Robustness, Low Risk-Free Rates, and Consumption Volatility in General Equilibrium

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Abstract

This paper develops a tractable continuous-time recursive utility (RU) version of the Huggett (1993) model to explore how the preference for robustness (RB) interacts with intertemporal substitution and risk aversion and then affects the interest rate, the dynamics of consumption and income, and the welfare costs of model uncertainty in general equilibrium. We show that RB reduces the equilibrium interest rate and the relative volatility of consumption growth to income growth when the income process is stationary, but our benchmark model cannot match the observed relative volatility. An extension to an RU-RB model with a risky asset is successful at matching this ratio. The model implies that the welfare costs of uncertainty are very large.

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Keywords: Robustness, Precautionary Savings, the Permanent Income Hypothesis, Low Interest Rates, Consumption and Income Inequality, General Equilibrium.

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1 Introduction

Hansen and Sargent (1995) first formally introduced the preference for robustness (RB, a concern for model misspecification) into linear-quadratic-Gaussian (LQG) economic models.[1] In robust control problems, agents are concerned about the possibility that their true model is misspecified in a manner that is difficult to detect statistically; consequently, they make their optimal decisions as if the subjective distribution over shocks is chosen by an evil agent in order to minimize their expected lifetime utility.[2] As showed in Hansen, Sargent, and Tallarini (1999), Luo and Young (2010), and Luo, Nie, and Young (2012), robustness models can produce precautionary savings even within the class of discrete-time LQG models, which leads to analytical simplicity. The precautionary savings induced by a concern about robustness emerges because the agent makes pessimistic forecasts of future income and wants to protect himself against mistakes in specifying conditional means of the income shocks. Unfortunately, if we consider problems outside the LQG setting (e.g., when the utility function is constant-absolute-risk-averse, i.e., CARA, or constant-relative-risk-averse, i.e., CRRA), RB-induced worst-case distributions are generally non-Gaussian, which generally renders the model analytically intractable.[3]

In intertemporal consumption-savings problems, *prudent* households save today for three reasons: (i) they anticipate future declines in income (saving for a rainy day), (ii) they face uninsurable risks (precautionary savings), and (iii) they are patient relative to the interest rate. For example, the "permanent income hypothesis" (PIH) of Friedman (1957) emphasizes motive (i) in which consumption is solely determined by permanent income (the annuity value of total wealth including both financial wealth and human wealth) and follows a random walk (see Hall 1978).[4] In contrast, Caballero (1990) examined a precautionary saving motive due to the interaction of risk aversion

[1] See Hansen and Sargent (2007) for a textbook treatment on robustness. It is worth noting that we can use either robust decision-making or recursive multiple-prior utility (Chen and Epstein 2002) due to ambiguity aversion to capture the same idea that the decision maker is concerned that their model is misspecified and thus considers a range of models when making decisions (Gilboa and Schmeidler 1989). In this paper, for simplicity, we follow the Hansen and Sargent (2007) to use the robust control specification to model the basic idea of the multiple-priors utility model.

[2] Many empirical and experimental studies have repeatedly confirmed Ellsberg's conjecture that a preference for robustness (or ambiguity aversion) would lead to a violation of the Savage axioms. For example, Ahn, Choi, Gale, and Kariv (2014) used a rich experiment data set to estimate a portfolio-choice model and found that about 40 percent of subjects display either statistically significant pessimism or ambiguity aversion.

[3] See Chapter 1 of Hansen and Sargent (2007) for discussions on the computational difficulties in solving non-LQG RB models, and Bidder and Smith (2012) and Young (2012) for numerical methods to compute the worst-case distributions.

[4] This statement holds, for example, if households have quadratic utility and have access to a single risk-free bond with a constant return. If utility is not quadratic, the random walk nature of consumption is only approximately true, but the PIH still holds.

and unpredictable future income uncertainty when the consumer has convex marginal utility. The Caballero model leads to a constant precautionary savings demand and a constant dissavings term due to relative impatience. Wang (2003) showed in a Caballero-Huggett equilibrium model that the precautionary saving demand and the impatience dissavings term cancel out in general equilibrium and the PIH reemerges.

Conceptually, the precautionary savings coming from a concern about robustness differs in structure from that coming from the interaction between future uncertainty and the convex marginal utility. Although Hansen and Sargent (1995, 2007) showed that the max-min robustness model is observationally equivalent to the corresponding risk-sensitive model in which the EIS is set to be one, conceptually, the RB models mentioned above do not explicitly model the interactions between robustness, risk aversion and intertemporal substitution when examining consumption, saving and investment decisions. This paper therefore fills the gap by providing a general recursive utility framework to explore the general equilibrium implications of the preference for robustness for the risk-free rate and the cross-sectional dispersion of consumption (relative to income).[5] We investigate both the theoretical mechanism (how RB influences the equilibrium interest rate and the relative dispersion/volatility of consumption to income) and the empirical performance (whether plausibly calibrated values of RB lead the model to fit the data better). Specifically, the main goal of this paper is to construct a tractable continuous-time dynamic stochastic general equilibrium (DSGE) precautionary saving model in which RB consumers have recursive utility with exponential risk and intertemporal attitudes and face uninsurable labor income.[6] We disentangle two distinct aspects of preferences: the agent's elasticity of intertemporal substitution (EIS; attitudes towards variation in consumption across time) with the coefficient of absolute risk aversion (CARA; attitudes towards variation in consumption across states).[7]

As the first contribution of this paper, we show that this continuous-time DSGE model featuring incomplete markets, RB, and the separation of risk aversion and intertemporal substitution can be solved explicitly. We find that the effective coefficient of absolute risk aversion ($\widetilde{\gamma}$) is determined by the interaction between the true CARA (γ), the EIS (ψ), and the degree of RB (ϑ) via the

[5] For applications of recursive utility (RU) in intertemporal consumption-portfolio chice and asset pricing, see, for example, Epstein and Zin (1989), Campbell (2003), Vissing-Jorgensen and Attanasio (2003), and Guvenen (2006).

[6] See Cagetti, Hansen, Sargent, and Williams (2002), Anderson, Hansen, and Sargent (2003), Maenhout (2004), Kasa (2006), and Kasa and Lei (2017) for the applications of robustness in continuous-time models.

[7] Constant-relative-risk-aversion (CRRA) utility functions are more common in macroeconomics, mainly due to balanced-growth requirements. CRRA utility would greatly complicate our analysis because the intertemporal consumption model with CRRA utility and stochastic labor income has no explicit solution and leads to non-linear consumption rules. See Kasa and Lei (2017) for a recent application of RB in a continuous-time Blanchard-Yaari model with CRRA utility and wealth hetergeneity.

following formula:

$$\widetilde{\gamma} = \gamma + \frac{\vartheta}{\psi}.$$

It is clear from this expression that the EIS affects individual consumption-saving-portfolio rules.

Second, we show that a general equilibrium under RB can be constructed in the vein of Bewley (1986) and Huggett (1993).[8] We also show that the general equilibrium is unique in such an RB-RU economy. Furthermore, we find that the EIS plays an important role in affecting the equilibrium properties of the model economy. In particular, we show that an increase in EIS affects the equilibrium interest rate through two channels: (i) it increases the relative importance of the impatience-induced dissaving effect (*the direct channel*) and (ii) it reduces the precautionary saving amount by reducing the effect of RB (*the indirect channel*). In general equilibrium, for given EIS and CARA, we find that the interest rate decreases with the degree of RB. The intuition is that the stronger the preference for RB, the greater the amount of model uncertainty, leading to strong precautionary savings effects and therefore low interest rates. In addition, we show that the relative volatility of consumption growth to income growth is determined only by the equilibrium interest rate and the persistence coefficient of the income process; the relative volatility decreases with RB if the income process is stationary.

Third, after calibrating the RB parameter using the detection error probabilities (DEP), we find that RB has significant effects on the equilibrium interest rate and consumption volatility. In the U.S. economy the average real risk-free interest rate averaged roughly 1.87 percent between 1981 and 2010, and is only about 1.37 percent if the sample is from 1981 to 2015.[9] The full-information rational expectations (FI-RE) model requires the coefficient of risk aversion parameter to be 24 to match this rate if the EIS is 0.8, and requires the coefficient to be 15 if the EIS is 0.5.[10] In contrast, when consumers take into account model uncertainty, the model can generate a low equilibrium interest rate with much lower values of the coefficient of risk aversion.[11] In addition, we find that as income uncertainty increases, the relative volatility of consumption growth to income growth decreases through the general equilibrium interest rate channel.[12] We also show that if the

[8] Wang (2003) constructs a general equilibrium under full-information rational expectations (FI-RE) in the same Bewley-Huggett type model economy with CARA utility that we study. Angeletos and Calvet (2006) characterize a closed-form recursive general equilibrium in a neoclassical growth model with idiosyncratic production risk and incomplete markets.

[9] Here the numbers are computed using CPI to measure inflation. Using PCE leads to similar results. See Table 1 for different measures of the risk-free rates.

[10] We normalize the mean consumption level to be 1, so the coefficient of relative risk aversion is equal to the coefficient of absolute risk aversion.

[11] Barillas, Hansen, and Sargent (2009) showed that most of the observed high market price of risk in the U.S. can be reinterpreted as a market price of model uncertainty and the risk-aversion parameter can thus be reinterpreted as measuring the representative agent's doubts about the model specification.

[12] This theoretical result might provide a potential explanation for the empirical evidence documented in Blundell,

benchmark model generates the observed low risk-free rate, the model's predicted relative volatility of consumption to income is well below the empirical counterpart. The reason was noted in the previous paragraph – the relative volatility depends only on the interest rate and the persistence of income changes, and this persistence is too low relative to the low interest rate to generate adequate consumption volatility.

To correct this anomaly we extend our benchmark model to a model with one risky asset. The presence of the risky asset affects equilibrium precautionary saving through two channels: (i) the risky asset can be used to hedge the labor income risk and (ii) it increases the amount of total uncertainty when the net supply of the risky asset is positive. More importantly, we find that the relative volatility of consumption growth to income growth is increasing in the supply of the risky asset and the risk-free rate is decreasing. For plausibly calibrated parameter values of RB, we find that the extended model can simultaneously generate the observed low risk free rate and high relative volatility of consumption to income in the US economy.

Finally, we use our model to assess the welfare gains associated with eliminating model uncertainty – what would an agent pay in order to find out exactly (and with complete confidence) the stochastic process affecting his income? We find that these numbers are very large – the cost can be as large as 25 percent of permanent income. These costs are increasing in the aversion to model uncertainty and decreasing in the elasticity of intertemporal substitution (EIS). In addition, we find that the general equilibrium effects increase the cost relative to partial equilibrium with a fixed interest rate. In GE, as model uncertainty is eliminated precautionary savings falls, leading to higher interest rates; in turn, higher interest rates facilitate consumption smoothing, leading to larger welfare gains.

This paper is organized as follows. Section 2 presents a robustness version of the Caballero–Bewley-Huggett type model with incomplete markets and precautionary savings. Section 3 discusses the general equilibrium implications of RB for the interest rate and the joint dynamics of consumption and income. Section 4 presents our quantitative results after estimating the income process and calibrating the RB parameter. Section 5 considers the extension to the multiple-asset case. Section 6 concludes.

Pistaferri, and Preston (2008) that income and consumption inequality diverged over the sampling period they study.

2 A Continuous-time Heterogeneous-Agent Economy with Robustness

2.1 The Full-information Rational Expectations Model with Recursive Utility and Precautionary Savings

In this section, we first consider a full-information rational expectations (FI-RE) recursive utility model with labor income and precautionary savings. Although the expected power utility model has many attractive features, that model implies that the elasticity of intertemporal substitution is the reciprocal of the coefficient of relative risk aversion. Conceptually risk aversion (attitudes towards atemporal risks) and intertemporal substitution (attitudes towards shifts in consumption over time) capture two distinct aspects of decision-making and need not be so tightly connected.[13] In contrast, the class of recursive utility functions (Kreps and Porteus 1978; Epstein and Zin 1989; Duffie and Epstein 1992) enable one to disentangle risk aversion from intertemporal substitution. In this paper, we assume that agents in our model economy have recursive preferences of the Kreps-Porteus/Epstein-Zin type: for every stochastic consumption stream, $\{c_t\}_{t=0}^{\infty}$, the utility stream, $\{f(U_t)\}_{t=0}^{\infty}$, is recursively defined by

$$f(U_t) = \left(1 - e^{-\delta\Delta t}\right) f(c_t) + e^{-\delta\Delta t} f(CE_t[U_{t+\Delta t}]). \tag{1}$$

where Δt is the time interval, $\delta > 0$ is the agent's subjective discount rate, $f(c_t) = (-\psi)\exp(-c_t/\psi)$, $f(U_t) = (-\psi)\exp(-U_t/\psi)$,

$$CE_t[U_{t+\Delta t}] = g^{-1}(E_t[g(U_{t+\Delta t})]), \tag{2}$$

is the certainty equivalent of U_{t+1} conditional on the period t information, and $g(U_{t+\Delta t}) = -\exp(-\gamma U_{t+\Delta t})/\gamma$. In (1), $\psi > 0$ governs the elasticity of intertemporal substitution (EIS), while $\gamma > 0$ governs the coefficient of absolute risk aversion (CARA).[14] A high value of ψ corresponds to a strong willingness to substitute consumption over time, and a high value of γ implies a low willingness to substitute consumption across states of nature. Note that when $\psi = 1/\gamma$, the functions f and g are the same and the recursive utility reduces to the standard time-separable expected utility function used in Caballero (1990) and Wang (2003).

We assume that there is only one risk-free asset in the model economy and there are a continuum of consumers who face uninsurable labor income. The evolution of the financial wealth (w_t) of a

[13] Risk aversion describes the agent's reluctance to substitute consumption across different states of the world and is meaningful even in a static setting. In contrast, intertemporal substitution describes the agent's willingness to substitute consumption over time and is meaningful even in a deterministic setting.

