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NONLINEAR ROBIN PROBLEMS WITH LOCALLY DEFINED REACTION

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ABSTRACT. We consider a nonlinear Robin problem driven by a $p-$ Laplacian. The reaction consistes of two terms. The first one is parametric and only locally defined, while the second one is $(p-1)$- superlinear. Using cutt-off techniques together with critical point theory and critical groups, we show that for big values of the parameter $\lambda > 0$, the problem has at least three nontrivial solutions, all with sign information (positive, negative and nodal). In the semilinear case ($p = 2$), we produce a second nodal solution, for a total of four nontrivial solutions, all with sign information.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2- boundary $\partial\Omega$. In this paper we study the following parametric nonlinear Robin problem

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \lambda f(z,u(z)) + g(z,u(z)) \\ \hfill \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega, \end{cases}$$

with $\lambda > 0$, $1 < p < \infty$. By $\triangle_p$ we denote the $p-$Laplace differential operator defined by

$$\Delta_p u = div\left(|Du|^{p-2} Du\right), \text{ for all } u \in W^{1,p}(\Omega),$$

where $|\cdot|$ denotes the norm in $\mathbb{R}^N$. The potential function ξ satisfies $\xi \in L^\infty(\Omega)$ and $\xi(z) \geq 0$ for a.a. $z \in \Omega$. The reaction of the problem (right-hand side) consists of two terms. One is the parametric term $\lambda f(z,x)$ with $\lambda > 0$ being the parameter. The other one is a perturbation $g(z,x)$. Both functions f and g are Carathéodory functions (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ and $z \to g(z,x)$ are measurable functions, while for a.a. $z \in \Omega$, $x \to f(z,x)$ and $x \to g(z,x)$ are continuous). The interesting feature of our work here, is that the parametric term $\lambda f(z,\cdot)$ is only locally defined, namely the conditions imposed on $f(z,\cdot)$ concern only its behavior near zero. There are no hypotheses on $f(z,\cdot)$ for large values of $x \in \mathbb{R}$.

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In the boundary condition, $\frac{\partial u}{\partial n_p}$ denotes the conormal derivative of u corresponding to the $p-$Laplacian and is interpreted using the nonlinear Green's identity (see Papageorgiou-Radulescu-Repovs [?], Corollary 1.5.17, p.35). Specifically, for $u \in C^1\left(\overline{\Omega}\right)$, we have

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}\,\frac{\partial u}{\partial n},$$

where $n\,(.)$ is the outward unit normal on $\partial\Omega$. Using cut-off techniques together with variational tools based on the critical point theory and Morse theory (critical groups), we show that for all $\lambda > 0$ big, problem (P_λ) has at least three nontrivial smooth solutions, all with sign information. More precisely, we prove that there exist two solutions with fixed sign (one positive and the other negative) and a third solution which is nodal (that is, sign changing). In the semilinear case (that is, $p = 2$), by strengthening the regularity of the functions $f\,(z,\cdot)$ and $g\,(z,\cdot)$ (we assume that both are C^1 functions), we produce a second nodal solution, for a total of four nontrivial solutions, all with sign information. Finally, for both the nonlinear and the semilinear problems, we show that the solutions produced converge to zero in $C^1\left(\overline{\Omega}\right)$ as $\lambda \to \infty$.

The first paper dealing with equations which have reaction terms that are only locally defined is the work of Wang [?]. In that paper, the author deals with a semilinear Dirichlet equation driven by the Laplacian and with a reaction of the form $x \to \lambda\,|x|^{q-2}\,x + g\,(z,x)$, where $1 < q < 2$. So, in the reaction we encounter a parametric concave term and a perturbation $g \in C\,(\Omega \times \mathbb{R}, \mathbb{R})$, which is odd in $x \in \mathbb{R}$ for $|x|$ small, and $\lim_{x\to 0}\frac{g(z,x)}{|x|^{q-2}x} = 0$ uniformly for a.a. $z \in \Omega$. No other conditions are imposed on g. In particular, there are no conditions on $g\,(z,\cdot)$ for $|x|$ big. The symmetry of the reaction near zero permits the use of a symmetric mountain pass theorem, and so the author shows that for all $\lambda > 0$, the problem has a sequence $\{u_n\}_{n\geq 1} \subseteq H_0^1\,(\Omega)$ of weak solutions such that $\|u_n\|_\infty \to 0$ as $n \to \infty$. No sign information is given for the solutions produced. Later, Li-Wang [?] extended the result to Schrödinger equations, and in addition proved that the solutions are nodal.

More recently, Papageorgiou-Radulescu [?] and Papageorgiou-Radulescu-Repovs [?] extended the aforementioned works to nonlinear, nonhomogeneous Robin problems, while very recently Aizicovici-Papageorgiou-Staicu [?] obtained similar results for anisotropic (p,q)-equations. All these papers impose a local symmetry condition on the reaction, which permits the use of some version of the symmetric mountain pass theorem. No such symmetry condition is employed here.

2. Mathematical Background - Hypotheses

In the analysis of problem (P_λ) we will use the the Sobolev space $W^{1,p}(\Omega)$, $1 < p < \infty$, and the Banach space $C^1(\overline{\Omega})$. By $\|.\|$ we will denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p\right]^{\frac{1}{p}} \text{ for all } u \in W^{1,p}(\Omega),$$

where $\|.\|_p$ stands for the L^p-norm. The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \left\{u \in C^1\left(\overline{\Omega}\right) : u(z) \geq 0 \text{ for all } z \in \Omega\right\}.$$

This cone has a nonempty interior given by|

$$int\, C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\right\},$$

If $u,\ v \in W^{1,p}(\Omega$ and $u(z) \leq v(z)$ for a.a. $z \in \Omega$, then we define

$$[u,v] = \left\{y \in W^{1,p}(\Omega) : u(z) \leq y(z) \leq v(z) \text{ for a.a. } z \in \Omega\right\}.$$

Also by $int_{C^1(\overline{\Omega})}[u,v]$ with denote the interior in $C^1(\overline{\Omega})$ of $[u,v] \bigcap C^1(\overline{\Omega})$.

On $\partial\Omega$ we consider the $(N-1)$ $-$dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Having this measure, we can define in the usual way the boundary Lebesgue spaces $L^s(\partial\Omega)$ $(1 \leq s \leq \infty)$. We recall that there exists a unique continuous linear linear map $\gamma_0 : W^{1,p}(\Omega \to L^p(\partial\Omega)$ known as the "trace map", such that

$$\gamma_0(u) = u\,|_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \bigcap C(\overline{\Omega}).$$

So, the trace map extends to all Sobolev functions the notion of boundary value. We know that γ_0 is compact from $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$, $\operatorname{Im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)$ $(\frac{1}{p}+\frac{1}{p'}=1)$ and $ker\gamma_0 = W_0^{1,p}(\Omega)$

In the sequel for the sake of notational simplicity, we drop the use of the trace map γ_0. All restrictions of Sobolev functions to $\partial\Omega$ are understood in the sense of traces.

If $x \in \mathbb{R}$, then we set

$$x^{\pm} = \max\{\pm x, 0\}.$$

For $u \in W^{1,p}(\Omega)$, we define $u^{\pm}(z) = u(z)^{\pm}$ for a.a. $z \in \Omega$. We know that

$$u^{\pm} \in W^{1,p}(\Omega),\ u = u^+ - u^- \text{ and } |u| = u^+ + u^-.$$

Given a Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$, we say that it satisfies the Ambrosetti-Rabinowitz condition (the AR-condition for short), if there exist $M > 0$ and $q > p$ such that:

$$0 < qF_0(z,x) \leq f_0(z,x)\,x \text{ for a.a. } z \in \Omega,\ \text{all } |x| \geq M,$$

where $F_0(z,x) = \int_0^x f_0(z,s)\,ds$, and

$$0 < \operatorname*{essinf}_{\Omega} F_0(\cdot, \pm M).$$

This condition is very convenient for the verification of the Palais-Smale condition (the PS-condition for short).

