(2)] *Let $N \leq M$. Then $u.dim(N) \leq u.dim(M)$, and they coincide when $N \leq_e M$.*

Proof. Immediate from Corollary 1.3.7. The equality follows from the definition of uniform dimension. □

Proposition 1.3.10. [24, Proposition 6.12] *The following statements are equivalent:*

(a) *A module M has finite uniform dimension n.*

(b) *The injective hull of M, $E(M)$ is a direct sum of n many indecomposable injective modules E_i.*

Proof. (a) ⇒ (b) Assume that $u.dim(M) = n$. Then M contains an essential submodule $V = U_1 \oplus \cdots \oplus U_n$, where each U_i is uniform. By Proposition 1.1.21 we have $E(M) = E(V) = E(U_1 \oplus \cdots \oplus U_n) = E(U_1) \oplus \cdots \oplus E(U_n)$. So we are done if we manage to prove that $E(U_i)$ is indecomposable, for each U_i. Assume $E(U_i) = K \oplus N$, with $K, N \neq 0$. Then, as $U_i \leq_e E(U_i)$, we have $U_i \cap K, U_i \cap N \neq 0$. But U_i is uniform, so $U_1 \cap K, U_1 \cap N \leq_e U_i$, contradicting $(U_1 \cap K) \cap (U_i \cap N) = 0$.

(b) ⇒ (a) By the previous lemma $u.dim(M) = u.dim(E(M))$, so without loss of generality we may assume M is injective such that $M = M_1 \oplus \cdots \oplus M_n$, where each M_i is indecomposable and injective. Now by the definition of uniform module, each M_i is uniform. Therefore $u.dim(M) = n$. □

1.4 Torsion Subgroups and Singular Submodules

In this section we introduce and develop the notions of torsion subgroup and singular submodule.

Definition. Let G be an (additive) abelian group. The *order* of an element g in G is the smallest $m \in \mathbb{N}$ such that $mg = 0$. In this case we say that g has *finite order*, and represent this by $o(g) = m$. Otherwise, $o(g) = \infty$, and we say g has *infinite order*.

Definition. The *(maximal) torsion subgroup,* $T(G)$ of G, consists of all elements of finite order. If $G = T(G)$ we say G is a *torsion group*.

Definition. A subgroup of G is said to be *torsion-free* if all it elements, with the exception of zero (order 1), have infinite order.

Definition. If every non-zero element of a group G has order p^n for some $n \in \mathbb{N}$, we say that G is a *p-group* or a *p-primary group*.

Definition. Given a group G, for each prime p we define the *p-primary component of G* by

$$T_p(G) = \{a \in G : p^k a = 0, \text{ for some } k \in \mathbb{N}\}.$$

Lemma 1.4.1. [20, Chapter 2, Lemma 1.1] *Let $G = \langle g \rangle$ be a finite cyclic group where $o(g) = m = p_1^{r_1} \dots p_k^{r_k}$ with different primes p_i. Then G has a decomposition into a direct sum $G = \langle g_1 \rangle \oplus \dots \oplus \langle g_k \rangle$, where $o(g_i) = p_i^{r_i}$, with uniquely determined summands.*

Proof. For $i = 1, \dots, k$, define $m_i = mp_i^{-r_i}$ and $g_i = m_i g$. Then all $m_i's$ are relatively prime, so there exist $s_i \in \mathbb{Z}$ such that $s_1 m_1 + \dots + s_k m_k = 1$. Hence $g = s_1 m_1 g + \dots + s_k m_k g = s_1 g_1 + \dots + s_k g_k$, so the $g_i's$ generate G. Furthermore, $\langle g_i \rangle$ has order $p_i^{r_i}$ and is disjoint from $\langle g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_k \rangle$ which has order m_i. Thus $G = \langle g_1 \rangle \oplus \dots \oplus \langle g_k \rangle$. The uniqueness of the summands $\langle g_i \rangle$ follows from the fact that $\langle g_i \rangle$ is the only subgroup of G containing all the elements whose orders are powers of p_i. □

The following proposition has a fundamental role in the study of abelian groups. It gives us a characterization of a torsion group in function of the direct sum of its p-primary components.

Proposition 1.4.2. [20, Chapter 2, Theorem 1.2] *A torsion group G is the direct sum of its p-groups $T_p(G)$ with different primes p, i.e., $T(G) = \bigoplus_{p \in \mathbb{P}} T_p(G)$.*

Proof. Let G be a torsion group. Clearly for each p prime, $T_p(G)$ is non-empty, because $0 \in T_p(G)$. Now take $a, b \in T(G)$, so $p^n a = 0 = p^m b$, for some $m, n \in \mathbb{N}$. Then $p^{n+m}(a - b) = 0$, so $a - b \in T_p(G)$. This implies that $T_p(G)$ is a subgroup of G. Now take $p_1, \dots, p_j$ distinct primes different from p. By definition, every element of $T_{p_1}(G) + \dots + T_{p_j}(G)$ is annihilated by a product of powers of $p_1, \dots, p_j$. Therefore $T_p(G) \cap (T_{p_1}(G) + \dots + T_{p_j}(G)) = 0$. Thus the $T_p(G)'s$ generate their direct sum in G. By the previous lemma, this must be the whole G.

If G has a different decomposition, into p-groups (for different primes p), say $\bigoplus_{p \in \mathbb{P}} S_p(G)$, then by the maximality of T_p, we have that $S_p(G) \leq T_p(G)$, for each $p \in \mathbb{P}$. If we had $S_p(G)$ strictly contained in $T_p(G)$, then $\bigoplus_{p \in \mathbb{P}} S_p(G)$ would not be equal to G. Therefore this decomposition is uniquely determined. □

Definition. Given a right R-module M, the following subset of M is a submodule of M called the *singular submodule* of M :

$$Z(M) = \{m \in M : ann(m) \leq_e R_R\} = \{m \in M : mI = 0, \text{ for some } I \leq_e R_R\}$$

A module M is said to be a *singular module* if $Z(M) = M$. If $Z(M) = 0$, M is said to be *nonsingular*. A ring R *is right nonsingular*, if R_R *is a* nonsingular module.

The following result is clear from the definition.

Lemma 1.4.3. *Let N be a submodule of M. Then $Z(N) = N \cap Z(M)$.*

Remark. By this lemma it follows that if M is nonsingular (resp. singular), then all its submodules are nonsingular (resp. singular).

Definition. We define the *second singular submodule* or *Goldie torsion submodule of* M, $Z_2(M)$, by the equality $Z_2(M)/Z(M) = Z(M/Z(M))$.

Lemma 1.4.4. *Let M be a module over a ring R. Then $Z(M) \leq_e Z_2(M)$. Furthermore, $Z_2(M)$ is a closed submodule of M.*

Proof. Let $H \leq Z_2(M)$ such that $H \cap Z(M) = 0$. Then H is nonsingular. On the other hand, $H \cong (H \oplus Z(M))/Z(M) \leq Z_2(M)/Z(M) = Z(M/Z(M))$. Then H is both singular and nonsingular, so $H = 0$. ☐

Proposition 1.4.5. [22, Proposition 1.5] *Let M be an R-module.*

(a) *M is singular if and only if $M \cong N/K$, where $K \leq_e N$.*

(b) *If $K \leq N$ and $Z(N) = 0$, then N/K is singular if and only if $K \leq_e N$.*

Proof. (a) ($\Rightarrow$) Assume M is a singular module, and take an arbitrary $x \in M$. As M is singular, $x \in Z(M)$, and there exists $I_x \leq_e R_R$ such that $xI_x = 0$. Let $N = \oplus_{x\in M} R$ and let $f : N \to M$ be the epimorphism defined in Proposition 1.3.1. Take $K = \ker f$. Clearly $\bigoplus_{x\in M} I_x$ is a submodule of K, and so by Proposition 1.1.10(4) it follows that $\oplus I_x \leq_e N$. Therefore $K \leq_e N$. Now by the First Isomorphism Theorem we conclude that $M \cong N/K$.

($\Leftarrow$) Suppose $M \cong N/K$ with $K \leq_e N$. We take $x \in N$ and define the homomorphism, $f : R \to N$, $f(r) = xr$ and by Proposition 1.1.10(3) we infer that $f^{-1}(K) \leq_e R$. Now we have $x(f^{-1}(K)) \leq K$, and so N/K is singular, meaning M is singular.

(b) Assume N/K is singular and take $x \in N\backslash 0$, As N/K is singular, $xI \leq K$, for some $I \leq_e R_R$. But since N is nonsingular it follows that $xI \neq 0$, and by Lemma 1.1.9 $K \leq_e N$. The converse is clear by (a). ☐

Lemma 1.4.6. [22, Proposition 1.6] *For any module M over a nonsingular ring R, $Z(M/Z(M)) = 0$.*

Proof. By Lemma 1.4.4, $Z(M/Z(M)) = Z_2(M)/Z(M)$ with $Z(M) \leq_e Z_2(M)$ Now assume there exists $x \in Z_2(M)\backslash Z(M)$. This would mean that $ann(x)$ is not essential in R, meaning there exists a non-zero right ideal I of R_R such that $ann(x) \cap I = 0$. Now $xI \cong I$. Then $xI \neq 0$ must be nonsingular as well, so $xI \cap Z(M) = 0$. However this contradicts the fact that $Z(M)$ is an essential submodule of $Z_2(M)$. We have shown that $Z_2(M) = Z(M)$, and I is nonsingular, because R is nonsingular, therefore $Z(M/Z(M)) = 0$. ☐

Proposition 1.4.7. [22, Proposition 1.7] Let R be a nonsingular ring.

(a) *A module M is singular if and only if $Hom_R(M, B) = 0$, for every nonsingular module B.*

(b) *A module B is nonsingular if and only if $Hom_R(M, B) = 0$, for every singular module M.*

Proof. (a) By the previous lemma, $M/Z(M) = 0$, and $Hom_R(A, A/Z(A)) = 0$, only if A is singular.

(b) If $Hom_R(M, B) = 0$, then B must be nonsingular. ☐

1.5 The Socle

Definition. The *socle* of an R-module M is the sum of all simple submodules of M,and is denoted by $Soc(M)$.

Remark. Note that M is semisimple if and only if $M = Soc(M)$. Furthermore, the socle of a module is semisimple, by Proposition 1.1.1.

We will use the following proposition as an analogous to the definition.

Proposition 1.5.1. [4, Proposition 9.7] Let M be a module over R. Then

$$Soc(M) = \bigcap\{L \leq M : L \textit{ is essential in } M\}.$$

Proof. Let T be a simple submodule of M. If we take another arbitrary submodule L such that $L \leq_e M$, then by definition of essential submodule it follows that $T \cap L \neq 0$ and so $T \leq L$. Therefore $Soc(M)$ is contained in every essential submodule of M.

For the other inclusion, let H be the intersection of all essential submodules of M. Take $N \leq H$ and N' a complement submodule of N in M (a complement submodule of N is a maximal submodule of M such that $N \cap N' = 0$). Now let us prove that $N \oplus N'$ is an essential submodule of M. Take $0 \neq S \leq M$ such that $(N \oplus N') \cap S = 0$. It follows that $N \cap (N' + S) = 0$ and this contradicts the maximality of N', hence $N \oplus N' \leq_e M$. Note that $N \leq H \leq N \oplus N'$ and by Modular Law we have $H = H \cap (N \oplus N') = N \oplus (H \cap N')$. Therefore N is a direct summand of H, meaning H is semisimple and we conclude that $H \leq Soc(M)$. □

Proposition 1.5.2. [4, Corollary 9.9] *Let K be a submodule of M. Then $Soc(K) = K \cap Soc(M)$. In particular, $Soc(Soc(M)) = Soc(M)$.*

Proof. The simple submodules of K are exactly the simple submodules of M that lie in K, so $Soc(K) \leq Soc(M)$. Since $Soc(M)$ is semisimple, then by Proposition 1.1.8, $K \cap Soc(M)$ is semisimple, and therefore contained in $Soc(K)$. □

We now present some relevant results relating the Noetherian and Artinian conditions to the socle.

Proposition 1.5.3. [4, Corollary 10.11(1)] *Let M be an non-zero module. If M is Artinian, then M has a simple submodule. In fact, $Soc(M)$ is an essential submodule of M.*

Proof. By Proposition 1.2.2 we are guaranteed a minimal submodule. Now take N a non-zero submodule of M such that $N \cap Soc(M) = 0$. Again by the Artinian hypothesis, if N is not itself minimal then it admits a minimal submodule H. However since H is minimal, it is contained in $Soc(M)$. Hence $H \cap Soc(M) \neq 0$. Therefore $N = 0$. Thus $Soc(M)$ is an essential submodule. □

Lemma 1.5.4. [22, Proposition 1.2] *Let M be a module. Then M is semisimple if and only if M has no proper essential submodules.*

Proof. ($\Rightarrow$) Since M is semisimple, then $M = Soc(M)$. Hence M does not admit a proper essential submodule.

($\Leftarrow$) First we prove that given a submodule N of M and K a maximal submodule of M such that $N \cap K = 0$, then $N \oplus K$ is essential in M. Take $L \leq M$ such that $(N \oplus K) \cap L = 0$, so $N \cap (K \oplus L) = 0$. Now by the maximality of K we know that $L \leq K$, hence $L = (N \oplus K) \cap L = 0$. By the definition of essential submodule we infer that $N \oplus K \leq_e M$ and by hypothesis $N \oplus K = M$. Thus M is semisimple. □

The concept of semiartinian will prove itself important in the latter chapters.

Definition. A module M is said to be *semiartinian,* if each of its non-zero factor modules has a simple module. A ring R is said to be right semiartinian if R_R is semiartinian.

Proposition 1.5.5. *Let M be a module. If M is Noetherian and semiartinian, then M is Artinian.*

Proof. Assume $M \neq 0$. By hypothesis, $Soc(M) \neq 0$, meaning there is a simple non-zero submodule S_1 of M. If $S_1 = M$ we are done. Otherwise, M/S_1 is non-zero, so there exists a non-zero submodule S_2 of M such that S_2/S_1 is simple. Now we may assume there is a non-zero submodule S_n such that S_n/S_{n-1} is simple. If $S_n \neq M$, then S_{n+1}/S_n is simple, for some non-zero submodule S_{n+1} such that S_{n+1}/S_n is simple. Now we have a proper ascending chain $S_0 < S_1 < \cdots < S_n < \cdots$, where each S_i is Artinian, by Proposition 1.2.3(2). By the Noetherian hypothesis, this chain must eventually stabilize, i.e. $M = S_N$, for some $N \in \mathbb{N}$. Therefore M is Artinian. $\square$

Definition. Given a commutative right R-module M. The set

$$X(M) = \{x \in M : \text{every prime ideal containing } ann(x) \text{ is maximal}\}$$

is the *maximal component* of M. If $M = X(M)$ we say that M has *maximal orders*.

The proof of the proposition is beyond the scope of this thesis.

Proposition 1.5.6. [29, Proposition 3] *Let R be a commutative Noetherian ring and M a module. Then the following are equivalent:*

(a) *M is Artinian*

(b) *M is a submodule of $E_1 \oplus \ldots \oplus E_n$, where $E_i = E(R/I_i)$ with I_i a maximal ideal of R.*

(c) *M has maximal orders and finitely generated socle.*

The following is a way of "horizontally slicing" a module using its socle.

Definition. We define the *socle series* of a module M as follows:

$$0 \leq Soc_1(M) \leq Soc_2(M) \leq \cdots$$

where Soc_n is defined recursively by

$$Soc_1(M) = Soc(M) \text{ and } Soc_{n+1}(M)/Soc_n(M) = Soc(M/Soc_n(M)), \text{ for } n \geq 1.$$

Remark. This series can be finite and does not need to be stable.

Proposition 1.5.7. [12, Lemma 4] *For any module M, any submodule $Soc_n(M)$ in the socle series of M is fully invariant.*

Proof. Suppose by contradiction that the result does not hold. Then there exists $n \geq 1$ such that $Soc_n(M)$ is not fully invariant. Let $\phi \in End(M)$, so $\phi(Soc_{n-1}(M)) \leq Soc_{n-1}(M)$ and we have the induced homomorphism $\phi' : M' \to M'$, with $M' = M/Soc_{n-1}(M)$. Now $\phi'(Soc_1(M')) \leq Soc_1(M')$. Therefore $\phi(Soc_n(M)) \leq Soc_n(M)$, meaning $Soc_n(M)$ is fully invariant, a contradiction. $\square$

1.6 The Radical and the Jacobson Radical

We start by introducing the dual definition of essential submodule.

Definition. Let R be a ring and N a submodule of M. The submodule N is said to be *superfluous* or *small in* M, if for every $L \leq M$ such that $N + L = M$, implies $M = L$. We denote this by $N \leq_s M$.

Dual to the socle we have the following notion.

Definition. Let M be an R-module. The *radical* of the module M is the intersection of all maximal submodules of M, denoted by $Rad(M)$.

The following is interchangeable with the definition.

Proposition 1.6.1. [4, Proposition 9.13] *For a module* M,

$$Rad(M) = \sum\{L \leq M \mid L \textit{ is superfluous in } M\}.$$

Proof. Let $L \leq_s M$ and take K a maximal submodule of M. If $L \not\leq K$, thus $M = K + N$. However by the superfluous condition we must have $M = K$,which means K is not maximal, a contradiction. So every superfluous submodule of M is contained in $Rad(M)$. For the other inclusion, take $x \in M$. If $K \leq M$ with $xR + K = M$. Therefore $K = M$, or there exists a maximal su bmodule N of M such that $K \leq N$ and $x \notin N$. If $x \in Rad(M)$. Then the maximal submodule N cannot exist in these conditions, thus $xR \leq_s M$. □

The following propositions is clear by definition, and it tells us that the radical of a module is fully invariant.

Proposition 1.6.2. [4, Proposition 9.14] *Let* M *and* N *be modules and* $f : M \to N$ *a homomorphism. Then*

$$f(Rad(M)) \leq Rad(N).$$

Proposition 1.6.3. [4, Proposition 9.15] *If* $f : M \to N$ *is an epimorphism and* $\ker f \leq Rad(M)$, *then* $Rad(N) = f(Rad(M))$. *In particular,* $Rad(M/Rad(M)) = 0$.

Proof. To prove the first part, by Proposition 1.6.2, it is enough to prove that $Rad(N)$ is a submodule of $f(Rad(M))$. We start by proving that for a maximal submodule K of M, we have $f^{-1}(f(K)) = K$. In general, $f^{-1}(f(K)) = K + \ker f$, and by hypothesis $\ker f \leq Rad(M) \leq K$. Therefore $K + \ker f = K$. Now we show $f(K)$ is a maximal submodule of N. Assume $f(K)$ is not proper, i.e., $f(K) = N$. Then $K = f^{-1}(f(K)) = f^{-1}(N) = M$, a contradiction. Now take L a submodule of N such that $f(K) \leq L$. Then $K \leq f^{-1}(L)$, and so $f^{-1}(L)$ is K or M. Hence $L = f(f^{-1}(L))$ is $f(K)$ or $f(M) = N$. Thus, $f(K)$ is a maximal submodule of N.

Take $y \in Rad(N)$. Since f is an epimorphism, there exists $y \in M$ such that $y = f(x)$. We claim that $x \in Rad(M)$. Let K be a maximal submodule of M. We have already proved that $f(K)$ is a maximal submodule of N, so that $y \in f(K)$. Therefore $x \in f^{-1}(f(K)) = K$. We conclude that $x \in Rad(M)$

For the second half, take the canonical epimorphism $f : M \to M/Rad(M)$, by the first half we know that $Rad(M/Rad(M)) = f(Rad(M))$. But $\ker f = Rad(M)$, hence $f(Rad(M)) = 0$. □

Proposition 1.6.4. [4, Proposition 9.18] *If every proper submodule of* M *is contained in a maximal submodule of* M, *then* $Rad(M)$ *is the unique largest superfluous submodule of* M.

Proof. Let L be a proper submodule of M and let K be a maximal submodule such that $L \leq K$. Then by Proposition 1.6.1, $L + Rad(M) \leq K \neq M$. □

The proof of the proposition is beyond the scope of this thesis.

