Do banks need a supervisor?

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Abstract

The paper studies a simple microeconomic stochastic model of a bank operating in a competitive environment. The model allows us to describe the conditions on the model parameters that generate both the formation of bubbles in the credit market and the formation of stable banks with self-restrictive behavior, that do not require the intervention of the regulator. The comparative statics of equilibria is studied with respect to the basic parameters of the model, a theoretical assessment is carried out of the probability of bank default based on the values of exogenous factors in both the short and long term.

Key words: Banking microeconomics, Credit bubble, Probability of default, Capital adequacy ratio

JEL codes: G21, G28, G32, G33

Introduction

Fractional reserve banking causes a credit expansion due to the money multiplier effect. Credit bubble can burst, causing a credit crunch. There were hundreds of credit crises in world history (see, e.g., [2, pp. 344-347]). In turn, the credit crunch could trigger an economic crisis. For example, the burst of the US subprime mortgage bubble caused a global crisis. Credit crises lead to high social costs - unemployment, impoverishment of the population, social instability. To avoid such severe consequences, the regulator limits the expansion of banks by macro-prudential policies, in particular, limiting from below by CAR (capital adequacy ratio).

CAR shows a huge variability (see Figure 1). Instead of maximally using an equity and increasing risk assets up to the level corresponding to the capital adequacy requirements imposed by the regulator, many banks demonstrate two opposite types of behavior: some aggressively exceed this level, falsifying

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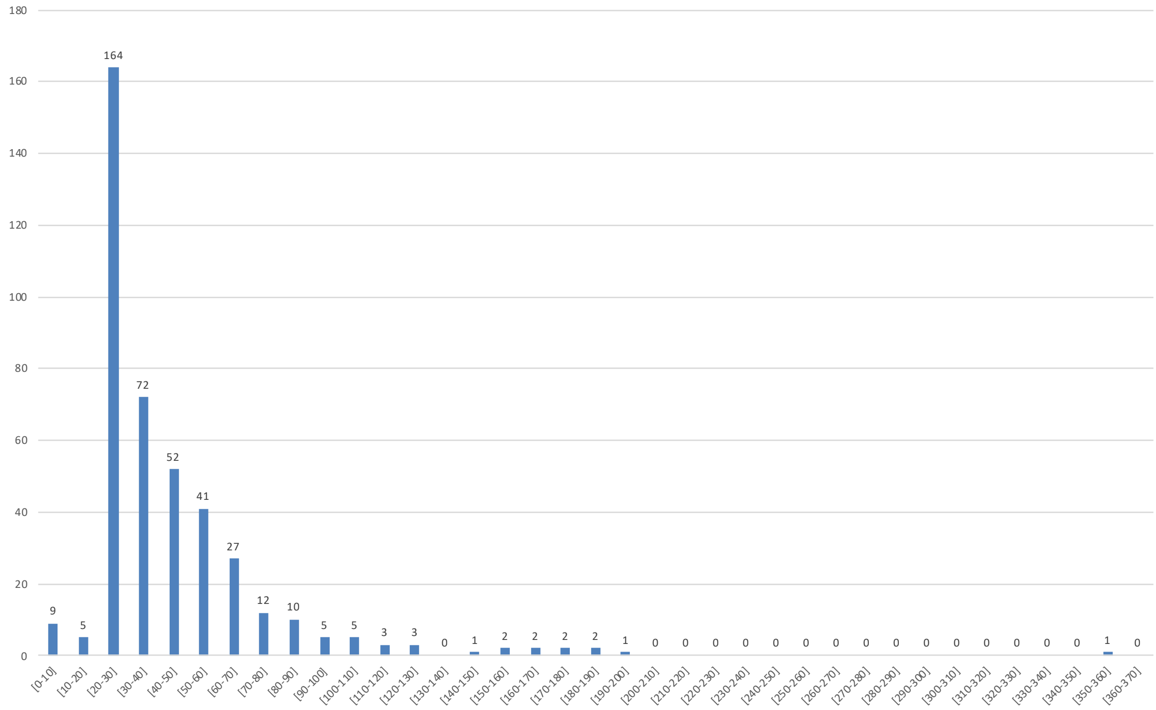

Figure 1: CAR distribution of Russian banks as of May 1, 2019 (%). Source: Bank of Russia data, authors' calculations.

the equity, and others, on the contrary, use an excess equity, sufficiently exceeding the regulatory threshold.

The problem of the equity falsifying is particularly acute in developing countries with high level of corruption, weak institutions and a poor quality of supervision and regulation of the banking sector.

The paper discusses a simple microeconomic model of a bank that takes into account the stochastic nature of borrowers' defaults. To simulate a stochastic process, we use the uniform distribution function, which allows us to obtain solutions in an analytical form, as well as the Vasicek distribution function used in Basel III [1]. The Vasicek distribution allows the taking into account the correlation of the borrowers' assets. Although in reality the credit risk is often accompanied by the risk of outflow of liabilities, in the framework of this paper we do not allow for the liquidity risk, leaving it for a future research.

Using the model, we study the mechanism of CAR choice. It is allowed for a bank to choose negative CAR, while the conditions guaranteeing a positive CAR are found. We study also the comparative statics of the banker's decision and of the probability of the bank's default with respect to the model parameters: the interest rates of attraction and allocation of resources, the correlation of borrowers' assets and other factors. There are identified the four areas in the space of exogenous parameters within which the banks are capable of self-restraint, and when they choose the unlimited expansion. This stratification of the model parameters, according to the nature of the equilibrium solution, opens a new way to determine whether the regulator is needed to intervene in banking, what are the boundaries of this intervention and its effectiveness. On the one hand, the concept of laissez-faire can be destructive, since the banking market failure entails far-reaching negative consequences, not so much even for the bank owner as for its many clients. If the bank is large enough, then its default can cause a domino effect.

On the other hand, practice shows that the tightening of regulation is faced with the problem of low efficiency of regulatory measures, since banks have ample opportunities to manipulate information, creating the appearance of compliance with regulatory constraints. Another negative effect of over-regulation is a decrease in the efficiency of banks and the economy as a whole. As regulation is tightened, bankers spend too much time and efforts on compliance, instead of doing business. In this regard, it is worth to recall the general economic principle, according to which an economic individual can bypass external constraints, but cannot ignore his/her own incentives.

The classification of the solutions obtained in this paper allows us to identify cases when a state intervention in the bank activity is superfluous, since the decision satisfying the regulator is supported by internal stimuli. If the decision falls into another class, for example, it is characterized by an excessively low or even negative CAR, then this intervention is inevitable.

Rating agencies use to apply the empirical econometric models to calculate the credit ratings. Our model allows to consider the prospect of calculating bank credit ratings not with empirical econometric models, but with micro-based modeling. Microeconomic models can be useful for evaluating implications of banking regulation.

The paper is organized as follows. Section 1 presents the basic concepts and notation, a simplified linear model is constructed, demonstrating the mechanism of the emergence of credit bubbles, which, within the framework of this simplified model, turn out to be almost inevitable. In Section 2, a more realistic nonlinear model is constructed that takes into account the partial impairment of bank assets in the event of a bank default, and the conditions for the existence and uniqueness of the equilibria are obtained. The mechanism of the formation of an equilibrium market credit rate in a competitive environment of risk-neutral players is considered. In Section 3, we consider the parametrized classification of the equilibrium states based on CAR and study the comparative statics of equilibrium characteristics both analytically and using computer simulation. Section 4 is devoted to a more visual graphical classification of decisions on the main parameters. The most important result is the determination of compliance with the requirements of Basel III. Finally, Section 5 is devoted to the multi-period extension of this model, in particular, the assessment of the probability of a bank's default in the long term is found. The main results and conclusions of the work are formulated in the Conclusion.

1 Linear model

It is assumed that the bank operates in a completely competitive environment, being a price taker, that is, the interest rates on deposits and loans are formed outside the bank. Consider a single-period three-phase model in which a bank is created in the initial phase $s = -1$ with an initial equity $E_{-1} > 0$. Then the phase $s = 0$ begins, during which the bank makes strategic decisions: it attracts deposits $D_0 \geq 0$ at the interest rate R and places borrowed and its own funds in uniform loans of the same size at the interest rate r before the onset of the final phase $s = 1$, leaving the cash $M_0 \geq 0$. It is assumed that the inequalities $r > R > 0$ hold. The supply of loans and the demand for deposits are satisfied in full, and interest rates are exogenous parameters, due to the assumption of perfectly competitive environment. Next, the net loans (loans after deducting the loan loss provision) are $C_0 = (1 - \lambda)(E_{-1} + D_0 - M_0)$, where λ is the loan loss provision coverage ratio. It is natural to assume that the inequality $C_0 \geq 0$ holds. In any phase s, the assets are equal to the sum of cash M_s and loans C_s, and, in turn, coincide with the sum of deposits D_s and equity E_s:

$$M_s + C_s = D_s + E_s,$$

where equity $E_0 = E_{-1} - \lambda(E_{-1} + D_0 - M_0)$. At the phase $s = 1$ all loans are repaid, except for those defaulted, hence $C_1 = 0$, and all assets acquire the form of cash M_1. After the deposits are returned and interest is paid at the rate R, the bank's equity becomes equal to $E_1 = M_1 - D_1$.

Consider the case when the bank provided n loans, and the probability of default of each loan $i \in \{1, \ldots, n\}$ is the same and equals $PD \in [0, 1]$. It is assumed that the losses given default $LGD = 1$. Random variable L_i takes two possible values: $L_i = 1$, if credit i is defaulted (with probability PD), and $L_i = 0$ (with probability $1-PD$) otherwise. Generally speaking, the random variables L_i $i \in \{1, \ldots, n\}$ are not independent. Then a random variable

$$L = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} L_i$$

characterizes the share of nonperforming loans, taking values in the interval $[0, 1]$. It is obvious that $\mathbb{E}(L) = PD$, while the distribution law of L is ambiguous due to possible dependence of various L_i.

