MAXIMAL INDEPENDENT SETS, VARIANTS OF CHAIN/ANTICHAIN PRINCIPLE AND COFINAL SUBSETS WITHOUT AC

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Abstract. In set theory without the Axiom of Choice (AC), we observe new relations of the following statements with weak choice principles.

- $\mathcal{P}_{lf,c}$ (Every locally finite connected graph has a maximal independent set).
- $\mathcal{P}_{lc,c}$ (Every locally countable connected graph has a maximal independent set).
- $CAC_1^{\aleph_\alpha}$ (If in a partially ordered set all antichains are finite and all chains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$) if $\aleph_\alpha$ is regular.
- CWF (Every partially ordered set has a cofinal well-founded subset).
- If $G = (V_G, E_G)$ is a connected locally finite chordal graph, then there is an ordering $<$ of V_G such that $\{w < v : \{w, v\} \in E_G\}$ is a clique for each $v \in V_G$.

1. INTRODUCTION

As usual, ZF denotes the Zermelo-Fraenkel set theory without the Axiom of Choice (AC), and ZFA is ZF with the axiom of extensionality weakened to allow the existence of atoms. In this note, we observe new relations of some combinatorial statements with weak choice principles.

1.1. Maximal independent sets.

Friedman [[Fri11], **Theorem 6.3.2, Theorem 2.4**] proved that AC is equivalent to the statement *'Every graph has a maximal independent set'* (abbreviated here as $\mathcal{P}$) in ZF. Spanring [Spa14] gave a different argument to prove the result. Consider the following weaker formulations of $\mathcal{P}$.

- Fix $n \in \omega\backslash\{0,1\}$. We denote by P_{K_n}, the class of those graphs whose only components are K_n (complete graph on n vertices). We denote by $\mathcal{P}_n$ the statement *'Every graph from the class P_{K_n} has a maximal independent set'*.
- We denote by $\mathcal{P}_{lf,c}$ the statement *'Every locally finite connected graph has a maximal independent set'*.
- We denote by $\mathcal{P}_{lc,c}$ the statement *'Every locally countable connected graph has a maximal independent set'*.

In this note, we observe the following.

1. AC_n (Every family of n element sets has a choice function) is equivalent to $\mathcal{P}_n$ for every $n \in \omega\backslash\{0,1\}$ in ZF (c.f. [§**3**, **Observation 3.1**]).
2. AC^{ω}_{fin} (Every denumerable family of non-empty finite sets has a choice function) is equivalent to $\mathcal{P}_{lf,c}$ in ZF (c.f. [§**3**, **Observation 3.2**]).
3. $UT(\aleph_0, \aleph_0, \aleph_0)$ (The union of any countable family of countable sets is countable) implies $\mathcal{P}_{lc,c}$, and $\mathcal{P}_{lc,c}$ implies $AC^{\aleph_0}_{\aleph_0}$ (Every denumerable family of denumerable sets has a choice function) in ZF (c.f. [§**3**, **Observation 3.3**]).

1.2. A variant of Chain/Antichain principle.

A famous application of the infinite Ramsey's theorem is the *Chain/Antichain principle* (abbreviated here as "CAC"), which states that *'Any infinite partially ordered set contains either an infinite chain or an infinite antichain'*. Tachtsis

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[Tac16] investigated the possible placement of CAC in the hierarchy of weak choice principles. Komjáth–Totik [KT06] proved the following generalized versions of CAC, applying Zorn's lemma.

- *If in a partially ordered set all antichains are finite and all chains are countable, then the set is countable* (c.f. [[KT06], **Chapter 11**, **Problem 8**]).
- *If in a partially ordered set all chains are finite and all antichains are countable, then the set is countable* (c.f. [[KT06], **Chapter 11**, **Problem 7**]).

For each regular $\aleph_\alpha$, we denote by $CAC_1^{\aleph_\alpha}$ the statement *'if in a partially ordered set all antichains are finite and all chains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$'* and we denote by $CAC^{\aleph_\alpha}$ the statement *'if in a partially ordered set all chains are finite and all antichains have size $\aleph_\alpha$, then the set has size $\aleph_\alpha$'*. In [BG20], we observed that for any regular $\aleph_\alpha$ and any $2 \leq n < \omega$, $CAC^{\aleph_\alpha}$ does not imply AC_n^- (Every infinite family of n-element sets has a partial choice function) in ZFA. In [BG20], we also observed that $CAC^{\aleph_\alpha}$ does not imply *'there are no amorphous sets'* in ZFA. In this note, we observe the following.

