Classical Conformal Blocks and Accessory Parameters from Isomonodromic Deformations

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Abstract: Classical conformal blocks appear in the large central charge limit of 2D Virasoro conformal blocks. In the AdS_3/CFT_2 correspondence, they are related to classical bulk actions and used to calculate entanglement entropy and geodesic lengths. In this work, we discuss the identification of classical conformal blocks and the Painlevé VI action showing how isomonodromic deformations naturally appear in this context. We recover the accessory parameter expansion of Heun's equation from the isomonodromic τ-function. We also discuss how the $c = 1$ expansion of the τ-function leads to a novel approach to calculate the 4-point classical conformal block.

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Contents

1 Introduction

Classical conformal blocks [1–3] are essential pieces of the holographic duality between AdS_3 gravity and 2D conformal field theory (CFT) [4–6]. Holographic CFTs are assumed to have a *sparse light spectrum* and to contain only correlators dominated by the identity channel in the semiclassical limit [5, 7], usually called classical vacuum blocks. These blocks can be used to study thermal aspects of 3D gravity [6, 8], holographic entanglement entropy [5, 9], to calculate bulk geodesic lengths [10–14] and Lyapunov exponents of out-of-time-order correlators [15].

Classical conformal blocks are defined by the conformal block exponentiation conjecture in the large central charge limit[1] [1, 6, 16]

$$\langle V_{\Delta_0}(0)V_{\Delta_x}(x)\Pi_\Delta V_{\Delta_1}(1)V_{\Delta_\infty}(\infty)\rangle_{c\to\infty} \sim \exp\left(\frac{c}{6}f_\delta(\delta_0,\delta_x,\delta_1,\delta_\infty;x)\right). \tag{1.1}$$

We denote the central charge by c and $\Delta_i = \frac{c}{6}\delta_i$, $i = 0, x, 1, \infty$, are the conformal dimensions of the chiral primary operators V_{Δ_i}, with δ_i being the classical dimensions. Π_Δ is the projection operator to the intermediate channel with weight $\Delta = \frac{c}{6}\delta$. In broad terms, we say that an operator $\mathcal{O}$ is *light* if its weight $\Delta_\mathcal{O} \ll c$ and it is called *heavy* if $\Delta_\mathcal{O} \sim c$ as $c \to \infty$. The function $f_\delta(\{\delta_i\}; x)$ is called the *classical conformal block* [2, 3]. No closed CFT expression is known for this special function. It can be written as a series expansion in x using a direct CFT approach [18] or Zamolodchikov's recurrence formula [1, 19]. The CFT approach quickly gets too cumbersome to find the explicit coefficients at higher orders in x. A formal resummation of the recurrence formula was presented in [20], but the full classical conformal block is still out of reach.

A more promising direction is to obtain classical conformal blocks via the AGT correspondence [21]. The Nekrasov partition function Z_{Nek} encodes information about the moduli space of vacua and its non-perturbative corrections in supersymmetric gauge theories [22]. According to the AGT correspondence, for a certain class of $\mathcal{N} = 2$ SUSY theories, Z_{Nek} is identified with a 2D Liouville conformal block. One can show that the so-called Nekrasov-Shatashvili limit [23] is equivalent to the large central charge limit of Liouville theory. This fact was used in [24, 25] to obtain an expression for the classical conformal block in terms of the $\mathcal{N} = 2$ *twisted superpotential*, calculated at the saddle-point of the partition function. The saddle-point condition then has to be solved order-by-order in x. The twisted superpotential can be understood in terms of the symplectic geometry of the moduli space of $\mathrm{SL}(2,\mathbb{C})$ flat-connections and the Bethe/gauge correspondence [26], which gives extra hints on the deeper mathematical structure of classical conformal blocks. For a review on exact results in $\mathcal{N} = 2$ field theories, see [27].

In a parallel development, Litvinov *et al* [3] discussed how the 4-point classical conformal block is related to the classical action of the Painlevé VI (PVI) equation [28]. In this approach, the derivative of the PVI action, evaluated on a PVI solution with certain boundary conditions, gives the accessory parameter of a Fuchsian differential equation with 4 regular singular points, also known as Heun's equation [29]. The Heun equation is obtained by the

[1]There is no rigorous CFT proof of this conjecture, only plausibility and numerical arguments [1]. It is also compatible with classical saddle-point arguments in Liouville theory [16, 17].

classical limit of the 5-point conformal block with a level-2 *light* degenerate insertion

$$\langle\varphi_L(z)V_{\Delta_0}(0)V_{\Delta_x}(x)\Pi_\Delta V_{\Delta_1}(1)V_{\Delta_\infty}(\infty)\rangle_{c\to\infty}\sim\psi(z,x)\exp\left(\frac{c}{6}f_\delta(\{\delta_i\};x)\right). \tag{1.2}$$

The level-2 null vector equation for $\varphi_L(z)$, also known as level-2 BPZ equation [18], reduces to the normal form of Heun's equation

$$\left[\partial_z^2-\frac{t(t-1)H_x}{z(z-1)(z-x)}+\frac{\delta_0}{z^2}+\frac{\delta_1}{(z-1)^2}+\frac{\delta_x}{(z-x)^2}+\frac{\delta_\infty-\delta_0-\delta_1-\delta_x}{z(z-1)}\right]\psi(z,x)=0, \tag{1.3}$$

where the accessory parameter is given by

$$H_x=-\partial_x f_\delta(\{\delta_i\};x). \tag{1.4}$$

On the other hand, the semiclassical limit of the 5-point conformal block with a level-2 heavy degenerate insertion

$$\langle\varphi_H(\lambda)V_{\Delta_0}(0)V_{\Delta_x}(x)\Pi_\Delta V_{\Delta_1}(1)V_{\Delta_\infty}(\infty)\rangle_{c\to\infty}\sim\exp\left(\frac{c}{6}S_\delta(\{\delta_i\};\lambda,x)\right) \tag{1.5}$$

obeys a BPZ equation equivalent to the Hamilton–Jacobi equation of the PVI action. This means that the BPZ equation implies that λ must be a solution of the PVI equation.

In order to recover the 4-point classical conformal block from the PVI action, the authors of [3] set $\lambda=\infty$ in (1.5) and obtain an integral formula for the classical block. Taking the derivative of this formula then leads to the accessory parameter of Heun's equation (1.4) in terms of the initial condition for λ, fixed by $\lambda=\infty$. With a clever usage of a double series expansion of the PVI solution [28], the authors of [3] managed to solve the condition $\lambda=\infty$ order by order in x and then substituted the result into the accessory parameter formula. However, their procedure relies on substituting the double series expansion into the PVI equation, which is a complicated second order non-linear differential equation, to obtain the doubles series expansion terms also order by order in x.

In this paper, we show how the isomonodromic τ-function [30–32], also known as PVI τ-function [33, 34], can be used to find the accessory parameter expansion discussed in [3] in a more straightforward way. Both PVI solutions and the accessory parameter can be written in terms of the τ-function. Since the works of Sato, Jimbo and Miwa [35–39], it is known that the PVI τ-function is related to a $c=1$ correlator of monodromy fields. In fact, the isomonodromic approach effectively solves the Riemann-Hilbert problem of $\mathrm{SL}(2,\mathbb{C})$ Fuchsian systems [30–32]. For a long time, only the asymptotics of the τ-function was known, limiting its scope of applications. Its full expansion was constructed only recently in [34] and proved in [40]. This expansion is given by a linear combination of $c=1$ conformal blocks, which are written in closed form via AGT correspondence. There are two integration constants (σ,s)

for the τ-function, labeling irreducible representations of the 4-point monodromy group, which we review in section 3.3. The relevant expansion in terms of $c = 1$ conformal blocks [34] is presented in 4.1. This τ-function expansion is what allows us to solve the initial condition for the isomonodromic flow and then find the accessory parameter expansion.

The way we solve the accessory parameter expansion clarifies two important things. First, the initial condition used by [3] can be uniquely defined in terms of isomonodromic deformations. Second, the recent result [34] on the $c = 1$ expansion of the PVI τ-function allow us to present the accessory parameter expansion in a more systematic way, providing a practical algorithm on how to fully solve this problem.

Let us now give the outline of this paper. In section 2, we review the relationship between the semiclassical limit of BPZ equations and Fuchsian equations. We introduce a slightly more general derivation than [3], using a 6-point conformal block with 2 degenerate insertions instead of a 5-point conformal block. This introduces the relationship between the classical conformal block (1.5) and the Painlevé VI action.

In Section 3, we review the standard setup of isomonodromic deformations, the τ-function definition and the associated ordinary differential equation (ODE) with one extra apparent singularity. We then review the connection between the semiclassical limit of CFT correlators and isomonodromic deformations [41, 42]. We move on to discuss the relationship between Fuchsian equations and the monodromy group of the 4-punctured sphere, summarizing how the different parameters in this paper relate with each other. Finally, we show how the BPZ equations of section 2 can be encoded in a Fuchsian system when $c = 1$. The conclusion is that the monodromy data of a $c = \infty$ conformal block can be encoded in a $c = 1$ Fuchsian system. This makes the connection between classical conformal blocks, Painlevé VI and isomonodromic deformations explicit.

In section 4, we present our algorithm to calculate the accessory parameter (1.4) using the isomonodromic τ-function [33, 34]. The algorithm consists of three steps, described in detail in this section. The crucial step is to solve a special initial condition to the isomonodromic flow. We can only solve this condition order-by-order in x, in a similar fashion to [3] or the AGT approach of [25]. This constrains the general moduli space of the monodromy group to a subspace with only one parameter, the composite monodromy σ. Our approach gives new analytic insights on classical conformal blocks compared to [3].

In Section 5, we show how the PVI action can be written only in terms of τ-functions. In principle, this gives a formula for the 4-point classical conformal block, up to the solution of the appropriate PVI initial condition. However, as we discuss in the linear dilaton case, the PVI τ-function simplifies in certain special cases [34, 43]. We leave the study of the different special cases for future work.

The relationship between isomonodromic deformations and the CFT semiclassical limit has been previously explored in [41, 42]. Although these papers recognize the importance of isomonodromic deformations, the relevance of the τ-function and its detailed implementation has only been discussed here. Therefore, our main contribution is to give a unified prescription on how to use the isomonodromic τ-function to obtain the accessory parameter of Heun's equation and the associated classical conformal block in 2D CFT.

2 Classical Conformal Blocks and Accessory Parameters

In this section, we review how the semiclassical limit of a special 6-point conformal block leads to a Fuchsian equation with 4 singular points and one apparent singularity. The method generalizes to arbitrary n-point conformal blocks with an appropriate number of extra degenerate insertions [41, 42].

Our derivation relies only on the 2D conformal symmetry and the definition of conformal blocks. Although we use the standard Liouville notation for the conformal dimensions and central charge, we do not make any particular assumption about the spectrum. This comes *a posteriori* and it is the main point of the bootstrap program [2, 6, 7, 44–47]. For reviews on Liouville theory and CFT, we suggest [48, 49].

A chiral primary operator $V_{\Delta(P)}$ has conformal dimension

$$\Delta(P) = \frac{Q^2}{4} + P^2, \tag{2.1}$$

where P is the momentum of the operator and Q parametrizes the central charge as

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}, \tag{2.2}$$

with $b \in \mathbb{C}$. The spectrum is dual under $b \to 1/b$, and we choose $b \to 0$ to denote the semiclassical limit.

Let us consider the chiral 6-point correlator with two degenerate insertions

$$\langle \varphi_{\Delta_L}(z)\varphi_{\Delta_H}(\lambda)V_{\Delta_1}(z_1)V_{\Delta_2}(z_2)V_{\Delta_3}(z_3)V_{\Delta_4}(z_4)\rangle = \sum_{\boldsymbol{P}} \mathcal{C}_{\boldsymbol{P}}\, \mathcal{F}_{\boldsymbol{P}}(\Delta_L, \Delta_H, \boldsymbol{\Delta}|z, \lambda, \boldsymbol{z}), \tag{2.3}$$

where φ_{Δ_L} and φ_{Δ_H} stands for *light* and *heavy* level-2 degenerate operators, respectively. $\boldsymbol{\Delta} = (\Delta_1, \ldots, \Delta_4)$ stands for the conformal dimensions of generic heavy fields at positions $\boldsymbol{z} = (z_1, \cdots, z_4)$. The conformal dimensions can be written as

$$\Delta_L = -\frac{1}{2} - \frac{3b^2}{4}, \quad \Delta_H = -\frac{1}{2} - \frac{3}{4b^2}, \quad \Delta_i \equiv \Delta(P_i) = \frac{Q^2}{4} + P_i^2, \; (i = 1, \ldots, 4). \tag{2.4}$$

$\mathcal{C}_{\boldsymbol{P}}$ represents the appropriate products of structure constants. With the ordering given in figure 1, the conformal block $\mathcal{F}_{\boldsymbol{P}}(\Delta_L, \Delta_H, \boldsymbol{\Delta}|z, \lambda, \boldsymbol{z})$ is labeled by the intermediate momenta $\boldsymbol{P} = (P, P + \frac{is_1}{2b}, P + \frac{is_1}{2b} + \frac{is_2 b}{2})$, $s_1, s_2 = \pm 1$.

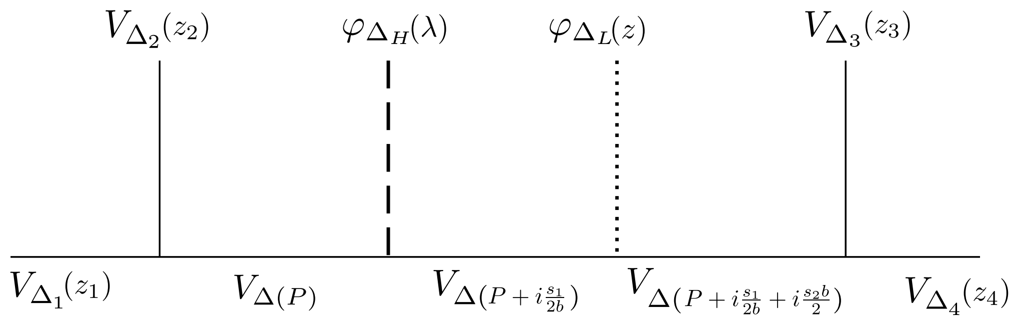

Figure 1. 6-point conformal block with one heavy and one light insertion. The intermediate momenta are labeled by P and two integers $s_1, s_2 = \pm 1$.