Balance sheet items		$s=-1$	$s=0$	$s=1$
Assets	Cash	$M_{-1}=E_{-1}$	$M_0 \geq 0$	M_1
	Loans	$C_{-1}=0$	$C_0 \geq 0$	$C_1=0$
Liabilities		$D_{-1}=0$	$D_0 \geq 0$	$D_1=(1+R)D_0$
Equity		$E_{-1}>0$	$E_0=M_0+C_0-D_0$	$E_1=M_1-D_1$

Table 1: Control variables and dependencies between variables

The banker's problem

In the simplest setting of model, the banker solves the problem of maximizing the mathematical expectation of the future equity

$$\max_{M_0 \geq 0, D_0 \geq 0} \mathbb{E}(E_1)$$

s.t.

$$C_0 = (1-\lambda)(E_{-1}+D_0-M_0) \geq 0.$$

The final cash

$$M_1 = (1+r)(E_{-1}+D_0-M_0)(1-L)+M_0$$

is a random variable. Substituting it into the expression for the final equity

$$E_1 = M_1 - D_1$$

taking into account

$$D_1 = (1+R)D_0,$$

will get

$$E_1 = (1+r)(E_{-1}+D_0-M_0)(1-L)+M_0-(1+R)D_0. \quad (1.1)$$

All balance sheet items, taking into account interdependence and timing, are summarized in Table 1.

The solution of the problem

Let's determine the loan risk-adjusted interest rate

$$\tilde{r} = r-(1+r)PD.$$

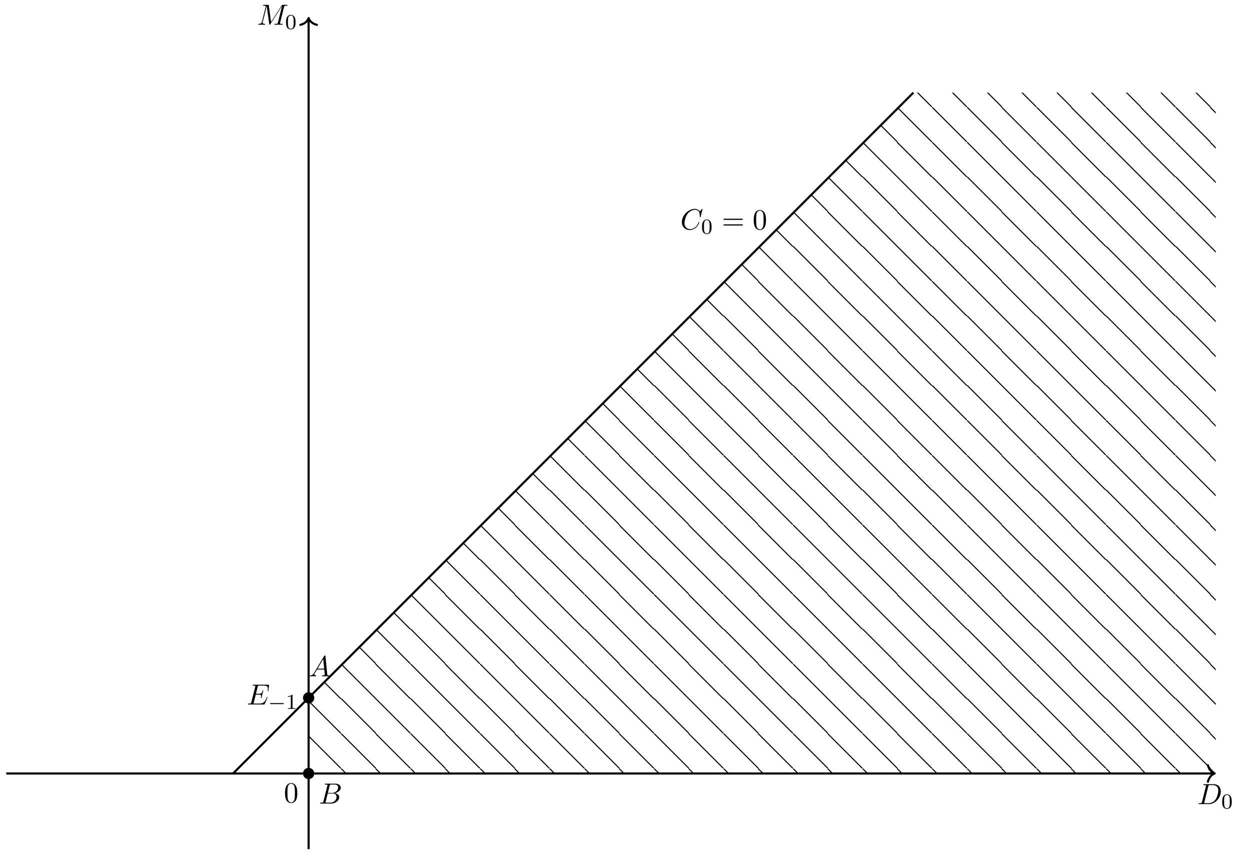

Figure 2: Feasible solutions (D_0, M_0)

Note that this value may be negative, if the probability of default $PD > r/(1+r)$. Using this notion, we can represent the banker's objective function as follows

$$\mathbb{E}(E_1) = (\tilde{r} - R)D_0 - \tilde{r}M_0 + (1+\tilde{r})E_{-1}, \tag{1.2}$$

thus, the banker's problem takes on the form

$$\max_{M_0 \geq 0, D_0 \geq 0} (\tilde{r} - R)D_0 - \tilde{r}M_0 \tag{1.3}$$

s.t.

$$D_0 - M_0 + E_{-1} \geq 0. \tag{1.4}$$

The set of feasible solutions of this problem is presented in Figure 2.

Suppose that the state does not limit the credit activity of the bank. If the probability of default PD is very high, so the loan risk-adjusted interest rate $\tilde{r}$ is negative, then the optimal strategy of the bank is to stop any activity: the deposits are not attracted and the funds are not placed in loans, which means that the solution is the point A with coordinates

$$D_0^* = 0,\ \ M_0^* = E_{-1}.$$

If the probability of default PD is relatively high and loan risk-adjusted interest rate is already positive, but so far below the deposit interest rate $0 < \tilde{r} < R$, the solution is the point B with coordinates

$$D_0^* = 0,\ \ M_0^* = 0,$$

that is, the bank does not attract deposits and does not leave cash, the initial equity is placed in loans.

Suppose now that the probability of default PD is as low as the loan risk-adjusted interest rate $\tilde{r} > R$. Then, in the absence of external regulation, the objective function (1.3) becomes unbounded from above on the set of feasible solutions. This case corresponds to the unlimited expansion of the bank: the bank attracts an infinitely large amount of deposits and places them in the loan portfolio of infinitely large size.

Thus, solutions exist only in an "abnormal" situation, when the borrowers' probability of default PD is high enough, while the bank's optimal strategies are very primitive — from lending solely through equity capital to a complete cessation of activity.

2 Nonlinear model

Suppose now that the bank default is connected with additional losses, moreover, the banker is "responsible" i.e., assumes all costs in the event of bank failure. More precisely, if random amount of bank equity at the end of the period

$$E_1 = (1+r)(E_{-1} + D_0 - M_0)(1-L) + M_0 - (1+R)D_0$$

takes positive value, then this amount goes to the banker unchanged. In that case, i.e., under the condition $E_1 \leq 0$, the bank sells a loan portfolio with a discount $0 \leq d \leq 1$. As result, the final equity is equal to

$$E_1^b = (1-d)(1+r)(E_{-1} + D_0 - M_0)(1-L) + M_0 - (1+R)D_0 \leq E_1.$$

Note that in the case of $d = 0$ we get exactly the linear model of the bank, considered in the previous section.

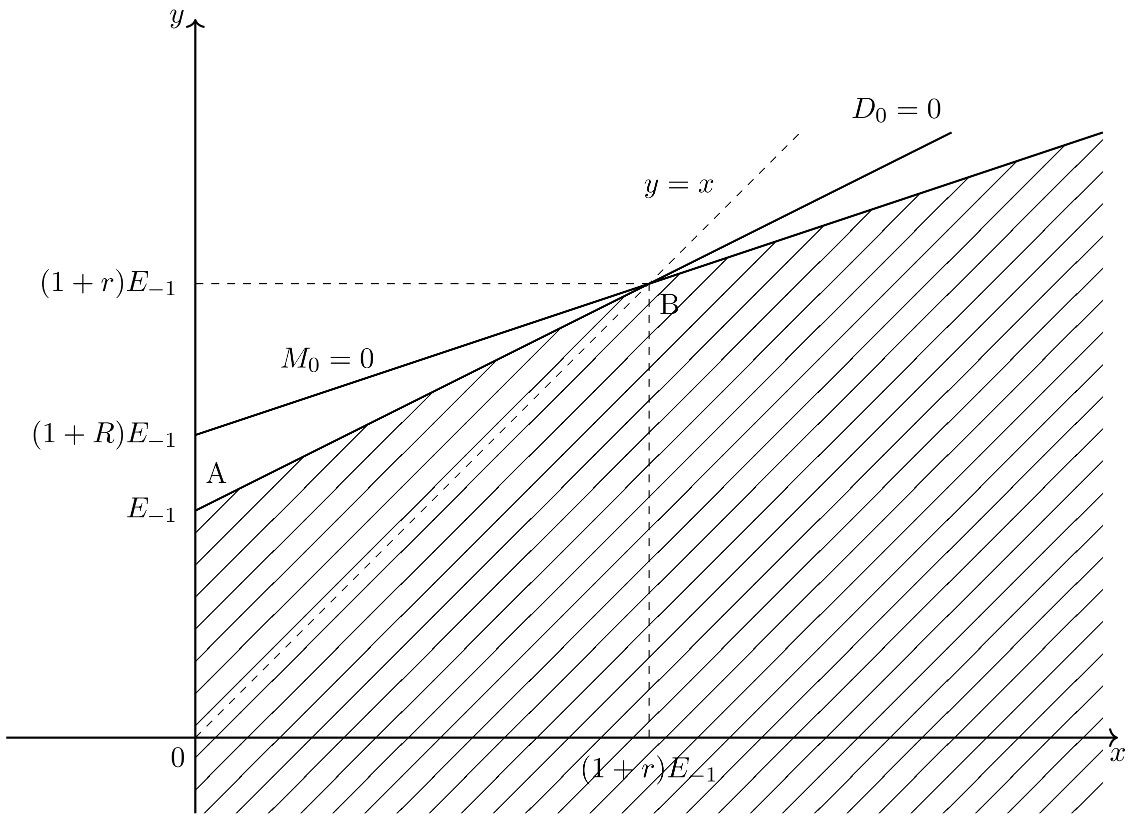

Figure 3: Feasible solutions (x, y).