1. For any regular $\aleph_\alpha$, and any $2 \leq n < \omega$, $CAC_1^{\aleph_\alpha}$ does not imply AC_n^- in ZFA (c.f. [§**4**, **Theorem 4.3**]). In particular, for any regular $\aleph_\alpha$, $CAC_1^{\aleph_\alpha}$ holds in the model constructed in the proof of [[HT20], **Theorem 8**].
2. For any regular $\aleph_\alpha$, $CAC_1^{\aleph_\alpha}$ does not imply *'there are no amorphous sets'* in ZFA (c.f. [§**4**, **Theorem 4.4**]). In particular, for any regular $\aleph_\alpha$, $CAC_1^{\aleph_\alpha}$ holds in the basic Fraenkel model.
3. $CAC_1^{\aleph_0}$ implies $PAC_{fin}^{\aleph_1}$ in ZF if we denote by $PAC_{fin}^{\aleph_1}$ the statement *'Every infinite $\aleph_1$-sized family $\mathcal{A}$ of non-empty finite sets has a $\aleph_1$-sized subfamily $\mathcal{B}$ with a choice function'* (c.f. [§**4**, **Theorem 4.6**]).
4. DC (Dependent choice) does not imply $CAC_1^{\aleph_0}$ in ZFA (c.f. [§**4**, **Theorem 4.7**]).

1.3. **Cofinal well-founded subsets and improving the choice strength of a result.** Tachtsis [[Tac18], **Theorem 10(ii)**] proved that CWF (Every partially ordered set has a cofinal well-founded subset) holds in the basic Fraenkel model. In [[THS16], **Theorem 3.26**], Tachtsis, Howard, and Saveliev proved that CS (Every partially ordered set without a maximal element has two disjoint cofinal subsets) holds in the basic Fraenkel model. Halbeisen–Tachtsis [[HT20], **Theorem 10(ii)**] proved that LOC_2^- (Every infinite linearly orderable family of 2-element sets has a partial choice function) does not imply $LOKW_4^-$ (Every infinite linearly orderable family $\mathcal{A}$ of 4-element sets has a partial Kinna–Wegner selection function) in ZFA. We construct a model of ZFA and observe the following.

1. (LOC_2^- + CS + CWF) does not imply LOC_n^- in ZFA if $n \in \omega$ such that $n = 3$ or $n > 4$ (c.f. [§**5**, **Theorem 5.2**]).
2. (LOC_2^- + CS + CWF) does not imply $CAC_1^{\aleph_0}$ in ZFA (c.f. [§**5**, **Corollary 5.3**]).

Fix $n \in \omega\backslash\{0,1\}$, and $k \in \omega\backslash\{0,1,2\}$. The authors of [CHHKR08] proved that AC_n holds in $\mathcal{N}_2^*(k)$ (generalised version of Howard's model $\mathcal{N}_2^*(3)$ from [HR98]) if k has no divisors less than or equal to n (c.f. [[CHHKR08], **Theorem 4.8**]). We observe that it is possible to improve the choice strength of the result if k is a prime applying the methods of [HT13]. In particular, we observe the following.

1. Fix any prime $p \in \omega\backslash\{0,1,2\}$, and any $n \in \omega\backslash\{0,1\}$. If p is not a divisor of n, then AC_n holds in $\mathcal{N}_2^*(p)$. Moreover, CWF holds in $\mathcal{N}_2^*(p)$ (c.f. [§**5**, **Theorem 5.4**]).

We also remark that CWF holds in the Second Fraenkel's model (labeled as Model $\mathcal{N}_2$ in [HR98]), and $\mathcal{N}_{22}(p)$ (the model from [[HT13], §**4.4**]) for any prime $p \in \omega\backslash\{0,1,2\}$ (c.f. [§**5**, **Remark 5.5**, **Remark 5.6**]).

1.4. **Locally finite connected chordal graphs.** Fulkerson–Gross [FG65] proved that a finite graph $G = (V_G, E_G)$ is chordal if and only if there is an ordering $<$ of V_G such that $\{w < v :$

$\{w, v\} \in E_G\}$ is a clique for each $v \in V_G$ (c.f. [[Kom15], **Lemma 1**]). We apply the result to observe the following.

(1) AC^{ω}_{fin} implies the statement *'If* $G = (V_G, E_G)$ *is a connected locally finite chordal graph, then there is an ordering* $<$ *of* V_G *such that* $\{w < v : \{w, v\} \in E_G\}$ *is a clique for each* $v \in V_G$*'* in ZF (c.f. [§**3**, **Observation 3.5**]).