The correlator (2.3) obeys a light BPZ equation in the variable z and a heavy BPZ equation in the variable λ. Because of linearity, the conformal blocks obey the same equations

$$\left[\frac{1}{b^2}\partial_z^2 + \frac{\Delta_H}{(z-\lambda)^2} + \frac{\partial_\lambda}{z-\lambda} + \sum_{i=1}^{4}\left(\frac{\Delta_i}{(z-z_i)^2} + \frac{\partial_{z_i}}{z-z_i}\right)\right]\mathcal{F}_{\boldsymbol{P}}(z,\lambda,\boldsymbol{z}) = 0, \tag{2.5a}$$

$$\left[b^2\partial_\lambda^2 + \frac{\Delta_L}{(\lambda-z)^2} + \frac{\partial_z}{\lambda-z} + \sum_{i=1}^{4}\left(\frac{\Delta_i}{(\lambda-z_i)^2} + \frac{\partial_{z_i}}{\lambda-z_i}\right)\right]\mathcal{F}_{\boldsymbol{P}}(z,\lambda,\boldsymbol{z}) = 0, \tag{2.5b}$$

where we omitted the conformal dimensions in $\mathcal{F}_{\boldsymbol{P}}(z,\lambda,\boldsymbol{z})$ for convenience. We can simplify these equations using global conformal transformations and the Ward identity

$$\begin{aligned}&\langle T(w)\varphi_L(z)\varphi_H(\lambda)V_{\Delta_1}(z_1)\cdots V_{\Delta_4}(z_4)\rangle = \\ &= \sum_{i=L,H,1,\ldots,4}\left(\frac{\Delta_i}{(w-z_i)^2} + \frac{\partial_{z_i}}{w-z_i}\right)\langle\varphi_L(z)\varphi_H(\lambda)V_{\Delta_1}(z_1)\cdots V_{\Delta_4}(z_4)\rangle,\end{aligned} \tag{2.6}$$

where $T(w)$ is the 2D stress tensor, with $z_L = z$ and $z_H = \lambda$. A straightforward consequence of the asymptotic behavior $T(w) \sim w^{-4}$ as $w \to \infty$ is that

$$\oint_{w=\infty} dw\ \epsilon(w)\langle T(w)\varphi_L(z)\varphi_H(\lambda)V_1(z_1)\cdots V_4(z_4)\rangle = 0 \tag{2.7}$$

with $\epsilon(w) = \prod_{i=1}^{3}(w-z_i)/(w-z)$ [48]. Using (2.6) and (2.7) in (2.5a), we choose $z_1 = 0, z_2 = t, z_3 = 1, z_4 = \infty$ and relabeling the Δ's accordingly, we get

$$\begin{aligned}&\left[b^{-2}\partial_z^2 - \left(\frac{1}{z} + \frac{1}{z-1}\right)\partial_z + \frac{\Delta_0}{z^2} + \frac{\Delta_1}{(z-1)^2} + \frac{\Delta_t}{(z-t)^2} + \frac{\Delta_H}{(z-\lambda)^2} + \frac{t(t-1)\partial_t}{z(z-1)(z-t)} + \right.\\ &\left. + \frac{\lambda(\lambda-1)\partial_\lambda}{z(z-1)(z-\lambda)} + \frac{\Delta_\infty - \Delta_L - \Delta_H - \Delta_0 - \Delta_1 - \Delta_t}{z(z-1)}\right]\mathcal{F}_{\boldsymbol{P}}(z,\lambda,t) = 0.\end{aligned} \tag{2.8}$$

Let us now analyze the semiclassical limit. Assuming $P_k = i\theta_k/b$ and $P = i\sigma/b$ as $b \to 0$, we have

$$\Delta_L \to -\frac{1}{2}, \quad \Delta_H \to -\frac{3}{4b^2}, \quad \Delta(P) \to \frac{\delta_\sigma}{b^2}, \quad \Delta_k \to \frac{\delta_k}{b^2}, \quad (k = 0, x, 1, \infty), \tag{2.9}$$

where

$$\delta_\sigma = \frac{1}{4} - \sigma^2, \quad \delta_k = \frac{1}{4} - \theta_k^2, \quad (k = 0, x, 1, \infty). \tag{2.10}$$

Assuming heavy-light factorization and exponentiation[2] [2, 3, 16], we write the semiclassical limit of the 6-point conformal block when $b \to 0$ as

$$\mathcal{F}_\sigma^\pm(z, \lambda, t) \sim \psi(z, \lambda, t) \exp\left(\frac{1}{b^2} S_\sigma^\pm(\lambda, t)\right). \tag{2.11}$$

We simplified the notation to $\boldsymbol{P} \to (\sigma, \pm)$ because the light field $\varphi_L(z)$ does not contribute to the intermediate momenta in the semiclassical limit. The function $\psi(z, \lambda, t)$ encodes the fusion (monodromy) information of the light field, as we are going to see below. Fusing the light degenerate field with any of the other fields, we end up with the semiclassical limit of the 5-point block with all insertions being heavy

$$\mathcal{F}_\sigma^\pm(\lambda, t) \sim \exp\left(\frac{1}{b^2} S_\sigma^\pm(\lambda, t)\right). \tag{2.12}$$

Substituting (2.11) in (2.8) gives

$$\begin{aligned}\Bigg[\partial_z^2 &+ \frac{t(t-1)C_t}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)C_\lambda}{z(z-1)(z-\lambda)} + \\ &+ \frac{\delta_0}{z^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_t}{(z-t)^2} - \frac{\frac{3}{4}}{(z-\lambda)^2} + \frac{\delta_\infty - \delta_0 - \delta_1 - \delta_t + \frac{3}{4}}{z(z-1)}\Bigg]\psi(z, \lambda, t) = 0,\end{aligned} \tag{2.13}$$

which is a Fuchsian equation with 5 singular points and *accessory parameters*

$$C_t = \partial_t S_\sigma^\pm, \quad C_\lambda = \partial_\lambda S_\sigma^\pm. \tag{2.14}$$

Following the same procedure for (2.5b), we get

$$\begin{aligned}\Bigg[b^2\partial_\lambda^2 &- \left(\frac{1}{\lambda} + \frac{1}{\lambda-1}\right)\partial_\lambda + \frac{\Delta_0}{\lambda^2} + \frac{\Delta_1}{(\lambda-1)^2} + \frac{\Delta_t}{(\lambda-t)^2} + \frac{\Delta_L}{(\lambda-z)^2} + \frac{t(t-1)\partial_t}{\lambda(\lambda-1)(\lambda-t)} + \\ &+ \frac{z(z-1)\partial_z}{\lambda(\lambda-1)(\lambda-z)} + \frac{\Delta_\infty - \Delta_L - \Delta_H - \Delta_0 - \Delta_1 - \Delta_t}{\lambda(\lambda-1)}\Bigg]\mathcal{F}_{\boldsymbol{P}}(z, \lambda, t) = 0.\end{aligned} \tag{2.15}$$

[2]The exponentiation can be understood in Liouville field theory. In principle, the semiclassical limit of a correlator is given by a classical saddle-point of the Liouville action, if this saddle-point is unique.

The semiclassical limit of this equation gives a constraint on the accessory parameters (2.14)

$$(\partial_\lambda S_\sigma^\pm)^2 - \left(\frac{1}{\lambda}+\frac{1}{\lambda-1}\right)\partial_\lambda S_\sigma^\pm + \frac{t(t-1)\partial_t S_\sigma^\pm}{\lambda(\lambda-1)(\lambda-t)} + \\ + \frac{\delta_0}{\lambda^2} + \frac{\delta_1}{(\lambda-1)^2} + \frac{\delta_t}{(\lambda-t)^2} + \frac{\delta_\infty-\delta_0-\delta_1-\delta_t+\frac{3}{4}}{\lambda(\lambda-1)} = 0. \quad (2.16)$$

This is exactly the condition for $z=\lambda$ to be an apparent singularity of (2.13) [41, 42, 50]. This means that $\psi(z,\lambda,t)$ has integer monodromy around $z=\lambda$ but no logarithmic behavior. Thus $z=\lambda$ is *not* a singular point of the solution. Moreover, (2.16) can be interpreted as a Hamilton–Jacobi equation for $S_\sigma^\pm(\lambda,t)$

$$\frac{\partial S_\sigma^\pm}{\partial t} + H\left(\lambda, \frac{\partial S_\sigma^\pm}{\partial\lambda}, t\right) = 0, \quad (2.17)$$

where

$$H(\lambda,p,t) = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left[p^2 - \left(\frac{1}{\lambda}+\frac{1}{\lambda-1}\right)p + \right. \\ \left. + \frac{\delta_0}{\lambda^2} + \frac{\delta_1}{(\lambda-1)^2} + \frac{\delta_t}{(\lambda-t)^2} + \frac{\delta_\infty-\delta_0-\delta_1-\delta_t+\frac{3}{4}}{\lambda(\lambda-1)}\right]. \quad (2.18)$$

The equation of motion for $\lambda(t)$ obtained from this Hamiltonian is the PVI equation, therefore $S_\sigma^\pm(\lambda,t)$ is the PVI action. The heavy BPZ equation gives a saddle-point condition for $S_\sigma^\pm(\lambda,t)$ [3].

If we define the Hamiltonian system $(\lambda(t),p(t))$ evolving under the PVI Hamiltonian (2.18), it is possible to show that the monodromy data of the Fuchsian equation (2.13) *does not change* as we change t in the complex plane. Therefore, (2.13) is the *isomonodromic deformation* of a 4-point Fuchsian equation, a deformed Heun's equation, with $z=\lambda$ being an apparent singularity and not contributing to the monodromy data [41, 42, 50]. This means that isomonodromic deformations naturally emerge in CFT. We will review the standard setup of isomonodromic deformations in the next section. We also discuss how isomonodromic deformations relate the monodromy group of the 4-punctured sphere and the moduli space of Fuchsian equations (also called *opers* in the literature [51]).

3 Isomonodromic Deformations and the Semiclassical Limit

In the previous section, we saw that the conformal block exponentiation (2.11) effectively transforms the light BPZ equation into a linear ODE for $\psi(z,\lambda,t)$. The classical conformal block $S_\sigma^\pm(\lambda,t)$ turns out to be the PVI action. In this section, we review the isomonodromic setup from a Fuchsian system and how to obtain the semiclassical equations (2.13) and (2.17) in this approach. We will then see that (2.17) is equivalent to the definition of the isomonodromic τ-function when (λ,p) are PVI solutions. We also make a digression about the monodromy group and the moduli space of flat connections, summarizing how the different objects introduced in this paper can be labeled by the two PVI integration constants (σ,s). Finally, we finish this section arguing that the monodromy data of solutions of both heavy and light BPZ equations can be encoded in a Fuchsian system. We show that this can be consistently done only if $c=1$. This has a two-fold purpose: first, to argue that the isomonodromic τ-function can be understood as a $c=1$ correlator and, second, to show that the monodromy data of $c=\infty$ conformal blocks and $c=1$ correlators can be encoded in the same Fuchsian system.

3.1 Isomonodromic Deformations and the Garnier System

In this section, we recover the standard $c=1$ setup for isomonodromic deformations of Fuchsian systems [30–32, 34]. Let us start with the following Fuchsian system for the vector $\Psi(\boldsymbol{a}|z,\lambda)=(\psi_1(\boldsymbol{a}|z,\lambda),\ \psi_2(\boldsymbol{a}|z,\lambda))^T$

$$\partial_z\Psi=A(z)\Psi, \tag{3.1a}$$

$$\partial_\lambda\Psi=-A(\lambda)\Psi, \tag{3.1b}$$

$$\partial_{a_i}\Psi=-\left(\frac{\lambda-z}{\lambda-a_i}\right)\frac{A_i}{z-a_i}\Psi, \tag{3.1c}$$

where the A_i are $\mathfrak{gl}(2,\mathbb{C})$ matrices with

$$\operatorname{Tr}A_i=2\theta_i,\quad \operatorname{Tr}A_i^2=0. \tag{3.2}$$

We call this choice of $\operatorname{Tr}A_i$ the *canonical gauge*. The Fuchsian system above has n singular points $\boldsymbol{a}=(a_1,\ldots,a_n)$ and corresponding monodromy coefficients $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_n)$ (see section 3.3 for details on the monodromy group). The integrability conditions of (3.1) are the *Schlesinger equations* [34, 52]

$$\partial_{a_i}A_j=\frac{\lambda-a_j}{\lambda-a_i}\frac{[A_i,A_j]}{a_i-a_j},\qquad i\neq j, \tag{3.3a}$$

$$\partial_{a_j}A_j=-\sum_{i\neq j}\frac{[A_i,A_j]}{a_i-a_j},\qquad \partial_\lambda A_j=-\sum_{i\neq j}\frac{[A_i,A_j]}{\lambda-a_i}. \tag{3.3b}$$

These equations represent *isomonodromic deformations* of the Fuchsian system (3.1), as they generate a flow changing the positions of the singular points $\boldsymbol{a}$ without changing the monodromies. In fact, taking the trace of equations (3.3), it is clear that $\mathrm{Tr}\, A_i$ do not change under the flow. The isomonodromic τ-function $\tau_S = \tau_S(\boldsymbol{\theta}; \boldsymbol{a})$ is defined by

$$d \log \tau_S = \sum_{i<j}^{n} \mathrm{Tr}(A_i A_j) d \log(a_i - a_j), \tag{3.4}$$

which is a closed 1-form provided that the Schlesinger equations are satisfied. This is the generating function of the isomonodromic Hamiltonians

$$H_{S,i} = \partial_{a_i} \log \tau_S = \sum_{j \neq i} \frac{\mathrm{Tr}(A_i A_j)}{a_i - a_j}, \quad i = 1, \ldots, n. \tag{3.5}$$

One can show from the Schlesinger equations (3.3) that these Hamiltonians generate the isomonodromic flow for $\lambda = \lambda(\boldsymbol{a})$ [30, 50].

Let us focus now on the $n = 4$ case. Applying a Möbius transformation, we fix the singular points to $\boldsymbol{a} = (a_0, a_t, a_1, a_\infty) = (0, t, 1, \infty)$ and monodromy parameters $\boldsymbol{\theta} = (\theta_0, \theta_t, \theta_1, \theta_\infty)$. Then we have the Fuchsian system

$$\partial_z \Psi = A(z)\Psi, \tag{3.6a}$$

$$\partial_\lambda \Psi = -A(\lambda)\Psi, \tag{3.6b}$$

$$\partial_t \Psi = -\left(\frac{\lambda - z}{\lambda - t}\right) \frac{A_t}{z - t} \Psi, \tag{3.6c}$$

with

$$A(z) = \sum_{i=0,1,t} \frac{A_i}{z - a_i}, \quad A_\infty = -\sum_{i=0,1,t} A_i = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \tag{3.7}$$

where $2\theta_\infty = \kappa_1 - \kappa_2 - 1$ and $\kappa_1 + \kappa_2 = -2(\theta_0 + \theta_1 + \theta_t)$. These last conditions can be solved to

$$\kappa_1 = \theta_\infty + \frac{1}{2} - \sum_{i=0,1,t} \theta_i, \quad \kappa_2 = -\theta_\infty - \frac{1}{2} - \sum_{i=0,1,t} \theta_i. \tag{3.8}$$

Notice that we used the $\mathrm{SL}(2,\mathbb{C})$ gauge freedom to fix A_∞ to be in diagonal form. The solution Ψ has monodromies on the complex z-plane given by the eigenvalues of $A_i \sim \mathrm{diag}(2\theta_i, 0)$ (see section 3.3). A convenient parameterization for the A_i was given by [31]

$$A_i = \begin{pmatrix} p_i + 2\theta_i & p_i q_i \\ -\frac{(p_i + 2\theta_i)}{q_i} & -p_i \end{pmatrix}, \quad i = 0, 1, t, \tag{3.9}$$

where p_i and q_i are functions of (λ, t) and the fixed monodromy parameters $\boldsymbol{\theta}$. The diagonal form of A_∞ in (3.7) implies the constraints

$$\sum_{i=0,1,t} p_i = \kappa_2, \quad \sum_{i=0,1,t} p_i q_i = 0, \quad \sum_{i=0,1,t} \frac{(p_i + 2\theta_i)}{q_i} = 0. \tag{3.10}$$

The second equation above implies that $A_{12}(z)$ must have a simple zero in z and, for consistency with (3.6), it has to be at $z=\lambda$

$$A_{12}(z) = k\frac{\lambda - z}{z(z-1)(z-t)}, \quad k \in \mathbb{C}. \tag{3.11}$$

We can then solve for the q_i's in (3.10) via

$$p_i q_i = \operatorname*{Res}_{z=a_i}\left[k\frac{\lambda - z}{z(z-1)(z-t)}\right] = k\frac{\lambda - a_i}{f'(a_i)}, \quad f(z) \equiv z(z-1)(z-t). \tag{3.12}$$

We have only two equations left in (3.10) for the three p_i's, so we introduce the variable

$$\mu = \sum_{i=0,1,t} \frac{p_i + 2\theta_i}{\lambda - a_i} \tag{3.13}$$

and solve the constraints for them in terms of (λ, μ, t). The expressions for the p_i's are not particularly enlightening to display here and can be found in [31].