Recall that if X is a Banach space and $\varphi \in C^1(X,\mathbb{R})$, then we say that φ satisfies the PS-condition, if every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty$$

admits a strongly convergent subsequence.

By $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ we denote the nonlinear operator defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2} (Du, Dh)_{\mathbb{R}^N} \, dz \text{ for all } u,\ h \in W^{1,p}(\Omega).$$

This operator has the following properties (see Gasinski-Papageorgiou [?], Problem 2.192, p.279):

- it is bounded (that is, it maps bounded sets to bounded sets);
- it is continuous and monotone (hence maximal monotone too);
- it is of type $(S)_+$, that is, *for every sequence* $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ *such that* $u_n \xrightarrow{w} u$ *in* $W^{1,p}(\Omega)$ *and*
$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0,$$
one has
$$u_n \to u \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty.$$

Here $\xrightarrow{w}$ designates the weak convergence in $W^{1,p}(\Omega)$ and $\langle \cdot, \cdot \rangle$ denotes the duality brackets for the pair $\left(W^{1,p}(\Omega)^*, W^{1,p}(\Omega)\right)$.

Let $\mathcal{S} \subseteq W^{1,p}(\Omega)$. We say that S is downward directed (resp. upward directed), if for all $u_1,\ u_2 \in \mathcal{S}$ we can find $\widehat{u} \in \mathcal{S}$ such that $\widehat{u} \leq u_1$ and $\widehat{u} \leq u_2$ (resp. for all $v_1,\ v_2 \in \mathcal{S}$, we can find $\widehat{v} \in \mathcal{S}$ such that $v_1 \leq \widehat{v}$ and $v_2 \leq \widehat{v}$).

Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:

$$K_\varphi = \left\{u \in X : \varphi'(u) = 0\right\} \text{ (the critical set of } \varphi),$$

and

$$\varphi^c = \{u \in X : \varphi(u) \leq c\} \text{ (the sublevel of } \varphi \text{ at } c).$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subset Y_1 \subset X$. For every $k \in \mathbb{N}_0$, by $H_k(Y_1, Y_2)$ we denote the k^{th}- relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Recall that for $k \in -\mathbb{N}$ we have $H_k(Y_1, Y_2)$. Suppose $u \in K_\varphi$ is isolated and let $c = \varphi(u)$. Then the *critical groups of* φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{u\}) \text{ for all } k \in \mathbb{N}_0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology theory implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Now suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the $PS-$condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the *critical groups of* φ *at infinity* are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \in \mathbb{N}_0.$$

By the second deformation theorem (see Papageorgiou-Radulescu-Repovs [?], Theorem 5.3.12, p.386), this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. Indeed if $c' < c < \inf \varphi(K_\varphi)$, then $\varphi^{c'}$ is a strong deformation retract of φ^c (see [?], p.386) and so,

$$H_k(X, \varphi^c) = H_k\left(X, \varphi^{c'}\right) \text{ for all } k \in \mathbb{N}_0$$

(see [?], Corollary 6.1.24, p.468).

Suppose that K_φ is finite. We introduce the following quantities:

$$M(t,u) = \sum_{k\in\mathbb{N}_0} rank\ C_k(\varphi,u)\, t^k \text{ for all } t\in\mathbb{R},\ \text{all } u\in K_\varphi,$$

$$P(t,\infty) = \sum_{k\in\mathbb{N}_0} rank\ C_k(\varphi,\infty)\, t^k \text{ for all } t\in\mathbb{R}.$$

Then the "Morse relation" says that

$$\sum_{u\in K_\varphi} M(t,u) = P(t,\infty) + (1+t)\, Q(t), \tag{2.1}$$

where

$$Q(t) = \sum_{k\in\mathbb{N}_0} \beta_k t^k$$

is a formal series in $t\in\mathbb{R}$ with nonnegative integer coefficients.

Now we introduce the hypotheses on the data of problem (P_λ).

$\mathbf{H}(\xi)$: $\xi\in L^\infty(\Omega)$, $\xi(z)\geq 0$ for a.a. $z\in\Omega$;

$\mathbf{H}(\beta)$: $\beta\in C^{0,\alpha}(\Omega)$ with $\alpha\in(0,1)$, $\beta(z)\geq 0$ for all $z\in\Omega$;

$\mathbf{H}_0$: $\xi\not\equiv 0$ or $\beta\not\equiv 0$.

Remark: If $\beta\equiv 0$, then we recover the Neumann problem.

$\mathbf{H}(f)$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for a.a. $z\in\Omega$ and

(i) there exists $r\in(p,p^*)$ such that

$$\lim_{x\to 0}\frac{f(z,x)}{|x|^{r-2}x} = 0 \text{ uniformly for a.a. } z\in\Omega,$$

where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } \ p<N \\ +\infty & \text{if } \ N\leq p; \end{cases}$$

(ii) if $F(z,x)=\int_0^x f(z,s)\,ds$, then there exists $\tau\in(r,p^*)$ such that

$$\lim_{x\to\infty}\frac{F(z,x)}{x^\tau} = +\infty \text{ uniformly for a.a. } z\in\Omega.$$

Remarks: We emphasize that this reaction term is only locally defined. No conditions are imposed on $f(z,x)$ for $|x|$ big. We also point out that no sign condition is imposed on $f(z,\cdot)$.

$\mathbf{H}(g)$: $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $g(z,0)=0$ for a.a. $z\in\Omega$ and

(i) there exist $a\in L^\infty(\Omega)$ and $1<p<d<p^*$ such that

$$|g(z,x)|\leq a(z)\left[1+|x|^{d-1}\right] \text{ for a.a. } z\in\Omega,\ \text{all } x\in\mathbb{R};$$

(ii) If $G(z,x) = \int_0^x g(z,s)\,ds$, then there exists $q \in (p,r)$ (see hypothesis $\mathbf{H}(f)(i)$) and $M > 0$ such that

$$0 < qG(z,x) \leq g(z,x)\,x \text{ for a.a. } z \in \Omega, \text{ all } |x| \geq M,$$

and

$$0 \leq \operatorname*{essinf}_{\Omega} G(\cdot, \pm M);$$

(iii) there exists $c_0 > 0$ such that

$$0 \leq g(z,x)\,x \leq c_0 |x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Remarks: We see that for a.a. $z \in \Omega$, $g(z,\cdot)$ satisfies the AR-condition (see $\mathbf{H}(g)(ii)$). Moreover, $g(z,\cdot)$ satisfies a global sign condition (see $\mathbf{H}(g)(iii)$).

In what follows by $\gamma : W^{1,p}(\Omega) \to \mathbb{R}$ we denote the C^1-functional defined by

$$\gamma(u) = \|Du\|_p^p + \int_\Omega \xi(z)\,|u|^p\,dz + \int_{\partial\Omega} \beta(z)\,|u|^p\,d\sigma \text{ for all } u \in W^{1,p}(\Omega).$$

Hypotheses $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0$ together with Lemma 4.11 of Mugnai-Papageorgiou [**?**] and Proposition 2.3 of Gasinski-Papageorgiou [**?**] imply that

$$C_1 \|u\|^p \leq \gamma(u) \text{ for some } C_1 > 0, \text{ all } u \in W^{1,p}(\Omega). \tag{2.2}$$

On account of hypotheses $\mathbf{H}(f)(i), (ii)$, we can find $\delta_0 > 0$ such that

$$|f(z,x)| \leq |x|^{r-1}, \ |F(z,x)| \leq \tfrac{1}{r}|x|^r, \ F(z,x) \geq |x|^\tau \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0. \tag{2.3}$$

Let $\theta \in (0,\delta_0)$ and consider the cut-off function $\eta \in C_c^1(\mathbb{R})$ such that

$$\operatorname{supp}\eta \subseteq [-\theta,\theta], \ 0 \leq \eta \leq 1, \ \eta\,|_{[-\frac{\theta}{2},\frac{\theta}{2}]} \equiv 1. \tag{2.4}$$

