Relating Kleene algebras with pseudo uninorms

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Abstract. This paper explores a strict relation between two core notions of the semantics of programs and of fuzzy logics: Kleene Algebras and (pseudo) uninorms. It shows that every Kleene algebra induces a pseudo uninorm, and that some pseudo uninorms induce Kleene algebras. This connection establishes a new perspective on the theory of Kleene algebras and provides a way to build (new) Kleene algebras. The latter aspect is potentially useful as a source of formalism to capture and model programs acting with fuzzy behaviours and domains.

1 Introduction

The adoption of algebraic structures and techniques to model and reason about programs has a long tradition in Computer Science and is the basis of some of its main pillars, including Process Algebra and Abstract Data Types Specification. In particular, *Algebras of Programs*, coming from regular languages and automata theory, have been widely considered as a suitable framework to support the rigorous semantics for analysis of algorithms and the design and development of complex systems. On the basis of this field is the notion of Kleene Algebra [14], today accepted as the standard abstraction of a computational system. Among its examples, an algebraic framework for coherent confluence proofs, in rewriting theory, for a higher dimensional generalisation of modal Kleene algebra proposes in [3] and the algebra of the regular languages, traces of programs and the algebra of relations on which the program states transitions are modelled as binary relations on the set of states. For instance, by starting from the atomic programs represented in the transition systems of Fig. 1 we can adopt the Kleene algebra of binary relations to interpret composed programs. For instance, the sequential composition $A_\pi; A_{\pi'}$ that consists in executing one computation step in A_π followed by another in $A_{\pi'}$, is interpreted by the standard relational composition, as represented in Fig 2. Moreover, operations of non deterministic choice + and iteration closure $*$, the ones needed to encode any imperative program, are also provided by the mentioned Kleene algebra.

If the mentioned above models plays a relevant role in the current formal development and design processes, the emergence of new computational paradigms and scenarios, as Fuzzy and Probabilistic programming, entails not only the definition of new Kleene algebra models, but also some variants and generalisations.

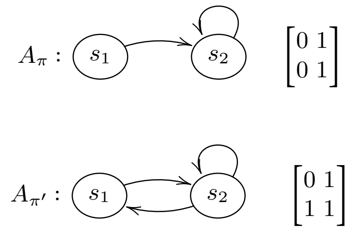

Fig. 1. Examples of abstract programs

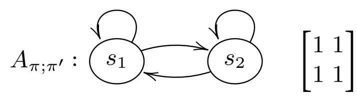

Fig. 2. Examples of abstract programs

As examples of the latter efforts, we can point out our recent development on the study of Kleene algebras to deal with "intervals as programs" [22], in order to deal with situations where the precise values of the transitions weights are not provided (e.g. entailed by the machine representation of an irrational number).

This paper develops a novel algebraic study on Kleene algebras, based on pseudo uninorms defined over partial orders. As is well known, we can easily obtain a Kleene algebra from any Boolean algebra, by taking the operation $\star$ as the constant function $x\star = \top$, where $\top$ is the top element of the algebra. Following this intuition, we abstract the infimum operation as pseudo uninorms defined over partial orders, in order to build algebras for fuzzy programs. As expected such new algebras generalises the standard case.

We investigate how fuzzy programs, i.e., elements of these structures, behaves with respect to the Kleene operations. At this level, classic choice is maintained, but the notions of sequential composition and Kleene closure are abstracted as specific uninorms, defined over meet semilattices.

Building new Kleene algebras from other Kleene algebras can be also very useful. The work of Conway plays a very relevant role. He introduces in [4] some matricial constructions that preserve the Kleene algebra structure. In other words, he introduces a method, with which, given a a Kleene algebra over a set K, it construct a Kleene algebra over the squared matrices $\mathrm{M}_n(K)$. For instance the Kleene algebra of relations used bellow (cf. Fig 1), which elements are the adjacency matrices, can be taken with this method from the Kleene algebra defined by the two-elements boolean algebra (with $0^\star = 1^\star = 1$).

In this work we introduce an operator to construct new Kleene algebras from other Kleene algebras based in the notion of automorphism. These maps re-interprets programs and the programs operations of a Kleene algebra into a new Kleene algebra, by contributing with an alternative source of program algebras.

Context and Contributions.

In [18] Menger introduced triangular norms (t-norms) in order to provide triangle inequality for distances on probabilistic metric spaces. Since Menger's definition is weak, Schweizer and Sklar [23] provided a new definition for t-norms adding new axioms such as associativity and taking 1 as the neutral element. In [24], they introduced the notion of t-conorm by simply taking 0 as the neutral element instead of 1.

The axioms of T-norms (T-conorms) was, then, changed by abolishing some of its conditions. Those weakening gave rise to the so called **pseudo t-norm**. In [9] (see also [7]) Siegfried Gottwald considered the notion of pseudo t-norm by abolishing the commutativity property. On the other hand, further authors such as abolished other axioms — see [11, 15, 29]. In particular, in [11] Sándor Jenei suppressed the commutativity property and the left side of the isotonicity property; and in [15], in addition to these two properties, Hua-Wen Liu suppressed the associativity. In [17], M. Mas, M. Monserrat and J. Torrens introduced the notion of left uninorms and right uninorms. One year after W. Sander [21] introduced the notion of pseudo uninorm as a bivariated function on the unit interval that is associative, isotone and has a neutral element. This notion coincides with the functions which are, both, left and right uninorms. In [28] the notion of left uninorms, right uninorms and pseudo uninorms was extended for lattice-valued sets. Two pseudo uninorms having the same neutral element is called of the **same type**. In [27] the notion of pseudo uninorm was extended for complete lattices and here we generalize them for posets. Recently, the papers [16, 25, 26] consider lattice-valued and $[0,1]$-valued pseudo uninorms.

In this paper, we investigate the notion of pseudo uninorms and show how they can be used to build Kleene Algebras, which is a kind of algebra used to model some computational systems.

Outline. This paper is organized in the following way: Section 2 introduces the notions of pseudo uninorms and Kleene algebras. Section 3 provide some new results and construction on pseudo uninorms. Section 4 shows how Kleene algebras are built from certain pseudo uninorms and that every Kleene algebra is related to a pseudo uninorm. The section also studies automorphisms on this structures and how they can generate new pseudo uninorm based Kleene algebras.

2 Preliminaries

Let $\langle P, \leq\rangle$ be a poset and $e \in P$. Then, trivially, $\langle P_e, \leq_e\rangle$ and $\langle P^e, \leq^e\rangle$ are poset with a greater and least, respectively, element when $P_e = \{x \in P : x \leq e\}$, $P^e = \{x \in P : e \leq x\}$ and $\leq_e$ and $\leq^e$ are the restriction of $\leq$ to P_e and P^e, respectively [3]. Let $\Delta_P = \{a \in P : \text{for each } x \in P \ a \leq x \text{ or } x \leq a\}$. $\langle P, \leq\rangle$ is a **total order set**, whenever $\Delta_P = P$. $\langle P, \leq\rangle$ is a **meet (join) semilattice** if every

[3] The reader can also find in the literature $\downarrow e$ and $\uparrow e$, respectively.

$x, y \in P$ have an infimum (supremum) in P, denoted by $x \wedge y$ ($x \vee y$). $\langle P, \leq \rangle$ is a lattice if it is both: meet and join semilattice.