Let us consider the second order ODE for the first component of $\Psi = (\psi_1\,,\,\psi_2)^T$

$$\partial_z^2\psi_1 - (\operatorname{Tr} A(z) + \partial_z \log A_{12}(z))\partial_z\psi_1 + \left(\det A(z) + A_{11}(z)\partial_z \log \frac{A_{12}(z)}{A_{11}(z)}\right)\psi_1 = 0. \tag{3.14}$$

Writing (p_i, q_i) in terms of (λ, μ, t) and using the parameterization (3.9) above, we get the deformed Heun equation in canonical form

$$\partial_z^2\psi_1 + g_1(z)\partial_z\psi_1 + g_2(z)\psi_1 = 0, \tag{3.15a}$$

$$g_1(z) = \frac{1-2\theta_0}{z} + \frac{1-2\theta_1}{z-1} + \frac{1-2\theta_t}{z-t} - \frac{1}{z-\lambda}, \tag{3.15b}$$

$$g_2(z) = \frac{\kappa_1(\kappa_2+1)}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)}, \tag{3.15c}$$

with the accessory parameter $K = K(\boldsymbol{\theta}; \lambda, \mu, t)$ given by

$$K(\boldsymbol{\theta}; \lambda, \mu, t) \;=\; \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left[\mu^2 - \left(\frac{2\theta_0}{\lambda} + \frac{2\theta_1}{\lambda-1} + \frac{2\theta_t - 1}{\lambda - t}\right)\mu + \frac{\kappa_1(\kappa_2+1)}{\lambda(\lambda-1)}\right]. \tag{3.16}$$

Notice that (3.8) implies in

$$\kappa_1(\kappa_2+1) = \left(\sum_{i=0,1,t}\theta_i - \frac{1}{2}\right)^2 - \theta_\infty^2. \tag{3.17}$$

As we saw above, the integrability conditions for the Fuchsian system are equivalent to the isomonodromic equations (3.3). Using the parameterization above in terms of (λ, μ, t), the isomonodromic equations reduce to the *Garnier system* [50, 53]

$$\dot{\lambda} = \frac{\partial K}{\partial \mu}, \quad \dot{\mu} = -\frac{\partial K}{\partial \lambda}. \tag{3.18}$$

We denote a solution of the isomonodromic flow as $(\lambda(t), \mu(t))$. The second order equation for $\lambda(t)$ is the Painlevé VI equation [31]. The PVI solutions are, in general, transcendental, i.e., cannot be reduced to simple algebraic or special functions. Jimbo has used the isomonodromic technique [33] to find the asymptotics of the τ-function (3.5) and, consequently, of the PVI transcendents, near its critical points $t = 0, 1, \infty$. The full expansion of the PVI τ-function was found only recently in [34]. We will review this formula in section 4.

3.2 Semiclassical BPZ Equations from the Fuchsian System

We claimed in section 2 that a certain heavy-light 6-point correlator naturally encodes isomonodromic equations. To show this explicitly, we obtain the semiclassical Fuchsian equation (2.13) from the Fuchsian system (3.6). Applying the transformation

$$\psi_1(t|z, \lambda) = (z - \lambda)^{\frac{1}{2}} \prod_{i=0,1,t} (z - a_i)^{-\frac{1}{2}+\theta_i} \psi(z, \lambda, t) \tag{3.19}$$

to (3.15), we find the semiclassical Fuchsian equation

$$\partial_z^2 \psi + \left(-\frac{t(t-1)H}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)p}{z(z-1)(z-\lambda)} + \right. \\ \left. + \sum_{i=0,1,t} \frac{\delta_i}{(z-a_i)^2} + \frac{-\frac{3}{4}}{(z-\lambda)^2} + \frac{\delta_\infty - \sum_{i=0,1,t} \delta_i + \frac{3}{4}}{z(z-1)} \right) \psi = 0, \tag{3.20}$$

where the monodromy parameters are encoded by

$$\delta_i \equiv \delta(\theta_i) = \frac{1}{4} - \theta_i^2 \tag{3.21}$$

and the accessory parameters are

$$p = \mu + \sum_{i=0,1,t} \frac{1 - 2\theta_i}{2(\lambda - a_i)}, \tag{3.22}$$

$$H(\boldsymbol{\theta}; \lambda, p, t) = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)} \left[p^2 - \left(\frac{1}{\lambda} + \frac{1}{\lambda - 1} \right) p + \right. \\ \left. + \frac{\delta_0}{\lambda^2} + \frac{\delta_1}{(\lambda-1)^2} + \frac{\delta_t}{(\lambda-t)^2} + \frac{\delta_\infty - \delta_0 - \delta_1 - \delta_t + \frac{3}{4}}{\lambda(\lambda-1)} \right]. \tag{3.23}$$

Now comparing (3.20) and (3.23) with (2.13) and (2.16), we find

$$H = -\partial_t S_\sigma^\pm(\lambda, t), \quad p = \partial_\lambda S_\sigma^\pm(\lambda, t), \tag{3.24}$$

and thus we have recovered the semiclassical BPZ equations of section 2 from the isomonodromic Fuchsian system.

At this point, probably it is not clear to the reader what is the relation between the classical intermediate momentum σ in the semiclassical action and the Fuchsian system parameters (λ, μ, t) (or (λ, p, t) in the Fuchsian equation (3.20)). We will clarify this point in the rest of this section by discussing the relationship between the monodromy group, the moduli space of flat connections and the semiclassical action.

3.3 Monodromy Group, Flat Connections and the Semiclassical Action

In this section, we will consider the Fuchsian system (3.6) in the $\mathrm{SL}(2,\mathbb{C})$ gauge, where $\mathrm{Tr}\, A_i = 0$. This can be obtained by the gauge transformation (3.39) discussed below. Assuming that λ and t are fixed, the formal solution of (3.6) is given by

$$\Psi(z) = \mathcal{P} \exp\left(\int^z A\right) \Psi(z_0), \tag{3.25}$$

where $\mathcal{P}$ represents a path-ordered exponential and z_0 is an arbitrary base point. A consequence of this formula is that the poles of the gauge connection $A(z)$ correspond to branch points of $\Psi(z)$. If we do the analytic continuation of $\Psi(z)$ around a closed path γ, enclosing one or more singular points, the solution will change by a *monodromy matrix* M_γ, i.e., $\Psi_\gamma = M_\gamma \Psi$. Elementary paths enclosing only one singular point a_i have monodromy matrix M_i and we label those matrices by their trace $\mathrm{Tr}\, M_i = 2\cos(2\pi\theta_i)$. The four-point monodromy group is then generated by three out of four $\mathrm{SL}(2,\mathbb{C})$ matrices obeying the monodromy identity

$$M_0 M_t M_1 M_\infty = \mathbb{1}. \tag{3.26}$$

As $\Psi(z)$ is analytic everywhere except at the branch cuts, the knowledge of the related monodromy data essentially determines the solution next to these points. The global information on how to connect different local solutions is encoded in the composite monodromies, obtained when a path encloses two singular points (two or more points for $n > 4$). For example, $\Psi_{\gamma_{0t}} = M_0 M_t \Psi$ has composite monodromy parameter defined by $\mathrm{Tr}\, M_0 M_t = -2\cos(2\pi\sigma_{0t})$ (see figure 2). All representations of the 4-point monodromy group are labeled by 4 elementary monodromies $\boldsymbol{\theta} = (\theta_0, \theta_t, \theta_1, \theta_\infty)$ and 3 composite monodromies $\boldsymbol{\sigma} = (\sigma_{0t}, \sigma_{1t}, \sigma_{01})$, as this is the sufficient data to generate all possible loops around singular points in the 4-punctured sphere. We call $(\boldsymbol{\theta}; \boldsymbol{\sigma})$ the *monodromy data* associated to $\Psi(z)$.

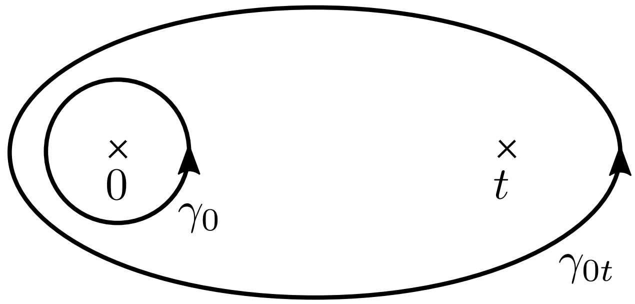

Figure 2. An elementary path γ_0 has monodromy θ_0 and a composite path γ_{0t} enclosing two singular points has composite monodromy σ_{0t}.

Let us define the monodromy parameters by

$$p_i = \mathrm{Tr}\, M_i = 2\cos 2\pi\theta_i, \quad p_{ij} = \mathrm{Tr}\, M_i M_j = 2\cos 2\pi\sigma_{ij}, \quad i,j = 0,1,t,\infty. \tag{3.27}$$

Assuming that the p_i's are fixed, irreducible representations of the monodromy group are labeled by three composite monodromies (p_{0t}, p_{1t}, p_{01}). For $\mathrm{SL}(2,\mathbb{C})$ matrices, the monodromy parameters also obey the Fricke-Jimbo relation [33]

$$\begin{aligned} p_{0t}p_{1t}p_{01} + p_{0t}^2 + p_{1t}^2 + p_{01}^2 + p_0^2 + p_t^2 + p_1^2 + p_0p_1p_tp_\infty = \\ (p_0p_t + p_1p_\infty)p_{0t} + (p_1p_t + p_0p_\infty)p_{1t} + (p_0p_1 + p_tp_\infty)p_{01} + 4, \end{aligned} \tag{3.28}$$

and thus only two composite monodromies are independent of each other. Following [33, 40], if we fix the $\boldsymbol{\theta}$ and σ_{0t}, the Fricke-Jimbo relation can be parametrized in terms of s_{0t} as

$$\left(p_{0t}^2 - 4\right) p_{1t} = D_{t,+}\, s_{0t} + D_{t,-}\, s_{0t}^{-1} + D_{t,0}, \tag{3.29a}$$

$$\left(p_{0t}^2 - 4\right) p_{01} = D_{u,+}\, s_{0t} + D_{u,-}\, s_{0t}^{-1} + D_{u,0}, \tag{3.29b}$$

with coefficients given by

$$D_{t,0} = p_{0t}\left(p_0p_1 + p_tp_\infty\right) - 2\left(p_0p_\infty + p_tp_1\right), \tag{3.30a}$$

$$D_{u,0} = p_{0t}\left(p_tp_1 + p_0p_\infty\right) - 2\left(p_0p_1 + p_tp_\infty\right), \tag{3.30b}$$

$$D_{t,\pm} = 16 \prod_{\epsilon=\pm} \sin\pi\left(\theta_t \mp \sigma_{0t} + \epsilon\theta_0\right) \sin\pi\left(\theta_1 \mp \sigma_{0t} + \epsilon\theta_\infty\right), \tag{3.30c}$$

$$D_{u,\pm} = -D_{t,\pm} e^{\mp 2\pi i\sigma_{0t}}. \tag{3.30d}$$

Solving the system (3.29) for s_{0t} when

$$\sigma_{ij} + \epsilon\theta_i + \epsilon'\theta_j \notin \mathbb{Z}, \quad \epsilon, \epsilon' = \pm 1, \tag{3.31}$$

we get

$$
\begin{aligned}
s_{0t}^{\pm}&(\cos 2\pi(\theta_t \mp \sigma_{0t}) - \cos 2\pi\theta_0)(\cos 2\pi(\theta_1 \mp \sigma_{0t}) - \cos \pi\theta_\infty) \\
&= (\cos 2\pi\theta_t \cos 2\pi\theta_1 + \cos 2\pi\theta_0 \cos 2\pi\theta_\infty \pm i \sin 2\pi\sigma_{0t} \cos 2\pi\sigma_{01}) \\
&- (\cos 2\pi\theta_0 \cos 2\pi\theta_1 + \cos 2\pi\theta_t \cos 2\pi\theta_\infty \mp i \sin 2\pi\sigma_{0t} \cos 2\pi\sigma_{1t}) e^{\pm 2\pi i \sigma_{0t}}.
\end{aligned} \tag{3.32}
$$

The special cases when (3.31) is not true correspond to reducible representations, which are all listed in the context of PVI solutions in [28]. In those cases, to find s we should go back to the Fricke-Jimbo relation (3.28). In conclusion, we can label irreducible representations of the 4-point monodromy group by two parameters (σ_{0t}, s_{0t}). This parametrization is essentially the same under the permutation of the composite monodromies and its related to the number of independent ways to slice the 4-punctured sphere into two pairs of pants [40].

3.3.1 Summary of Parameters

In section 3.1, we parametrized the gauge connection $A(z)$ in terms of elementary monodromies $\boldsymbol{\theta}$ and two extra parameters (λ, μ) (or (λ, p)). Therefore, for fixed t, the *moduli space* of flat connections $A(z)$ can be labeled by $(\boldsymbol{\theta}; \lambda, \mu)$. From (3.25), it is clear that there should be a map between the moduli parameters (λ, μ) and the monodromy parameters (σ_{0t}, s_{0t}). This is the Riemann-Hilbert map for Fuchsian systems, the map between irreducible representations of the monodromy group and the moduli space of flat connections [40, 50]. As explicitly shown in [33], isomonodromic deformations define such map via the integration constants (σ_{0t}, s_{0t}). Jimbo obtained the asymptotics of $A(z; \lambda(t), \mu(t), t)$ for small t and showed that the formulas only depend on $(\boldsymbol{\theta}; \sigma_{0t}, s_{0t})$. This can then be used to find the asymptotics of the τ-function and the PVI solutions $(\lambda(t), \mu(t))$ in terms of $(\boldsymbol{\theta}; \sigma_{0t}, s_{0t})$. This will become clear in section 4 when we discuss the isomonodromic τ-function expansion.