Using this cut-off function, we introduce the following modification of the parametric, locally defined reaction term

$$\widehat{f}_\lambda(z,x) = \eta(x)\,\lambda f(z,x) + [1-\eta(x)]\,|x|^{r-2}\,x. \tag{2.5}$$

This is a Carathéodory function. We consider the positive and negative truncations of $\widehat{f}_\lambda(z,\cdot)$, namely the Carathéodory functions

$$\widehat{f}_\lambda^{\pm}(z,x) = \widehat{f}_\lambda\left(z, \pm x^{\pm}\right).$$

We set

$$\widehat{F}_\lambda^{\pm}(z,x) = \int_0^x \widehat{f}_\lambda^{\pm}(z,s)\,ds.$$

Also, we introduce the positive and negative truncations of $g(z,\cdot)$, namely the Carathéodory functions

$$g_\pm(z,x) = g\left(z, \pm x^{\pm}\right).$$

We set

$$G_{\pm}(z,x)=\int\limits_0^x g_{\pm}(z,x)\,ds.$$

Finally we define

$$\widehat{\zeta}_\lambda^{\pm}(z,x)=\widehat{f}_\lambda^{\pm}(z,x)+g_{\pm}(z,x)\ \text{ for }\ (z,x)\in\Omega\times\mathbb{R}.$$

These are Carathéodory functions.

Proposition 2.1. *If hypotheses* $\mathbf{H}(f)$, $\mathbf{H}(g)$ *hold, then for every* $\lambda>0$, *the functions* $\widehat{\zeta}_\lambda^{\pm}(z,\cdot)$ *satisfy the AR condition.*

Proof. On account of hypothesis $\mathbf{H}(g)(ii)$, it suffices to show that $\widehat{f}_\lambda^{+}(z,\cdot)$ satisfies the AR condition. First we note that (**??**), (**??**) and (**??**) imply

$$\left|\widehat{f}_\lambda(z,x)\right|\le C_2|x|^{r-1}\ \text{ for a.a. }\ z\in\Omega,\ \text{ all }x\in\mathbb{R}, \tag{2.6}$$

with $C_2=C_2(\lambda)>0$, hence

$$\left|\widehat{F}_\lambda(z,x)\right|\le \frac{C_2}{r}|x|^{r}\ \text{ for a.a. }\ z\in\Omega,\ \text{ all }x\in\mathbb{R}. \tag{2.7}$$

Let $x>\theta$. We have

$$\begin{aligned}\widehat{F}_\lambda^{+}(z,x)&=\int\limits_0^x\widehat{f}_\lambda^{+}(z,s)\,ds=\int\limits_0^x\widehat{f}_\lambda(z,s)\,ds\\&=\int\limits_0^x\left[\eta(s)\lambda f(z,s)+[1-\eta(s)]s^{r-1}\right]ds\ (\text{see } (\mathbf{??}))\\&=\int\limits_0^\theta\left[\eta(s)\lambda f(z,s)+[1-\eta(s)]s^{r-1}\right]ds+\int\limits_\theta^x s^{r-1}ds\ (\text{see } (\mathbf{??}))\\&\le C_3\lambda\theta^r+\tfrac{1}{r}x^r\ \text{ for some } C_3>0.\end{aligned} \tag{2.8}$$

Since $x>\theta$, from (**??**) and (**??**) it follows that

$$\widehat{f}_\lambda^{+}(z,x)=x^{r-1}. \tag{2.9}$$

Then with $q\in(p,r)$ as in hypothesis $\mathbf{H}(g)(ii)$, we have

$$\widehat{f}_\lambda^{+}(z,x)\,x-q\widehat{F}_\lambda^{+}(z,x)\ge\left[1-\frac{q}{r}\right]x^r-qC_3\lambda\theta^r\ (\text{see } (\mathbf{??}),(\mathbf{??})). \tag{2.10}$$

Choose $M_+>\max\{M,\theta\}$ (see $\mathbf{H}(g)(ii)$) big such that

$$\left[1-\frac{q}{r}\right]M_+^r>qC_2\lambda\theta^r\ (\text{recall } q<r).$$

So, from (**??**) we have

$$\widehat{f}_\lambda^{+}(z,x)\,x\ge q\widehat{F}_\lambda^{+}(z,x)\ \text{ for a.a. }\ z\in\Omega,\ \text{ all }x\ge M_+.$$

Also note that for $x \geq M_+$, we have

$$\widehat{F}_\lambda^+(z,x) = \int_0^\theta \widehat{f}_\lambda^+(z,s)\,ds + \int_\theta^x \widehat{f}_\lambda^+(z,s)\,ds$$

$$\geq -C_2 \int_0^\theta s^{r-1} ds + \frac{1}{r}\left[x^r - \theta^r\right] \ \text{(see (??) and (??))}$$

$$= \frac{1}{r} x^r - \frac{C_4}{r}\theta^r \text{ for some } C_4 > 0.$$

Choosing M_+ even bigger if necessary, we may assume that

$$M_+^r > C_4 \theta^r.$$

Therefore we have

$$\operatorname*{essinf}_{\Omega} \widehat{F}_\lambda^+(\cdot, M_+) > 0 \text{ and } \widehat{F}_\lambda^+(z,x) > 0 \text{ for a.a. } z \in \Omega, \text{ all } x \geq M_+.$$

This proves that $\widehat{\zeta}_\lambda^+(z,\cdot)$ satisfies the AR condition. Similarly we show that $\widehat{\zeta}_\lambda^-(z,\cdot)$ satisfies the AR condition. □

3. Nonlinear problems

Let by $\widehat{\varphi}_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functionals defined by

$$\widehat{\varphi}_\lambda^\pm(u) = \frac{1}{p}\gamma(u) - \int_\Omega \left[\widehat{F}_\lambda^\pm(z,x) + G^\pm(z,u)\right] dz \text{ for all } u \in W^{1,p}(\Omega).$$

Proposition 3.1. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g)$ *hold and* $\lambda \geq 1$, *then we can find* $\rho_\lambda > 0$ *and* $\widehat{m}_\lambda > 0$ *such that*

$$\widehat{\varphi}_\lambda^\pm(u) \geq \widehat{m}_\lambda > 0 \text{ for all } u \in W^{1,p}(\Omega) \text{ with } \|u\| = \rho_\lambda.$$

Proof. Using (??), (??), hypothesis $\mathbf{H}(g)(ii)$ and the fact that $\lambda \geq 1$, we obtain

$$\widehat{\varphi}_\lambda^\pm(u) \geq C_1 \|u\|^p - \lambda C_5 \|u\|^r \text{ for some } C_5 > 0, \text{ all } u \in W^{1,p}(\Omega),$$

hence

$$\widehat{\varphi}_\lambda^\pm(u) \geq \left[C_1 - \lambda C_5 \|u\|^{r-p}\right] \|u\|^p.$$

Therefore if $\rho_\lambda \in \left(0, \left(\frac{C_1}{\lambda C_5}\right)^{\frac{1}{r-p}}\right)$, then

$$\widehat{\varphi}_\lambda^\pm(u) \geq \widehat{m}_\lambda := \rho_\lambda^p \left[C_1 - \lambda C_5^{r-p} \rho_\lambda^{r-p}\right] > 0$$
$$\text{for all } u \in W^{1,p}(\Omega) \text{ with } \|u\| = \rho_\lambda.$$

□

Proposition 3.2. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g)$ *hold, then there exist* $\widetilde{u} \in W^{1,p}(\Omega)$, $\widetilde{u} \geq 0$ *and* $\widetilde{\lambda}_1 \geq 1$ *such that for all* $\lambda \geq \widetilde{\lambda}_1$ *we have*

$$\widehat{\varphi}_\lambda^\pm(\pm\widetilde{u}) < 0 \text{ and } \|\widetilde{u}\| > \rho_\lambda.$$

Proof. Let $\widetilde{u} = \frac{\theta}{2} \in W^{1,p}(\Omega)$. Then from (**??**), (**??**) and hypothesis $\mathbf{H}(g)(iii)$, we have

$$\begin{aligned}\widehat{\varphi}_\lambda^{\pm}(\widetilde{u}) &\leq \frac{\widetilde{u}^p}{p}\left[\|\xi\|_\infty |\Omega|_N + \|\beta\|_{L^\infty(\partial\Omega)}\sigma(\partial\Omega)\right] - \int_\Omega \lambda F(z,\widetilde{u})\,dz \\ &\leq C_6\widetilde{u}^p - \lambda\widetilde{u}^\tau \text{ for some } C_6 > 0 \text{ (see (??))}.\end{aligned}$$

Here by $|\cdot|_N$ we denote the Lebesgue measure in $\mathbb{R}^N$.