Closure operators play an important role in several fields of the mathematics; e.g. in Algebra, Logic and Topology. In this paper a closure operator will be required to develop this work:

Definition 1. *Let $\langle P, \leq \rangle$ be a poset. A* **closure operator** *on P is a function $\star : P \to P$ such that for each $x, y \in P$*

(C1) *if $x \leq y$ then $x^\star \leq y^\star$ — Isotonicity,*
(C2) *$x \leq x^\star$ — inflation, and*
(C3) *$(x^\star)^\star = x^\star$ — idempotency.*

Definition 2. *Let $\langle P, \leq \rangle$ be a poset. A function $U : P \times P \to P$ is a* **pseudo uninorm** *on P, whenever, for each $w, x, y, z \in P$ it satisfies:*

1. *$U(x, U(y, z)) = U(U(x, y), z)$ —* ***Associativity****,*
2. *$w \leq x$ and $y \leq z$ then $U(w, y) \leq U(x, z)$ —* ***Isotonicity****, and*
3. *there is $e \in P$ s.t. $U(x, e) = U(e, x) = x$ —* ***has neutral element****.*

$\mathfrak{U}_P^e$ is the set of all pseudo uninorms on P with neutral element e. If e is the greater (least) element of P then U is called of ***pseudo t-norm*** *(****pseudo t-conorm****).*

Commutative pseudo uninorms are called of uninorm on P in [12]. Uninorms on poset $[0, 1]$ were introduced in [6], but the name uninorm only was coined in [30].

Remark 1. If $\langle P, \leq \rangle$ is a meet-semilattice, then the infimum, i.e. $\wedge$, is a pseudo t-norm iff P has a top element. Analogously, if $\langle P, \leq \rangle$ is a join-semilattice then the supremum, i.e. $\vee$, is a pseudo t-conorm iff P has a bottom element.

Remark 2. The set $\mathfrak{U}_P^e$ endowed with the following binary relation is a partial order:

$$U_1 \preceq_e U_2 \text{ iff } \forall x, y \in P,\ \ U_1(x, y) \leq U_2(x, y).$$

If $U \in \mathfrak{U}_P^e$ then $U(x, y) \leq x \leq e$ and $U(x, y) \leq y$ whenever $x, y \in P_e$, $U(x, y) \geq x \geq e$ and $U(x, y) \geq y$ whenever $x, y \in P^e$, and $x \leq U(x, y) \leq y$ (and also $x \leq U(y, x) \leq y$) whenever $x \in P_e$ and $y \in P^e$.

Remark 3. Let $\langle P, \leq \rangle$ be a bounded poset and $e \in P$. If $U \in \mathfrak{U}_P^e$ then $U(\top, \top) \geq U(\top, e) = \top$ and therefore $U(\top, \top) = \top$. Analogously it is possible to prove that $U(\bot, \bot) = \bot$.

We recall in the notion of Kleene algebra. This algebraic structure represents the abstract notion of a computational systems where programs can be modelled. Namely it is constituted by an universe of programs K that can be operated by a (non deterministic) choice $+$, by a sequential composition ; and by an iterative closure $*$. The algebra of regular languages, of binary relations and of program traces are well known instantiations of such structure.

Definition 3. *An algebra* $\langle K, +, \cdot, \star, 0, 1\rangle$ *of type* $(2, 2, 1, 0, 0)$ *is a Kleene algebra if for each* $a, b, c \in K$ *satisfy the following axioms:*

- ***(KA1)*** $a + (b + c) = (a + b) + c$;
- ***(KA2)*** $a + b = b + a$;
- ***(KA3)*** $a + a = a$;
- ***(KA4)*** $a + 0 = 0 + a = a$;
- ***(KA5)*** $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;
- ***(KA6)*** $a \cdot 1 = 1 \cdot a = a$;
- ***(KA7)*** $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$;
- ***(KA8)*** $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$;
- ***(KA9)*** $a \cdot 0 = 0 \cdot a = 0$;
- ***(KA10)*** $1 + (a \cdot a^\star) \leq a^\star$;
- ***(KA11)*** $1 + (a^\star \cdot a) \leq a^\star$;
- ***(KA12)*** *If* $a \cdot b \leq b$ *then* $a^\star \cdot b \leq b$; *and*
- ***(KA13)*** *If* $a \cdot b \leq a$ *then* $a \cdot b^\star \leq a$.

Where $\leq$ *is the natural partial order on* K *defined by*

$$a \leq b \text{ if and only if } a + b = b. \tag{1}$$

Remark 4. In fact $\langle K, \leq\rangle$ is a join-semilattice with 0 as least element [14].

Lemma 1. *[14] Let* $\langle K, +, \cdot, \star, 0, 1\rangle$ *be a Kleene algebra. Then*

- ***(KO1)*** *If* $a \leq b$ *then* $a^\star \leq b^\star$.
- ***(KO2)*** $0^\star = 1$.
- ***(KO3)*** $1 + a \cdot a^\star = a^\star$.
- ***(KO4)*** $(a^\star)^\star = a^\star$.

3 Some new results and construction on pseudo uninorms

Proposition 1. *Let* $\langle P, \leq\rangle$ *be a poset with a bottom element* $\perp$. *For each* $e \in P$, *if* $\langle P^e, \leq^e\rangle$ *is a join-semilattice, then* $\langle \mathfrak{U}_P^e, \preceq_e\rangle$ *has a bottom element.*

Proof. Let $\perp$ be the least element of P. Then the function

$$\underline{U}_e(x, y) = \begin{cases} \perp & \text{if } x, y \notin P^e \\ x \vee y & \text{if } x, y \in P^e \\ x & \text{if } x \notin P^e \text{ and } y \in P^e \\ y & \text{if } x \in P^e \text{ and } y \notin P^e \end{cases}$$

is the bottom element of $\langle \mathfrak{U}_P^e, \preceq_e\rangle$.

Proposition 2. *Let* $\langle P, \leq\rangle$ *be a poset with a greater element* $\top$. *For each* $e \in P$ *if* $\langle P_e, \leq_e\rangle$ *is a meet-semilattice then* $\langle \mathfrak{U}_P^e, \preceq_e\rangle$ *has a greater element.*

Proof. The function

$$\overline{U}_e(x,y) = \begin{cases} x \wedge y & \text{if } x, y \in P_e \\ \top & \text{if } x, y \notin P_e \\ x & \text{if } x \notin P_e \text{ and } y \in P_e \\ y & \text{if } x \in P_e \text{ and } y \notin P_e \end{cases}$$

is the greater element of $\langle \mathfrak{U}_P^e, \preceq_e \rangle$.

Proposition 3. *Let $\langle P, \leq \rangle$ be a poset, $e \in P$ and $U \in \mathfrak{U}_P^e$. Then the restriction, $U_{/P_e}$, is a pseudo t-norm on $\langle P_e, \leq_e \rangle$ and $U_{/P^e}$ is a pseudo t-conorm on $\langle P^e, \leq^e \rangle$.*

Proof. Straightforward.

Corollary 1. *Let $\langle P, \leq \rangle$ be a poset, $e \in P$, $U \in \mathfrak{U}_P^e$. Then for each isotone bijection $\phi : P_e \to P$, the function $T : P \times P \to P$ defined by*

$$T(x,y) = \phi(U(\phi^{-1}(x), \phi^{-1}(y)))$$

is a pseudo t-norm on P.

Corollary 2. *Let $\langle P, \leq \rangle$ be a poset, $e \in P$, $U \in \mathfrak{U}_P^e$. Then for each isotone bijection $\psi : P^e \to P$, the function $S : P \times P \to P$ defined by*

$$S(x,y) = \psi(U(\psi^{-1}(x), \psi^{-1}(y)))$$

is a pseudo t-conorm on P.