[14] It is well-known that the CARA utility specification is tractable for deriving optimal policies and constructing general equilibrium in different settings. See Caballero (1990), Wang (2003, 2009), and Angeletos and Calvet (2006).

typical consumer is

$$dw_t = (rw_t + y_t - c_t)\, dt; \tag{3}$$

r is the return to the risk-free asset and c_t and y_t are consumption and labor income at time t, respectively. Uninsurable labor income (y_t) follows an Ornstein-Uhlenbeck process:

$$dy_t = \rho\left(\frac{\kappa}{\rho} - y_t\right) dt + \sigma_y dB_t, \tag{4}$$

where the unconditional mean and variance of income are $\overline{y} = \kappa/\rho$ and $\sigma_y^2/\left(2\rho\right)$, respectively, the persistence coefficient ρ governs the speed of convergence or divergence from the steady state, B_t is a standard Brownian motion on the real line $\mathcal{R}$, and σ_y is the unconditional volatility of the income change over an incremental unit of time. To present the model more compactly, we define a new state variable, s_t:

$$s_t \equiv w_t + h_t,$$

where h_t is human wealth at time t and is defined as the expected present value of current and future labor income discounted at the risk-free interest rate r,

$$h_t \equiv E_t\left[\int_t^\infty \exp\left(-r\left(s-t\right)\right) y_s ds\right].$$

For the given the income process, (4), $h_t = y_t/\left(r+\rho\right) + \mu/\left(r\left(r+\rho\right)\right)$.[15] Using s_t as the unique state variable, we can rewrite (3) as

$$ds_t = \left(rs_t - c_t\right) dt + \sigma_s dB_t, \tag{5}$$

where $\sigma_s = \sigma_y/\left(r+\rho\right)$ is the unconditional variance of the innovation to s_t.[16]

The optimization problem can thus be written as

$$f\left(J_t\right) = \max_{c_t}\left\{\left(1-e^{-\delta\Delta t}\right) f\left(c_t\right) + e^{-\delta\Delta t} f\left(CE_t\left[J_{t+\Delta t}\right]\right)\right\}, \tag{6}$$

subject to (5). An educated guess is that $J_t = As_t + A_0$. The J function at t time $t+\Delta t$ can thus be written as

$$J\left(s_{t+\Delta t}\right) = As_{t+\Delta t} + A_0 \approx As_t + A\left(rs_t - c_t\right)\Delta t + A\sigma_s \Delta B_t + A_0,$$

[15] If $\rho > 0$, the income process is stationary and deviations of income from the steady state are temporary; if $\rho \leq 0$, income is non-stationary. The last case captures the essence of Hall and Mishkin (1982)'s specification of individual income that includes a non-stationary component. The $\rho = 0$ case corresponds to a simple Brownian motion without drift. The larger ρ is, the less y tends to drift away from $\overline{y}$. As ρ goes to ∞, the variance of y goes to 0. We need to impose the restriction that $r > -\rho$ to guarantee the finiteness of human wealth.

[16] In the next section, we will introduce robustness directly into this "reduced" precautionary savings model. It is not difficult to show that the reduced univariate model and the original multivariate model are equivalent in the sense that they lead to the same consumption and saving functions, because the financial wealth part of total wealth is deterministic between periods. The detailed proof is available from the corresponding author by request.

where $\Delta s_t \equiv s_{t+\Delta t} - s_t$ and $\Delta s_t \approx (rs_t - c_t)\,\Delta t + \sigma_s \Delta B_t$ where $\Delta B_t = \sqrt{\Delta t}\epsilon$ and ϵ is a standard normal.

In the benchmark full-information rational expectations (FI-RE) model, we assume that the consumer trusts the model, i.e., no model uncertainty. The Hamilton-Jacobi-Bellman (HJB) equation is then

$$\delta f(J_t) = \sup_{c_t \in \mathcal{C}} \{\delta f(c_t) + \mathcal{D} f(s_t)\} \tag{7}$$

where

$$\mathcal{D} f(s_t) = f'(J_t)\left(A(rs_t - c_t) - \frac{1}{2}\gamma A^2 \sigma_s^2\right), \tag{8}$$

$\mathcal{C}$ is the set of admissible values for the consumption choice, and the transversality condition, $\lim_{t\to\infty}\{E\,|\exp(-\delta t)\,f_t|\} = 0$, holds at the optimum. Solving the HJB equation subject to (5) leads to the consumption function

$$c_t = rs_t + \Psi - \Gamma, \tag{9}$$

where

$$\Psi \equiv \psi\left(\frac{\delta}{r} - 1\right) \tag{10}$$

is the savings demand due to relative patience (if $\delta < r$, this term is negative and so savings rises) and

$$\Gamma \equiv \frac{1}{2} r\gamma\sigma_s^2, \tag{11}$$

is the consumer's precautionary saving demand.[17] From (10), it is clear that if the consumer is impatient relative to the market ($\delta > r$), the higher the EIS, the stronger the demand for consumption. If $\delta > r$ households want consumption to fall over time, and a higher EIS implies that consumption will be allowed to fall faster for a given value of $\frac{\delta}{r}$; as a result, consumption must initially be high. Following the literature on precautionary savings, we measure the demand for precautionary saving as the amount of saving induced by the combination of uninsurable labor income risk and risk aversion. From (11), one can see that the precautionary saving demand is larger for a higher value of the coefficient of absolute risk aversion (higher γ), a more volatile income innovation (higher σ_y), and a larger persistence coefficient (lower ρ).[18] Holding other parameters constant, we can see from (9) to (11) that intertemporal substitution and risk aversion have opposing effects on consumption and saving decisions if $\delta > r$ (which will be the case in general equilibrium).[19]

[17] See Appendix 7.2 for the derivations.

[18] As argued in Caballero (1990) and Wang (2003, 2009), a more persistent income shock takes a longer time to wear off and thus induces a stronger precautionary saving demand by a prudent forward-looking consumer.

[19] As a side note, incomplete markets generally imply that aggregate dynamics depend on the wealth distribution, this "curse of dimensionality" is circumvented by our CARA-Gaussian specification since savings functions are linear.

2.2 Incorporating Fear of Model Uncertainty

To introduce aversion to model uncertainty into our model (and thus generate a demand for robust decision rules), we follow the continuous-time methodology proposed by Anderson, Hansen, and Sargent (2003) (henceforth, AHS) and adopted in Maenhout (2004). Households take Equation (5) as the approximating model. The corresponding set of distorting models can thus be obtained by adding endogenous distortions $\upsilon(s_t)$ to (5):

$$ds_t = (rs_t - c_t)\,dt + \sigma_s\left(\sigma_s \upsilon(s_t)\,dt + dB_t\right). \tag{12}$$

As shown in AHS (2003), the objective $\mathcal{D}J$ defined in (8) can be thought of as $E[dJ]/dt$ and plays a key role in generating robustness. A key insight of AHS (2003) is that this differential expectations operator reflects a particular underlying model for the state variable because this operator is determined by the stochastic differential equations of the state variables. Consumers accept (5) as the best approximating model, but are still concerned that the model is misspecified. They therefore want to consider a range of models (the distorted models (12)) surrounding the approximating model when computing the continuation payoff. A preference for robustness manifests by having the agent guard against the distorting model that is reasonably close to the approximating model. The drift adjustment $\upsilon(s_t)$ is chosen to minimize the sum of (i) the expected continuation payoff adjusted to reflect the additional drift component in (12) and (ii) an entropy penalty:

$$\inf_{\upsilon}\left[\mathcal{D}f + f'(J_t)\,A\upsilon(s_t)\,\sigma_s^2 + \frac{1}{\vartheta_t}\mathcal{H}\right], \tag{13}$$

where the first two terms are the expected continuation payoff when the state variable follows (12), i.e., the alternative model based on drift distortion $\upsilon(s_t)$, $\mathcal{H} = \left(\upsilon(s_t)\,\sigma_s\right)^2/2$ is the relative entropy or the expected log likelihood ratio between the distorted model and the approximating model and measures the distance between the two models, and $1/\vartheta_t$ is the weight on the entropy penalty term.[20] ϑ_t is fixed and state independent in AHS (2003), whereas it is state-dependent in Maenhout (2004). The role of the state-dependent counterpart to ϑ_t in Maenhout (2004) is to assure the homotheticity or scale invariance of the decision problem under a CRRA utility function.[21] Note that the evil agent's minimization problem, (13), is invariant to the scale of total resources s_t under the state-dependent specification for $\vartheta_t(s_t)$, which we use as well so that the demand for robustness does not disappear as the value of total wealth increases.

[20] The last term in (13) is due to the consumer's preference for robustness. Note that the $\vartheta_t = 0$ case corresponds to the standard expected utility case. This robustness specification is called the *multiplier (or penalty)* robust control problem. We will discuss another closely related robustness specification, the *constraint* robust control problem, in the next subsection. See AHS (2003) and Hansen, Sargent, Turmuhambetova, and Williams (2006) (henceforth, HSTW) for detailed discussions on these two robustness specifications.

[21] See Maenhout (2004) for detailed discussions on the appealing features of "homothetic robustness".

We can then obtain the HJB equation for the RB model:

$$\delta f\left(J_{t}\right)=\sup_{c_{t}\in\mathcal{C}}\inf_{\upsilon_{t}}\left\{\delta f\left(c_{t}\right)+\mathcal{D}f\left(s_{t}\right)+\upsilon\left(s_{t}\right)\sigma_{s}^{2}J_{s}+\frac{1}{\vartheta\left(s_{t}\right)}\mathcal{H}\right\}. \tag{14}$$

Solving first for the infimization part of (14) yields

$$\upsilon\left(s_{t}\right)^{*}=-\vartheta\left(s_{t}\right)f_{s},$$

where $\vartheta\left(s_{t}\right)=-\vartheta/f\left(s_{t}\right)>0$ (see Appendix 7.3 for the derivation). Following Uppal and Wang (2003) and Liu, Pan, and Wang (2005), here we can also define "$1/f\left(s_{t}\right)$" in the $\vartheta\left(s_{t}\right)$ specification as a *normalization* factor that is introduced to convert relative entropy (the distance between the approximating model and a given distorted model) into units of utility so that it is consistent with the units of the expected future value function evaluated with the distorted model. Adopting a slightly more general specification, $\vartheta\left(s_{t}\right)=-\varphi\vartheta/f\left(s_{t}\right)$ where φ is a constant, does not affect the main results of the paper, as we can just define a new constant, $\widetilde{\vartheta}=\varphi\vartheta$, and $\widetilde{\vartheta}$, rather than ϑ, will enter the decision rules. It is worth noting that this state-dependent robustness specification is similar to the AR(1) ambiguity shocks proposed in Bhandari, Borovička, and Ho (2016). They identified AR(1) ambiguity shocks using survey data from the Surveys of Consumers and the Survey of Professional Forecasters, and found that in the data, the ambiguity shocks are an important source of variation in labor market variables.

Since $\vartheta\left(s_{t}\right)>0$, the perturbation adds a negative drift term to the state transition equation because $J_{s}>0$. Substituting for υ^{*} in (14) gives

$$\delta f\left(U_{t}\right)=\sup_{c_{t}\in\mathcal{C}}\left\{\delta f\left(c_{t}\right)+f'\left(J_{t}\right)A\left(rs_{t}-c_{t}-\frac{1}{2}\gamma A\sigma_{s}^{2}+\frac{\vartheta}{f\left(U_{t}\right)}Af'\left(U_{t}\right)\sigma_{s}^{2}\right)-\frac{\vartheta}{2f\left(J_{t}\right)}A^{2}\left(f'\left(J_{t}\right)\right)^{2}\sigma_{s}^{2}\right\}. \tag{15}$$

2.3 The Robust Consumption Function and Precautionary Saving

We can now solve (15) and obtain the consumption rule under robustness. The following proposition summarizes the solution.

Proposition 1 *Under robustness, the consumption function and the saving function are*

$$c_{t}^{*}=rs_{t}+\Psi-\Gamma, \tag{16}$$

and

$$d_{t}^{*}=x_{t}+\Gamma-\Psi, \tag{17}$$

respectively, where $x_{t}\equiv\rho\left(y_{t}-\overline{y}\right)/\left(r+\rho\right)$ *is the demand for savings "for a rainy day",*

$$\Psi\left(r\right)\equiv\psi\left(\delta/r-1\right)$$

captures the saving demand of relative patience,

$$\Gamma \equiv \frac{1}{2} r \widetilde{\gamma} \sigma_s^2 \tag{18}$$

is the demand for precautionary savings due to the interaction of income uncertainty, intertemporal substitution, and risk and uncertainty aversion, and

$$\widetilde{\gamma} \equiv \gamma + \frac{\vartheta}{\psi} \tag{19}$$

is the effective coefficient of absolute risk aversion. The corresponding value function is

$$f_t = -\delta \frac{\psi}{r} \exp \left(- \left(\frac{\delta}{r} - 1 - \frac{1}{2} \frac{r^*}{\psi} \widetilde{\gamma} \sigma_s^2 \right) - \frac{r}{\psi} s_t \right). \tag{20}$$

Finally, the worst possible distortion is

$$\upsilon^* = -r \frac{\vartheta}{\psi}. \tag{21}$$

Proof. See Appendix 7.3. ■

From (16), it is clear that robustness does not change the marginal propensity to consume out of permanent income (MPC), but does affect the amount of precautionary savings (Γ). In continuous time consumption is less sensitive to unanticipated income shocks than in the discrete-time robust LQG-PIH model of Hansen, Sargent, and Tallarini (1999) (henceforth, HST); in discrete time the MPC increases with the amount of model uncertainty, causing consumption to become respond more to changes in permanent income (as noted in Luo 2008 and Luo and Young 2010, robust control exacerbates the excess sensitivity puzzle). Expression (18) shows that the precautionary savings demand now depends on the effective coefficient of risk aversion $\widetilde{\gamma}$ which is a function of the EIS (ψ), the CARA (γ), and the degree of robustness (ϑ). Specifically, it increases with γ and ϑ, whereas it decreases with ψ.

Another interesting question here is the relative importance of RB (ϑ) and CARA (γ) in determining the precautionary savings demand, holding other parameters constant. We can use the elasticities of precautionary saving as a measure of their importance.

Proposition 2 *The relative sensitivity of precautionary saving to risk aversion (γ), intertemporal substitution (ψ), robustness (ϑ) can be measured by*

$$\mu_{\gamma\vartheta} \equiv \frac{e_\gamma}{e_\vartheta} = \frac{\gamma}{\vartheta/\psi}, \tag{22}$$

$$\mu_{\psi\vartheta} \equiv \frac{e_\psi}{e_\vartheta} = -1. \tag{23}$$

respectively, where $e_\gamma \equiv \frac{\partial\Gamma/\Gamma}{\partial\gamma/\gamma}$, $e_\psi \equiv \frac{\partial\Gamma/\Gamma}{\partial\psi/\psi}$, and $e_\vartheta \equiv \frac{\partial\Gamma/\Gamma}{\partial\vartheta/\vartheta}$ are the elasticities of the precautionary saving demand to CARA, EIS, and RB, respectively.

Proof. The proof is straightforward. ■

The interpretation of (22) is that the precautionary savings demand is more sensitive to the actual coefficient of (absolute) risk aversion (γ) than it is to RB (ϑ) if the actual CARA is greater than RB amplified by the inverse of the EIS, i.e., $\gamma > \vartheta/\psi$. Of course, it is not exactly clear how to interpret a proportional change in either parameter since they do not have units, but we report this result to show that risk aversion does not clearly dominate the motives of the agents in the model.