We also list some other graph-theoretical statements restricted to locally finite connected graphs, which follows from AC^{ω}_{fin} in ZF (c.f. [§**3**, **Remark 3.6**]).

2. Notations, definitions, and known results

Definition 2.1. ***(Graph-theoretical definitions, and notations).*** A graph $G = (V_G, E_G)$ is *locally finite* if every vertex of G has finite degree. We say that a graph is *locally countable* if every vertex has denumerable set of neighbours. Given a non-negative integer n, a *path of length* n in the graph $G = (V_G, E_G)$ is a one-to-one finite sequence $\{x_i\}_{0\leq i\leq n}$ of vertices such that for each $i < n$, $\{x_i, x_{i+1}\} \in E_G$; such a path joins x_0 to x_n. The graph G is *connected* if any two vertices are joined by a path of finite length. An *independent set* is a set of vertices in a graph, no two of which are connected by an edge. A set $W_G \subseteq V_G$ is called a *maximal independent set* in $G = (V_G, E_G)$ if and only if it is independent and there is no independent set W'_G such that $W_G \subsetneq W'_G$ (c.f.[Spa14]). A *clique* is a set of vertices in a graph, such that any two of them are joined by an edge. We denote by K_n, the complete graph on n vertices.

Definition 2.2. ***(Chain, antichain, cofinal well-founded subsets).*** Let P be a set. A binary relation $\leq$ on P is called a *partial order* on P if $\leq$ is reflexive, antisymmetric, and transitive. The ordered pair $(P, \leq)$ is called a *partially ordered set* or *poset.* A subset $D \subseteq P$ is called a *chain* if $(D, \leq\restriction D)$ is linearly ordered. A subset $A \subseteq P$ is called an *antichain* if no two elements of A are comparable under $\leq$. A subset $C \subseteq P$ is called *cofinal* in P if for every $x \in P$ there is an element $c \in C$ such that $x \leq c$. An element $p \in P$ is *minimal* if for all $q \in P$, $(q \leq p)$ implies $(q = p)$. A subset $W \subseteq P$ is *well-founded* if every non-empty subset V of W has a $\leq$-minimal element.

Definition 2.3. ***(Amorphous sets).*** An innite set X is called *amorphous* if X cannot be written as a disjoint union of two innite subsets.

Definition 2.4. ***(A list of forms).***

(1) The ***Axiom of Choice*****, AC (Form 1 in** [HR98]**)**: Every family of nonempty sets has a choice function.
(2) AC^{ω}_{fin}**(Form 10 in** [HR98]**)**: Every denumerable family of non-empty finite sets has a choice function. We recall two equivalent formulations of AC^{ω}_{fin}.
  - $UT(\aleph_0, fin, \aleph_0)$ **(Form 10 A in** [HR98]**)**: The union of denumerably many pairwise disjoint finite sets is denumerable.
  - PAC^{ω}_{fin}**(Form 10 E in** [HR98]**)**: Every denumerable family of finite sets has an infinite subfamily with a choice function.
(3) $AC^{\aleph_0}_{\aleph_0}$ **(Form 32 A in** [HR98]**)**: Every denumerable set of denumerable sets has a choice function. We recall the following equivalent formulation of $AC^{\aleph_0}_{\aleph_0}$.
  - $PAC^{\aleph_0}_{\aleph_0}$ **(Form 32 B in** [HR98]**)**: Every denumerable set of denumerable sets has an infinite subset with a choice function.
(4) AC_2 **(Form 88 in** [HR98]**)**: Every family of pairs has a choice function.
(5) AC_n **for each** $n \in \omega, n \geq 2$ **(Form 61 in** [HR98]**)**: Every family of n element sets has a choice function. We denote by AC^-_n the statement *'Every infinite family of* n*-element sets has a partial choice function' (****Form 342(n)*** *in* [HR98]*, denoted by* C^-_n *in* ***Definition 1 (2)*** *of* [HT20]*).*
(6) LOC^-_n **for each** $n \in \omega, n \geq 2$ **(see** [HT20]**)**: Every infinite linearly orderable family of n-element sets has a partial choice function. We denote by $LOKW^-_n$ the statement

'Every infinite linearly orderable family $\mathcal{A}$ of n-element sets has a partial Kinna–Wegner selection function' (c.f. ***Definition 1 (2)*** *of* [HT20]*).*

(7) The ***Van Douwens Choice Principle*, vDCP** (see [HT13]): Every family $X = \{(X_i, \leq_i) : i \in I\}$ of linearly ordered sets isomorphic with $(\mathbb{Z}, \leq)$ ($\leq$ is the usual ordering on $\mathbb{Z}$) has a choice function.