The semiclassical equation (3.20) is parametrized by λ and the accessory parameters H and p, for fixed t and $\boldsymbol{\theta}$. As H is a function of (λ, p, t), these parameters label the possible equations. If we assume that $(\lambda(t), p(t))$ are solutions of the isomonodromic equations, i.e.

$$
\dot{\lambda} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial \lambda}, \tag{3.33}
$$

then $(\lambda(t), p(t))$ are PVI solutions and can be labeled by the monodromy parameters (σ_{0t}, s_{0t}). Therefore, for fixed $(t, \boldsymbol{\theta})$, we have that $(\lambda(t), p(t)) \sim (\sigma_{0t}, s_{0t})$.

We also showed in (3.24) that the accessory parameters are given by derivatives of the semiclassical action $S_\sigma(\lambda(t), t)$. To complete the picture relating the semiclassical equation (3.20) and the monodromy parameters, we need to show that the classical intermediate momentum σ is the *composite monodromy* σ_{0t}. Let us assume that, for fixed $\lambda \neq 0, 1, t, \infty$, the solution $\psi(z, \lambda, t)$ of (3.20) has a small t expansion as

$$
\psi(z, \lambda, t) = z^{\frac{1}{2} - \sigma_{0t}} f(z, \lambda) + \mathcal{O}(t). \tag{3.34}
$$

Assuming that (3.34) is well-defined at both $z = 0$ and $z = t$, for t small enough, σ_{0t} represents the composite monodromy parameter. Substituting (3.34) in (3.20), we find at lowest order

$$H_0 = \delta_0 + \delta_t - \delta_{\sigma_{0t}} \tag{3.35}$$

where $H_0 = \lim_{t\to 0} t(1-t)H$. From CFT, we know that for small t

$$S_\sigma(\lambda, t) \sim (\delta_\sigma - \delta_0 - \delta_t)\log(t) \tag{3.36}$$

and thus

$$H(\lambda, p, t) = -\partial_t S_\sigma(\lambda, t) \sim \frac{\delta_0 + \delta_t - \delta_\sigma}{t}, \tag{3.37}$$

which agrees with (3.35) if $\sigma = \sigma_{0t}$. From here and in the rest of the paper, we define $(\sigma, s) \equiv (\sigma_{0t}, s_{0t})$, unless otherwise stated.

In summary, for fixed $(t, \boldsymbol{\theta})$, we have the following set of parameters

$$\begin{aligned} \text{Monodromy representations:} &\quad (\sigma, s) \\ \text{Semiclassical Action:} &\quad (\lambda, \sigma) \\ \text{Flat Connection:} &\quad (\lambda, \mu) \quad (\text{or } (\lambda, p)) \end{aligned}$$

The assumption that $(\lambda(t), \mu(t))$ is a solution of the isomonodromic flow connects the different parameters, since all quantities of interest can be phrased in terms of (σ, s).

3.4 Fuchsian System and $c = 1$ BPZ equations

A natural question is whether it is possible to encode the level-2 heavy and light BPZ equations into a Fuchsian system for any value of c. This was proved for a single $c = 1$ BPZ equation in [54]. As we show below, we can recover each level-2 BPZ equation separately for arbitrary c from an appropriate Fuchsian system. However, we can only consistently recover both BPZ equations if $c = 1$. The relationship is true if the associated linear system allows for isomonodromic deformations.

Let us first change the gauge of (3.1) by applying the transformation

$$\Psi = \prod_{i=1}^{n} [(z - a_i)(\lambda - a_i)]^{\theta_i} \Phi, \quad A_i = B_i + \theta_i \mathbb{1} \tag{3.39}$$

to (3.40), we get

$$\epsilon_1 \partial_z \Phi = B(z)\Phi, \tag{3.40a}$$

$$\epsilon_2 \partial_\lambda \Phi = B(\lambda)\Phi, \tag{3.40b}$$

$$\alpha \partial_{a_i} \Phi = \left(\frac{\lambda - z}{\lambda - a_i}\right) \frac{B_i}{z - a_i} \Phi, \quad (i = 1, \ldots, n) \tag{3.40c}$$

where ϵ_1, ϵ_2 and α are three arbitrary constants and

$$B(z) = \sum_{i=1}^{n} \frac{B_i}{z - a_i} = \begin{pmatrix} B_{11}(z) & B_{12}(z) \\ B_{21}(z) & B_{22}(z) \end{pmatrix}, \quad \sum_{i=1}^{n} B_i = 0, \tag{3.41}$$

such that

$$\mathrm{Tr}\, B_i = 0, \quad \mathrm{Tr}\, B_i^2 = -2\epsilon_1\epsilon_2\Delta_i. \tag{3.42}$$

The choice of $\mathrm{Tr}\, B_i$ sets the $\mathrm{SL}(2,\mathbb{C})$ gauge for the Fuchsian system.

To derive a second order equation for Φ, we take the derivative of (3.40a) with respect to z

$$\epsilon_1 \partial_z^2 \Phi = (\partial_z B + \frac{B^2}{\epsilon_1})\Phi. \tag{3.43}$$

Using the relation for $\mathfrak{sl}(2,\mathbb{C})$ matrices

$$B_i B_j + B_j B_i = \mathrm{Tr}(B_i B_j)\mathbb{1}, \tag{3.44}$$

we can easily show that

$$B^2 = \sum_{i=1}^{n} \left(\frac{-\epsilon_1\epsilon_2\Delta_i}{(z-a_i)^2} + \frac{H_i}{z - a_i} \right)\mathbb{1}, \tag{3.45}$$

where the accessory parameters are defined by

$$H_i \equiv \sum_{j \neq i} \frac{\mathrm{Tr}(B_i B_j)}{a_i - a_j}. \tag{3.46}$$

From (3.40c) and (3.40b), we obtain

$$\sum_{i=1}^{n} \frac{B_i}{(z-a_i)^2} = \frac{-\epsilon_1 \partial_z \Phi + \epsilon_2 \partial_\lambda \Phi}{z - \lambda} + \sum_{i=1}^{n} \frac{\alpha \partial_{a_i} \Phi}{z - a_i}. \tag{3.47}$$

Plugging (3.45) and (3.47) into (3.43) and dividing by ϵ_2, we get

$$\frac{\epsilon_1}{\epsilon_2}\left(\partial_z^2 \Phi - \frac{1}{z-\lambda}\partial_z \Phi \right) + \frac{\partial_\lambda \Phi}{z - \lambda} + \sum_{i=1}^{n} \left(\frac{\Delta_i}{(z-a_i)^2} + \frac{\frac{\alpha}{\epsilon_2}\partial_{a_i} - \frac{1}{\epsilon_1\epsilon_2} H_i}{z - a_i} \right) \Phi = 0. \tag{3.48}$$

Now we apply the transformation

$$\Phi(\boldsymbol{a}|z,\lambda) = (z-\lambda)^{\frac{1}{2}} \left[h(\boldsymbol{a})\right]^{\frac{1}{\alpha\epsilon_1}} \chi(\boldsymbol{a}|z,\lambda) \tag{3.49}$$

in (3.48), with the choice

$$H_i(\boldsymbol{a}) = \partial_{a_i} \log h(\boldsymbol{a}). \tag{3.50}$$

Notice that (3.46) and (3.5) imply that $h(\boldsymbol{a})$ is the τ-function up to a overall function of t. Then we get the following equation for χ

$$\left[\frac{\epsilon_1}{\epsilon_2}\partial_z^2 + \frac{\Delta_H}{(z-\lambda)^2} + \frac{\partial_\lambda}{z-\lambda} + \sum_{i=1}^{n}\left(\frac{\Delta_i}{(z-a_i)^2} + \frac{\frac{\alpha}{\epsilon_2}\partial_{a_i}}{z-a_i}\right)\right]\chi(\boldsymbol{a}|z,\lambda) = 0, \tag{3.51}$$

where $\Delta_H = -\frac{3\epsilon_1}{4\epsilon_2} - \frac{1}{2}$. Setting $\alpha = \epsilon_2$ and $b^2 = \epsilon_2/\epsilon_1$, this equation becomes the generalization of the BPZ equation (2.5a) for a correlator with n arbitrary insertions at $\boldsymbol{z} = \boldsymbol{a}$ and two degenerate insertions, one at z and another at λ. However, if we repeat the same procedure for (3.40b), we only get the second BPZ equation (2.5b) if we choose $\alpha = -\epsilon_1$. This means that $\alpha = \epsilon_2 = -\epsilon_1$ is a sufficient condition for the Fuchsian system (3.40) consistently reproduce the two BPZ equations. Accordingly, $b = i$ and thus $c = 1$ in the CFT interpretation. Using the parametrization (2.1) and (2.2), the $c = 1$ BPZ equations are thus

$$\left[-\partial_z^2 + \frac{\frac{1}{4}}{(z-\lambda)^2} + \frac{\partial_\lambda}{z-\lambda} + \sum_{i=1}^{n}\left(\frac{\theta_i^2}{(z-a_i)^2} + \frac{\partial_{a_i}}{z-a_i}\right)\right]\chi(\boldsymbol{a}|z,\lambda) = 0, \tag{3.52a}$$

$$\left[-\partial_\lambda^2 + \frac{\frac{1}{4}}{(\lambda-z)^2} + \frac{\partial_z}{\lambda-z} + \sum_{i=1}^{n}\left(\frac{\theta_i^2}{(\lambda-a_i)^2} + \frac{\partial_{a_i}}{\lambda-a_i}\right)\right]\chi(\boldsymbol{a}|z,\lambda) = 0, \tag{3.52b}$$

where $\Delta_i = \theta_i^2$ is the $c = 1$ conformal weight. This shows that the Fuchsian system (3.40) with $\epsilon_2 = -\epsilon_1$ simultaneously encodes the monodromy data of the two $c = 1$ BPZ equations above. Moreover, this fact is the starting point of the original argument of why the isomonodromic τ-function is equivalent to a $c = 1$ correlator [34]. For a proof of this fact, see [40].

4 Accessory parameter from the Isomonodromic τ-function

We previously discussed how the accessory parameter $H(\lambda(t), p(t), t)$ in (3.20) can be written in terms of the monodromy data $(\boldsymbol{\theta}; \sigma, s)$, given that $(\lambda(t), p(t))$ are solutions of the isomonodromic flow. Now we want to use this fact to write the accessory parameter expansion H_x of Heun's equation (1.3) in terms of the monodromy data. The key point is to impose a special initial condition for the isomonodromic flow, which we discuss next. Then we will present our algorithm on how to solve this initial condition and obtain H_x using the τ-function expansion of [34]. We review the τ-function expansion in section 4.2, solve the initial condition in section 4.3 and finally present the accessory parameter expansion in section 4.4.

4.1 Accessory Parameter from Initial Conditions

Let us focus on the $n = 4$ isomonodromic system (3.6). Here we discuss how to recover the accessory parameter of Heun's equation (1.3) from the deformed Heun's equation (3.15) by an appropriate initial condition on $(\lambda(t),\mu(t))$ at $t=x$. The canonical form of (1.3), obtained by the transformation $\psi=\prod_{i=0,1,x}(z-a_i)^{(1-2\theta_i)/2}\,y$, is given by

$$y''(z)+\left(\frac{1-2\theta_0}{z}+\frac{1-2\theta_1}{z-1}+\frac{1-2\theta_x}{z-x}\right)y'(z)+\left(\frac{\kappa_1(\kappa_2+1)}{z(z-1)}-\frac{x(x-1)K_x}{z(z-1)(z-x)}\right)y(z)=0, \tag{4.1}$$

where the accessory parameters of both equations are related by

$$H_x=K_x+\frac{(1-2\theta_0)(1-2\theta_x)}{2x}+\frac{(1-2\theta_1)(1-2\theta_x)}{2(x-1)}. \tag{4.2}$$

Consider the deformed Heun equation (3.15), which we repeat here for convenience to the reader,

$$\begin{aligned}y''+\left(\frac{1-2\theta_0}{z}+\frac{1-2\theta_1}{z-1}+\frac{1-2\theta_t}{z-t}-\frac{1}{z-\lambda(t)}\right)y'+\\ +\left(\frac{\kappa_1(\kappa_2+1)}{z(z-1)}-\frac{t(t-1)K(\boldsymbol{\theta};\lambda(t),\mu(t),t)}{z(z-1)(z-t)}+\frac{\lambda(\lambda-1)\mu(t)}{z(z-1)(z-\lambda(t))}\right)y=0,\end{aligned} \tag{4.3}$$

with

$$K(\boldsymbol{\theta};\lambda,\mu,t)\;=\;\frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left[\mu^2-\left(\frac{2\theta_0}{\lambda}+\frac{2\theta_1}{\lambda-1}+\frac{2\theta_t-1}{\lambda-t}\right)\mu+\frac{\kappa_1(\kappa_2+1)}{\lambda(\lambda-1)}\right]. \tag{4.4}$$

We recover (4.1) from (4.3) by applying the following initial condition for the isomonodromic flow to (4.3) [55, 56]

$$t=x,\quad \lambda(x)=x,\quad \mu(x)=-\frac{K_x}{2\theta_t},\quad \theta_t=\theta_x-\frac{1}{2}. \tag{4.5}$$

We show in Appendix A that this condition is well-posed with respect to the isomonodromic equations. As we discussed before, $(\lambda(t),\mu(t))$ have (σ,s) as integration constants, so imposing (4.5) seems equivalent to fix the values of (σ,s) separately. However, as we are going to see next, due to the special structure of the isomonodromic flow, the condition $\lambda(x)=x$ implies a non-trivial relation $s=s(\sigma,x)$ between the composite monodromy parameters. This means that Heun's equation (4.1) lives in a proper subspace of all possible Fuchsian opers with 4 singulars points.

Litvinov *et al* [3] found the accessory parameter expansion of H_x by regularizing the PVI action via the solution of PVI equation with $\lambda(x)=\infty$. Here we present an alternative way to calculate the accessory parameter using the isomonodromic τ-function expansion [34] with the condition $\lambda(x)=x$. This gives a clearer implementation of the proposal [3] and, in addition, suggests a deeper connection between $c=\infty$ and $c=1$ conformal blocks. Notice that the condition $\lambda(x)=\infty$ is equivalent to our choice by a bi-rational transformation [28].

Our algorithm to find the accessory parameter expansion consists of three steps

1. Write the accessory parameter H_x in terms of the isomonodromic τ-function;
2. Solve the initial condition $\lambda(x) = x$ to obtain a monodromy constraint $s = s(\sigma, x)$;
3. Substitute the constraint $s = s(\sigma, x)$ in the τ-function to obtain H_x.

We will show below that the condition $\lambda(x) = x$ can be phrased in terms of the τ-function, which is essential to find $s(\sigma, x)$ as a series expansion in x. We remind the reader that the well-posedness of the initial condition is discussed in appendix A. After obtaining $s(\sigma, x)$, to obtain the accessory parameter expansion is just a matter of straightforward calculation. This calculation is cumbersome at higher orders, so we show it explicitly only up to order x^0 in this section. In Appendix C, we reproduce the analytic CFT formulas up to order x^2 and present numerical evidence up to order x^5, compared to the direct CFT approach.

