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ULAM–HYERS STABILITY OF FOUR-POINT BOUNDARY VALUE PROBLEM FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH A PARAMETER#

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Abstract. Fractional calculus is considered to be a powerful tool in describing complex systems with a wide range of applicability in many fields of science and engineering. The behavior of many systems can be described by using fractional differential equations with boundary conditions. In this sense, the study on the stability of fractional boundary value problems is of high importance.
The main goal of this paper is to investigate the Ulam–Hyers stability and Ulam–Hyers–Rassias stability of a class of fractional four-point boundary value problem involving Caputo derivative and with a given parameter. By using contraction principles, sufficient conditions are obtained to guarantee the uniqueness of solution. Therefore, we obtain sufficient conditions for the stability of that class of nonlinear fractional boundary value problems on the space of continuous functions. The presented results improve and extend some previous research. Finally, we construct some examples to illustrate the theoretical results.

Key words: fractional boundary value problem, Caputo derivative, Ulam–Hyers stability, Ulam–Hyers–Rassias stability.

AMS Subject Classification: 26A33, 34B15, 34D20, 47H10.

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1. Introduction

In the last decades, fractional calculus and fractional boundary value problems have been getting increase attention. In part, this is the case due to the wide range of new applications that they have in several different areas such as aerodynamics, biology, biophysics, blood flow phenomena, chemistry, control theory, economics, physics, signal and image processing (cf., e.g., [1–4]). In some cases, it turns to be clear that very particular properties are better modelled by using fractional derivatives (e.g. when describing long-memory processes of many materials). One important and interesting subarea of research within fractional differential equations is their stability analysis (cf., e.g., [2; 5–11]).

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There are different types of stability. The notion of Ulam–Hyers stability was first introduced by Ulam in 1940 within a problem focusing on the stability of functional equations of group homomorphisms (cf. [12]). Ulam posed a question of the stability of functional equations which was answered, one year later, by Hyers in the context of Banach spaces (cf. [13]). In 1978, Rassias proved a considerable generalized result of Hyers (cf. [14]). It is clear that when facing a system that is stable in the Ulam–Hyers and Ulam–Hyers–Rassias sense, most of the times we do not need to reach exact solutions. This is quite useful in different situations (and in the obvious case when the exact solution is not known). In this sense, those stabilities are of high importance in several areas such as fluid dynamics, numerical analysis, optimization, biology, economics and social sciences, etc. (cf., e.g., [2, 5, 15]).

A pioneer work on the Ulam stability and data dependence for fractional differential equations with Caputo derivative was published in 2011 by Wang et al. (cf. [16]), where the following fractional differential equation was considered:

$$^{C}\mathscr{D}^{\alpha}_{a+}x(t) = f(t, x(t)), \quad t \in [a, \infty).$$

After that, new works have emerged with focus on the stability theory for fractional differential equations. Among other works, we would like to highlight some of the results obtained in the last years involving Caputo derivatives. For instance, in 2019, Ali et. al (cf. [17]) investigated the existence and uniqueness of positive solution for the fractional boundary value problem:

$$\begin{cases} ^{C}\mathscr{D}^{\alpha}_{0+}x(t) - f(t, x(t)) = 0, \ 1 < \alpha < 2, \ t \in [0, 1], \\ \lambda_1 x(0) + \nu_1 x(1) = g_1(u), \\ \lambda_2 x'(0) + \nu_2 x'(1) = g_2(u), \end{cases}$$

where g_k, $k = 1, 2$, are continuous functions, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a nonlinear continuous function and $\lambda_k, \nu_k \in \mathbb{R}$ such that $\lambda_k + \nu_k \neq 0$, $k = 1, 2$. The authors presented necessary and sufficient conditions for four types of Ulam's stability (Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias). In the same way, in 2020, Ali et al. (cf. [18]), studied the existence and uniqueness of the solution of a three-point boundary-value problem for a differential equation of fractional order:

$$\begin{cases} ^{C}\mathscr{D}^{\alpha}_{0+}x(t) = f\left(t, x(t), {}^{C}\mathscr{D}^{\alpha}_{0+}x(t)\right), & 1 < \alpha < 2, \, t \in [0, 1], \\ x(0) = 0, \ x(1) = \delta x(\eta), & \delta, \eta \in (0, 1), \end{cases}$$

where $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Four types of Ulam's stability were studied for the considered problem. In another paper, Ali et al. (cf. [19]), using classical fixed point theory due to Schauder and Banach, derived some results on the existence of solutions and established four types of Ulam's stability for the following class of fractional order differential equations under anti-periodic boundary conditions:

$$\begin{cases} ^{C}\mathscr{D}^{\alpha}_{0+}x(t) = f\left(t, x(t), x(\lambda t), {}^{C}\mathscr{D}^{\alpha}_{0+}x(t)\right), \quad 2 \leqslant \alpha \leqslant 2, \ t \in [0, T], \\ x(0) = -x(T), \ \ {}^{C}\mathscr{D}^{p}_{0+}x(0) = -{}^{C}\mathscr{D}^{p}_{0+}x(T), \ \ {}^{C}\mathscr{D}^{q}_{0+}x(0) = -{}^{C}\mathscr{D}^{q}_{0+}x(T), \end{cases}$$

where $0 < \lambda < 1$, $0 < p < 1$, $1 < q < 2$ and $f : [0, T] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. In 2020, Dai et. al (cf. [20]) considered the class of fractional differential equations with an integral boundary condition:

$$\begin{cases} x'(t) + {}^{C}\mathscr{D}^{\alpha}_{0+}x(t) = f(t, x(t)), & 0 < \alpha < 1, \ t \in [0, 1], \\ x(1) = I^{\alpha}_{0+}x(\eta), & \beta > 0, \ \eta \in (0, 1], \end{cases}$$

where $x \in C^1([0,1])$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The authors proved the existence and uniqueness of solutions of the problem and investigated Ulam–Hyeres and Ulam–Hyers–Rassias stabilities by means of the Bielecki-type metric and the Banach fixed point theorem. In 2020, Palaniappan (cf. [21]) proved the Ulam–Hyers–Rassias stability of a nonlinear fractional differential equation with three point integrable boundary conditions:

$$\begin{cases} {}^C\mathscr{D}^{\alpha}_{0+}x(t) = f(t, x(t)), & 1 < \alpha < 2,\ t \in [0,1], \\ x(0) = 0,\ x(1) = a\int\limits_0^{\nu} x(s)\,\mathrm{d}s, & 0 < \nu < 1, \end{cases}$$

