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On balanced games with infinitely many players: Revisiting Schmeidler's result

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ABSTRACT

We consider transferable utility cooperative games with infinitely many players and the core understood in the space of bounded additive set functions. We show that, if a game is bounded below, then its core is non-empty if and only if the game is balanced. This finding generalizes Schmeidler (1967) "On Balanced Games with Infinitely Many Players", where the game is assumed to be non-negative. We also generalize Schmeidler's (1967) result to the case of restricted cooperation too.

1. Introduction

The core [16,8] is one of the most important solution concepts of cooperative game theory. It is important not only from the theory viewpoint, but for its simple and easy to understand nature, it also helps to solve various problems arising in practice.

In the transferable utility setting (henceforth *TU games*) the Bondareva-Shapley Theorem [2,17,5] provides a necessary and sufficient condition for the non-emptiness of the core of a finite TU game; it states that the core of a finite TU game with our without restricted cooperation is not empty if and only if the TU game is balanced. The textbook proof of the Bondareva-Shapley Theorem goes by the strong duality theorem of linear programs, see, e.g., Peleg and Sudhölter [12].

Schmeidler [15], Kannai [9,10], and Pintér [13], among others, considered TU games with infinitely many players. All these papers studied the case when the core consists of bounded additive set functions. Schmeidler [15] and Kannai [9] showed that the core of a non-negative TU game with infinitely many players is not empty if and only if the TU game is balanced.

In this paper we consider infinite sign unrestricted TU games with infinite many players with and without restricted cooperation. Particularly, we follow Schmeidler [15] and assume that the allocations are bounded additive set functions.

Applications of infinite signed TU games go back in times at least as early as Shapley and Shubik [18] (economic systems with externalities), which generalize market games [19]. Further applications are (semi-) infinite transportation games [14,20], infinite sequencing games [6], and somehow less directly the line of literature represented by, e.g., Montrucchio and Semeraro [11] among others.

While we can analyze the non-emptiness of the core in the finite setting by using the aforementioned Bondareva-Shapley Theorem [2,17,5], we have been missing an appropriate tool for such TU games with infinitely many players.

Our contribution is an extension of Schmeidler's result [15] saying a non-negative infinite TU game without restricted cooperation has a non-empty core if and only if it is balanced, to the general case saying an infinite TU game bounded below with or without restricted cooperation has a non-empty core if and only if it is balanced (Theorems 4 and 8).

It is worth mentioning that neither Schmeidler's [15] nor Kannai's [9,10] approach (proof) can be applied to achieve our generalization (Theorems 4 and 8). Our approach is different from the previous ones.

The set-up of this paper is as follows. In Sections 2 and 3, we introduce basic notions of TU games with infinitely many players, including the core and balancedness, and we present our main result (Theorem 4). In Sections 4 and 5, we recall some useful con-

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cepts pertaining to functional spaces, topology and compactness, and we prove our main result. We additionally give an example to show the tightness of our main result and we also mention an interesting "limiting" property of the core. Finally, in Section 6, we discuss the case of restricted cooperation and give our second main result (Theorem 8).

2. Preliminaries of infinite TU games

We consider transferable utility cooperative games with a finite or infinite set N of players. A *coalition* is a subset $S \subseteq N$, so the power set $\mathcal{P}(N) = \{S : S \subseteq N\}$ is the collection of all coalitions that can be considered. Let $\mathcal{A} \subseteq \mathcal{P}(N)$ be the collection of all *feasible coalitions*, which are those that can potentially form. When there are no restrictions on coalition formations, we assume that $\mathcal{A}$ is a field of sets over N; that is, the collection $\mathcal{A}$ is such that $\emptyset \in \mathcal{A}$ and, if $S, T \in \mathcal{A}$, then $N \setminus S \in \mathcal{A}$ and also $S \cup T \in \mathcal{A}$. In the case of restricted cooperation, we assume only that $\emptyset, N \in \mathcal{A}$.

Then a *transferable utility cooperative game* (henceforth *game* for short) is represented by its *coalition function*, which is a mapping $v\colon \mathcal{A} \to \mathbb{R}$ such that $v(\emptyset) = 0$. For any coalition $S \in \mathcal{A}$, the value $v(S)$ is understood as the payoff that the coalition S receives if it is formed.

Assume that the players form the grand coalition $N \in \mathcal{A}$. Then $v(N) \in \mathbb{R}$ is the value of the grand coalition N, and the issue is to allocate this value among the players. Following Schmeidler [15], we define the allocations as bounded additive set functions $\mu\colon \mathcal{A} \to \mathbb{R}$; that is, a function such that $|\mu(S)| \le C$ for all $S \in \mathcal{A}$ for some constant $C \in \mathbb{R}$ and $\mu(S \cup T) = \mu(S) + \mu(T)$ for any disjoint $S, T \in \mathcal{A}$. Let $\mathrm{ba}(\mathcal{A}) = \{\mu\colon \mathcal{A} \to \mathbb{R} : \mu$ is a bounded additive set function$\}$ denote the space of all bounded additive set functions on $\mathcal{A}$. Then the *core* of the game represented by v is the set

$$\text{ba-core}(v) = \Big\{ \mu \in \mathrm{ba}(\mathcal{A}) : \mu(N) = v(N)\,, \quad \mu(S) \ge v(S) \quad \text{for all } S \in \mathcal{A} \setminus \{N\} \Big\}.$$

In words, the core consists of all the allocations of the value $v(N)$ among the players (efficiency) such that any coalition $S \in \mathcal{A} \setminus \{N\}$ that could potentially emerge gets by the proposed allocations at least as much as the value $v(S)$ (coalitional rationality), see Shapley [16], Gillies [8], Kannai [10], and Zhao [21].