To simplify further calculations, we make a linear substitution of variables

$$x = (1+r)(E_{-1} + D_0 - M_0), \; y = (1+r)E_{-1} + (r-R)D_0 - rM_0. \tag{2.1}$$

It is easy to see that the mapping $(D_0, M_0) \to (x, y)$ is nonsingular, the inverse transformation is as follows

$$M_0 = \frac{1}{R}\left[(1+R)E_{-1} + \frac{r-R}{1+r}x - y\right], \; D_0 = \frac{1}{R}\left[E_{-1} + \frac{r}{1+r}x - y\right]. \tag{2.2}$$

The variable x can be interpreted as a sum of payed back loans with interest in case of all borrowers meet their obligations, i.e., $L \equiv 0$. Moreover,

$$x = \frac{1+r}{1-\lambda}C_0$$

implies that the constraint $C_0 \geq 0$ is equivalent to $x \geq 0$. The variable y can be interpreted as an final equity in case of all borrowers meet their obligations, i.e., $L \equiv 0$. The set of feasible controls in terms of (x, y) is as presented in Figure 3.

Taking into account (2.1), the objective function of banker may be rewritten in new terms as follows

$$U(x,y) = \mathbb{E}(E_1 | E_1 > 0) + \mathbb{E}(E_1^b | E_1 \leq 0) = \mathbb{E}(y - x \cdot L | y > x \cdot L) + \mathbb{E}(y - x \cdot (d + (1-d)L) | y \leq x \cdot L),$$

where $x \geq 0$. Obviously, $x = 0$ implies $U(0, y) = y$, moreover, in case of $y \geq x$ the bank's default is an impossible event, therefore $U(x, y) = y - PD \cdot x$ for all $y \geq x \geq 0$.

Now we shall study the behavior of function $U(x, y)$ in the open convex area $x > 0$, $x > y$. To simplify calculations, we shall use the notion $w = y/x$, which satisfies $w < 1$ in the considered area. Let $F(z)$, $f(z) = F'(z)$ be, respectively the cumulative distribution function (CDF) and the probability density function (PDF) of random variable L, respectively. In this case

$$U(x, y) = x\left[\mathbb{E}(w - \varepsilon | w > \varepsilon) + \mathbb{E}(w - d - (1 - d)\varepsilon) | w \leq \varepsilon)\right] = y - PD \cdot x - d \cdot \int_{y/x}^{1} (1 - z) f(z) \mathrm{d}z. \quad (2.3)$$

Note that $y = x$ implies $U(x, x) = x - PD \cdot x$, which is consistent with expression of $U(x, y)$ for $y \geq x$. Therefore, $U(x, y)$ is continuous in area $x \geq 0$. Let us show that for the positive loan risk-adjusted interest rate $\tilde{r} > 0$ (which is equivalent to $PD < \frac{r}{1+r}$), all of the optimum solutions of banker's problem, if any, belong to the line

$$M_0 = 0 \iff y = (1 + R)E_{-1} + \frac{r - R}{1 + r}x$$

on the right of B.

Lemma 1. *Let $\tilde{r} > 0$ holds, then any optimum solution of the maximization problem of function $U(x, y)$ on the set of feasible solutions satisfies an identity*

$$y = (1 + R)E_{-1} + \frac{r - R}{1 + r}x$$

for some $x \geq (1 + r)E_{-1}$.

Proof. Note that the derivative $\frac{\partial U}{\partial y} > 0$. This is obvious for $y \geq x$, when $U(x, y) = y - PD \cdot x$, while for $x > y \iff w < 1$ we obtain

$$\frac{\partial U}{\partial y} = 1 + d \cdot \left(f(w) \frac{1}{x} - w f(w) \frac{1}{x} \right) x = 1 + d \cdot (1 - w) f(w) > 0. \quad (2.4)$$

In particular, this means that the maximums of function $U(x, y)$, if any, must belong to the upper bound of the feasible solutions set. It is easy to see that the function $U(x, y) = y - PD \cdot x$ strictly increases along an interval $[A, B]$, moving from A to B, provided that $PD < \frac{r}{1+r}$. This implies that the maximums can't belong to the set $[A, B)$. Q.E.D.

Remark. As a simple consequence of inequality $\frac{\partial U}{\partial y} > 0$ is that we can discard values $y < 0$ without loss of generality.

Lemma 1 implies that the maximization problem for function $U(x, y)$ can be reduced to the maximization problem for the function of one variable

$$\widetilde{U}(x) = U(x, y(x)) = (1+R)E_{-1} + \frac{\tilde{r}-R}{1+r}x - d \cdot x \cdot \int_{w(x)}^{1} (1-z)f(z)\mathrm{d}z, \tag{2.5}$$

where

$$y(x) = (1+R)E_{-1} + \frac{r-R}{1+r}x$$

and

$$w(x) = \frac{y(x)}{x} = \frac{(1+R)E_{-1}}{x} + \frac{r-R}{1+r} \in \left(\frac{r-R}{1+r}, 1\right]. \tag{2.6}$$

s.t. $x \geq (1+r)E_{-1}$.

Lemma 2. *Let the function* $(1-z)f(z)$ *is strictly decreasing in interval* $\frac{r-R}{1+r} < z < 1$, *then* $\widetilde{U}''(x) < 0$ *for all* $x \geq (1+r)E_{-1}$. *Moreover, if inequalities* $\tilde{r} > R$ *and*

$$d > \frac{\tilde{r}-R}{(1+r)\int_{\frac{r-R}{1+r}}^{1}(1-z)f(z)dz} \tag{2.7}$$

hold, then there is unique solution of the banker's problem.

Proof. Differentiating the function (2.5), we obtain the following expressions for the first and second derivatives

$$\widetilde{U}'(x) = \frac{\tilde{r}-R}{1+r} - d \cdot \left(\int_{w(x)}^{1} (1-z)f(z)\mathrm{d}z + \left(w(x) - \frac{r-R}{1+r}\right)(1-w(x))f(w(x))\right), \tag{2.8}$$

$$\widetilde{U}''(x) = d \cdot \frac{(1+R)^2E_{-1}^2}{x^3}\left[(1-w(x))f'(w(x)) - f(w(x))\right] = d \cdot \frac{(1+R)^2E_{-1}^2}{x^3}\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z)f(z)\right]\big|_{z=w(x)},$$

which immediately implies the first statement of the lemma. Furthermore,

$$\widetilde{U}'((1+r)E_{-1}) = \frac{\tilde{r}-R}{1+r} - d\left(1 - F(1) - PD + \int_0^1 zf(z)\mathrm{d}z\right) = \frac{\tilde{r}-R}{1+r} > 0,$$

$$\lim_{x\to\infty} \widetilde{U}'(x) = \frac{\tilde{r}-R}{1+r} - d \cdot \int_{\frac{r-R}{1+r}}^{1} (1-z)f(z)\mathrm{d}z,$$

because of $F(1) = 1$, $\int_0^1 zf(z)\mathrm{d}z = PD$. Taking into account an inequality $\widetilde{U}''(x) < 0$, we obtain that the necessary and sufficient condition for existence and uniqueness of the first order condition $\widetilde{U}'(x) = 0$ is an inequality

$$\lim_{x\to\infty} \widetilde{U}'(x) < 0 \iff d > \frac{\tilde{r}-R}{(1+r)\int_{\frac{r-R}{1+r}}^{1}(1-z)f(z)\mathrm{d}z}.$$

Q.E.D.

Example. Let's consider two examples of distributions, which meet conditions of Lemma 2. The first one is a uniform distribution on interval $[0, 1]$:

$$f(x) = \begin{cases} 0 & x \notin [0,1] \\ 1 & x \in [0,1] \end{cases}, \quad F(x) = \begin{cases} 0 & x < 0 \\ x & x \in [0,1] \\ 1 & x > 1 \end{cases}.$$

In this case, the probability of default $PD = 1/2$, which implies that inequality $\tilde{r} > R$ holds if and only if $r > 1 + 2R$. Moreover, the function $(1 - z)f(z) = 1 - z$ strictly decreases, thus $\widetilde{U}''(x) < 0$, and inequality (2.7) boils down to

$$d > \frac{r - (1 + 2R)}{\frac{(r-R)^2}{1+r} - (r - (1 + 2R))}. \tag{2.9}$$

Obviously the inequalities

$$0 \leq \frac{r - (1 + 2R)}{\frac{(r-R)^2}{1+r} - (r - (1 + 2R))} \leq 1$$

hold if and only if the interest rates R and r satisfy

$$0 \leq r - (1 + 2R) \leq \frac{(r - R)^2}{2(1 + r)}.$$

Therefore, the banker's problem has solution in case of uniform distribution under the very specific conditions for the model parameters. Otherwise, the banker's objective function unrestrictedly increases with respect to x, which corresponds to the unbounded credit expansion.

As another example, we consider the Vasicek distribution of the loan losses (see [3]) with CDF

$$F(z; PD, \rho) = N\left(\frac{\sqrt{1-\rho}N^{-1}(z) - N^{-1}(PD)}{\sqrt{\rho}}\right), \tag{2.10}$$

where PD is a probability of a borrower's default, ρ is a correlation coefficient of a borrower's assets. The corresponding PDF is as follows

$$f(z; PD, \rho) = \sqrt{\frac{1-\rho}{\rho}} \exp\left(-\frac{1}{2\rho}(\sqrt{1-\rho}N^{-1}(z) - N^{-1}(PD))^2 + \frac{1}{2}(N^{-1}(z))^2\right). \tag{2.11}$$

For $\rho < 1/2$ the PDF (2.11) is unimodal with mode at

$$z_{\text{mode}} = N\left(\frac{\sqrt{1-\rho}}{1-2\rho}N^{-1}(PD)\right),$$

(see, e.g., [3]), moreover, in this case $z_{\text{mode}} < PD$ when $PD < 1/2$. Therefore, if the loan risk-adjusted interest rate $\tilde{r}$ exceeds the deposit interest rate R, which is equivalent to inequality $PD < \frac{r-R}{1+r}$, then for $\rho < 1/2$, $PD < 1/2$ the inequalities

$$z_{\text{mode}} < PD < \frac{r-R}{1+r}$$

hold, which implies that the Vasicek PDF $f(z; PD, \rho)$, as well as function $(1-z)f(z; PD, \rho)$, decreases on interval $\frac{r-R}{1+r} < z < 1$. This implies the uniqueness of solution of the banker's problem, while the existence requires an additional inequality for discount

$$d > \frac{\tilde{r} - R}{(1+r)\int_{\frac{r-R}{1+r}}^{1}(1-z)f(z; PD, \rho)\mathrm{d}z},$$

otherwise, the banker's objective function is unrestricted on the set of feasible solutions.