$f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $a \in \mathbb{R}$ is such that $a \neq \frac{2}{\nu^2}$. In [22], based on some fixed point techniques, El-hady et al. studied the Ulam–Hyers and Ulam–Hyers–Rassias of a class of fractional differential equation of the type:

$${}^C\mathscr{D}^{\alpha}_{a+}x(t) = f(t, x(t)), \quad 0 < \alpha < 1,$$

where $x : [a,b] \to X$ is a continuous function and X a normed space.

Motivated by the papers [9, 11], we recall the uniqueness of solution, and investigate Ulam–Hyers stability and Ulam–Hyers–Rassias stability for a class of Caputo fractional differential equations with boundary conditions. More precisely, we consider a class of four-point fractional boundary value problem (FBVP) of the form

$$\begin{cases} {}^C\mathscr{D}^{\alpha}_{0+}x(t) + \lambda f(t, x(t)) = 0, \\ x'(0) - \beta x(\xi) = 0,\ \ x'(1) + \gamma x(\eta) = 0, \end{cases} \tag{1}$$

where $1 < \alpha \leqslant 2$, $0 \leqslant \xi \leqslant \eta \leqslant 1$, $0 \leqslant \beta, \gamma \leqslant 1$, λ is a positive constant, $f(t,x) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and increasing in x for each $t \in [0,1]$ and ${}^C\mathscr{D}^{\alpha}_{0+}$ is the Caputo fractional derivative of order α. Moreover, we shall consider

$$\Delta = \beta(1 + \gamma\eta - \gamma\xi) + \gamma < (\alpha - 1)(1 - \beta\xi). \tag{2}$$

To the best of our knowledge, there is no paper dealing with the Ulam–Hyers and Ulam–Hyers–Rassias stability of solutions of Caputo fractional differential equation with those type of initial value conditions.

2. Preliminary Notions and Uniqueness of Solutions

In this section, we will introduce some basic definitions, notations and lemmas that will be used in the proofs of the main results. This includes some statements about the uniqueness of solutions.

In what follows, we will denote by $C(I, \mathbb{R})$ the Banach space of all continuous functions from I to $\mathbb{R}$ with the finite norm

$$\|x\| = \sup_{t \in I} |x(t)|.$$

Definition 2.1. The Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of a function $u(t)$ is defined by

$$I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t - s)^{\alpha - 1} u(s)\,\mathrm{d}s$$

provided the right side is pointwise defined on $(0,\infty)$, and where Γ is the Euler Gamma function defined by $\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}\,\mathrm{d}t$, $\alpha>0$.

DEFINITION 2.2. The Caputo fractional derivative of order $\alpha>0$ of a continuous function $u(t)$ is given by

$$ {}^C\mathscr{D}^{\alpha}_{0+}u(t)=\frac{1}{\Gamma(n-\alpha)}\int\limits_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s, $$

where $n=[\alpha]+1$ (with $[\alpha]$ denoting the integer part of the real number α), provided that the right side is pointwise defined on $(0,\infty)$.

In the recent years, an increasing number of Gronwall's inequality generalizations have been obtained with the aim of overtaking some of the difficulties encountered in studying differential equations. A classical form of this inequality is the following:

Lemma 2.1 [23, 24]**.** *Let $y(t)$ and $h(t)$ be nonnegative and continuous functions on $0\leqslant t\leqslant\tau$ for which the inequality*

$$ y(t)\leqslant k+\int\limits_0^t h(s)\,y(s)\,\mathrm{d}s,\quad 0\leqslant t\leqslant\tau, $$

holds, where k is a nonnegative constant. Then

$$ y(t)\leqslant ke^{\int\limits_0^t h(s)\,\mathrm{d}s}. $$

The next theorem is an important classical result in fixed point theory and fundamental in what follows to prove the stability of the problem under study.

Theorem 2.1 (Banach Contraction Principle)**.** *Let (X,d) be a generalized complete metric space, and consider a mapping $T:X\to X$ which is a strictly contractive operator, that is*

$$ d(Tx,Ty)\leqslant Ld(x,y)\quad(\forall\,x,y\in X) $$

for some constant $0\leqslant L<1$. Then

(a) *The mapping T has a unique fixed point $x^*=Tx^*$;*

(b) *The fixed point x^* is globally attractive, namely, for any starting point $x\in X$, the following holds:*

$$ \lim_{n\to\infty}T^nx=x^*; $$

(c) *We have the following inequalities:*

$$ \begin{gathered} d(T^nx,x^*)\leqslant L^n\,d(x,x^*),\quad n\geqslant 0,\ x\in X;\\ d(T^nx,x^*)\leqslant\frac{1}{1-L}\,d\left(T^nx,T^{n+1}x\right),\quad n\geqslant 0,\ x\in X;\\ d(x,x^*)\leqslant\frac{1}{1-L}\,d(x,Tx),\quad x\in X. \end{gathered} $$

We turn now to what is known about the uniqueness of solutions.