It is worth noticing that if the class of feasible coalitions is a field, then any additive set function defined on the field can be extended onto the power set. Therefore, it is not misleading to call a game where the class of feasible coalitions is a field game without restricted cooperation.

The case when the class of feasible coalitions is not a field, however, leads to the very same features of the core as restricted cooperation leads in the finite setting, see Faigle [5], explaining why we call this case restricted cooperation.

The key question is whether the core is non-empty. An answer is provided by the Bondareva-Shapley Theorem.

3. The Bondareva-Shapley Theorem

Consider a game having finitely many players without restricted cooperation. In this case, we have $N = \{1, 2, \dots, n\}$ for some natural number n and $\mathcal{A} = \mathcal{P}(N)$. Moreover, in this setting, the allocations of the value $v(N)$ among the players are given by *payoff vectors*, any of them is an n-tuple $a = (a_i)_{i=1}^n \in \mathbb{R}^N$ of real numbers; the number a_i means the payoff allocated to player i for $i = 1, 2, \dots, n$. Then the *core* of this game is defined to be the set

$$\mathrm{core}(v) = \Big\{ a \in \mathbb{R}^N : \textstyle\sum_{i \in N} a_i = v(N)\,, \quad \sum_{i \in S} a_i \ge v(S) \quad \text{for all } S \in \mathcal{P}(N) \setminus \{N\} \Big\}.$$

The intuitive meaning of the core(v) is the same as that of the ba-core(v), see above. Clearly, given a payoff vector $a \in \mathbb{R}^N$, we can define the corresponding additive set function $\mu\colon \mathcal{P}(N) \to \mathbb{R}$ by $\mu(S) = \sum_{i \in S} a_i$ for any $S \in \mathcal{P}(N)$. Conversely, given an additive set function $\mu\colon \mathcal{P}(N) \to \mathbb{R}$, we can define the corresponding payoff vector $a \in \mathbb{R}^N$ by $a_i = \mu(\{i\})$ for $i = 1, 2, \dots, n$. Here any additive set function $\mu\colon \mathcal{P}(N) \to \mathbb{R}$ is bounded as the number of the players is finite. We thus have a one-to-one correspondence between the ba-core(v) and the core(v). Hence, the notion of bounded additive function $\mu \in \mathrm{ba}(\mathcal{A})$ naturally extends the concept of the payoff vector $a \in \mathbb{R}^N$ when the set N of the players is infinite.

Regarding the question whether the core(v) is non-empty, for any coalition $S \subseteq N$, define its *characteristic vector* to be the row vector $\chi_S = \big(\chi_S(1) \;\; \chi_S(2) \;\; \dots \;\; \chi_S(n)\big)$ with $\chi_S(i) = 1$ if $i \in S$, and with $\chi_S(i) = 0$ if $i \notin S$, for $i = 1, 2, \dots, n$. We say that a collection $\mathcal{S} = \{S_1, S_2, \dots, S_r\} \subseteq \mathcal{P}(N)$ of coalitions is *balanced* if there exist non-negative real numbers $\lambda_1, \lambda_2, \dots, \lambda_r$, called balancing weights, such that

$$\sum_{p=1}^{r} \lambda_p \chi_{S_p} = \chi_N\,. \tag{1}$$

Moreover, we say that the game represented by v is *balanced* if

$$\sum_{p=1}^{r} \lambda_p v(S_p) \le v(N) \tag{2}$$

for every balanced collection $\{S_1, S_2, \dots, S_r\} \subseteq \mathcal{P}(N)$ of coalitions. The following result due to Bondareva [2] and Shapley [17], later extended by Faigle [5] to the restricted cooperation case, has become classical:

Theorem 1 *(Bondareva-Shapley Theorem). Consider a game with finitely many players, with or without restricted cooperation, represented by a coalition function $v : \mathcal{P}(N) \to \mathbb{R}$. Then the* core$(v)$ *is non-empty if and only if the game is balanced.*

Consider now a general game without restricted cooperation; that is, the set N of the players can be finite or infinite and the class of feasible coalitions $\mathcal{A} \subseteq \mathcal{P}(N)$ is a field of sets over N. Concerning the question whether the ba-core(v) is non-empty, we follow Schmeidler [15], who proceeds analogously as in the classical case; that is:

For any subset $S \subseteq N$, define its *characteristic function* $\chi_S : N \to \{0, 1\}$ by letting $\chi_S(i) = 1$ if $i \in S$, and $\chi_S(i) = 0$ if $i \notin S$, for every $i \in N$. We say that a collection $\mathcal{S} = \{S_1, S_2, \dots, S_r\} \subseteq \mathcal{A}$ of coalitions is *balanced* if there exist non-negative real numbers $\lambda_1, \lambda_2, \dots, \lambda_r$, called balancing weights, such that

$$\sum_{p=1}^{r} \lambda_p \chi_{S_p} = \chi_N\,. \tag{3}$$

Furthermore, we say that the game represented by v is *balanced* if

$$\sum_{p=1}^{r} \lambda_p v(S_p) \le v(N) \tag{4}$$

for every balanced collection $\{S_1, S_2, \dots, S_r\} \subseteq \mathcal{A}$ of coalitions.

Remark 2. Actually, Schmeidler [15] uses different notation – the player set N and the field of all feasible coalitions $\mathcal{A}$ is denoted by S and Σ, respectively, in the following quotations – and defines balancedness in a slightly different way: "A game is balanced if $\sup \sum_i a_i v(A_i) \le v(S)$ when the sup is taken over all finite sequences of a_i and A_i, where the a_i are non-negative numbers, the A_i are in Σ, and $\sum_i a_i \chi_{A_i} \le \chi_S$." Considering non-negative games, Schmeidler explains that his definition is different from the "definition with equality" formally: "It is easy to verify that this sup does not change even if it is constrained by $\sum_i a_i \chi_{A_i} = \chi_S$ (instead of the inequality); also, for balanced games, the sup equals $v(S)$." — See Schmeidler [15, p. 1]. In the case of non-negative games Schmeidler's definition of balancedness is equivalent with the "definition with equality"; however, in the general, signed case, those are different.

Then Schmeidler [15] proves the following result, see Kannai [9] for another proof:

Theorem 3 *(Bondareva-Shapley Theorem, Schmeidler [15]). Given a finite or infinite set N of the players and a field of sets $\mathcal{A} \subseteq \mathcal{P}(N)$ over N, consider a game represented by a coalition function $v\colon \mathcal{A} \to \mathbb{R}$. If the game is non-negative; that is,*

$$\forall S \in \mathcal{A}\colon \quad v(S) \ge 0\,,$$

then the ba-core(v) *is non-empty if and only if the game is balanced.*

It is easy to see that Theorem 3 is a generalization of Theorem 1 if the game is non-negative. Our goal, nonetheless, is to establish the following result:

Theorem 4 *(Bondareva-Shapley Theorem, a generalization). Given a finite or infinite set N of the players and a field of sets $\mathcal{A} \subseteq \mathcal{P}(N)$ over N, consider a game represented by a coalition function $v\colon \mathcal{A} \to \mathbb{R}$. If the game is bounded below; that is,*

$$\exists L \in \mathbb{R} \quad \forall S \in \mathcal{A}\colon \quad v(S) \ge L\,,$$

then the ba-core(v) *is non-empty if and only if the game is balanced.*

Notice that Theorem 4 directly generalizes both Theorems 1 and 3 because a game with finitely many players is always bounded below.

Notice also the following meaning of Schmeidler's [15] balancedness condition (3) and (4). Let N be an infinite set of players, let $\mathcal{A} \subseteq \mathcal{P}(N)$ be an infinite field of feasible coalitions over N, and let $v\colon \mathcal{A} \to \mathbb{R}$ represent a bounded below cooperative game. Assume that ba-core$(v) = \emptyset$; that is, the *infinite* system of linear relations $\mu(N) = v(N)$ and $\mu(S) \ge v(S)$ for all $S \in \mathcal{A} \setminus \{N\}$, which defines the ba-core(v), has no solution. Then, by Theorem 4, the game is not balanced; that is, there exist a collection $\mathcal{S} = \{S_1, S_2, \dots, S_r\} \subseteq \mathcal{A}$ and non-negative balancing weights $\lambda_1, \lambda_2, \dots, \lambda_r$ such that (3) holds and (4) does not hold. Notice that this fact implies that the *finite* subsystem $\mu(N) = v(N)$ and $\mu(S_p) \ge v(S_p)$ for $p = 1, 2, \dots, r$ has no solution. It is no loss of generality to assume that the collection $\mathcal{S}$ is a field of sets over N. Letting $v_{|\mathcal{S}}$ be the restriction of v onto $\mathcal{S}$, we have that ba-core$(v_{|\mathcal{S}}) = \emptyset$ too. Although the set N of the players is infinite, notice that $v_{|\mathcal{S}}$ represents a *finite* game in fact, and its core is empty. This "compactness" property behind Schmeidler's [15] balancedness condition (3) and (4) is essential and we use it in our proof of Theorem 4 in Section 5.

Before we present our proof of Theorem 4, we find it appropriate to introduce and recall several notions and concepts.