Proposition 4. *Let $\langle P, \leq \rangle$ be a poset, $\mathfrak{U}_P$ the set of all pseudo uninorms on P and "$\preceq$" the following binary relation:*

$$U_1 \preceq U_2 \text{ iff } \forall x, y \in P,\; U_1(x,y) \leq U_2(x,y).$$

Then

1. *$\langle \mathfrak{U}_P, \preceq \rangle$ is a poset;*
2. *Let $U_1, U_2 \in \mathfrak{U}_P$ be pseudo uninorms with neutral elements e_1 and e_2, respectively. If $U_1 \preceq U_2$ then $e_2 \leq e_1$;*
3. *Let $U_1, U_2 \in \mathfrak{U}_P$ be pseudo uninorms with neutral elements e_1 and e_2, respectively. If neither $e_1 \leq e_2$ nor $e_2 \leq e_1$ then neither $U_1 \preceq U_2$ nor $U_2 \preceq U_1$.*
4. *If $\langle P, \leq \rangle$ has a greater and a least element then $\langle \mathfrak{U}_P, \preceq \rangle$ also have a greater and a least element.*

Proof. 1. Straightforward.
2. $e_2 = U_1(e_2, e_1) \leq U_2(e_2, e_1) = e_1$.
3. If $U_1 \preceq U_2$ then by previous item $e_2 \leq e_1$. Analogously, if $U_2 \preceq U_1$ then by previous item $e_1 \leq e_2$. Therefore, if e_1 and e_2 are not comparable, then also are not comparable U_1 with U_2.

4. Let $\perp$ and $\top$ be the least and the greater element of P, respectively. Then, let

$$U_\top(x,y)=\begin{cases} x & \text{if } y=\perp\\ y & \text{if } x=\perp\\ \top & \text{otherwise}\end{cases} \quad U_\perp(x,y)=\begin{cases} x & \text{if } y=\top\\ y & \text{if } x=\top\\ \perp & \text{otherwise}\end{cases}$$

It is easy to prove that $U_\perp, U_\top \in \mathfrak{U}_P$. Let $U \in \mathfrak{U}_P$ with $e \in P$ as neutral element and $x,y \in P$. Then if $x=\top$ then $U_\perp(\top,y)=y=U(e,y)\le U(\top,y)$. Analogously, for the case $y=\top$. Now, if $x\ne\top$ and $y\ne\top$ then $U_\perp(x,y)=\perp\le U(x,y)$. So, $U_\perp\le U$. Analogously, it is proven that $U\le U_\top$.

Corollary 3. *Let $\langle P,\le\rangle$ be a poset. Then,*

1. *If T is a pseudo t-norm on P then, $T(x,y)\le x$ and $T(x,y)\le y$.*
2. *If S is a pseudo t-conorm on P then, $x\le S(x,y)$ and $y\le S(x,y)$.*

It is obvious that uninorms, pseudo t-norms and pseudo t-conorms on a bounded lattice are pseudo uninorms on the same lattice. But, there exist pseudo uninorms which are neither uninorms, pseudo t-norms nor pseudo t-conorms. The following proposition provides an infinite family of such pseudo uninorms.

The following results generalize the Proposition 2.1 and 2.2 of [5].

Proposition 5. *Let $\langle P,\le\rangle$ be a join-semilattice with top element and $T: P\times P\to P$ be a pseudo t-norm on P. Then for any $e\in P$ and isotone bijection $\phi: P_e\to P$, the mapping $U_e: P\times P\to P$ defined by:*

$$U_e(x,y)=\begin{cases} \phi^{-1}(T(\phi(x),\phi(y))) & \textit{if } x,y\in P_e\\ x\vee y & \textit{if } x,y\notin P_e\\ x & \textit{if } x\notin P_e \textit{ and } y\in P_e\\ y & \textit{if } x\in P_e \textit{ and } y\notin P_e\end{cases} \tag{2}$$

is a pseudo uninorm on P with e as neutral element.

Proof. Let $x,y,z\in P$. If $x,y,z\in P_e$, then

$$\begin{aligned} U_e(x,U_e(y,z)) &= \phi^{-1}(T(\phi(x),T(\phi(y),\phi(z))))\\ &= \phi^{-1}(T(T(\phi(x),\phi(y)),\phi(z)))\\ &= U_e(U_e(x,y),z).\end{aligned}$$

In any other case, $U_e(x,U_e(y,z))=\vee\{x,y,z\}\cap\overline{P_e}=U_e(U_e(x,y),z)$. Therefore, U_e is associative.

Let $x\in P$. If $x\in P_e$, then $U_e(x,e)=\phi^{-1}(T(\phi(x),\phi(e)))=x=\phi^{-1}(T(\phi(e),\phi(x)))=U_e(e,x)$. If $x\notin P_e$ then $U_e(x,e)=x=U_e(e,x)$. Therefore e is a neutral element of U_e.

Let $x,y,z\in P$ such that $y\le z$. If $x\notin P_e$ we have the following cases:

1. $y\notin P_e$: then $z\notin P_e$ and therefore, $U_e(x,y)=x\vee y\le x\vee z=U_e(x,z)$.
2. $z\in P_e$: then $y\in P_e$ and therefore, $U_e(x,y)=x=U_e(x,z)$.
3. $y\in P_e$ and $z\notin P_e$ and therefore, $U_e(x,y)=x\le x\vee z=U_e(x,z)$.

If $x \in P_e$ the we have three cases:

1. $y \notin P_e$: then $z \notin P_e$ and therefore,
   $U_e(x,y) = y \leq z = U_e(x,z)$.
2. $z \in P_e$: then $y \in P_e$ and therefore,
   $U_e(x,y) = \phi^{-1}(T(\phi(x),\phi(y))) \leq \phi^{-1}(T(\phi(x),\phi(z))) = U_e(x,z)$.
3. $y \in P_e$ and $z \notin P_e$ and therefore,
   $U_e(x,y) = \phi^{-1}(T(\phi(x),\phi(y))) \leq y \leq z = U_e(x,z)$.

Therefore, U_e is isotone in the second component. The prove that is isotone in the first component is analogous.

Proposition 6. *Let $\langle P, \leq\rangle$ be a meet-semilattice with bottom element and $S : P \times P \to P$ be a pseudo t-conorm on P. Then for any $e \in P$ and isotone bijection $\psi : P^e \to P$, the mapping $U^e : P \times P \to P$ defined by:*

$$U^e(x,y) = \begin{cases} \psi^{-1}(S(\psi(x),\psi(y))) & \textit{if } x,y \in P^e \\ x \wedge y & \textit{if } x,y \notin P^e \\ x & \textit{if } x \notin P^e \textit{ and } y \in P^e \\ y & \textit{if } x \in P^e \textit{ and } y \notin P^e \end{cases} \tag{3}$$

is a pseudo uninorm on P with e as neutral element.

Proof. Analogous to Proposition 5.

Proposition 7. *Let $\langle P, \leq\rangle$ be a poset, $e \in P$ and $U_1, U_2 \in \mathfrak{U}^e_P$. Then the mapping $U_1 \rtimes U_2 : P \times P \to P$ defined by*

$$U_1 \rtimes U_2(x,y) = \begin{cases} U_1(x,y) & \textit{if } x,y \in P_e \\ U_2(x \vee e, y \vee e) & \textit{if } x,y \notin P_e \\ x & \textit{if } x \notin P_e \textit{ and } y \in P_e \\ y & \textit{if } x \in P_e \textit{ and } y \notin P_e \end{cases} \tag{4}$$

is a pseudo uninorm on P with e as neutral element.

Proof. Let $x,y,z \in P$ such that $y \leq z$. If $x \notin P_e$ then:

- Case $z \in P_e$: then $y \in P_e$ and therefore $U_1 \rtimes U_2(x,y) = x = U_1 \rtimes U_2(x,z)$.
- Case $y,z \notin P_e$: then $U_1 \rtimes U_2(x,y) = U_2(x \vee e, y \vee e) \leq U_2(x \vee e, z \vee e) = U_1 \rtimes U_2(x,z)$.
- Case $y \in P_e$ and $z \notin P_e$: then, by Remark 2, $U_1 \rtimes U_2(x,y) = x \leq x \vee e \leq U_2(x \vee e, z \vee e) = U_1 \rtimes U_2(x,z)$.