Lemma 2.2 [11, Lemma 2.4]**.** *If $f_x(t)\in C([0,1],\mathbb{R}^+)$, $1<\alpha\leqslant 2$, then the unique solution of the FBVP*

$$ \begin{cases} {}^C\mathscr{D}^{\alpha}_{0+}x(t)+f_x(t)=0,\\ x'(0)-\beta x(\xi)=0,\ \ x'(1)+\gamma x(\eta)=0, \end{cases} $$

is

$$x(t) = -I_{0+}^{\alpha}(f_x(s))(t) + \frac{\beta(1+\gamma\eta-\gamma t)}{\Delta} I_{0+}^{\alpha}(f_x(s))(\xi) + \frac{\beta(1-\beta\xi+\beta t)}{\Delta} I_{0+}^{\alpha-1}(f_x(s))(1) + \frac{\gamma(1-\beta\xi+\beta t)}{\Delta} I_{0+}^{\alpha}(f_x(s))(\eta) = \int\limits_0^1 G(t,s) f_x(s)\,\mathrm{d}s,$$

where $f_x(s) = f(x, x(s))$ *and*

$$G(t,s) = \begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\beta(1+\gamma\eta-\gamma t)(\xi-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} + \frac{(1-\beta\xi+\beta t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} + \frac{\gamma(1-\beta\xi+\beta t)(\eta-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & s \leqslant \xi,\ s \leqslant t, \\ \frac{\beta(1+\gamma\eta-\gamma t)(\xi-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} + \frac{(1-\beta\xi+\beta t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} + \frac{\gamma(1-\beta\xi+\beta t)(\eta-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & s \leqslant \xi,\ t \leqslant s, \\ -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-\beta\xi+\beta t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} + \frac{\gamma(1-\beta\xi+\beta t)(n-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & \xi \leqslant s \leqslant \eta,\ s \leqslant t, \\ \frac{(1-\beta\xi+\beta t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} + \frac{\gamma(1-\beta\xi+\beta t)(n-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & \xi \leqslant s \leqslant \eta,\ t \leqslant s, \\ -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-\beta\xi+\beta t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)}, & \eta \leqslant s,\ s \leqslant t, \\ \frac{(1-\beta\xi+\beta t)(1-s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)}, & \eta \leqslant s,\ t \leqslant s. \end{cases}$$

From Lemma 2.2, we conclude that $x(t)$ is the solution of FBVP (1) if and only if

$$x(t) = \lambda \int\limits_0^1 G(t,s) f(s, x(s))\,\mathrm{d}s,$$

where $G(t,s)$ is given as in Lemma 2.2.

Lemma 2.3 [11, Lemma 2.5]**.** *If* $(\alpha-1)(1-\beta\xi) > \Delta$, *then the function* $G(t,s)$ *satisfies the following conditions:*

(1) $G \in C([0,1]\times[0,1))$ *and* $0 \leqslant G(t,s) \leqslant M(1-s)^{\alpha-2}$, *for* $t, s \in (0,1)$,

(2) *There exists a positive number* ρ *such that*

$$\min_{\frac{1}{4}\leqslant t\leqslant\frac{3}{4}} G(t,s) \geqslant \rho M (1-s)^{\alpha-2}, \quad 0 < s < 1,$$

where

$$M = \frac{\beta(1+\gamma\eta) + (\alpha-1)(1-\beta\xi+\beta) + \gamma(1-\beta\xi+\beta)}{\Delta\Gamma(\alpha)}. \tag{3}$$

3. Ulam–Hyers–Rassias Stability Analysis

In this section, we discuss the Ulam–Hyers and Ulam–Hyers–Rassias stabilities for the FBVP (1). Let ϵ be a positive real number and $\varphi : [0,1] \to \mathbb{R}^+$ be a continuous function. We consider the following inequalities with boundary conditions:

$$\begin{cases} \left|{}^C\mathscr{D}_{0+}^{\alpha} y(t) + \lambda f(t, y(t))\right| \leqslant \epsilon, & t \in [0,1], \\ y'(0) - \beta y(\xi) = 0, \quad y'(1) + \gamma y(\eta) = 0, \end{cases} \tag{4}$$

$$\begin{cases} \left|{}^C\mathscr{D}_{0+}^{\alpha} y(t) + \lambda f(t, y(t))\right| \leqslant \epsilon\varphi(t), & t \in [0,1], \\ y'(0) - \beta y(\xi) = 0, \quad y'(1) + \gamma y(\eta) = 0. \end{cases} \tag{5}$$

Definition 3.1. The problem (1) is Ulam–Hyers stable if there exists a real constant $k > 0$ such that, for every $\epsilon > 0$ and for every solution $y \in C([0,1],\mathbb{R}^+)$ of (4), there exists a solution $x \in C([0,1],\mathbb{R}^+)$ of the problem (1) with

$$|y(t) - x(t)| \leqslant k\epsilon, \quad t \in [0,1].$$

Definition 3.2. The problem (1) is generalized Ulam–Hyers stable if there is $\psi \in C(\mathbb{R}^+,\mathbb{R}^+)$ and $\psi(0) = 0$ such that for every solution $y \in C([0,1],\mathbb{R}^+)$ of (4) there exists a solution $x \in C([0,1],\mathbb{R}^+)$ of problem (1) which satisfies the following inequality

$$|y(t) - x(t)| \leqslant \psi(\epsilon), \quad t \in [0,1].$$

Definition 3.3. The problem (1) is Ulam–Hyers–Rassias stable with respect to $\varphi : [0,1] \to \mathbb{R}^+$ if there exists a real constant $k_\varphi > 0$ such that, for every $\epsilon > 0$ and for every solution $y \in C([0,1],\mathbb{R}^+)$ of (5), there exists a solution $x \in C([0,1],\mathbb{R}^+)$ of the problem (1) with

$$|y(t) - x(t)| \leqslant k_\varphi \epsilon\varphi(t), \quad t \in [0,1].$$

Remark 3.1. A function $y \in C([0,1],\mathbb{R}^+)$ is a solution of (4) if and only if there exists a function $g \in C([0,1],\mathbb{R}^+)$, which depends on y such that

- $|g(t)| \leqslant \epsilon$, $t \in [0,1]$, $\epsilon > 0$,
- ${}^C\mathscr{D}^\alpha_{0+}y(t) + \lambda f(t,y(t)) = g(t)$, $t \in [0,1]$,
- $y'(0) - \beta y(\xi) = 0$, $y'(1) + \gamma y(\eta) = 0$.