Proposition 1.6.5. [17, Proposition 7.32 A] *The following statements are equivalent:*

(a) *Each simple right R-module is injective.*

(b) *Each right ideal is the intersection of maximal right ideals.*

(c) $Rad(M) = 0,$ *for all* $M \in$ *Mod-R.*

Definition. A ring R is a *right V-ring* if every simple right R-module is injective.

The Jacobson radical is the radical of R_R (or ${}_RR$).

Definition. We define the *Jacobson radical* of a ring R as follows

$$J(R) = \bigcap \{I_i : I_i \text{ is a maximal right ideal of } R\}.$$

If we want to define this in terms of the radical, it is simply, $Rad(R_R) = J(R)$.

Claim. To be precise we should define the left and right Jacobson radical. However the left and right Jacobson radicals coincide even in the non-commutative case. See [25, §4] for the development of this equivalence.

Next we obtain an equivalent characterization for the Jacobson radical in function of the annihilator. However we require the following lemma.

Lemma 1.6.6. [25, Lemma 4.1] *For $y \in R$, the following statements are equivalent:*

(a) $y \in J(R)$.

(b) $1 - yx$ *is right invertible, for any* $x \in R$.

(c) $My = 0,$ *for any simple R-module M.*

Proof. (a) ⇒ (b) Assume $y \in J(R)$. If $1-yx$ is not right invertible, for some $x \in R$, then $(1-yx)R \subsetneq R$ is contained in a maximal right ideal I of R. However $1 - yx \in I$ and $y \in I$, together imply that $1 \in I$. Hence $I = R$, which is a contradiction.

(b) ⇒ (c) Assume $my \neq 0$, for some $m \in M \backslash 0$, then $(my)R = M$. In particular, for some $x \in R$ we have $m = (my)x$, so $m(1 - yx) = 0$. But, by hypothesis, $(1 - yx)$ is right invertible, therefore $m = 0$, a contradiction.

(c) ⇒ (a) Let I be a right maximal ideal of R, then (R/I)-module is simple. Therefore, by hypothesis, $(R/I)y = 0$ thus $y \in I$. By definition $y \in J(R)$. □

If we combine the definition and (c) of the previous lemma, the Jacobson radical of a ring R becomes the intersection of the annihilators of M, where M is any simple R-module.

Corollary 1.6.7. [25, Corollary 4.2] *Let R be a ring and M a module. Then*

$$J(R) = \bigcap \{ann(M),\ M \text{ is simple}\}.$$

In particular, $J(R)$ is an ideal of R.

The proof of the following is clear.

Lemma 1.6.8. [25, Proposition 4.6] *Let R be a ring and $I \subseteq J(R)$ an ideal. Then $J(R/I) = J(R)/I$. In particular, $J(R/J(R)) = 0$.*

The following useful result combines the Artinian condition and the Jacobson radical.

Proposition 1.6.9. [25, Theorem 4.14] *A ring R is semisimple if and only if R is right Artinian and $J(R) = \{0\}$.*

Proof. ($\Rightarrow$) Let us assume R is semisimple. By Corollary 1.2.8 we know that R is Artinian, meaning we are left to prove that $J(R) = 0$. Since R is semisimple there exists a right ideal I of R such that $R = J(R) \oplus I$. If $J(R) \neq 0$, then $I \neq R$ so there exists a maximal right ideal I' of R such that $I \subseteq I'$. Now since $J(R) \subseteq I'$ it follows that $R \subseteq I'$, so we have arrived at a contradiction, therefore $J(R)$ must be trivial.

($\Leftarrow$) Now assume that R is right Artinian and that $J(R) = \{0\}$. Let I_1 be a minimal right ideal of R (note that this ideal exists since the ring is Artinian). Now, since $J(R) = \{0\} \subsetneq I_1$, then there exists a right maximal ideal I_1^* of R such that $I_1^* \cap I_1 = \{0\}$. Otherwise the minimality of I_1 implies that $I_1 = I_1 \cap I^* \subseteq I^*$, for all right maximal ideals $I^*$of R, and therefore $I_1 \subseteq J(R) = 0$. From the maximality of I_1^* it follows that $R = I_1 \oplus I_1^*$. If $I_1^* = \{0\}$ then $R = \mathfrak{R}_1$ and R is semisimple and we conclude the proof. If $I_1^* \neq \{0\}$ then again by the Artinian hypothesis there exists a right minimal ideal I_2 of R such that $I_2 \subseteq I_1^*$. Repeating the same argument there exists a right maximal ideal I_2^* of R such that $R = I_2 \oplus I_2^*$. Therefore

$$I_1^* = I_1^* \cap (I_2 \oplus I_2^*) = I_2 \oplus (I_1^* \cap I_2^*)$$

and

$$R = I_1 \oplus I_2 \oplus (I_1^* \cap I_2^*).$$

If $I_1^* \cap I_2^* = \{0\}$, then $R = I_1 \oplus I_2$ and R is semisimple, otherwise we keep repeating the process. After repeating this process, we obtain $I_1, I_2, ..., I_n$ minimal right ideals, and $I_1^*, I_2^*, ..., I_n^*$ maximal right ideals such that $I_m \subseteq I_{m-1}^*$ and $R = I_1 \oplus I_2 \oplus ..., \oplus I_m \oplus (I_1^* \cap I_2^* \cap ... \cap I_m^*)$, for every $m \leq n$. Furthermore we have the descending chain of ideals

$$I_1^* \supsetneq I_1^* \cap I_2^* \supsetneq \cdots \supsetneq I_1^* \cap \cdots \cap I_n^* \supseteq \cdots$$

which is stable, since R is Artinian. Therefore $R = I_1 \oplus I_2 \oplus \cdots \oplus I_n$, for some $n \in \mathbb{N}$. Hence by Proposition 1.1.2, we conclude R is semisimple. ☐

Definition. A ring R is said to be *semiprimary*, if $J(R)$ is nilpotent ($J(R)^n = 0$, for some $n \in \mathbb{N}$) and $R/J(R)$ is semisimple.

Semiprimary rings do not show anywhere else in this dissertation, so with the intent of not introducing more results we leave the following as a claim.

Claim. [25, Theorems 4.12 and 4.14] Any Artinian ring is semiprimary.

We are now ready to establish an important relation between Artinian and Noetherian rings.

Theorem 1.6.10. (Hopkins-Levitzki) [25, Theorem 4.15] Let R be a semiprimary ring. Then for any $M \in$ Mod-R the following statements are equivalent:

(a) *M is Noetherian.*

(b) *M is Artinian.*

(c) *M has a composition series.*

In particular, (i) R is right (resp. left) Artinian if and only if it is right (resp. left) Noetherian and semiprimary. (ii) Any finitely generated right module over a right Artinian ring has a composition series.

Proof. We have already seen in Proposition 1.2.7 that (c) implies (a) and (b), so it is enough to show $(a), (b) \Rightarrow (c)$.

Assume M is Noetherian or Artinian. By hypothesis, we fix an $n \in \mathbb{N}$ such that $J(R)^n = 0$. Consider the chain

$$M \geq M.J(R) \geq M.J(R)^2 \geq \cdots \geq M.J(R)^n = 0.$$

It is enough to show that every factor $M.J(R)^i / M.J(R)^{i+1}$ admits a composition series, but each factor is either Noetherian or Artinian as a $R/J(R)$-module. Now by assumption $R/J(R)$ is semisimple, so that each factor $M.J(R)^i / M.J(R)^{i+1}$ is the direct sum of simple $R/J(R)$-modules. By the chain condition, this direct sum must be finite, meaning $M.J(R)^i / M.J(R)^{i+1}$ admits a composition series as a $R/J(R)$-module.

To prove (i) consider the equivalence between (a) and (b) applied to R_R (resp. ${}_RR$). For (ii), observe that a finitely generated (right) module over a (right) Artinian ring is also Artinian. □

Remark. This result implies that a right Artinian ring is always right Noetherian.

We finish this section by defining local and semilocal rings, whose definitions strongly correlate with the Jacobson radical.

Proposition 1.6.11. [25, Theorem 19.1] *Let $R \neq 0$ a ring. The following are equivalent:*

(a) *R has a unique maximal right ideal*

(b) *R has a unique maximal left ideal.*

(c) *$R/J(R)$ is a division ring.*

Proof. By symmetry, it is enough to show that (a) and (c) are equivalent.

(a) $\Rightarrow$ (c) By hypothesis $J(R)$ is the unique maximal right ideal of R. Therefore $R/J(R)$ has two right ideals, the zero ideal and itself. Hence $R/J(R)$ is a division ring.

(c) $\Rightarrow$ (a) By definition, $J(R)$ is contained in any maximal right ideal. Now since $R/J(R)$ is a division ring, the only maximal right ideal must be $J(R)$. □

Definition. A non-zero ring R is *local* if it has a unique maximal right or left ideal.

Definition. A ring R is *semilocal* if $R/J(R)$ is semisimple. Equivalently, R is semilocal if $R/J(R)$ is right (or left) Artinian.

Remark. By Proposition 1.6.9, we know that $R/J(R)$ is semisimple if and only if $R/J(R)$ is Artinian and $J(R/J(R)) = 0$. This equality always holds by Lemma 1.6.8, therefore the definitions are equivalent.

Artinian and local rings are both semilocal. If R is Artinian, then $R/J(R)$ is Artinian. If R is local, then by the previous proposition $R/J(R)$ is a division ring, hence simple, thus Artinian.

1.7 More Classes of Modules and Rings

This section aims to gather the definitions and results regarding rings and modules that do not fit elsewhere in a natural way.

We begin with the definition of hereditary ring, as the class of hereditary Noetherian rings will prove itself fundamental in Section 3.3.

Definition. A ring R is said *right hereditary,* if all of its right ideals are projective. If R is both left and right hereditary, we say that R is *hereditary*.

Definition. A ring R is *right semihereditary*, if all of its right ideals are finitely generated. If R is both left and right hereditary, we say that R is *semihereditary*.

Remark. Let R be an arbitrary ring. We have the following:

(a) A right semisimple ring R is right hereditary, because its right ideals are summands of R, hence projective.

(b) A right Principal Ideal Domain is also a right hereditary ring. Note that in an integral domain for any $x \in R$ we can define the isomorphism, $R \to xR$, $r \mapsto xr$, so any principal right ideal is free, hence projective. In particular $\mathbb{Z}$ is an hereditary ring.

(c) Clearly any semihereditary ring is hereditary.

We have a particular interest in the class of Artinian serial rings, as we will see in Sections 3.2 and 3.4.

Definition. A module M is said to be a *uniserial module* if all its submodules are totally ordered by inclusion i.e., for any submodules A and B of M, either $A \subseteq B$ or $B \subseteq A$.

Definition. A module is said to be *serial* if it is the direct sum of uniserial modules. A ring R is *right (resp. left) serial* if R_R (resp. ${}_RR$) is a serial module. If both conditions hold we say R is a *serial ring*.

Remark. It is trivial that a simple module is uniserial, so a semisimple module is a serial module.

The following proposition gives us an important characterization. Its proof is beyond the scope of this thesis. Note that by the following proposition indecomposable modules are uniserial.

Proposition 1.7.1. [16, Theorem 5.6] *Let R be an Artinian serial ring. Then every module M is a direct sum of cyclic uniserial modules, and any two direct sum decompositions of M into direct sums of non-zero uniserial modules are isomorphic.*

We now move onto the concept of the dual module, but first let us state this clear result.

Lemma 1.7.2. *Let R be a ring and M a right R-module. The additive group $Hom_R(M, R)$ is a left R-module, with the following scalar multiplication: if $f \in Hom_R(M, R)$ and $r \in R$ we have $rf : M \to R$, $(rf)(x) = rf(x)$.*

The left R-module as defined above is said to be the *dual module of* M *(over R)* and we denote it by $M^* = Hom_R(M, R)$. Its elements are said to be the *linear forms on* M. For every ordered pair of elements $x \in M$ and $f \in M^*$, the element $f(x)$ of R is denoted by $< x, f >$.

Definition. Consider a module M and its dual $M^* = Hom_R(M, R)$. An element $x \in M$ and an element $f \in M^*$ are said *orthogonal*, if $< x, f >= 0$.

A subset N of M and a subset $N^*$of M^* are *orthogonal sets* if, for all $x \in N,\ f \in N^*$, x and f are orthogonal.

Definition. Two modules are called orthogonal if they have no non-zero isomorphic submodules.

We define yet another important class of rings, present throughout Chapter 2.

Definition. A ring R is said to be a *right SI-ring* if all its singular right R-modules are injective.

Proposition 1.7.3. [22, Proposition 3.1] *For a ring R, the following are equivalent:*

(a) *R is a right SI-ring.*

(b) *All singular right R-modules are semisimple.*

(c) *R/I is semisimple, for all essential ideals I.*

Proof. (a) ⇒ (b) Let M be an arbitrary singular right R-module. Then by definition of right SI-ring, M is injective, and all of its submodules are also injective, meaning they are all summands of M. Hence M is semisimple.

(b) ⇒ (c) Since $I \leq_e R$, it follows that R/I is singular, hence by hypothesis R/I is semisimple.

(c) ⇒ (a) Suppose that M is a singular right R-module and let us prove that M is R-injective, and therefore injective. Let I be an esssential ideal and $f : I \to M$ a homomorphism. Note that $I/\ker f$ is singular, because it is isomorphic to $f(I) \leq M$. Then $\ker f$ is an essential ideal, which means by hypothesis that $R/\ker f$ must be semisimple. Thus $I/\ker f$ is a summand of $R/\ker f$ and this means that f can be extended to some $\bar{f} : R_R \to M$ and we are done. □

Proposition 1.7.4. [22, Proposition 3.6] *If R is a right SI-ring, then $R/Soc(R_R)$ is right Noetherian.*

Proof. It is enough to prove that if $J = Soc(R_R)$ and $J \leq I \leq R_R$, then I/J is finitely generated. We choose $K \leq I$ such that $J \oplus K \leq_e I$. Then $I/(J \oplus K)$ is singular and by SI hypothesis injective, hence a summand of $R/(J \oplus K)$. Therefore $I/(J \oplus K)$ is cyclic, and we are done if we manage to prove that K is finitely generated. Let us start by seeing that K has finite dimension, otherwise there exists an infinite sequence $\{K_1, K_2, \ldots\}$ of independent non-zero submodules of K. Since $K \cap J = 0$, this means none of the $K_i \leq K$ are semisimple, and by Lemma 1.5.4 each K_i has a proper essential submodule that we represent by H_i. Since $H_i \leq_e K_i$ for every i, then $\oplus H_i \leq_e \oplus K_i$, by Proposition 1.1.10(4). Now by Proposition 1.4.5(a), $(\oplus K_i)/(\oplus H_i)$ is singular, so $(\oplus K_i)/(\oplus H_i)$ is injective, and so a summand of $R/(\oplus H_i)$. Thus it is cyclic, contradicting the fact that it is an infinite direct sum of non-zero modules. Hence K has finite dimension. Now take a maximal independent family $\{E_i\}_{i \in I}$ of non-zero cyclic submodules of K, so $E = \oplus_{i \in I} E_i$ is finitely generated, since K has finite dimension. Furthermore, from the maximality of $\{E_i\}_{i \in I}$ we infer that $E \leq_e K$. Thus K/E is singular, once again injective, and we conclude that K/E is a summand of R/E. Therefore K/E is cyclic, meaning K must be finitely generated. □

The proof of the following is beyond the scope of this thesis. Furthermore, for the definition of Morita equivalent, see A.3.

Proposition 1.7.5. [22, Theorem 3.11] *A ring R is a right SI-ring if and only if there is a ring decomposition $R = K \times R_1 \times \ldots \times R_n$ such that $K/Soc(K_K)$ is a semisimple ring and each R_i is Morita equivalent to a right SI-domain.*

Definition. A ring R is said to be a *right PCI-ring* if every cyclic right module not isomorphic to R is injective.

Claim 1.7.6. If R is a domain, then the notion of right PCI-domain and right SI-domain are equivalent.

A ring that satisfies any of the following conditions is said to be a *quasi-Frobenius ring*, which we represent by QF-ring.

Theorem 1.7.7. [24, Theorem 15.1] *For any ring R the following conditions are equivalent*

(a) *R is right Noetherian on one side and right self-injective.*

(b) *R is left Noetherian and right self-injective.*

(c) *R is right Noetherian and satisfies the following conditions:*

(i) $ann_r(ann_l(I)) = I$, *for any right ideal $I \subseteq R$.*

(ii) $ann_r(ann_l(J)) = J$, *for any left ideal $J \subseteq R$.*

(d) *R is Artinian on both sides and satisfies (i) and (ii).*

Chapter 2

Poor Modules

The main focus of this work is to study pauper modules, a concept directly derived from poor modules. With that in mind, this chapter does not present a full study of poor modules, instead being dedicated to presenting necessary and/or illustrative results for the next chapter.

Another way to state that a (right) R-module is injective is to say that its injectivity domain is maximum, i.e., $\mathfrak{I}n^{-1}(M) =$ Mod-R. In this case, the injectivity domain of M is “wealthy”, so the notion of “poor” modules arises from the injectivity domain of a module being minimum. In Proposition 1.1.12 we have established that SSMod-R is contained in the injectivity domain of any module. It turns out the intersection of all injectivity domains in Mod-R is indeed SSMod-R, by Proposition 2.1.3.

The first section is dedicated to develop these ideas, while providing some useful results going forward. The second section justifies in a way our pursuit of the study of poor modules, illustrating the usefulness of this concept in the characterization of different classes of rings.

2.1 Definitions and general results

The results and definitions in this section follow from [1, 3, 8, 15]. Please note that some authors use the term "semisimple" to mean the ring has a trivial Jacobson radical. For Artinian rings, Proposition 1.6.9 guarantees that this notion coincides with the notion of semisimple we introduced in Chapter 1. Therefore, we will use "semisimple Artinian ring" to eliminate that ambiguity, as in the papers referenced.

Poor modules are in their essence opposite to injective modules.

Definition. A right module M is *poor* if $\mathfrak{I}n^{-1}(M) =$ SSMod-R.

Lemma 2.1.1. *A module M is poor if and only if every cyclic module $xR \in \mathfrak{I}n^{-1}(M)$ is semisimple.*

Proof. ($\Rightarrow$) This implication follows from the definition of poor module.

($\Leftarrow$) Take an arbitrary module $N \in \mathfrak{I}n^{-1}(M)$. By Proposition 1.1.16 we know that M being N-injective is equivalent to M being xR-injective for all of the cyclic submodules of N. Now since every module is spanned by the set of its cyclic submodules and these are semisimple by hypothesis, then so must be N. Therefore M is poor. $\square$

Naturally, we are concerned with the existence of poor modules.

Theorem 2.1.2. [15, Proposition 1] *Every ring R has a poor module.*

Proof. We start by considering a complete set $\{M_\omega \mid \omega \in \Omega\}$ of representatives of isomorphism classes of non-semisimple cyclic right R-modules. For every $\omega \in \Omega$, since M_ω is not semisimple, then by Lemma

1.5.4 we can choose a proper essential submodule N_ω of M_ω. Now take $S = \oplus_{\omega\in\Omega} N_\omega$. If S is not poor, by Lemma 2.1.1 we can choose A to be a non-semisimple cyclic module such that S is A-injective. Since the set defined above is complete, then for some $\omega \in \Omega$ we have that $A \cong M_\omega$. So A admits a proper essential submodule $K \lneq_e A$, which is isomorphic to N_ω. As N_ω is a direct summand of S, so K is also A-injective. Thus K is a direct summand of A, contradicting $K \lneq_e A$. Therefore S must be a poor module. □

The following proposition justifies the definition of poor modules and establishes SSMod-R as the lower bound for the injectivity domain of an arbitrary right R-module.

Proposition 2.1.3. [1, Proposition 3.1] *We have.*

$$\bigcap_{M\in \text{ Mod-R}} \mathfrak{I}n^{-1}(M) = \textit{SSMod-R}.$$

Proof. We have already seen that SSMod-R is contained in the injectivity domain of any R-module. For the other inclusion, take N in the intersection of the injectivity domains of every $M \in$ Mod-R and $T \leq N$. Then in particular T is N-injective, so the inclusion $T \to N$ splits, by Proposition 1.1.14, and T is a direct summand of N. Thus N is a semisimple right R-module. □

The following proposition makes it apparent that direct sums are fundamental in the study of poor modules.