HST (1999) showed that the discount factor and the concern about robustness are observationally equivalent in the sense that they lead to the same consumption and investment decisions in a discrete-time LQG representative-agent permanent income model. The reason for this result is that introducing a concern about robustness increases savings in the same way as increasing the discount factor, so that the discount factor can be changed to offset the effect of a change in RB on consumption and investment.[22] In contrast, in our continuous-time CARA-Gaussian model, we have a more general observational equivalence result between δ, γ, and ϑ:

Proposition 3 *Let*

$$\gamma^{fi} = \gamma + \frac{\vartheta}{\psi}, \tag{24}$$

where γ^{fi} is the coefficient of absolute risk aversion in the FI-RE model. Then consumption and savings are identical in the FI-RE and RB models, holding other parameter values constant. Furthermore, let $\delta = r$ in the RB model, and

$$\delta^{fi} = r - \frac{1}{2}\vartheta\left(\frac{r}{\psi}\right)^2 \sigma_s^2, \tag{25}$$

where δ^{fi} is the discount rate in the FI-RE model. Then consumption and savings are identical in the FI-RE and RB models, ceteris paribus.

Proof. Using (16) and (18), the proof is straightforward. ■

Expression (24) means that a consumer with a preference for robustness (ϑ) and recursive utility with EIS (ψ) and CARA (γ) is observationally equivalent to a consumer with full-information and recursive utility with EIS (ψ) and CARA ($\gamma + \vartheta/\psi$). In contrast, within a Merton model with recursive utility, Maenhout (2004) showed that an agent with a preference for robustness and Epstein-Zin recursive utility with EIS (ψ) and CRRA (γ) is observationally equivalent to an agent with full-information and recursive utility with EIS (ψ) and CARA ($\gamma + \vartheta$). In other words, in Maenhout's model, the effective coefficient of relative risk aversion ($\gamma + \vartheta$) does not depend on the EIS (ψ).

[22] As shown in HST (1999), the two models have different implications for asset prices because continuation valuations would alter as one alters the values of the discount factor and the robustness parameter within the observational equivalence set.

3 General Equilibrium Implications of RB

3.1 Definition of the General Equilibrium

As in Huggett (1993) and Wang (2003), we assume that the economy is populated by a continuum of *ex ante* identical, but *ex post* heterogeneous agents, with each agent having the saving function, (18). In addition, we also assume that the risk-free asset in our model economy is a pure-consumption loan and is in zero net supply.[23] The key insights can be also obtained in a CARA-Gaussian production economy with incomplete markets (as in Angeletos and Calvet 2006) using a neoclassical production function with capital and bonds as saving instruments. We consider the simpler Huggett-type endowment economy for two reasons. First, in the endowment economy, we can directly compare the model's predictions on the dynamics of individual consumption and income with its empirical counterpart, and do not need to infer the idiosyncratic productivity shock process. Second, the endowment economy allows us to solve the models explicitly, and thus helps us identify distinct channels via which RB interacts with risk aversion, discounting, and intertemporal substitution and affects the consumption-saving behavior.

In the model economy, the initial cross-sectional distribution of income is assumed to be its stationary distribution $\Phi(\cdot)$. By the law of large numbers in Sun (2006), provided that the spaces of agents and the probability space are constructed appropriately, aggregate income and the cross-sectional distribution of permanent income $\Phi(\cdot)$ will be constant over time.

Proposition 4 *The total savings demand "for a rainy day" in the precautionary savings model with RB equals zero for any positive interest rate. That is,* $F_t(r) = \int_{y_t} x_t(r)\, d\Phi(y_t) = 0$*, for* $r > 0$*.*

Proof. Given that labor income is a stationary process, the LLN can be directly applied and the proof is the same as that in Wang (2003). ■

This proposition states that the total savings "for a rainy day" is zero, at any positive interest rate; with a constant income distribution and linear decision rules, agents in the stationary wealth distribution follow the 'American dream and American nightmare' path, where any rise in income today is eventually offset by a future decline. Therefore, from (17), for $r > 0$, the expression for total savings under RB in the economy at time t can be written as

$$D(\vartheta, r) \equiv \Gamma(\vartheta, r) - \Psi(r). \tag{26}$$

where $\Gamma = r(\gamma + \vartheta/\psi)\sigma_s^2/2$ is the demand for precautionary savings, and $\Psi(r) = \psi(\delta - r)/r$ captures the saving demand of relative patience. We can now define a general equilibrium.

Definition 5 *Given (26), a general equilibrium under RB is defined by an interest rate* r^* *satisfying*

$$D(\vartheta, r^*) = 0. \tag{27}$$

[23] We can easily generalize to fixed positive net savings, as in a Lucas-style tree model. Nothing would change.

3.2 Theoretical Results

The following proposition shows that an equilibrium exists and provides a sufficient condition for uniqueness. We also show that, in any equilibrium, the PIH is satisfied.

Proposition 6 *There exists at least one equilibrium interest rate $r^* \in (0, \delta)$ in the precautionary-savings model with RB; if $\delta < \rho$ the equilibrium interest rate is unique on $(0, \delta)$. In equilibrium, each consumer's optimal consumption is described by the PIH, in that*

$$c_t^* = r^* s_t. \tag{28}$$

Furthermore, the evolution equations of wealth and consumption are

$$dw_t^* = x_t dt, \tag{29}$$

$$dc_t^* = \frac{r^*}{r^* + \rho} \sigma_y dB_t, \tag{30}$$

respectively, where $x_t = \rho \left(y_t - \overline{y}\right) / \left(r^* + \rho\right)$.

Proof. If $r > \delta$, both $\Gamma(\vartheta, r)$ and $\Psi(r)$ in the expression for total savings $D(\vartheta, r)$ are positive, which contradicts the equilibrium condition $D(\vartheta, r) = 0$. Since $\Gamma(\vartheta, r) - \Psi(r) < 0$ (> 0) when $r = 0$ ($r = \delta$), the continuity of the expression for total savings implies that there exists at least one interest rate $r^* \in (0, \delta)$ such that $D(\vartheta, r^*) = 0$. To establish the conditions under which this equilibrium is unique, we take the derivative

$$\frac{\partial D(\vartheta, r)}{\partial r} = \left(\gamma + \frac{\vartheta}{\psi}\right) \frac{\sigma^2}{(r+\rho)^2} \left(\frac{1}{2} - \frac{r}{r+\rho}\right) + \frac{\delta \psi}{r^2}$$

and note a sufficient condition for this derivative to be positive for any $r > 0$ is

$$\frac{1}{2} - \frac{r}{r+\rho} > 0 \Leftrightarrow r < \rho.$$

Therefore, if $\rho > \delta$ there is only one equilibrium in $(0, \delta)$. From Expression (16), we can obtain the individual's optimal consumption rule under RB in general equilibrium as $c_t^* = r^* s_t$. Substituting (28) into (3) yields (29). Using (5) and (28), we can obtain (30). ■

The intuition behind this proposition is similar to that in Wang (2003). With an individual's constant total precautionary savings demand $\Gamma(\vartheta, r)$, for any $r > 0$, the equilibrium interest rate r^* must be at a level with the property that individual's dissavings demand due to impatience is exactly balanced by their total precautionary-savings demand, $\Gamma(\vartheta, r^*) = \Psi(r^*)$. We can see from (27) that EIS affects the equilibrium interest rate via two channels: (i) the precautionary saving channel and (ii) the impatience-induced dissaving channel. As EIS decreases, it increases the precautionary saving demand via increasing the effective coefficient of risk aversion and also reduces the impatience-induced dissaving effect; both channels drive down the equilibrium interest rate. It is also clear from (27) that a high value of ψ would amplify the relative importance of the dissaving effect $\Psi(r)$ for the equilibrium interest rate. The intuition behind this result is simple. When ψ is higher, consumption growth responds less to changes in the interest rate. In order to clear the market, the consumer must be offered a higher equilibrium risk free rate in order to be induced to save more and making his consumption tomorrow even more in excess of what it is today (less smoothing).

From the equilibrium condition,

$$\frac{1}{2}r^*\left(\gamma+\frac{\vartheta}{\psi}\right)\frac{\sigma^2}{(r^*+\rho)^2}-\psi\left(\frac{\delta}{r^*}-1\right)=0, \tag{31}$$

it is straightforward to show that

$$\frac{dr^*}{d\vartheta}=-\frac{r^*\sigma_s^2}{\psi}\left(\widetilde{\gamma}\sigma_s^2\frac{\rho-r^*}{\rho+r^*}+\frac{2\delta\psi}{r^{*2}}\right)^{-1}<0 \tag{32}$$

for plausible parameter values of ρ. It is clear from this expression that r^* is decreasing in the degree of RB, ϑ. In addition, it is clear that

$$\frac{dr^*}{d\gamma}<0 \text{ and } \frac{dr^*}{d\psi}>0.$$

That is, the equilibrium interest rate decreases with the degree of risk aversion and increases with the degree of intertemporal substitution. From (30) and (29), we can conclude that although both the CARA model and the LQ model lead to the PIH in general equilibrium, both risk aversion and intertemporal substitution play roles in affecting the dynamics of consumption and wealth in the CARA model via the equilibrium interest rate channel.

Following Caballero (1991) and Wang (2003), we set $\gamma=2.5$, $\psi=0.5$, $\sigma_y=0.1823$, and $\rho=0.0834$.[24] Figure 1 shows that the aggregate saving function $D(\vartheta,r)$ is increasing with the interest rate for different values of ϑ when $\delta=0.04$, and there exists a unique interest rate r^* for every given ϑ such that $D(\vartheta,r^*)=0$.[25]

The magnitude of the EIS (ψ) is an open and unresolved question, as the literature has found a very wide range of values. Parker (2002) and Vissing-Jorgensen and Attanasio (2003) estimate the EIS to be well in excess of one, while Hall (1988) and Campbell (2003) estimate a value well below one (and possibly zero). Guvenen (2006) finds that stockholders have a higher EIS (around

[24] In Section 4.1, we will provide more details about how to estimate the income process using the U.S. panel data. The main result here is robust to the choices of these parameter values.

[25] We ignore negative interest rate equilibria because the resulting consumption function does not make economic sense. It is easy to see that D has the same zeroes as a cubic function, so that there exist conditions under which the equilibrium is globally unique, but these conditions are not amenable to analysis.

1.0) than non-stockholders (around 0.1); he argues that this disparity can explain the differences between the two results above, since Parker (2002) and Vissing-Jorgensen and Attanasio (2003) use micro data (dominated by the consumption of nonstockholders) and Hall (1988) and Campbell (2003) use macro Euler equations (where only stockholder consumption matters). Havránek (2015) surveys the vast literature and suggests that a range around $0.3-0.4$ is appropriate after correcting for selective reporting bias, while Crump et al. (2015) find that the EIS is precisely and robustly estimated to be around 0.8 in the general population using the newly released FRBNY Survey of Consumer Expectations (SCE). Here we choose $\psi = 0.5$ for illustrative purposes and will examine how EIS affects the general equilibrium under RI in Section 4.

The following result is immediate.

Proposition 7 *The relative volatility of consumption growth to income growth is*

$$\mu \equiv \frac{\mathrm{sd}\left(dc_t^*\right)}{\mathrm{sd}\left(dy_t\right)} = \frac{r^*}{r^* + \rho}. \tag{33}$$

Figure 1 also shows how RB (ϑ) affects the equilibrium interest rate (r^*). It is clear from the figure that the stronger the preference for robustness, the lower the equilibrium interest rate. From (33), we can see that RB can affect the volatility of consumption by reducing the equilibrium interest rate. The following proposition summarizes the results about how the persistence coefficient of income changes the effect of RB on μ.

Proposition 8 *Using (33), we have*

$$\frac{\partial \mu}{\partial \vartheta} = \frac{\rho}{\left(r^* + \rho\right)^2} \frac{\partial r^*}{\partial \vartheta} < 0$$

because $\rho > 0$ and $\partial r^* / \partial \vartheta < 0$.

Proof. The proof is straightforward. ■

In the next section, we explore the effects of RB on equilibrium interest rates and consumption dynamics.

4 Quantitative Analysis

In this section, we first describe how we estimate the income process and calibrate the robustness parameter. We then present quantitative results on how RB affects the equilibrium interest rate and relative volatility of consumption to income.

4.1 Estimation of the Income Process

To implement the quantitative analysis, we need to first estimate ρ and σ_y in the income process specification (4). We use micro data from the Panel Study of Income Dynamics (PSID). Following Blundell, Pistaferri, and Preston (2008), we define the household income as total household income (including wage, financial, and transfer income of head, wife, and all others in household) minus financial income (defined as the sum of annual dividend income, interest income, rental income, trust fund income, and income from royalties for the head of the household only) minus the tax liability of non-financial income. This tax liability is defined as the total tax liability multiplied by the non-financial share of total income. Tax liabilities after 1992 are not reported in the PSID and so we estimate them using the TAXSIM program from the NBER. Details on sample selection are reported in Appendix 7.1.

To exclude extreme outliers, following Floden and Lindé (2001) we normalize both income and consumption measures as ratios of the mean of each year, and exclude households in the bottom and top 1 percent of the distribution of those ratios. To eliminate possible heteroskedasticity in the income measures, we regress each on a series of demographic variables to remove variation caused by differences in age and education. We next subtract these fitted values from each measure to create a panel of income residuals. We then use this panel to estimate the household income process as a stationary AR(1) process with Gaussian innovations:

$$y_t = \phi_0 + \phi_1 y_{t-1} + \sigma\varepsilon_t,\ t \geq 1,\ \ |\phi_1| < 1, \tag{34}$$

where $\varepsilon_t \sim N(0,1)$, $\phi_0 = (1-\phi_1)\,\overline{y}$, $\overline{y}$ is the mean of y_t, and the initial level of labor income y_0 are given. Once we have estimates of ϕ_1 and σ, we can recover the drift and diffusion coefficients in the Ornstein-Uhlenbeck process specified in (4) by rewriting (34) in the time interval $[t, t+\Delta t]$ as

$$y_{t+\Delta t} = \phi_0 + \phi_1 y_t + \sigma\sqrt{\Delta t}\varepsilon_{t+\Delta t}, \tag{35}$$

where $\phi_0 = \kappa\,(1-\exp(-\rho\Delta t))\,/\,(\rho\Delta t)$, $\phi_1 = \exp(-\rho\Delta t)$, $\sigma = \sigma_y\sqrt{(1-\exp(-2\rho\Delta t))\,/\,(2\rho\Delta t)}$, and $\varepsilon_{t+\Delta t}$ is the time-$(t+\Delta t)$ standard normal distributed innovation to income.[26] As the time interval, Δt, converges to 0, (35) reduces to the Ornstein-Uhlenbeck process, (4). The estimation results and the recovered persistence and volatility coefficients in (4) are reported in Table 2.