From now on, we shall suggest that all considered distributions of the loan losses satisfy the basic assumption of Lemma 2, which guarantees the second order condition for function $\widetilde{U}(x)$.

Assumption. The PDF $f(z)$ satisfies the following condition: function $(1-z)f(z)$ strictly decreases for all $\frac{r-R}{1+r} < z < 1$.

Proposition 1. *Let Assumption holds then the maximum value of banker's objective function is as follows*

$$V(E_{-1}) \equiv \widetilde{U}(x^*) = (1+R)\left(1 + d \cdot (1-w^*)f(w^*)\right)E_{-1}, \tag{2.12}$$

where w^* is a solution of the following equation

$$\frac{\tilde{r} - R}{1+r} - d \cdot \left(\int_{w}^{1}(1-z)f(z)dz + \left(w - \frac{r-R}{1+r}\right)(1-w)f(w)\right) = 0. \tag{2.13}$$

See for the Proof the Appendix A.1.

2.1 Non-arbitrage condition and the endogenous credit interest rate

Based on the assumption of a perfect competitive environment of a bank, we considered yet the interest rates R and r as exogenous constants. In this subsection, a possible market mechanism is proposed, that determines the endogenous loan interest rate r. This mechanism is based on the assumption of the risk neutrality of bankers, which assumes that the expected return of banking activities will be equal to the return of risk-free assets.

Given this assumption, bank starts with initial equity $E_{-1} > 0$ possessing two alternative options:

1) to invest E_{-1} in risk-free assets with return $r_f > R$, which allows to obtain $(1+r_f)E_{-1}$ at the end of period, or, 2) to act as a regular banker. Due to Proposition 1, the second options provides the maximum expected equity

$$V(E_{-1}) = (1+R)\left(1+d\cdot(1-w^*))f(w^*)\right)E_{-1}$$

at the end of period. Therefore, the non-arbitrage condition is as follows

$$1+r_f = (1+R)\left(1+d\cdot(1-w^*))f(w^*)\right),$$

which is equivalent to

$$(1-w^*(r))f(w^*(r)) = \frac{r_f - R}{d\cdot(1+R)}, \tag{2.14}$$

where $w^*(r)$ is an implicit function of r, determined by the following equation

$$\frac{\tilde{r}-R}{1+r} - d\cdot\left(\int_w^1 (1-z)f(z)\mathrm{d}z + \left(w - \frac{r-R}{1+r}\right)(1-w)f(w)\right) = 0. \tag{2.15}$$

Considering (2.14) and (2.15) as a system of equations, we may first solve the equation[1]

$$(1-w)f(w) = \frac{r_f - R}{d\cdot(1+R)},$$

with respect to w, and then substitute this solution $\bar{w} \equiv \bar{w}(R, r_f)$ into (2.15). The resulting equation is linear with respect to r, thus, its solution is as follows

$$r^* = \frac{R + PD + d\cdot\int_{\bar{w}}^1 (1-z)f(z)\mathrm{d}z - (\bar{w}+R)\frac{r_f-R}{1+R}}{1 - PD - d\cdot\int_{\bar{w}}^1 (1-z)f(z)\mathrm{d}z - (1-\bar{w})\frac{r_f-R}{1+R}}. \tag{2.16}$$

In case of the uniform distribution with $f(z) = 1$, $PD = 1/2$, the formula (2.16) boils down to

$$r^* = \frac{1 + 2R + d\cdot(1-\bar{w}^2) - 2(\bar{w}+R)\frac{r_f-R}{1+R}}{1 - d\cdot(1-\bar{w}^2) - 2(1-\bar{w})\frac{r_f-R}{1+R}},$$

where

$$\bar{w} = 1 - \frac{r_f - R}{d\cdot(1+R)}.$$

[1] Due to Assumption the function $(1-z)f(z)$ is strictly decreasing, therefore, this solution $\bar{w}(R, r_f)$ is unique.

3 Capital adequacy ratio

Lemma 1 has a simple and straightforward interpretation in original terms. Its statement implies the zero value of an equilibrium cash M_0^*, because in frameworks of the presented study,we don't consider the risk of an early outflow of liabilities, which means that there is no need to "freeze" the money instead of using them alternatively with profits. The leftmost point B of the line

$$y = (1+R)E_{-1} + \frac{r-R}{1+r}x,$$

with $x_B = (1+r)E_{-1}$, corresponds to the "doing nothing" point $M_0 = 0$, $D_0 = 0$, and satisfies inequality $\widetilde{U}'(x_B) > 0$, provided that $\tilde{r} > R$. Indeed, the positive return of this operation provides incentives for banker to increase the size of the loan portfolio C_0, which coincides with x, up to a positive constant multiplier. However, increasing of the loan portfolio entails the larger risk of default, while the burden of losses depends on discount value d. Consider two possible cases, which cause different types of the banker's behavior.

Case 1. Let the discount d be sufficiently small, so that condition (2.7) is violated. This implies an unrestricted increasing of objective function $\widetilde{V}(x)$ when $x \to +\infty$, which means that banker has incentives for unrestricted expansion of the loan portfolio C_0. Formally, we obtain the same outcome when banker does not want to pay his/her liabilities in case of default. Indeed, such type of behavior corresponds to the following banker's objective function

$$\mathbb{E}(E_1|\ E_1 > 0) = x\mathbb{E}(w - L|w > L),$$

which obviously exceeds the function

$$U(x,y) = x\mathbb{E}(w - L|w > L) + x\mathbb{E}(w - d - (1-d)L)|w \leq L),$$

because the second term of this sum is negative.

To prevent this negative tendency, a regulator restricts the lending activity by imposing of the *required* capital adequacy ratio k and by requirement to satisfy the following conditin

$$E_0/C_0 \geq k,$$

where

$$E_0 = M_0 + C_0 - D_0 = M_0 - D_0 + (1-\lambda)(E_{-1} + D_0 - M_0).$$

Here we assume that the risk weight of loans is equal to 1. Using (2.1) we obtain the equivalent form of the regulator's restriction in terms of variable x as follows

$$x \leq x(k) = \frac{(1+r)E_{-1}}{1-(1-\lambda)(1-k)}. \tag{3.1}$$

It is obvious that in this case, the modified banker problem with additional constraint (3.1) has solution in $(x(k), y(k))$, where

$$y(k) = y(x(k)) = (1+R)E_{-1} + \frac{(r-R)E_{-1}}{1-(1-\lambda)(1-k)}.$$

Case 2. Now assume that the discount d is sufficiently large to satisfy the condition (2.7) and, therefore, there exists finite number x^*, which maximizes $\widetilde{V}(x)$. This situation may be interpreted as if the banker endogenously imposes self-restriction $x \leq x(k^*)$, so that the optimum solution x^* make this constraint feasible, i.e., $x^* = x(k^*)$. Solving this linear equation with respect to k^*, we obtain the formula for internally determined CAR

$$k^* = \frac{(1+r)E_{-1} - \lambda \cdot x^*}{(1-\lambda)x^*}. \tag{3.2}$$

Moreover, an identity (2.6) implies the following equivalent form of (3.2), which associates CAR k^* with equilibrium value w^*

$$k^* = \frac{1}{1-\lambda}\left[\frac{1+r}{1+R}w^* - \frac{r-R}{1+R} - \lambda\right]. \tag{3.3}$$

The value of CAR k^* may be negative, which corresponds to the negative value of equity E_0. These negative values, however, are bounded from below, because $x(k) > 0$ implies

$$1-(1-\lambda)(1-k) > 0 \iff k > -\frac{\lambda}{1-\lambda}.$$

The positive values of CAR are provided by the following result.

Theorem. *Let function $f(z)$ satisfies the Assumption and the following inequality*

$$d > \frac{\tilde{r} - R}{(1+r)\left(\int_{w_0}^{1}(1-z)f(z)dz + \frac{(1+R)\lambda}{1+r}(1-w_0)f(w_0)\right)}, \tag{3.4}$$

where

$$w_0 = w(x(0)) = \frac{r - R + (1+R)\lambda}{1+r}.$$

Then the value of equilibrium CAR $k^* \in (0,1)$.

Proof. The inequality $k^* < 1$ is obviously satisfied, because $k = 1$ corresponds to $x(1) = (1+r)E_{-1}$, while $\widetilde{U}'(x(1)) = \frac{\tilde{r}-R}{1+r} > 0$. Since $\widetilde{V}'(x)$ is a strictly decreasing function, to ensure the positive value of CAR k^* is necessary an sufficient to satisfy condition $\widetilde{U}'(x(0)) = \widetilde{U}'\left(\frac{(1+r)E_{-1}}{\lambda}\right) < 0$, which is equivalent to (3.4). Q.E.D.

3.1 Comparative statics of equilibrium

Now we shall study how the equilibrium reacts to the changes in basic parameters d, R and r. Besides the rather technical equilibrium values w^* and x^*, our main interest focuses on the size of loan portfolio C_0^* and of attracted deposits D_0^*, as well as value of CAR k^*.

Lemma 3. *Signs of partial derivatives of the equilibrium values of* w^*, x^*, C_0^*, D_0^*, k^* *with respect to* d, R, *and* r *are as follows*

Functions \ *Parameters*	$\frac{\partial}{\partial d}$	$\frac{\partial}{\partial R}$	$\frac{\partial}{\partial r}$
w^*	> 0	> 0	< 0
x^*	< 0	< 0	> 0
C_0^*	< 0	< 0	> 0
D_0^*	< 0	< 0	> 0
k^*	> 0	> 0	< 0

See Proof in Appendix A.2.

It is easy to see that these results completely comply with intuitive expectations. For example, increasing of discount d suppress the banker's activity, forcing to reduce the loan portfolio C_0^* and the attraction of deposits, while CAR k^* became larger, which means the higher degree of self-restriction. As expected, an increasing in the deposit interest rate R reduce the demand of deposit, while increasing in the loan interest rate r results in increasing of supply of loans, etc.