Remark 3.2. A function $y \in C([0,1],\mathbb{R}^+)$ is a solution of (5) if and only if there exists a function $h \in C([0,1],\mathbb{R}^+)$, which depends on y such that

- $|h(t)| \leqslant \epsilon\varphi(t)$, $t \in [0,1]$, $\epsilon > 0$,
- ${}^C\mathscr{D}^\alpha_{0+}y(t) + \lambda f(t,y(t)) = h(t)$, $t \in [0,1]$,
- $y'(0) - \beta y(\xi) = 0$, $y'(1) + \gamma y(\eta) = 0$.

Theorem 3.1. *Suppose $\Delta < (\alpha-1)(1-\beta\xi)$ and let $f : [0,1]\times\mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and non decreasing function such that:*

(i) $|f(t,y) - f(t,x)| \leqslant L|y - x|$, *for all* $t \in [0,1]$, $x, y \in \mathbb{R}^+$ *and where* $L > 0$*;*

(ii) $\sup_{t\in[0,1]} f(t,x) = N$, *with* $0 < N < \infty$.

Then the FBVP (1) is Ulam–Hyers stable and, consequently, generalized Ulam–Hyers, provided that $L < \frac{\alpha-1}{\lambda M}$, *with M as defined in (7).*

$\lhd$ Recall that from Lemma 2.2, $x(t)$ is a solution of the FBVP (1) if and only if

$$x(t) = \lambda \int_0^1 G(t,s)f(s,x(s))\,\mathrm{d}s.$$

Consider the operator T defined on $C([0,1])$ by

$$(Tx)(t) = \lambda \int_0^1 G(t,s)f(s,x(s))\,\mathrm{d}s, \quad \lambda > 0.$$

We transform problem (1) into a fixed point problem, $x = Tx$, observing that the eventual fixed points of the operator T are solutions of problem (1). Applying the Banach contraction mapping principle, we shall show that T has a unique fixed point.

Let $\sup_{t\in[0,1]} f(t,x) = N < \infty$ and choose a positive constant $r \geqslant \frac{\lambda MN}{\alpha-1}$. Let us show that $TB_r \subset B_r$, where $B_r = \{x \in C([0,1]) : \|x\| \leqslant r\}$. For any $x \in B_r$ we have:

$$|(Tx)(t)| = \left|\lambda \int_0^1 G(t,s) f(s,x(s))\,\mathrm{d}s\right| \leqslant \lambda \int_0^1 |G(t,s)|\,|f(s,x(s))|\,\mathrm{d}s$$
$$\leqslant \lambda MN \int_0^1 (1-s)^{\alpha-2}\,\mathrm{d}s = \frac{\lambda MN}{\alpha-1} \leqslant r.$$

Thus, we conclude that $TB_r \subset B_r$. By the continuity of the functions $f(t,x(t))$ and $G(t,s)$, we have that $Tx \in C([0,1],\mathbb{R}^+)$ for any $x \in C([0,1],\mathbb{R}^+)$, which proves that T maps $C([0,1],\mathbb{R}^+)$ into itself.

Let us prove that T is strictly contractive. For $x, y \in C([0,1],\mathbb{R}^+)$,

$$|Ty - Tx| = \lambda \left|\int_0^1 G(t,s)(f(s,y) - f(s,x))\,\mathrm{d}s\right| \leqslant \lambda \int_0^1 |G(t,s)|\,|f(s,y) - f(s,x)|\,\mathrm{d}s$$
$$\leqslant \lambda L|y-x| \int_0^1 M(1-s)^{\alpha-2}\,\mathrm{d}s = \frac{\lambda ML}{\alpha-1}|y-x|,$$

where $\frac{\lambda ML}{\alpha-1} < 1$ since $L < \frac{\alpha-1}{\lambda M}$. Thus, the operator T is a contraction. By Theorem 2.1 it has a unique fixed point, which is the unique solution of the problem (1).

Let $y \in C([0,1],\mathbb{R}^+)$ be the solution of (4) and let $x \in C([0,1],\mathbb{R}^+)$ be the unique solution of the FBVP (1).

Let us consider the notation $f_x(t) = f(t,x(t))$ and $f_y(t) = f(t,y(t))$. From Remark 3.1, we have

$${}^{C}\mathscr{D}^{\alpha}_{0+} y(t) + \lambda f_y(t) - g(t) = 0,$$

with $g(t) \in C([0,1],\mathbb{R})$, $|g(t)| \leqslant \epsilon$ and $\lambda f_y(t) - g(t) \in C([0,1],\mathbb{R})$. From Lemma 2.2, we conclude that

$$y(t) = \int_0^1 G(t,s)\,(\lambda f_y(s) - g(s)) = \lambda \int_0^1 G(t,s) f_y(s)\,\mathrm{d}s - \int_0^1 G(t,s) g(s)\,\mathrm{d}s.$$

Thus, using condition (ii). and Lemma 2.3, we get

$$|x(t) - y(t)| = \left|\lambda \int_0^1 G(t,s) f_x(s)\,\mathrm{d}s - \lambda \int_0^1 G(t,s) f_y(s)\,\mathrm{d}s + \int_0^1 G(t,s) g(s)\,\mathrm{d}s\right|$$
$$\leqslant \left|\lambda \int_0^1 G(t,s) f_x(s)\,\mathrm{d}s - \lambda \int_0^1 G(t,s) f_y(s)\,\mathrm{d}s\right| + \left|\int_0^1 G(t,s) g(s)\,\mathrm{d}s\right|$$

$$
\begin{aligned}
&\leqslant \lambda \int_0^1 |G(t,s)|\,|f_x(s)-f_y(s)|\,\mathrm{d}s + \int_0^1 |G(t,s)||g(s)|\,\mathrm{d}s\\
&\leqslant \lambda ML \int_0^1 (1-s)^{\alpha-2}|x(s)-y(s)|\,\mathrm{d}s + M\epsilon \int_0^1 (1-s)^{\alpha-2}\,\mathrm{d}s\\
&= \frac{M\epsilon}{\alpha-1} + \lambda ML \int_0^1 (1-s)^{\alpha-2}|x(s)-y(s)|\,\mathrm{d}s.
\end{aligned}
$$