Lemma 2.1.4. *The injectivity domain of the direct sum of two modules is the intersection of the injectivity domains of the summands, i.e., for arbitrary right R-modules M_1, M_2 we have*

$$\mathfrak{I}n^{-1}(M_1 \oplus M_2) = \mathfrak{I}n^{-1}(M_1) \cap \mathfrak{I}n^{-1}(M_2).$$

Proof. When a set of indexes I is finite, we know that the direct product and the direct sum coincide. Therefore the result follows directly from Proposition 1.1.18. □

The next result is not just a useful consequence of the previous proposition, it also shows that when it comes to direct sums the poor condition acts as a sort of absorption property. In fact this corollary is the inspiration behind the definition of a pauper module (see Chapter 3).

Corollary 2.1.5. [1, Remark 2.4] *If M is a poor module, then for any module N we have that $M \oplus N$ is poor.*

Proof. Let $K \in \mathfrak{I}n^{-1}(M \oplus N)$. By Lemma 2.1.4, we have $K \in \mathfrak{I}n^{-1}(M) \cap \mathfrak{I}n^{-1}(N)$. Thus $K \in \mathfrak{I}n^{-1}(M)$ and since M is poor, K must be semisimple. Hence $M \oplus N$ is poor. □

In general, the corollary above is not an equivalence. When one of the summands is an injective module, we have the following result.

Proposition 2.1.6. [3, Lemma 2.4] *Let M be an arbitrary module and E an injective module. Then $E \oplus M$ is poor if and only if M is poor.*

Proof. The "if" part is immediate from the corollary above. For the converse, suppose $E \oplus M$ is poor and take $N \in \mathfrak{I}n^{-1}(M)$. As E is injective, $N \in \mathfrak{I}n^{-1}(E)$. By lemma 2.1.4, $N \in \mathfrak{I}n^{-1}(E \oplus M)$. Hence, by hypothesis, N is semisimple. Therefore, M is poor. □

The proof of Theorem 2.1.2 already gave us a "hint" on how to build poor modules, and the following proposition gives us an explicit way to do so.

Proposition 2.1.7. [15, Proposition 2] *Let $M = \oplus_{N\in\Gamma} N$, where Γ is any complete set of representatives of cyclic right R-modules. Then M is poor.*

Proof. Let K be an arbitrary module such that M is K-injective. Then all cyclic submodules of factors of K are K-injective. Thus, by Proposition 1.1.11, it follows that K must be semisimple. Therefore M is poor. □

The following is the first example of the usefulness of poor modules as a tool to characterize a ring.

Proposition 2.1.8. [1, Proposition 3.7] *If a (right) nonsingular ring R has a nonsingular poor module, then R is an SI-ring.*

Proof. Let M be a nonsingular poor module and take an arbitrary singular module N. Lemma 1.4.7(a) tells us we have $Hom_R(N, M) = 0$, so $N \in \mathfrak{In}^{-1}(M)$. Since M is poor, N must be semisimple. Therefore by the equivalence established in Proposition 1.7.3, we conclude that R is a right SI-ring. □

Proposition 2.1.9. [8, Proposition 5.1] *Let R be a semilocal ring. Then $R/J(R)$ is a poor module.*

Proof. Let $I = R/J(R)$ and suppose that $B \in \mathfrak{In}^{-1}(I)$, where B is a cyclic right R-module. We want to prove that $Rad(B) = 0$. So, by contradiction, suppose there is a non-zero $x \in Rad(B)$. Let $f : xR \to I$ be a non-zero homomorphism. So, f can be extended to a homomorphism $g : B \to I$, because I is B-injective. Then we have that $f(xR) = g(xR) \leq g(Rad(B)) \leq Rad(I)$, by Proposition 1.6.2, but also $Rad(I) = 0$, by Proposition 1.6.3. However this would imply that $f(xR) = 0$, which contradicts our assumption. Therefore $Rad(B) = 0$, hence $BJ(R) = 0$ and $J(R) \subseteq ann(B)$. Now by the semilocal hypothesis, I_R is semisimple. As $B = bR \cong R/ann(b)$, for some $b \in B$, then B is also semisimple. Therefore I is poor. □

The following example is a direct application of the previous proposition.

Example 2.1.10. $\mathbb{Z}/6\mathbb{Z}$ is a poor $\mathbb{Z}/12\mathbb{Z}$-module.

We take $R = \mathbb{Z}/12\mathbb{Z}$ which is Artinian, because it is finite. By the remark following the definition of semilocal we know this ring is semilocal.

Let us compute $J(R)$. It is easy to see that the maximal ideals of R are $2\mathbb{Z}/12\mathbb{Z}$ and $3\mathbb{Z}/12\mathbb{Z}$. So by definition $J(R) = (2\mathbb{Z}/12\mathbb{Z}) \cap (3\mathbb{Z}/12\mathbb{Z}) = 6\mathbb{Z}/12\mathbb{Z}$. Now by the Third Isomorphism Theorem

$$R/J(R) = (\mathbb{Z}/12\mathbb{Z})/(6\mathbb{Z}/12\mathbb{Z}) \cong \mathbb{Z}/6\mathbb{Z}.$$

The two following results lead us to a fundamental example of a poor module. Its proof is beyond the scope of this thesis.

Proposition 2.1.11. [1, Proposition 3.4] *Let R be an hereditary Noetherian domain and let M be a semisimple module that contains exactly one copy of each simple R-module. Then M is either poor or injective.*

In particular, if R admits only one simple module (up to isomorphism), then that module is either injective or poor. If a ring R and module M satisfy these conditions, then M is poor unless R is a V-ring.

Corollary 2.1.12. [1, Corollary 3.5] *Let R be an hereditary Noetherian domain. If there exists a non-simple and non-zero uniserial module U, then every semisimple module M that contains every simple R-module is poor.*

Proof. Take modules M and U as above. By the previous proposition, M is injective or poor. By hypothesis U contains a simple submodule $N \lneq U$. Also by hypothesis we can embed N into M. Assuming M is injective, then N is U-injective. Now by Proposition 1.1.14, N is a direct summand of U, which contradicts the uniserial hypothesis. Therefore, M is poor. $\square$

This example is fundamental in the study of pauper abelian groups.

Example 2.1.13. Let $R = \mathbb{Z}$, and $M = \oplus_{p\in\mathbb{P}}(\mathbb{Z}/p\mathbb{Z}) \cong \oplus_{p\in\mathbb{P}}\mathbb{Z}_p$.

Note that by definition M contains exactly one copy of each of its simple R-modules. Furthermore, we have already seen that $\mathbb{Z}$ is an hereditary domain, and it is also well known that $\mathbb{Z}$ is Noetherian. So by the previous corollary it follows that M is a poor module.

2.2 Rings with no middle class

This section presents important results that hold their own significance, although their importance will become more clear in Chapter 3. Most of the results presented here are from [15]. They are utilized in order to prove the main result of this section, Theorem 2.2.14.

Definition. For a class of right R-modules $\mathcal{A}$, we say that R has *no $\mathcal{A}$-middle class*, if every module in $\mathcal{A}$ is either poor or injective.

So we have two extreme cases:

- When all modules in $\mathcal{A}$ are poor we say that the ring R is *$\mathcal{A}$-destitute*.
- When no module in $\mathcal{A}$ is poor we say that the ring R is an *$\mathcal{A}$-utopia*.

If $\mathcal{A} =$ Mod-R in one of the cases described above, we omit $\mathcal{A}$ and simply state that the ring R is destitute (resp. utopia). If we take $\mathcal{A} = \{\text{simple } R\text{-modules}\}$ we say R is *simple-destitute* (resp. *simple-utopia*), and analogous for other classes of rings.

An immediate application of the definition.

Proposition 2.2.1. [1, Remark 2.3] *For a ring R, the follow conditions are equivalent:*

(a) *R is semisimple Artinian.*

(b) *R is destitute.*

(c) *There exists an injective poor module.*

(d) *$\{0\}$ is a poor module.*

Proof. (a) $\Rightarrow$ (b) From Proposition 1.1.23 we infer that every right R-module is semisimple. Take M and N arbitrary right R-modules such that M is N-injective. Since N is semisimple, M is poor. By the arbitrary choice of M, Mod-R is destitute.

(b) $\Rightarrow$ (d) Obvious by definition.

(d) $\Rightarrow$ (c) It is clear, since $\{0\}$ is injective.

(c) $\Rightarrow$ (a) Let M be an injective poor module. Then M is R-injective and, because M is poor, R is semisimple. $\square$

Most of the following results are fairly self-explanatory, in terms of the characterization they offer for certain classes of rings. They also illustrate the "strength" of the no middle class condition.

Proposition 2.2.2. [1, Proposition 3.2] *Let R be a right PCI-domain. Then R has no middle class and R_R is a poor module.*

Proof. If R is a division ring, then R_R is a simple module. By Proposition 2.2.1, R is destitute and all modules are poor.

Now let us assume that R is not a division ring. Then, by definition of PCI-domain, the only cyclic modules that are not injective are isomorphic to R_R. Now by Claim 1.7.6, the injective cyclic modules are all singular, therefore semisimple by Proposition 1.7.3. Hence N is a non-injective module M must be poor. □

Proposition 2.2.3. [1, Theorem 4.1] *Let R be a semilocal ring such that $J(R)$ is simple and an essential right ideal of R. Then R has no middle class. In particular, $J(R)$ is a poor R-module.*

Proof. Let R be a ring in the conditions described above. Also take M a module which is not injective and aR a cyclic module. If M is aR-injective and since M is not injective, it follows that $ann(a) \neq 0$. By hypothesis of $J(R)$ being essential and simple it follows that $0 \neq J(R) \leq ann(a)$. Now by the semilocal hypothesis, we have that aR is semisimple, since $aR \cong R/ann(a) \leq (R/J(R))/(ann(a)/J(R))$. Therefore M is poor and R has no middle class. In particular, $J(R)$ is a poor module. □

The following is an useful structural result.

Lemma 2.2.4. [15, Lemma 1] *The property of having no middle class is inherited by factor rings.*

Proof. Let R be a ring with no right middle class and take I an ideal of R. Let M be a right (R/I)-module which is not poor. Thus, there exists a non-semisimple module $N_{R/I}$ such that $M_{R/I}$ is $N_{R/I}$-injective. So M_R is N_R-injective. Now N_R is non-semisimple and R has no middle class, then M_R is injective. Therefore $M_{R/I}$ is injective. □

Note that (i) is a sort of complementary result to Proposition 1.7.3(b), by adding the no (right) middle class condition.

Lemma 2.2.5. [15, Lemma 2] *Let R be a ring with no middle class that is not a right SI-ring. Then the following conditions hold:*

(i) *Every nonsingular right R-module is injective (hence semisimple).*

(ii) *The second singular submodule splits in any right R-module.*

(iii) *There exists a ring direct sum $R = S \oplus T$ such that S is a semisimple Artinian ring and T_T has essential socle with $Z(T_T) = Soc(T_T)$.*

(iv) *$Soc(R_R)$ is an essential submodule of R_R.*

Proof. (i) Assume that R is not a right SI-ring and that it has no right middle class. Then there exists a singular right module which is not injective. So by Proposition 1.1.22, $M_R \lneq_e E(M)$. Furthermore, by Proposition 1.5.4 we infer that $E(M)$ is not semisimple. So every nonsingular module is $E(M)$-injective. Then by hypothesis, every nonsingular module is injective, as well as semisimple.

(ii) Let M be a module. We have shown in Lemma 1.4.4, that $Z(M)$ is essential in $Z_2(M)$, and $Z_2(M)$ is a closed submodule of M. This means that there exists a submodule $N \leq M$ such that $(N \oplus Z_2(M))/Z_2(M)$ is essential in $M/Z_2(M)$. Now by (i) N must be injective, as it is nonsingular. Finally $M = N \oplus Z_2(M)$ i.e., $Z_2(M)$ splits in M.

(iii) From (ii) we have $R_R = I_R \oplus Z_2(R_R)$, for some right ideal I of R. So $I \cong R/Z_2(R)$, thus I_R is nonsingular, and by (i) it follows that I_R is injective and semisimple. Note that for any $r \in R$ there is an isomorphism between rI and a direct summand of I_R, hence rI is also nonsingular, which implies that $Z_2(R_R)I = 0$. Therefore I is a two-sided ideal. Now by contradiction, assume that $Z(R_R)$ is not semisimple. Again by (i) this implies that $Z(R_R) \neq 0$. thus $Z(E(R_R))$ is not semisimple either. In general the singular submodule is a fully invariant submodule of a module. In particular $Z(E(R_R))$ is a fully invariant submodule of $E(R_R)$. Now by Proposition 1.1.25, $Z(E(R_R))$ is quasi-injective, thus not poor. By the no middle class hypothesis, $Z(E(R_R))$ must be injective. Therefore $Z(R_R) = Z(E(R_R)) \cap R_R$ is a closed submodule of R_R. From this we infer that $Z_2(R_R) = Z(R_R)$, which by (ii) implies that $Z(R_R)$ splits in R_R. This leads us to conclude that $Z(R_R) = 0$, which is a contradiction. So $Z(R_R)$ must be semisimple. Take $S = I_R$ and $T = Z_2(R_R)$, so $R = S \oplus T$ and $Z(T_T) = Z(R_R) = Soc(T_T)$. Hence, by Proposition 1.4.4 we have $Soc(T_T) \leq_e T_T$.

(iv) Consequence of (iii), since $Soc(R) = Soc(S) \oplus Soc(T) = S \oplus Soc(T) \leq_e S \oplus T = R$. □

Lemma 2.2.6. [15, Lemma 3] *Let R be a ring with no right middle class such that $Soc(R)$ is singular and essential. Then R is an indecomposable ring.*

Proof. By contradiction, assume that $R = I_1 \oplus I_2$, with $I_1, I_2 \neq 0$ ideals. Also let J_1 and J_2 be ideals contained in $I_1$and I_2 respectively. If the map $f : J_1 \to J_2$ is an R-homomorphism, then we have $f(J_1) = f(J_1)I_2 = f(J_1I_2) = f(0) = 0$, which implies that every right ideal contained in I_2 is I_1-injective. In particular, $Soc(I_2)$ is I_1-injective. Now by the no middle class hypothesis we have that, either $soc(I_2)$ is injective or I_1 is semisimple. In the first case, the sum of simple submodules of I_2 is injective, hence I_2 is semisimple. In the latter case, we conclude that R has a simple direct summand, contradicting the fact that $Soc(R_R)$ is singular. Therefore $I_1 = 0$ or $I_2 = 0$, so R is indecomposable. □

Lemma 2.2.7. [15, Lemma 4] *Let R be a ring with no middle class. If R has a non-semisimple and Notherian right module, then R is a right Notherian ring.*

Proof. Let M be a Noetherian and non-semisimple module. Take a set of injective right R-modules $\{E_i \mid i \in I\}$. Then by Proposition 1.2.4 we have that $\bigoplus_{i \in I} E_i$ is M-injective. Now since M is non-semisimple and R has no middle class, we infer that $\bigoplus_{i \in I} E_i$ is injective. Therefore by Proposition 1.2.5, we conclude R is right Noetherian. □

The following proposition reduces the study of no right middle class rings to the following two classes.

Proposition 2.2.8. [15, Lemma 5] *Any ring with no right middle class is either right semiartinian or right Noetherian.*

Proof. Assume R is not a right semiartinian ring. Take $\mathcal{I}$ to be the non-zero union of the right socle series of R, and consider the union ring $R/\mathcal{I}$. Now by the hypothesis of not being semiartinian, for some module M in the socle series we have that $Soc(M) = 0$, hence $Soc(R/\mathcal{I}) = 0$. Furthermore, by Lemma 2.2.4 we also know that $R/\mathcal{I}$ has no middle class. Now, if $R/\mathcal{I}$ was not an SI-ring, then by Lemma 2.2.5(iv) we would have $Soc(R_R) \leq_e R_R$, so we infer that $R/\mathcal{I}$ is a right SI-ring. Therefore by Proposition 1.7.4, we conclude that $R/\mathcal{I}$ is a right Noetherian ring. Thus $R/\mathcal{I}$ is a Noetherian right R-module, as well as non-semisimple. By the previous lemma, R is a right Noetherian ring. □

Lemma 2.2.9. [15, Lemma 6] *A ring R with no right middle class that has singular right socle is right Noetherian.*

Proof. Assume that R is in the conditions as described. In order to see that R is right Noetherian, we know it is enough to show that it is not semiartinian, by the previous proposition. Let us assume the contrary, that R is semiartinian. This means that for every non-zero module $Soc(M) \neq 0$. In particular, we know there exists a simple right ideal $I \subseteq R$. Note that since $Soc(R_R)$ is singular, I cannot be a direct summand of R_R and hence is not injective. Therefore we have that I is properly contained in $E(I)$, so $E(I)/I$ admits a simple submodule. Then I is maximal in a submodule I' of $E(I)$. Hence I' is clearly Noetherian, and non-semisimple. Now by Lemma 2.2.7, we conclude that R is right Noetherian. $\square$

Lemma 2.2.10. [15, Lemma 7] *Let R be a ring with no middle class, with non-zero singular right socle. Then R is right Artinian.*

Proof. By the previous lemma we know that R is right Noetherian. To prove that R is Artinian we know that by Proposition 1.5.5, it is enough to prove that R is semiartinian. Let us assume R is not right semiartinian and reach a contradiction. We denote the union of the right socle series of R by U, and take $\bar{R} = R/U$. By the not semiartinian assumption, we have $Soc(\bar{R}_R) = 0$ and $\bar{R} \neq 0$, this implies that $Soc(\bar{R}_R)$ is not an essential submodule of $\bar{R}$. So, by Proposition 1.5.3, $\bar{R}$ is not Artinian. Now by the no middle class assumption and Lemma 2.2.4 we know $\bar{R} = R/U$ has no middle class either, hence $\bar{R}_R$ is injective or poor. If $\bar{R}_R$ is injective, then the ring $\bar{R}$ is self-injective. Also $\bar{R}$ is right Noetherian, by Proposition 1.2.3(1). This means $\bar{R}$ is a QF-ring, hence $\bar{R}$ is Artinian, by Proposition 1.7.7, a contradiction. Thus $\bar{R}$ is poor.

Now we restrict our focus to a cyclic submodule. Assume we have a non-semiartinian cyclic N module. We take V to be the union of the socle series of N and let $\bar{N} = N/V$. Again we have $Soc(\bar{N}_R) = 0$ and $\bar{N} \neq 0$.

First we claim that $\bar{N}$ has a submodule $\bar{W} = W/V$, where W a submodule of N containing V such that $\bar{N}/\bar{W} \cong N/W$ is not semiartinian. By contradiction, assume that every factor $\bar{N}/X$ is semiartinian, with $0 \neq X \leq \bar{N}$. Then $\bar{R}$ is $\bar{N}/X$-injective, and poor, by the previous paragraph, hence $\bar{N}/X$ is semisimple. Take K to be a simple right ideal of R, which exists by the non-zero socle assumption. Furthermore, since $Soc(R_R)$ is singular, K cannot be injective. Therefore by no right middle class assumption, K is poor.

Now we claim that K is $\bar{N}$-injective. Let G be a submodule of $\bar{N}$, and let $f : G \to K$ be a non-zero homomorphism. As $Soc(\bar{N})$, then $Soc(G) = 0$, hence $\ker f \neq 0$. By the argument done above, we infer that $\bar{N}/\ker f$ is semisimple, and so $\bar{N}/\ker f \cong (G/\ker f) \oplus (Y/\ker f)$, for some $Y \leq \bar{N}$. Now take the projections $g_1 : \bar{N} \to \bar{N}/\ker f$, $g_2 : (G/\ker f) \oplus (Y/\ker f) \to \bar{N}$, and the induced isomorphism $\bar{f} : G/\ker f \to K$. Therefore $\bar{f} g_1 g_2 : \bar{N} \to K$ extends f, hence K is $\bar{N}$-injective. However, K is poor, so $\bar{N}$ must be semisimple, a contradiction. We have just proven the claim of the previous paragraph. Now if we take $N = R$, we have a non-zero right ideal A_1 of R such that R/A_1 is not semiartinian. If $N = R/A_1$, then there is a non-zero right ideal A_2 of A_1 such that R/A_2 is not semiartinian. By recursion, we obtain an infinite strictly ascending chain $A_1, A_2, \ldots$ of right ideals of R, where each R/A_i is not semiartinian. But this contradicts the fact that R is Noetherian, thus R is semiartinian, hence R is Artinian. $\square$

Building upon Proposition 2.2.8 we have.