4.2 Calibration of the Robustness Parameter

We adopt the calibration procedure outlined in Hansen, Sargent, and Wang (2002) and AHS (2003) to calibrate the value of the RB parameter (ϑ) that governs the degree of robustness. Specifically, we calibrate ϑ by using the method of detection error probabilities (DEP) that is based on a

[26] Note that here we use the fact that $\Delta B_t = \varepsilon_t\sqrt{\Delta t}$, where ΔB_t represents the increment of a Wiener process.

statistical theory of model selection. We can then infer what values of ϑ imply reasonable fears of model misspecification for empirically-plausible approximating models. The model detection error probability denoted by p is a measure of how far the distorted model can deviate from the approximating model without being discarded; low values for this probability mean that agents are unwilling to discard many models, implying that the cloud of models surrounding the approximating model is large. In this case, it is easier for the consumer to distinguish the two models. The value of p is determined by the following procedure. Let model P denote the approximating model, (5) and model Q be the distorted model, (12). Define p_P as

$$p_P = \text{Prob}\left(\ln\left(\frac{L_Q}{L_P}\right) > 0 \middle| P\right), \tag{36}$$

where $\ln\left(\frac{L_Q}{L_P}\right)$ is the log-likelihood ratio. When model P generates the data, p_P measures the probability that a likelihood ratio test selects model Q. In this case, we call p_P the probability of the model detection error. Similarly, when model Q generates the data, we can define p_Q as

$$p_Q = \text{Prob}\left(\ln\left(\frac{L_P}{L_Q}\right) > 0 \middle| Q\right). \tag{37}$$

Given initial priors of 0.5 on each model and the length of the sample is N, the detection error probability, p, can be written as:

$$p(\vartheta; N) = \frac{1}{2}(p_P + p_Q), \tag{38}$$

where ϑ is the robustness parameter used to generate model Q. Given this definition, we can see that $1-p$ measures the probability that econometricians can distinguish the approximating model from the distorted model.

The general idea of the calibration procedure is to find a value of ϑ such that $p(\vartheta; N)$ equals a given value after simulating model P, (5), and model Q, (12).[27] In the continuous-time model with the iid Gaussian specification, $p(\vartheta; N)$ can be easily computed. Since both models P and Q are arithmetic Brownian motions with constant drift and diffusion coefficients, the log-likelihood ratios are Brownian motions. The logarithm of the Radon-Nikodym derivative of the distorted model (Q) with respect to the approximating model (P) can be written as

$$\ln\left(\frac{L_Q}{L_P}\right) = \int_0^t \overline{\upsilon} dB_s - \frac{1}{2}\int_0^t \overline{\upsilon}^2 ds, \tag{39}$$

where

$$\overline{\upsilon} \equiv \upsilon^* \sigma_s = -\frac{\vartheta}{\psi} r^* \sigma_s. \tag{40}$$

[27] The number of periods used in the calculation, N, is set to be 31, the actual length of the data ($1980 - 2010$).

Similarly, the logarithm of the Radon-Nikodym derivative of the approximating model (P) with respect to the distorted model (Q) is

$$\ln\left(\frac{L_P}{L_Q}\right) = -\int_0^t \overline{\upsilon} dB_s + \frac{1}{2}\int_0^t \overline{\upsilon}^2 ds. \tag{41}$$

Using (36)-(41), it is straightforward to derive $p(\vartheta; N)$:

$$p(\vartheta; N) = \Pr\left(x < \frac{\overline{\upsilon}}{2}\sqrt{N}\right), \tag{42}$$

where x follows a standard normal distribution. From the expressions of $\overline{\upsilon}$, (40), and $p(\vartheta; N)$, (42), it is clear that the value of p is decreasing with the value of ϑ. Under the observational equivalence condition between the multiplier and constraint robustness formulations, (42) can be rewritten as $p(\vartheta; N) = \Pr\left(x < -\sqrt{2\eta}\sqrt{N}\right)$, where η is the upper bound on the distance between the two models and measures the consumer's tolerance for model misspecification.

We first explore the relationship between the DEP (p) and the value of the RB parameter, ϑ. A general finding is a negative relationship between these two variables. The upper panels of Figure 2 illustrates how DEP (p) varies with the value of ϑ for different values of EIS (ψ) and CARA (γ).[28] We can see from the figures that the stronger the preference for robustness (higher ϑ), the less the DEP (p) is. For example, let $\gamma = 2$ and $\psi = 0.8$, then $p = 0.432$ and $r^* = 3.10$ percent when $\vartheta = 1$, while $p = 0.108$ and $r^* = 2.31$ percent when $\vartheta = 9$.[29] Both values of p are reasonable as argued in AHS (2002), HSW (2002), Maenhout (2004), and Hansen and Sargent (Chapter 9, 2007). In other words, a value of ϑ below 9 is reasonable in this case in which $\gamma = 2$ and $\psi = 0.8$. Furthermore, from the two upper panels of Figure 2, we can also see that the DEP increases with both ψ and γ for given values of ϑ, and the impact of a change in ψ is much larger than that of a change in γ. The intuition is that a change in the EIS has two channels to affect the values of $\overline{\upsilon}$ and p: (i) the direct channel and (ii) the indirect channel via affecting the general interest rate channel (r^*), and the direct channel dominates the indirect channel. In contrast, a change in CARA only affects the values of $\overline{\upsilon}$ and p via the indirect equilibrium interest rate channel, which is relatively weak. Furthermore, using (22), in this case, we have $\mu_{\gamma\vartheta} = 5.625$ and 0.625 when we set $\vartheta = 1$ and 9, respectively. That is, the relative importance of risk aversion to RB in determining the precautionary savings demand decreases with the value of ϑ, holding other parameters constant.

The two lower panels of Figure 2 illustrate how DEP (p) varies with ϑ for different values of

[28] Based on the estimation results, we set $\overline{y} = 1$, $\sigma_y = 0.182$, and $\rho = 0.083$. The implied coefficient of relative risk aversion (CRRA) in our CARA utility specification can be written as either γc or γy. Given that the value of the CRRA is very stable and $\overline{\upsilon}$ can be expressed as $r(\vartheta/\psi)\sigma_y/(r+\rho)$, proportional changes in the mean and standard deviation of y do not change our calibration results because their effects on γ and σ_y cancel.

[29] Caballero (1990) and Wang (2009) also consider the $\gamma = 2$ case.

σ_y and ρ if $\psi = 0.5$ and $\gamma = 2$.[30] It also shows that the higher the value of ϑ, the less the DEP (p). In addition, to calibrate the same value of p, smaller values of σ_y (less volatile labor income processes) or higher values of ρ (less persistent income processes) lead to higher values of ϑ. The intuition behind this result is that σ_s and ϑ have opposite effects on $\overline{v}$ and then p (see (40)).

As emphasized in Hansen and Sargent (2007), in the robustness model, p is a measure of the amount of model uncertainty, whereas ϑ is a measure of the agent's aversion to model uncertainty. If we keep p constant when recalibrating ϑ for different values of γ, ρ, or σ_y, the amount of model uncertainty is held constant – that is, the set of distorted models with which we surround the approximating model does not change. In contrast, if we keep ϑ constant, p will change accordingly if the values of γ, ρ, or σ_y change; in this case, the amount of model uncertainty is "elastic" and will change accordingly as the agent's aversion to uncertainty changes.

4.3 Effects of RB on the Equilibrium Interest Rate and Consumption Volatility

The equilibrium interest rate and relative volatility of consumption to income are jointly determined by the degree of robustness, risk aversion, intertemporal substitution, and the income process. To better see how RB affects the equilibrium interest rate and the relative volatility, we present two quantitative exercises here. The first exercise fixes the parameters of the income process at the estimated values and allows the risk aversion and intertemporal substition parameters to change, while the second exercise fixes the risk aversion and intertemporal substitution parameters and allows the key income process parameter to vary.

Figure 3 shows that the equilibrium interest rate and the equilibrium relative consumption volatility decrease with the calibrated value of ϑ for different values of ψ and γ when $\sigma_y = 0.182$ and $\rho = 0.083$. For example, if ϑ is increased from 1 to 4 (p decreases from 0.40 to 0.20), r^* falls from 2.77 percent to 2.18 percent and μ falls from 0.249 to 0.207, given $\psi = 0.5$ and $\gamma = 2$.[31] In addition, the figure also shows that the interest rate and the relative volatility decrease with γ and increase with ψ for different values of ϑ.

Our model has the potential to explain the observed low real interest rate in the U.S. economy; see Laubach and Williams (2015) or Hall (2016) for evidence on low real rates. One of our theoretical results shows that a stronger aversion to model uncertainty lowers the equilibrium real interest rate. In the US, the average real risk-free interest rate has been about 1.87 percent between 1981 and 2010 if we use CPI to measure inflation, and about 1.96 percent if we use PCE to measure inflation.[32]

[30] Since $\sigma_s = \sigma_y/(r+\rho)$, both changes in the persistence coefficient (ρ) and changes in volatility coefficient (σ_y) will change the value of σ_s.

[31] In the FI-RE case, $r^* = 3.11$ percent and $\mu = 0.272$.

[32] Following Campbell (2003), we calculate the average of the real 3-month Treasury yields. Here we choose the $1981-2010$ period because it is more consistent with our sample period of the panel data in estimating the joint consumption and income process. When we consider an extended period from 1981 to 2015, the real interest rate is

Therefore, depending on what inflation index is used, the risk-free rate is between 1.87 and 1.96 percent. In our following discussion, we set the risk free rate to be 1.91 percent which is the average of the two real interest rates under CPI and PCE. Using the equilibrium condition, we find that the full-information RE model without RB requires the coefficient of risk aversion parameter to be 24 to match this rate if $\psi = 0.8$, and requires the coefficient to be 15 if $\psi = 0.5$.[33]

In contrast, when consumers take into account model uncertainty, the model can generate an equilibrium interest rate of 1.91 percent with much lower values of the coefficient of risk aversion.[34] Figure 4 shows the relationship between γ and ϑ for interest rates equal to 1.91 percent for different values of ψ. For example, if $\psi = 0.5$, the RB model with $\gamma = 5$ and $\vartheta = 5$ leads to the same interest rate as in the FI-RE model with $\gamma = 15$. Using the same calibration procedure discussed in Section 4.1, we find that the corresponding DEP is $p = 0.174$. In other words, agents tolerate a 17.4 percent probability that they cannot distinguish the distorted model from the approximating model. We have summarized these results in Table 3. As argued in Hansen and Sargent (2007) and in Section 4.2, this value is viewed as reasonable in the literature.

The explanation that agents have become more concerned about model misspecification after the $2007-09$ financial crisis does not seem unreasonable given the long and deep recession which generated skepticism (at least in the popular press) about whether the standard macro models fully capture the key features of the economy.[35] To provide a numerical example, under our calibrated parameter values and $\gamma = 2$, an increase in model uncertainty reflected by a reduction in the DEP from $p = 0.319$ to $p = 0.117$ (an increase in ϑ from 2 to 6.3) leads to a reduction in the equilibrium interest rate from 2.53 percent to 1.91 percent.

The explanation of a lower equilibrium real interest rate due to higher savings is of course not new. Bernanke (2005), Summers (2014), and Blanchard, Furceri, and Pescatori (2014) also argue that increases in global savings could be a reason for a lower equilibrium real interest rate in the US and other advanced economies. These explanations for higher savings rely on either demographic trends (such as an aging population) or capital flows from emerging economies to advanced economies, in contrast to our story about enhanced risk aversion; Hall (2016) tells a

1.37 percent when using CPI and is 1.75 percent when using PCE. Hall (2016) finds that real rates (computed using TIPS) have been consistently falling for several decades, so we are overstating the current rate.

[33] Note that since we set the mean income level to be 1, the coefficient of relative risk aversion (CRRA) evaluated at this level is equal to the coefficient of absolute risk aversion (CARA).

[34] This result is comparable to that obtained in Barillas, Hansen, and Sargent (2009). They found that most of the observed high market price of uncertainty in the U.S. can be reinterpreted as a market price of model uncertainty rather than the traditional market price of risk.

[35] Boyarchenko (2012) uses data on CDS spreads to show that the amount of ambiguity in the economy rose during the financial crisis but the aversion to ambiguity did not change; in our model the same parameter controls both features.

related story that involves wealth being redistributed from more risk tolerant agents to more risk averse ones, which leads to a rise in average risk aversion.[36]

To examine how RB affects the relative volatility of consumption to income ($\mu = \mathrm{sd}\left(dc_t^*\right) / \mathrm{sd}\left(dy_t\right)$), we follow Luo *et al.* (2016) and construct a panel data set which contains both consumption and income at the household level.[37] Figure 5 shows the relative volatility of consumption to income between 1980 and 2000.[38] From the figure, the average empirical value of the relative volatility (μ) is 0.377 for the $1980-1996$ period, and is 0.326 for the period from 1980 to 2010. The minimum and maximum values of the empirical relative volatility from 1980 to 2010 are 0.195 (year 2006) and 0.55 (year 1982), respectively. From the expression for the equilibrium relative volatility (33), we can see that when the real interest rate is low, it is impossible for the model to generate sufficiently high relative volatility of consumption to income without using an implausible value for ρ. For example, when $r^* = 1.91$ percent we obtain $\mu = 0.19$, which is well below the average value $\mu = 0.326$; to get $\mu = 0.326$ we would need $\rho = 0.039$, a value that can be rejected given our estimated value of $\rho = 0.082$. Because this moment matters for the welfare calculations that are the focus of the paper, in the next section we will resolve the disparity between data and model using a risky asset in positive net supply.

4.4 The Welfare Cost of Model Uncertainty

We can quantify the effects of RB on the welfare cost of volatility in the general equilibrium using the Lucas elimination-of-risk method (see Lucas 1987 and Tallarini 2000). We define the total welfare cost of volatility as the percentage of permanent income the consumer is willing to give up in the initial period to be as well off in the FI-RE economy as he is in the RB economy.[39] That is, define

$$\widetilde{f}\left(s_0\left(1-\Delta\right)\right) = f\left(s_0\right), \tag{43}$$

where

$$\widetilde{f}\left(s_0\left(1-\Delta\right)\right) = -\frac{\delta}{\widetilde{\alpha}_1}\exp\left(-\widetilde{\alpha}_0 - \widetilde{\alpha}_1 s_0\left(1-\Delta\right)\right) \text{ and } f\left(s_0\right) = -\frac{\delta}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1 s_0\right)$$

[36] Hall's evidence for this shift is indirect – he notes the rise in the volume of assets and liabilities held by "risk-splitting intermediaries".

[37] Appendix 7.1 presents details on how the panel is constructed.

[38] See Appendix 7.1 for more details on how the panel was constructed.