The key to understand the relation $s = s(\sigma, x)$ is to phrase the initial conditions in terms of the isomonodromic τ-function. We will do this in the next subsection. Here we express H_x in terms of the τ-function. For convenience we define

$$\boldsymbol{\theta}_{s_1,s_2} = (\,\theta_0, \theta_1, \theta_t + \tfrac{1}{2}s_1, \theta_\infty + \tfrac{1}{2}s_2), \quad s_1, s_2 = 0, \pm, \tag{4.6}$$

and $\boldsymbol{\theta}_{0,0} \equiv \boldsymbol{\theta}$. The definition of the isomonodromic Hamiltonian (3.5) for our particular 4-point case is given by

$$H_{S,t} = \frac{\mathrm{Tr}(A_0 A_t)}{t} + \frac{\mathrm{Tr}(A_1 A_t)}{t-1}. \tag{4.7}$$

Using the parameterization (3.9) for the A_i's in (4.7), we find that

$$H_{S,t}(\boldsymbol{\theta}_{0,+}; \lambda, \mu, t) = K(\boldsymbol{\theta}_{+,+}; \lambda, \mu, t) + \frac{4\theta_0\theta_t}{t} + \frac{4\theta_1\theta_t}{t-1}. \tag{4.8}$$

Imposing the initial conditions (4.5) in (4.8), we obtain

$$K_x = \frac{d}{dt} \log[t^{-4\theta_0\theta_t}(1-t)^{-4\theta_1\theta_t}\tau_S(\boldsymbol{\theta}_{0,+}; t)]\bigg|_{t=x}, \tag{4.9}$$

where we used the τ-function definition

$$H_{S,t}(\boldsymbol{\theta}_{0,+}; \lambda(t), \mu(t), t) = \frac{d}{dt} \log \tau_S(\boldsymbol{\theta}_{0,+}; t). \tag{4.10}$$

Notice that (4.9) substitutes the initial condition for $\mu(x)$ in (4.5), as it gives K_x in terms of the monodromy data. Together with (4.2), we can then find H_x in terms of the τ-function.

4.2 CFT Expansion of the τ-function

In order to proceed, we need to introduce the τ-function expansion of [34]. For later convenience, we define a slightly different τ-function

$$\tau_S(\boldsymbol{\theta};t) = t^{2\theta_0\theta_t}(1-t)^{4\theta_1\theta_t}\, t^{\sigma^2-\theta_0^2-\theta_t^2}\tau(\boldsymbol{\theta};t). \tag{4.11}$$

This new τ-function changes the Hamiltonian, but does not change the equations of motion, as it is multiplied by a pure function of t. The complete expansion of the Painlevé VI τ-function, adapted to our definition (4.11), is given by

$$\tau(\boldsymbol{\theta};t) = \sum_{n\in\mathbb{Z}} C(\boldsymbol{\theta},\sigma+n)s^n t^{n(n+2\sigma)}\mathcal{B}(\boldsymbol{\theta},\sigma+n;t), \tag{4.12}$$

where we assume that the real part of sigma $\Re\sigma$ obeys

$$0 \le \Re\sigma < \frac{1}{2}. \tag{4.13}$$

The structure constants are given in terms of Barnes functions[3]

$$C(\boldsymbol{\theta},\sigma) = \frac{\prod_{\epsilon,\epsilon'=\pm} G(1+\theta_t+\epsilon\theta_0+\epsilon'\sigma)G(1+\theta_1+\epsilon\theta_\infty+\epsilon'\sigma)}{\prod_{\epsilon=\pm} G(1+2\epsilon\sigma)}, \tag{4.14}$$

and the $\mathcal{B}$'s are the $c=1$ conformal blocks, given by the AGT combinatorial series

$$\mathcal{B}(\boldsymbol{\theta},\sigma;t) = \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta},\sigma)t^{|\lambda|+|\mu|}, \tag{4.15}$$

summing over pairs of Young diagrams λ,μ with

$$\begin{aligned}\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta},\sigma) = \prod_{(i,j)\in\lambda} &\frac{((\theta_t+\sigma+i-j)^2-\theta_0^2)((\theta_1+\sigma+i-j)^2-\theta_\infty^2)}{h_\lambda^2(i,j)(\lambda'_j+\mu_i-i-j+1+2\sigma)^2}\times \\ &\prod_{(i,j)\in\mu}\frac{((\theta_t-\sigma+i-j)^2-\theta_0^2)((\theta_1-\sigma+i-j)^2-\theta_\infty^2)}{h_\mu^2(i,j)(\lambda_i+\mu'_j-i-j+1-2\sigma)^2},\end{aligned} \tag{4.16}$$

where (i,j) denotes the box in the Young diagram λ, λ_i the number of boxes in row i, λ'_j the number of boxes in column j and $h_\lambda(i,j)=\lambda_i+\lambda'_j-i-j+1$ its hook length. A proof for this expansion is available in [40] and alternatively in [57].

[3] Defined by the functional relation $G(z+1)=\Gamma(z)G(z)$, with $\Gamma(z)$ being the Euler gamma function. For further properties, see appendix A of [43].

4.3 Solving the condition $\lambda(x) = x$

Inspired by [3], our approach to find H_x is to solve the condition $\lambda(x) = x$ by assuming a series expansion for $s = s(\sigma, x)$ in x. The Painlevé property implies that the PVI solution is regular at $t = x$ for any $x \neq 0, 1, \infty$ [3, 28, 50]. As we are going to see below, this fact is consistent with the τ-function also being regular at $t = x$. Okamoto's formula [58] gives $\lambda(t)$ in terms of a ratio of τ-functions

$$\lambda(t) - t = \frac{t(t-1)}{2\theta_\infty}\frac{d}{dt}\log\frac{\tau(\boldsymbol{\theta}_{0,-};t)}{\tau(\boldsymbol{\theta}_{0,+};t)}, \tag{4.17}$$

where $\tau(\boldsymbol{\theta}, t)$ is given by (4.12). Evaluated at $t = x$, the left hand side vanishes because of our initial condition. Assuming that $\theta_\infty \neq 0$ and neither of the τ-functions above vanishes or blows up at $\lambda(x) = x$, we have that

$$\left.\left[\frac{d}{dt}\left[\tau\left(\boldsymbol{\theta}_{0,-};t\right)\right]\tau\left(\boldsymbol{\theta}_{0,+};t\right) - \frac{d}{dt}\left[\tau\left(\boldsymbol{\theta}_{0,+};t\right)\right]\tau\left(\boldsymbol{\theta}_{0,-};t\right)\right]\right|_{t=x} = 0. \tag{4.18}$$

In the following, we rewrite the condition (4.18) as a double series expansion. To do this, let us introduce some definitions. The isomonodromic Hamiltonian (4.10) and the accessory parameter expansion (4.9) do not depend on the normalization of the τ-function, so we normalize the structure constants as

$$\bar{C}_n(\boldsymbol{\theta}, \sigma) := \frac{C(\boldsymbol{\theta}, \sigma + n)}{C(\boldsymbol{\theta}, \sigma)}. \tag{4.19}$$

As we prove in Appendix B, these ratios can be factorized as

$$\bar{C}_n(\boldsymbol{\theta}, \sigma) = \mathcal{C}_n(\boldsymbol{\theta}, \sigma) A(\boldsymbol{\theta}, \sigma)^n, \quad (n > 0) \tag{4.20}$$

where $\mathcal{C}_n(\boldsymbol{\theta}, \sigma)$ is given by (B.7) and $A(\boldsymbol{\theta}, \sigma)$ is given by (B.4). Another important formula is

$$\bar{C}_n(\boldsymbol{\theta}_{0,-}\,, \sigma) = f_n(\boldsymbol{\theta}\,, \sigma)\bar{C}_n(\boldsymbol{\theta}_{0,+}\,, \sigma), \quad (n > 0) \tag{4.21}$$

where

$$f_n(\boldsymbol{\theta}, \sigma) = \prod_{k=1}^{|n|}\frac{\left(\sigma + k - \frac{1}{2} - \theta_\infty\right)^2 - \theta_1^2}{\left(\sigma + k - \frac{1}{2} + \theta_\infty\right)^2 - \theta_1^2}, \quad f_0(\boldsymbol{\theta}, \sigma) = 1. \tag{4.22}$$

For $n < 0$, we change $\sigma \to -\sigma$ in the formulas above.

Plugging (4.19) and (4.20) into (4.12), we get

$$\tau(\boldsymbol{\theta}; t) = \sum_{n\in\mathbb{Z}} \mathcal{C}_n(\boldsymbol{\theta}, \sigma)\mathcal{B}(\boldsymbol{\theta}, \sigma + n; t)t^{n^2}[X(\boldsymbol{\theta}, \sigma, s; t)]^n, \tag{4.23}$$

where

$$X(\boldsymbol{\theta}, \sigma, s; t) := A(\boldsymbol{\theta}, \sigma) s\, t^{2\sigma}. \tag{4.24}$$

We can rewrite the $c = 1$ blocks (4.15) in terms of levels L

$$\mathcal{B}(\boldsymbol{\theta}, \sigma + n; t) = \sum_{L=0}^{\infty} \mathcal{B}_n^{(L)}(\boldsymbol{\theta}, \sigma)\, t^L, \tag{4.25}$$

where

$$\mathcal{B}_n^{(L)}(\boldsymbol{\theta}, \sigma) = \sum_{|\lambda|+|\mu|=L} \mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta}, \sigma + n) \tag{4.26}$$

is a restricted sum at level L of the coefficients (4.16). We will suppress the monodromy arguments in what follows, for simplicity, except the ones that are shifted. Using the definitions (4.23) and (4.25), we get

$$\tau\,(t) = \sum_{n\in\mathbb{Z}} \mathcal{C}_n \sum_{L=0}^{\infty} \mathcal{B}_n^{(L)}\, [X(t)]^n\, t^{n^2+L}, \tag{4.27}$$

$$\frac{d}{dt}\tau\,(t) = \sum_{n\in\mathbb{Z}} \mathcal{C}_n \sum_{L=0}^{\infty} (n\,(n + 2\sigma) + L)\, \mathcal{B}_n^{(L)}\, [X(t)]^n\, t^{n^2+L-1}, \tag{4.28}$$

where we used

$$\frac{dX(t)}{dt} = \frac{2\sigma}{t} X(t). \tag{4.29}$$

Using (4.27) and (4.28) in (4.18), with the properly shifted monodromies and considering (4.21), the equation (4.18) can be rewritten as

$$\sum_{n,p\in\mathbb{Z}} \sum_{L,M=0}^{\infty} D \begin{bmatrix} L & M \\ n & p \end{bmatrix} x^{n^2+p^2+L+M-1} [X(x)]^{n+p} = 0, \tag{4.30}$$

where

$$\begin{aligned} D \begin{bmatrix} L & M \\ n & p \end{bmatrix} \equiv\, & \mathcal{C}_n(\boldsymbol{\theta}_{0,+}, \sigma)\, \mathcal{C}_p(\boldsymbol{\theta}_{0,+}, \sigma) \left[f_n(\boldsymbol{\theta}_{0,+}, \sigma)\, \mathcal{B}_n^{(L)}\,(\boldsymbol{\theta}_{0,-}, \sigma)\, \mathcal{B}_p^{(M)}\,(\boldsymbol{\theta}_{0,+}, \sigma) - \right. \\ & \left. -\ f_p(\boldsymbol{\theta}_{0,+}, \sigma)\, \mathcal{B}_n^{(L)}\,(\boldsymbol{\theta}_{0,+}, \sigma)\, \mathcal{B}_p^{(M)}\,(\boldsymbol{\theta}_{0,-}, \sigma) \right] \times [n\,(n + 2\sigma) + L]\,. \end{aligned} \tag{4.31}$$

Given that $\lambda(t)$ admits a regular expansion near $t = x$, so does the right hand side of (4.17). This implies that $X(x)$ also admits a Taylor expansion for x small

$$X(\boldsymbol{\theta}, \sigma, s(\sigma, x); x) = \sum_{k=0}^{\infty} X_k(\boldsymbol{\theta}, \sigma) x^k, \tag{4.32}$$

where we assume that $s = s(\sigma, x)$ is the source of the series expansion. Our goal is to extract the coefficients $X_k = X_k(\boldsymbol{\theta}, \sigma)$ by solving (4.30) order by order in x. Therefore, this procedure gives us the coefficients of the $s = s(\sigma, x)$ series solution.

We rewrite (4.30) as a double series condition

$$\sum_{\alpha=0}^{\infty} \sum_{\beta=-n_\alpha}^{n_\alpha} e_{\alpha,\beta}\, x^\alpha [X(x)]^\beta = 0, \tag{4.33}$$

with

$$e_{\alpha,\beta} = {\sum_{n,p,L,M}}' D \begin{bmatrix} L & M \\ n & p \end{bmatrix}, \tag{4.34}$$

where the $\sum'$ means that the summation is over all $n, p \in \mathbb{Z}$; $L, M = 0, 1, 2, \ldots$ such that $n^2 + p^2 + L + M - 1 = \alpha$ and $n + p = \beta$. From (4.31), we see that

$$D \begin{bmatrix} 0 & M \\ 0 & p \end{bmatrix} = 0, \quad \forall\, p, M. \tag{4.35}$$

This means that the $\alpha = -1$ term does not contribute to (4.30). For fixed α, we have that $-n_\alpha \leq \beta \leq n_\alpha$, where $n_\alpha = \lfloor \sqrt{\alpha + 1} \rfloor$, with $\lfloor x \rfloor$ being the floor function of x. To explain the β constraint, let us consider both n and p to be positive (or both negative). For fixed α, the upper bound on β can be reached when $L = M = 0$ so that $\alpha + 1 = n^2 + p^2 \geq (n + p)^2 = \beta^2$.

Let us now go back to (4.30). For $\alpha = 0$, we have $-1 \leq \beta \leq 1$ and thus

$$D \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} X_0^{-1} + D \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + D \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} X_0 = 0,$$

which, using (4.31), (4.26), (4.22) and (B.7), we can show it is equivalent to

$$\frac{(\theta_t + \sigma)^2 - \theta_0^2}{4\sigma^2} X_0^{-1} + \left(\frac{\theta_0^2 - \sigma^2 - \theta_t^2}{2\sigma^2} \right) + \frac{(\theta_t - \sigma)^2 - \theta_0^2}{4\sigma^2} X_0 = 0. \tag{4.36}$$

This is essentially the same equation presented in [3] obtained from the PVI solution[4]. It gives two solutions for X_0 and we choose $X_0 = 1$ at this order, for consistency with the asymptotics of the PVI solution, as discussed in [3]. Plugging back $X(x) = 1 + X_1 x + \ldots$ to (4.33), the higher order equations in x will give only one solution for the coefficients X_k. The next two coefficients are given in Appendix C in terms of σ and δ's.