Proposition 2.2.11. [15, Lemma 9] *Let R be a right semiartinian ring with no right middle class. Then R is either a right V-ring or a right Artinian ring.*

Proof. Assume R is not a right V-ring. So there exists a simple right submodule $N \leq R_R$ which is not injective. But since N is not injective, as in the proof of Lemma 2.2.9, we infer that there exists a simple submodule of $E(N)/N$, hence R is right Noetherian. Therefore by the semiartinian hypothesis we conclude that R is Artinian, by Proposition 1.5.5. $\square$

Definition. The socle of a module M is said to be *homogeneous* if all of its simple submodules are isomorphic to one another. In other words, for arbitrary simple submodules, V_1, V_2 in $Soc(M)$ we have $V_1 \cong V_2$.

Lemma 2.2.12. [15, Lemma 8] *Let R be a right nonsingular ring with no right middle class. Then there exists a ring direct sum of the form $R = S \oplus T$, where S is a semisimple Artinian ring and T is a ring with homogeneous right socle (that can be zero).*

Proof. Assume that R is a right nonsingular ring with no right middle class. We want to start by showing that $Soc(R_R)$ does not contain a submodule of the form $A \oplus B$, where both A and B are infinitely generated orthogonal submodules (see Section 1.7). So with a contradiction in mind let us assume that A and B do exist in these conditions. As A and B are infinitely generated, then A and B must be non-injective. Now because A and B are orthogonal, it follows that for any $f \in Hom_R(E(B), E(A))$ we have $f(B) = 0$. Since $B \leq \ker f \leq E(B)$, then by Proposition 1.1.10(1), $\ker f \leq_e E(B)$. Furthermore, $\operatorname{im} f \cong E(B)/\ker f$, hence $\operatorname{im} f$ is singular, by Proposition 1.4.5. Thus $f = 0$, since R is nonsingular. Therefore A is $E(B)$-injective. Since $E(B)$ is non-semisimple, because it is infinitely generated, then A cannot be poor, which by the no middle class hypothesis means it is injective, contradicting our assumption.

Now consider two non-isomorphic simple modules $S_1$and S_2 contained in $Soc(R_R)$. By the same argument above, we can assume without loss of generality that $E(S_1)$ is $E(S_2)$-injective and vice-versa. So at least one of them would have to be injective. The same argument also applies to a simple right ideal S which is orthogonal to an infinitely generated semisimple right ideal I.

From this we are able to conclude that $Soc(R_R)$ can only have a finite number of homogeneous components, where only one of them may be infinitely generated. Once again by the same argument we are able to conclude that every other components must be injective. This allows us to construct a summand of the form $R = S \oplus T$, as stated in the lemma. $\square$

Lemma 2.2.13. [15, Lemma 10] *Let R be a non-semisimple right SI-ring with no right middle class such that $R/Soc(R_R)$ is semisimple. Then R has a unique simple singular right R-module.*

Proof. Assume that R is a non-semisimple right SI-ring with no middle class and that $R/Soc(R_R)$ is semisimple. Note that $0 \neq Soc(R_R) \leq_e R_R$ and R is semiartinian. By hypothesis we have $R/Soc(R_R) = \bigoplus_{i=1}^{n}(B_i/Soc(R_R))$ for some right ideals B_i, where each $B_i/Soc(R_R)$ is a simple submodule of $R/Soc(R_R)$. Since R is not semisimple Artinian, we can infer by Lemma 1.5.4 that R admits an essential maximal right ideal. Now by Proposition 1.4.5 there exists a simple singular module S. Since S is simple, it must be generated by a single element, $s \in S$, such that $S = sR$. By definition of singular module, for this generator s we have that $ann(s) \leq_e R_R$. Now by Proposition 1.5.1, $Soc(R_R) \leq ann(s)$. Since $S = sR = \sum_{i=1}^{n} sB_i$ and S is simple, we must have that $S = sB_i$, for some $i \in \{i, \ldots, n\}$, therefore $S \cong B_i/Soc(R_R)$. By the arbitrary choice of S it follows that, any other simple singular module would be isomorphic to some $B_i/Soc(R_R)$. So if we manage to prove that all the modules of the form $B_i/soc(R_R)$ are isomorphic to each other we are done.

Let $i \neq j$. There exists $a \in B_j$, such that $B_j = aR + Soc(R_R)$. We already know that $Soc(R_R) \leq_e R_R$, so by Proposition 1.1.10(1), each B_k is also essential. By Lemma 1.1.21(2) this implies that $E(R_R) =$

$E(B_k)$. Now since B_i is not semisimple and R has no middle class, we infer that the quasi-injective module $\sum_{f\in Hom_R(B_j,E(B_j))} f(B)$ is not semisimple, because it cointains B_j, and coincides with $E(B_j) = E(R_R)$. This means that B_i generates $E(R_R)$. Thus there exists an epimorphism $f : B_j^{(\Gamma)} \to E(R_R)$, for some index set Γ. Now choose an element x in a finite subsum of $B_i^{(\Gamma)}$ such that $f(x) = a \in B_j$. We now restrict $f's$ domain to this finite subsum of $B_i^{(\Gamma)}$,obtaining a homomorphism of the form $g : B_j^m \to E(R_R)$, for some $m \in \mathbb{N}$, such that $aR \leq \text{im } g$. Let $C = g^{-1}(aR)$. Since R is a right SI-ring, then it is right hereditary, by [22, Proposition 3.3]. The right ideal aR is projective, so by Proposition 1.3.3 we know that $g_{|C}$ splits, thus aR can be embedded in C, as well as embedded in B_i^m.

Let $a = (b_1, \ldots, b_m)$ with $b_k \in B_i$, for every $k \in \{i, \ldots, m\}$. Now $aR \subseteq b_1R \oplus \ldots \oplus b_mR$, and since aR is not semisimple we have for some $z \in \{1, \ldots, m\}$ that $b_z \notin Soc(R_R)$. Therefore $B_j = b_zR + Soc(R_R)$. Since $ann_r(a) \subseteq ann_r(b_z)$, then by the hereditary condition it follows that we have a splitting epimorphism $h : aR \to b_zR$, hence $aR = \ker h \oplus L$ with L a submodule of aR. From this sum we extrapolate that $(\ker h/soc(\ker h)) \oplus (L/Soc(L)) \cong (aR/Soc(aR))$. Now applying The Second Isomorphism Theorem we have $aR/Soc(aR) \cong (aR+Soc(R_R))/(Soc(R_R))$ and this is clearly isomorphic to $B_j/Soc(R_R)$. Since L is not semisimple and $B_j/Soc(R_R)$ is simple, we deduce that $\ker h$ is semisimple. Therefore $L + Soc(R_R) = B_j$. Again by the Second Isomorphism Theorem we obtain $(B_j/Soc(R_R)) = (L + Soc(R_R))/Soc(R_R) \cong L/Soc(L)$ and also $(L/Soc(L) \cong (b_zR/Soc(b_zR)) \cong (B_i/soc(R_R)$. We have just shown that any two simple singular modules are isomorphic, this means that a simple singular module is unique up to isomorphism. □

We now proceed to the main result of this section, whose proof we shall split into two propositions.

Theorem 2.2.14. [15, Theorem 2] *If R is a ring with no right middle class, then $R \cong S \times T$, where S is a semisimple Artinian ring, and T is zero or it belongs to one of the following classes:*

(a) *T is Morita equivalence to a right PCI-domain, or*

(b) *T is an indecomposable right SI-ring satisfying the following conditions:*

(i) *T is either a right Artinian or a right V-ring,*

(ii) *T has homogeneous essential right socle and*

(iii) *there is a unique simple singular right T-module up to isomorphism, or*

(c) *T is an indecomposable right Artinian ring satisfying the following conditions:*

(i) *$Soc(T_T) = Z(T_T) = J(T)$,*

(ii) T has homogeneous right socle, and

(iii) there is a unique non-injective simple right T-module up to isomorphism.

In (c) T is either a QF-ring with $J(T)^2 = 0$, or poor as a right module.

Proposition 2.2.15. [15, Proposition 3] *Let R be a ring with no middle class such that R is not right SI-ring. Then R is the direct sum of a semisimple Artinian ring S and a ring T satisfying the conditions of Theorem 2.2.14(c).*

Proof. Immediately applying Lemma 2.2.5(iii) we have that $R = S \oplus T$, where S is semisimple Artinian and T has essential right socle with $Z(T_T) = Soc(T_T)$. By Lemma 2.2.4 we know T has no middle class. Assume that $T \neq 0$. We have that T is indecomposable as a ring and that it must be right Artinian, by Lemmas 2.2.6 and 2.2.10 respectively.

Now let E be an injective T-module. We have seen in Proposition 1.6.2 that $Rad(E)$ is a fully invariant submodule of E. Thus by Proposition 1.1.25, $Rad(E)$ is quasi-injective. Since T is Artinian, then by Hopkins-Levitzki Theorem, it is also Noetherian. Therefore by Proposition 1.6.4, $Rad(E) \leq_s E$. Now by no middle class assumption $Rad(E)$ must be semisimple. Note that $J(T) \leq Rad(E(T_T))$ is also semisimple as a T-module. As $Z(T_T) = Soc(T_T)$, every simple right ideal of T_T is contained in $J(T)$. And we conclude that $J(T) = Z(T_T) = Soc(T_T)$.

Choose an arbitrary simple right ideal T_1 of T. Then clearly T_1 is also a singular right ideal, hence it is non-injective. Now let us verify uniqueness (up to isomorphism). Let T_2 be an arbitrary non-injective simple right T-module. Since T is a right Artinian ring, it admits a chain $T_2 \subseteq T_2' \subseteq E(T_2)$ of modules such that T_2 is maximal, and essential in T_2'. Since T_1 is non-injective it is poor, thus T_1 is a proper submodule of $\sum_{f \in Hom_R(T_2', E(T_1))} f(T_2')$ by Proposition 1.1.25. So there exists a T-homomorphism $f : T_2' \to E(T_1)$ such that $f(T_2') \nsubseteq T_1$. Hence T_1 is a proper submodule of $f(T_2')$, implying that the composition length of $f(T_2')$ is greater than one, which means that f is clearly a monomorphism. From this it is clear that $T_1 \cong T_2$. Therefore we have found a unique (up to isomorphism) non-injective simple right T-module and that $Soc(T_T)$ is homogeneous. □

Proposition 2.2.16. [15, Proposition 4] *Let R be a right SI-ring with no right middle class. Then R is the ring direct sum of a semisimple Artinian ring S and a ring T, which is Morita equivalent to a right PCI-domain or is as described in Theorem 2.2.14(b).*

Proof. Let us assume R is a right SI-ring with no right middle class. Now by Lemma 2.2.12 we have $R = S \oplus T$, where S is a semisimple Artinian ring, and T is a ring such that $Soc(T_T)$ is homogeneous (it can be zero). Since R has no middle class, neither does T, by Lemma 2.2.4. Assume $T = T_1 \oplus T_2$, where T_1 and T_2 are ideals of T such that T_1 is not a semisimple Artinian ring. Then every right ideal of T_2 is T_1-injective, as a T-module, and by the no middle class asusmption, it is injective. Therefore T_2 must be semisimple Artinian. This means that T cannot be decomposed into two rings that are not semisimple Artinian.

Since R is a right SI-ring, then T is also a right SI-ring (we are just restricting the condition from R to a subring T). Now by Proposition 1.7.5, and since T cannot be decomposed into two rings that are not semisimple Artinian, we infer that T is Morita equivalent to a right PCI-domain, or $T/Soc(T_T)$ is semisimple. The first case is exactly Theorem 2.2.14(a).

Assume $T/Soc(T_T)$ is semisimple. Hence T is right semiartinian. Now let us prove, by contradiction, that under these conditions T is indecomposable. So assume $T = T_1 \oplus T_2$, where $T_1, T_2 \neq 0$ are two-sided ideals. Since $Soc(T_T)$ is essential, then for each $i = 1, 2$, there is a simple right ideal U_i of T contained in T_i. Now we have $U_1 T_2 = 0$ and $V_2 T_2 = V_2$, but this would imply that $Soc(T_T)$ is not homogeneous, a contradiction. Therefore T must be indecomposable. Now by Proposition 2.2.11 we know that T is either a right V-ring or a right Artinian ring. If T is a semisimple Artinian V-ring we are done. So assume T is not semisimple Artinian. It follows by Lemma 2.2.13 that T admits a unique (up to isomorphism) simple singular right module, concluding the proof. □

Example 2.2.17. We will now take and develop a bit further examples given in [15] that illustrate the three possibilities of Theorem 2.2.14.

(i) By Proposition 2.2.2, it is clear that any right PCI-domain is in the conditions of Theorem 2.2.14(a).

(ii) Let $R = \begin{pmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{pmatrix}$, where $\mathbb{K}$ is a field and take $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Let M be a module such that M is A-injective, where A is a non-semisimple cyclic module. It is not hard to see that R is Artinian and serial. Then by Proposition 1.7.1 we have $A = A_1 \oplus \ldots \oplus A_n$, where A_i are cyclic uniserial modules. Thus each A_i is isomorphic to e_1R, e_2R, or $e_1R/Soc(e_1R)$. Since A is non-semisimple, then for some $i \in \{1, \ldots, n\}$ we have $A_i \cong e_1R$. Therefore M is injective. By the arbitrary choice of M we conclude that R has no middle class. Now R admits two simple R-modules, $e_2R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{K} \end{pmatrix}$ and $S = R/\begin{pmatrix} 0 & \mathbb{K} \\ 0 & \mathbb{K} \end{pmatrix}$, so $Soc(R_R) = \begin{pmatrix} 0 & \mathbb{K} \\ 0 & \mathbb{K} \end{pmatrix}$ is homogeneous and essential and e_2R is singular. Thus R satisfies Theorem 2.2.14(b).

(iii) Choose $R = \mathbb{Z}/p^2\mathbb{Z}$, for a prime p. It is straightforward to see that R has a unique maximal ideal, $J(R) = p\mathbb{Z}/p^2\mathbb{Z}$. This ideal is clearly simple (semisimple) and essential. Now $R/J(R) = \mathbb{Z}/p\mathbb{Z}$ is also simple (semisimple), hence R is semilocal. Then by Proposition 2.2.3, R is a ring with no middle class in the conditions of Theorem 2.2.14(c).

We have proven the main result of this section. We will end this section by showing that a commutative ring with no middle class is Artinian. We shall skip the proof of most of the remaining results, whose purpose is to prove Theorem 2.2.22.

Lemma 2.2.18. [15, Proposition 7] *Let R be a right Artinian ring with unique (up to isomorphism) local module of length two, and homogeneous $Soc(R_R) = J(R)$. Then R has no (right) middle class. In particular R is in the conditions of Theorem 2.2.14(c).*

Lemma 2.2.19. [27, Corollary 2.14] *Let R be a right Artinian ring. Then, R has no right middle class if and only if $J(R)$ contains no non-trivial ideal of R.*

Lemma 2.2.20. [5, Lemma 2.4] *Suppose that we have a ring of the form $R = S \oplus T$, a direct sum of two rings, where S is semisimple Artinian. Then R has no (simple) middle class if and only if T has no (simple) middle class.*

The following proposition shows us the strength of the no middle class condition.

Proposition 2.2.21. [5, Proposition 4.2] *If R is a commutative Notherian ring with no middle class, then R is Artinian.*

Remark. The purpose of [9] is self-explanatory by its title "A right PCI ring is right Noetherian". We shall use the fact that a ring that is Morita equivalent to a PCI–ring (domain) is a Noetherian ring (domain).

Theorem 2.2.22. [5, Theorem 4.3] *A commutative ring R has no middle class if and only if there is a ring decomposition $R = S \oplus T$, where S is semisimple Artinian and T is zero or a local ring whose maximal ideal is minimal.*

Proof. ($\Rightarrow$) Under these conditions, we apply Theorem 2.2.14. So we have a ring decomposition $R = S \oplus T$, with S semisimple Artinian and T is either zero (we would be done if that were the case), or it fits one of the following cases.

(i) T is Morita equivalent to a right PCI-domain T'. In this case, T' is a right Noetherian domain by [9]. Now, since R is commutative with no middle class, by the previous proposition T' is also Artinian, which implies it is a simple ring. A commutative simple ring is a field, concluding the proof.

(ii) T is an indecomposable SI-ring, which is either Artinian or a V-ring. Let us start by assuming T is Artinian. Then T is a finite product of local rings. Now the indecomposable hypothesis allows us to infer that T is a commutative Artinian local ring. If T is not a field, then there is a minimal non-zero ideal I of T, and by the local hypothesis $I \cong T/J(T)$ is a division ring. Since T is an SI-ring and $T/J(T)$ is singular as a T-module, it follows by the SI-ring hypothesis that I is injective, meaning that I is a direct summand of T. However this contradicts the indecomposable assumption of T. So, under these conditions, as T is Artinian it would have to be a field. Now assume T is a V-ring, and without loss of generality that T is not Noetherian. Note that, if it was Noetherian we would just be going back to Case (i). Since we have no middle class we infer that T is semiartinian, by Proposition 2.2.8. Let I be a non-zero minimal ideal of T. Now since $J(T) = 0$, then T admits a maximal ideal $\mathcal{M}$ which does not contain I. Thus T is a sum of the form $I \oplus \mathcal{M}$, but since T is indecomposable we conclude that $T = I$. Therefore T is a field.

(iii) T is an indecomposable Artinian ring, with $Soc(T_T) = J(T)$. In this case, R is once again a local ring. Then we know by Corollary 2.2.19, that the maximal ideal $J(T)$ is also minimal.

($\Leftarrow$) For the converse, assume T is a commutative local ring whose maximal ideal is simple, then T has a unique (up to isomorphism) local module of length two, which is itself. Furthermore, it also has homogeneous socle, which is equal to $J(T)$. Now from Proposition 2.2.18, it follows that T has no middle class. Finally we apply Lemma 2.2.20 and we are done. $\square$

We have already seen that a no middle class ring is either Noetherian or semiartinian, in the latter V-ring or Artinian. So by the previous Theorem and Proposition 2.2.21 we have the following.

Corollary 2.2.23. [5, Corollary 4.4] *Any commutative ring R with no middle class is Artinian.*

Chapter 3

Pauper Modules

Throughout the previous chapter we have developed the concept of poor module. Furthermore, we have studied several classes of modules regarding their injectivity domains. Now we seek to do the same for indecomposable poor modules. Most of what we will do in this chapter is seeing which classes of modules satisfy certain conditions.

As in Chapter 2, please note that some authors use the term "semisimple" to mean the ring has a trivial Jacobson radical. For Artinian rings, Proposition 1.6.9 guarantees that this notion coincides with the notion of semisimple we introduced in Chapter 1. Therefore, we will use "semisimple Artinian ring" to eliminate that ambiguity, as in the papers referenced.

Unless otherwise stated the results in this chapter follow from [3].

3.1 Definitions and general results

We have seen in Corollary 2.1.5 that the sum of a poor module with any arbitrary module is itself poor, meaning there may not be much interest in the study of these types of sums. Therefore the need of an intrinsic characterization of poor modules, and the notion of pauper module arises.

Definition. A module is said to be *pauper*, if it is poor and no proper direct summand of it is poor.

Remark. An indecomposable poor module is pauper. However this does not mean a pauper module is necessarily indecomposable, or that it even has finite dimension. In fact, Examples 3.1.4(i) and (ii) are counterexamples of just that.

We want to characterize rings and different classes of modules in terms of their (or lack thereof) pauper modules.

Let R be a ring and $\mathcal{A}$ a class of right R-modules.