3.2 How CAR k^* depends on the correlation ρ – numeric analysis

In this subsection we consider the case of Vasicek distribution of the loan losses (2.10), which is characterized by the specific parameter ρ – the correlation between assets of various borrowers. From intuitive point of view, the larger is correlation ρ, the higher is risk of the domino effect for borrowers defaults, thus, the more restrictive banking policy is required. In other words, $k^*(\rho)$ should be increasing function, but it is not clear, whether the presented model catches this effect? The analytical

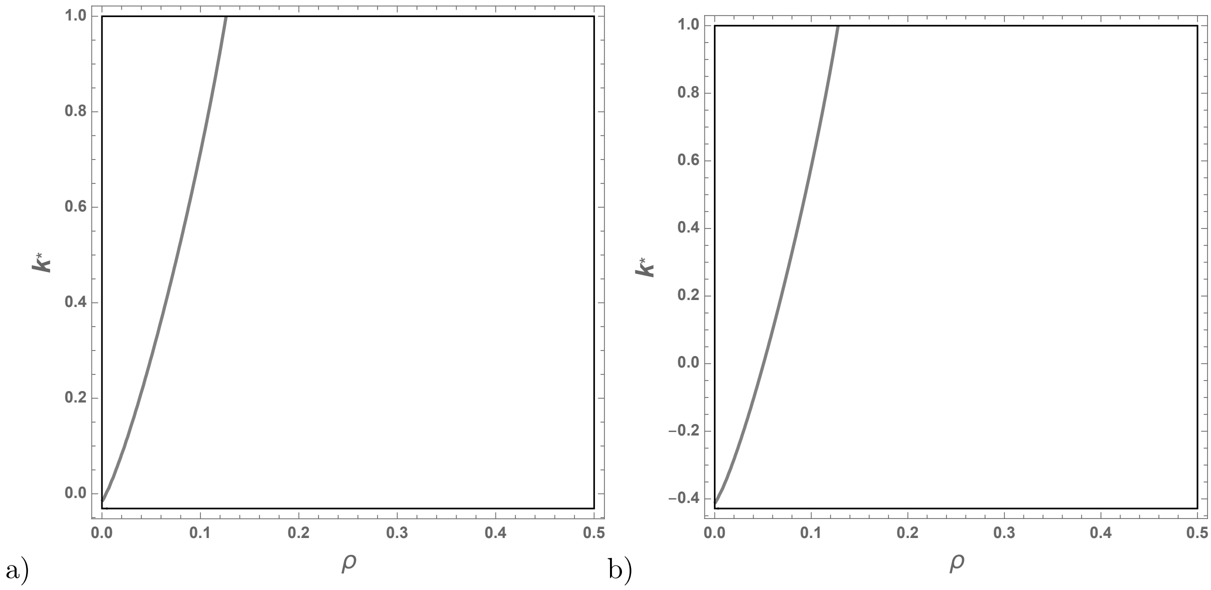

Figure 4: CAR k^* as a function of ρ

way, like in Lemma 3, failed due to very tedious calculations, thus, the Figures 4 a) and b) show two examples of computer simulations.

Let $r = 0.1$, $R = 0.05$, $PD = 0.03$, the discount value $d = 0.5$. In this case the inequality $\tilde{r} > R$ holds. Given these values, the Vasicek CDF is parametrized by correlation $\rho \in (0, 0.5)$. The Figures 4 a) and b) depict the plot of the implicit function $k^*(\rho)$, which is determined by equation

$$\frac{\tilde{r} - R}{1 + r} - d \cdot \left(\int_{w(k)}^{1} (1 - z) f(z; PD, \rho) \mathrm{d}z + \left(w(k) - \frac{r - R}{1 + r} \right) (1 - w(k)) f(w(k); PD, \rho) \right) = 0, \quad (3.5)$$

where

$$w(k) = \frac{r - R}{1 + r} + \frac{1 + R}{1 + r} (\lambda + (1 - \lambda) k)$$

is an inverse function to (3.3). For the case of Figure 4 a) we use value of the loan loss provision $\lambda = 0.03$, while Figure 4b) corresponds to the value $\lambda = 0.3$. In both cases for all ρ sufficiently small, CAR $k^*(\rho)$ strictly increases, while ρ became sufficiently large, there is no discount $0 \leq d \leq 1$, which is sufficient for "self-restriction" of banker. This means that we get Case 1 from Section 3, which requires the regulation. The only difference between Figures 4a) and 4b) is based on value of the loan loss provision coverage ratio. For $\lambda = 0.03$ we obtain

$$-\frac{\lambda}{1 - \lambda} \approx -0.031,$$

which implies that CAR $k^*(\rho)$ is almost always positive, with exception of very small ρ, while for $\lambda = 0.3$ the range of negative CARs is more substantial.

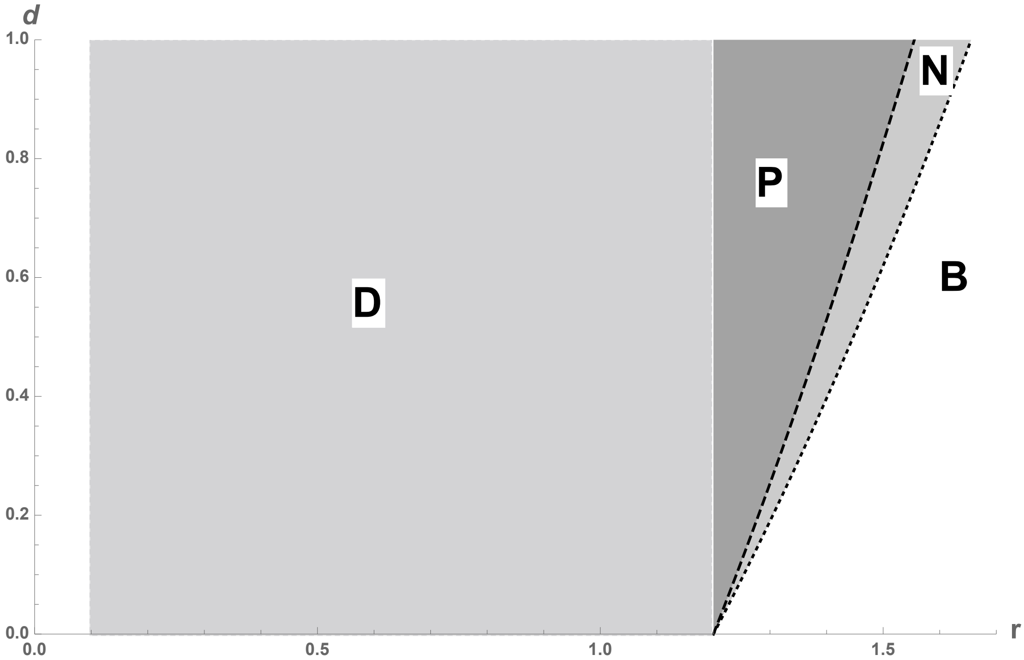

Figure 5: Uniform distribution of loan losses: $R = 0.1$, $\lambda = 0.5$

4 Parametric stratification by the solution types

The main aim of the present section is to visualize the various types of equilibria in terms of the model primitives. First, assume that the deposit interest rate R, the loan loss provision λ, and CDF $F(z)$ for the loan losses L are given. Consider the set $\mathcal{S}$ of feasible points $r > R$, $0 \leq d \leq 1$ of the parameter plane (r, d). With any point of this set we associate specific type of equilibrium, which corresponds to the whole set of parameters, including the given ones. Figures 5 and 6 show two examples of such stratification of $\mathcal{S}$ for specific types of CDFs – uniform and Vasicek. Despite the very different properties of these CDFs, the resulting segmentations are quite similar and consist of the four areas:

I. Bubble area B corresponds to the unrestricted credit expansion. It consists of points $(r, d) \in \mathcal{S}$, that violate condition (2.7), in other words, these points of $\mathcal{S}$ satisfy the inequality $d \leq d_N(r)$, where

$$d_N(r) \equiv \frac{\tilde{r} - R}{(1 + r) \int_{\frac{r-R}{1+r}}^{1} (1 - z) f(z) \mathrm{d}z}. \tag{4.1}$$

II. Negative area N corresponds to case when the bank attracts deposits and places funds to the loan portfolio of the limited size, while equity is negative. It consists of points $(r, d) \in \mathcal{S}$ that satisfy condition (2.7), though violate condition (3.4), which means that these points satisfy

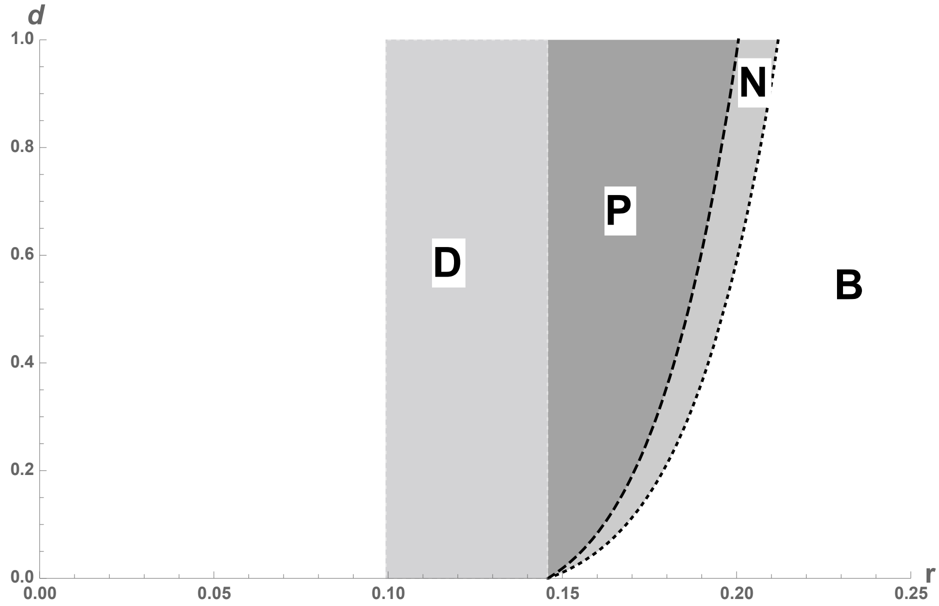

Figure 6: Vasicek distribution of loan losses: $R = 0.1$, $PD = \lambda = 0.04$, $\rho = 0.1$

inequalities $d_N(r) < d \le d_W(r)$, where

$$d_W(r) \equiv \frac{\tilde{r} - R}{(1+r)\left(\int_{w_0(r)}^{1}(1-z)f(z)\mathrm{d}z + \frac{(1+R)\lambda}{1+r}(1-w_0(r))f(w_0(r))\right)}, \tag{4.2}$$
$$w_0(r) = \frac{r - R + (1+R)\lambda}{1+r}$$

III. Positive area P corresponds to case when the bank attracts deposits and places funds to the loan portfolio of the limited size, while equity is negative. It consists of points $(r, d) \in \mathcal{S}$ that satisfy condition (3.4), which means that these points satisfy inequalities $d > d_W(r)$, $\tilde{r} > R$.