[39] This approach is also used in Epaulard and Pommeret (2003) to examine the welfare cost of volatility in a representative-agent model with recursive utility. In their model, the total welfare cost of volatility is defined as the percentage of capital the representative agent is ready to give up at the initial period to be as well off in a certain economy as he is in a stochastic one.

are the value functions under FI-RE and RB, respectively, Δ is the compensating amount measured as a percentage of s_0,

$$\alpha_1 = \frac{r^*}{\psi}, \widetilde{\alpha}_1 = \frac{\widetilde{r}^*}{\psi}, \alpha_0 = \frac{\delta}{r^*} - 1 - \frac{1}{2}\frac{r^*}{\psi}\left(\gamma + \frac{\vartheta}{\psi}\right)\sigma_s^2, \widetilde{\alpha}_0 = \frac{\delta}{\widetilde{r}^*} - 1 - \frac{1}{2}\frac{\widetilde{r}^*}{\psi}\left(\gamma + \frac{\vartheta}{\psi}\right)\widetilde{\sigma}_s^2,$$

and r^* and $\widetilde{r}^*$ are the equilibrium interest rates in the RB and FI-RE economies, respectively.[40] We state our measurement of the welfare costs formally in the next proposition.

Proposition 9 *The welfare costs due to model uncertainty are given by*

$$\Delta = \frac{s_0\left(\widetilde{\alpha}_1 - \alpha_1\right) - \ln\left(\widetilde{\alpha}_1/\alpha_1\right)}{\widetilde{\alpha}_1 s_0} = \left(1 - \frac{r^*}{\widetilde{r}^*}\right) - \frac{\psi}{\widetilde{c}_0}\ln\left(\frac{\widetilde{r}^*}{r^*}\right), \tag{44}$$

where $\widetilde{c}_0 = \widetilde{r}^* s_0$ *is optimal consumption under FI-RE.*

Proof. Substituting (27) into the expressions of α_0 and $\widetilde{\alpha}_0$ in the value functions under FI-RE and RB, we obtain that $\alpha_0 = \widetilde{\alpha}_0 = 0$. Combining these results with (43) yields (44). ■

To understand how the welfare cost varies with the degree of uncertainty aversion, we note that

$$\frac{\partial\Delta}{\partial\vartheta} = \frac{\partial\Delta}{\partial r^*}\frac{\partial r^*}{\partial\vartheta}.$$

The second term is negative, $\partial r^*/\partial\vartheta < 0$, for the reasons we have already discussed. The first term is

$$\frac{\partial\Delta}{\partial r^*} = -\frac{1}{\widetilde{r}^*}\left(1 - \frac{\psi}{r^* s_0}\right);$$

for reasonable values we expect this term to be negative, so that higher model uncertainty leads to larger welfare costs.

To do quantitative welfare analysis, we set s_0 such that $c_0 = \widetilde{r}^* s_0 = y_0$. Figure 6 illustrates how the welfare cost of model uncertainty varies with ϑ for different values of γ and σ_y if $y_0 = 1$, $\psi = 2/3$, and $\rho = 0.083$.[41] We can see from this figure that the welfare costs of model uncertainty are nontrivial and increasing in γ and σ_y. The intuition behind this result is that higher income uncertainty leads to higher induced model uncertainty. For example, if $\psi = 0.5$ and $\vartheta = 2$, the welfare cost of model uncertainty Δ is 8.46 percent. If ϑ increases from 2 to 4, Δ increases from 8.46 percent to 12.23 percent. The figure also shows that an increasing income volatility can significantly increase the welfare cost of model uncertainty. For example, when $\gamma = 2$, $\vartheta = 2$, and income volatility σ_y increases from 0.18 to 0.23, Δ increases from 8.46 percent to 10.25 percent. One policy implication stemming from this finding is that macro policies aiming to reduce income volatility and inequality are more beneficial in an economy in which consumers have a greater

[40] See Appendix 7.3 for the derivation of the value functions. Note that $\Delta = 0$ when $\vartheta = 0$.

[41] When generating the left and right panels of this figure, we set $\sigma_y = 0.182$ and $\gamma = 2$, respectively.

aversion to model uncertainty, both because they reduce risk but also because they mitigate costly precautionary saving.[42]

5 Extension to an RU-RB Model with Multiple Financial Assets

In this section, we follow Merton (1971), Maenhout (2004), and Wang (2009), and assume that consumers can assess two financial assets: one risk free asset and one risky asset. Our aim here is to resolve the anomaly from the benchmark model regarding the relative volatility of consumption to income at low interest rates.

5.1 Model Specification

The consumer can purchase both a risk-free asset with a constant interest rate r and a risky asset (the market portfolio) with a risky return r_t^e. The instantaneous return dr_t^e of the risky market portfolio over dt is given by

$$dr_t^e = (r + \pi)\, dt + \sigma_e dB_{e,t}, \tag{45}$$

where π is the market risk premium; $B_{e,t}$ is a standard Brownian motion; and σ_e is the standard deviation of the market return. Let ρ_{ye} be the contemporaneous correlation between the labor income process and the return of the risky asset. If $\rho_{ye} = 0$, the labor income risk is purely idiosyncratic, so the risky asset does not provide a hedge against labor income declines. The agent's financial wealth evolution is then given by

$$dw_t = (rw_t + y_t - c_t)\, dt + \alpha_t\, (\pi dt + \sigma_e dB_{e,t})\,, \tag{46}$$

where α_t denotes the amount of wealth that the investor allocates to the market portfolio at time t.

As in the benchmark model, we define a new state variable, s_t: $s_t \equiv w_t + h_t$, where h_t is human wealth at time t and is defined as the expected present value of current and future labor income discounted at the risk-free interest rate r: $h_t \equiv E_t \left[\int_t^\infty \exp\left(-r\,(s-t)\right) y_s ds\right]$. Following the same state-space-reduction approach used in the benchmark model, the budget constraint can be written as:

$$ds_t = (rs_t - c_t + \pi\alpha_t)\, dt + \sigma dB_t, \tag{47}$$

where $\sigma dB_t = \sigma_e \alpha_t dB_{e,t} + \sigma_s dB_{y,t}$, $\sigma_s = \sigma_y / (r + \rho)$, and

$$\sigma = \sqrt{\sigma_e^2 \alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t} \tag{48}$$

[42] Ellison and Sargent (2015) found that idiosyncratic consumption risk has a greater effect on the cost of business cycles when agents fear model misspecification. In addition, they showed that endowing agents with fears about misspecification leads to greater welfare costs caused by the idiosyncratic consumption risk. The underlying reasons are the same: the enhanced risk aversion created by uncertainty aversion.

is the unconditional variance of the innovation to s_t.

5.2 Consumption and Saving Rules under RB

To introduce robustness into the above recursive utility model, we follow the same procedure as in Section 2.2 and write the distorting model by adding an endogenous distortion $\upsilon(s_t)$ to the law of motion of the state variable s_t, (47),

$$ds_t = (rs_t - c_t + \pi\alpha_t)\,dt + \sigma\left(\sigma\upsilon(s_t)\,dt + dB_t\right). \tag{49}$$

The drift adjustment $\upsilon(s_t)$ is chosen to minimize the sum of the expected continuation payoff, but adjusted to reflect the additional drift component in (49), and an entropy penalty:

$$\inf_{\upsilon}\left[\mathcal{D}f + f'(J_t)\,A\upsilon(s_t)\,\sigma^2 + \frac{1}{\vartheta_t}\mathcal{H}\right], \tag{50}$$

where the first two terms are the expected continuation payoff when the state variable follows (49), i.e., the alternative model based on drift distortion $\upsilon(s_t)$, $\mathcal{H} = (\upsilon(s_t)\,\sigma)^2/2$ is the relative entropy or the expected log likelihood ratio between the distorted model and the approximating model and measures the distance between the two models, and $1/\vartheta_t$ is the weight on the entropy penalty term. Solving this infimization part of (50) yields

$$\upsilon(s_t)^* = -\vartheta(s_t)\,f_s,$$

where $\vartheta(s_t) = -\vartheta/f(s_t) > 0$. Substituting for υ^* in (50) gives

$$\delta f(U_t) = \sup_{c_t\in\mathcal{C}}\left\{\delta f(c_t) + f'(J_t)\,A\left(rs_t - c_t + \pi\alpha_t - \frac{1}{2}\gamma A\sigma_s^2 + \frac{\vartheta}{f(U_t)}Af'(U_t)\,\sigma^2\right) - \frac{\vartheta}{2f(J_t)}A^2\left(f'(J_t)\right)^2\sigma^2\right\}. \tag{51}$$

The following proposition summarizes the solution to this dynamic program.

Proposition 10 *Under robustness, the consumption function, the portfolio rule, and the saving function are*

$$c_t^* = r\left(s_t - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{r\sigma_e^2}\right) + \Psi - \Gamma + \Pi, \tag{52}$$

$$\alpha^* = \frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \tag{53}$$

and

$$d_t^* = x_t + \Gamma - \Psi + \Pi, \tag{54}$$

respectively, where $x_t \equiv \rho(y_t - \overline{y})/(r+\rho)$ is the demand for savings "for a rainy day",

$$\Gamma \equiv \frac{1}{2}r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)\sigma_s^2 \tag{55}$$

is the demand for precautionary savings due to the interactions of income uncertainty, intertemporal substitution, and risk and uncertainty aversion,

$$\Psi \equiv \psi \left(\frac{\delta}{r} - 1 \right) \tag{56}$$

captures the saving demand of relative patience,

$$\Pi \equiv \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2} \tag{57}$$

is the additional saving demand due to the higher expected return of the risky asset, and $\widetilde{\gamma} \equiv \gamma + \vartheta/\psi$ *is the effective coefficient of absolute risk aversion. Finally, the worst possible distortion is* $\upsilon^* = -r\,(\vartheta/\psi)$*.*

Proof. See Appendix 7.4. ■

Expression (16) shows that the presence of the risky asset in the agent's investment opportunity has two effects on current consumption. First, it reduces the risk-adjusted certainty equivalent human wealth by $\pi\rho_{ye}\sigma_s\sigma_e / \left(r\sigma_e^2\right)$ because the agent faces more risk when holding the risky asset. Second, it increases current consumption because it offers a higher expected return (see Merton 1971). In general equilibrium, the second effect dominates the first effect. Furthermore, from the definition of individual saving, we can see that the presence of $\pi\alpha^*$ term has the potential to increase saving because it offers a higher expected return. Combining these two effects, it is easy to show that the net effect of the risky asset on current saving is governed by $\Pi > 0$ defined in (57). In addition, since the risky asset can be used to hedge labor income risk (provided the correlation is not zero), it will reduce the precautionary saving demand arising from income uncertainty by a factor $1 - \rho_{ye}^2 \in (0, 1)$. From (54), it is clear that there are four saving motives in the model with a risky asset. The first three saving motives, x_t, Γ, and Ψ, are the same as that mentioned in our benchmark model. The fourth term captures the additional saving demand due to the higher expected return of the risky asset and obviously does not appear in the benchmark model.

Since the effective coefficient of absolute risk aversion depends on both the EIS and the degree of RB, it is clear from (53) that even if the consumer only has a constant investment opportunity set, the optimal share invested in the risky asset not only depends on risk aversion, but also depends on intertemporal substitution if $\vartheta > 0$.[43] Svensson (1989) showed in a FI-RE recursive utility model that when the investment opportunity set is constant, the optimal share invested in the risky asset depends on the risk aversion parameter, but not on the EIS.

[43] A constant investment opportunity set means a constant interest rate, a constant expected return on risky assets, and a constant volatility the returns on risky assets.

5.3 General Equilibrium Implications

We first consider the equilibrium in the market for the risky asset. Assuming that the net supply of the risky asset is $\overline{\alpha} \geq 0$, the equilibrium condition in the market for the risky asset is

$$\overline{\alpha} = \frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} \tag{58}$$

for a given risk free rate, r.

Using the individual saving function (54) and following the same aggregation procedure used in the previous section, we have the following result on savings:

Proposition 11 *The total demand of savings "for a rainy day" equals zero for any positive interest rate. That is,* $F_t(r) = \int_{y_t} x_t(r)\, d\Phi(y_t) = 0$*, for* $r > 0$*.*

Proof. The proof uses the LLN and is the same as that in Wang (2003). ■

Using this result, from (54), after aggregating across all consumers, the expression for total savings can be written as

$$D(\vartheta, r) \equiv \Gamma(\vartheta, r) - \Psi(r) + \Pi(\vartheta, r), \tag{59}$$

where $\Gamma(\vartheta, r)$, $\Psi(r)$, and $\Pi(\vartheta, r)$ are given in (55), (56), and (57), respectively. To compare $D(\vartheta, r)$ with the aggregate saving function obtained in the benchmark model, we rewrite $D(\vartheta, r)$ as follows:

$$D(\vartheta, r) = \widetilde{\Gamma}(\vartheta, r) - \Psi(r) + \widetilde{\Pi}(\vartheta, r), \tag{60}$$

where $\widetilde{\Gamma}(\vartheta, r) = r\widetilde{\gamma}\sigma_s^2/2$, $\widetilde{\Pi}(\vartheta, r) = \overline{\alpha}r\widetilde{\gamma}\rho_{ye}\sigma_s\sigma_e + \overline{\alpha}^2 r\widetilde{\gamma}\sigma_e^2/2$, and $\overline{\alpha}$ is determined by (58). Comparing the two aggregate saving functions, $\widetilde{\Pi}(\vartheta, r)$ is an additional term due to the positive net supply of the risky asset in this model. As in the benchmark model, an equilibrium interest rate r^* satisfies $D(\vartheta, r^*) = 0$. The following proposition proves that an equilibrium exists and that the PIH is satisfied.

Proposition 12 *There exists an equilibrium with an interest rate* $r^* \in (0, \delta)$ *and*

$$\pi^* = r\widetilde{\gamma}\sigma_e(\rho_{ye}\sigma_s + \overline{\alpha}\sigma_e), \tag{61}$$

and if $\rho > \delta$ *and* $\rho_{ye} \geq 0$ *this equilibrium is unique. In any such equilibrium, each consumer's optimal consumption-portfolio rules are described by:*

$$c_t^* = r^* s_t, \tag{62}$$

and

$$\alpha^* = \overline{\alpha}, \tag{63}$$

respectively. Furthermore, in this equilibrium, the evolution equation of s_t is

$$ds_t = \left(\frac{\pi^2}{r\widetilde{\gamma}\sigma_e^2}\right) dt + \sigma dB_t, \tag{64}$$

if the true economy is governed by the approximating model.