(8) The ***Axiom of Multiple Choice*, MC (Form 67 in** [HR98]**)**: Every family $\mathcal{A}$ of non-empty sets has a multiple choice function, i.e., there is a function f with domain $\mathcal{A}$ such that for every $A \in \mathcal{A}$, $f(A)$ is a non-empty finite subset of A.

(9) **MC(n) where $n \geq 2$ is an integer (see** [HT13]**)**: For every family $\{X_i : i \in I\}$ of non-empty sets, there is a function F with domain I such that for all $i \in I$, we have that $F(i)$ is a finite subset of X_i and $gcd(n, |F(i)|) = 1$.

(10) **LW (Form 90 in** [HR98]**)**: Every linearly-ordered set can be well-ordered.

(11) AC^{WO} **(Form 40 in** [HR98]**)**: Every well-ordered set of non-empty sets has a choice function.

(12) DC_κ **for an innite well-ordered cardinal κ (Form 87(κ) in** [HR98]**)**: Let κ be an innite well-ordered cardinal (i.e., κ is an aleph). Let S be a non-empty set and let R be a binary relation such that for every $\alpha < \kappa$ and every α-sequence $s = (s_\epsilon)_{\epsilon<\alpha}$ of elements of S there exists $y \in S$ such that sRy. Then there is a function $f : \kappa \to S$ such that for every $\alpha < \kappa$, $(f \restriction \alpha)Rf(\alpha)$. We note that $DC_{\aleph_0}$ is a reformulation of DC (the principle of Dependent Choices **(Form 43 in** [HR98]**)**). We denote by $DC_{<\lambda}$ the assertion $(\forall\eta < \lambda)DC_\eta$.

(13) **UT(WO, WO, WO) (Form 231 in** [HR98]**)**: The union of a well-ordered collection of well-orderable sets is well-orderable.

(14) $(\forall\alpha)UT(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ **(Form 23 in** [HR98]**)**: For every ordinal α, if A and every member of A has cardinality $\aleph_\alpha$, then $|\cup A| = \aleph_\alpha$.

(15) **$\aleph_1$ is regular (Form 34 in** [HR98]**)**.

(16) ***Dilworths decomposition theorem for infinite posets of finite width,* DT (c.f.**[Tac19]**)**: If $\mathbb{P}$ is an arbitrary poset, and k is a natural number such that $\mathbb{P}$ has no antichains of size $k+1$ while at least one k-element subset of $\mathbb{P}$ is an antichain, then $\mathbb{P}$ can be partitioned into k chains.

(17) The ***Chain/Antichain Principle*, CAC (Form 217 in** [HR98]**)**: Every infinite poset has an infinite chain or an infinite antichain.

(18) **There are no amorphous sets (Form 64 in** [HR98]**)**.

(19) **CS** (see [THS16]): Every poset without a maximal element has two disjoint cofinal subsets.

(20) **CWF** (see [Tac18]): Every poset has a cofinal well-founded subset.

(21) **A weaker form of Łoś's lemma, LT (Form 253 in** [HR98]**)**: If $\mathcal{A} = \langle A, \mathcal{R}^{\mathcal{A}}\rangle$ is a non-trivial relational $\mathcal{L}$-structure over some language $\mathcal{L}$, and $\mathcal{U}$ be an ultrafilter on a non-empty set I, then the ultrapower $\mathcal{A}^I/\mathcal{U}$ and $\mathcal{A}$ are elementarily equivalent.