Applying Lemma 2.1, we obtain that

$$
|x(t)-y(t)| \leqslant \frac{M\epsilon}{\alpha-1} e^{\lambda ML \int_0^1 (1-s)^{\alpha-2}\,\mathrm{d}s} = \epsilon \frac{M}{\alpha-1} e^{\frac{\lambda ML}{\alpha-1}},
$$

and we conclude that the FBVP (1) is Ulam–Hyers stable. Moreover, if we set $\psi(\epsilon) = \epsilon\frac{M}{\alpha-1}e^{\frac{\lambda ML}{\alpha-1}}$, then we conclude that the FBVP (1) is generalized Ulam–Hyers stable. $\rhd$

Theorem 3.2. *Assume $\Delta < (\alpha-1)(1-\beta\xi)$ and let $f : [0,1]\times\mathbb{R}^+ \to \mathbb{R}^+$ be a non decreasing and continuous function satisfying*

a) $|f(t,y)-f(t,x)| \leqslant L|y-x|$ *for all $t\in[0,1]$, $y,x\in\mathbb{R}^+$ and where $L>0$;*

b) $\sup_{t\in[0,1]} f(t,x) = N$ *with $0<N<\infty$.*

Then the FBVP (1) *is Ulam–Hyers–Rassias stable with respect to $\varphi : [0,1]\to\mathbb{R}^+$ provided that $L < \frac{\alpha-1}{\lambda M}$, with M as defined in* (7).

$\lhd$ As in the proof of Theorem 3.1, we transform problem (1) into a fixed point problem, $x = Tx$ and, assuming the conditions of the theorem, we conclude that T has a unique fixed point, which is the unique solution of the FBVP (1).

Let $y \in C([0,1],\mathbb{R}^+)$ be any solution of (5). By Remark 3.2,

$$
{}^{C}\mathscr{D}^{\alpha}_{0+} y(t) + \lambda f_y(t) - h(t) = 0, \quad |h(t)| \leqslant \epsilon\varphi(t). \tag{6}
$$

From Lemma 2.2, we can write

$$
y(t) = \lambda \int_0^1 G(t,s) f_y(s)\,\mathrm{d}s - \int_0^1 G(t,s) h(s)\,\mathrm{d}s.
$$

Fixing the notation

$$
M_1(t) = \frac{\beta(1+\gamma\eta-\gamma t)}{\Delta}, \quad M_2(t) = \frac{\beta(1-\beta\xi+\beta t)}{\Delta}, \quad M_3(t) = \frac{\gamma(1-\beta\xi+\beta t)}{\Delta},
$$

also according to Lemma 2.2, we have that

$$
\begin{aligned}
\int_0^1 G(t,s)\,h(s)\,\mathrm{d}s = \big| &-I^{\alpha}_{0+}(h(s))(t) + M_1(t) I^{\alpha}_{0+}(h(s))(\xi)\\
&+ M_2(t) I^{\alpha-1}_{0+}(h(s))(1) + M_3(t) I^{\alpha}_{0+}(h(s))(\eta)\big|,
\end{aligned}
$$

and thus,

$$|y(t) - Ty(t)| = \big| -I^{\alpha}_{0+}(h(s))(t) + M_1(t) I^{\alpha}_{0+}(h(s))(\xi) \\ + M_2(t) I^{\alpha-1}_{0+}(h(s))(1) + M_3(t) I^{\alpha}_{0+}(h(s))(\eta) \big| .$$

Since, for $t \in [0,1]$, there exists positive real numbers k_1, k_2, k_3 such that

$$|M_1(t)| \leqslant k_1, \quad |M_2(t)| \leqslant k_2, \quad |M_3(t)| \leqslant k_3, \tag{7}$$

we have

$$|y(t) - Ty(t)| \leqslant \epsilon\varphi(t) \left[\frac{1}{\Gamma(\alpha+1)} + k_1 \frac{\xi}{\Gamma(\alpha+1)} + k_2 \frac{1}{\Gamma(\alpha)} + k_3 \frac{\eta}{\Gamma(\alpha+1)} \right].$$

Thus, it follows that

$$|y(t) - Ty(t)| \leqslant K\epsilon\varphi(t),$$

where $K = \frac{1}{\Gamma(\alpha+1)} + k_1 \frac{\xi}{\Gamma(\alpha+1)} + k_2 \frac{1}{\Gamma(\alpha)} + k_3 \frac{\eta}{\Gamma(\alpha+1)}$. Finally, applying Banach contraction principle (Theorem 2.1), we obtain that

$$|y(t) - x(t)| \leqslant \frac{1}{1-L} K\epsilon\varphi(t) < \frac{\lambda M K}{\lambda M - \alpha + 1} \epsilon\varphi(t)$$

and we conclude that the FBVP (1) is Ulam–Hyers–Rassias stable. ▷

4. Examples

In this section, we present two examples to illustrate the previous theory.