Proof. If $r > \delta$, $\Gamma(\vartheta, r)$, $-\Psi(r)$, and $\Pi(\vartheta, r)$ in the expression for total savings $D(\vartheta, r^*)$ are positive, which is inconsistent with the equilibrium condition $D(\vartheta, r^*) = 0$. Since $\widetilde{\Gamma}(\vartheta, r) - \Psi(r) + \widetilde{\Pi}(\vartheta, r) < 0$ (> 0) if $r = 0$ ($r = \delta$), the continuity of the expression for total savings implies that there exists at least one interest rate $r^* \in (0, \delta)$ such that $D(\vartheta, r^*) = 0$. To establish the uniqueness condition, we again study the derivative

$$\frac{\partial D(\vartheta, r)}{\partial r} = \left(\gamma + \frac{\vartheta}{\psi}\right)\frac{\sigma^2}{(r+\rho)^2}\left(\frac{1}{2} - \frac{r}{r+\rho}\right) + \frac{\delta\psi}{r^2} + \frac{\rho}{r+\rho}\overline{\alpha}\widetilde{\gamma}\rho_{ye}\sigma_s\sigma_e + \frac{1}{2}\overline{\alpha}^2\widetilde{\gamma}\sigma_e^2;$$

as before sufficient conditions for the derivative to be always positive are $r < \rho$ and $\rho_{ye} \geq 0$. Therefore, there is only one equilibrium in $(0, \delta)$ if $\rho > \delta$. From Expression (16), we can obtain the individual's optimal consumption rule under RB in general equilibrium as $c_t^* = r^* s_t$. ■

Figure 7 shows that the aggregate saving function $D(\vartheta, r)$ is increasing with the interest rate for different values of $\overline{\alpha}$.[44] It clearly shows that there exists a unique interest rate r^* for every given $\overline{\alpha}$ such that $D(\vartheta, r^*) = 0$, and a higher supply of the risky asset leads to a lower equilibrium interest rate given ϑ.[45] Although the presence of the risky asset helps hedge labor income risk, it also makes consumers face greater total uncertainty (see Expression 57), leading to higher overall saving and lower returns.

The following result is an immediate implication on how the presence of the risky asset affects the relative volatility of consumption growth to income growth under RB.

Proposition 13 *The relative volatility of consumption growth to income growth is*

$$\mu \equiv \frac{\mathrm{sd}(dc_t^*)}{\mathrm{sd}(dy_t)} = r^*\sqrt{\left(\frac{\sigma_e}{\sigma_y}\right)^2\overline{\alpha}^2 + \left(\frac{1}{r^*+\rho}\right)^2 + 2\frac{\rho_{ye}}{r^*+\rho}\frac{\sigma_e}{\sigma_y}\overline{\alpha}}. \tag{65}$$

Comparing (33) with (65), it is clear that the positive net supply of the risky asset will be helpful at increasing the relative volatility of consumption to income while keeping the real interest rate at a low level. To quantitatively examine the effects of RB on the relative volatility of consumption growth to income growth, we first use the observed risk premium of 6 percent to calibrate the value of $\overline{\alpha}$ using (58). Estimating the correlation between individual labor income and the equity return is

[44] Here we use the same parameter values in the benchmark model.

[45] The idea is similar to Caballero, Farhi, and Gourinchas (2008) – it is the relatively low supply of risk-free securities that leads to low interest rates.

complicated by the lack of panel data on household portfolio choice, and we found several estimates in the literature: Davis and Willen (2000) estimated that the correlation is between 0.1 and 0.3 for college-educated males, and is 0.25 or more for college-educated women, while Heaton and Lucas (1999) found that the correlation between entrepreneurial earnings and equity returns was about 0.2, so $\rho_{ye} = 0.15$ seems like a reasonable value. In our model, if $\gamma = 1.5$, $\vartheta = 2.8$, $\psi = 0.55$, $\delta = 0.04$, and $\rho_{ye} = 0.15$, the corresponding DEP (p) is 0.295, the equilibrium interest rate (r^*) is 1.91 percent, and the relative volatility (μ) is 0.37, which equals the empirical counterpart for the sample from 1980 to 2010. If $\gamma = 2.5$, $\vartheta = 3.135$, $\psi = 0.55$, $\delta = 0.04$, and $\rho_{ye} = 0.15$, the model leads to $p = 0.295$, $r^* = 1.91$ percent, and $\mu = 0.31$, which are the empirical counterparts for the sample from 1980 to 1996. These results are summarized in Table 3.

5.4 Welfare Costs of Uncertainty Revisited

Following the same procedure adopted in the benchmark model, we define the total welfare cost of volatility as the percentage of total wealth the consumer is ready to give up at the initial period to be as well off in the FI-RE economy as he is in the RB economy:

$$\widetilde{f}\left(s_0\left(1-\Delta\right)\right) = f\left(s_0\right), \tag{66}$$

where

$$\widetilde{f}\left(s_0\left(1-\Delta\right)\right) = -\frac{\delta}{\widetilde{\alpha}_1}\exp\left(-\widetilde{\alpha}_0 - \widetilde{\alpha}_1 s_0\left(1-\Delta\right)\right) \text{ and } f\left(s_0\right) = -\frac{\delta}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1 s_0\right)$$

are the value functions under FI-RE and RB, respectively, Δ is the compensating amount measured as a percentage of s_0,

$$\alpha_1 = \frac{r^*}{\psi}, \widetilde{\alpha}_1 = \frac{\widetilde{r}^*}{\psi},$$
$$\alpha_0 = \frac{\delta}{r^*} - 1 - \frac{1}{2}\frac{r^*}{\psi}\left(\gamma + \frac{\vartheta}{\psi}\right)\left(\sigma_s^2 - \overline{\alpha}^2\sigma_e^2\right), \widetilde{\alpha}_0 = \frac{\delta}{\widetilde{r}^*} - 1 - \frac{1}{2}\frac{\widetilde{r}^*}{\psi}\left(\gamma + \frac{\vartheta}{\psi}\right)\left(\widetilde{\sigma}_s^2 - \overline{\alpha}^2\sigma_e^2\right),$$

and r^* and $\widetilde{r}^*$ are the equilibrium interest rates in the FI-RE and RB economies, respectively.[46] As noted above, when $\overline{\alpha} = 0$, our multiple-asset model reduces to the benchmark.

The following proposition summarizes the result about how RB affects the welfare costs in general equilibrium.

Proposition 14 *The welfare costs due to model uncertainty are given by*

$$\Delta = \frac{s_0\left(\widetilde{\alpha}_1 - \alpha_1\right) - \ln\left(\widetilde{\alpha}_1/\alpha_1\right)}{\widetilde{\alpha}_1 s_0} = \left(1 - \frac{r^*}{\widetilde{r}^*}\right) + \frac{\psi}{\widetilde{c}_0}\ln\left(\frac{\widetilde{r}^*}{r^*}\right) + \frac{\overline{\alpha}\left(\widetilde{\pi} - \pi\right)}{\widetilde{c}_0}, \tag{67}$$

[46] When we compare welfare in these two economies, we assume that the asset supply is the same across the two economies.

where $\widetilde{c}_0 = \widetilde{r}^* s_0$ *is optimal consumption under FI-RE,*

$$\widetilde{\pi} = \widetilde{r}^* \widetilde{\gamma} \sigma_e \left(\rho_{ye} \sigma_s + \overline{\alpha} \sigma_e \right) \text{ and } \pi = r^* \widetilde{\gamma} \sigma_e \left(\rho_{ye} \sigma_s + \overline{\alpha} \sigma_e \right)$$

are the risk premia in the FI-RE and RB economies, respectively.

Proof. Substituting the equilibrium condition (27) into the expressions of α_0 and $\widetilde{\alpha}_0$ in the value functions under FI-RE and RB, we obtain that

$$\alpha_0 = \frac{1}{\psi} \overline{\alpha} \pi, \widetilde{\alpha}_0 = \frac{1}{\psi} \overline{\alpha} \widetilde{\pi}.$$

Combining these results with (**??**) yields (67). ■

To do quantitative welfare analysis, we set $c_0 = \widetilde{r}^* s_0 = 1$ as we did in our benchmark model. The left panel of Figure 6 illustrates how the welfare cost of model uncertainty varies with ϑ for different values of ψ and $\overline{\alpha} = 8$.[47] We can see from this figure that the welfare costs of model uncertainty are nontrivial and decreasing in ψ. The intuition behind this result is that the lower the EIS, the larger the effect of RB on the equilibrium interest rate and therefore the larger the welfare costs. The right panel of Figure 6 shows that the welfare cost of model uncertainty decreases with the supply of the risky asset.[48] The reason behind this result is that the risky asset provides a hedging tool for the consumer as long as $\rho_{ye} \neq 0$, and higher supply of the asset means that agents inefficient precautionary savings motives are weaker.

6 Conclusions

This paper has developed a tractable continuous-time continuous-time recursive utility (RU) version of the Huggett (1993) model to explore how the preference for robustness (RB) interacts with intertemporal substitution and risk aversion and then affects the interest rate, the dynamics of consumption and income, and the welfare costs of model uncertainty in general equilibrium. We found that for moderate risk aversion and plausibly calibrated parameter values of RB, our benchmark model can generate the observed low risk free rate in the US economy. However, the model cannot generate the observed high relative volatility of consumption to income. However, if we allow for a positive net supply of a risky asset (interpreted as the market portfolio), our model is able to reconcile low interest rates, moderate risk aversion, and relatively high volatility of consumption to income. The resulting model implies that the welfare costs of model uncertainty are large.

[47] When generating the left and right panels of this figure, we set $\rho = 0.083$, $\sigma_y = 0.182$ and $\gamma = 2$.

[48] Here we set $\psi = 0.8$, and need to impose a condition, $\sigma_s^2 - \overline{\alpha}^2 \sigma_e^2 > 0$, such that the total uncertainty has negative impact on the lifetime welfare of the consumer.

7 Appendix

7.1 Description of Data

This appendix describes the data we use to estimate the income process as well as the method we use to construct a panel of both household income and consumption for our empirical analysis.

We use micro data from the Panel Study of Income Dynamics (PSID). Our household sample selection closely follows that of Blundell, Pistaferri, and Preston (2008) as well.[49] We exclude households in the PSID low-income and Latino samples. We exclude household incomes in years of family composition change, divorce or remarriage, and female headship. We also exclude incomes in years where the head or wife is under 30 or over 65, or is missing education, region, or income responses. We also exclude household incomes where non-financial income is less than \$1000, where year-over-year income change is greater than \$90,000, and where year-over-year consumption change is greater than \$50,000. Our final panel contains 7,220 unique households with 54,901 yearly income responses and 50,422 imputed nondurable consumption values.[50]

The PSID does not include enough consumption expenditure data to create full picture of household nondurable consumption. Such detailed expenditures are found, though, in the Consumer Expenditure Survey (CEX) from the Bureau of Labor Statistics. But households in this study are only interviewed for four consecutive quarters and thus do not form a panel. To create a panel of consumption to match the PSID income measures, we use an estimated demand function for imputing nondurable consumption created by Guvenen and Smith (2014). Using an IV regression, they estimate a demand function for nondurable consumption that fits the detailed data in the CEX. The demand function uses demographic information and food consumption which can be found in both the CEX and PSID. Thus, we use this demand function of food consumption and demographic information (including age, family size, inflation measures, race, and education) to estimate nondurable consumption for PSID households, creating a consumption panel.

In order to estimate the income process, we narrow the sample period to the years $1980-1996$, due to the PSID survey changing to a biennial schedule after 1996. To further restrict the sample to exclude households with dramatic year-over-year income and consumption changes, we eliminate household observations in years where either income or consumption has increased more than 200 percent or decreased more than 80 percent from the previous year.

[49] They create a new panel series of consumption that combines information from PSID and CEX, focusing on the period when some of the largest changes in income inequality occurred.

[50] There are more household incomes than imputed consumption values because food consumption - the main input variable in Guvenen and Smith's nondurable demand function - is not reported in the PSID for the years 1987 and 1988. Dividing the total income responses by unique households yields an average of $7-8$ years of responses per household. These years are not necessarily consecutive as our sample selection procedure allows households to be excluded in certain years but return to the sample if they later meet the criteria once again.

7.2 Solving the FI-RE Recursive Utility Model

The optimizing problem can be written as:

$$f\left(J_t\right)=\max_{c_t}\left\{\left(1-e^{-\delta\Delta t}\right)f\left(c_t\right)+e^{-\delta\Delta t}f\left(CE_t\left[J_{t+\Delta t}\right]\right)\right\}, \tag{68}$$

where $f\left(J_t\right)$ is the value function. An educated guess is that $J_t=As_t+A_0$. The J function at t time $t+\Delta t$ can thus be written as

$$J\left(s_{t+\Delta t}\right)=As_{t+\Delta t}+A_0\approx As_t+A\left(rs_t-c_t\right)\Delta t+A\sigma_s\Delta B_t+A_0,$$

where $\Delta s_t\equiv s_{t+\Delta t}-s_t$ and $\Delta s_t\approx\left(rs_t-c_t\right)\Delta t+\sigma_s\Delta B_t$. (Here $\Delta B_t=\sqrt{\Delta t}\epsilon$ and ϵ is a standard normal distributed variable.)

Using the definition of the certainty equivalent of $J_{t+\Delta t}$, we have

$$\begin{aligned}\exp\left(-\gamma CE_t\right)&=E_t\left[\exp\left(-\gamma J\left(s_{t+\Delta t}\right)\right)\right]\\&=\exp\left(-\gamma AE_t\left[s_{t+\Delta t}\right]+\frac{1}{2}\gamma^2A^2\operatorname{var}_t\left[s_{t+\Delta t}\right]-\gamma A_0\right)\\&=\exp\left(-\gamma A\left[s_t+\left(rs_t-c_t\right)\Delta t\right]+\frac{1}{2}\gamma^2A^2\sigma_s^2\Delta t-\gamma A_0\right),\end{aligned}$$

which means that

$$CE_t=A\left[s_t+\left(rs_t-c_t-\frac{1}{2}\gamma A\sigma_s^2\right)\Delta t\right]+A_0. \tag{69}$$

Substituting these expressions back into (68) yields:

$$f\left(J_t\right)=\max_{c_t}\left\{\delta f\left(c_t\right)\Delta t+f\left(J_t\right)+f'\left(J_t\right)\left(A\left(rs_t-c_t\right)-\frac{1}{2}\gamma A^2\sigma_s^2\right)\Delta t-\delta\Delta tf\left(J_t\right)\right\}$$

where we use the facts that $e^{-\delta\Delta t}=1-\delta\Delta t$,

$$J_{t+\Delta t}\approx J_t+J_t'\left(rs_t-c_t\right)\Delta t=J_t+A\left(rs_t-c_t\right)\Delta t,$$

and

$$f\left(J_t+A\left(rs_t-c_t\right)\Delta t+\frac{1}{2}A^2\sigma_s^2\Delta t\right)\approx f\left(J_t\right)+f'\left(J_t\right)\left(A\left(rs_t-c_t\right)-\frac{1}{2}\gamma A^2\sigma_s^2\right)\Delta t.$$

Dividing both sides by Δt, the Bellman equation can then be simplified as:

$$\delta f\left(J_t\right)=\max_{c_t}\left\{\delta f\left(c_t\right)+f'\left(J_t\right)\left(A\left(rs_t-c_t\right)-\frac{1}{2}\gamma A^2\sigma_s^2\right)\right\}. \tag{70}$$

The FOC for c is then

$$\delta f'\left(c_t\right)=f'\left(J_t\right)A,$$

which implies that

$$c_t = -\psi \ln\left(\frac{A}{\delta}\right) + (As_t + A_0). \tag{71}$$

Substituting this expression for c back to the Bellman equation and matching the coefficients, we have:

$$A = r$$

and

$$A_0 = \psi\left(\frac{\delta - r}{r}\right) + \psi \ln\left(\frac{r}{\delta}\right) - \frac{1}{2}\gamma r \sigma_s^2 - \frac{\vartheta}{2\psi} r \sigma_s^2.$$

Substituting these coefficients into (71) gives the consumption function, (9), in the main text.