[4]With the corresponding Bäcklund transformation $\theta_0 \to \theta_1$ and $\theta_t \to \theta_\infty + \frac{1}{2}$.

4.4 Accessory Parameter Expansion

Substituting the definition (4.11) into (4.9) and then into (4.2), we get H_x in terms of the τ-function

$$H_x = \frac{\delta_0 + \delta_x - \delta_\sigma}{x} + \frac{(1-2\theta_1)(1-2\theta_x)}{2(x-1)} + \frac{d}{dt}\log\tau(\boldsymbol{\theta}_{0,+};t)\Big|_{t=x}. \tag{4.37}$$

Notice that the first term in (4.37) is already the expected answer from CFT, as it comes from the leading behavior of the classical conformal block. The final step in our algorithm is to take the series solution for $X(x)$ obtained above and plug it back into the τ-function in (4.37), with the substitution $\theta_t = \theta_x - \frac{1}{2}$. This computation is straightforward but demanding, so we will not discuss it here in general form. We will show below how the computation works for the first non-trivial term. This will make it clear how to proceed to obtain higher-order terms.

Remembering that $X(t) = Ast^{2\sigma}$, under the assumption $0 < \Re\sigma < \frac{1}{2}$, we have the small t expansion of (4.23)

$$\tau(t) = 1 + \left(\mathcal{C}_1 X(t) + \mathcal{C}_{-1}X(t)^{-1} + \mathcal{B}_0^{(1)}\right)t + \mathcal{O}(t^{2(1\pm\Re\sigma)}),$$

and

$$\frac{d}{dt}\tau(t) = (1-2\sigma)\mathcal{C}_{-1}X(t)^{-1} + (1+2\sigma)\mathcal{C}_1 X(t) + \mathcal{B}_0^{(1)} + \mathcal{O}(t^{1\pm2\Re\sigma}), \tag{4.38}$$

The τ-function has no well-defined Taylor expansion around $t=0$ because its first derivative diverges as $t^{-2\sigma}$. On the other hand, it is possible to do such expansion near a regular point $t = x$, for x sufficiently close to zero. In this sense, we are allowed to write

$$\frac{d}{dt}\log\tau(t) \sim \frac{(1-2\sigma)\mathcal{C}_{-1}X(t)^{-1} + (1+2\sigma)\mathcal{C}_1X(t) + \mathcal{B}_0^{(1)}}{1 + \left(\mathcal{C}_1X(t) + \mathcal{C}_{-1}X(t)^{-1} + \mathcal{B}_0^{(1)}\right)t}. \tag{4.39}$$

Using that $X(t=x) = 1 + X_1 x + \ldots$ and $\mathcal{B}_0^{(1)} = \mathcal{B}_{0,1} + \mathcal{B}_{1,0}$, we have

$$\frac{d}{dt}\log\tau(t)\Big|_{t=x} = (1-2\sigma)\mathcal{C}_{-1} + (1+2\sigma)\mathcal{C}_1 + \mathcal{B}_{0,1} + \mathcal{B}_{1,0} + \mathcal{O}(x) \tag{4.40}$$

to leading order in x. From (B.7), we see that

$$\begin{aligned}&(1+2\sigma)\mathcal{C}_1 + (1-2\sigma)\mathcal{C}_{-1} = \\ &\quad = \frac{\left[(\theta_t-\sigma)^2 - \theta_0^2\right]\left[(\theta_\infty+\sigma)^2 - \theta_1^2\right]}{4\sigma^2(1+2\sigma)} + \frac{\left[(\theta_t+\sigma)^2 - \theta_0^2\right]\left[(\theta_\infty-\sigma)^2 - \theta_1^2\right]}{4\sigma^2(1-2\sigma)}\end{aligned} \tag{4.41}$$

and from (4.16)

$$\mathcal{B}_{1,0}+\mathcal{B}_{0,1}=$$
$$=\frac{\left[(\theta_t-\sigma)^2-\theta_0^2\right]\left[(\theta_1-\sigma)^2-\theta_\infty^2\right]}{4\sigma^2}+\frac{\left[(\theta_t+\sigma)^2-\theta_0^2\right]\left[(\theta_1+\sigma)^2-\theta_\infty^2\right]}{4\sigma^2}. \quad (4.42)$$

If we carefully sum (4.41) and (4.42), substituting $\theta_i^2 = 1/4-\delta_i$ and the shifted monodromy values $\theta_t \to \theta_x - \frac{1}{2}, \theta_\infty \to \theta_\infty + \frac{1}{2}$, we get

$$\frac{(\delta_\sigma+\delta_x-\delta_0)(\delta_\sigma+\delta_1-\delta_\infty)}{2\delta_\sigma}+(1-2\theta_1)(1-2\theta_x). \quad (4.43)$$

The second term above cancels with the second term of (4.37) and the first term above matches the standard CFT result [3]. The next order calculation is much more complicated, so we show here the result for (4.37) only up to first order in x

$$H_x=\frac{\delta_0+\delta_x-\delta_\sigma}{x}+\frac{(\delta_0-\delta_\sigma-\delta_x)(\delta_\sigma+\delta_1-\delta_\infty)}{2\delta_\sigma}+\mathcal{O}(x). \quad (4.44)$$

We present the next two terms in the expansion above in appendix C.

We implemented the algorithm of this section in a computer algebra program, as we do not have the explicit series solution for the constraint (4.33). We tested the accessory parameter expansion up to order x^2 analytically by comparing with the semiclassical limit of the CFT conformal block [3, 25] using the inverse gram matrix approach [43, 59]. We also checked this algorithm numerically, substituting the θ's with some fixed numbers from the beginning, up to order x^5. More details are given in appendix C. This gives strong evidence that the isomonodromic approach presented here reproduces the classical conformal block expansion. Although we do not have a proof to all orders in x, we believe that our discussion above on the semiclassical limit and isomonodromic deformations, given the exponentiation hypothesis and the assumptions on the conformal weights, is enough mathematical evidence for the τ-function expansion.

In addition, according to our tests, it is numerically faster to calculate the accessory parameter expansion with our approach than inverting the Gram matrix and taking the semiclassical limit. We did not compare our approach with the Zamolodchikov recurrence formula, but, then again, the exact large c limit is also demanding in this case, while our algorithm already gives the analytic coefficients in the semiclassical limit.

5 Isomonodromic Approach to Classical Conformal Blocks

Now that we found the accessory parameter using the isomonodromic τ-function, an interesting question is if we can find an analogous formula for classical conformal blocks. One way to approach this problem is to use the symplectic structure of isomonodromic deformations [3, 55]. As we discussed above, $S_\sigma(\lambda,t)$ is the action of the isomonodromic Hamiltonian system

$$\dot{\lambda}=\frac{\partial H}{\partial p},\quad \dot{p}=-\frac{\partial H}{\partial\lambda},\tag{5.1}$$

determined by the Hamilton–Jacobi equation (2.17). The associated symplectic structure is given by

$$\Omega=dp\wedge d\lambda-dH\wedge dt.\tag{5.2}$$

Defining the one-form

$$\omega=pd\lambda-Hdt,\tag{5.3}$$

we recover $\Omega=d\omega$. We can define the action as the generating function of the canonical transformation from (p,λ) to action-angle coordinates (σ,ν)

$$dS_\sigma=pd\lambda-Hdt+\nu d\sigma.\tag{5.4}$$

Here $\sigma=\sigma_{0t}$ and ν is the canonically conjugate variable to σ [3, 26]. These coordinates parameterize the moduli space of $\mathrm{SL}(2,\mathbb{C})$ flat-connections, similarly to (σ,s). The transformation

$$p(\mu)=\mu+\sum_{i=0,1,t}\frac{1-2\theta_i}{2(\lambda-a_i)},\tag{5.5}$$

$$\begin{aligned}H(\boldsymbol{\theta};\lambda,p(\mu),t)=K(\boldsymbol{\theta};\lambda,\mu,t)+&\\+\frac{(1-2\theta_0)(1-2\theta_t)}{2t}+\frac{(1-2\theta_1)(1-2\theta_t)}{2(t-1)}+\frac{1-2\theta_t}{2(\lambda-t)}&\end{aligned}\tag{5.6}$$

between the normal form (3.20) and the canonical form (3.15) is canonical with respect to Ω [50]. This induces a transformation

$$S_\sigma=S_\sigma^c+g(\lambda,t)\tag{5.7}$$

by the function

$$g(\lambda,t)=\log\left[\prod_{i=0,1}(t-a_i)^{-2(\frac{1}{2}-\theta_i)(\frac{1}{2}-\theta_t)}\prod_{i=0,1,t}(\lambda-a_i)^{\frac{1}{2}-\theta_i}\right]\tag{5.8}$$

and thus

$$dS_\sigma^c=\mu d\lambda-Kdt+\nu d\sigma.\tag{5.9}$$

Integrating (5.9) over a solution of the isomonodromic system, as the action-angle variables are constant on the orbits, we get

$$S^c_\sigma(\lambda(t),t) = \int_{(\lambda(0),0)}^{(\lambda(t),t)} (\mu d\lambda' - K dt') = \int_0^t \left(\mu \frac{d\lambda'}{dt'} - K\right) dt'. \tag{5.10}$$

Let us try to simplify the PVI Lagrangian as much as possible. First, we notice that

$$\dot\lambda = \frac{\partial K}{\partial \mu} = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left[2\mu - \left(\frac{2\theta_0}{\lambda} + \frac{2\theta_1}{\lambda-1} + \frac{2\theta_t - 1}{\lambda - t}\right)\right] \tag{5.11}$$

and therefore

$$\mu\dot\lambda - K = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\mu^2 - \frac{\kappa(\lambda-t)}{t(t-1)}. \tag{5.12}$$

Using Okamoto's formula (4.17), we can integrate the second term in (5.12)

$$S^c_\sigma(\lambda(t),t) = \int_0^t \frac{\lambda(t')(\lambda(t')-1)(\lambda(t')-t')}{t'(t'-1)}\mu(t')^2\, dt' - \frac{\kappa}{2\theta_\infty}\log\frac{\tau(\boldsymbol{\theta}_{0,-};t)}{\tau(\boldsymbol{\theta}_{0,+};t)}\bigg|_0^t. \tag{5.13}$$

As both τ-functions have the same leading behaviour as t goes to zero, only the upper limit contributes to the second term above. Another equation proved by Okamoto in [58] is

$$\begin{aligned}\lambda(\lambda-1)\mu &= -\kappa_1(\lambda-t) + t(t-1)\frac{d}{dt}\log\left[\frac{\tau(\boldsymbol{\theta}_{-,0};t)}{\tau(\boldsymbol{\theta}_{0,+};t)}t^{\theta_0+\theta_t-\frac14}\right]\\ &= \frac{t(t-1)}{2\theta_\infty}\frac{d}{dt}\log\left[\left(\frac{\tau_+}{\tau_-}\right)^{\kappa_1}\left(t^{\theta_0+\theta_t-\frac14}\frac{\tau_0}{\tau_+}\right)^{2\theta_\infty}\right],\end{aligned} \tag{5.14}$$

where we used (4.17) in the second line and

$$\tau_+ \coloneqq \tau(\boldsymbol{\theta}_{0,+};t), \quad \tau_- \coloneqq \tau(\boldsymbol{\theta}_{0,-};t), \quad \tau_0 \coloneqq \tau(\boldsymbol{\theta}_{-,0};t). \tag{5.15}$$

It also follows from (4.17) that

$$\lambda = \frac{t(t-1)}{2\theta_\infty}\frac{d}{dt}\log\left[(t-1)^{2\theta_\infty}\frac{\tau_-}{\tau_+}\right], \quad \lambda - 1 = \frac{t(t-1)}{2\theta_\infty}\frac{d}{dt}\log\left[t^{2\theta_\infty}\frac{\tau_-}{\tau_+}\right]. \tag{5.16}$$

We then use (5.14) and (5.16) to express the action only in terms of τ-functions

$$S^c_\sigma(\lambda(t),t) = \frac{1}{2\theta_\infty}\int_0^t I[\tau_\pm,\tau_0;t']\, dt' - \frac{\kappa}{2\theta_\infty}\log\left(\frac{\tau_-}{\tau_+}\right), \tag{5.17}$$

where

$$I[\tau_\pm,\tau_0;t]=\frac{\frac{d}{dt}\log\left(\frac{\tau_-}{\tau_+}\right)\left\{\frac{d}{dt}\log\left[\left(\frac{\tau_+}{\tau_-}\right)^{\kappa_1}\left(t^{\theta_0+\theta_t-\frac{1}{4}}\frac{\tau_0}{\tau_+}\right)^{2\theta_\infty}\right]\right\}^2}{\frac{d}{dt}\log\left(t^{2\theta_\infty}\frac{\tau_-}{\tau_+}\right)\frac{d}{dt}\log\left((t-1)^{2\theta_\infty}\frac{\tau_-}{\tau_+}\right)}. \tag{5.18}$$

Although (5.17) is still complicated, substitution of (4.23) into (5.18) should give a formula for it only in terms of known functions of the monodromies. However, to get a closed form for the 4-point classical conformal block (1.1), we still need to impose the condition $\lambda(x)=x$ into the resulting formula, following the procedure of the previous section. As we can only solve this condition order-by-order in x, a closed form is still out of reach. But, as we are going to see below, some special cases can be tractable. We leave the detailed study of this formula for future work.

5.1 Recovering the 4-point Classical Conformal Block

The 5-point semiclassical block (2.12) is given by

$$\begin{aligned}&\langle V_{\delta_0}(0)V_{\delta_t}(t)\,\Pi_\sigma\,\varphi_H(\lambda(t))\Pi_{\sigma\pm\frac{1}{2}}V_{\delta_1}(1)V_{\delta_\infty}(\infty)\rangle\sim\exp\left(\frac{1}{b^2}S^{\pm}_{\sigma}(\lambda(t),t)\right)\\&=(\lambda(t)-t)^{(\frac{1}{2}-\theta_t)/b^2}\prod_{i=0,1}\left[(t-a_i)^{-2(\frac{1}{2}-\theta_i)(\frac{1}{2}-\theta_t)/b^2}(\lambda-a_i)^{(\frac{1}{2}-\theta_i)/b^2}\right]\exp\left(\frac{1}{b^2}S^{c,\pm}_{\sigma}(\lambda(t),t)\right),\end{aligned} \tag{5.19}$$

where we used (5.7). We label the fields by their classical weights δ_i and Π_σ corresponds to the projection operator onto the intermediate state with momentum $P=i\sigma/b$. Notice that because $\lambda(t)$ is a solution of isomonodromic deformations, we can change the position of the heavy degenerate field without changing the other monodromies. This has a nice AdS_3 interpretation, as degenerate insertions are conical defects in the bulk [60–62]. The PVI action governs the evolution of this conical defect in a way that the monodromies do not change. Therefore, we impose the boundary condition for the isomonodromic flow at $t=x$ to be $\lambda(x)=x$. This entails to taking the fusion

$$\varphi_H(\lambda(t))V_{\delta_t}(t)=C_+(\lambda(t)-t)^{(\frac{1}{2}-\theta_t)/b^2}V_{\delta(\frac{1}{2}+\theta_t)}(t)+C_-(\lambda(t)-t)^{(\frac{1}{2}+\theta_t)/b^2}V_{\delta(\frac{1}{2}-\theta_t)}(t). \tag{5.20}$$

Our choice (5.19) clearly corresponds to taking $C_-=0$ so that the leading term in the OPE cancels with the appropriate term in right hand side of (5.19). Therefore, we end up with

$$\langle V_{\delta_0}(0)V_{\delta_x}(x)\Pi_\nu V_{\delta_1}(1)V_{\delta_\infty}(\infty)\rangle=\prod_{i=0,1}(x-a_i)^{-2(\frac{1}{2}-\theta_i)(\frac{1}{2}-\theta_x)/b^2}\exp\left(\frac{1}{b^2}S^{c,\pm}_{\sigma}(x;\boldsymbol{\theta})\right), \tag{5.21}$$

where we defined $\theta_t = \theta_x - \frac{1}{2}$ and $\nu = \sigma \pm \frac{1}{2}$. The identification of the classical conformal block with the τ-function in (5.17) is a new result with important technical consequences, as the τ-function is a linear combination of $c = 1$ conformal blocks, as described in [34].