We choose $\widetilde{\lambda}_0 \geq 1$ such that

$$\widehat{\varphi}_\lambda^{\pm}(\widetilde{u}) < 0 \text{ for all } \lambda \geq \widetilde{\lambda}_0. \tag{3.1}$$

From the proof of Proposition **??**, we know that

$$\rho_\lambda \to 0+ \text{ as } \lambda \to \infty.$$

So, we can find $\widetilde{\lambda}_1 \geq \widetilde{\lambda}_0 \geq 1$ such that

$$\|\widetilde{u}\| > \rho_\lambda \text{ for all } \lambda \geq \widetilde{\lambda}_1.$$

We conclude that for $\widetilde{u} = \frac{\theta}{2} \in int\ C_+$ and for $\lambda \geq \widetilde{\lambda}_1$ we have

$$\widehat{\varphi}_\lambda^{\pm}(\pm\widetilde{u}) < 0 \text{ and } \|\widetilde{u}\| > \rho_\lambda.$$

□

From Proposition **??**, we know that the integrands $\widehat{\zeta}_\lambda^{\pm}(\cdot,\cdot)$ satisfy the AR-condition. So, we have the following result (see Ambrosetti-Rabinowitz [**?**]):

Proposition 3.3. *If hypotheses $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g)$ hold, then for every $\lambda > 0$, the functionals $\widehat{\varphi}_\lambda^{\pm}$ satisfy the PS-condition.*

We consider the following nonlinear parametric Robin problem

$$(Q_\lambda) \qquad \begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \widehat{f}_\lambda(z,u(z)) + g(z,u(z)) \\ \hfill \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega,\ \lambda > 0,\ 1 < p < \infty. \end{cases}$$

Using variational tools, we can show the existence of constant sign solutions of (Q_λ) when $\lambda \geq 1$ is big.

Proposition 3.4. *If hypotheses $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g)$ hold, and $\lambda \geq \widetilde{\lambda}_1$ (see Proposition **??**), then problem (Q_λ) has at least two constant sign solutions $u_\lambda \in int\ C_+$ and $v_\lambda \in -int\ C_+$.*

Proof. Propositions **??**, **??** and **??** permit the use of the mountain pass theorem [**?**]. So, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$u_\lambda \in K_{\widehat{\varphi}_\lambda^+} \text{ and } \widehat{\varphi}_\lambda^+(0) = 0 < \widehat{m}_\lambda \leq C_\lambda = \widehat{\varphi}_\lambda^+(u_\lambda). \tag{3.2}$$

From (**??**) we have that $u_\lambda \neq 0$ and

$$\left(\widehat{\varphi}_\lambda^+\right)'(u_\lambda) = 0.$$

Hence

$$\begin{aligned}&\langle A(u_\lambda), h\rangle + \int_\Omega \xi(z)|u_\lambda(z)|^{p-2}u_\lambda(z)hdz \\ &+\int_{\partial\Omega}\beta(z)|u_\lambda(z)|^{p-2}u_\lambda(z)hd\sigma \\ &= \int_\Omega\left[\widehat{f}_\lambda^+(z,u_\lambda)+g_+(z,u_\lambda)\right]hdz \text{ for all } h\in W^{1,p}(\Omega).\end{aligned} \tag{3.3}$$

In (**??**) we choose $h=-u_\lambda^-\in W^{1,p}(\Omega)$. We obtain

$$C_1\left\|u_\lambda^-\right\|^p\leq 0 \text{ (see (\textbf{??})),}$$

therefore

$$u_\lambda\geq 0,\ u_\lambda\neq 0.$$

Then from (**??**) we have

$$\begin{cases} -\Delta_p u_\lambda(z)+\xi(z)u_\lambda(z)^{p-1}=\widehat{f}_\lambda(z,u_\lambda(z))+g(z,u_\lambda(z)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\text{for a.a. } z\in\Omega, \\ \frac{\partial u_\lambda}{\partial n_p}+\beta(z)u_\lambda^{p-1}=0 \text{ on } \partial\Omega.\end{cases} \tag{3.4}$$

From (**??**) and Proposition 2.10 of Papageorgiou-Radulescu [**?**], we infer that $u_\lambda\in L^\infty(\Omega)$. Then we apply Theorem 2 of Lieberman [**?**] and obtain that

$$u_\lambda\in C_+\backslash\{0\}.$$

From (**??**) it follows

$$\Delta_p u_\lambda(z)\leq\left[\|\xi\|_\infty+2\|u_\lambda\|_\infty^{r-p}\right]u_\lambda(z)^{p-1} \text{ for a.a. } z\in\Omega$$

(see (**??**), (**??**) and hypothesis $\mathbf{H}(g)(iii)$) and by the nonlinear maximum principle we get

$$u_\lambda\in int\ C_+.$$

Similarly, working this time with $\widehat{\varphi}_\lambda^-$, we produce a negative solution

$$v_\lambda\in -int\ C_+.$$

□

Next we determine the behavior of u_λ and v_λ as $\lambda\to\infty$.

Proposition 3.5. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta)$, $\mathbf{H}_0$, $\mathbf{H}(f)$, $\mathbf{H}(g)$ *hold, then*

$$u_\lambda\to 0 \text{ and } v_\lambda\to 0 \text{ in } C^1\left(\overline{\Omega}\right) \text{ as } \lambda\to+\infty.$$

Proof. Let $\lambda_n\to+\infty$ and consider $u_n=u_{\lambda_n}\in int\ C_+$ be positive solutions of problem (Q_{λ_n}), $n\in\mathbb{N}$. From the proof of Proposition **??**, we know that

$$\widehat{m}_{\lambda_n}\leq C_{\lambda_n}=\widehat{\varphi}_{\lambda_n}^+(u_n)=\inf_{\gamma\in\Gamma}\max_{0\leq s\leq 1}\widehat{\varphi}_{\lambda_n}^+(\widetilde{\gamma}(s)), \tag{3.5}$$

where

$$\Gamma=\left\{\widetilde{\gamma}\in C\left([0,1],W^{1,p}(\Omega)\right):\widetilde{\gamma}(0)=0,\widetilde{\gamma}(1)=\widetilde{u}\right\}$$

From (**??**) we have

$$\widehat{\varphi}_{\lambda_n}^+(u_n)\leq\max_{0\leq s\leq 1}\widehat{\varphi}_{\lambda_n}^+(s\widetilde{u}). \tag{3.6}$$

Also (**??**) , (**??**) , (**??**) and hypothesis $\mathbf{H}(g)(iii)$ imply that

$$\widehat{\varphi}_{\lambda_n}(s\widetilde{u}) \leq C_7 s^p - \lambda_n C_8 s^\tau \text{ for some } C_7 > 0,\ C_8 > 0.$$

We consider the function

$$\mu_{\lambda_n}(s) = C_7 s^p - C_8 s^\tau \text{ for all } s \geq 0, \text{ with } n \in \mathbb{N}.$$

Evidently since $p < \tau$, we can find $s_0 > 0$ such that

$$0 < \mu_{\lambda_n}(s_0) = \max_{s \geq 0} \mu_{\lambda_n}(s),$$

hence

$$\mu'_{\lambda_n}(s_0) = 0,$$

therefore

$$s_0 = s_0(\lambda_n) = \left[\frac{pC_7}{\lambda_n \tau C_8}\right]^{\frac{1}{\tau - p}}. \tag{3.7}$$