If $x \in P_e$ then:

- Case $y,z \in P_e$: Then we have that $U_1 \rtimes U_2(x,y) = U_1(x,y) \leq U_1(x,z) = U_1 \rtimes U_2(x,z)$.
- Case $y \in P_e$ and $z \notin P_e$: Then, by Remark 2, we have that $U_1 \rtimes U_2(x,y) = U_1(x,y) \leq y \leq z = U_1 \rtimes U_2(x,z)$.
- Case $y \notin P_e$: then $z \notin P_e$ and therefore $U_1 \rtimes U_2(x,y) = y \leq z = U_1 \rtimes U_2(x,z)$.

Therefore, $U_1 \rtimes U_2$ is isotone in the second component. The prove that is isotone in the first component is analogous.

Let $x, y, z \in P$. Case $x, y, z \in P_e$ or $x, y, z \notin P_e$ we have that by associativity of U_1 and U_2, $U_1 \rtimes U_2(x, U_1 \rtimes U_2(y, z)) = U_1 \rtimes U_2(U_1 \rtimes U_2(x, y), z)$. The other cases:

1. Case $x, y \in P_e$ and $z \notin P_e$: $U_1 \rtimes U_2(x, U_1 \rtimes U_2(y, z)) = z = U_1 \rtimes U_2(U_1 \rtimes U_2(x, y), z)$.
2. Case $x \in P_e$ and $y, z \notin P_e$: $U_1 \rtimes U_2(x, U_1 \rtimes U_2(y, z)) = U_2(y \vee e, z \vee e) = U_1 \rtimes U_2(U_1 \rtimes U_2(x, y), z)$.
3. Case $x, y \notin P_e$ and $z \in P_e$: $U_1 \rtimes U_2(x, U_1 \rtimes U_2(y, z)) = U_2(x \vee e, y \vee e) = U_1 \rtimes U_2(U_1 \rtimes U_2(x, y), z)$.
4. Case $x \notin P_e$ and $y, z \in P_e$: $U_1 \rtimes U_2(x, U_1 \rtimes U_2(y, z)) = x = U_1 \rtimes U_2(U_1 \rtimes U_2(x, y), z)$.

Therefore, $U_1 \rtimes U_2$ is associative, and since e is clearly a neutral element then $U_1 \rtimes U_2$ is a pseudo uninorm.

Proposition 8. *Let $\langle P, \leq \rangle$ be a poset, $e \in P$ and $U_1, U_2 \in \mathfrak{U}_P^e$. Then the mapping $U_1 \ltimes U_2 : P \times P \to P$ defined by*

$$U_1 \ltimes U_2(x, y) = \begin{cases} U_1(x \wedge e, y \wedge e) & \text{if } x, y \notin P^e \\ U_2(x, y) & \text{if } x, y \in P^e \\ x & \text{if } x \notin P^e \text{ and } y \in P^e \\ y & \text{if } x \in P^e \text{ and } y \notin P^e \end{cases} \tag{5}$$

is a pseudo uninorm on P with e as neutral element.

Proof. Analogous.

As corollary we have the following generalization of the Theorem 1 in [8] (see also Theorem 2.1 in [27]).

Proposition 9. *Let $\langle L, \leq \rangle$ be a bounded lattice and $T, S : L \times L \to L$ be a pseudo t-norm and a pseudo t-conorm on L, respectively. Then, for any $e \in L$ and isotone bijections $\phi : L_e \to L$ and $\psi : L^e \to L$, the mappings $U_1, U_2 : L \times L \to L$ defined by:*

$$U_1(x, y) = \begin{cases} \phi^{-1}(T(\phi(x), \phi(y))) & \text{if } x, y \in L_e \\ \psi^{-1}(S(\psi(x), \psi(y))) & \text{if } x, y \in L^e \\ x & \text{if } x \notin L_e \text{ and } y \in L_e \\ y & \text{if } x \in L_e \text{ and } y \notin L_e \\ (x \wedge y) \vee e & \text{otherwise} \end{cases} \tag{6}$$

$$U_2(x, y) = \begin{cases} \phi^{-1}(T(\phi(x), \phi(y))) & \text{if } x, y \in L_e \\ \psi^{-1}(S(\psi(x), \psi(y))) & \text{if } x, y \in L^e \\ x & \text{if } x \notin L^e \text{ and } y \in L^e \\ y & \text{if } x \in L^e \text{ and } y \notin L^e \\ (x \vee y) \wedge e & \text{otherwise} \end{cases} \tag{7}$$

are pseudo uninorms on L with e as neutral element.

Proof. Let $x, y, z \in L$. If $x, y, z \in L_e$ then

$$\begin{aligned} U_e(x, U_e(y,z)) &= \phi^{-1}(T(\phi(x), T(\phi(y), \phi(z)))) \\ &= \phi^{-1}(T(T(\phi(x), \phi(y)), \phi(z))) \\ &= U_e(U_e(x,y), z). \end{aligned}$$

Analogously, if $\{x, y, z\} \subseteq L^e$ then

$$\begin{aligned} U_e(x, U_e(y,z)) &= \psi^{-1}(S(\psi(x), S(\psi(y), \psi(z)))) \\ &= \psi^{-1}(S(S(\psi(x), \psi(y)), \psi(z))) \\ &= U_e(U_e(x,y), z). \end{aligned}$$

$U_e^T(x, U_e^T(y,z)) = \top = U_e^T(U_e^T(x,y), z)$. Therefore, U_e^T is associative.

Let $x \in L$. If $x \in L_e$, then $U_e^T(x,e) = \phi^{-1}(T(\phi(x), \phi(e))) = \phi^{-1}(T(\phi(x), \top)) = x$ and, analogously, $U_e^T(e,x) = x$. If $x \in L^e$ then $U_e^T(x,e) = x \vee e = x = U_e^T(e,x)$. Finally, if $x \int L_e \cup L^e$ then $U_e^T(x,e) = e = U_e^T(e,x)$. Therefore e is a neutral element of P_e^T.

Let $x, y, z \in [0,1]$ such that $y \leq z$. If $x \leq e$ then $P_e^T(x,y) = y \leq z = P_e^T(x,z)$ and if $x > e$ then $P_e^T(x,y) = \max(x,y) \leq \max(x,z) = P_e^T(x,z)$. So, P_e^T is isotonic in the second argument. The prove that it is isotonic in the first component is analogous.

Hence, P_e^T is a pseudo uninorm.

3.1 Annihilators of pseudo uninorms

In the literature, an element a of a set A is called **an annihilator for a function** $F : A \times A \to A$, whenever "$F(a,x) = F(x,a) = a$ for each $x \in A$". For example, zero is an annihilator for the usual multiplication. It is not difficult to see that an annihilator for F is unique and also that bottom, $\bot$, and top, $\top$, elements are annihilators for pseudo t-norm and pseudo t-conorm, respectively.

From this point on, the expression $U(\bot, \top)$ will be denoted by a_U.

Theorem 1. *Let $\langle P, \leq, \bot, \top \rangle$ be a bounded poset and $e \in P$. If $U \in \mathfrak{U}_P^e$ has an annihilator, then it is a_U.*

Proof. Suppose that a is the annihilator of U. Then $U(\bot, \top) \leq U(a, \top) = a$ and $U(\bot, \top) \geq U(\bot, a) = a$. Therefore, $a = a_U$. On the other hand, $U(\top, \bot) \leq U(\top, a) = a$ and $U(\top, \bot) \geq U(a, \bot) = a$. Therefore, $a = U(\top, \bot) = U(\bot, \top) = a_U$.