Linear Dilaton Case

In some special cases, the PVI solutions dramatically simplify. The simplest example is obtained by assuming

$$\sum_{i=0,1,t} \theta_i + \theta_\infty = \frac{1}{2} \tag{5.22}$$

and $\sigma_{ij} = \theta_i + \theta_j$, $i, j = 0, 1, t$. This implies that $\kappa_2 + 1 = 0$ (see (3.8)) and thus $\mu(t) = 0$ is consistent with the equations of motion (3.18) [50]. Therefore, (5.13) gives $S^c_\sigma = 0$ and the PVI action is given by

$$S_\sigma(\lambda(t), t) = \log \left[(t(t-1))^{\theta_t - \frac{1}{2}} \prod_{i=0,1,t} (\lambda(t) - a_i)^{\frac{1}{2} - \theta_i} \right] \tag{5.23}$$

leading to the 4-point conformal block (5.21)

$$\langle V_{\delta_0}(0) V_{\delta_x}(x) \Pi_\nu V_{\delta_1}(1) V_{\delta_\infty}(\infty) \rangle = x^{-2\alpha_0 \alpha_x} (x-1)^{-2\alpha_1 \alpha_x}, \tag{5.24}$$

where $\nu = \theta_0 + \theta_x$ and

$$\alpha_i = \left(\frac{1}{2} - \theta_i \right) \frac{1}{b^2}, \tag{5.25}$$

according to the alternative Liouville definition $\Delta_i = \alpha_i(Q - \alpha_i)$. This corresponds to the semiclassical limit of the linear dilaton correlator obeying the screening condition [48]

$$\sum_{i=0,1,x,\infty} \alpha_i = Q \to \frac{1}{b^2} \quad \text{as} \quad b \to 0. \tag{5.26}$$

Notice that, in the $c = 1$ interpretation, the τ-function is a hypergeometric function [43, 63], showing the non-triviality of this case in comparison to the $c = \infty$ result.

6 Conclusions

In this work, we discussed the deep mathematical relation between the large central charge limit of conformal blocks and the isomonodromic τ-function. We recovered the accessory parameter (4.37) of the 4-point Fuchsian equation (1.3) using the isomonodromic τ-function, with the additional fusion constraint $\lambda(x) = x$. We believe our approach gives a more straightforward algorithm than the one presented in [3].

The isomonodromic approach is relevant to applications of the theory of differential equations, as there is no need to take any semiclassical limit, in comparison to the CFT approach to calculate the accessory parameters. Thus, it can be applied to any particular problem that is governed by a Heun's equation. Recently, Piatek and Pietrykowski [64] recovered Floquet solutions of Heun's equation using CFT. This result nicely complements our discussion of accessory parameters. These techniques might be used in many concrete applications. In particular, the isomonodromic method has already proved useful for scattering problems in black hole physics [55, 56, 65].

We can also use it to calculate accessory parameters of confluent cases of Heun's equation, using the corresponding τ-functions described in [43, 66]. Those are connected to the so-called irregular conformal blocks [67–69], which deserve to be better understood in CFT applications. Finally, we notice that it is straightforward to generalize the isomonodromic method for Fuchsian equations with any number of singular points by using the appropriate τ-function [40, 70]. Our algorithm can then be generalized to find the accessory parameters and classical conformal blocks of n-point correlators.

The integral formula (5.17) for the PVI action is the first step to a closed expression for the 4-point classical conformal block (1.1). Using the τ-function $c = 1$ expansion, it should be possible to fully integrate the PVI action and impose the fusion constraint to obtain (1.1). This has many potential applications, in particular, related to the AdS/CFT correspondence: the emergence of AdS_3 gravity backgrounds [6, 8], calculations of entanglement entropy [5] and the bulk computation of classical blocks from the geodesic approach [10–14]. In fact, the isomonodromic approach generalizes the monodromy method of [5, 71]. The main obstruction is solving the PVI boundary condition in closed form. We gave a simple example where the PVI solution simplifies and recovers the $c = \infty$ limit of a linear dilaton conformal block [48]. Many other special limits remain to be explored, for example, the classical, algebraic and Riccati solutions of PVI [34, 43]. Moreover, the connection between the isomonodromic and the AGT approaches [25, 72] can give further insights on all these applications.

Although the relationship between classical conformal blocks, Painlevé VI equation and isomonodromic deformations has been discussed before [3, 41, 42], the importance of the τ-function has been fully appreciated only here. This relation is relevant not only from the technical point of view, but also highlights the intriguing map between the $c = 1$ and $c = \infty$ conformal blocks. Coulomb Gas (Dotsenko–Fateev) integral [59] and Fredholm determinant [70, 73, 74] representations of conformal blocks might also be useful to understand this map.

Another important question is what is the quantum counterpart of the $c = 1$ isomonodromic structure described here. A related development is that the canonical quantization of the isomonodromic equations is equivalent to the Knizhnik–Zamolodchikov (KZ) equations (equivalently the isomonodromic equations are the classical limit of the KZ equations) [75, 76]. Solutions of BPZ equations can be related to solutions of KZ equations [77–79], connecting Liouville theory to WZNW models. This web of relations, in connection to AGT correspondence and Hitchin systems, was reviewed in [41, 42]. These papers also briefly mention the role of the isomonodromic τ-function in the semiclassical CFT limit. More recently, the authors of [80] have proposed a generalization of the isomonodromy/CFT correspondence to arbitrary central charge using q-Painlevé conformal blocks [81] and cluster algebras. It would be interesting to study how the classical limit studied here emerges from this quantum description.

The physical meaning of the $c = 1$ and $c = \infty$ relationship, if any, deserves further exploration in the future and we hope our work will be helpful in this direction too.

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A Well-posedness of Initial Conditions

Here we show that the boundary conditions

$$\lambda(x) = x, \quad \mu(x) = -\frac{K_x}{2\theta_t}, \tag{A.1}$$

are well-posed with respect to the isomonodromic flow. The Garnier system (3.33) is given by

$$\dot{\lambda} = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}\left(2\mu - \frac{2\theta_0}{\lambda} - \frac{2\theta_1}{\lambda-1} - \frac{2\theta_t-1}{\lambda-t}\right), \tag{A.2a}$$

$$\begin{aligned}\dot{\mu} = -\frac{1}{t(t-1)}\Big[& (\lambda(3\lambda-2)+t(1-2\lambda))\mu^2 \\ & +(2\theta_0(-2\lambda+t+1)+2\theta_1(t-2\lambda)-(2\lambda-1)(2\theta_t-1))\mu+\kappa_1(\kappa_2+1)\Big].\end{aligned} \tag{A.2b}$$

This system has three critical points at $t = 0, 1, \infty$. Close to these singular points, the asymptotics of $\lambda(t)$ can be obtained via Painlevé VI solutions [28]. Our initial condition is defined close to a regular point $t = x$, with $x \neq 0, 1, \infty$. Therefore, the solution of the

Garnier system can be expressed as Taylor series in a neighborhood around $t = x$. If we substitute the series solution around $t = x$

$$\lambda(t) = x + \lambda_1(t-x) + \frac{\lambda_2}{2}(t-x)^2 + \cdots, \tag{A.3}$$

$$\mu(t) = \mu_0 + \mu_1(t-x) + \cdots \tag{A.4}$$

into (A.2), we get that $\lambda_1 = 1 - 2\theta_t$ and that λ_2 and μ_1 are determined explicitly in terms of x, θ's and by μ_0. This means that μ_0 can be taken as any finite constant, for example, the initial condition (A.1). Therefore, the initial conditions (A.1) are consistent with a series solution of the Garnier system around $t = x$.

B Ratio of Painlevé VI Structure Constants

Let $C_n \equiv C(\boldsymbol{\theta}, \sigma + n)$, where $C(\boldsymbol{\theta}, \sigma)$ is given by (4.14). A useful formula is the ratio of two structure constants [34]

$$\begin{aligned}\frac{C_{n\pm1}}{C_n} = &-\frac{\Gamma^2\left(1 \mp 2(\sigma+n)\right)}{\Gamma^2\left(1 \pm 2(\sigma+n)\right)} \prod_{\epsilon=\pm} \frac{\Gamma\left(1+\epsilon\theta_0+\theta_t \pm (\sigma+n)\right)\Gamma\left(1+\epsilon\theta_\infty+\theta_1 \pm (\sigma+n)\right)}{\Gamma\left(1+\epsilon\theta_0+\theta_t \mp (\sigma+n)\right)\Gamma\left(1+\epsilon\theta_\infty+\theta_1 \mp (\sigma+n)\right)} \times \\ &\times \frac{\left(\theta_0^2 - (\theta_t \mp (\sigma+n))^2\right)\left(\theta_\infty^2 - (\theta_1 \mp (\sigma+n))^2\right)}{4\left(\sigma+n\right)^2\left(1 \pm 2(\sigma+n)\right)^2},\end{aligned} \tag{B.1}$$

which can be derived from (4.14). Using that

$$\Gamma(z+n) = (z)_n\Gamma(z), \quad \Gamma(z-n) = \frac{(-1)^n\Gamma(z)}{(1-z)_n}, \tag{B.2}$$

where $(z)_n = z(z+1)\ldots(z+n-1)$ is the Pochhammer symbol, we have

$$\begin{aligned}\frac{C_{n\pm1}}{C_n} = &\left(\prod_{\epsilon=\pm} \frac{(1+\epsilon\theta_0+\theta_t+\sigma)_n(-\epsilon\theta_0-\theta_t+\sigma)_n(1+\epsilon\theta_\infty+\theta_1+\sigma)_n(-\epsilon\theta_\infty-\theta_1+\sigma)_n}{(2\sigma)_{2n\pm1}(1\pm2\sigma)_{2n\pm1}}\right)^{\pm1} \\ &\left(\theta_0^2 - (\theta_t \mp (\sigma+n))^2\right)\left(\theta_\infty^2 - (\theta_1 \mp (\sigma+n))^2\right)\left(\frac{\theta_\infty+\theta_1+\sigma}{\theta_\infty+\theta_1-\sigma}\right)^{\pm1}(-A)^{\pm1}\end{aligned} \tag{B.3}$$

where

$$[A(\boldsymbol{\theta}, \sigma)]^{\pm1} := \frac{4\sigma^2(1\pm2\sigma)^2}{[(\theta_\infty \pm \sigma)^2 - \theta_1^2][(\theta_t \mp \sigma)^2 - \theta_0^2]}\frac{C_{\pm1}}{C_0}. \tag{B.4}$$

Finally, we define the ratios

$$\bar{C}_n \equiv \frac{C_n}{C_0} = \prod_{k=0}^{|n|-1} \frac{C_{(k+1)\,\mathrm{sgn}(n)}}{C_{k\,\mathrm{sgn}(n)}} \tag{B.5}$$

and, if we use (B.3), we get

$$\bar{C}_n(\boldsymbol{\theta},\sigma) \coloneqq \mathcal{C}_n(\boldsymbol{\theta},\operatorname{sgn}(n)\sigma)A(\boldsymbol{\theta},\sigma)^n, \tag{B.6}$$

where

$$\begin{aligned}
&\mathcal{C}_n(\sigma) \equiv \\
&\prod_{k=0}^{|n|-1}\left(\prod_{\epsilon=\pm}\frac{(1+\epsilon\theta_0+\theta_t+\sigma)_k(-\epsilon\theta_0-\theta_t+\sigma)_k(1+\epsilon\theta_\infty+\theta_1+\sigma)_k(-\epsilon\theta_\infty-\theta_1+\sigma)_k}{(2\sigma)_{2k+1}(1+2\sigma)_{2k+1}}\right)\times \\
&\qquad\times\left(\frac{\sigma+\theta_\infty+\theta_1}{\sigma-\theta_\infty-\theta_1}\right)\left(\theta_0^2-(\theta_t-\sigma-k)^2\right)\left(\theta_\infty^2-(\theta_1-\sigma-k)^2\right).
\end{aligned} \tag{B.7}$$