4. Several notions and concepts

Let N be a set and let $\mathcal{A} \subseteq \mathcal{P}(N)$ be a field of sets over N. Then the pair $(N, \mathcal{A})$ is called *chargeable space*. Recall that, for any $S \subseteq N$, the symbol χ_S denotes the characteristic function $\chi_S\colon N \to \{0,1\}$ of the set S. Given a function $f\colon N \to \mathbb{R}$, we say it is a *simple function* if $f = \lambda_1 \chi_{S_1} + \lambda_2 \chi_{S_2} + \dots + \lambda_r \chi_{S_r}$ for some natural number r, for some real numbers $\lambda_1, \lambda_2, \dots, \lambda_r$, and for some sets $S_1, S_2, \dots, S_r \in \mathcal{A}$. Let $\Lambda(\mathcal{A}) = \{\, f\colon N \to \mathbb{R} : f$ is a simple function$\,\}$ denote the vector (i.e. linear) space of all simple functions defined over $(N, \mathcal{A})$, where the sum of two functions and the multiplication of a function by a constant are both defined in the usual way, i.e. pointwise. For a simple function $f \in \Lambda(\mathcal{A})$, define its norm to be

$$\|f\| = \sup_{i \in N} \big| f(i) \big|\,,$$

so $\Lambda(\mathcal{A})$ is a normed linear space.

Likewise, notice that the space ba$(\mathcal{A})$ of all bounded additive set functions on $\mathcal{A}$ is also a vector space; for a $\mu \in$ ba$(\mathcal{A})$, define its norm to be

$$\|\mu\| = \sup_{\substack{r \in \mathbb{N} \\ S_1, S_2, \dots, S_r \in \mathcal{A} \\ S_1 \cup S_2 \cup \dots \cup S_r = N \\ S_i \cap S_j = \emptyset,\ i \ne j}} \big|\mu(S_1)\big| + \big|\mu(S_2)\big| + \dots + \big|\mu(S_r)\big|. \tag{5}$$

It is well-known that the topological dual $(\Lambda(\mathcal{A}))^*$ of the vector space $\Lambda(\mathcal{A})$, which is the space of all continuous linear functionals on $\Lambda(\mathcal{A})$, is isometrically isomorphic to the space ba$(\mathcal{A})$ (see, e.g., Dunford and Schwartz [4], Theorem IV.5.1, p. 258). Indeed, a continuous linear functional $\mu' \in (\Lambda(\mathcal{A}))^*$ induces a bounded additive set function $\mu \in$ ba$(\mathcal{A})$ by letting $\mu(S) = \mu'(\chi_S)$ for $S \in \mathcal{A}$, and, conversely, a bounded additive set function $\mu \in$ ba$(\mathcal{A})$ induces a continuous linear functional $\mu' \in (\Lambda(\mathcal{A}))^*$ by letting

$$\mu'(f) = \lambda_1 \mu(S_1) + \lambda_2 \mu(S_2) + \dots + \lambda_r \mu(S_r) \tag{6}$$

for any simple function $f = \lambda_1 \chi_{S_1} + \lambda_2 \chi_{S_2} + \dots + \lambda_r \chi_{S_r} \in \Lambda(\mathcal{A})$. This is the reason why, for simplicity, we shall identify the space $(\Lambda(\mathcal{A}))^*$ with ba$(\mathcal{A})$.

Consider now a game represented by a coalition function $v\colon \mathcal{A} \to \mathbb{R}$, and let the game be bounded below; that is, there exists a constant $L \in \mathbb{R}$ such that $v(S) \ge L$ for all $S \in \mathcal{A}$. Assume that a $\mu \in$ ba-core(v). Let $S_1, S_2, \dots, S_r \in \mathcal{A}$ be pairwise disjoint and such that $N = S_1 \cup S_2 \cup \dots \cup S_r$. Then

$$\begin{aligned}
\sum_{p=1}^{r} \big|\mu(S_p)\big| &= \sum_{\substack{p=1 \\ \mu(S_p) \ge 0}}^{r} \mu(S_p) - \sum_{\substack{p=1 \\ \mu(S_p) < 0}}^{r} \mu(S_p) \\
&= \mu\Bigg(\bigcup_{\substack{p=1 \\ \mu(S_p) \ge 0}}^{r} S_p \Bigg) - \mu\Bigg(\bigcup_{\substack{p=1 \\ \mu(S_p) < 0}}^{r} S_p \Bigg) \\
&= \mu\Bigg(N \setminus \bigcap_{\substack{p=1 \\ \mu(S_p) \ge 0}}^{r} (N \setminus S_p) \Bigg) - \mu\Bigg(\bigcup_{\substack{p=1 \\ \mu(S_p) < 0}}^{r} S_p \Bigg) \\
&= \mu(N) - \mu\Bigg(\bigcap_{\substack{p=1 \\ \mu(S_p) \ge 0}}^{r} (N \setminus S_p) \Bigg) - \mu\Bigg(\bigcup_{\substack{p=1 \\ \mu(S_p) < 0}}^{r} S_p \Bigg) \\
&\le \mu(N) - 2L = v(N) - 2L\,.
\end{aligned}$$

By taking the definition (5) of the norm into account, it follows the ba-core(v) is contained in the closed ball $B_R = \big\{\, \mu \in$ ba$(\mathcal{A})$:

$\|\mu\| \le v(N) - 2L\}$ of radius $R = v(N) - 2L$. (Notice that $v(N) - 2L \ge 0$, for if we had $v(N) < 2L$, then the ba-core(v) would obviously be empty, contradicting the assumption that $\mu \in$ ba-core(v).)