The next proposition is a generalization of the Lemma 1 in [8].

Proposition 10. *Let $\langle P, \leq, \bot, \top \rangle$ be a bounded poset and $e \in P$. If $U \in \mathfrak{U}_P^e$, then*

1. $U(a_U, x) \leq a_U \leq U(x, a_U)$ *for all* $x \in P$*;*
2. $U(a_U, x) = a_U$ *for all* $x \in P^e$*;*
3. $U(x, a_U) = a_U$ *for all* $x \in P_e$*.*

Proof. Let $x \in P$, then $U(a_U, x) \leq U(a_U, \top) = U(U(\bot, \top), \top) = U(\bot, U(\top, \top)) = U(\bot, \top) = a_U$ and $U(x, a_U) \geq U(\bot, U(\bot, \top)) = U(U(\bot, \bot), \top) = U(\bot, \top) = a_U$. Therefore, $U(a_U, x) \leq a_U \leq U(x, a_U)$ for all $x \in P$.

If $x \geq e$ then $U(\top, x) \geq U(\top, e) = \top$ and so $U(\top, x) = \top$. Therefore, $U(a_U, x) = U(U(\bot, \top), x) = U(\bot, U(\top, x)) = U(\bot, \top) = a_U$ and $U(x, a_U) = U(x, U(\bot, \top)) = U(U(x, \bot), \top) \geq U(\bot, \top) = a_U$.

If $x \leq e$ then $U(x, \bot) \leq U(e, \bot) = \bot$ and so $U(x, \bot) = \bot$. Therefore, $U(x, a_U) = U(x, U(\bot, \top)) = U(U(x, \bot), \top) = U(\bot, \top) = a_U$ and $U(a_U, x) = U(U(\bot, \top), x) = U(\bot, U(\top, x)) \leq U(\bot, \top) = a_U$. Hence, $U(x, a_U) = a_U \geq U(a_U, x)$.

Corollary 4. *Let $\langle P, \leq, \bot, \top \rangle$ be a bounded poset and $e \in P$. If $U \in \mathfrak{U}_P^e$ is commutative then U has an annihilator element.*

The next proposition is stronger, since the commutativity is relaxed whereas the existence of an annihilator is maintained.

Proposition 11. *Let $\langle P, \leq, \bot, \top \rangle$ be a bounded poset and $e \in P$. If $U \in \mathfrak{U}_P^e$ is such that $U(\bot, \top) = U(\top, \bot)$, then*

- *a_U is annihilator;*
- *$a_U = \bot$ or $a_U = \top$ or a_U incomparable with e.*

Proof. Let $x \in P$. Then, $U(a_U, x) \geq U(a_U, \bot) = U(U(\bot, \top), \bot) = U(U(\top, \bot), \bot) = U(\top, U(\bot, \bot)) = U(\top, \bot)$ and $U(x, a_U) \leq U(\top, U(\bot, \top)) = U(\top, U(\top, \bot)) = U(U(\top, \top), \bot) = U(\top, \bot)$. Therefore, by Proposition 10, $U(\top, \bot) \leq U(a_U, x) \leq a_U \leq U(x, a_U) \leq U(\top, \bot) = U(\top, \bot)$ and, consequently, $U(a_U, x) = a_U = U(x, a_U)$. Hence, a_U is an annihilator of U.

If $a_U \leq e$ then $a_U = U(\bot, a_U) \leq U(\bot, e) = \bot$ and so $a_U = \bot$. If $a_U \geq e$ then $a_U = U(\top, a_U) \geq U(\top, e) = \top$ and so $a_U = \top$. Therefore, $a_U = \bot$ or $a_U = \top$ or a_U incomparable with e.

Proposition 12. *Let $\langle P, \leq, \bot, \top \rangle$ be a bounded poset and $e \in P$. If $U \in \mathfrak{U}_P^e$ is such that $U(\bot, \top) = U(\top, \bot) = \bot$, then $\bot$ is the annihilator of U.*

Proof. Let $x \in P$. Since U is isotone then $U(\bot, x) \leq U(\bot, \top) = \bot$ and $U(x, \bot) \leq U(\top, \bot) = \bot$. Therefore, $U(x, \bot) = U(\bot, x) = \bot$ for each $x \in P$.

Proposition 13. *Let $\langle P, \leq, \bot, \top \rangle$ be a bounded poset and $e \in P$. If $U \in \mathfrak{U}_P^e$ is such that $U(\bot, \top) = U(\top, \bot) = \top$ then $\top$ is an annihilator of U.*

Proof. Let $x \in P$. Since U is isotone then $U(x, \top) \geq U(\bot, \top) = \top$ and $U(\top, x) \geq U(\top, \bot) = \top$. Therefore, $U(x, \top) = U(\top, x) = \top$ for each $x \in P$.

3.2 Idempotency

In the literature, an operation $F : A \times A \to A$ is called **idempotent** whenever for each $x \in A$, $F(x, x) = x$. In this section we will confront the notion of pseudo uninorms with such property.

Proposition 14. *Let $\langle P, \leq \rangle$ be a poset such that $\langle P_e, \leq_e \rangle$ is a meet-semilattice and $\langle P^e, \leq^e \rangle$ is a join-semilattice. $U \in \mathfrak{U}_P^e$ is idempotent iff for each $x, y \in L$,*

$$U(x,y) = \begin{cases} x \wedge y & \text{if } x, y \in P_e \\ x \vee y & \text{if } x, y \in P^e \\ U(x,y) \in [x \wedge y, x \vee y] & \text{otherwise} \end{cases}$$

Proof. ($\Rightarrow$) If $x, y \in P_e$ then, by one hand, $U(x,y) \leq U(x,e) = x$ and $U(x,y) \leq U(e,y) = y$ and therefore, $U(x,y) \leq x \wedge y$. On the other hand, $x \wedge y = U(x \wedge y, x \wedge y) \leq U(x,y)$. Therefore, $U(x,y) = x \wedge y$.

If $x, y \in P^e$ then, by one hand, $U(x,y) \geq x \vee y$ and by the other hand, $x \vee y = U(x \vee y, x \vee y) \geq U(x,y)$. Therefore, $U(x,y) = x \vee y$.

In other case:

– If x and y are comparable, then by a symmetric argument it is sufficient to consider the case $x \in P_e$ and $y \in P^e$, and therefore $x \leq y$. Thereby, $x = U(x,x) \leq U(x,y)$ and $U(x,y) = U(U(x,x),y) = U(x,U(x,y)) \leq U(x,e) = x$, i.e. $U(x,y) = x \wedge y$.
– If x and y are not comparable, then $x \in P_e$ and $y \notin P_e \cup P^e$, or, $x \in P^e$ and $y \notin P_e \cup P^e$. In the first case, $U(x,y) \leq U(e,y) = y \leq x \vee y$ and $U(x,y) \geq U(x, y \wedge e) = x \wedge y \wedge e = x \wedge y$. Analogously, in the second case, $U(x,y) \geq U(e,y) = y \geq x \wedge y$ and $U(x,y) \leq U(x, y \vee e) = x \vee y \vee e = x \vee y$. So, in both cases, $U(x,y) \in [x \wedge y, x \vee y]$.

($\Leftarrow$) Straightforward.

Corollary 5. *$\langle P, \leq \rangle$ be a meet-semilattice with a top element, denoted by $\top$. Then $U \in \mathfrak{U}_P^\top$ is idempotent iff $U(x,y) = x \wedge y$ for each $x, y \in P$.*

Corollary 6. *$\langle P, \leq \rangle$ be a join-semilattice with a bottom element, denoted by $\perp$. Then $U \in \mathfrak{U}_P^\perp$ is idempotent iff $U(x,y) = x \vee y$ for each $x, y \in P$.*

3.3 Join morphism

In this section we show how pseudo uninorms behave with respect to distributivity over supremum or just a join morphism.