IV. Degenerate area D consists of points $(r, d) \in \mathcal{S}$ that satisfy the inequalities $\tilde{r} \le R$, $0 \le d \le 1$, which means that condition (2.7) hold in trivial way, though the banker's optimum solution is degenerate: the bank does not attracts deposits and, in case of $\tilde{r} < 0$, does not place funds in the loans.

For both examples Bubble area **B** is unrestricted with respect to r, Negative **N** and Positive **P** areas are the adjacent curvilinear triangles with mutual vertex $\left(\frac{PD+R}{1-PD}, 0\right)$ and with adjacent bases, which belong to the line $d = 1$, Degenerate area **D** always is a rectangle $R < r \le \frac{R+PD}{1-PD}$, $0 \le d \le 1$.

Proposition 2. *Let the Assumption holds, then the structure of areas* **B**, **N**, **P**, **D** *is persisting.*

See Proof in Appendix A.3.

Remark. For any given positive value of discount $d_+ > 0$, no matter how small is it, we obtain the nonempty intersection of the line $d = d_+$ with all four areas. In case of $d = 0$, corresponding to the linear model of Section 1, areas **N** and P vanish and we obtain only two generic cases — Degenerate area **D** and Bubble area **B**, which agrees with result obtained in Section 1.

4.1 The bank probability of default and Basel III requirements

The considered above optimum bank management is based on the risk-neutral behavior, targeted to maximize the expected equity $\mathbb{E}(E_1)$, which is nominally greater than initial equity E_{-1}, due to Proposition 1. However, the risk of default persists even if the management decisions are optimal. The probability of event $E_1 \leq 0$ may be calculated as follows

$$p = \mathbb{P}(y^* - x^* L \leq 0) = \mathbb{P}(L \geq w^*) = \int_{w*}^{1} f(z)\mathrm{d}z = 1 - F(w^*), \tag{4.3}$$

where w^* is solution of equation (2.13).

Function $1 - F(w)$ obviously strictly decreases with respect to w, therefore, Lemma 3 implies that

$$\frac{\partial p}{\partial d} < 0, \ \frac{\partial p}{\partial R} < 0, \ \frac{\partial p}{\partial r} > 0,$$

which is quite intuitive, and p does not depend on initial equity E_{-1}.

The Basel requires that the probability of the bank's default must not exceed 0.001, which means that the inequality $F(w^*) \geq 0.999$, or, equivalently, $w^* \geq F^{-1}(0.999) = \mathrm{VaR}_{99.9}$, must hold. To determine the corresponding value of the CAR, let us substitute

$$x(k) = \frac{(1+r)E_{-1}}{1-(1-\lambda)(1-k)}$$

into (2.6). As result, we obtain the following formula

$$w(k) = \frac{r-R}{1+r} + \frac{1+R}{1+r}(\lambda + (1-\lambda)k).$$

Therefore, the threshold value $\hat{k}$ of CAR, guaranteeing the Basel III requirements, may be found from condition

$$w(\hat{k}) = \mathrm{VaR}_{99.9},$$

which is equivalent to

$$\hat{k} = \frac{1}{1-\lambda}\left[\frac{1+r}{1+R}\mathrm{VaR}_{99.9} - \frac{r-R}{1+R} - \lambda\right].$$

The Basel III analysis is based on the Vasicek loan losses distribution (2.10), therefore,

$$\hat{w}(PD,\rho) \equiv \text{VaR}_{99.9} = F^{-1}(0.999; PD, \rho) = N\left(\sqrt{\frac{\rho}{1-\rho}}N^{-1}(0.999) + \sqrt{\frac{1}{1-\rho}}N^{-1}(PD)\right), \quad (4.4)$$

which allows to calculate the corresponding required value of CAR. Now we are going to identify sets of the bank parameters d, r, R, PD, ρ, λ, which guarantee that the banker complies voluntarily with Basel III requirements, or, on the contrary, the external regulation is needed. Substituting $\hat{w} = \hat{w}(PD,\rho)$ in equation (2.13), we can determine the value of discount d_B guaranteeing the precise discharge of Basel III requirements, as follows

$$d_B(r; R, \hat{w}) = \frac{\tilde{r} - R}{(1+r)\left(\int_{\hat{w}}^{1}(1-z)f(z)\mathrm{d}z + \left(\hat{w} - \frac{r-R}{1+r}\right)(1-\hat{w})f(\hat{w})\right)}. \quad (4.5)$$

Moreover, $\frac{\partial w^*}{\partial d} > 0$ due to Lemma 3, which implies that for all $d > d_B(r; R, \hat{w})$ the corresponding equilibrium value $w^*(d) > \hat{w}$, i.e., banker complies voluntarily with Basel III requirements.

Let parameters R, PD, ρ, λ be given, consider the curve $d = d_B(r)$ in the parameter plane (r, d). Obviously it starts at point $d = 0$, $r = \frac{PD+R}{1-PD}$, as well as the previously considered curves $d = d_N(r)$ and $d = d_W(r)$, moreover, function $d_B(r)$ strictly increases with respect to r, because function $\frac{\tilde{r}-R}{1+r}$ is increasing, and

$$\int_{\hat{w}}^{1}(1-z)f(z)\mathrm{d}z + \left(\hat{w} - \frac{r-R}{1+r}\right)(1-\hat{w})f(\hat{w})$$

strictly decreases with respect to r, because $\hat{w}$ does not depend on r by definition (4.4). Finally, the realistic value of parameters imply that required CAR $\hat{k}$ is positive, which means that graph of $d_B(r)$ belongs to Positive area $\mathbf{P}$, i.e., $d_B(r) > d_W(r)$ and the "Basel curve" $d = d_B(r)$ divides $\mathbf{P}$ into two sub-areas: $\mathbf{P}_A$, where Basel III requirements are violated, and $\mathbf{P}_B$, where they are complied. To illustrate this division, consider the following example with $R = 0.1$, $\lambda = PD = 0.04$, $\rho = 0.01$, presented on Figure 7.

As we can see, the "Basel friendly" combination of parameters admits an arbitrary value of discount d, while the loan interest rates should not be too large. The existence of area $\mathbf{P}_B$ may explain the paradoxical dispersion of real values of CAR, observed on Figure 1.

This approach can be easily generalized from specific Basel III requirements to the case of an arbitrary distribution of the loan losses, satisfying the basic Assumption from Section 2. Let's consider the probability of the bank's default p as a parameter with possible values from interval $[0, 1]$. For any given p, the equation $p = 1 - F(w)$ determines the threshold value $w^p = F^{-1}(1-p)$, such that, for any equilibrium solution of the banker's problem, that satisfy the inequality $w^* \geq w^p$, the requirements

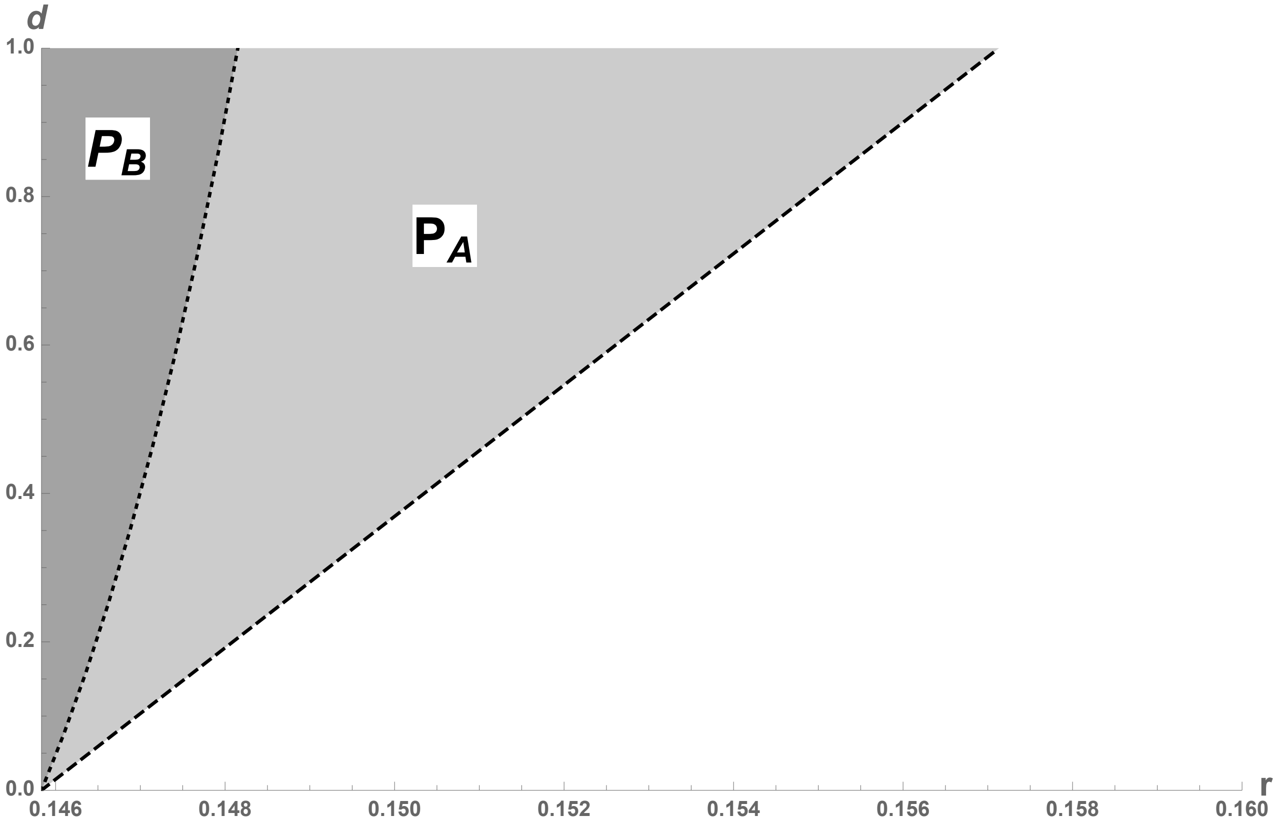

Figure 7: Dichotomy "self-restriction – external restriction"

of "generalized Basel" are complied, thus regulation is not needed, otherwise, in case of $w^* < w^p$, regulation is required. Note that solvability of the banker's problem implies $w^* \in \left(\frac{r-R}{1+r}, 1\right]$, due to (2.6), which means that for the sufficiently large probability

$$p \geq 1 - F\left(\frac{r-R}{1+r}\right) \iff \frac{r-R}{1+r} \geq w^p,$$

we get Bubble area. Now assume that $p < 1 - F\left(\frac{r-R}{1+r}\right)$, guaranteeing inequality $w^p > \frac{r-R}{1+r}$, then the curve separating areas of "self-restriction" and "external restriction" looks similar to (4.5):

$$d = d_B(r,p) \equiv \frac{\tilde{r} - R}{(1+r)\left(\int_{w^p}^{1}(1-z)f(z)\mathrm{d}z + \left(w^p - \frac{r-R}{1+r}\right)(1-w^p)f(w^p)\right)}.$$