7.3 Solving the Benchmark RU-RB Model

Following AHS (1999), Uppal and Wang (2003) and Maenhout (2004), we introduce robustness into the above otherwise standard model as follows:

$$\delta f(J_t) = \max_c \min_\upsilon \left\{\delta f(c_t) + f'(J_t) A\left(rs_t - c_t - \frac{1}{2}\gamma A \sigma_s^2 + \upsilon_t \sigma_s^2\right) + \frac{1}{2\vartheta_t}\upsilon_t^2 \sigma_s^2\right\} \tag{72}$$

subject to the distorting equation, (12). Solving first for the infimization part of the problem yields

$$\upsilon^*(s_t) = -\vartheta_t A f'(J_t).$$

Given that $\vartheta(s_t) > 0$, the perturbation adds a negative drift term to the state transition equation because $f'(J_t) > 0$. Substituting it into the above HJB equation yields:

$$\delta f(J_t) = \max_c \left\{\delta f(c_t) + f'(J_t) A\left(rs_t - c_t - \frac{1}{2}\gamma A \sigma_s^2 - \vartheta_t A f'(U_t) \sigma_s^2\right) + \frac{1}{2\vartheta_t}\left(\vartheta_t A f'(J_t)\right)^2 \sigma_s^2\right\} \tag{73}$$

Following Uppal and Wang (2003) and Maenhout (2004), we assume that

$$\vartheta_t = -\frac{\vartheta}{f(U_t)}.$$

The HJB equation reduces to

$$\delta f(U_t) = \max_c \left\{\delta f(c_t) + f'(J_t) A\left(rs_t - c_t - \frac{1}{2}\gamma A \sigma_s^2 + \frac{\vartheta}{f(U_t)} A f'(U_t) \sigma_s^2\right) - \frac{\vartheta}{2f(J_t)} A^2 \left(f'(J_t)\right)^2 \sigma_s^2\right\}.$$

The FOC for c is then

$$\delta f'(c_t) = f'(J_t) A,$$

which implies that

$$c_t = -\psi \ln\left(\frac{A}{\delta}\right) + (As_t + A_0). \tag{74}$$

Substituting this expression for c back to the Bellman equation and matching the coefficients, we have:

$$A = r \text{ and } A_0 = \psi\left(\frac{\delta - r}{r}\right) + \psi \ln\left(\frac{r}{\delta}\right) - \frac{1}{2}\gamma r \sigma_s^2 - \frac{\vartheta}{2\psi} r \sigma_s^2.$$

Substituting these coefficients into (74) gives the consumption function, (16), and the value function, (20), in the main text.

Finally, we check if the consumer's transversality condition (TVC),

$$\lim_{t\to\infty} E\left[\exp\left(-\delta t\right)\left|f\left(s_t\right)\right|\right] = 0, \tag{75}$$

is satisfied. Substituting the consumption function, c_t^*, into the state transition equation for s_t yields

$$ds_t = \widetilde{A}dt + \sigma_s dB_t,$$

where $\widetilde{A} = -\frac{\psi(\delta - r)}{r} + \frac{1}{2} r \widetilde{\gamma} \sigma_s^2$ under the approximating model. This Brownian motion with drift can be rewritten as

$$s_t = s_0 + \widetilde{A}t + \sigma\left(B_t - B_0\right), \tag{76}$$

where $B_t - B_0 \sim N\left(0, t\right)$. Substituting (76) into $E\left[\exp\left(-\delta t\right)\left|f\left(s_t\right)\right|\right]$ yields:

$$\begin{aligned} E\left[\exp\left(-\delta t\right)\left|f\left(s_t\right)\right|\right] &= \frac{1}{\alpha_1} E\left[\exp\left(-\delta t - \alpha_0 - \alpha_1 s_t\right)\right] \\ &= \frac{1}{\alpha_1} \exp\left(E\left[-\delta - \alpha_0 - \alpha_1 s_t\right] + \frac{1}{2} \operatorname{var}\left(\alpha_1 s_t\right)\right) \\ &= \frac{1}{\alpha_1} \exp\left(-\delta t - \alpha_0 - \alpha_1\left(s_0 + \widetilde{A}t\right) + \frac{1}{2}\alpha_1^2 \sigma_s^2 t\right) \\ &= \left|J\left(s_0\right)\right| \exp\left(-\left(\delta + \alpha_1 \widetilde{A} - \frac{1}{2}\alpha_1^2 \sigma_s^2\right) t\right) \end{aligned}$$

where $\left|J\left(s_0\right)\right| = \frac{1}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1 s_0\right)$ is a positive constant and we use the facts that $s_t - s_0 \sim N\left(\widetilde{A}t, \sigma_s^2 t\right)$. Therefore, the TVC, (75), is satisfied if and only if the following condition holds:

$$\delta + \alpha_1 \widetilde{A} - \frac{1}{2}\alpha_1^2 \sigma_s^2 = r + \frac{1}{2}\left(\frac{r}{\psi}\right)^2 \left(\frac{\gamma}{\psi} - 1 + \vartheta\right) \sigma_s^2 > 0.$$

Given the parameter values we consider in the text, it is obvious that the TVC is always satisfied in both the FI-RE and RB models. It is straightforward to show that the TVC still holds under the distorted model in which $\widetilde{A} = -\frac{\psi(\delta - r)}{r} + \frac{1}{2} r \widetilde{\gamma} \sigma_s^2 - \frac{r}{\psi}\vartheta \sigma_s^2$ for plausible values of ϑ.

7.4 Solving the RU-RB Model with a Risky Asset

The robust HJB equation for the RU-RB model with multiple financial assets can be written as:

$$\delta f\left(J_t\right) = \max_c \min_\upsilon \left\{\delta f\left(c_t\right) + f'\left(U_t\right) A\left(r s_t - c_t + \pi \chi_t - \frac{1}{2}\gamma A \sigma^2 + \upsilon_t \sigma^2\right) + \frac{1}{2\vartheta_t}\upsilon_t^2 \sigma^2\right\}, \tag{77}$$

subject to the distorting equation, (49). Solving first for the infimization part of the problem yields

$$\upsilon^*(s_t) = -\vartheta_t A f'(J_t).$$

Given that $\vartheta(s_t) > 0$, the perturbation adds a negative drift term to the state transition equation because $f'(J_t) > 0$. Substituting it into the above HJB equation yields:

$$\delta f(J_t) = \max_c \left\{ \delta f(c_t) + f'(U_t) A \left(r s_t - c_t + \pi \alpha_t - \frac{1}{2}\gamma A \sigma^2 - \vartheta_t A f'(U_t) \sigma^2 \right) + \frac{1}{2\vartheta_t} \left(\vartheta_t A f'(U_t) \right)^2 \sigma^2 \right\}$$

Following Uppal and Wang (2003) and Maenhout (2004), we assume that $\vartheta_t = -\vartheta / f(U_t)$. The HJB equation reduces to

$$\delta f(U_t) = \max_c \left\{ \delta f(c_t) + f'(U_t) A \left(r s_t - c_t + \pi \alpha_t - \frac{1}{2}\gamma A \sigma^2 + \frac{\vartheta f'(U_t)}{f(U_t)} A \sigma^2 \right) - \frac{\vartheta \left(f'(U_t) \right)^2}{2 f(J_t)} A^2 \sigma^2 \right\}.$$

Using the fact that $f(U_t) = (-\psi)\exp(-U_t/\psi)$, the HJB reduces to

$$\delta f(U_t) = \max_c \left\{ \delta f(c_t) + f'(U_t) A \left(r s_t - c_t + \pi \alpha_t - \frac{1}{2}\left(\gamma + \frac{\vartheta}{\psi} \right) A \sigma^2 + \frac{\vartheta f'(U_t)}{f(U_t)} A \sigma^2 \right) \right\}.$$

The FOC for c is then

$$\delta f'(c_t) = f'(J_t) A,$$

which implies that

$$c_t = -\psi \ln\left(\frac{A}{\delta} \right) + (A s_t + A_0). \tag{78}$$

The FOC for α_t is

$$\alpha_t = \frac{\pi}{r(\gamma + \vartheta/\psi)\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \tag{79}$$

which is just (53). Substituting this expression for c back to the Bellman equation and matching the coefficients, we have:

$$A = r \tag{80}$$

and

$$A_0 = \psi\left(\frac{\delta}{r} - 1 \right) - \frac{1}{2} r \widetilde{\gamma} \left(1 - \rho_{ye}^2 \right) \sigma_s^2 + \frac{\pi^2}{2 r \widetilde{\gamma} \sigma_e^2} - \frac{\pi \rho_{ye} \sigma_s \sigma_e}{\sigma_e^2} + \psi \ln\left(\frac{r}{\delta} \right). \tag{81}$$

Substituting these coefficients into (78) gives the consumption function, (52) in the main text.

Given the optimal consumption-portfolio rules, the individual saving function can be written as

$$
\begin{aligned}
d_t^* &= ra_t + y_t - c_t^* + \pi\alpha^* \\
&= ra_t + y_t - \left(rs_t + \Psi - \Gamma - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} + \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2} \right) + \pi \left(\frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} \right) \\
&= \left[r\left(a_t + \frac{1}{r+\rho_1} y_t + \frac{\rho_1}{r(r+\rho_1)}\overline{y} \right) - r\left(\frac{1}{r+\rho_1} y_t + \frac{\rho_1}{r(r+\rho_1)}\overline{y} \right) \right] + y_t \\
&- \left(rs_t + \Psi - \Gamma - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} + \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2} \right) + \pi \left(\frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} \right) \\
&= \frac{\rho}{r+\rho}\left(y_t - \overline{y} \right) + \Gamma - \Psi + \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} - \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2} + \pi \left(\frac{\pi}{r\widetilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} \right) \\
&= \frac{\rho}{r+\rho}\left(y_t - \overline{y} \right) + \Gamma - \Psi + \Pi,
\end{aligned}
$$

where $\Pi = \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}$.

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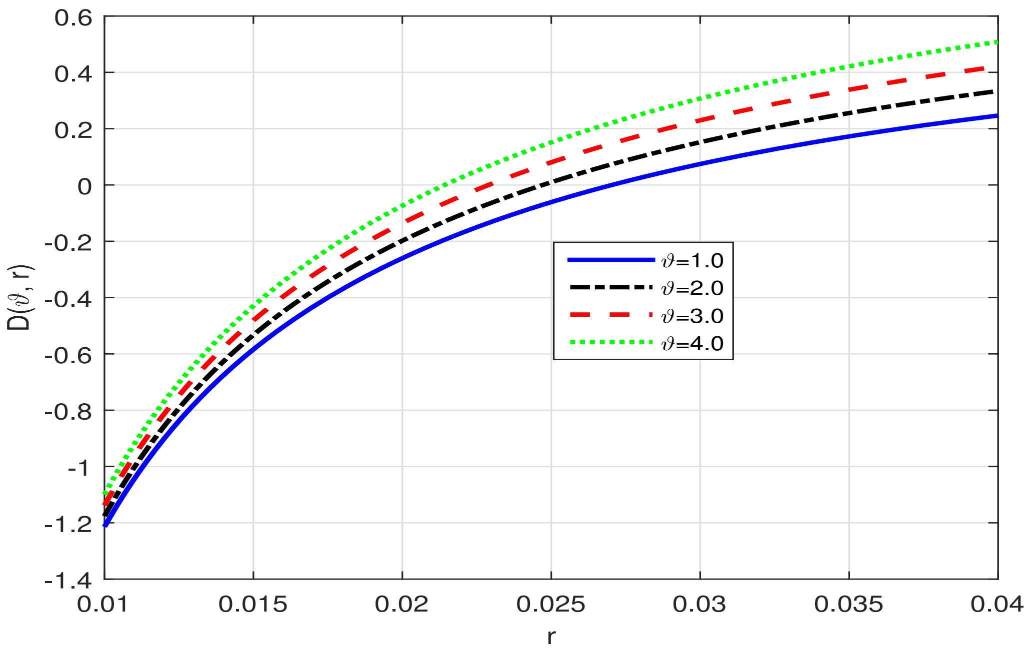