- (Existence) We say the class of modules $\mathcal{A}$ satisfies (E) if it contains pauper modules.
- (Ubiquity) We say the class of modules $\mathcal{A}$ satisfies (U) if for every poor module P in $\mathcal{A}$ there exists a pauper module M in $\mathcal{A}$ such that $M \subseteq^{\oplus} P$, i.e., M is a direct summand of P.

Finding out if a class of modules satisfies (U) is in general not something easy to accomplish. In light of that we define a weaker property (U') that under certain conditions is equivalent to (U) as we will see.

- A class of modules $\mathcal{A}$ satisfies (U') if for every poor module P in $\mathcal{A}$ there exists a pauper $M \in \mathcal{A}$ such that M is a pure submodule of P (see Section A.2 for the definition of pure submodule).

When $\mathcal{A}$ =Mod-R, we say that R satisfies (E), (U) or (U') when $\mathcal{A}$ does.

We have shown in Proposition 2.1.2 that every ring has at least one poor module. The next question seems to be if all rings have at least one pauper module. The answer is no, as shown in the following counterexample.

Proposition 3.1.1. *Let R be a (non-semisimple) right semiartinian right V-ring. Then R has no pauper right module.*

Proof. By Theorem 2.1.2 we know there exists a a poor R-module M. Since R is semiartinian we have $Soc(M) \neq 0$. Take a simple submodule N of M. By the V-ring hypothesis, N is injective, hence $M = N \oplus N'$ for some $N' \leq M$. Furthermore, N' is poor, by Lemma 2.1.6. We have just shown that any poor R-module contains a proper poor direct summand. Therefore R has no pauper modules. □

Corollary 3.1.2. *Let R be a (non-semisimple) right V-ring. If R has a pauper module M, then $Soc(M) = 0$. Furthermore, every semiartinian right module is semisimple.*

Proof. Let M be a pauper module. We must have $Soc(M) = 0$, otherwise M would not be pauper as we have shown in the previous proposition. Take an arbitrary semiartinian module N and K a submodule of N. Clearly $Hom_R(K, M) = 0$. Therefore M is N-injective. Since M is poor, then N is semisimple. □

The following theorem is fundamental in the study of pauper abelian groups. Its proof will be done in Section 3.5, where it is more thematically appropriate. Recall that $\mathbb{P}$ denotes the set of prime numbers.

Theorem 3.1.3. [2, Theorem 3.1] *An abelian group G is poor (as a $\mathbb{Z}$-module) if and only if its torsion part $T(G)$ has a direct summand isomorphic to $\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$.*

Example 3.1.4. Going forward we will represent $\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$ by G_0.

(i) It is clear that $T(G_0) = G_0$, by Proposition 1.4.2. Therefore G_0 is poor by the previous theorem, and pauper since no direct summand of it is poor.

(ii) Define $G = \prod_{p\in\mathbb{P}} \mathbb{Z}_p$, again by Proposition 1.4.2 and the previous theorem, we infer that $T(G) = G_0$, and G is poor. Now assume that $G = H \oplus K$, for some poor submodule $H \leq_{\mathbb{Z}} G$. Once again by Theorem 3.1.3, G_0 is contained in H. Furthermore, $H \cap K = 0$, implies that $K \cap G_0 = 0$. We have $G/G_0 \cong \bigoplus_{i\in I} \mathbb{Q}_i$ divisible and $Rad(G) = 0$. Now $G/G_0 \cong (K \oplus H)/G_0 \cong K \oplus (H/G_0)$. Therefore K is also divisible. This means that for every $n \in \mathbb{Z}\backslash 0$ we have $nK = K$. Hence $Rad(K) = K$. Since $Rad(K) = K \subseteq Rad(G) = 0$, we infer that $K = 0$. Finally we conclude that $G = H$ is pauper.

(iii) Take the ring $R = \mathbb{Z}_{30}$. Since $\mathbb{Z}_{30}$ is finite, then it is clearly Artinian. It is also clear that its maximal ideals are $\langle 2\rangle$, $\langle 3\rangle$ and $\langle 5\rangle$. Therefore $J(\mathbb{Z}_{30}) = \{0\}$, and by Proposition 1.6.9, we conclude $\mathbb{Z}_{30}$ is semisimple. Now $\mathbb{Z}_{30}$ is semisimple Artinian, then the trivial module $\{0\}$ is poor, by Proposition 2.2.1. It is also clear that $\{0\}$ is indecomposable, so that it is pauper. This falls under a general case. A ring R is semisimple Artinian if and only if $\{0\}$ is pauper. Therefore semisimple Artinian rings satisfy (U).

Remark. By the previous theorem the ring of integers satisfies condition (U').

Let us see more classes of modules that satisfy (E).

Proposition 3.1.5. *Over arbitrary rings, finite uniform dimensional poor modules have pauper direct summands.*

Proof. Let M be a finite uniform dimensional poor module and assume M is not pauper. Then there exist non–zero submodules $N_1, K_1 \leq M$ such that $M = N_1 \oplus K_1$, where N_1 is a poor module. If N_1 is pauper we are done, otherwise there exist non-zero submodules $N_2, K_2 \leq N_1$ such that $N_1 = N_2 \oplus K_2$, with N_2 poor. As $u.dim(M) < \infty$, by Corollary 1.3.7, M does not contain an infinite direct summand. This means that after a finite number of repetitions of this process, we must reach a pauper submodule of M. □

Proposition 3.1.6. *Let R be a semilocal ring. Then the right R-module $R/J(R)$ has a pauper direct summand.*

Proof. We have already seen in Proposition 2.1.9 that $R/J(R)$ is poor. Now by the semilocal hypothesis, $R/J(R)$ is semisimple. Hence it has finite uniform dimension, which by the previous proposition lets us conclude that $R/J(R)$ admits pauper direct summands. □

Example 3.1.7. In light of the two previous propositions.

(i) By Proposition 1.3.8(a), Noetherian (resp. Artinian) modules have finite uniform dimension, hence Noetherian (resp. Artinian) poor modules admit pauper direct summands.

(ii) Take $R = \mathbb{Z}$ and $M = \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}_m$ $(m > 0)$. In this case the uniform dimension of M is characterized by the number of distinct prime divisors of m. This means the $\mathbb{Z}$-module $\mathbb{Z}_m$ that is poor, admits pauper direct summands.

(iii) We have already seen that $\mathbb{Z}_{30}/J(\mathbb{Z}_{30}) \cong \mathbb{Z}_{30}$ is semisimple, hence semilocal. In general, any finite ring or any finite dimension algebra over a field $\mathbb{K}$ is semilocal. Therefore, it admits pauper direct summands.

(iv) If R is a semilocal ring, then $A = \mathbb{M}_n(R)$ is also semilocal. This follows from the fact that $Rad(A) = \mathbb{M}_n(Rad(R))$. So now we get

$$A/Rad(A) \cong \mathbb{M}_n(R/Rad(R)).$$

By hypothesis, $R/Rad(R)$ is semisimple. So $\mathbb{M}_n(R/Rad(R))$ is also semisimple. Therefore A is semilocal, thus $A/J(A)$ admits pauper direct summands, by Proposition 3.1.6.

The following theorem is a strong indicator of the significance of the existence of pauper modules.

Theorem 3.1.8. *Let R be a ring with no right middle class. The following statements are equivalent:*

(a) *R is right Noetherian.*

(b) *R has a pauper right module.*

(c) *Every non-injective right R-module has a pauper submodule.*

(d) *Every non-injective right R-module has a cyclic pauper submodule.*

Proof. (d) ⇒ (c) ⇒ (b) are immediate.

(b) ⇒ (a) Assume that R is not right Noetherian. Then by Proposition 2.2.8, R must be right semiartinian. Furthermore, by Proposition 2.2.11, R is either a V-ring or Artinian. However R cannot be right Artinian, because this would imply R is right Noetherian, by the Hopkins-Levitzki's Theorem. Therefore R is a right semiartinian right V-ring. In Proposition 3.1.1 we have shown that this specific class of rings does not satisfy (E), a contradiction. Thus R must be Noetherian.

(a) ⇒ (d) Assume R is right Noetherian, and let M be a non-injective R-module. By Proposition 1.2.5, we may assume that M does not have any non-zero injective submodule. Let $0 \neq N \leq M$ be a cyclic submodule of M. Now, by the no right middle class hypothesis, N must be poor. It is clear that N has finite uniform dimension, so by Proposition 3.1.5, N has a pauper direct summand, which must be cyclic, since it is a submodule of a cyclic module. □

Theorem 3.1.8 has the following clear consequence.

Theorem 3.1.9. *Let R be a right V-ring with no right middle class. The following are equivalent:*

(a) *R is semisimple Artinian.*

(b) *R has a pauper right R-module.*

(c) *Every right R-module contains a pauper submodule.*

The following is a Corollary combining Theorem 2.2.14 and the previous two theorems.

Corollary 3.1.10. *Let R be a ring with no right middle class. Then R has a pauper right module if and only if $R = S \times T$, where S is semisimple and T is Noetherian.*

3.2 Rings with no indecomposable middle class

It seems natural to combine what was done in Section 2.2 with the indecomposable condition.

If R has no right middle class, a module M is pauper if and only if M is indecomposable and non-injective.

Definition. A ring R has *no indecomposable right middle class* if, every indecomposable right R-module is poor or injective.

Remark. From the definition it is clear that a ring with no (right) middle class is, in particular, a ring with no indecomposable (right) middle class.

Remark. A semisimple V-ring is a somewhat trivial example of a ring with no indecomposable middle class, since over a semisimple ring every indecomposable module is simple. We have the following weaker condition.

Note that indecomposable modules are simple and injective, so we have the following result.

Proposition 3.2.1. *Let R be a right semiartinian V-ring. Then R is a ring with no indecomposable middle class.*

Similar to Lemma 2.2.4 we have.

Proposition 3.2.2. *Let R be a ring with no indecomposable right middle class and I an ideal of R. Then the ring R/I has no indecomposable right middle class.*

Proof. Let M be an indecomposable right (R/I)-module. Then M is also indecomposable, as a right R-module. Since by hypothesis R has no indecomposable right middle class, then M_R is either injective or poor, thus $M_{R/I}$ is injective or poor. $\square$

Lemma 3.2.3. *Let R be a commutative ring, and A and B be R-modules of composition length two and isomorphic simple socle V. Then $A/V \cong B/V \cong V$ and, in fact, $A \cong B$. Moreover, A is A-injective.*

Proof. Since A has a chain of length two, it is of the form $\{0\} \subsetneq K \subsetneq A$ such that K is simple, so $K = Soc(A) \cong V$. This means that A is not a semisimple R-module, otherwise the socle of A would be itself. The same is valid for B. Assume that $A/V \cong U$ and $U \ncong V$, for some simple submodule U of A. Define $P = ann_R(V)$ and $Q = ann_R(U)$. Now we have $PQA = 0$ and A is a (R/PQ)-module. Also note that R/P and R/Q are simple (R/PQ)-modules, so that $R/PQ \cong (R/P) \oplus (R/Q)$ is semisimple as an (R/PQ)-module, and by Proposition 1.1.23(c), A is also a semisimple (R/PQ)-module. However this would imply that A semisimple as a R-module, which is a contradiction. Hence $A/V \cong V$, we have $V \leq_e A$ and $V \leq_e B$, and by Proposition 1.6.4, $Rad(A) = V \leq_s A$ and $Rad(B) = V \leq_s B$. This implies that $A = aR$ and $B = bR$, for some $a \in A$ and $b \in B$, in other words A and B are cyclic modules. Clearly $ann_R(A) = ann_R(B)$ and we conclude that $A = aR \cong R/I \cong bR = B$ for some ideal I of R. Hence $A \cong B$.

We are left to prove that A is A-injective. It is enough to show that for a submodule X of A such that $X \cong V$, then any homomorphism $f : X \to A$ can be extended to a homomorphism $g : A \to A$. Assume without loss of generality, that $X = V$. We have already seen that A and V are cyclic modules, which we represent by $A = aR$ and $V = xR$. Now since $f(V) \leq V$, then for some $s \in R$ we have $f(x) = xs$. Therefore it is enough to define $g : A \to A$, by $g(b) = bs$, for each $b \in A$. Thus $g(x) = xs = f(x)$, so g extends f. $\square$

Lemma 3.2.4. *For a simple module V over a commutative Noetherian ring, the properties of injectivity, projectivity and flatness are equivalent.*

Proof. The necessity of the commutative condition follows from [36, Lemma 2.6], which states that if V is a simple module, then V is flat if and only if it is injective. Note that finitely presented modules are in particular finitely related, so combining Lemmas A.2.4 and A.2.5 we conclude that for a Noetherian ring, flatness and projectivity are equivalent. $\square$

Theorem 3.2.5. *Let R be a commutative Noetherian ring. Then R has no indecomposable middle class if and only if $R = S \times T$, where S semisimple Artinian and $T = 0$ or T is a local ring whose maximal ideal is minimal. In other words, R is the direct product of finitely many fields and at most one ring of composition length two.*

Proof. Let us start with the "Only if" part. Assume R has no indecomposable middle class. Take $S = \sum_{i \in I} J_i$, where J_i are the injective minimal ideals of R. Since R is Noetherian, then by Proposition 1.2.5, S is injective. Hence $R = S \oplus T$, for some ideal T of R. Suppose $T \neq 0$, then T cannot be semisimple, otherwise it would be contained in S.

Let us show that T is a local ring. By contradiction, assume T is not local. By Lemma 3.2.4 and $S \cap T = 0$, we infer that T has no injective (projective) simple factor. Take K_1 and K_2 two distinct maximal ideals of T. Thus $U = T/K_1$ and $U' = T/K_2$ are non-injective simple modules. So by the no indecomposable middle class assumption, U and U' must be poor. Now by Proposition 1.5.6, the injective hull of U is Artinian, hence $Soc(E(U)/U) \neq 0$, which means there exists $V \leq E(U)$ such that V/U is simple. We also know U and U' are not isomorphic, which by Proposition 1.1.4 implies $Hom(U, U') = 0$. So U' is V-injective, contradicting the poorness of U'. Therefore T is a local ring.

Let U be the unique simple T-module and V as defined above. Then by Lemma 3.2.3, V is V-injective. Furthermore, since V indecomposable, it must also be injective, by hypothesis. Since $E(U)$ is the smallest injective module containing U, we have $V = E(U)$. Now we want to show that T is Artinian. Since T is Noetherian, by Propositon 1.5.5, we know it is enough to see that T is semiartinian. Again we shall do this by contradiction. Assume $Soc(M) = 0$, with M_T a cyclic module. Furthermore, consider M to be a non-injective module (note that if M is injective we can take a non-injective cyclic submodule of M). Since M is generated by a single element, then by the Noetherian hypothesis, it must be indecomposable as well, which by hypothesis, means that M is poor. However, we also know that $Hom(V, M) = 0$, because $Soc(M) = 0$, which implies that M is V-injective. Thus M is injective, a contradiction because M is poor. Therefore T is semiartinian, hence Artinian.

Now T is Artinian, so by Proposition 1.5.6 it follows that $Soc(T_T)$ is finitely generated, so $Soc(T_T) \cong U^n$, for some $n \in \mathbb{N}$. We have shown in Proposition 1.5.3 that $Soc(T_T)$ is essential, i.e., $Soc(T_T) \leq_e T \leq V^n$. Henceforth, the quotient module $T/Soc(T_T)$ can be embedded in $(V/U)^n$, which is clearly semisimple, thus $T/Soc(T_T)$ is semisimple. Since T is local and $V/U \cong U$, then $T/Soc(T_T) \cong U$, therefore $J(T) = Soc(T_T)$. Now by this equality and Lemma 1.6.6(c) we have $0 = J(T).Soc(T_T) = (Soc(T_T))^2$. We have already seen that V is non-semisimple and has composition length two, so it is cyclic, $V = vT$ with $v \in V$. Note that, $V = vT \cong T/ann_R(v)$, so if we manage to prove that $ann_R(v) = 0$, then $V \cong T$. Let $t \in T \backslash 0$ such that $tv = 0$. A left zero divisor can never be a unit, so $t \in Soc(T)$. As T is local and its socle is homogeneous, $tT = U^n$, for some $n \in \mathbb{N}$. Therefore $Uv = 0$, and $Soc(T).v = 0$. Hence V is a $T/Soc(T)$-module. Now since $T/Soc(T)$ is simple, V is semisimple, which is a contradiction. Thus $ann(v) = 0$ and so $f : T \to V$, $f(t) = vt$ is an isomorphism.

Let us prove the "If" part. Take M a non-injective and indecomposable right R-module. So by hypothesis $M = M_1 \oplus M_2$, where M_1 is an S-module and M_2 is a T-module. Now since M is indecomposable and non-injective, M_1 must be trivial. Hence $M = M_2 \cong U$, where U is the unique (up to isomorphism) simple T-module. Clearly every local ring is in particular semilocal, by Proposition 3.1.6 we conclude that M is poor and indecomposable. ☐

The following two results establish that for commutative Noetherian rings and Artinian serial rings, the "no middle class" condition is equivalent to the "no indecomposable middle class" condition.

Corollary 3.2.6. *Let R be a commutative Noetherian ring. Then R has no middle class if and only if R has no indecomposable middle class.*

Proof. By the previous theorem and Theorem 2.2.22. ☐

The proof of the following theorem will not be done here. It is the combination of several results of [15] and [27, Corollary 3.2].

Lemma 3.2.7. [5, Theorem 3] *Let R be a ring. Then R has no right middle class if and only if $R \cong S \oplus T$, where S is semisimple Artinian and T satisfies one of the following conditions:*

(a) *T is Morita equivalent to a right PCI-domain, or*

(b) *T is a right SI-ring, V-ring with the following properties:*

(i) T *has essential homogeneous right socle and*

(ii) *for any submodule* A *of* Q_T*, which does not contain the right socle of* T *as a proper submodule,* $QA = Q$*, where* Q *is the maximal right quotient ring of* T, *or*

(c) T *is a right Artinian ring, such that* $J(T)$ *properly contains no non-zero ideals.*

Theorem 3.2.8. *Let* R *be an Artinian serial ring. The following characterizations are equivalent:*

(a) R *has no indecomposable middle class.*

(b) R *has no right middle class.*

(c) $R = S \times T$, *where* S *is semisimple Artinian and* T *is a ring such that* $J(T)^2 = 0$.

Proof. (a) $\Rightarrow$ (b) Take a non-injective right module M. Since R is a serial ring, Proposition 1.7.1 gives us a characterization of the form $M = \oplus_{i \in I} U_i$, with U_i uniserial modules for $i \in I$. As M is non-injective, then for some $i_0 \in I$ we must have some U_{i_0} non-injective. Thus by no indecomposable middle class assumption U_{i_0} is poor, which by Corollary 2.1.5, means that M is poor.

(b) $\Rightarrow$ (a) Evident.

(b) $\Leftrightarrow$ (c) Follows from Lemma 3.2.7. □

Proposition 3.2.9. [8, Proposition 5.7] *Let* R *be a right semiartinian ring with no simple middle class. Then* R *is a right V-ring or, there is a ring direct sum* $R = S \oplus T$, *where* S *is semisimple Artinian and* T *has a unique non-injective simple right module up to isomorphism, and* $Soc(T)$ *is homogeneous.*

Proof. Let us assume R is not a V-ring. Take U_R to be a non-injective simple module. By the hypothesis of no simple middle class, U is poor. Now by a similar argument to the one done in Proposition 2.2.15, we infer that R has a unique non-injective simple right module (up to isomorphism). Let S be the sum of the injective simple right ideals of R. By contradiction, assume that S is not injective, and consider its injective hull $E(S)$. By semiartinian hypothesis it follows that $Soc(E(S)/S) \neq 0$. Now take X/S to be a simple submodule of $E(S)/S$. Note that $S \leq_e X$, by Proposition 1.1.10(1). Let us show that U is X-injective. Choose $0 \neq Y \leq X$, and let $f : Y \to U$ be a homomorphism. If $Y \leq S$, then since Y is semisimple, and U is poor (non-injective), so $Hom_R(Y, U) = 0$. Therefore f extends to X trivially. Now if $Y \nsubseteq S$, since S is a maximal submodule of X, it follows that $Y + S = X$, and S is semisimple, so $S = Y \cap S \oplus S'$, where S' is a submodule of S. Since Y does not intersect S it follows that $X = Y \oplus S'$. Considering the projection $\pi : X \to Y$ we infer that $f\pi : X \to U$ extends f. Hence U is X-injective. However we have already seen that U is poor. Thus X is semisimple. so S is a direct summand of X. But this implies that S is not essential in X, a contradiction. Thus $R = S \oplus T$, for some right ideal T of R. By the choice of S we have that $Hom(S, T) = 0$ and $Hom(T, S) = 0$. Therefore, both S and T are two-sided ideals, hence $R = S \oplus T$ is a ring direct sum. Since R admits a unique non-injective simple module, T also has a unique non-injective simple right module, which implies that $Soc(T_T)$ is homogeneous. □

Theorem 3.2.10. *Let* R *be a right Artinian ring. Suppose* R *has no indecomposable middle class. Then* $R \times S$, *where* S *is semisimple Artinian,* $Soc(T_T)$ *is homogeneous,* $J(T) = Soc(T_T)$ *and* $J(T)^2 = 0$.