Note that in case of $p = 1 - F\left(\frac{r-R}{1+r}\right)$ we obtain

$$d_B\left(r, 1 - F\left(\frac{r-R}{1+r}\right)\right) = \frac{\tilde{r} - R}{(1+r)\int_{\frac{r-R}{1+r}}^{1}(1-z)f(z)\mathrm{d}z} = d_N(r).$$

It is easy to see that the function $d_B(r,p)$ strictly increases with respect to r and strictly decreases with p. Moreover, for any given $p > 0$, the graph of the function $d_B(r,p)$ starts in $\left(\frac{R+PD}{1-PD}, 0\right)$ and for two different values of parameter $p > p'$ we have $d_B(r,p) < d_B(r,p')$ for all $r > \frac{R+PD}{1-PD}$. Furthermore, $p \to 0$ implies $w^p \to 1$, $d_B(r,p) \to \infty$ for any $r > \frac{R+PD}{1-PD}$, which means that the graph of $d = d_B(r,p)$ asymptotically approaches to the vertical line $r = \frac{R+PD}{1-PD}$. This implies that graphs of functions $d_B(r,p)$ for all admissible p wholly fill in a union of areas $\mathbf{N} \cup \mathbf{P}$. For realistic regulation policy the value of p will be chosen sufficiently close to 0, therefore, the corresponding “generalized Basel curve” dissects the Positive area into two pieces, as in Figure 7. In this case, the areas **N** and **B** will be automatically added to the external regulation zone, while **D**, as usual, does not require any regulation, because the lending activity is bounded here due to the disadvantageous economic situation. This completes the generalized Basel dichotomy of the parameter area.

5 Multi-period extension of the model

Consider generalization of our model to the multi-period case $t = 1, \ldots, T$. For each period t, let's denote an initial equity as $E_{-1}^{(t)} > 0$, the control variables, i.e., the deposits and the cash, as $D_0^{(t)} \geq 0$, $M_0^{(t)} \geq 0$, respectively, the loan portfolio as

$$C_0^{(t)} = (1-\lambda)\left(E_{-1}^{(t)} + D_0^{(t)} - M_0^{(t)}\right) \geq 0,$$

which depend on chosen controls, and on the random equity at the end of period as

$$E_1^{(t)} = (1+r)(E_{-1}^{(t)} + D_0^{(t)} - M_0^{(t)})(1-L_t) + M_0^{(t)} - (1+R)D_0^{(t)},$$

where random variable L_t for all t are independent and identically-distributed. When random variable $E_1^{(t)} \leq 0$, then the bank default occurs, otherwise, for $E_1^{(t)} > 0$, we define the initial equity for the next period as follows $E_{-1}^{(t+1)} = E_1^{(t)}$. For the first period let $E_{-1}^{(1)} = E_{-1} > 0$ be an initial equity of the banker. The other model parameters, e.g., r, R, etc, are assumed to be identical across all periods.

Let's construct the time-consistent solution for this extension, taking into account that for any period t the “current” solution is conditional — the bank should be “alive” to this moment, i.e., $E_{-1}^{(t)} > 0$, to be able to make the current decision. In particular, the banker problem in the last period T is (conditionally) identical to the one-period problem with given initial equity $E_{-1}^{(T)} > 0$, therefore, due to Proposition 1, the expected maximum value of equity is equal to

$$\max \mathbb{E}_T(E_1^{(T)}) = V(E_{-1}^{(T)}) = (1+R)\left(1 + d \cdot (1-w^*)\right) f(w^*)) \, E_{-1}^{(T)},$$

wherew^* is (unique) solution of equation

$$\frac{\tilde{r}-R}{1+r} - d\cdot\left(\int_w^1 (1-z)f(z)\mathrm{d}z + \left(w - \frac{r-R}{1+r}\right)(1-w)f(w)\right) = 0. \tag{5.1}$$

The larger is $E_{-1}^{(T)} = E_1^{(T-1)}$, the greater will be value of $V(E_{-1}^{(T)})$, therefore, the aim of the banker in previous period $T-1$ is to maximize the expected equity value $E_1^{(T-1)}$, which, in turn, implies the same choice of optimum w^* in period $T-1$, because the equation (5.1) does not depend on period. Using the inverse induction, we obtain that the only time-consistent solution of the multi-period model is a T-times replicated optimum solution of the one-period model.

Note that the coefficient $(1+R)(1+d\cdot(1-w^*))f(w^*)) > 1$, which means that the sequence of random variable of equity $E_1^{(t)}$, provided that the banker made an optimum choice in every period, is a sub-martingale with respect to sequence $E_{-1}^{(t)}$, in other words, the "Game of banker" is profitable. However, it should be mentioned that this consideration does not take into account the inter-temporal discounting of equity. Now assume that the discount factor $\beta = \frac{1}{1+r_f}$, where r_f is risk-free profitability, while the interest loan rate r satisfy non-arbitrage condition (2.14), then in this case the "Game of banker" become zero-profitable and the sequence of discounted equities $\beta^t E_1^{(t)}$ is a martingale.

5.1 Probability of the bank's default in multi-period case

The uniform banker's choice across all periods implies that the *conditional* probability of the bank default is the same in each period t and equal to $p = 1 - F(w^*)$, where w^* is solution of equation (5.1), under condition $E_1^{(t-1)} = E_{-1}^{(t)} > 0$, i.e., that the bank survived in the course of all previous $t-1$ periods. Probability of this condition is, in turn, $(1-p)^{t-1} = F(w^*)^{t-1}$. Therefore, the probability of the bank default, occuring *precisely* in period t, is equal to $(1-F(w^*))F(w^*)^{t-1}$, while the total probability of the bank's default in all T periods is

$$p_T = (1-F(w^*))\sum_{t=1}^{T} F(w^*)^{t-1} = 1 - F(w^*)^T. \tag{5.2}$$

If $F(w^*) < 1$, then $p_T \to 1$ when $T \to \infty$, i.e., every bank eventually will default[2]. Moreover, Lemma 3 and (5.2) imply that for any given $T \geq 1$ the probability p_T satisfies the following conditions

$$\frac{\partial p_T}{\partial d} < 0,\ \frac{\partial p_T}{\partial R} < 0,\ \frac{\partial p_T}{\partial r} > 0,$$

[2]This is indirectly confirmed by banking history: although the first Italian bank was established in 1157, after which many banks were established in the XII-XV centuries, the oldest bank that survived to this day is Banca Monte dei Paschi di Siena SpA, is tracing its history to a mount of piety founded in 1472, but was founded in its present form in 1624.

and does not depend on initial equity E_{-1}. This result is quite intuitive, because the increasing in discount d and/or deposit interest rate R induces the banker to be more safe, or, conservative, while increasing in the loan interest rate r stimulates more risky, or, aggressive, behavior of the banker.

6 Conclusion

The banking is one of the most over-regulated and over-supervised industries, and the pressure on banks continues to grow. A natural question arises: can banks do without a regulator - at least in some aspects of their activities that are now under strict regulation and supervision?

For example, can banks limit their credit expansion on their own, without intervention of a regulator? To answer this question, we built a simple microeconomic model with one stochastic factor – the share of non-performing loans. It turned out that if, in the event of a bank default, a loan portfolio can be sold without a discount, then the banker has no incentives to limit the credit expansion, even despite the prospect of incurring of huge losses. This means that in this case, banking cannot do without a regulator, only the state can restrict the credit expansion.

The situation changes drastically, when we assume that in the event of a bank failure, its loan portfolio is sold at some non-zero discount. In this case, when certain limitations on the model parameters are satisfied, an endogenous restriction of credit expansion arises. Unlike external restrictions that banks have learned to successfully circumvent, these restrictions are internal, and deceiving oneself is usually not beneficial. However, from the point of view of the regulator, which evaluates the result in terms of CAR, the level of bank self-restraint may seem unacceptable, for example, if the ratio has a negative or too low positive value.

There is a problem of identifying the outcome in terms of the basic parameters of the model. This task received a comprehensive solution. A procedure has been formulated and justified, which makes it possible to unambiguously determine the type of outcome according to the model exogenous parameters and the known loss distribution function. It was shown that with sufficiently weak and natural restrictions on the loss distribution function, that the parameter space is divided into 4 non-empty regions in which one of the four possible outcomes is realized: **B** ("Bubble") - there are no bounded solutions (i.e., we get an analogue of the linear model with zero discount); **N** ("Negative") - limited solutions exist, but the bank's equity is negative, which automatically means that there is a high risk of bank default; **P** ("Positive") limited solutions are characterized by positive equity; and, finally, **D** - degenerate solutions - deposits are not attracted, loans are placed maximum within their own funds, as a result, credit expansion does not occur due to unfavorable conditions. In addition, a more subtle identification of compliance with the requirements established by Basel III in the area **P** was carried out.

The influence of exogenous factors on solutions, both analytically and, in particularly complex cases, using computer simulations, has been studied. In all cases, the results of the study of comparative statics are consistent with intuitive expectations.