We endow the space ba($\mathcal{A}$) with the weak* topology with respect to $\Lambda(\mathcal{A})$. The topology will be introduced if we describe all the neighborhoods of a point. A set $U \subseteq$ ba($\mathcal{A}$) is a *weak* neighborhood* of a $\mu_0 \in$ ba($\mathcal{A}$) if there exist a natural number r and functions $f_1, f_2, \dots, f_r \in \Lambda(\mathcal{A})$ such that $\bigcap_{p=1}^{r}\{\mu \in \mathrm{ba}(\mathcal{A}) : |\mu'(f_p) - \mu_0'(f_p)| < 1\} \subseteq U$, where μ' and μ_0' are continuous linear functionals induced by μ and μ_0, respectively, see (6). By Alaoglu's Theorem (see, e.g., Aliprantis and Border [1], Theorem 6.21, p. 235), the closed ball B_R is compact in the weak* topology. That is, if $G_i \subseteq$ ba($\mathcal{A}$) are weakly* open sets for $i \in I$, where I is an index set, such that $\bigcup_{i\in I} G_i \supseteq B_R$, then $\bigcup_{j=1}^{n} G_{i_j} \supseteq B_R$ for some natural number n and for some $i_1, i_2, \dots, i_n \in I$.

Let $F_i \subseteq B_R$ be weakly* closed sets for $i \in I$, where I is an index set. We say the collection $\{F_i\}_{i\in I}$ is a *centered system* of sets if $\bigcap_{j=1}^{n} F_{i_j} \neq \emptyset$ for any natural number n and for any $i_1, i_2, \dots, i_n \in I$. By considering the complements ($G_i = \mathrm{ba}(\mathcal{A}) \setminus F_i$), it follows $\bigcap_{i\in I} F_i \neq \emptyset$.

In our proof of Theorem 4 we consider the weakly* closed sets

$$F_S = \left\{ \mu \in \mathrm{ba}(\mathcal{A}) : \mu(N) = v(N) \text{ and } \mu(S) \ge v(S) \text{ and } \|\mu\| \le R \right\}$$

for $S \in \mathcal{A}$. The main idea is to show that, if the game v is balanced, then the system $\{F_S\}_{S\in\mathcal{A}}$ is centered. Noticing that ba-core(v) $= \bigcap_{S\in\mathcal{A}} F_S \neq \emptyset$, the proof will be done.

We are now ready to present our proof of Theorem 4.

5. Proof of Theorem 4

Below we give our proof of Theorem 4. The notions and concepts introduced in Section 4 are utilized in the proof, with compactness playing the crucial role.

Proof of Theorem 4. Assume that the given coalition function $v: \mathcal{A} \to \mathbb{R}$ is bounded below by L. We are going to show that ba-core(v) $\neq \emptyset$ if and only if the given game is balanced. The “only if” part is obvious. Assume that $\mu \in$ ba-core(v) and let $\mathcal{S} = \{S_1, S_2, \dots, S_r\} \subseteq \mathcal{A}$ be a balanced collection of coalitions, so that (3) holds for some non-negative balancing weights $\lambda_1, \lambda_2, \dots, \lambda_r$. Then $\sum_{p=1}^{r} \lambda_p v(S_p) \le \sum_{p=1}^{r} \lambda_p \mu(S_p) = \mu(N) = v(N)$, so (4) is satisfied, and the game is balanced. It remains to prove the “if” part.

Take any sets $S_0, S_1, \dots, S_n \in \mathcal{A}$. We want to show that $\bigcap_{j=0}^{n} F_{S_j} \neq \emptyset$. We can assume w.l.o.g. that the sets $S_0, \dots, S_n$ are distinct with $S_0 = \emptyset$ and $S_n = N$, and that the collection $\{S_0, \dots, S_n\} \subseteq \mathcal{A}$ is a field of sets. (Roughly speaking, the more sets we take, the smaller the intersection $\bigcap_{j=0}^{n} F_{S_j}$ is. Having to prove that the intersection is non-empty anyway, we can include the empty and the grand coalition among the sets. Moreover, we can add further sets from $\mathcal{A}$ so that the collection $\{S_0, \dots, S_n\}$ becomes a finite field of sets.)

We can also assume w.l.o.g. that $S_1, \dots, S_{n'}$ are all the atoms of the field; that is, they are all minimal elements in the collection $\{S_1, \dots, S_n\}$. Obviously, the atoms $S_1, \dots, S_{n'}$ are pairwise disjoint, and it holds $n = 2^{n'} - 1$.