C Analytical and Numerical Checks of Accessory Parameter Expansion

As discussed in section 4.3, the solution of the condition $\lambda(x)=x$ gives a series expansion $X(x)=1+X_1x+X_2x^2\dots$ for the PVI integration constant $s(x)$ encoded in $X(x)$. We present analytically the first

$$X_1=-\sigma\frac{(\delta_\sigma^2+(\delta_1-\delta_\infty)(\delta_0-\delta_x))}{\delta_\sigma^2},$$

and second order coefficients of this expansion

$$
\begin{aligned}
X_2 =\ & \sigma^2 \frac{\left(\delta_\sigma^2+(\delta_1-\delta_\infty)(\delta_0-\delta_x)\right)^2}{2\delta_\sigma^4}+\\
&+\frac{\sigma}{8\delta_\sigma^4(4\delta_\sigma+3)^2}\Big\{\delta_1^2\left[\delta_\sigma^2(4\delta_\sigma(\delta_\sigma+6)+9)+3\left(20\delta_\sigma^2-6\delta_\sigma-9\right)\delta_x^2-6\delta_\sigma^2(8\delta_\sigma+3)\delta_x\right]+\\
&+\delta_\sigma^2\left[-\delta_\sigma^2(26\delta_\sigma(2\delta_\sigma+3)+27)+(4\delta_\sigma(\delta_\sigma+6)+9)\delta_x^2-6\delta_\sigma^2\delta_x\right]-\\
&-6\delta_\sigma^2\delta_\infty\left[\delta_\sigma^2+(8\delta_\sigma+3)\delta_x^2+2(4\delta_\sigma(\delta_\sigma+2)+3)\delta_x\right]+\\
&+\delta_0^2\left[6\delta_1\left[\left(-20\delta_\sigma^2+6\delta_\sigma+9\right)\delta_\infty-\delta_\sigma^2(8\delta_\sigma+3)\right]-6\delta_\sigma^2(8\delta_\sigma+3)\delta_\infty+\right.\\
&\left.+3\left(20\delta_\sigma^2-6\delta_\sigma-9\right)\delta_\infty^2+3\delta_1^2\left(20\delta_\sigma^2-6\delta_\sigma-9\right)+\delta_\sigma^2(4\delta_\sigma(\delta_\sigma+6)+9)\right]+\\
&+2\delta_0\left[6\delta_1\left[\delta_\infty\left[(8\delta_\sigma+3)\delta_\sigma^2+\left(20\delta_\sigma^2-6\delta_\sigma-9\right)\delta_x\right]+\delta_\sigma^2\left(-4\delta_\sigma^2+8\delta_\sigma(\delta_x-1)+3(\delta_x-1)\right)\right]-\right.\\
&-3\delta_\infty^2\left((8\delta_\sigma+3)\delta_\sigma^2+\left(20\delta_\sigma^2-6\delta_\sigma-9\right)\delta_x\right)+2\delta_\sigma^2\delta_\infty(4\delta_\sigma(5\delta_\sigma+6)+3(8\delta_\sigma+3)\delta_x+9)-\\
&\left.-3\delta_1^2\left((8\delta_\sigma+3)\delta_\sigma^2+\left(20\delta_\sigma^2-6\delta_\sigma-9\right)\delta_x\right)+\delta_\sigma^2\left(-3\delta_\sigma^2-(4\delta_\sigma(\delta_\sigma+6)+9)\delta_x\right)\right]+\\
&+2\delta_1\left(-(4\delta_\sigma(\delta_\sigma+6)+9)\delta_\sigma^2\delta_\infty-3\delta_\sigma^4+2\delta_\sigma^2\delta_x(4\delta_\sigma(5\delta_\sigma+6\delta_\infty+6)+9(\delta_\infty+1))+\right.\\
&\left.+3\delta_x^2\left(\left(-20\delta_\sigma^2+6\delta_\sigma+9\right)\delta_\infty-\delta_\sigma^2(8\delta_\sigma+3)\right)\right)+\\
&+\delta_\infty^2\left(+\left(\delta_\sigma^2(4\delta_\sigma(\delta_\sigma+6)+9)+3\left(20\delta_\sigma^2-6\delta_\sigma-9\right)\delta_x^2-6\delta_\sigma^2(8\delta_\sigma+3)\delta_x\right)\right)\Big\}\,.
\end{aligned}
$$

Plugging back the $X(x)$ series into the logarithm of the τ-function (4.37), following the procedure of section 4.4, we get the series expansion of the accessory parameter

$$
H_x=\frac{\delta_0+\delta_x-\delta_\sigma}{x}+\frac{(\delta_0-\delta_\sigma-\delta_x)(\delta_\sigma+\delta_1-\delta_\infty)}{2\delta_\sigma}+\sum_{n=1}H_n x^n. \tag{C.1}
$$

We present here the next two terms in (C.1)

$$
\begin{aligned}
H_1=-\frac{x}{8\delta_\sigma^3(4\delta_\sigma+3)}\Big\{&\delta_0^2\left(\delta_\sigma^2(3-6\delta_\infty)+5\delta_\sigma\delta_\infty^2-2\delta_1\left(5\delta_\sigma\delta_\infty+3\delta_\sigma^2-3\delta_\infty\right)+\delta_\sigma^3\right.\\
&\left.+\delta_1^2(5\delta_\sigma-3)-3\delta_\infty^2\right)-2\delta_0\left(2\delta_1\left(3\delta_\sigma^3-3\delta_\sigma^2(\delta_\infty+\delta_x-1)-5\delta_\sigma\delta_\infty\delta_x+3\delta_\infty\delta_x\right)\right.\\
&+(\delta_\sigma-\delta_\infty)\left(\delta_\sigma^2(7\delta_\sigma-3\delta_\infty+6)+\delta_x\left(\delta_\sigma(3-5\delta_\infty)+\delta_\sigma^2+3\delta_\infty\right)\right)+\delta_1^2\left(3\delta_\sigma^2+(5\delta_\sigma-3)\delta_x\right)\Big)\\
&+(\delta_\sigma-\delta_\infty)\left(\delta_\sigma^2\left(-\delta_\sigma(\delta_\infty-9)+13\delta_\sigma^2-3\delta_\infty\right)+\delta_x^2\left(\delta_\sigma(3-5\delta_\infty)+\delta_\sigma^2+3\delta_\infty\right)\right.\\
&\left.+6\delta_\sigma^2\delta_x(3\delta_\sigma+\delta_\infty+2)\right)\\
&-2\delta_1\left(\delta_\sigma^2\left(\delta_\sigma(\delta_\infty-6)-9\delta_\sigma^2+3\delta_\infty\right)+\delta_x^2\left(5\delta_\sigma\delta_\infty+3\delta_\sigma^2-3\delta_\infty\right)-2\delta_\sigma^2\delta_x(5\delta_\sigma+3\delta_\infty+3)\right)\\
&+\delta_1^2\left(\delta_\sigma^2(\delta_\sigma+3)+(5\delta_\sigma-3)\delta_x^2-6\delta_\sigma^2\delta_x\right)\Big\}\,,
\end{aligned}
$$

$$
\begin{aligned}
H_2 = \frac{1}{16\delta_\sigma^5\left(4\delta_\sigma^2+11\delta_\sigma+6\right)} & \Big(\Big(\left(9\delta_\sigma^2-19\delta_\sigma+6\right)\delta_1^3+\left(-14\delta_\sigma^3-3\left(9\delta_\infty-4\right)\delta_\sigma^2+57\delta_\infty\delta_\sigma-18\delta_\infty\right)\delta_1^2 \\
& + \left(5\delta_\sigma^4+7\delta_\sigma^3+3\left(9\delta_\infty^2-2\right)\delta_\sigma^2-57\delta_\infty^2\delta_\sigma+18\delta_\infty^2\right)\delta_1 \\
& - \delta_\infty\left(5\delta_\sigma^4+\left(7-14\delta_\infty\right)\delta_\sigma^3+3\left(3\delta_\infty^2+4\delta_\infty-2\right)\delta_\sigma^2-19\delta_\infty^2\delta_\sigma+6\delta_\infty^2\right)\Big)\delta_0^3 \\
& + \Big(\left(2\left(6-7\delta_\sigma\right)\delta_\sigma^2-3\delta_x\left(9\delta_\sigma^2-19\delta_\sigma+6\right)\right)\delta_1^3 \\
& + \left(5\delta_\sigma^4+21\left(2\delta_\infty+2\delta_x-1\right)\delta_\sigma^3+9\left(-4\delta_\infty+\left(9\delta_\infty-4\right)\delta_x+2\right)\delta_\sigma^2-171\delta_x\delta_\infty\delta_\sigma+54\delta_\infty\delta_x\right)\delta_1^2 \\
& - 3\Big(\delta_x\left(5\delta_\sigma^4+7\delta_\sigma^3+3\left(9\delta_\infty^2-2\right)\delta_\sigma^2-57\delta_\infty^2\delta_\sigma+18\delta_\infty^2\right) \\
& - 2\delta_\sigma^2\left(2\delta_\sigma^3+\left(5\delta_\infty+4\right)\delta_\sigma^2-7\left(\delta_\infty-1\right)\delta_\infty\delta_\sigma+6\left(\delta_\infty-1\right)\delta_\infty\right)\Big)\delta_1 \\
& - \left(\delta_\sigma-\delta_\infty\right)\Big(3\delta_\sigma^5-3\left(7\delta_\infty-5\right)\delta_\sigma^4+\left(14\delta_\infty^2-3\left(5\delta_x+11\right)\delta_\infty+18\right)\delta_\sigma^3 \\
& + 3\delta_\infty\left(-4\delta_\infty+\left(9\delta_\infty-7\right)\delta_x+6\right)\delta_\sigma^2+3\delta_\infty\left(6-19\delta_\infty\right)\delta_x\delta_\sigma+18\delta_\infty^2\delta_x\Big)\Big)\delta_0^2 \\
& + \Big(\left(3\left(9\delta_\sigma^2-19\delta_\sigma+6\right)\delta_x^2+\delta_\sigma^2\left(5\delta_\sigma^2+7\delta_\sigma-6\right)\right)\delta_1^3 \\
& - 3\Big(\left(14\delta_\sigma^3+3\left(9\delta_\infty-4\right)\delta_\sigma^2-57\delta_\infty\delta_\sigma+18\delta_\infty\right)\delta_x^2-2\delta_\sigma^2\left(5\delta_\sigma^2+7\delta_\sigma-6\right)\delta_x \\
& - \delta_\sigma^2\left(\delta_\sigma+2\right)\left(4\delta_\sigma^2-5\delta_\infty\delta_\sigma+3\delta_\infty\right)\Big)\delta_1^2+3\Big(\left(5\delta_\sigma^4+7\delta_\sigma^3+3\left(9\delta_\infty^2-2\right)\delta_\sigma^2-57\delta_\infty^2\delta_\sigma+18\delta_\infty^2\right)\delta_x^2 \\
& - 4\left(3\delta_\sigma^2+5\delta_\infty\delta_\sigma-3\delta_\infty\right)\delta_\sigma^2\left(\delta_\sigma+2\right)\delta_x+\left(7\delta_\sigma^3+\left(9-12\delta_\infty\right)\delta_\sigma^2+5\delta_\infty^2\delta_\sigma-3\delta_\infty^2\right)\delta_\sigma^2\left(\delta_\sigma+2\right)\Big)\delta_1 \\
& + \left(\delta_\sigma-\delta_\infty\right)\Big(3\delta_\infty\left(-5\delta_\sigma^3+\left(9\delta_\infty-7\right)\delta_\sigma^2+\left(6-19\delta_\infty\right)\delta_\sigma+6\delta_\infty\right)\delta_x^2 \\
& + 6\left(\delta_\sigma^2+\left(3-5\delta_\infty\right)\delta_\sigma+3\delta_\infty\right)\delta_\sigma^2\left(\delta_\sigma+2\right)\delta_x \\
& + \left(26\delta_\sigma^3+\left(24-19\delta_\infty\right)\delta_\sigma^2+\delta_\infty\left(5\delta_\infty-3\right)\delta_\sigma-3\delta_\infty^2\right)\delta_\sigma^2\left(\delta_\sigma+2\right)\Big)\Big)\delta_0 \\
& - \delta_1\Big(2\left(19\delta_\sigma^2-3\left(\delta_\infty-4\right)\delta_\sigma-9\delta_\infty\right)\left(\delta_\sigma+2\right)\delta_\sigma^4 \\
& - 6\delta_x^2\left(4\delta_\sigma^3+\left(5\delta_\infty+8\right)\delta_\sigma^2+7\delta_\infty\left(\delta_\infty+1\right)\delta_\sigma-6\delta_\infty\left(\delta_\infty+1\right)\right)\delta_\sigma^2 \\
& + 3\left(15\delta_\sigma^3+3\left(4\delta_\infty+3\right)\delta_\sigma^2+5\delta_\infty^2\delta_\sigma-3\delta_\infty^2\right)\delta_x\left(\delta_\sigma+2\right)\delta_\sigma^2 \\
& + \left(5\delta_\sigma^4+7\delta_\sigma^3+3\left(9\delta_\infty^2-2\right)\delta_\sigma^2-57\delta_\infty^2\delta_\sigma+18\delta_\infty^2\right)\delta_x^3\Big) \\
& - \left(\delta_\sigma-\delta_\infty\right)\Big(\left(23\delta_\sigma^2-3\left(\delta_\infty-5\right)\delta_\sigma-9\delta_\infty\right)\left(\delta_\sigma+2\right)\delta_\sigma^4 \\
& + \left(3\delta_\sigma^3+\left(15-9\delta_\infty\right)\delta_\sigma^2+\left(-14\delta_\infty^2-9\delta_\infty+18\right)\delta_\sigma+6\delta_\infty\left(2\delta_\infty+3\right)\right)\delta_x^2\delta_\sigma^2 \\
& + \left(38\delta_\sigma^3+\left(17\delta_\infty+24\right)\delta_\sigma^2+\delta_\infty\left(5\delta_\infty-3\right)\delta_\sigma-3\delta_\infty^2\right)\delta_x\left(\delta_\sigma+2\right)\delta_\sigma^2 \\
& + \delta_\infty\left(-5\delta_\sigma^3+\left(9\delta_\infty-7\right)\delta_\sigma^2+\left(6-19\delta_\infty\right)\delta_\sigma+6\delta_\infty\right)\delta_x^3\Big) \\
& + \delta_1^2\Big(3\left(\delta_\sigma^2+5\delta_\sigma+6\right)\delta_\sigma^3-3\delta_x\left(\delta_\sigma+2\right)\left(7\delta_\sigma^2+\left(5\delta_\infty-3\right)\delta_\sigma-3\delta_\infty\right)\delta_\sigma \\
& + \left(14\delta_\sigma^3+3\left(9\delta_\infty-4\right)\delta_\sigma^2-57\delta_\infty\delta_\sigma+18\delta_\infty\right)\delta_x^2\Big)\left(\delta_x-\delta_\sigma\right) \\
& - \delta_1^3\delta_x\left(\left(9\delta_\sigma^2-19\delta_\sigma+6\right)\delta_x^2-2\delta_\sigma^2\left(7\delta_\sigma-6\right)\delta_x+\delta_\sigma^2\left(5\delta_\sigma^2+7\delta_\sigma-6\right)\right)\,.
\end{aligned}
$$

We tested our results numerically from order x^3 and only up to order x^5, as it is time consuming to simplify the expressions in terms of δ's. The table 1 shows the values of the coefficients H_n if we substitute the θ's by numbers in our algorithm from the beginning. The numbers presented here all match the classical conformal block calculated via the inverse Gram matrix CFT approach, also substituting numbers from the beginning.

Notice that we tested some special transformations of conformal blocks by permutations of the values of the δ's, in the rows 3 to 8 of the table. In the rows 3 and 4, and also 7 and 8, we

δ_σ	δ_0	δ_1	δ_x	δ_∞	H_3	H_4	H_5
1	1	1	1	1	−1.07867	−1.14311	−1.18935
1	0.4	0.4	0.4	0.4	−0.492431	−0.491792	−0.491431
2	1	1	2	2	−1.2865	−1.30949	−1.31137
2	2	2	1	1	−1.2865	−1.30949	−1.31137
0.5	0.1	0.2	0.3	0.4	−0.201122	−0.20042	−0.199985
0.5	0.2	0.1	0.4	0.3	−0.201122	−0.20042	−0.199985
0.4	2	1	2.1	1.2	−1.7517	−1.73116	−1.65458
0.4	2.1	1.2	2	1	−1.8517	−1.83116	−1.75458

Table 1. Numerical coefficients from order x^3 to order x^5 for different values of δ's.

permute $\delta_0 \leftrightarrow \delta_x, \delta_1 \leftrightarrow \delta_\infty$. The conformal block is not invariant under this transformation, as we can see in rows 7 and 8. However, in the lines 3 and 4, the weights remain the same and this becomes a symmetry. Finally, in rows 5 and 6, we test the symmetry $\delta_0 \leftrightarrow \delta_1, \delta_x \leftrightarrow \delta_\infty$.

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