Example 1. Consider the fractional boundary value problem

$$\begin{cases} {}^C\mathscr{D}^{\frac{3}{2}}_{0+} x(t) + \sqrt{\pi} f(t, x(t)) = 0, \\ x'(0) - \frac{1}{5} x\left(\frac{1}{4}\right) = 0, \;\; x'(1) + \frac{1}{5} x\left(\frac{1}{2}\right) = 0. \end{cases} \tag{8}$$

It follows that $\alpha = \frac{3}{2}$, $\beta = \gamma = \frac{1}{5}$, $\xi = \frac{1}{4}$, $\eta = \frac{1}{2}$ and $\lambda = \sqrt{\pi}$ when considering the previous notation. Thus,

$$\Delta = \beta(1 + \gamma\eta - \gamma\xi) + \gamma = 0{,}41 < (\alpha - 1)(1 - \beta\xi) = 0{,}475.$$

Let $f(t, x(t)) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$f(t, x(t)) = \frac{1}{20}\, t\, |x(t)|, \quad t \in [0,1],$$

which is continuous and increasing in x for each $t \in [0,1]$. Moreover, we have that $\sup_{t\in[0,1]} f(t, x(t)) = \frac{1}{20} \sup_{t\in[0,1]} |x(t)| < \infty$ since $\sup_{t\in[0,1]} |x(t)| < \infty$. Consider now the continuous function

$$y(t) = \begin{cases} \frac{1}{5} t + \frac{19}{20}, & t \leqslant \frac{1}{2}, \\ -\frac{21}{100} t + \frac{231}{200}, & t > \frac{1}{2}. \end{cases}$$

Note that the function y satisfies the conditions:

$$|y(t)| \leqslant 1{,}05, \quad y'(0) - \frac{1}{5} y\left(\frac{1}{4}\right) = 0, \quad y'(1) + \frac{1}{5} y\left(\frac{1}{2}\right) = 0.$$

For $t \leqslant \frac{1}{2}$, we have

$$\left|{}^{C}\mathscr{D}^{\frac{3}{2}}_{0+}y(t) + \sqrt{\pi}f(t,y(t))\right| = \left|\frac{\sqrt{\pi}t}{20}\left(\frac{1}{5}t + \frac{19}{20}\right)\right| \leqslant \frac{21\sqrt{\pi}}{800} < 0,047.$$

For $t > \frac{1}{2}$, we have

$$\left|{}^{C}\mathscr{D}^{\frac{3}{2}}_{0+}y(t) + \sqrt{\pi}f(t,y(t))\right| = \left|-\frac{\sqrt{\pi}}{20}\left(-\frac{21}{100}t^2 + \frac{231}{200}t\right)\right| \leqslant \frac{189\sqrt{\pi}}{4000} < 0,084.$$

Thus, we conclude that, for $t \in [0,1]$

$$\left|{}^{C}\mathscr{D}^{\frac{3}{2}}_{0+}y(t) + \sqrt{\pi}f(t,y(t))\right| < 0,084.$$

Moreover, one recognize that

$$|f(t,y_1(t)) - f(t,y_2(t))| = \left|\frac{1}{20}ty_1(t) - \frac{1}{20}ty_2(t)\right| = \left|\frac{1}{20}t(y_1(t) - y_2(t))\right| \leqslant \frac{1}{20}|y_1(t) - y_2(t)|.$$

Thus, $L = \frac{1}{20}$. Since $M = \frac{5}{\sqrt{\pi}}$, we verify that $L < \frac{\alpha-1}{\lambda M} = \frac{1}{10}$. Thereby, from Theorem 3.1, there exists a unique solution $x(t) \in C([0,1],\mathbb{R}^+)$ of the FBVP (8) such that

$$|y(t) - x(t)| \leqslant \frac{21}{80}e^{\frac{1}{2}},$$

and we conclude that the problem (8) has the stability in the Ulam–Hyers sense.

Additionally, we can also observe that

$$\left|{}^{C}\mathscr{D}^{\frac{3}{2}}_{0+}y(t) + \sqrt{\pi}f(t,y(t))\right| \leqslant \frac{1}{200}(15t+3) = \frac{1}{200}\varphi(t), \quad t \in [0,1]$$

(see Fig. 1).

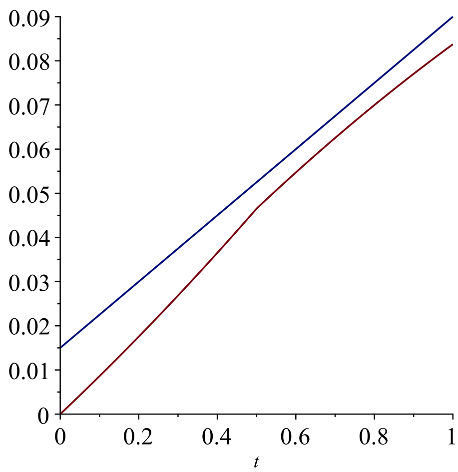

Fig. 1. $\left|{}^{C}\mathscr{D}^{\frac{3}{2}}_{0+}y(t) + \sqrt{\pi}f(t,y(t))\right|$ and $\frac{1}{200}\varphi(t) = \frac{1}{200}(15t+3)$, $t \in [0,1]$.

Thus, we conclude that $|x(t) - y(t)| \leqslant \frac{k_\varphi}{200}\varphi(t)$ for some $k_\varphi > 0$, which means that the FBVP (8) has the stability in the Ulam–Hyers–Rassias sense with respect to $\varphi(t) = 15t + 3$.

EXAMPLE 2. Consider the fractional boundary value problem

$$\begin{cases} {}^{C}\mathscr{D}_{0+}^{\frac{3}{2}}x(t) + \frac{1}{2}f(t,x(t)) = 0, \\ x'(0) - \frac{1}{5}x\left(\frac{1}{2}\right) = 0, \quad x'(1) + \frac{1}{5}x\left(\frac{1}{2}\right) = 0. \end{cases} \tag{9}$$

It follows that $\alpha = \frac{3}{2}$, $\beta = \frac{1}{6}$, $\gamma = \frac{1}{5}$, $\xi = \frac{1}{2}$, $\eta = \frac{1}{2}$ and $\lambda = \frac{1}{2}$. In this case, we have that

$$\Delta = \beta(1 + \gamma\eta - \gamma\xi) + \gamma = \frac{11}{30} < (\alpha - 1)(1 - \beta\xi) = \frac{11}{24}.$$