Now, the sets $\emptyset = S_0, S_1, \dots, S_n$ being fixed, we apply balancedness to the sets $S_1, \dots, S_n$:

$$\begin{aligned} \forall \lambda_1, \dots, \lambda_n \ge 0 :\ & \lambda_1 \chi_{S_1} + \dots + \lambda_n \chi_{S_n} = \chi_N \\ & \Longrightarrow \lambda_1 v(S_1) + \dots + \lambda_n v(S_n) \le v(N). \end{aligned} \tag{7}$$

By using the Bondareva-Shapley Theorem for finite games (Theorem 1), we show that the system of relations

$$\begin{aligned} &\mu(N) = v(N), \\ &\mu(S_1) \ge v(S_1), \\ &\dots\dots\dots\dots\dots \\ &\mu(S_n) \ge v(S_n) \end{aligned} \tag{8}$$

has a solution $\mu \in$ ba($\mathcal{A}$) such that $\|\mu\| \le R$.

Consider a new finite game $v' : \mathcal{P}(N') \to \mathbb{R}$ with the player set $N' = \{1, \dots, n'\}$. Define the game as follows. Recall first that the collection $\{S_0, \dots, S_n\}$ is a field of sets and that $S_1, \dots, S_{n'}$ are all its atoms, which are pairwise disjoint. Now, for an $S' \subseteq N'$, let $S = \bigcup_{i'\in S'} S_{i'}$, notice $S \in \{S_0, \dots, S_n\}$, and set $v'(S') = v(S)$. The new finite game v' has thus been defined.

Now, condition (7) equivalently says that the new game v' is balanced. By the Bondareva-Shapley Theorem (Theorem 1), its core is non-empty: there exist $a_1, \dots, a_{n'} \in \mathbb{R}$ such that $\sum_{i'=1}^{n'} a_{i'} = v'(N')$ and $\sum_{i'\in S'} a_{i'} \ge v'(S')$ for any $S' \subseteq N'$.

Since the atoms $S_1, \dots, S_{n'}$ are non-empty sets, there exist elements $x_{i'} \in S_{i'}$ for $i' = 1, \dots, n'$. Consider the measure

$$\mu = a_1 \delta_{x_1} + \dots + a_{n'} \delta_{x_{n'}},$$

where $\delta_{x_{i'}}$ is the Dirac measure concentrated at $x_{i'}$. We have $\mu(N) = \mu(S_1 \cup \dots \cup S_{n'}) = a_1 + \dots + a_{n'} = v(N)$. For any $j = 1, \dots, n$, let $S_j' = \{i' \in N' : S_{i'} \subseteq S_j\}$. Then $S_j = \bigcup_{i'\in S_j'} S_{i'}$, and $\mu(S_j) = \sum_{i'\in S_j'} a_{i'} \ge v'(S_j') = v(S_j)$. We have shown thus that μ is a solution to the system of inequalities (8).

Finally, let us calculate the norm $\|\mu\|$ of the solution, see (5). For a $T \in \mathcal{A}$, we observe that

$$\mu(T) = \sum_{\substack{i'=1 \\ x_{i'} \in T}}^{n'} a_{i'}.$$

Given pairwise disjoint sets $T_1, \dots, T_s \in \mathcal{A}$ such that $N = T_1 \cup \dots \cup T_s$, and recalling $\sum_{i'=1}^{n'} a_{i'} = v(N)$, we have

$$\begin{aligned} \sum_{q=1}^{s} |\mu(T_q)| &= \sum_{q=1}^{s} \left| \sum_{\substack{i'=1 \\ x_{i'} \in T_q}}^{n'} a_{i'} \right| \le \sum_{q=1}^{s} \sum_{\substack{i'=1 \\ x_{i'} \in T_q}}^{n'} |a_{i'}| \\ &= \sum_{i'=1}^{n'} |a_{i'}| = \sum_{\substack{i'=1 \\ a_{i'} \ge 0}}^{n'} a_{i'} - \sum_{\substack{i'=1 \\ a_{i'} < 0}}^{n'} a_{i'} \\ &= v(N) - 2 \sum_{\substack{i'=1 \\ a_{i'} < 0}}^{n'} a_{i'} \le v(N) - 2v\left(\bigcup_{\substack{i'=1 \\ a_{i'} < 0}}^{n'} S_{i'} \right) \\ &\le v(N) - 2L = R. \end{aligned}$$

It follows that $\|\mu\| \le R$. To conclude, we have a $\mu \in$ ba($\mathcal{A}$) such that it is a solution to (8) and $\|\mu\| \le R$, which means $\mu \in \bigcap_{j=1}^{n} F_{S_j}$. Since $F_{S_j} \subseteq F_\emptyset$ for $j = 1, \dots, n$, it holds $\mu \in \bigcap_{j=0}^{n} F_{S_j}$. We have shown thus that the system $\{F_S\}_{S\in\mathcal{A}}$ is centered. As the closed R-ball B_R is weakly* compact, we have ba-core(v) $= \bigcap_{S\in\mathcal{A}} F_S \neq \emptyset$. $\square$

The following example demonstrates that Theorem 4 cannot be generalized any further. We are going to construct a game unbounded below that is balanced, but its core is empty.