Proposition 15. *Let $\langle P, \leq \rangle$ be a join-semilattice. If $U \in \mathfrak{U}_P$, then for each $x, y, z \in P$:*

1. *$U(x, y \vee z) \geq U(x,y) \vee U(x,z)$, and*
2. *$U(y \vee z, x) \geq U(y,x) \vee U(z,x)$.*

Proof. For each $x, y, z \in P$ we have that once $U(x,y) \leq U(x, y \vee z)$ and $U(x,z) \leq U(x, y \vee z)$ then $U(x,y) \vee U(x,z) \leq U(x, y \vee z)$. The prove that $U(y \vee z, x) \geq U(y,x) \vee U(z,x)$ is analogous.

Definition 4. *Let $\langle P, \leq \rangle$ be a join-semilattice and $U \in \mathfrak{U}_P$ be a pseudo uninorm. U is a join morphism if for each $x, y, z \in P$,*

1. $U(x, y \vee z) = U(x, y) \vee U(x, z)$, *and*
2. $U(y \vee z, x) = U(y, x) \vee U(z, x)$.

Proposition 16. *Let $\langle P, \leq \rangle$ be a join-semilattice and $U \in \mathfrak{U}_P$ be a pseudo uninorm such that:*

1. *For each $w, x, y, z \in P$, if $y \leq z$ and $U(x, y) \leq w \leq U(x, z)$, then there exists $u \in P$ such that $U(x, u) = w$,*
2. *for each $w, x, y, z \in P$, if $y \leq z$ and $U(y, x) \leq w \leq U(z, x)$, then there exists $u \in P$ such that $U(u, x) = w$, and*
3. *for each $x, y, z \in P$, if $U(x, y) \leq U(x, z)$ or $U(y, x) \leq U(z, x)$, then $y \leq z$.*

Then U is join morphism.

Proof. By Proposition 15,

$$U(x, y) \vee U(x, z) \leq U(x, y \vee z) \tag{8}$$

Since, $U(x, y) \leq U(x, y) \vee U(x, z)$ and $U(x, z) \leq U(x, y) \vee U(x, z)$ then by property 1., there exist $u \in P$ such that $U(x, u) = U(x, y) \vee U(x, z)$ and therefore, by Eq. (8), $U(x, u) \leq U(x, y \vee z)$. So, by property 3., $u \leq y \vee z$. Thus, because $u \geq y$ and $u \geq z$, we have that $u = y \vee z$ and consequently $U(x, u) = U(x, y \vee z)$. Hence, $U(x, y) \vee U(x, z) = U(x, u) = U(x, y \vee z)$.

The prove that $U(y \vee z, x) = U(y, x) \vee U(z, x)$ is analogous.

Proposition 17. *Let $\langle P, \leq \rangle$ be a totally ordered set. Each pseudo uninorm on P is a join morphism.*

Proof. Let $x, y, z \in P$. Since, P is totally ordered, by a symmetric argument, it is sufficient just consider that $y \leq z$. So, $U(x, y) \leq U(x, z)$ and $U(y, x) \leq U(z, x)$. Therefore, $U(x, y) \vee U(x, z) = U(x, z) = U(x, y \vee z)$ and $U(y, x) \vee U(z, x) = U(z, x) = U(y \vee z, x)$.

4 Kleene algebras based on pseudo uninorms

In this section we show how Kleene algebras are built by using pseudo uninorms under some conditions. In order to achieve that we propose the notion of Kleene operator based on a pseudo uninorm:

Definition 5. *Let $\langle P, \leq \rangle$ be a join-semilattice, $e \in P$ and $U \in \mathfrak{U}_P^e$ a pseudo uninorm. A Kleene operator based on U is a function $\star : P \to P$ such that for each $x, y \in P$ satisfy:*

- ***(K1)*** $e \vee U(x, x^\star) \leq x^\star$,
- ***(K2)*** $e \vee U(x^\star, x) \leq x^\star$,
- ***(K3)*** *If $U(x, y) \leq y$ then $U(x^\star, y) \leq y$, and*
- ***(K4)*** *If $U(y, x) \leq y$ then $U(y, x^\star) \leq y$.*

Proposition 18. *Let $\langle P, \leq \rangle$ be a join-semilattice, $e \in P$ and $U \in \mathfrak{U}_P^e$ such that either U is a join morphism or $e \in \Delta_P$. If $U(x,x) \leq x$ for each $x \in P^e$ then the operator $x^\star = x \vee e$ is a Kleene operator for U.*

Proof. Observe that $U(x,x) \leq x$ for each $x \in P_e$. So, the condition "$U(x,x) \leq x$ for each $x \in P^e$" is equivalent to "$U(x,x) \leq x \vee e$ for each $x \in P$". Let $x \in P$, then

- **(K1)** Since $x \leq x^\star$ and $x^\star \in P^e$ then $e \vee U(x, x^\star) \leq e \vee U(x^\star, x^\star) \leq e \vee x^\star = x^\star$.
- **(K2)** Analogous to **(K1)**.
- **(K3)** If U is a join morphism and $U(x,y) \leq y$ then $U(x^\star, y) = U(x \vee e, y) = U(x,y) \vee U(e,y) = U(x,y) \vee y = y$. On the other hand, if $e \in \Delta_P$ and $U(x,y) \leq y$ then when $x \leq e$ we have that $U(x^\star, y) = U(x \vee e, y) = U(e,y) = y$ and when $e \leq x$ we have that $U(x^\star, y) = U(x \vee e, y) = U(x,y) \leq y$. Therefore, in both cases, the operator $\star$ satisfy **(K3)**.
- **(K4)** Analogous to **(K3)**.

Theorem 2. *Let $\langle P, \leq \rangle$ be a join-semilattice, $e \in P$ and $U \in \mathfrak{U}_P^e$ such that either U is a join morphism or $e \in \Delta_P$. Then the operator $x^\star = x \vee e$ is a Kleene operator for U iff for each $x, y \in P^e$, $U(x,y) = x \vee y$*

Proof. Straighhtforward. ($\Rightarrow$) if $x \in P^e$, then $U(x,x) \geq U(x,e) = x$ and so, $x = x^\star \geq e \vee U(x, x^\star) = e \vee U(x,x) = U(x,x)$. Therefore, $U(x,x) = x$ for each $x \geq e$, i.e. $U_{/P^e}$ is idempotent. So, by Proposition 3, $U_{/P^e}$ is an idempotent pseudo t-conorm on $\langle P^e, \leq^e \rangle$. Hence, by Proposition 14, $U(x,y) = U_{/P^e}(x,y) = x \vee y$ for each $x, y \in P^e$.

($\Leftarrow$) Straightforward of the Proposition 18.