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A Appendix

Notations and abbreviations

E_s	equity
D_s	deposits
M_s	cash
C_s	net loans
r	loan interest rate
R	deposit interest rate
λ	loan loss provision coverage ratio
L	share of nonperforming loans (the portfolio percentage gross loss)
$PD = \mathbb{E}(L)$	probability of default of a borrower
$\tilde{r} = r - (1 + r)PD$	loan risk-adjusted interest rate
d	discount of loan nominal value in case of selling of the loan
$x = (1 + r)(E_{-1} + D_0 - M_0)$	auxiliary variables
$y = x - (1 + R)D_0 + M_0$	
$w(x, y) = y/x$	
$F(x)$	CDF (cumulative density function)
$f(x) = F'(x)$	PDF (probability density function)
k	CAR (capital adequacy ratio)
ρ	correlation coefficient of borrower's assets
$U(x, y)$	objective function of banker
r_f	risk-free rate
β	discount factor

A.1 Proof of Proposition 1

Indeed, the first order condition $\widetilde{U}'(x^*) = 0$ implies

$$0 = x^* \cdot \widetilde{U}'(x^*) = \frac{\tilde{r} - R}{1 + r} x^* - d \cdot \left(\int_{w(x^*)}^{1} (1 - z) f(z) \mathrm{d}z \right) x^* -$$

$$-d \cdot \left(w(x^*) - \frac{r - R}{1 + r} \right) x^* \cdot (1 - w(x^*)) f(w(x^*)) = \left(\frac{r - R}{1 + r} x^* + (1 + R) E_{-1} \right) - (1 + R) E_{-1} -$$

$$-PDx^* - d \cdot \left(\int_{w(x^*)}^{1} (1 - z) f(z) \mathrm{d}z \right) x - d \cdot (1 + R) E_{-1} \cdot (1 - w(x^*)) f(w(x^*)) =$$

$$= \widetilde{U}(x^*) - (1 + R) E_{-1} \left(1 + d \cdot (1 - w^*) f(w^*) \right),$$

where

$$w^* = \frac{(1 + R) E_{-1}}{x^*} + \frac{r - R}{1 + r}$$

satisfies the equation

$$\frac{\tilde{r} - R}{1 + r} - d \cdot \left(\int_{w}^{1} (1 - z) f(z) \mathrm{d}z + \left(w - \frac{r - R}{1 + r} \right) (1 - w) f(w) \right) = 0.$$

Q.E.D.

A.2 Proof of Lemma 3

Equilibrium value of w^* is determined by the following equation

$$u(w) \equiv \frac{\tilde{r} - R}{1 + r} - d \cdot \left(\int_{w}^{1} (1 - z) f(z) \mathrm{d}z + \left(w - \frac{r - R}{1 + r} \right) (1 - w) f(w) \right) = 0,$$

as an implicit function of all parameters. The corresponding derivative with respect to an arbitrary parameter $a$is as follows

$$\frac{\partial w^*}{\partial a} = -\frac{\partial u}{\partial a} / \frac{\partial u}{\partial w},$$

where

$$\frac{\partial u}{\partial w} = d \cdot \left(w - \frac{r - R}{1 + r} \right) (f(w) - (1 - w) f'(w)) = -d \cdot \left(w - \frac{r - R}{1 + r} \right) \frac{\mathrm{d}}{\mathrm{d}w} [(1 - w) f(w)] > 0,$$

because $(1 - w) f(w)$ is decreasing function. Moreover,

$$\frac{\partial u}{\partial d} = -\left(\int_{w}^{1} (1 - z) f(z) \mathrm{d}z + \left(w - \frac{r - R}{1 + r} \right) (1 - w) f(w) \right) < 0,$$

which implies $\frac{\partial w^*}{\partial d} > 0$. Taking into account that

$$w = \frac{(1+R)E_{-1}}{x} + \frac{r-R}{1+r} \iff x = \frac{(1+R)E_{-1}}{w - \frac{r-R}{1+r}},$$

$$C_0 = \frac{1-\lambda}{1+r}x,$$

and

$$D_0 = \frac{1}{R}\left[E_{-1} + \frac{r}{1+r}x - y(x)\right] = \frac{x}{1+r} - E_{-1},$$

we obtain $\frac{\partial x^*}{\partial d} < 0$, $\frac{\partial C_0^*}{\partial d} < 0$ and $\frac{\partial D_0^*}{\partial d} < 0$.

The following derivative

$$\frac{\partial u}{\partial R} = -\frac{1}{1+r} - d \cdot \frac{(1-w)f(w)}{1+r} < 0,$$

which implies $\frac{\partial w^*}{\partial R} > 0$. As a simple consequence, we obtain

$$\frac{\partial x^*}{\partial R} = \frac{E_{-1}\left(w - \frac{r-R}{1+r}\right) - (1+R)E_{-1}\left(\frac{\partial w}{\partial R} + \frac{1}{1+r}\right)}{\left(w - \frac{r-R}{1+r}\right)^2} = -\frac{E_{-1}\left((1-w) + (1+R)\frac{\partial w}{\partial R}\right)}{\left(w - \frac{r-R}{1+r}\right)^2} < 0,$$

$\frac{\partial C_0^*}{\partial R} < 0$ and $\frac{\partial D_0^*}{\partial R} < 0$.

Furthermore, the inequality

$$\frac{\partial u}{\partial r} = \frac{1+R}{(1+r)^2} + d \cdot \frac{(1+R)(1-w)f(w)}{(1+r)^2} > 0$$

implies $\frac{\partial w^*}{\partial r} < 0$. Moreover,

$$\frac{\partial x^*}{\partial r} = -\frac{(1+R)E_{-1}\left(\frac{\partial w^*}{\partial r} - \frac{1+R}{(1+r)^2}\right)}{\left(w^* - \frac{r-R}{1+r}\right)^2} > 0$$

$$\frac{\partial C_0^*}{\partial r} = (1-\lambda)\frac{\partial}{\partial r}\left(\frac{x^*}{1+r}\right),$$

$$\frac{\partial D_0^*}{\partial r} = \frac{\partial}{\partial r}\left(\frac{x^*}{1+r}\right)$$

Applying formula (2.6), we obtain that

$$\frac{x^*(r)}{1+r} = \frac{(1+R)E_{-1}}{z(r)+R},$$

where $z(r) = (1+r)w^*(r) - r$. Moreover,

$$z'(r) = -(1 - w^*(r)) + (1+r)\frac{\partial w^*}{\partial r} < 0,$$

because of $w^*(r) < 1$ and $\frac{\partial w^*}{\partial r} < 0$. This means that $\frac{\partial}{\partial r}\left(\frac{x^*}{1+r}\right) > 0$, which implies $\frac{\partial C_0^*}{\partial r} > 0$ and $\frac{\partial D_0^*}{\partial r} > 0$.

Finally, due to formula (3.3), the capital adequacy ratio may be represented as follows

$$k^* = \frac{1}{1-\lambda}\left[\frac{1+r}{1+R}w^* - \frac{r-R}{1+R} - \lambda\right].$$

This implies immediately that

$$\frac{\partial k^*}{\partial d} = \frac{1+r}{(1-\lambda)(1+R)}\frac{\partial w^*}{\partial d} > 0.$$

Moreover,

$$\frac{\partial k^*}{\partial R} = \frac{1}{1-\lambda}\left[-\frac{1+r}{(1+R)^2}w^* + \frac{1+r}{1+R}\frac{\partial w^*}{\partial d} + \frac{1+r}{(1+R)^2}\right] = \frac{1}{1-\lambda}\left[\frac{1+r}{(1+R)^2}(1-w^*) + \frac{1+r}{1+R}\frac{\partial w^*}{\partial R}\right] > 0$$

and

$$\frac{\partial k^*}{\partial r} = \frac{1}{(1-\lambda)(1+R)}\left[(1+r)\frac{\partial w^*}{\partial r} - (1-w^*)\right] < 0$$

because $w^* \leq 1$, $\frac{\partial w^*}{\partial R} > 0$ and $\frac{\partial w^*}{\partial r} < 0$. Q.E.D.

A.3 Proof of Proposition 2

It is obvious that for any given R and PD the "degeneration" area **D** is a rectangle. This shape does not changes under varying of parameter R and CDF $F(z)$, anyway two rectangles always are homothetic. The rest is to show the robustness of shapes of areas **B**, **N**, and **P**. Let's show that the functions (4.1) and (4.2)

$$d_N(r) = \frac{\tilde{r} - R}{(1+r)\int_{\frac{r-R}{1+r}}^{1}(1-z)f(z)\mathrm{d}z},$$

$$d_P(r) = \frac{\tilde{r} - R}{(1+r)\left(\int_{w_0(r)}^{1}(1-z)f(z)\mathrm{d}z + \frac{(1+R)\lambda}{1+r}(1-w_0(r))f(w_0(r))\right)},$$

where

$$w_0(r) = \frac{r - R + (1+R)\lambda}{1+r}$$

satisfy the following conditions:

1. $d_N\left(\frac{PD+R}{1-PD}\right) = d_P\left(\frac{PD+R}{1-PD}\right) = 0$, i.e., these curves initiate in the same point,

2. $d_P(r) > d_N(r)$ for all $r > \frac{PD+R}{1-PD}$, i.e., area **P** resides to the left on area **N**,

3. $d_N(r)\,,\ d_P(r)$ strictly increase for all $r > \frac{PD+R}{1-PD}$.

The first statement is obvious by due to definition of $d_N(r)\,,\ d_P(r)$, furthermore,

$$w_0 = \frac{r - R + (1+R)\lambda}{1+r} > \frac{r-R}{1+r},$$

while integral $\int_w^1 (1-z)f(z)\mathrm{d}z$ decreases obviously with respect to w, and additionally an inequality $\frac{(1+R)\lambda}{1+r}(1-w_0)f(w_0) > 0$ holds. This implies that $d_P(r) > d_N(r)$ holds for all $r > \frac{PD+R}{1-PD}$. Then, representing the function $d_N(r)$ as follows

$$d_N(r) = \frac{\tilde{r} - R}{1+r} \cdot \frac{1}{\int_{\frac{r-R}{1+r}}^1 (1-z)f(z)\mathrm{d}z},$$

and taking into account that the functions $\frac{\tilde{r}-R}{1+r}$, $\frac{r-R}{1+r}$ are strictly increasing and positive, we obtain that the function $d_N(r)$ is also strictly increasing. Similarly to the previous considerations, we can prove that the increasing of $d_P(r)$ follows from decreasing of the following function

$$\int_{w_0(r)}^1 (1-z)f(z)\mathrm{d}z + \frac{(1+R)\lambda}{1+r}(1-w_0(r))f(w_0(r))$$

with respect to r. The latter statement is an obvious consequence of the following facts: function $w_0(r)$ strictly increases with respect to r, integral $\int_w^1 (1-z)f(z)\mathrm{d}z$ decreases with respect to w, and function $(1-z)f(z)$ strictly decreases with respect to z. Finally, the functions $d_N(r)\,,\ d_P(r)$ are unrestrictedly increasing with $r \to \infty$, because

$$\frac{\tilde{r} - R}{1+r} \to 1 - PD,\ \frac{r-R}{1+r} \to 1,\ w_0(r) \to 1.$$

Therefore, their graphs intersect the line $d = 1$ in finite points $r_N > r_P > \frac{PD+R}{1-PD}$, which determine the bases of the curvilinear triangles **N** and **P**. Q.E.D.