Proof. Let R be a right Artinian ring with no indecomposable middle class. By the previous proposition we have $R = S \times T$, where S is semisimple Artinian and T has a unique non-injective simple module, and $Soc(T_T)$ is homogeneous. Consider U to be the non-injective simple T-module and take $E(U)$ to be its injective hull. By the proof of the previous proposition we are able to infer that no simple submodule of T is a direct summand of T. Therefore $Soc(T_T) \leq J(T)$. Now by the Artinian hypothesis, $Soc(E(U)/U) = A/U \neq 0$. Then by Propositions 1.5.7 and 1.1.25, we infer that A is quasi-injective. Since A is indecomposable in R, then A must be injective. Hence $A = E(U)$, and $E(U)/U$ is semisimple. Again by the Artinian condition, the $Soc(T_T)$ is finitely generated, thus $Soc(T_T) \cong U^n$, for some $n \in \mathbb{N}$. Therefore $T/Soc(T_T)$ can be embedded in the semisimple module $(E(U)/U)^n$, which means $T/Soc(T_T)$ is also semisimple, so $J(T)$ is a submodule of $Soc(T_T)$. Therefore $J(T) = Soc(T_T)$ and $(J(T))^2 = 0$. $\square$

3.3 Over Noetherian Rings

We have seen in Corollary 2.2.23 that commutative rings with no middle class are Artinian, hence Noetherian, by the Hopkins–Levitzki Theorem. The two major result of this section are Theorems 3.3.2 and 3.3.10. The first establishes that for Noetherian rings the conditions (U) and (U') are equivalent, while the latter shows that commutative hereditary Noetherian rings satisfy (U').

The following lemma is valid over any ring.

Lemma 3.3.1. [21, Lemma 8.4] *Let R be any ring and A, B, C, H, F, M be right R-modules. Suppose*

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "\gamma"] \arrow[d, "\phi"'] & F \arrow[r, "\delta"] \arrow[d, "\psi"'] \arrow[dl, dotted, "\sigma"'] & M \arrow[r] \arrow[d, "\eta"] \arrow[dl, dotted, "\rho"'] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram with exact arrows. There exists a map $\sigma : F \to A$ making the upper triangle commute (i.e., $\sigma\gamma = \phi$) if and only if there is a map $\rho : M \to B$ making the lower triangle commute (i.e., $\beta\rho = \eta$).

Proof. $(\Rightarrow)$ Assume the upper triangle commutes. Then $(\psi - \alpha\sigma)\gamma = \alpha\phi - \alpha\phi = 0$ implies there exists a homomorphism $\rho : M \to B$ such that $\rho\delta = \psi - \alpha\sigma$. Hence $\beta\rho\delta = \beta\psi - \beta\alpha\sigma = \eta\delta$, thus $\beta\rho = \eta$.

$(\Leftarrow)$ Now assume $\beta\rho = \eta$. Then for $\tau = \psi - \rho\delta$ we have $\beta\tau = \beta\psi - \beta\rho\delta = \eta\delta - \eta\delta = 0$. Now there exists a homomorphism $\sigma : F \to A$ such that $\alpha\sigma = \tau$. Therefore $\alpha\sigma\gamma = \psi\gamma - \rho\delta\gamma = \alpha\phi$, hence $\sigma\gamma = \phi$. $\square$

Theorem 3.3.2. *Let R be a right Noetherian ring and M be a right R-module. If P is a pure submodule of M and P is poor, then M is poor.*

Proof. Assume that M is N-injective, with N an arbitrary cyclic module. It is enough to prove that P is also N-injective. Let N_1 be a submodule of N, and take a homomorphism $f : N_1 \to P$. Now by definition of N-injective, we can take a homomorphism $g : N \to M$ such that $g_{|N_1} = f$. Hence we have the following diagram where both rows are exact

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] & M \arrow[r, "\pi_1"] & M/P \arrow[r] & 0 \\
0 \arrow[r] & N_1 \arrow[r] \arrow[u, "f"] & N \arrow[r, "\pi_2"'] \arrow[u, "g"] \arrow[ul, dotted, "v"] & N/N_1 \arrow[r] \arrow[u, "h"'] \arrow[ul, dotted, "u"] & 0.
\end{tikzcd}$$

Since R is Noetherian and N is cyclic, then N_1 must be finitely generated. Thus N/N_1 is finitely presented. Furthermore, the top row is pure exact, so by Lemma A.2.3 there exists a homomorphism

$u : N/N_1 \to M$ such that $\pi_1 u = h$. Now by the previous lemma, there exists a homomorphism $v : N \to P$ such that $v_{|N_1} = f$. Therefore P is N-injective. $\square$

In a commutative Noetherian ring, the irredundant complete sum of non-injective simples is pauper.

Proposition 3.3.3. *Let R be a commutative Noetherian ring and Γ be a complete set of representatives of non-injective simple R-modules. Then $S = \oplus_{S_i \in \Gamma} S_i$ is pauper. Moreover, for any poor R-module M, the singular submodule, $Z(M)$ contains a copy of S.*

Proof. Let us start by proving that S is poor. Suppose that S is A-injective, with A a cyclic module. Then we have $A \cong R/I$ where I is some ideal of R. Let S^* be the direct sum of a complete set of representatives of non-injective simple R/I-modules, By definition S^* is isomorphic to some direct summand of S. Since S is A-injective, S^* is also A-injective, thus S^* is R/I-injective, a contradiction, unless $S^* = 0$. If R/I has no non-injective simple submodules, then R/I must be a commutative Noetherian V-ring. Therefore by Proposition 1.1.11, R/I is semisimple. Hence A is semisimple, and we can infer that S is poor.

Now let us show that S is pauper. Take a proper direct summand N of S. So there exists a non-injective simple R-module T such that $Hom_R(T, N) = 0$. By Proposition 1.5.6 we know that the injective hull of T is Artinian. Since T is non-injective, then $E(T)$ must contain a submodule B of composition length two. Then B is not semisimple and N is B-injective. Thus N is not poor, hence S is pauper.

Finally let M be a poor R-module. By an analogous argument done in the previous paragraph, $Hom_R(V, M) \neq 0$, for some non-injective simple module V. Now by Lemma 3.2.4, V is not projective, so $V \subseteq Z(M)$, thus $S \subseteq Z(M)$. $\square$

Corollary 3.3.4. *Let R be a commutative Noetherian ring. Then any module N such that $S = \oplus_{S_i \in \Gamma} S_i \leq N \leq \prod_{S_i \in \Gamma} S_i$ is poor.*

Proof. Let N be in the conditions described above. We have seen in Proposition 3.3.3, that S is poor. Furthermore, we also know that S is a pure submodule of $\prod_{S_i \in \Gamma} S_i$. Thus S is also pure submodule of N. Then by Theorem 3.3.2 we are done. $\square$

Lemma 3.3.5. *Let R be a commutative Noetherian ring and $A = \{S_i\}_{i \in I}$ be a complete set of representatives of non-isomorphic simple R-modules. Then the module $M = \prod_{i \in I} S_i / \oplus_{i \in I} S_i$ has no maximal submodules, i.e., $Rad(M) = M$.*

Proof. Take $K = \oplus_{i \in I} S_i$, and let $\mathcal{I}$ be an arbitrary maximal ideal of R. Then for some $S_i \in A$ we have $\mathcal{I} = ann_R(S_i)$, and $\mathcal{I} S_j = S_j$ for all $S_j \in A$ such that $S_i \ncong S_j$. Therefore $\mathcal{I}(\prod_{i \in I} S_i) = \prod_{j \neq i} S_j$. Now by the Second Isomorphism Theorem, $M = (\prod_{i \in I} S_i + K)/K$, thus $\mathcal{I} M = (\prod_{j \neq i} S_j + K)/K = M$, and $Rad(M) = \bigcap QM = M$ where Q ranges over the set of maximal ideals of R. Hence M has no maximal ideals. $\square$

Proposition 3.3.6. *Let R be a commutative hereditary Noetherian ring, and let $\{S_i\}_{i \in I}$ be a complete set of representatives of non-isomorphic simple R-modules. Then any module N such that $\oplus_{i \in I} S_i \leq N \leq \prod_{i \in I} S_i$ and $Rad(N/(\oplus_{i \in I} S_i)) = N/(\oplus_{i \in I} S_i)$ is pauper. In particular, $\prod_{i \in I} S_i$ is pauper.*

Proof. Since all $S_i's$ are simple, then $Rad(\prod_{i \in I} S_i) = 0$. Assume $N = A \oplus B$, where A is poor. By Proposition 3.3.3, it follows that $\oplus_{i \in I} S_i \subseteq A$. Now let $\pi : N \to B$ be the natural projection. Then $\oplus_{i \in I} S_i \subseteq \ker \pi$. So we have the epimorphism $\bar{\pi} : N/(\oplus_{i \in I} S_i) \to B$. Now by Proposition 1.6.2 and hypothesis it follows that $\operatorname{im} \bar{\pi} \subseteq Rad(B) = 0$, thus $B = 0$. Therefore N is pauper. For the last statement, we know by Lemma 3.3.5 that $\prod_{i \in I} S_i/(\oplus_{i \in I} S_i)$ has no maximal submodules, i.e, $Rad(\prod_{i \in I} S_i/ (\oplus_{i \in I} S_i)) = \prod_{i \in I} S_i/(\oplus_{i \in I} S_i)$. $\square$

Proposition 3.3.7. *Let M be a right R-module. Suppose that for every non-injective simple module V, M has a direct summand isomorphic to V. Then M has a pure submodule isomorphic to S, where S is the irredundant complete direct sum of non-injective simple R-modules.*

Proof. Let U and V be two non-isomorphic, non-injective simple R-modules, that are direct summands of M. So $M = U \oplus N$, with $N \leq M$. Now define the natural projection $\pi : M \to U$, where $\ker \pi = N$. Since $U \ncong V$, then by Proposition 1.1.4, $\pi(V) = 0$, hence $V \leq N$. Now for some submodule Y of M, we have that $M = V \oplus Y$ and $N = (U \oplus Y) \cap N = V \oplus (Y \cap N)$, so V is a direct summand of N. Therefore $M = U \oplus V \oplus K$, for some $K \leq M$. Furthermore, we can extend this argument, by induction, to a finite set of simple all non-isomorphic submodules of M. Let $\{U_i\}_{i\in I}$ be a complete set of representatives of the non-injective simple R-modules of M, with each U_i a direct summand of M. Therefore for every finite $J \subseteq I$ we have $N_J = \oplus_{i\in J} U_i$ is a direct summand of M, hence a pure submodule of M as well. As N_J is pure, then $S = \oplus_{i\in I} S_i = \underset{\rightarrow}{lim} N_J$ is pure, since the direct limit of a pure submodule is itself pure. ☐

Lemma 3.3.8. *Let R be a commutative hereditary Noetherian ring. Let M be an R-module and V a simple submodule of M. The following are equivalent.*

(a) *V is a closed submodule in M.*

(b) *$QV = V \cap QM$ for each maximal ideal Q of R.*

(c) *V is a direct summand of M.*

Proof. (a) $\Leftrightarrow$ (b) In the effort of not introducing even more notation we refer the reader to [35, Theorem 4.5.1].

(b) $\Rightarrow$ (c) Define $P = ann_R(V)$. So by hypothesis, $0 = PV = V \cap PM$, thus $V \nsubseteq PM$. Since M/PM is a semisimple, then PM must be an intersection of maximal submodules of M. Then there exists a maximal submodule $K \leq M$ such that $V + K = M$. As V is simple, we infer that $V \cap K = 0$, hence $M = V \oplus K$.

(c) $\Rightarrow$ (a) Obvious. ☐

Proposition 3.3.9. *Let R be a commutative hereditary Noetherian ring. An R-module M is poor if and only if, for every non-injective simple module V, M has a direct summand isomorphic to V.*

Proof. ($\Leftarrow$) By Proposition 3.3.7, M admits a pure submodule N, which is isomorphic to the irredundant complete sum of non-injective simple R-modules S. Now by Proposition 3.3.3, N is pauper (poor). So by Proposition 3.3.2 we conclude that M is poor.

($\Rightarrow$) By negation, assume that V is a non-injective simple module such that M has no summand isomorphic to V. As V is non-injective we can take a submodule A of $E(V)$ with a composition series of length two such that $V \leq_e A$ and $A/V \cong V$. Now we must distinguish between two cases: ☐

Case 1. If $Hom_R(V, M) = 0$, as $0 \leq V \leq A$ is a composition series for A with $A/V \cong V$, then for every $A_1 \leq A$ we have $Hom_R(A_1, M) = 0$. Hence M is A-injective, thus M is not poor, a contradiction.

Case 2. If $Hom_R(V, M) \neq 0$, let A_1 be a non-trivial submodule of A, and let $f : A_1 \to M$ be any non-zero homomorphism. Without loss of generality, we may assume that $A_1 = V$. Now if $f(V)$ is a closed submodule of M, then it follows by the previous proposition, that $f(V)$

is a direct summand of M. This means that M has a direct summand isomorphic to V, a contradiction. So $f(V)$ cannot be a closed submodule of M. Then there is a submodule N of M, with composition length two such that $V \leq_e N$. Thus by Lemma 3.2.3 $A \cong B$, and B is A-injective. Therefore M is clearly B-injective, a contradiction.

We are finally ready to prove the main result of this section. Note that the following result is a generalization of Theorem 3.1.3.

Theorem 3.3.10. *A commutative hereditary Noetherian ring R satisfies (U') and the following statements are equivalent, for every R-module M:*

(a) *M is poor.*

(b) *$Z(M)$ is poor.*

(c) *For every non-injective simple module V, M has a direct summand isomorphic to V.*

(d) *M has a pure submodule isomorphic to S, where S is the direct sum of non-isomorphic and non-injective simple R-modules.*

Proof. (a) $\Rightarrow$ (b) Let us assume that M is poor, and let V be a non-injective simple module. Then by the previous proposition, M admits a direct summand U which is isomorphic to V. Furthermore, by Proposition 3.3.3 we know that $U \leq Z(M)$. So U is also a direct summand of $Z(M)$, and again by Proposition 3.3.9, $Z(M)$ is poor.

(b) $\Rightarrow$ (a) By the hereditary hypothesis, we apply Lemma A.2.6, so that $M/Z(M)$ is flat. By definition $(M/Z(M) \otimes_R -)$ is exact, thus $Z(M)$ is a pure submodule of M. Therefore, by Theorem 3.3.2, M must be poor.

(a) $\Leftrightarrow$ (c) This is the previous proposition.

(c) $\Rightarrow$ (d) By Proposition 3.3.7.

(d) $\Rightarrow$ (c) By Proposition 3.3.3, S is poor. Now since S is a pure submodule of M, then by Proposition 3.3.2 we conclude the proof. $\square$

3.4 Over Semiartinian Rings

We will prove in Proposition 3.4.2 that the irredundant complete sum of non-injective simple modules is pauper, over commutative semiartinian ring, Furthermore, commutative semiartinian rings with zero radical satisfy (U'), by Proposition 3.4.3. This section will be concluded by proving that Artinian serial rings satisfy (U).

Lemma 3.4.1. [8, Lemma 5.9] *Let R be a commutative ring and let V be a simple R-module. If V is N-injective for some R-module N, then $V^{(I)}$ is N-injective for every index set I.*

Proof. Let $P = ann(V)$, and I an index set. Since V is N-injective, then by Proposition 1.1.18, V^I is also N-injective. Furthermore, R is commutative and $V^I P = 0$. Therefore V^I is an (R/P)-module, and P is maximal, hence R/P is a field, which implies that V^I is a semisimple (R/P)-module. Then V^I is also semisimple as a R-module. Therefore $V^{(I)}$ is a direct summand of V^I. Thus $V^{(I)}$ is also N-injective. $\square$

Proposition 3.4.2. *Let R be a commutative semiartinian ring. Then the irredundant complete sum of non-injective simple right R-module, S is pauper. Moreover, any poor module contains an isomorphic copy of each non-injective simple R-module.*

Proof. Take M to be an arbitrary cyclic R-module, and assume the irredundant complete sum of non-injective simple R-modules S is M-injective. By the proof of Proposition 1.5.3, we know that the semiartinian condition is sufficient in order for us to have $Soc(M) \leq_e M$. Furthermore, assume that M is not semisimple. Now by contradiction, suppose that $Soc(M)$ has infinite length. So either M contains an infinite direct summand of S, denoted by N, or it contains a direct summand isomorphic to $V^{(I)}$, where V is a non-injective simple module and I is an infinite index set. In the first case, N is M-injective. Then by Proposition 1.1.14, the inclusion $N \to M$ splits, a contradiction. In the latter case, since V is M-injective, by Lemma 3.4.1 we have that $V^{(I)}$ is also M-injective. Again by Proposition 1.1.14, the inclusion $V^{(I)} \hookrightarrow M$ splits, a contradiction. Therefore $Soc(M)$ has finite length, thus $Soc(M)$ is M-injective. Once again, by Proposition 1.1.14, $Soc(M)$ is a direct summand of M, a contradiction. Thus M must be semisimple, hence S is poor.

The last part follows by analogous arguments of those used in Proposition 3.3.3. $\square$

The equivalence between (a) and (b) in the following proposition is exactly the fact that for a commutative semiartinian ring R, the class of R-modules for which the radical is zero satisfies condition (U')

Proposition 3.4.3. *Let R be a commutative semiartinian ring and M a right R-module such that $Rad(M) = 0$. Then the following are equivalent:*

(a) *M is poor.*

(b) *M has a pure submodule isomorphic to S, where S is the direct sum of non-injective simple modules.*

(c) *For every non-injective simple R-module V, M has a direct summand isomorphic to V.*

Proof. (a) $\Rightarrow$ (b) By the previous proposition, we know the irredundant complete sum of non-injective simple R-modules is pauper. Moreover, M contains a summand isomorphic to some non-injective simple module in S. Therefore by Proposition 3.3.7, M admits a pure submodule isomorphic to S.

(b) $\Rightarrow$ (a) Analogous proof to the first paragraph of the previous proposition. As S is pauper it follows that M is poor.

(b) $\Rightarrow$ (c) Let V be a non-injective simple R-module. As $Rad(M) = 0$, then there exists a maximal submodule N of M such that $V + N = M$. Now $V \cap N = 0$, because V is simple. Thus $M = V \oplus N$.

(c) $\Rightarrow$ (b) Immediate from Proposition 3.3.7. $\square$

Corollary 3.4.4. *Let R be a commutative semiartinian ring and Γ a complete irredundant set of non-injective simple modules. Then $\prod_{S_i \in \Gamma} S_i$ is poor. Moreover, any submodule N of $\prod_{S_i \in \Gamma}$ containing $\oplus_{S_i \in \Gamma} S_i$ is poor.*

Corollary 3.4.5. *Let R be a commutative ring. Suppose R is Noetherian or semiartinian. Then any poor module has a pauper subm odule.*

Proof. In Propositions 3.3.3 and 3.4.2, we have shown that the irredundant complete sum of non-injective simple R-modules S, is a pauper module, over commutative Noetherian rings and commutative semiartinian rings respectively. In either case, any poor module has a submodule isomorphic to S. $\square$

Now we turn our attention to Artinian rings.