Let $f(t, x(t)) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$f(t, x(t)) = t + \frac{1}{10}|x(t)|, \quad t \in [0,1].$$

This function is continuous and increasing in x for each $t \in [0,1]$. Moreover, we have that $\sup_{t\in[0,1]} f(t,x(t)) = 1 + \frac{1}{10}\sup_{t\in[0,1]}|x(t)| < \infty$. Consider the continuous function

$$y(t) = \cos(\pi t).$$

The function y satisfies the conditions:

$$|y(t)| \leqslant 1, \quad y'(0) - \frac{1}{6}y\left(\frac{1}{2}\right) = 0, \quad y'(1) + \frac{1}{5}y\left(\frac{1}{2}\right) = 0.$$

We have that

$$\left|{}^{C}\mathscr{D}_{0+}^{\frac{3}{2}}y(t) + \frac{1}{2}f(t,y(t))\right| \leqslant 4,616, \quad t \in [0,1].$$

Moreover,

$$|f(t,y_1(t)) - f(t,y_2(t))| = \left|t\frac{1}{10}ty_1(t) - t - \frac{1}{10}ty_2(t)\right| \leqslant \frac{1}{10}|y_1(t) - y_2(t)|.$$

Thus, $L = \frac{1}{10}$. Since $M = \frac{157}{22\sqrt{\pi}}$, then $L < \frac{\alpha-1}{\lambda M} = \frac{22\sqrt{\pi}}{157}$.

From Theorem 3.1, we conclude that there exists a unique solution $x(t) \in C([0,1],\mathbb{R}^+)$ of the FBVP (8) such that

$$|y(t) - x(t)| \leqslant 4,616k,$$

which means that the problem (8) has the stability in the Ulam–Hyers sense.

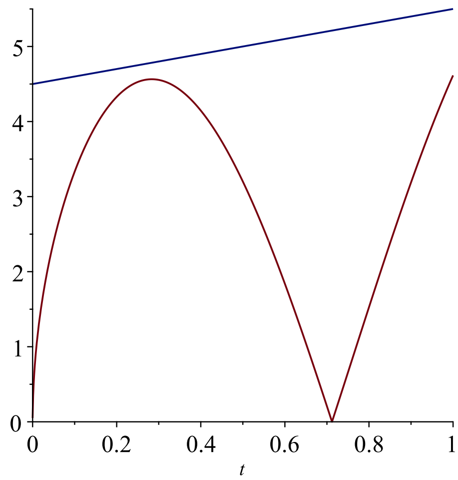

Fig. 2. $\left|{}^{C}\mathscr{D}_{0+}^{\frac{3}{2}}y(t) + \frac{1}{2}f(t,y(t))\right|$ and $\frac{1}{2}\varphi(t) = \frac{1}{2}(2t+9)$, $t \in [0,1]$.

Additionally, we can also observe that

$$\left|{}^{C}\mathscr{D}_{0+}^{\frac{3}{2}}y(t)+\frac{1}{2}f(t,y(t))\right|\leqslant\frac{1}{2}(2t+9)=\frac{1}{2}\varphi(t),\quad t\in[0,1]$$

(see Fig. 2) and thus, we conclude that $|x(t)-y(t)|\leqslant\frac{k_\varphi}{2}\varphi(t)$, for some $k_\varphi>0$, which means that the FBVP (8) has the stability in the Ulam–Hyers–Rassias sense with respect to $\varphi(t)=2t+9$.

5. Conclusions

Fractional differential equations using boundary conditions are extensively used in the modeling of a wide variety of real problems e.g. in chemistry, physics, economics. Knowledge about different types of stabilities of such problems is a key information for having additional information about the possible exact and approximate solutions of such problems. In this way, Ulam–Hyers and Ulam–Hyers–Rassias stabilities provide a strong information in that sense. Inspired in [9, 11], in this paper, we studied a general class of fractional differential equation using Caputo derivative and considering four-point boundary conditions. By means of Banach contraction mapping principle and other techniques, we exhibited sufficient conditions to have a unique solution of the considered problem. After that, we established, in the form of sufficient conditions, the Ulam–Hyers, the generalized Ulam–Hyers and the Ulam–Hyers–Rassias stabilities. At the end, two concrete examples were given to illustrate the obtained results.

References

1. *Oldham, K. B.* Fractional Differential Equations in Electrochemistry, *Advances in Engineering Software*, 2010, vol. 41, no. 1, pp. 9–12. DOI: 10.1016/j.advengsoft.2008.12.012.
2. *Podlubny, I. Fractional Differential Equations, Mathematics in Science and Engineering*, New York, Academic Press, 1999.
3. *Samko, S. G., Kilbas, A. A. and Marichev, O. I. Fractional Integrals and Derivatives: Theory and Applications*, Singapore, Gordon & Breach, 1993.
4. *Torvik, P. J. and Bagley, R-L.* On the Appearance of Fractional Derivatives in the Behavior of Real Materials, *Journal of Applied Mechanics*, 1984, vol. 51, no. 2, pp. 294–298. DOI: 10.1115/1.3167615.
5. *Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J. Theory and Applications of Fractional Differential Equations*, Amsterdam, Netherlands, Elsevier, 2016.
6. *Yang, C. and Zhai, C.* Uniqueness of Positive Solutions for a Fractional Differential Equation Via a Fixed Point Theorem of a Sum Operator, *Electronic Journal of Differential Equations*, 2021, vol. 70, pp. 1–8.
7. *Wang, C. and Xu, T.-Z.* Stability of the Nonlinear Fractional Differential Equations with the Right-Sided Riemann–Liouville Fractional Derivative, *Discrete and Continuous Dynamical Systems. Ser. S*, 2017, vol. 10, no. 3, pp. 505–521.
8. *Zhai, C. and Hao, M.* Fixed Point Theorems for Mixed Monotone Operators with Perturbation and Applications to Fractional Differential Equation Boundary Value Problems, *Nonlinear Analysis: Theory, Methods & Applications*, 2012, vol. 75, no. 4, pp. 2542–2551. DOI: 10.1016/j.na.2011.10.048.
9. *Zhai, C. and Xu, L.* Properties of Positive Solutions to a Class of Four-Point Boundary Value Problem of Caputo Fractional Differential Equations with a Parameter, *Communications in Nonlinear Science and Numerical Simulation*, 2014, vol. 19, no. 8, pp. 2820–2827. DOI: 10.1016/j.cnsns.2014.01.003.
10. *Zhai, C., Yan, W. and Yang, C.* A Sum Operator Method for the Existence and Uniqueness of Positive Solutions to Riemann–Liouville Fractional Differential Equation Boundary Value Problems. *Communications in Nonlinear Science and Numerical Simulation*, 2013, vol. 18, no. 4, pp. 858–866. DOI: 10.1016/j.cnsns.2012.08.037.