Using (**??**) we obtain

$$\mu_{\lambda_n}(s_0) \leq C_7 \left[\frac{pC_7}{\lambda_n \tau C_8}\right]^{\frac{p}{\tau - p}} = C_9 \lambda^{-\frac{p}{\tau - p}} \text{ for some } C_9 > 0, \text{ all } n \in \mathbb{N}. \tag{3.8}$$

From (**??**) we have

$$\widehat{\varphi}^+_{\lambda_n}(u_n) \leq \mu_{\lambda_n}(s_0) \leq C_9 \lambda^{-\frac{p}{\tau - p}} \text{ for all } n \in \mathbb{N} \text{ (see (??))},$$

hence

$$q\widehat{\varphi}^+_{\lambda_n}(u_n) + \left\langle \left(\widehat{\varphi}^+_{\lambda_n}\right)'(u_n), u_n \right\rangle \leq qC_9 \lambda^{-\frac{p}{\tau - p}} \text{ for all } n \in \mathbb{N},$$

therefore

$$\begin{aligned} &\left[\frac{q}{p} - 1\right] \gamma(u_n) \\ &+ \int_\Omega \left[\left(\widehat{f}^+_{\lambda_n}(z, u_n) + g_+(z, u_n)\right) u_n - q\widehat{F}^+_{\lambda_n}(z, u_n) + G_+(z, u_n)\right] dz \\ &\leq qC_9 \lambda^{-\frac{p}{\tau - p}}, \end{aligned}$$

and in view of Proposition **??** and hypothesis $\mathbf{H}(g)(ii)$ we conclude that

$$\|u_n\|^p \leq C_{10} \text{ for some } C_{10} > 0, \text{all } n \in \mathbb{N}.$$

Therefore $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. Then Proposition 2.10 of Papageorgiou-Radulescu [**?**] implies that we can find $C_{11} > 0$ such that

$$\|u_n\|_\infty \leq C_{11} \text{ for all } n \in \mathbb{N}$$

Invoking Theorem 2 of Lieberman [**?**], we can find $\alpha \in (0, 1)$ and $C_{12} > 0$ such that

$$u_n \in C^{1,\alpha}\left(\overline{\Omega}\right) \text{ and } \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_{12} \text{ for all } n \in \mathbb{N}.$$

We know that $C^{1,\alpha}\left(\overline{\Omega}\right)$ is compactly embedded in $C^1\left(\overline{\Omega}\right)$, so for at least a subsequence we have

$$u_n \to \overline{u} \text{ in } C^1\left(\overline{\Omega}\right) \text{ as } n \to \infty.$$

By (**??**) and (**??**) we infer

$$\widehat{\varphi}^+_{\lambda_n}(u_n) \to 0^+ \text{ as } n \to \infty. \tag{3.9}$$

Moreover, we have

$$\left\langle \left(\widehat{\varphi}_{\lambda_n}^{+}\right)'(u_n), h\right\rangle = 0 \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \tag{3.10}$$

Since $\lambda_n \to +\infty$, from (**??**) and (**??**) it follows that $\overline{u} = 0$. Therefore we conclude that

$$u_n \to 0 \text{ in } C^1\left(\overline{\Omega}\right) \text{ as } n \to \infty.$$

Similarly, working this time with $\widehat{\varphi}_{\lambda_n}^{-}(\cdot)$ we show that

$$v_{\lambda_n} \to 0 \text{ in } C^1\left(\overline{\Omega}\right) \text{ as } n \to \infty.$$

□

Now we will produce extremal constant sign solutions for problem (Q_λ), that is, we will show that for $\lambda > 0$ big, problem (Q_λ) has a smallest positive solution and a biggest negative solution

So, we consider the following two solution sets

$$\widehat{\mathcal{S}}_{\lambda}^{+} = \{u : u \text{ is a positive solution of } (Q_\lambda)\},$$

$$\widehat{\mathcal{S}}_{\lambda}^{-} = \{u : u \text{ is a negative solution of } (Q_\lambda)\}.$$

From Proposition **??** it follows that for $\lambda \geq \widetilde{\lambda}_1$

$$\varnothing \neq \widehat{\mathcal{S}}_{\lambda}^{+} \subseteq int\ C_+ \text{ and } \varnothing \neq \widehat{\mathcal{S}}_{\lambda}^{-} \subseteq -int\ C_+ .$$

Moreover, from Papageorgiou-Radulescu-Repovs [**?**] (see the proof of Proposition 7), we know that

$$\widehat{\mathcal{S}}_{\lambda}^{+} \text{ is downward directed}$$

and

$$\widehat{\mathcal{S}}_{\lambda}^{-} \text{ is upward directed.}$$

Proposition 3.6. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g),$ *hold and* $\lambda \geq \widetilde{\lambda}_1$*, then problem* (Q_λ) *has a smallest positive solution* $u_\lambda^* \in int\ C_+$ *and a biggest negative solution* $v_\lambda^* \in -int\ C_+$.

Proof. By Lemma 3.10, p.178 of Hu-Papageorgiou [**?**], we can find a decreasing sequence $\{u_n\}_{n\geq 1} \subseteq \widehat{\mathcal{S}}_{\lambda}^{+}$ such that

$$\inf_{n\geq 1} u_n = \inf \widehat{\mathcal{S}}_{\lambda}^{+}.$$

We have

$$\begin{aligned} &\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n(z)^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n(z)^{p-1} h d\sigma \\ &= \int_\Omega \left[\widehat{f}_\lambda(z, u_n) + g_+(z, u_n)\right] h dz \\ &\qquad \text{for all } n \in \mathbb{N}, \text{ all } h \in W^{1,p}(\Omega), \end{aligned} \tag{3.11}$$

$$0 \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N}. \tag{3.12}$$

In (??) we chose $h = u_n \in W^{1,p}(\Omega)$ and using (??) and (??), we infer that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \overset{w}{\to} u_\lambda^* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_\lambda^* \text{ in } L^p(\Omega) \text{ and } L^p(\partial\Omega). \tag{3.13}$$

In (??) we choose $h = u_n - u_\lambda^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (??). We obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_\lambda^*\rangle = 0,$$

hence

$$u_n \to u_\lambda^* \text{ in } W^{1,p}(\Omega) \tag{3.14}$$

(see Section 2). We pass to the limit as $n \to \infty$ in (??) and use (??). Then

$$\begin{aligned}\langle A(u_\lambda^*), h\rangle + \int_\Omega \xi(z)(u_\lambda^*)^{p-1} h dz + \int_{\partial\Omega} \beta(z)(u_\lambda^*)^{p-1} h d\sigma \\ = \int_\Omega \left[\widehat{f}_\lambda(z, u_\lambda^*) + g(z, u_\lambda^*)\right] h dz \text{ for all } h \in W^{1,p}(\Omega),\end{aligned}$$

hence $u_\lambda^* \in \widehat{\mathcal{S}}_\lambda^+ \cup \{0\}$. If we show that $u_\lambda^* \neq \{0\}$, then u_λ^* is the desired minimal positive solution 0f (Q_λ).