Theorem 3.4.6. *Let R be a right Artinian ring. Then every non-injective right module contains a pauper module if and only if there is a ring decomposition $R = S \oplus T$, where S is semisimple and $T = 0$, or T has a unique simple right module. In the case where $T \neq 0$, R has a unique pauper right module, namely the unique simple submodule of T.*

Proof. We start with the necessity condition. Since R is Artinian it follows that $R = e_1R \oplus e_2R \oplus \cdots \oplus e_nR \oplus f_1R \oplus \cdots \oplus f_mR$, where $e_1R, e_2R, \ldots, e_nR$ are the injective pauper right ideals of R. Let $S = e_1R \oplus e_2R \oplus \cdots \oplus e_nR$ and $T = f_1R \oplus \cdots \oplus f_mR$. Therefore $Hom_R(T, S) = Hom_R(S, T) = 0$, which implies $S \cap T = 0$. Hence $R = S \oplus T$ is a ring direct summand. If $T = 0$, then $R = S$ is semisimple, and we are done. Let $T \neq 0$. Now by contradiction, assume T admits two non-isomorphic simple right T-modules, U and V. By hypothesis, both U and V admit pauper submodules. If 0 is pauper, then $T = 0$, a contradiction. Therefore U and V must be both pauper. Now take a submodule A of $E(U)$ with composition length two. Then, V is A-injective. However, V is poor and A is not semisimple, a contradiction. Hence T has a unique simple submodule.

Now for the sufficiency. Assume $T = 0$. Thus R is semisimple and Artinian and by Proposition 2.2.1 we are done. Now assume that $T \neq 0$ and take U to be its unique simple right T-module. An Artinian ring is semilocal, so it follows by Proposition 2.1.9 that $T/J(T) \cong U^n$ is poor. as a T-module and as an R-module as well. Therefore U is a poor R-module. We take M a non-injective module, so $MT \neq 0$. Therefore M contains a simple submodule isomorphic to U. As U is poor and simple we are done. □

Lemma 3.4.7. *Let R be an Artinian serial ring and M, N indecomposable right R-modules. If $cl(M) \leq cl(N)$, then N is M-injective.*

Proof. Take A a submodule of M, let $f : A \to N$ be a homomorphism, consider the inclusion $i : N \hookrightarrow E(N)$, and define $h = if$. Then there is a homomorphism $g : M \to E(N)$ such that $g_{|A} = h$. Now by hypothesis, $cl(g(M)) \leq cl(M) \leq cl(N)$. Since N is uniserial, $E(N)$ is also uniserial. Therefore $g(M) \leq N$. Hence g extends f. Thus N is M-injective. □

Proposition 3.4.8. *Let R be an Artinian serial ring. Then the irredundant complete sum of non-injective simple right modules, S, is a pauper module. Moreover, any poor right R-module has a direct summand isomorphic to S. That is, any poor module has a pauper direct summand.*

Proof. Let us start by noting that any Artinian ring is in particular semilocal. So $R/J(R)$ is semisimple, which by Proposition 3.1.6 implies that S is poor. Now by contradiction, assume S is poor but not pauper. Then S admits a proper poor direct summand S'. Thus for some non-injective simple module U we have that $Hom_R(U, S') = 0$, by Proposition 1.1.4. As R is Artinian, there exists a non-semisimple submodule Y of $E(U)$ such that Y/U is simple. Furthermore, S' is Y-injective, but this contradicts the poorness of S'. Hence S must be pauper.

Let M be a poor module. By Proposition 1.7.1, $M = \oplus_{i \in I} U_i$ with each U_i a cyclic uniseral module. Let us prove that M has a summand isomorphic to S. This is equivalent to seeing that for each non-injective simple module K, there is a $i_0 \in I$ such that $K \cong U_{i_0}$. Again by contradiction, assume there exists a non-injective simple module T such that $T \ncong U_i$, for all $i \in I$. By the non-injective assumption, there exists $X \lneq E(T)$ such that X/T is simple and $cl(X) = 2$. As T is not isomorphic to any U_i, then for each U_j it follows that $Hom_R(T, U_j) \neq 0$. From this we infer that $cl(X) \leq cl(U_j)$. Hence by Lemma 3.4.7, U_j is X-injective. Since X is finitely generated, then M is also X-injective. However this contradicts the hypothesis that M is poor, so M must have a direct summand isomorphic to S. □

3.5 Pauper Abelian Groups

In Corollary 1.1.27 we established an equivalence between divisibility and injectivity for abelian groups. We start by pondering what does it mean for a group to be semisimple, the first definition and lemma offer an answer. After introducing some necessary concepts we will prove Theorem 3.1.3. Throughout this section the importance of Examples 3.1.4(i) and (ii) will become clearer.

The definitions and structural general results regarding abelian groups follow from [20]. In particular, Proposition 3.5.3 is key in the characterization of abelian groups. It gives us a decomposition of abelian groups in terms of its reduced and divisible (injective) subgroups.

In this section we will be showing that the class of torsion groups satisfies (U), (Corollary 3.5.6). The last result of this section is dedicated to proving that the class of abelian groups with torsion-free rank one also satisfies (U).

Definition. A positive integer is said to be *square-free* if it is not divisible by any element of the form p^2, with $p \in \mathbb{P}$.

Definition. An abelian group G is said to be *elementary*, if every element has a square-free order.

Naturally, an *elementary p-group* is an elementary group, whose elements all have order p, for some $p \in \mathbb{P}$

The following justifies why elementary groups are "semisimple" groups.

Lemma 3.5.1. [20, Chapter 2.1, Theorem 1.4] *An elementary group G is a direct sum of cyclic groups of prime orders (i.e. elementary p-groups).*

Proof. It is clear that an elementary group is a torsion group. Thus by Proposition 1.4.2 it is enough to consider the elementary p-subgroups of G. Note that an elementary p-group is a $\mathbb{Z}/p\mathbb{Z}$-vector space. Therefore it is the direct sum of one dimensional spaces, i.e., groups of order p. □

Definition. A group G is *bounded* if $nG = 0$, for some $n \in \mathbb{Z} \backslash 0$.

Remark. Clearly $\mathbb{Z}_p$ is bounded.

Lemma 3.5.2. [20, Chapter 3.5, Theorem 5.2] *A bounded group is a direct sum of cyclic groups.*

Proof. Take a bounded group A. Since A is bounded, then so are its p-primary components. Now [20, Theorem 5.1] states that an arbitrary p-group is a direct sum of cyclic groups if and only if, it is the union of an ascending chain of bounded subgroups

$$A_1 \leq A_2 \leq \ldots \leq A_n \leq \ldots, \; A = \bigcup_{n=1}^{\infty} A_n$$

Thus if we take the $A_i's$ to be the p-primary components of A we are done. □

Definition. A group G is said to be a *reduced group* if G has no divisible subgroups different from zero.

Definition. A subgroup H of G is said to be a *pure subgroup* in G, if an equation $nx = h \in H$ is solvable in G, this implies it is also solvable in H, i.e., if $nH = H \cap nG$, for every $n \in \mathbb{N}$.

Remark. In general, the torsion subgroup of a group G is pure. Take $g_0 \in T(G)$. For some $x \in G$ and $n \in \mathbb{N}$ we have $g_0 = nx$. Now by definition of torsion subgroup, it follows that g_0 has finite order, which implies by the equality that x also has finite order, i.e., $x \in T(G)$.

Remark. The sum of divisible groups is itself divisible. So the maximal divisible subgroup of G, D is the subgroup generated by all divisible subgroups of G. So if we take a reduced subgroup N of G, then $N \cap D = 0$.

The following results is essential in understanding the structure of abelian groups.

Proposition 3.5.3. [20, Chapter 4.2, Theorem 2.5] *Every group G is the direct sum of a divisible group D and a reduced group N, $G = D \oplus N$, where D is a uniquely determined subgroup of G and $N$is unique up to isomorphism.*

Proof. By the remark above, it is enough to show that D is a direct summand of G. Let D be an arbitrary divisible subgroup of G. Then by Corollary 1.1.27, D is injective, in particular D is G-injective. Therefore the identity map $id_D : D \to D$ extends to a homomorphism $f : G \to D$. Hence $G = \ker f \oplus D$, so D is a direct summand of G.

Now to prove the last statement. If $G = N \oplus D$, with D divisible and N reduced. Then D must be the unique maximal divisible subgroup of G. Note that N is the complement of D, which is always unique up to isomorphism. ☐

Combining the definitions of pure and p-group we have the following.

Definition. A subgroup H of G is *p-pure* if $p^k H = H \cap p^k G$, for every $k \in \mathbb{N}$.

Definition. Let $p \in \mathbb{P}$. A subgroup $H \leq G$ is said to be a *p-basic subgroup of G* if it satisfies the following conditions:

1. H is a direct sum of cyclic p-groups and infinite cyclic groups;
2. H is p-pure in G;
3. G/H is p-divisible.

Lemma 3.5.4. [20, Chapter 5.2, Lemma 2.1] *Let H be a subgroup of G such that H is a direct sum of cyclic groups, of the same order p^k. The following are equivalent:*

(a) *H is a pure (p-pure) subgroup of G.*

(b) $H \cap p^k G = 0$.

(c) *H is a direct summand of G.*

Proof. (a) $\Rightarrow$ (b) Assume H is a p-pure subgroup of G. Therefore $H \cap p^m G = p^m H$, for every $m \in \mathbb{N}$. If we choose $k = m$, then $p^k H = 0$.

(b) $\Rightarrow$ (c) Let K be a maximal subgroup of G such that $p^k G \leq K$ and $K \cap H = 0$. Let us prove that $G = H \oplus K$. Suppose that $g \in G \backslash (H \oplus K)$. Then we also have $pg \in H \oplus K$, thus $pg = h + k$, with $h \in H$ and $k \in K$. Hence $p^{k-1}h + p^{k-1}k = p^k g \in K$, which implies that $p^{k-1}h = 0$. By hypothesis there exists $h' \in H$ such that $ph' = h$. Furthermore, by the maximality of K, the subgroup $\langle K, g - h' \rangle$ contains $h_0 \in H \backslash 0$. Thus, for some $k' \in K$ and integer m we have $h_0 = k' + m(g - h')$. Since $H \cap K = 0$ and $p(g - h') = k \in K$, it follows that the greatest common divisor between m and p is 1. This implies that both $m(g - h\prime) = h_0 - k'$ and $p(g - h') = k$ are in $H \oplus K$. Therefore $g - h' \in H \oplus K$,so $g \in H \oplus K$, a contradiction. Hence $G = H \oplus K$.

(c) $\Rightarrow$ (a) is trivial by definition of p-pure subgroup. ☐

The following is a good illustration of the fact that certain types of pure subgroups are summands.

Lemma 3.5.5. [20, Chapter 5.2, Theorem 2.5] *A pure bounded subgroup H of G is a direct summand of G.*

Proof. Let H be a bounded subgroup of G. Then by Lemma 3.5.2, we have $H = H_1 \oplus C$, where H_1 is a direct sum of cyclic groups of order p^k, for some $k \in \mathbb{N}$. Furthermore, the least upper bound of the orders of elements in C are lesser than the orders of the elements of H. As H is pure in G, then H_1 is also pure, by Lemma 3.5.2. Once again by the previous lemma, $G = H_1 \oplus G_1$ for some $G_1 \leq G$. Therefore, $H = H_1 \oplus C_1$, with $C_1 = H \cap G_1 \cong C$. Thus C_1 is pure in G_1. Now by induction, C_1 is a direct summand of G_1. Therefore H is a direct summand of G. □

We are now ready to prove Theorem 3.1.3.

Theorem. *An abelian group G is poor (as a $\mathbb{Z}$-module) if and only if its torsion part $T(G)$ has a direct summand isomorphic to $\oplus_{p\in\mathbb{P}}\mathbb{Z}_p$.*

Proof. We start with the "Only if" part. Assume that a group G is poor and p is an arbitrary prime. If $T_p(G) = 0$, then for all $k \in \mathbb{N}_0 : p^k a = 0$, if and only every divisible p-subgroup H of G is a divisible. Therefore by Corollary 1.1.27 we have that G is H-injective, for every p-group H. However this contradicts the poorness of G. Thus $T_p(G) \neq 0$. If every element $g \in G$ of order p is divisible by p, then G is $\mathbb{Z}_{p^2}$-injective, because the only non-trivial subgroup of $\mathbb{Z}_{p^2}$ is $p\mathbb{Z}_p$. So there exists element, $a_p \in G$ of order p that is not divisible by p, i.e., $o(a_p) = p$ and $a \nmid p$. Therefore the cyclic group generated by $\langle a_p \rangle$ is a p-pure subgroup of $T_p(G)$, and a pure (bounded) subgroup of $T_p(G)$ as well. Now by Lemma 3.5.5, it follows that $\langle a_p \rangle$ is a direct summand of $T_p(G)$. Hence $\oplus_{p\in\mathbb{P}}\langle a_p \rangle$ is a direct summand of $\oplus_{p\in\mathbb{P}}T_p(G) = T(G)$. Therefore $\oplus_{p\in\mathbb{P}} < a_p > \cong G_0$.

For the "If" part, assume that $T(G)$ contains a direct summand that is isomorphic to G_0. We take A to be a summand of $T(G)$ such that $A \cong \mathbb{Z}_p$ (A is bounded). As $T(G)$ is pure in G, so must be A. Again by Lemma 3.5.5, A is a direct summand in G. So G has a direct summand isomorphic to $\mathbb{Z}_p$. Now let us assume G is H-injective, for some group H. Then for each $p \in \mathbb{P}$, $\mathbb{Z}_p$ is also H-injective. Now by contradiction, suppose that H is not elementary (semisimple). Thus there exists $h \in H$ such that h has infinite order, or $o(h) = p^n$, with $p \in \mathbb{P}$ and $n > 1$. If h has infinite order, then $\langle h \rangle = \mathbb{Z}$. If $o(h) = p^n$, this implies that $\langle h \rangle = \mathbb{Z}_{p^n}$. Therefore by Proposition 1.1.16 we infer that $\mathbb{Z}_p$ is either $\mathbb{Z}$-injective, or $\mathbb{Z}_{p^n}$-injective. Now a homomorphism $f : p\mathbb{Z} \to \mathbb{Z}_p$ such that $f(p) = 1$, does not admit an extension $g : \mathbb{Z} \to \mathbb{Z}_p$. Because if it did, we would have that $1 = f(p) = g(p) = pg(1) = 0$. On the other hand, the subgroup of $\mathbb{Z}_{p^n}$, generated by p^{n-1} is isomorphic to $\mathbb{Z}_p$. However $\langle p^{n-1} \rangle$ is not a direct summand of $\mathbb{Z}_{p^n}$. So in either case we arrive at a contradiction. Hence M must be semisimple. Thus G is poor. □

The following result is a consequence of Theorem 3.1.3. This corollary tells us that the class of torsion groups satisfies (U).

Corollary 3.5.6. [2, Corollary 3.2] *For an abelian group G, the following are equivalent:*

(a) *G is poor.*

(b) *The reduced part of G is poor.*

(c) *$T(G)$ is poor.*

(d) *For each prime p, G has a direct summand isomorphic to $\mathbb{Z}_p$.*

Proof. (a) $\Rightarrow$ (b) By Proposition 3.5.3 a group G is of the form $N \oplus D$, with N a reduced subgroup, and D a divisible subgroup. Since G is poor this means that N or D (or both) must be poor. However, D is divisible (injective), so by Lemma 2.1.6, N must be poor.

(b) $\Rightarrow$ (a) Again by Proposition 3.5.3 we know the reduced part of G is a summand of G. Hence we are done, by Corollary 2.1.5.

(a) $\Leftrightarrow$ (c) Immediate by Theorem 3.1.3.

(c) $\Rightarrow$ (d) We have already established in Proposition 1.4.2 that $T(G) = \oplus_{p\in\mathbb{P}} T_p(G)$. Furthermore, by Theorem 3.1.3 we know that the torsion group $T(G)$ admits a direct summand isomorphic to $\oplus_{p\in\mathbb{P}} T_p(G)$, which we know to be poor.

(d) $\Rightarrow$ (c) For each prime p, let G_p be a direct summand of G isomorphic to a summand of $T_p(G)$. Furthermore, each p-primary component $T_p(G)$ is isomorphic to $\mathbb{Z}_p$. Hence $T(G) = \oplus_{p\in\mathbb{P}} T_p(G) \cong G_0$, since G_0 is poor we are done. □

Theorem 3.5.7. *Let G be a pauper abelian group. Then* $T(G) \cong G_0$.

Proof. Let $T_p(G)$ be a p-primary component of G. As G is poor, then by Corollary 3.5.6, $T_p(G)$ has a direct summand V isomorphic to $\mathbb{Z}_p$. Furthermore, $T_p(G) = V \oplus H$, for some $H \leq T_p(G)$. We want to prove that $H = 0$, by contradiction assume that $H \neq 0$. Let A be a p-basic subgroup of H. Now by definition, $0 \neq A$ is a direct sum of cyclic p-groups. Let B be a cyclic direct summand of A. Therefore B is a pure bounded subgroup of G. Thus by Lemma 3.5.5, B is a direct summand of G. Hence $G = B \oplus C$ where C is poor, because by Theorem 3.1.3, $T(C)$ has a direct summand isomorphic to G_0. We have arrived at a contradiction. Therefore $H = 0$, so $T_p(G) = K$, which implies that $T(G) \cong G_0$. □

The following is an n obvious consequence of the previous theorem.

Corollary 3.5.8. *A torsion abelian group is pauper if and only if it is isomorphic to* G_0.

Lemma 3.5.9. *Let G be a group. Suppose that* $T(G) = G_0$ *and* G/G_0 *is indecomposable. If* G_0 *is not a direct summand in G, then G is pauper.*

Proof. By hypothesis, $T(G) = G_0$, thus G is poor, by Theorem 3.1.3. Now take H to be a poor direct summand of G. Since G_0 is not a direct summand, then $G_0 \subseteq H$. Furthermore, H/G_0 is a direct summand of G/G_0. However, G/G_0 is indecomposable. Hence $H = G$ or $H = G_0$. In either case G is indecomposable, so G is pauper. □

Remark. A pure subgroup of a divisible group is also divisible. Take a pure subgroup H of a divisible group G. Then by definition of divisibility, $H \ni h = nx$, with $n \in \mathbb{N}$ has a solution in G. Now since H is pure, $h = nx$ also has a solution in H. This holds for every $h \in H$ and $n \in \mathbb{N}$. Therefore H is divisible.

Proposition 3.5.10. *Let G be a pure subgroup of* $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$ *that contains* G_0. *Then G is pauper.*

Proof. Let $A = \prod_{p\in\mathbb{P}} \mathbb{Z}_p$, and assume G is a pure subgroup of A. Then $nG = G \cap nA$, for every $n \in \mathbb{N}$. If we take the quotient $nG/G_0 = (G \cap nA)/G_0$, it follows that $n(G/G_0) = G/G_0 \cap n(A/G_0)$. Therefore G/G_0 is a pure subgroup of A/G_0. We have already shown in Example 3.1.4(ii), that A/G_0 is divisible. Furthermore, since G/G_0 is pure, then by the previous remark, G/G_0 is also divisible, and $Rad(G/G_0) = G/G_0$. Now the result follows from Proposition 3.3.6. □

Let us switch our focus to torsion-free rank one groups, so we must give some definitions.

Definition. An abelian group G is said to be a *torsion-free* if all its elements, except for 0, are of infinite order.

Definition. The *torsion-free rank* of a group G,is the cardinality of a maximal linearly independent set.

Remark. The non-zero subgroups of $\mathbb{Q}$ are the torsion-free rank one groups. In other words, a torsion-free rank one group is a $\mathbb{Q}$-vector space.

Definition. We define the *p-height* of any $g \in G\backslash 0$ as the largest number k for which $g \in p^k G \backslash p^{k+1} G$. We denote this by $h_p(g)$. If no such p exists we denote the p-height by $h_p(g) = \infty$.