11. *Zhao, X., Chai, C. and Ge, W.* Positive Solutions for Fractional Four-Point Boundary Value Problems, *Communications in Nonlinear Science and Numerical Simulation*, 2011, vol. 16, no. 9, pp. 3665–3672. DOI: 10.1016/j.cnsns.2011.01.002.
12. *Ulam, S. M. Problems in Modern Mathematics*, New York, John Wiley & Sons, 1940.
13. *Hyers, D. H.* On the Stability of the Linear Functional Equation, *The National Academy of Sciences USA*, 1941, vol. 27, pp. 222–224.
14. *Rassias, T. M.* On the Stability of the Linear Mapping in Banach Spaces, *Proceedings of the American Mathematical Society*, 1978, vol. 72, no. 2 pp. 297–300. DOI: 10.1090/S0002-9939-1978-0507327-1.
15. *Podlubny, I. Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications*, Elsevier, 1998.
16. *Wang, J., Lv, L. and Zhou, Y.* Ulam Stability and Data Dependence for Fractional Differential Equations with Caputo Derivative, *Electronic Journal of Qualitative Theory of Differential Equations*, 2011, no. 63, pp. 1–10. DOI: 10.14232/ejqtde.2011.1.63.
17. *Ali, A., Shah, K. and Li, Y.* Topological Degree Theory and Ulam's Stability Analysis of a Boundary Value Problem of Fractional Differential Equations, *Frontiers in Functional Equations and Analytic Inequalities*, Springer, Anastassiou G., Rassias J. (eds), 2019, pp. 73–92. DOI: 10.1007/978-3-030-28950-8_4.
18. *Ali, A. and Shah, K.* Ulam–Hyers Stability Analysis of a Three-Point Boundary-Value Problem for Fractional Differential Equations, *Ukrainian Mathematical Journal*, 2020, vol. 72, no. 2, pp. 161–176. DOI: 10.1007/s11253-020-01773-2.
19. *Ali, A., Shah, K. and Abdeljawad, T.* Study of Implicit Delay Fractional Differential Equations under Anti-Periodic Boundary Conditions, *Advances in Difference Equations*, 2020, vol. 2020, Article no. 139, pp. 1–16. DOI: 10.1186/s13662-020-02597-x.
20. *Dai, Q., Gao, R., Li, Z. and Wang, C.* Stability of Ulam–Hyers and Ulam–Hyers–Rassias for a Class of Fractional Differential Equations, *Advances in Difference Equations*, 2020, vol. 2020, Article no. 103, pp. 1–15. DOI: 10.1186/s13662-020-02558-4.
21. *Palaniappan, M.* Hyers–Ulam–Rassias Stability of Nonlinear Fractional Differential Equation with Three Point Integral Boundary Conditions, *International Journal of Engineering Research and Technology*, 2020, vol. 13, no. 12, pp. 4679–4685.
22. *El-hady, E. and Öğrekçi, S.* On Hyers–Ulam–Rassias Stability of Fractional Differential Equations with Caputo Derivative, *Journal of Mathematics and Computer Science*, 2021, vol. 22, no. 4, pp. 325–332. DOI: 10.22436/jmcs.022.04.02.
23. *Gronwall, T. H.* Note on the Derivatives with Respect to a Parameter of the Solutions of a System of Differential Equations, *Annals of Mathematics*, 1919, vol. 20, no. 4, pp. 292–296. DOI: 10.2307/1967124.
24. *Bellman, R.* The Stability of Solutions of Linear Differential Equations, *Duke Mathematical Journal*, 1943, vol. 10, no. 4, pp. 643–647. DOI: 10.1215/S0012-7094-43-01059-2.

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УСТОЙЧИВОСТЬ ПО УЛАМУ — ХАЙЕРСУ ЧЕТЫРЕХТОЧЕЧНОЙ КРАЕВОЙ ЗАДАЧИ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ДРОБНОГО ПОРЯДКА КАПУТО С ПАРАМЕТРОМ

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Аннотация. Дробное исчисление является мощным инструментом описания сложных систем с широким диапазоном применимости во многих областях науки и техники. Поведение многих систем можно описать с помощью дифференциальных уравнений дробного порядка с граничными условиями. В этом смысле большое значение имеет исследование устойчивости дробных краевых задач. Основная цель данной работы — исследование устойчивости по Уламу — Хайерсу и устойчивости по Уламу — Хайерсу — Рассиасу класса дробных четырехточечных краевых задач, содержащих производную Капуто и с заданным параметром. Используя принцип сжимающих отображений, получаются достаточные условия, гарантирующие единственность решения. Таким образом, мы получаем достаточные условия устойчивости этого класса нелинейных дробных краевых задач в пространстве непрерывных функций. Представленные результаты улучшают и расширяют некоторые предыдущие исследования. Наконец, мы построим несколько примеров, иллюстрирующих полученные теоретические результаты.

Ключевые слова: дробная краевая задача, производная Капуто, устойчивость Улам — Хайерс, устойчивость Улам — Хайерс — Рассиас.

AMS Subject Classification: 26A33, 34B15, 34D20, 47H10.

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