We argue indirectly. So, suppose that $u_\lambda^* = 0$. Then $u_n \to 0$ in $W^{1,p}(\Omega)$ (see (??)). We set

$$y_n = \frac{u_n}{\|u_n\|},\ n \in \mathbb{N}.$$

Then

$$\|y_n\| = 1, y_n > 0 \text{ for all } n \in \mathbb{N}.$$

We may assume that

$$y_n \overset{w}{\to} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega) \text{ and } L^p(\partial\Omega). \tag{3.15}$$

From (??) we have

$$\begin{aligned}\langle A(y_n), h\rangle + \int_\Omega \xi(z) {y_n}^{p-1} h dz + \int_{\partial\Omega} \beta(z) {y_n}^{p-1} h d\sigma \\ = \int_\Omega \left[\frac{\widehat{f}_\lambda(z,u_n)}{\|u_n\|^{p-1}} + \frac{g(z,u_n)}{\|u_n\|^{p-1}}\right] h dz \text{ for all } h \in W^{1,p}(\Omega).\end{aligned} \tag{3.16}$$

By (??) and (??) we see that

$$\left\{\frac{\widehat{f}_\lambda(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{r'}(\Omega) \text{ is bounded, where } \frac{1}{r} + \frac{1}{r'} = 1. \tag{3.17}$$

Similarly from hypothesis $\mathbf{H}(g)(i)$ it follows that

$$\left\{\frac{g(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{r'}(\Omega) \text{ is bounded.} \tag{3.18}$$

If in (??) we choose $h = y_n - y \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (??), (??) and (??), we obtain

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0,$$

hence

$$y_n \to y \text{ in } W^{1,p}(\Omega) \text{ (see Section 2), with } \|y\| = 1. \tag{3.19}$$

On account of (**??**) , (**??**) , (**??**) , (**??**) and hypothesis $\mathbf{H}(g)(iii)$, we have

$$\frac{\widehat{f}_\lambda(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{w} 0 \text{ and } \frac{g(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{w} 0 \text{ in } L^{r'}(\Omega). \tag{3.20}$$

So, if in (**??**)we pass to the limit as $n \to \infty$ and use (**??**) and (**??**) , then

$$\langle A(y), h\rangle + \int_\Omega \xi(z) y^{p-1} h dz + \int_{\partial\Omega} \beta(z) y^{p-1} h d\sigma = 0 \text{ for all } h \in W^{1,p}(\Omega).$$

Let $h = y \in W^{1,p}(\Omega)$. Then

$$C_1 \|y\|^p \leq 0 \text{ (see (??))},$$

hence $y = 0$, which contradicts (**??**) . Therefore $u_\lambda^* \neq 0$ and so

$$u_\lambda^* \in \widehat{\mathcal{S}}_\lambda^+ \text{ and } u_\lambda^* = \inf \widehat{\mathcal{S}}_\lambda^+ .$$

Similarly, working with $\widehat{\mathcal{S}}_\lambda^-$, we produce $v_\lambda^* \in \widehat{\mathcal{S}}_\lambda^-$ with $v_\lambda^* = \sup \widehat{\mathcal{S}}_\lambda^-$. In this case, since $\widehat{\mathcal{S}}_\lambda^-$ is upward directed, we can find $\{v_n\}_{n\geq 1} \subseteq \widehat{\mathcal{S}}_\lambda^-$ increasing, such that

$$\sup_{n\geq 1} v_n = \sup \widehat{\mathcal{S}}_\lambda^-.$$

□

We will use these two extremal constant sign solutions in order to produce a nodal solution for problem (Q_λ) when λ is big enough.

Proposition 3.7. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta)$, $\mathbf{H}_0$, $\mathbf{H}(f)$, $\mathbf{H}(g)$ *hold, then there exists* $\widetilde{\lambda}_2 \geq \widetilde{\lambda}_1$ *such that for all* $\lambda \geq \widetilde{\lambda}_2$, *problem* (Q_λ) *has a nodal solution* $y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$.

Proof. Let $u_\lambda^* \in int\ C_+$ and $v_\lambda^* \in -int\ C_+$ be the two extremal constant sign solutions of problem (Q_λ) produced in Proposition **??**. We introduce the following Carathéodory function

$$\widehat{k}_\lambda(z,x) = \begin{cases} \widehat{f}_\lambda(z, v_\lambda^*(z)) + g(z, v_\lambda^*(z)) & \text{if } x < v_\lambda^*(z) \\ \widehat{f}_\lambda(z,x) + g(z,x) & \text{if } v_\lambda^*(z) \leq x \leq u_\lambda^*(z) \\ \widehat{f}_\lambda(z, u_\lambda^*(z)) + g(z, u_\lambda^*(z)) & \text{if } u_\lambda^*(z) < x. \end{cases} \tag{3.21}$$

We consider the positive and negative truncations of $\widehat{k}_\lambda(z,\cdot)$, namely the Carathéodory functions

$$\widehat{k}_\lambda^\pm(z,x) = \widehat{k}_\lambda\left(z, \pm x^\pm\right). \tag{3.22}$$

We set

$$\widehat{K}_\lambda(z,x) = \int_0^x \widehat{k}_\lambda(z,s)\, ds \text{ and } \widehat{K}_\lambda^\pm(z,x) = \int_0^x \widehat{k}_\lambda^\pm(z,s)\, ds$$

and introduce the C^1-functionals $\widehat{\psi}_\lambda$, $\widehat{\psi}_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_\lambda(u) = \frac{1}{p}\gamma(u) - \int_\Omega \widehat{K}_\lambda(z,u)\,dz \text{ for all } u \in W^{1,p}(\Omega)$$

and

$$\widehat{\psi}_\lambda^\pm(u) = \frac{1}{p}\gamma(u) - \int_\Omega \widehat{K}_\lambda^\pm(z,u)\,dz \text{ for all } u \in W^{1,p}(\Omega).$$

Using (**??**), (**??**) and the nonlinear regularity theory, we show easily that

$$K_{\widehat{\psi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}),\ K_{\widehat{\psi}_\lambda^+} \subseteq [0, u_\lambda^*] \cap C_+, K_{\widehat{\psi}_\lambda^-} \subseteq [v_\lambda^*, 0] \cap (-C_+).$$

The extremality of u_λ^*, v_λ^* implies that

$$K_{\widehat{\psi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}),\ K_{\widehat{\psi}_\lambda^+} = \{0, u_\lambda^*\},\ K_{\widehat{\psi}_\lambda^-} = \{0, v_\lambda^*\}. \tag{3.23}$$

Note that $\widehat{\psi}_\lambda^+$ is coercive (see (**??**), (**??**)). Also it is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\widetilde{u}_\lambda^* \in W^{1,p}(\Omega)$ such that

$$\widehat{\psi}_\lambda^+(\widetilde{u}_\lambda^*) = \inf\left\{\widehat{\psi}_\lambda^+(u) : u \in W^{1,p}(\Omega)\right\}. \tag{3.24}$$

Let

$$u_* = \min\left\{\frac{\theta}{2}, \min_{\overline{\Omega}} u_\lambda^*\right\} > 0$$

(recall that $u_\lambda^* \in int\ C_+$). Then

$$\widehat{\psi}_\lambda^+(u_*) \le C_{13}u_*^p - \lambda C_{14}u_*^\tau \text{ for some } C_{13},\ C_{14} > 0$$

(see (**??**), (**??**) and hypothesis $\mathbf{H}(g)(iii)$). So, we can find $\widetilde{\lambda}_2^+ \ge \widetilde{\lambda}_1$ such that

$$\widehat{\psi}_\lambda^+(u_*) < 0 \text{ for all } \lambda \ge \widetilde{\lambda}_2^+,$$

hence

$$\widehat{\psi}_\lambda^+(u_\lambda^*) < 0 = \widehat{\psi}_\lambda^+(0) \text{ for all } \lambda \ge \widetilde{\lambda}_2^+ \text{ (see (??))},$$

therefore

$$\widetilde{u}_\lambda^* \ne 0 \text{ for all } \lambda \ge \widetilde{\lambda}_2^+. \tag{3.25}$$

From (**??**) we have

$$\widetilde{u}_\lambda^* \in K_{\widehat{\psi}_\lambda^+},$$

hence

$$\widetilde{u}_\lambda^* = u_\lambda^* \in int\ C_+ \text{ (see (??), (??))}.$$

It is clear from (**??**) that

$$\widehat{\psi}_\lambda^+ |_{C_+} = \widehat{\psi}_\lambda |_{C_+},$$

hence u_λ^* is a local $C^1(\overline{\Omega})-$minimizer of $\widehat{\psi}_\lambda$, therefore

$$u_\lambda^* \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widehat{\psi}_\lambda \text{ for all } \lambda \ge \widetilde{\lambda}_2^+ \tag{3.26}$$

(see Papageorgiou-Radulescu [**?**], Proposition 2.12).