Definition. Given a torsion-free group G. The sequence of p-heights of $g \in G$,

$$\chi(g) = (h_{p_1}(g), h_{p_2}(g), ..., h_{p_n}(g), ...),$$

is called the *characteristic* of g.

Remark. [20, Chapter 12] Consider an arbitrary rational group G of torsion-free rank one. Then $A = G/G_0$ is a rational group, because G is of rank one. An element $a \in G$ is of the form $a = \frac{m}{n} b$, with $\frac{m}{n} \in \mathbb{Q}$ and $b \in G$. So we can describe any $a \in A\backslash 0$ in terms of its characteristic $\chi(a) = (h_2(a), h_3(a), \ldots, h_{p_n}(a), \ldots)$.

The proof of the following result is beyond the scope of this dissertation.

Theorem 3.5.11. *Let G be a torsion-free rank one subgroup of $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$ containing G_0 and $\chi(a) = (h_2(a), h_3(a), ..., h_p(a), ...)$, for some $0 \neq a \in A = G/G_0$. We have the following hypothesis:*

(i) *If $h_p(a) = 0$, for a finite number of primes p, then G is pauper.*

(ii) *If $h_p(a) \neq 0$, for a finite number of primes p, then G_0 is a direct summand of G, so G is not pauper.*

(iii) *If $h_p(a) = 0$, for an infinite number of primes p and $h_p(a) \neq 0$ for an infinite number of primes p, then G may or may not be pauper.*

Proof. [3, Theorem 6.6]. □

Although we have seen that the ring of integers satisfies (U'). It is still unknown if it satisfies (U). The last result of this section justifies the focus on the class of abelian groups of torsion-free rank one.

Corollary 3.5.12. *Let G be an abelian group in the conditions of the previous theorem such that G is not pauper. Then $G = G_0 \oplus H$, for some $0 \neq H \leq G$. It follows that, the class of torsion-free rank one groups satisfies (U).*

Proof. Suppose G is not pauper. Thus $G = K \oplus H$, where K is a proper and poor submodule of G. Since K is poor, then by Theorem 3.1.3 it follows that $G_0 \leq K$. Thus $H \cap G_0 = 0$. If $G_0 \lneq B$, then $G/G_0 \cong (K/G_0) \oplus H$. Hence G/G_0 is the sum of two non-zero groups. However G/G_0 is indecomposable, so $K = G_0$. □

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Appendix

A.1 Categories and Functors

The following definitions and results follows from [4].

Note that if A is a set and $\mathscr{C}$ is a class, then the indexed class $(A_C)_{C\in\mathscr{C}}$ in $\mathscr{P}(A)$ has a union and an intersection in A. Let $\mathscr{C}$ be a class, for each pair $A, B \in \mathscr{C}$, let $mor_{\mathcal{C}}(A, B)$ be a set; write the elements of $mor_{\mathcal{C}}(A, B)$ as "arrows" $f : A \to B$ for which A is called the *domain* and B the *codomain*. Finally, suppose that for each triple $A, B, C \in \mathscr{C}$ there is a function

$$\circ : mor_{\mathcal{C}}(B, C) \times mor_{\mathcal{C}}(A, B) \to mor_{\mathcal{C}}(A, C).$$

We denote the arrow assigned to a pair

$$g : B \to C \quad f : A \to B$$

by the arrow $gf : A \to C$. The system, $\mathcal{C} = (\mathscr{C}, mor_{\mathcal{C}}, \circ)$, consists of the class $\mathscr{C}$, the map $mor_{\mathcal{C}} : (A, B) \mapsto mor_{\mathcal{C}}(A, B)$ and the rule, $\circ$ is a *category* if:

(C.1) For every triple $h : C \to D$, $g : B \to C$, $f : A \to B$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(C.2) For each $A \in \mathscr{C}$, there is a unique $1_A \in mor_{\mathcal{C}}(A, A)$ such that if $f : A \to B$ and $g : C \to A$, then

$$f \circ 1_A = f \quad \text{and} \quad 1_A \circ g = g.$$

If $\mathcal{C}$ is a category, then the elements of the class $\mathscr{C}$ are called the *objects* of the category, the "arrows" $f : A \to B$ are called *morphisms*, the partial map $\circ$ is called the *composition*, and the morphisms 1_A are called the *identities* of the category.

A category $\mathcal{D} = (\mathscr{D}, mor_{\mathcal{D}}, \circ)$ is a *subcategory* of $\mathcal{C} = (\mathscr{C}, mor_{\mathcal{C}}, \circ)$ provided $\mathscr{D} \subseteq \mathscr{C}$, $mor_{\mathcal{D}}(A, B) \subseteq mor_{\mathcal{C}}(A, B)$ for each pair $A, B \in \mathscr{D}$, $\circ$ in $\mathcal{D}$ is the restriction of $\circ$ in $\mathcal{C}$. If in addition $mor_{\mathcal{D}}(A, B) = mor_{\mathcal{C}}(A, B)$ for each $A, B \in \mathscr{D}$, then $\mathcal{D}$ is a *full subcategory* of $\mathcal{C}$.

Informally a functor is a morphism between categories.

Definition. Let $\mathcal{C} = (\mathscr{C}, mor_{\mathcal{C}}, \circ)$ and $\mathcal{D} = (\mathscr{D}, mor_{\mathcal{D}}, \circ)$ be two categories.

A pair of functions $F = (F', F'')$ is a *covariant functor* from $\mathcal{C}$ to $\mathcal{D}$ if F' is a function from $\mathscr{C}$ to $\mathscr{D}$, F'' is a function from the morphisms of $\mathcal{C}$ to those of $\mathcal{D}$ such that for all $A, B, C \in \mathscr{C}$ and all $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$,

(F.1) $F''(f) : F'(A) \to F'(B)$ in $\mathcal{D}$;

(F.2) $F''(g \circ f) = F''(g) \circ F''(f)$;
(F.3) $F''(1_A) = 1_{F'(A)}$.
A *contravariant functor* is a pair $F = (F', F'')$ satisfying;
(F.1)* $F''(f) : F'(B) \to F'(A)$ in $\mathcal{D}$;
(F.2)* $F''(g \circ f) = F''(f) \circ F''(g)$;
(F.3) $F''(1_A) = 1_{F'(A)}$.
So a contravariant functor is "arrow reversing".

The category of abelian groups is represented by Ab. Furthermore, given a ring R we represent the category of left R-modules by R-Mod. We have a special interest in functors between module categories.

Definition. Let $\mathcal{C}$ be a full subcategory of R-modules and that $\mathcal{D}$ is a full subcategory of S-modules. Then a functor T from $\mathcal{C}$ to $\mathcal{D}$ is *additive* if for each M, N,modules in $\mathcal{C}$ and each pair $f, g : M \to N$ in $\mathcal{C}$,

$$T(f + g) = T(f) + T(g).$$

In particular, if T is additive and covariant, then the restriction

$$T : Hom_R(M, N) \to Hom_S(T(M), T(N))$$

is an abelian group homomorphism. If instead T is additive and contravariant, then the restriction

$$T : Hom_R(M, N) \to Hom_S(T(N), T(M))$$

is an abelian group homomorphism.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be full subcategories of categories of modules and let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor. If for every short exact sequence in $\mathcal{C}$

$$0 \to K \to M \to N \to 0$$

the sequence

$$0 \to F(K) \to F(M) \to F(N)$$

is exact in $\mathcal{D}$, then F is said to be *left exact*. If

$$F(K) \to F(M) \to F(N) \to 0$$

is exact in $\mathcal{D}$, then F is said to be *right exact.* In the contravariant case

$$0 \to G(N) \to G(M) \to G(K)$$

for left exact and

$$G(N) \to G(M) \to G(K) \to 0$$

for right exact. A functor that is both left and right exact is called an *exact functor*.

We may now construct the Hom functor.
Let $U =_R U_S$ be a bimodule. Let $f :_R M \to_R N$ be an R-homomorphism in $_RM$. Then for each

$\gamma \in Hom_R(U, M)$, we have $f\gamma \in Hom_R(U, N)$. We claim that

$$Hom(U, f) : \gamma \mapsto f\gamma$$

is an S-homomorphism.

$$Hom_R(U, f) : Hom_R(U, M) \to Hom_R(U, N).$$

For if $\gamma_1, \gamma_2 \in Hom_R(U, M)$ and $s_1, s_2 \in S$, then for all $u \in U$,

$$\begin{aligned} f \circ (s_1\gamma_1 + s_2\gamma_2)(u) &= f(\gamma_1(us_1) + \gamma_2(us_2)) \\ &= f\gamma_1(us_1) + f\gamma_2(us_2) \\ &= (s_1(f\gamma_1) + s_2(f\gamma_2))(u). \end{aligned}$$

Thus, we do have a function $Hom_R(U, -)$: R-Mod $\to$ S-Mod defined by

$$\begin{aligned} Hom_R(U, -) &: M \mapsto Hom_R(U, M) \\ Hom_R(U, -) &: f \mapsto Hom_R(U, f). \end{aligned}$$

The notation $Hom_R(U, f)$ can be akward, so if there is no ambiguity with the module U, we are likely to abbreviate

$$f_* = Hom_R(U, f).$$

Note that if $f : M \to N$ in R-Mod, then f_* is characterized by

$$\begin{tikzcd} M \arrow[rr, "f"] & & N \\ & U \arrow[ul, "\gamma"] \arrow[ur, "f_*(\gamma)"'] & \end{tikzcd}$$

Now it's an easy matter to check that $Hom_R(U, -)$ is actually an additive covariant functor from R-Mod to S-Mod. On the other hand, we can define a mapping

$$f^* = Hom_R(f, U) : Hom_R(N, U) \to Hom_R(M, U)$$

via

$$Hom_R(f, U) : \gamma \mapsto \gamma f.$$

It is straightforward to show that $f^* = Hom_R(f, U)$ is an S-homomorphism. For f^* we have

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "f^*(\gamma)"'] & & N \arrow[dl, "\gamma"] \\ & U & \end{tikzcd}$$

here then we have a function $Hom_R(-, U)$: R-Mod $\to$ Mod-S defined by

$$\begin{aligned} Hom_R(-, U) &: M \mapsto Hom_R(M, U) \\ Hom_R(-, U) &: f \mapsto Hom_R(f, U). \end{aligned}$$

Proposition A.1.1. [4, Theorem 16.1] Let R and S be rings and let $U =_R U_S$ be a bimodule. Then

$$Hom_R(U, -) : \text{R-Mod} \to \text{S-Mod}$$

is an additive covariant functor and

$$Hom_R(-, U) : \text{R-Mod} \to \text{Mod-S}$$

is an additive contravariant functor.

A.2 Tensor Product and Flat Modules

The definitions and constructions of this section follow from [4] until otherwise stated.

Definition. Given a right R-module M and a left R-module N and an abelian group A, a function

$$\beta : M \times N \to A$$

is said to be *R-balanced if for all* $m, m_i \in M$, $n, n_i \in N$ *and* $r \in R$

1. $\beta(m_1 + m_2, n) = \beta(m_1, n) + \beta(m_2, n)$;
2. $\beta(m, n_1 + n_2) = \beta(m, n_1) + \beta(m, n_2)$;
3. $\beta(mr, n) = \beta(m, rn)$.

There is a natural way to trade each R-balanced map in for a linear map by using the concept of a tensor product. Let M_R and ${}_RN$ be modules. A pair (T, τ) consisting of an abelian group T and an R-balanced map $\tau : M \times N \to T$ is a *tensor product of* M_R *and* ${}_RN$ in case for every abelian group A and every R-balanced map $\beta : M \times N \to A$ there is a unique $\mathbb{Z}$-homomorphism $f : T \to A$ such that the diagram

$$\begin{tikzcd} & M \times N \arrow[dl, "\tau"'] \arrow[dr, "\beta"] & \\ T \arrow[rr, "f"] & & A \end{tikzcd}$$

commutes. If (T, τ) is a tensor product of M_R and ${}_RN$, then clearly, $f \circ \tau$ is R-balanced for each homomorphism $f : T \to A$. Thus (T, τ) is a tensor product of M_R and ${}_RN$ if and only if for each abelian group A

$$f \leftrightarrow f \circ \tau$$

defines a one-to-one correspondence between $Hom_{\mathbb{Z}}(T, A)$ and the set of R-balanced maps $\beta : M \times N \to A$.

The tensor product exists and is unique up to isomorphism. For the uniqueness we have the following.

Proposition A.2.1. [4, Proposition 19.1] *If* (T, τ) *and* (T', τ') *are two tensor products of* $(M_{R,R} N)$ *then there exists a* $\mathbb{Z}$*-isomorphism* $f : T \to T'$ *such that* $\tau' = f\tau$.

Now let us construct a tensor product of $(M_{R,R} N)$ over R. Take $F = \mathbb{Z}^{(M \times N)}$ the free abelian group on $M \times N$. Then F has free basis $(x_\alpha)_{\alpha \in M \times N}$. For notational convenience let us simply write (m, n) for $x_{(m,n)}$. Then

$$F = \oplus_{M \times N} \mathbb{Z}(m, n).$$

Now let K be the subgroup of F generated by all the elements of the form

$$\begin{aligned}
&(m_1 + m_2, n) - (m_1, n) - (m_2, n), \\
&(m, n_1 + n_2) - (m, n_1) - (m, n_2), \\
&(mr, n) - (m, rn),
\end{aligned}$$

and set $T = F/K$. Define $\tau : M \times N \to T$ via

$$\tau(m, n) = (m, n) + K.$$

Proposition A.2.2. [4, Proposition 19.2] *With T and τ defined as above, (T, τ) is a tensor product of $(M_R, {}_R N)$ over R.*

Given $(M_R, {}_R N)$, let (T, τ) be the tensor product constructed above, we write $T = M \otimes_R N$ and for each $(m, n) \in M \times N$,

$$\tau(m, n) = m \otimes n.$$

We tend to be somewhat loose with our terminology and call $M \otimes_R N$ the tensor product of M and N.

Enroute to the tensor functors we next develop a theory of a tensor product $f \otimes g$ of two R-homomorphisms. Let M, M' be right R-modules and let N, N' be left R-modules. Suppose further that $f : M \to M'$ and $g : N \to N'$ are R-homomorphisms. Define a map $(f, g) : M \times N \to M' \otimes_R N'$ via

$$(f, g)(m, n) = f(m) \otimes g(n).$$

It is evident that (f, g) is R-balanced, so there is a unique $\mathbb{Z}$-homomorphism, which we shall denote by $f \otimes g$, from $M \otimes_R N$ to $M' \otimes N'$ such that the following diagram commutes:

$$\begin{tikzcd}
 & M \times N \arrow[dl, "\tau"'] \arrow[dr, "(f,g)"] & \\
M \otimes_R N \arrow[rr, "f \otimes g"'] & & M' \otimes_R N'
\end{tikzcd}$$

Thus, in particular, $f \otimes g$ is characterized via

$$(f \otimes g)(m \otimes n) = f(m) \otimes g(n).$$

Now we are ready to construct the tensor functor.

Let $U =_S U_R$ be a bimodule. Then it follows by [4, Propositions 19.7 and 19.8] that there is an additive covariant functor

$$(U \otimes_R -) : {}_R M \to_{\mathbb{Z}} M$$

defined by

$$(U \otimes_R -) : M \mapsto U \otimes_R M$$

$$(U \otimes_R -) : f \mapsto 1_U \otimes f.$$

By [4, Propositions 19.5] each $U \otimes_R M$ is a left S-module. We claim moreover that if $f : M \to M'$ is an R-homomorphism, then

$$U \otimes_R f :_S U \otimes_R M \to {}_SU \otimes_R M'$$

is an S-homomorphism. It is sufficient to check this on the generators $u \otimes m$ of $U \otimes_R M$. But for each $s \in S, u \in U$, and $m \in M$.

$$(U \otimes_R f)(su \otimes m) = (1_U \otimes f)(su \otimes m)$$

$$= su \otimes f(m) = s(u \otimes f(m))$$

$$= s((U \otimes_R f)(u \otimes m))$$

as claimed. Thus we may view this as an additive functor from R-Mod to S-Mod and write it

$$({}_SU \otimes_R -) : \text{R-Mod} \to \text{S-Mod}.$$

Similarly, there is an additive covariant functor

$$(- \otimes_S U_R) : \text{Mod-S} \to \text{Mod-R}$$

defined by

$$(- \otimes_S U_R) : N \mapsto N \otimes_S U_R$$

$$(- \otimes_S U_R) : g \mapsto g \otimes 1_U.$$

The remaining definitions of this section follow from [24].

Definition. A module P_R is said to be *finitely related* (abbreviated f.r.) if there exists an exact sequence

$$0 \to K \to F \to P \to 0$$

in Mod-R, where F is free (of arbitrary rank) and K is finitely generated.

A module P_R is said to be *finitely presented* (abbreviated f.p.) if there exists an exact sequence

$$0 \to K \to F \to P \to 0$$

in Mod-R, where F is free of finite rank and K is finitely generated (or equivalently, there exists an exact sequence $R^m \to R^n \to P \to 0$ with $m, n \in \mathbb{N}$).

Definition. A (short) exact sequence $\varepsilon : 0 \to A \xrightarrow{\varphi} B \to C \to 0$ in Mod-R is said to be *pure (exact)* if $\varepsilon \otimes_R C'$ is an exact sequence (of abelian groups) for any left R-module C'. (Of course only the injectivity of $A \otimes_R C' \to B \otimes_R C'$ is at stake.) If this is the case, we say that $\varphi(A)$ is a *pure submodule* of B (or that B is a *pure extension* of $\varphi(A)$).

Remark. Every direct summand of a module M is pure in M.

Lemma A.2.3. [33, Lemma 3.70] *Let $0 \to B' \xrightarrow{i} B \xrightarrow{p} B'' \to 0$ be a pure exact sequence, where i is the inclusion. If M is a finitely presented left R-module, then $p_* : Hom_R(M, B) \to Hom_R(M, B'')$ is surjective.*

Lemma A.2.4. [24, Proposition 4.29] *A ring R is right Noetherian iff all finitely generated (resp. cyclic) right R-modules are f.p.*

Definition. A right module P_R is said *flat* (or *R-flat)* if the functor $(P \otimes_R -)$ is exact on the category of left R-modules.

Lemma A.2.5. [24, Theorem 4.30] *Let P be a f.r. right module over any ring R. Then P is flat iff it is projective.*

Lemma A.2.6. [22, Proposition 2.3] *Let R be a commutative ring. Then all nonsingular R-modules are flat if and only if R is semihereditary.*

A.3 Morita Equivalences

The definitions and results of this section follow from [4].

Since every ring R has a natural R-module structure on itself, we often study a ring R by studying the category of R-modules. Two rings are said to be Morita equivalent if their module categories are equivalent. Let us define what we mean by equivalent categories.

Let $\mathcal{C}$ and $\mathcal{D}$ be arbitrary categories. Then the covariant functor $F : \mathcal{C} \to \mathcal{D}$ is a *category equivalence*, in case there is a functor (necessarely covariant) $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $GF \cong 1_{\mathcal{C}}$ and $FG \cong 1_{\mathcal{D}}$.

A functor G with this property is called an *inverse equivalence* of F. Two categories are *equivalent* in case there exists a category equivalence from one to the other. We write $\mathcal{C} \approx \mathcal{D}$ in case $\mathcal{C}$ and $\mathcal{D}$ are equivalent. It is easy to check that this defines an equivalence relation on the class of all categories.

We restrict our interest to module categories, so the functors between such categories are additive. Thus for two such categories to be equivalent there must be an additive equivalence from one to the other.

Two rings R and S are said to be *Morita equivalent*, abbreviated $R \approx S$, if R-Mod $\approx$ S-Mod, i.e., in case there are additive equivalences between these categories of modules. In [4, Corollary 22.3] it is shown that the categories R-Mod and S-Mod are equivalent if and only if Mod-R and Mod-S are equivalent.

These categorial equivalences, preserve many properties such as, projectivity, injectivity, simplicity, semisimplicity, finitely generated, finitely cogenerated, Artinian, Noetherian, indecomposable, as shown in [4, Proposition 21.6 & 21.8].