Theorem 3. *Let $\langle P, \leq, \bot, \top \rangle$ be a bounded join-semilattice, $e \in \Delta_P$ and $U \in \mathfrak{U}_P^e$ such that $U(\bot, \top) = U(\top, \bot) = \bot$ and $U(x,y) = x \vee y$ for each $x, y \in P^e$. Then $\langle P, \vee, U, \star, \bot, e \rangle$ where $x^\star = x \vee e$, is a Kleene algebra.*

Proof. The axioms **(KA1)** to **(KA4)** follows from definition of join-semilattice and least element, the axioms **(KA5)** and **(KA6)** from definition of pseudo uninorm, the axiom **(KA9)** from Proposition 12, and the axioms **(KA10)** to **(KA13)** from Proposition 18. Let $x, y, z \in P$. Then, since $e \in \Delta_P$, we have the following cases:

1. Case $x, y, z \in P^e$ then, from Theorem 2, we have that $U(x, y \vee z) = x \vee (y \vee z) = (x \vee y) \vee (x \vee z) = U(x,y) \vee U(x,z)$.
2. Case $y \in P_e$ and $z \in P^e$ then $y \leq z$ and therefore $U(x, y \vee z) = U(x,z) = U(x,y) \vee U(x,z)$.
3. Case $y \in P^e$ and $z \in P_e$ then $z \leq y$ and therefore $U(x, y \vee z) = U(x,y) = U(x,y) \vee U(x,z)$.
4. Case $y, z \in P_e$ then $U(x, y \vee z) \leq U(x, y \vee e) = U(x,y)$ and $U(x, y \vee z) \leq U(x, z \vee e) = U(x,z)$ and therefore, $U(x, y \vee z) \leq U(x,y) \vee U(x,z)$. So, because, trivially $U(x,y) \vee U(x,z) \leq U(x, y \vee z)$, then $U(x,y) \vee U(x,z) = U(x, y \vee z)$.

Therefore, the axiom **(KA7)** is satisfied for each $x, y, z \in P$. The axiom **(KA8)** can be proved in analogous way.

Theorem 4. *Let $\langle P, \leq, \bot, \top \rangle$ be a bounded join-semilattice, $e \in P$, $U \in \mathfrak{U}_P^e$ be a join morphism such that $U(\bot, \top) = U(\top, \bot) = \bot$ and $U(x, x) \leq x$ for each $x \in P^e$. Then $\langle P, \vee, U, \star, \bot, e \rangle$ where $x^\star = x \vee e$, is a Kleene algebra.*

Proof. The axioms **(KA1)** to **(KA4)** follows from definition of join-semilattice and least element, the axioms **(KA5)** and **(KA6)** from definition of pseudo uninorm,the axioms **(KA7)** and **(KA8)** because U is a join morphism, the axiom **(KA9)** from Proposition 12, and the axioms **(KA10)** to **(KA13)** from Proposition 18.

Theorem 5. *Let $\langle K, +, U, \star, 0, e \rangle$ be a Kleene algebra. Then*

1. *$U \in \mathfrak{U}_P^e$;*
2. *$e \leq x^\star$, for each $x \in K$;*
3. *$x^\star \geq e + x$, for each $x \in K$;*
4. *$\star$ is a closure operator on $\langle K, \leq \rangle$.*

Proof.
1. By the axioms **(KA5)** and **(KA6)**, U is associative and e is a neutral element. Suppose that $y \leq z$ then $U(x, z) = U(x, y + z) = U(x, y) + U(x, z)$ and therefore $U(x, y) \leq U(x, z)$. Analogously is proved that if $x \leq y$, then $U(x, z) \leq U(y, z)$.
2. Let $x \in K$. Then by **(KO1)** and **(KO2)**, once $0 \leq x$ we have that $e \leq x^\star$.
3. If $x \nleq e$ then by **(KO3)**, $U \in \mathfrak{U}_P^e$ and by previous item, $x^\star = e + U(x, x^\star) \geq e + U(x, e) = e + x$.
   Now, if $x \leq e$ then $U(x, e) \leq e$. So, by **(KA12)**, we have that $x^\star = U(x^\star, e) \leq e$. But, once by previous item $e \leq x^\star$, then $x^\star = e$. So, if $x \leq e$ then $x^\star = e = e + x$.
4. **(C1)** follows from **(KO1)**, **(C3)** follows from **(KO4)** and **(C2)** follows from previous item. In fact, $x \leq x + e \leq x^\star$.

4.1 Automorphisms on $[0, 1]$ acting on Kleene algebras

In fuzzy logic, a typical way of generating newer fuzzy connectives (t-norms, t-conorms and implications) from a fuzzy connective of the same type is obtained via automorphisms on the real unit interval $[0, 1]$, which are defined as bijective functions on $[0, 1]$ preserving natural ordering. Formally, a function $\phi : [0, 1] \to [0, 1]$ is an **automorphism** on $[0, 1]$ if it is bijective and isotone, i.e. $x \leq y \;\Rightarrow\; \phi(x) \leq \phi(y)$ [1, 13]. In [2] is considered the equivalent definition where automorphisms are continuous and strictly isotone function satisfying the boundary conditions $\phi(0) = 0$ and $\phi(1) = 1$.

Is clear that this notion can be generalize for arbitraries posets.

Definition 6. *A function $\phi : P \to P$ is an automorphism on a poset $\langle P, \leq \rangle$ if it is bijective and for each $x, y \in P$ we have that*

$$\phi(x) \leq \phi(y) \text{ if and only if } x \leq y. \tag{9}$$

We will denote the set of all automorphism on $\langle P, \leq \rangle$ by $Aut\langle P, \leq \rangle$.

Remark 5. Let $\phi, \psi \in Aut\langle P, \leq \rangle$.

1. The inverse of an automorphism is also an automorphism. In fact, the inverse of bijection also is a bijection and $x \leq y$ iff $\phi(\phi^{-1}(x)) \leq \phi(\phi^{-1}(y))$ iff $\phi^{-1}(x) \leq \phi^{-1}(y)$.
2. The composition of two automorphism is also an automorphism. In fact, the composition of bijective functions also is bijective and $x \leq y$ iff $\psi(x) \leq \psi(y)$ iff $\phi \circ \psi(x) \leq \phi \circ \psi(y)$.
3. The identity function Id_P on P is an automorphism. In addition, $\phi \circ Id_P = \phi = Id_P \circ \phi$.

Therefore, $\langle Aut\langle P, \leq \rangle, \circ \rangle$ is a group.

Let ϕ be an automorphism on P and $f : P^n \to P$. In algebra has been extensively study the actions of groups in order to interpret the elements of the group as "acting" on some space, but preserving the structure of that space [10, 20]. Here we study the action of the group $\langle Aut\langle P, \leq \rangle, \circ \rangle$ on pseudo uninorms, Kleene operators and Kleene algebras. In general, the **action of ϕ on a function** $f : P^n \to P$, denoted by f^ϕ, is defined as follows

$$f^\phi(x_1, \ldots, x_n) = \phi^{-1}(f(\phi(x_1), \ldots, \phi(x_n))). \tag{10}$$

In particular, the action of automorphism preserve the usual fuzzy connectives [1, 2, 19] and also pseudo uninorms on $[0, 1]$ [5, Theorem 3.1]. Here we generalize this last result by consider pseudo uninorms on an arbitrary poset $\langle P, \leq \rangle$.

Proposition 19. *Let U be a pseudo uninorm and ϕ be an automorphism on a poset $\langle P, \leq \rangle$. Then U^ϕ is also a pseudo uninorm. In addition,*

1. *if $\langle P, \leq \rangle$ is bounded (with $\bot$ and $\top$ as least and great elements) then $\phi(\bot) = \bot$ and $\phi(\top) = \top$.*
2. *if $\langle P, \leq \rangle$ is a join (meet) semilattice then $\phi(x \vee y) = \phi(x) \vee \phi(y)$ ($\phi(x \wedge y) = \phi(x) \wedge \phi(y)$), i.e. ϕ is a join (meet) morphism.*

Proof. Associativity: Let $x, y, z \in P$. Then, by equations (10) and (9), and the associativity of U,

$$\begin{aligned} & U^\phi(U^\phi(x, y), z) \\ & = \phi^{-1}(U(\phi(\phi^{-1}(U(\phi(x), \phi(y)))), \phi(z))) \\ & = \phi^{-1}(U(U(\phi(x), \phi(y)), \phi(z))) \\ & = \phi^{-1}(U(\phi(x), U(\phi(y)), \phi(z))) \\ & = \phi^{-1}(U(\phi(x), \phi(\phi^{-1}(U(\phi(y)), \phi(z))))) \\ & = U^\phi(x, U^\phi(y, z)) \end{aligned}$$

Isotonicity: Let $x_1, x_2, y_1, y_2 \in P$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$. Then by equation (10), Equation (9) and isotonicity of U,

$$\begin{aligned} U^\phi(x_1, y_1) & = \phi^{-1}(U(\phi(x_1), \phi(y_1))) \\ & \leq \phi^{-1}(U(\phi(x_2), \phi(y_2))) \\ & = U^\phi(x_2, y_2) \end{aligned}$$

Neutral element: Let $x \in P$. Then by equation (10) and the existence of neutral element for U (denoted by e), $U^{\phi}(x, \phi^{-1}(e)) = \phi^{-1}(U(\phi(x), e)) = x$. So, $\phi^{-1}(e)$ is the neutral element of U^{ϕ}.