Similarly, working this time with $\widehat{\psi}_\lambda^-$, we produce $\widetilde{\lambda}_2^- \ge \widetilde{\lambda}_1$ such that

$$v_\lambda^* \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widehat{\psi}_\lambda \text{ for all } \lambda \ge \widetilde{\lambda}_2^-. \tag{3.27}$$

Let

$$\widetilde{\lambda}_2 = \max\left\{\widetilde{\lambda}_2^+, \widetilde{\lambda}_2^-\right\}$$

and let $\lambda \geq \widetilde{\lambda}_2$. We may assume that

$$\widehat{\psi}_\lambda(v^*) \leq \widehat{\psi}_\lambda(u^*).$$

The reasoning is similar if the opposite inequality holds, using (**??**) instead of (**??**). Also, we may assume that

$$K_{\widehat{\psi}_\lambda} \text{ is finite.} \tag{3.28}$$

Otherwise, we already have an infinity of smooth nodal solutions.

Using (**??**), (**??**) and Theorem 5.7.6, p. 448, of Papageorgiou-Radulescu-Repovs [**?**], we can find $\rho \in (0,1)$ small, such that

$$\begin{aligned} &\widehat{\psi}_\lambda(v_\lambda^*) \leq \widehat{\psi}_\lambda(u_\lambda^*) < \inf\left\{\widehat{\psi}_\lambda(u) : \|u - u_\lambda^*\| = \rho\right\} =: \widehat{m}_\lambda, \\ &\|u_\lambda^* - v_\lambda^*\| > \rho. \end{aligned} \tag{3.29}$$

Evidently, $\widehat{\psi}_\lambda(\cdot)$ is coercive (see (**??**)). Therefore

$$\widehat{\psi}_\lambda \text{ satisfies the PS-condition} \tag{3.30}$$

(see Papageorgiou-Radulescu-Repovs [**?**], Proposition 5.1.15, p.369).

Then (**??**), (**??**) permit the use of the mountain pass theorem. So, we can find $y_\lambda \in W^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{\widehat{\psi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1\left(\overline{\Omega}\right), \ \widehat{m}_\lambda \leq \widehat{\psi}_\lambda(y_\lambda) \tag{3.31}$$

(see (**??**) and (**??**)). From (**??**) and (**??**) it follows that

$$y_\lambda \notin \{u_\lambda^*, v_\lambda^*\}. \tag{3.32}$$

Since y_λ is a critical point of $\widehat{\psi}_\lambda(\cdot)$ of mountain pass type, we have

$$C_1\left(\widehat{\psi}_\lambda, y_\lambda\right) \neq 0. \tag{3.33}$$

(see Papageorgiou-Radulescu-Repovs [**?**], Theorem 6.5.8, p.527).

On the other hand, if $u \in C^1\left(\overline{\Omega}\right)$ and

$$\|u\|_{C^1(\overline{\Omega})} \leq \rho_0 \leq \min\left\{\frac{\theta}{2}, \min\left\{\min_{\overline{\Omega}} u_\lambda^*, \min_{\overline{\Omega}}(-v_\lambda^*)\right\}\right\}$$

(recall that $u_\lambda^* \in int\ C_+$, $v_\lambda^* \in -int\ C_+$, see Proposition **??**), then

$$\begin{aligned} \widehat{\psi}_\lambda(u) &= \frac{1}{p}\gamma(u) - \int_\Omega [\lambda F(z,u) + G(z,u)]\,dz \text{ (see (??), (??), (??))} \\ &\geq \frac{1}{p}\gamma(u) - \frac{1}{r}[\lambda + C_0]\|u\|_r^r \text{ (see (??), and } \mathbf{H}(g)(iii) \\ &\geq \frac{C_1}{p}\|u\|^p - \frac{1}{r}[\lambda + C_0]\|u\|^r \text{ (see (??)).} \end{aligned}$$

Since $r > p$, for $\rho_0 \in (0,1)$ small, we have

$$\widehat{\psi}_\lambda(u) > 0 \text{ for all } 0 < \|u\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

hence $u = 0$ is a local $C^1(\overline{\Omega})$ −minimizer of $\widehat{\psi}_\lambda(\cdot)$, therefore $u = 0$ is a local $W^{1,p}(\Omega)$-minimizer of $\widehat{\psi}_\lambda(\cdot)$ (see [?]), and we conclude that

$$C_k\left(\widehat{\psi}_\lambda, 0\right) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \tag{3.34}$$

(where $\delta_{k,l}$ denotes the Kronecker symbol defined by $\delta_{k,l} = 1$ if $k = l$ and $\delta_{k,l} = 0$ if $k \neq l$). Comparing (??) and (??) , we infer that $y_\lambda \neq 0$ and so, $y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$ is a nodal solution of the problem (Q_λ) , for $\lambda \geq \widetilde{\lambda}_2$. □

In view of Proposition **??**, we arrive at:

Proposition 3.8. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g)$ *hold, then*

$$u_\lambda^*,\ v_\lambda^*,\ y_\lambda \to 0 \text{ in } C^1(\overline{\Omega}) \text{ as } \lambda \to +\infty.$$

Then Proposition **??** and (**??**) lead to the following multiplicity theorem for (P_λ) .

Theorem 3.9. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g)$ *hold, then there exists* $\widetilde{\lambda}_3 \geq \widetilde{\lambda}_2$ *such that for* $\lambda \geq \widetilde{\lambda}_3$*, problem* (P_λ) *has at least three nontrivial solutions*

$$u_\lambda \in int\ C_+,\ v_\lambda \in -int\ C_+ \text{ and } y_\lambda \in [v_\lambda, u_\lambda] \cap C^1(\overline{\Omega}),\ \text{nodal.}$$

Moreover,

$$u_\lambda,\ v_\lambda,\ y_\lambda \to 0 \text{ in } C^1(\overline{\Omega}) \text{ as } \lambda \to +\infty.$$

4. Semilinear problems

In the semilinear case ($p = 2$), under stronger regularity hypotheses on $f(z,\cdot)$ and $g(z,\cdot)$, we can improve Theorem **??** by producing a second nodal solution of (P_λ) for a total of four nontrivial solutions, all with sign information.

So, now the problem under consideration is the following

$$(SP_\lambda) \qquad \begin{cases} -\Delta u(z) + \xi(z) u(z) = \lambda f(z, u(z)) + g(z, u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z) u = 0 \text{ on } \partial\Omega,\ \lambda > 0. \end{cases}$$

The conditions on the two nonlinearities $f(z,x)$ and $g(z,x)$ are the following.

$\mathbf{H}(f)'$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $f(z,0) = 0$ for a.a. $z \in \Omega$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(*i*) there exists $r \in (2, 2^*)$ such that

$$\lim_{x\to 0} \frac{f(z,x)}{|x|^{r-2} x} = 0 \text{ uniformly for a.a. } z \in \Omega;$$

(*ii*) if $F(z,x) = \int_0^x f(z,s)\,ds$, then there exists $\tau \in (r, 2^*)$ such that

$$\lim_{x\to\infty} \frac{F(z,x)}{x^\tau} = +\infty \text{ uniformly for a.a. } z \in \Omega.$$

Remark: Hypothesis $\mathbf{H}(f)'(i)$ implies that

$$0 = f_x'(z,0) = \lim_{x\to 0} \frac{f(z,x)}{x} \text{ uniformly for a.a. } z \in \Omega.$$

H (g): $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $g(z,0) = 0$ for a.a. $z \in \Omega$, $g(z,\cdot) \in C^1(\mathbb{R})$ and

(i) there exist $a \in L^\infty(\Omega)$ and $2 < d < 2^*$ such that

$$\left|g_x'(z,x)\right| \le a(z)\left[1 + |x|^{d-2}\right] \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R};$$

(ii) If $G(z,x) = \int_0^x g(z,s)\,ds$, then there exist $q \in (2,r)$ and $M > 0$ such that

$$0 < qG(z,x) \le g(z,x)\,x \text{ for a.a. } z \in \Omega, \text{ all } |x| \ge M,$$

and

$$0 \le \operatorname*{essinf}_{\Omega} G(\cdot, \pm M);$$

(iii) there exists $c_0 > 0$ such that

$$0 \le g(z,x)\,x \le c_0 |x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Remark: Hypothesis $\mathbf{H}(g)'(iii)$ implies that

$$0 = g'(z,x) = \lim_{x \to 0} \frac{g(z,x)}{x} \text{ uniformly for a.a. } z \in \Omega.$$

$\mathbf{H}_1$: For every $\lambda > 0$ and $\rho > 0$, there exists $\xi_\rho^\lambda > 0$ such that for a.a. $z \in \Omega$, the function $x \to \lambda f(z,x) + g(z,x) + \xi_\rho^\lambda x$ is nondecreasing on $[-\rho, \rho]$.