Example 5. Let the player set be $N = \mathbb{N}$, and let $\mathcal{A} = \{S \subseteq N : S$ is finite or $N \setminus S$ is finite$\}$. Consider the game represented by the coalition function $v : \mathcal{A} \to \mathbb{R}$ defined as follows: for any $S \in \mathcal{A}$, let

$$v(S) = \begin{cases} 1 & \text{if } S = \{1\}, \\ 1 + \frac{1}{n} & \text{if } S = \{1, n\} \text{ for } n = 2, 3, \ldots, \\ -\sum_{n \in T} \frac{1}{n} & \text{if } S = N \setminus T \text{ for a finite } T \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that this game is balanced. Assuming that a $\mu \in \text{ba-core}(v)$, then $\mu(\{1\}) \geq v(\{1\}) = 1$ and $\mu(N \setminus \{1\}) \geq v(N \setminus \{1\}) = -1$. Since $\mu(\{1\}) + \mu(N \setminus \{1\}) = \mu(N) = v(N) = 0$, we have $\mu(\{1\}) = 1$. As $\mu(\{1, n\}) \geq v(\{1, n\}) = 1 + 1/n$, it follows $\mu(\{n\}) \geq 1/n$ for all $n = 2, 3, \ldots$. Summing up, we have $\mu(\{1, \ldots, n\}) \geq \ln(n+1)$, so $\mu \notin \text{ba}(\mathcal{A})$ because μ is not bounded. It follows $\text{ba-core}(v) = \emptyset$.

The following "limiting" property of the ba-core is interesting. It is obtained as a corollary of Theorem 4 by considering the balancedness condition (3) and (4).

Corollary 6. *Given a finite or infinite set N of the players and a field of sets $\mathcal{A} \subseteq \mathcal{P}(N)$ over N, let the game represented by a coalition function $v\colon \mathcal{A} \to \mathbb{R}$ be bounded below. For any $\varepsilon > 0$, define the coalition function $v_\varepsilon\colon \mathcal{A} \to \mathbb{R}$ as follows: let $v_\varepsilon(N) = v(N) + \varepsilon$ and $v_\varepsilon(S) = v(S)$ for all $S \in \mathcal{A} \setminus \{N\}$. If $\text{ba-core}(v_\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$, then $\text{ba-core}(v) \neq \emptyset$.*

Under the assumptions of Corollary 6, the converse statement is clear: if $\text{ba-core}(v) \neq \emptyset$, then $\text{ba-core}(v_\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. We thus conclude that a game represented by the coalition function v is balanced if and only if $\text{ba-core}(v_\varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. It follows hence that both the core non-emptiness and the game balancedness are monotone and closed prosperity properties in the sense of van Gellekom et al. [7].

6. Games with restricted cooperation

We now consider games with restricted cooperation. In general, the cooperation is restricted whenever the collection $\mathcal{A} \subseteq \mathcal{P}(N)$ of coalitions that can potentially form is a proper subset of $\mathcal{P}(N)$. In this sense, Theorem 4 covers the case of restricted cooperation, under the additional assumption that $\mathcal{A} \subseteq \mathcal{P}(N)$ is a field of sets over N, too. Now, let $\mathcal{A}' \subseteq \mathcal{P}(N)$ be the collection of all coalitions that can potentially emerge; the collection $\mathcal{A}'$ need not be a field of sets now. Assume only $\emptyset, N \in \mathcal{A}'$. Then any coalition function $v'\colon \mathcal{A}' \to \mathbb{R}$, such that $v(\emptyset) = 0$, represents a game with restricted cooperation.

To introduce the concept of core of this game with restricted cooperation, let $\mathcal{A} = \text{field}(\mathcal{A}')$ be the field hull of $\mathcal{A}'$; that is, the minimal collection $\mathcal{A} \supseteq \mathcal{A}'$ that is a field of sets over N. Then the core of a game v' with restricted cooperation is the set

$$\begin{aligned} &\text{ba-core}(v') = \\ &\Big\{ \mu \in \text{ba}\big(\text{field}(\mathcal{A}')\big) : \mu(N) = v'(N)\,, \\ &\qquad\qquad \mu(S) \geq v'(S) \qquad \text{for all} \quad S \in \mathcal{A}' \setminus \{N\} \Big\}. \end{aligned}$$

We again ask whether $\text{ba-core}(v') \neq \emptyset$.

The following example presents a non-negative game with restricted cooperation that is balanced as defined by (3) and (4), but only the feasible coalitions are considered. The core of this game is empty.