Figure 1: Effects of RB on Aggregate Savings

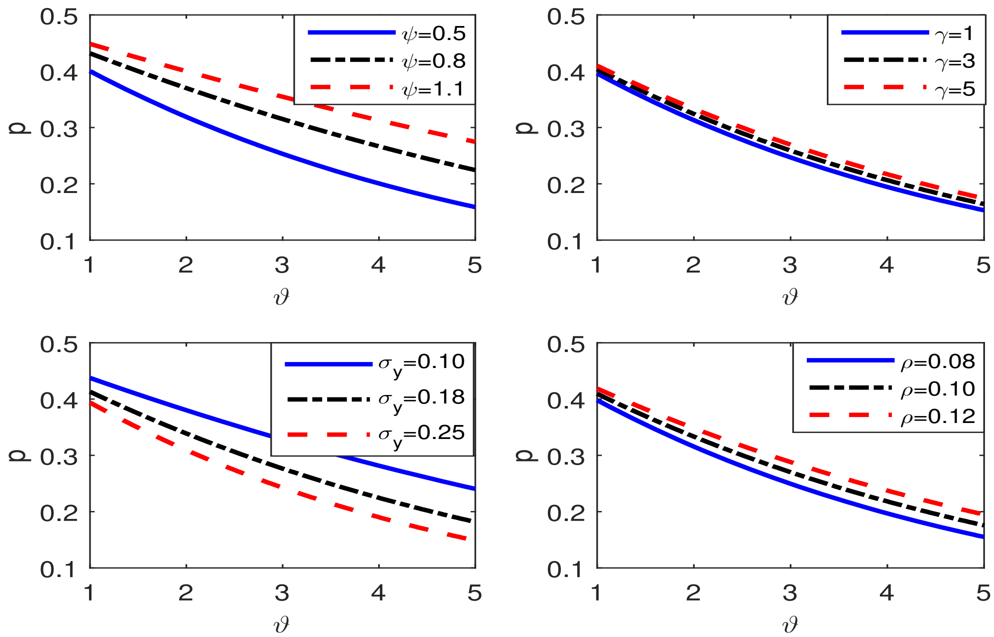

Figure 2: Relationship between ϑ and p

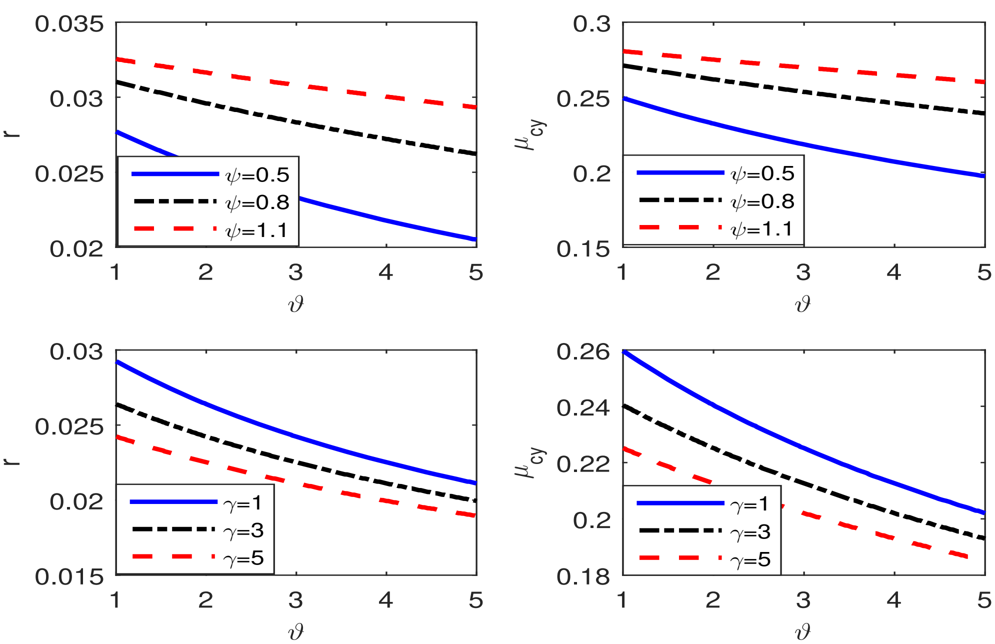

Figure 3: Effects of RB on the Interest Rate and Consumption Volatility

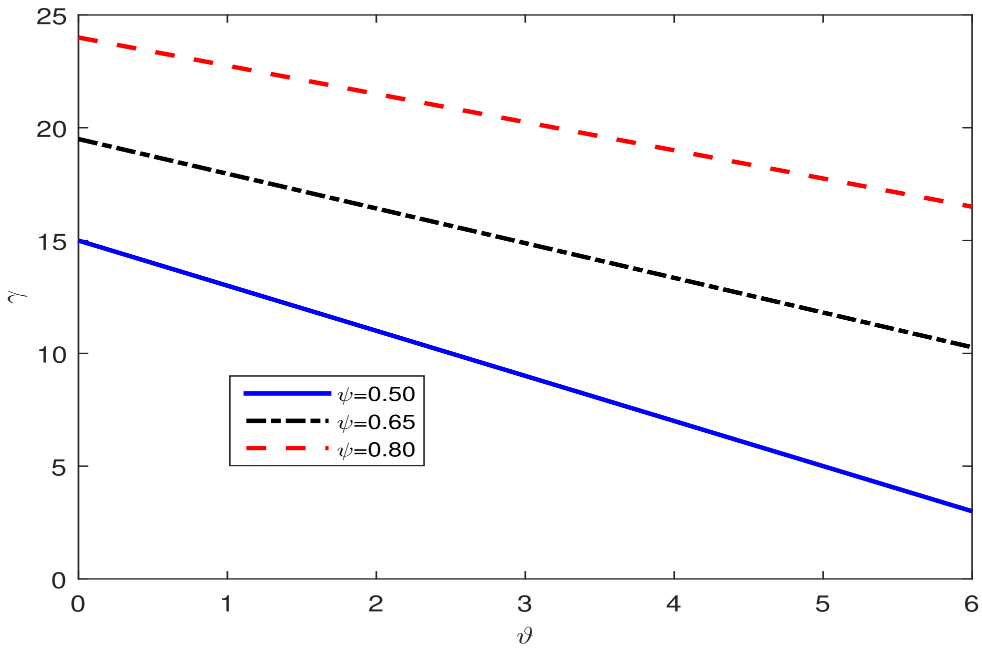

Figure 4: Relationship between γ and ϑ.

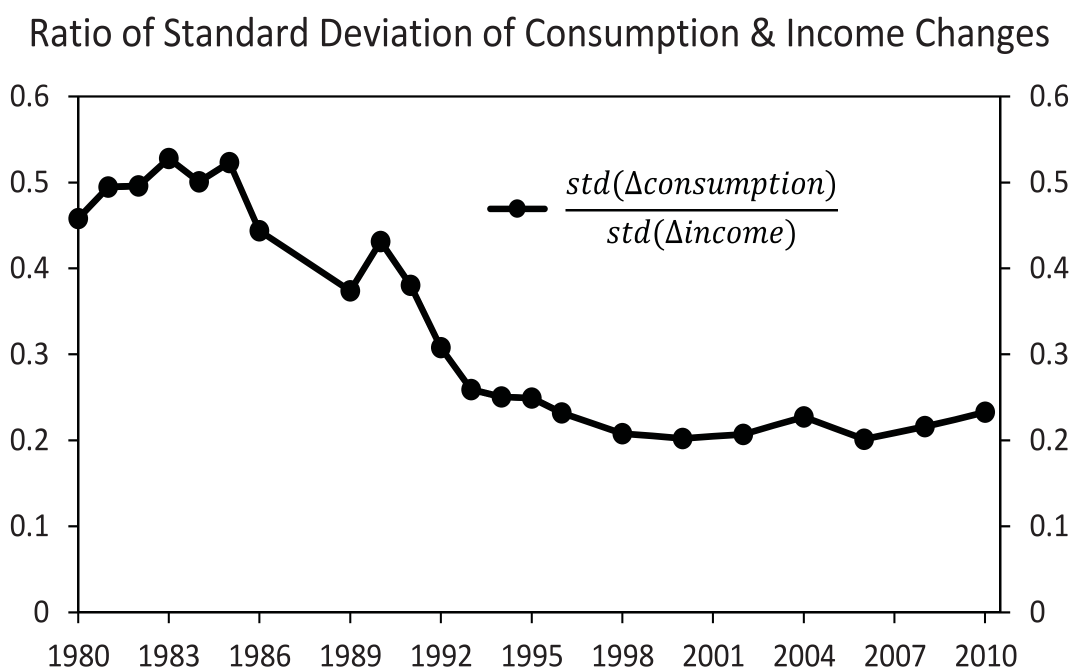

Figure 5: Relative Consumption Dispersion

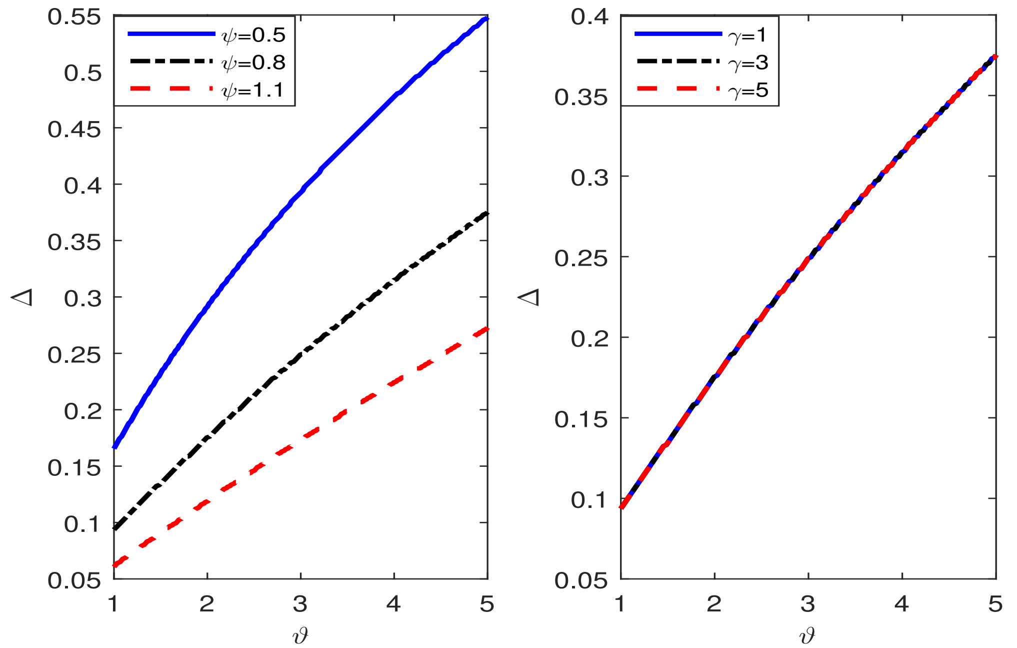

Figure 6: Effects of RB on the Welfare Cost

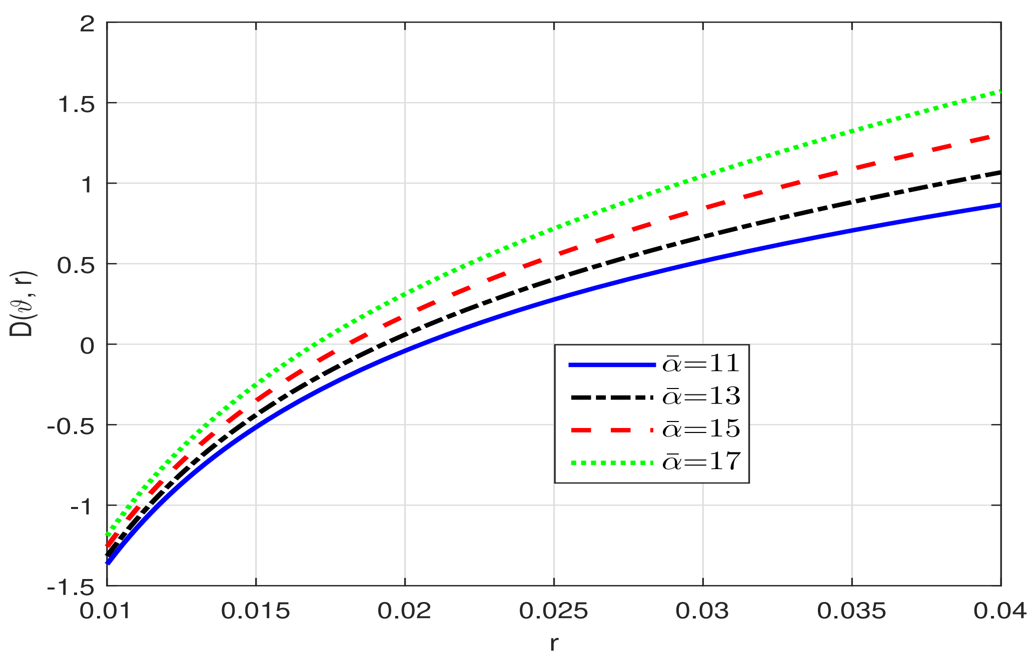

Figure 7: Effects of RB on Aggregate Savings

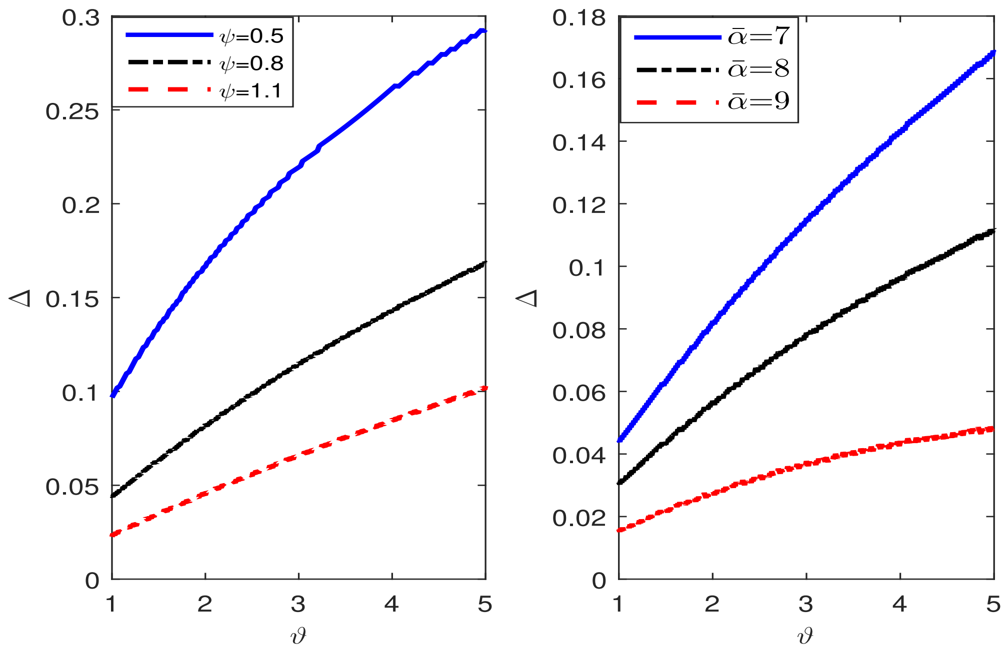

Figure 8: Effects of RB on the Welfare Cost

Table 1: Measures of the Risk Free Rate

	Three-month Nominal T-Bond	One-year Nominal T-Bond
CPI Inflation (1981 – 2010)	1.87%	2.33%
PCE Inflation (1981 – 2010)	1.96%	2.42%
CPI Inflation (1981 – 2015)	1.37%	1.78%
PCE Inflation (1981 – 2015)	1.75%	2.16%

Table 2: Estimation and Calibration Results

Labor Income	Parameter	Values
Discrete time specification		
constant	ϕ_0	0.0005
persistence	ϕ_1	0.919
std. of shock	σ	0.175
Continuous-time specification		
persistence	ρ	0.082
std. of income changes	σ_y	0.182

Table 3: Model Comparison with Key Parameter Values

Parameters Moments	Data	FI ($\psi = 0.55$)	RB (Benchmark) ($\psi = 0.55$)	RB ($\overline{\alpha} > 0$) ($\psi = 0.55$)	RB ($\overline{\alpha} > 0$) ($\psi = 0.55$)
γ		15	5	2.5	1.5
ϑ		0	5	3.1	2.8
p		0.5	0.175	0.295	0.313
$r^*(\%)$	1.91	1.91	1.91	1.91	1.91
μ	0.31 or 0.37	0.19	0.19	0.31	0.37