Remark: This is a lower Lipschitz condition. It is satisfied if for every $\lambda > 0$ and $\rho > 0$, there exists $\widehat{\xi}_\rho^\lambda > 0$ such that

$$\lambda f_x'(z,x) + g_x'(z,x) \ge -\widehat{\xi}_\rho^\lambda \text{ for a.a. } z \in \Omega., \text{ all } |x| \le \rho.$$

In what follows we set

$$\zeta_\lambda(z,x) = \widehat{f}_\lambda(z,x) + g(z,x), \quad \widehat{F}_\lambda(z,x) = \int_0^x \widehat{f}_\lambda(z,s)\,ds$$

and we consider the C^1-functional $\widehat{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\gamma(u) - \int_\Omega \left[\widehat{F}_\lambda(z,x) + G(z,u)\right] dz \text{ for all } u \in W^{1,p}(\Omega).$$

Theorem 4.1. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f)', \mathbf{H}(g)', \mathbf{H}_1$ *hold, then there exists* $\widetilde{\lambda}_3 \ge 1$ *such that for all* $\lambda \ge \widetilde{\lambda}_3$*, problem* (P_λ) *has at least four nontrivial solutions*

$$u_\lambda \in int\ C_+,\ v_\lambda \in -int\ C_+,\ \textit{and}\ y_\lambda,\ \widehat{y}_\lambda \in int_{C^1(\overline{\Omega})}[v_\lambda, u_\lambda],\ \textit{nodal.}$$

Proof. From Theorem **??**, we know that there exists $\widetilde{\lambda}_3 \ge 1$ such that for all $\lambda \ge \widetilde{\lambda}_3$ problem (P_λ).has at least three nontrivial solutions

$$u_\lambda \in int\ C_+,\ v_\lambda \in -int\ C_+ \text{ and } y_\lambda \in [v_\lambda, u_\lambda] \cap C^1\left(\overline{\Omega}\right) \text{ nodal.} \tag{4.1}$$

Let $\rho = \max\{\|u_\lambda\|_\infty, \|v_\lambda\|_\infty\}$ and let $\widehat{\xi}_\rho^\lambda > 0$ be as postulated by hypothesis $\mathbf{H}_1$. We have

$$\begin{aligned}
&-\Delta y_\lambda + \left[\xi(z) + \widehat{\xi}_\rho^\lambda\right] y_\lambda = \lambda f(z, y_\lambda) + g(z, y_\lambda) + \widehat{\xi}_\rho^\lambda y_\lambda \\
&\leq \lambda f(z, u_\lambda) + g(z, u_\lambda) + \widehat{\xi}_\rho^\lambda u_\lambda \text{ (see (\textbf{??}) and } \mathbf{H}_1) \\
&= -\Delta u_\lambda + \left[\xi(z) + \widehat{\xi}_\rho^\lambda\right] u_\lambda
\end{aligned}$$

hence

$$\Delta(u_\lambda - y_\lambda) \leq \left[\|\xi\|_\infty + \widehat{\xi}_\rho^\lambda\right](u_\lambda - y_\lambda),$$

therefore $u_\lambda - y_\lambda \in int\ C_+$ (by the Hopf boundary point theorem). Similarly we show that

$$y_\lambda - v_\lambda \in int\ C_+.$$

It follows that

$$y_\lambda \in int_{C^1(\overline{\Omega})}[v_\lambda, u_\lambda]. \tag{4.2}$$

Consider the homotopy

$$h_t(u) = h(t,u) = (1-t)\widehat{\psi}_\lambda(u) + t\widehat{\varphi}_\lambda(u) \text{ for all } (t,u) \in [0,1] \times H^1(\Omega).$$

Suppose that we could find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{y_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$t_n \to t \text{ in } [0,1],\ y_n \to y \text{ in } H^1(\Omega), h_t'(y_n) = 0 \text{ for all } n \in \mathbb{N}.$$

We have

$$\begin{aligned}
&\langle A(y_n), h\rangle + \int_\Omega \xi(z) y_n h dz + \int_{\partial\Omega} \beta(z) y_n h d\sigma \\
&= (1-t_n)\int_\Omega k_\lambda(z, y_n) h dz + t_n \int_\Omega \zeta_\lambda(z, y_n) h dz \text{ for all } h \in H^1(\Omega).
\end{aligned} \tag{4.3}$$

By (**??**), using standard regularity theory, we show that in fact we have

$$y_n \to y \text{ in } C^1\left(\overline{\Omega}\right)$$

hence

$$y_n \in [v_\lambda, u_\lambda] \text{ for all } n \geq n_0 \text{ (see (\textbf{??})).}$$

This contradicts (**??**). Then, the homotopy invariance property of critical groups (see Papageorgiou-Radulescu-Repovs [**?**], Theorem 6.3.8, p.505) implies that

$$C_k\left(\widehat{\psi}_\lambda, y_\lambda\right) = C_k(\widehat{\varphi}_\lambda, y_\lambda) \text{ for all } k \in \mathbb{N}_0, \tag{4.4}$$

hence

$$C_1(\widehat{\varphi}_\lambda, y_\lambda) \neq 0 \text{ (see (\textbf{??})).} \tag{4.5}$$

But $\widehat{\varphi}_\lambda \in C^2\left(H^1(\Omega), \mathbb{R}\right)$. So, by (**??**) and Theorem 6.5.11, p.530 of Papageorgiou-Radulescu-Repovs [**?**], we have

$$C_k(\widehat{\varphi}_\lambda, y_\lambda) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0,$$

hence

$$C_k\left(\widehat{\psi}_\lambda, y_\lambda\right) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0, \text{ (see (\textbf{??})).} \tag{4.6}$$

Recall that u_λ, v_λ are local minimizers of $\widehat{\psi}_\lambda(\cdot)$ (see the proof of Proposition **??**). Hence

$$C_k\left(\widehat{\psi}_\lambda, u_\lambda\right) = C_k\left(\widehat{\psi}_\lambda, v_\lambda\right) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{4.7}$$

Also from (**??**) we have

$$C_k\left(\widehat{\psi}_\lambda, 0\right) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{4.8}$$

The functional $\widehat{\psi}_\lambda(\cdot)$ is coercive (see (**??**)). Hence we obtain

$$C_k\left(\widehat{\psi}_\lambda, \infty\right) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{4.9}$$

Suppose that $K_{\widehat{\psi}_\lambda} = \{0, u_\lambda, v_\lambda, y_\lambda\}$. Then from (**??**), (**??**), (**??**), (**??**) and the Morse relation with $t = -1$ (see (**??**)) it follows

$$3(-1)^0 + (-1)^1 = (-1)^0,$$

therefore $(-1)^0 = 0$, a contradiction.

So, there exists $\widehat{y}_\lambda \in K_{\widehat{\psi}_\lambda}$, $\widehat{y}_\lambda \notin \{0, u_\lambda, v_\lambda, y_\lambda\}$, and since $\lambda \geq \widetilde{\lambda}_3$, this is the second nodal solution for problem (P_λ). Finally, using the Hopf boundary point theorem, we conclude that

$$\widehat{y}_\lambda \in int_{C^1(\overline{\Omega})}[v_\lambda, u_\lambda].$$

□

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