Bound preserving: Since ϕ is bijective, there exists $y \in P$ such that $\phi(y) = \bot$. Thus, once $\bot \leq y$ then by equation (9), $\phi(\bot) \leq \bot$. Analogously, we prove that $\top \leq \phi(\top)$.

Joint (meet) morphism: Since $x \leq x \vee y$ and $y \leq x \vee y$ then $\phi(x) \leq \phi(x \vee y)$ and $\phi(y) \leq \phi(x \vee y)$. So, $\phi(x) \vee \phi(y) \leq \phi(x \vee y)$. On the other hand, since ϕ is bijective, there exists $z \in P$ such that $\phi(z) = \phi(x) \vee \phi(y)$. Therefore, $\phi(z) \geq \phi(x)$ and $\phi(z) \geq \phi(y)$. Hence, by Equation (9), $z \geq x$ and $z \geq y$, i.e. $z \geq x \vee y$, and $z \leq x \vee y$. Consequently, $\phi(x) \vee \phi(y) = \phi(x \vee y)$. The proof that ϕ is a meet morphism (when $\langle P, \leq \rangle$ is a meet-semilattice) is analagous.

Proposition 20. *Let $\langle P, \leq \rangle$ be a join-semilattice, $e \in P$, $U \in \mathfrak{U}_P^e$ and $\phi \in Aut\langle P, \leq \rangle$. If $\star : P \to P$ is a Kleene operator for U then $\circledast : P \to P$, defined by $x^{\circledast} = \phi^{-1}(\phi(x)^{\star})$ is a Kleene operator based on U^{ϕ}.*

Proof. By Proposition 19, $U^{\phi} \in \mathfrak{U}_P^{\phi^{-1}(e)}$.

- **(K1)** Since, $\star$ is Klene opertaor for U and φ^{-1} is an automorphism and therefore is a join morphism and isotone, $\phi^{-1}(e) \vee U^{\phi}(x, x^{\circledast}) = \phi^{-1}(e) \vee \phi^{-1}(U(\phi(x), \phi(x)^{\star})) =$
$\phi^{-1}(e \vee U(\phi(x), \phi(x)^{\star})) \leq \phi^{-1}(\phi(x)^{\star}) = x^{\circledast}$.
- **(K2)** Analogous to **(K1)**.
- **(K3)** If $U^{\phi}(x, y) \leq y$ then $U(\phi(x), \phi(y)) \leq \phi(y)$. So, because $\star$ is a Kleene operator based on U, $U(\phi(x)^{\star}, \phi(y)) \leq \phi(y)$. Therefore, because $\phi^{-1} \in Aut\langle P, \leq \rangle$, we have that $U^{\phi}(x^{\circledast}, y) \leq y$.
- **(K4)** Analogous to **(K3)**.

Proposition 21. *Let $\langle K, +, \cdot, \star, 0, 1 \rangle$ be a Kleene algebras and $\phi \in Aut\langle K, \leq \rangle$ where $\leq$ is the partial order defined in Equation 1. Then $\langle K, +^{\phi}, \cdot^{\phi}, \circledast, 0, 1 \rangle$ also is Kleene algebra. In addition, for each $x, y \in K$ we have that $\phi(x+y) = \phi(x)+\phi(y)$, $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$, $\phi(0) = 0$ and $\phi(1) = 1$.*

Proof.

- **(KA1)** $a +^{\phi} (b +^{\phi} c) = \phi^{-1}(\phi(a) + (\phi(b) + \phi(c))) = \phi^{-1}((\phi(a) + \phi(b)) + \phi(c)) = (a +^{\phi} b) +^{\phi} c$.
- **(KA2)** $a +^{\phi} b = \phi^{-1}(\phi(a) + \phi(b)) = \phi^{-1}(\phi(b) + \phi(a)) = b +^{\phi} a$.
- **(KA3)** $a +^{\phi} a = \phi^{-1}(\phi(a) + \phi(a)) = \phi^{-1}(\phi(a)) = a$.
- **(KA4)** $a +^{\phi} 0 = \phi^{-1}(\phi(a) + 0) = \phi^{-1}(\phi(a)) = a$.
- **(KA5)** Analogous to **(KA1)**.
- **(KA6)** Analogous to **(KA4)**.
- **(KA7)** $a \cdot^{\phi} (b +^{\phi} c) = \phi^{-1}(\phi(a) \cdot (\phi(b) + \phi(c))) = \phi^{-1}((\phi(a) \cdot \phi(b)) + (\phi(a) \cdot \phi(c))) = (a \cdot^{\phi} b) +^{\phi} (a \cdot^{\phi} c)$.
- **(KA8)** Analogous to previous item.
- **(KA9)** Analogous to **(KA4)**.
- **(KA10)** $1 +^{\phi} (a \cdot^{\phi} a^{\circledast}) = \phi^{-1}(1 + (\phi(a) \cdot \phi(a)^{\star})) \leq \phi^{-1}(\phi(a)^{\star}) = a^{\circledast}$
- **(KA11)** Analogous to previous item.

(KA12) If $a \cdot^{\phi} b \leq b$ then $\phi^{-1}(\phi(a) \cdot \phi(b)) \leq b$ and so $\phi(a) \cdot \phi(b) \leq \phi(b)$. Hence, $\phi(a)^{\star} \cdot \phi(b) \leq \phi(b)$ and therefore $a^{\circledast} \cdot^{\phi} b \leq b$.

(KA13) Analogous to previous item.

In addition, since for each $x, y \in K$, we have that $x \leq x + y$ and $y \leq x + y$ then (*) $x \vee y \leq x + y$, where $x \vee y$ is the supremum of x and y w.r.t. $\leq$. By [14] we have that if $a \leq b$ then $a + c \leq b + c$, and therefore, since $x \leq x \vee y$ then $x + y \leq (x \vee y) + y$. But, because $y \leq x \vee y$ by Equation (1) $(x \vee y) + y = x \vee y$ and therefore (**) $x + y \leq x \vee y$. Hence, from (*) and (**), $x + y = x \vee y$ for each $x, y \in K$. Analogously we can prove that $x \cdot y = x \wedge y$ for each $x, y \in K$. Consequently, because ϕ is an automorphism $\phi(0) = 0$, $\phi(1) = 1$, and it is a join and meet morphism and so $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$.

5 Final remarks

In this paper we have shown the relation between the notions of Kleene algebras and pseudo uninorms. We have shown that every Kleene algebra induces a pseudo uninorm and that *some* pseudo uninorms induce Kleene algebras. This connection enables both: (1) another viewpoint on the theory of Kleene algebras and (2) indicates a way to build Kleene algebras in the fuzzy setting — since we provide the requirements to build Kleene algebras from pseudo uninorms.

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