Example 7. Let the player set be $N = \mathbb{N}$ and let $\mathcal{A}' = \{\emptyset\} \cup \big\{\{1, i\} : i = 1, 2, 3, \ldots\big\} \cup \{N \setminus \{1\}\} \cup \{N\}$. Consider the game represented by the coalition function $v'\colon \mathcal{A}' \to \mathbb{R}$ defined as follows: for any $S \in \mathcal{A}$, let

$$v'(S) = \begin{cases} 2 & \text{if } S = \{1\}, \\ 2 + \frac{1}{n} & \text{if } S = \{1, n\} \text{ for } n = 2, 3, \ldots, \\ 0 & \text{if } S = N \setminus \{1\} \text{ or } S = \emptyset, \\ 1 & \text{if } S = N. \end{cases}$$

Notice that this game is analogous to the one presented in Example 5. The field hull of $\mathcal{A}'$ is $\mathcal{A} = \{S \subseteq N : S \text{ is finite or } N \setminus S \text{ is finite}\}$. The fact that this game is balanced as defined by (3) and (4), where $\mathcal{A}$ and v are replaced with $\mathcal{A}'$ and v', respectively, is clear. To show that $\text{ba-core}(v') = \emptyset$, it is enough to follow the arguments presented in Example 5.

Because of Example 7, we have to introduce a new notion of balancedness in the case of restricted cooperation. A game represented by a coalition function v' defined on the class of feasible coalitions $\mathcal{A}'$ is *bounded-balanced* if there exists a bounded below balanced game v defined on $\mathcal{A} = \text{field}(\mathcal{A}')$ such that for every $S \in \mathcal{A}'$ it holds that $v(S) = v'(S)$. It is clear that if $\mathcal{A}'$ is a field and the game is bounded below (as in Theorem 4), then we get back the notion of balancedness applied in Theorem 4. Moreover, notice that for finite games bounded-balancedness and balancedness by Faigle [5] are equivalent.

The game in Example 7 above has an empty core because, even if it is non-negative, none of its bounded below "extensions" onto $\mathcal{A}$ is balanced and none of its balanced "extensions" onto $\mathcal{A}$ is bounded below.

Then the following theorem extends Theorem 4 to the class of games with restricted cooperation, hence it extends Theorem 4 in Faigle [5].

Theorem 8. *Consider a coalition function $v'\colon \mathcal{A}' \to \mathbb{R}$, where $\emptyset, N \in \mathcal{A}' \subseteq \mathcal{P}(N)$ and N is a finite or infinite set of the players. If v' is bounded below, then the $\text{ba-core}(v') \neq \emptyset$ if and only if v' is bounded-balanced.*

Proof. If $\mathcal{A}'$ is a field, then we are back at Theorem 4, hence there is nothing to do.

Suppose that $\mathcal{A}'$ is not a field. If v' is bounded-balanced, then take any game v that makes v' bounded-balanced and apply Theorem 4 to get $\emptyset \neq \text{ba-core}(v) \subseteq \text{ba-core}(v')$.

If $\text{ba-core}(v') \neq \emptyset$, then take an arbitrary $\mu \in \text{ba-core}(v')$, and let

$$v(S) = \begin{cases} v'(S) & \text{if } S \in \mathcal{A}', \\ \mu(S) & \text{if } S \in \text{field}(\mathcal{A}') \setminus \mathcal{A}'. \end{cases}$$

Since v' is bounded below, v is a bounded below game with non-empty core ($\text{ba-core}(v) \neq \emptyset$). Therefore, by Theorem 4, v is balanced. Finally, v makes v' bounded-balanced. $\square$

Notice that the game v' of Example 7 is not bounded-balanced, hence $\text{ba-core}(v') = \emptyset$.

Finally, the following phenomenon that may occur in the case of restricted cooperation is worth mentioning: If the collection $\mathcal{A}'$ is not a field, then there may exist allocations $\mu, \nu \in \text{ba-core}(v')$ such that $\mu(S) \geq \nu(S)$ for all coalitions $S \in \mathcal{A}'$ and $\mu \neq \nu$. In other words, the core allocation μ is more desirable than ν for at least one coalition. Define the *core of the collection* $\mathcal{A}'$ by

$$\begin{aligned} &\text{ba-Core}(\mathcal{A}') = \\ &\Big\{ \mu \in \text{ba}\big(\text{field}(\mathcal{A}')\big) : \mu(N) = 0\,, \\ &\qquad\qquad \mu(S) \geq 0 \qquad \text{for all} \quad S \in \mathcal{A}' \setminus \{N\} \Big\}. \end{aligned}$$

It is easy to see that, if the $\text{ba-core}(v')$ is not empty, then the aforementioned phenomenon occurs if and only if $\text{ba-Core}(\mathcal{A}') \neq$

$\{0\}$ because $\mu - \nu \in$ ba-Core$(\mathcal{A}')$. Derks and Reijnierse [3] study the conditions under which the ba-Core$(\mathcal{A}')$ is a singleton, a pointed cone, or a linear subspace in the case when the player set N is finite. The study of these conditions in the case of the core consisting of bounded additive functions, when the player set N is infinite, is a topic for further research.

Data availability

No data was used for the research described in the article.

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