2.1. **Group-theoretical facts.** A group $\mathcal{G}$ *acts on a set* X if for each $g \in \mathcal{G}$ there is a mapping $x \to gx$ of X into itself, such that $1x = x$ for every $x \in X$ and $h(gx) = (hg)x$ for every $g, h \in \mathcal{G}$. Alternatively, actions of a group $\mathcal{G}$ on a set X are the same as group homomorphisms from $\mathcal{G}$ to $Sym(X)$. Suppose that a group $\mathcal{G}$ acts on a set X. Let $Orb_{\mathcal{G}}(x) = \{gx : g \in \mathcal{G}\}$ be the orbit of $x \in X$ under the action of $\mathcal{G}$, and $Stab_{\mathcal{G}}(x) = \{g \in \mathcal{G} : gx = x\}$ be the stabilizer of x under the action of $\mathcal{G}$. The *Orbit-Stabilizer theorem* states that the size of the orbit is the index of the stabilizer, that is $|Orb_{\mathcal{G}}(x)| = [\mathcal{G} : Stab_{\mathcal{G}}(x)]$. We also recall that different orbits of the action are disjoint and form a partition of X i.e., $X = \bigcup\{Orb_{\mathcal{G}}(x) : x \in X\}$. An *alternating group* is the group of even permutations of a finite set. Let $\{G_i : i \in I\}$ be an indexed collection of groups. Define $\prod^{weak}_{i\in I} G_i = \{f : I \to \bigcup_{i\in I} G_i \mid (\forall i \in I)f(i) \in G_i, f(i) = 1_{G_i} \textit{ except finitely many } i\}$. The *weak direct product* of the groups $\{G_i : i \in I\}$ is the set $\prod^{weak}_{i\in I} G_i$ with the operation of component wise multiplicative defined for all $f, g \in \prod^{weak}_{i\in I} G_i$ by $(fg)(i) = f(i)g(i)$ for all $i \in I$.

2.2. Fraenkel–Mostowski permutation models.

We start with a ground model M of $ZFA+AC$ where A is a set of atoms. Each permutation of A extends uniquely to a permutation of M by ϵ-induction. A permutation model $\mathcal{N}$ of ZFA is determined by a group $\mathcal{G}$ of permutations of A and a normal filter $\mathcal{F}$ of subgroups of $\mathcal{G}$. Let $\mathcal{G}$ be a group of permutations of A and $\mathcal{F}$ be a normal filter of subgroups of $\mathcal{G}$. For $x \in M$, we denote the symmetric group with respect to $\mathcal{G}$ by $sym_{\mathcal{G}}(x) = \{g \in \mathcal{G} \mid g(x) = x\}$. We say x is *$\mathcal{F}$-symmetric* if $sym_{\mathcal{G}}(x) \in \mathcal{F}$ and x is *hereditarily $\mathcal{F}$-symmetric* if x is $\mathcal{F}$-symmetric and each element of transitive closure of x is symmetric. We define the permutation model $\mathcal{N}$ with respect to $\mathcal{G}$ and $\mathcal{F}$, to be the class of all hereditarily $\mathcal{F}$-symmetric sets and recall that $\mathcal{N}$ is a model of ZFA (c.f. [[Jec73], **Theorem 4.1**]). If $\mathcal{I} \subseteq \mathcal{P}(A)$ is a normal ideal, then the filter base $\{\text{fix}_{\mathcal{G}}E : E \in \mathcal{I}\}$ generates a normal filter over $\mathcal{G}$, where $\text{fix}_{\mathcal{G}}E$ denotes the subgroup $\{\phi \in \mathcal{G} : \forall y \in E(\phi(y) = y)\}$ of $\mathcal{G}$. Let $\mathcal{I}$ be a normal ideal generating a normal filter $\mathcal{F}_{\mathcal{I}}$ over $\mathcal{G}$. Let $\mathcal{N}$ be the permutation model determined by $M, \mathcal{G}$, and $\mathcal{F}_{\mathcal{I}}$. We say $E \in \mathcal{I}$ supports a set $\sigma \in \mathcal{N}$ if $\text{fix}_{\mathcal{G}}E \subseteq sym_{\mathcal{G}}(\sigma)$.

Lemma 2.5. The following hold.

1. In every Fraenkel–Mostowski permutation model, CS implies vDCP (c.f. [[THS16], **Theorem 3.15(3)**]).
2. In ZFA, CWF implies LW (c.f. [[Tac18], **Lemma 5**]).

Lemma 2.6. *(c.f. [*[HT13], ***Lemma 4.3****]). Assume P is a set of prime numbers, $\mathcal{M}$ is a Fraenkel-Mostowski permutation model determined by the set A of atoms, the group $\mathcal{G}$ of permutations of A, and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$. Assume further that*

1. *$\mathcal{G}$ is Abelian.*
2. *For every $x \in \mathcal{M}$, $Orb_{\mathcal{G}}(x)$ is finite.*
3. *There is a group $\mathcal{G}_0 \in \mathcal{F}$ such that for all $\phi \in \mathcal{G}_0$, if p is a prime divisor of the order of ϕ then $p \in P$.*

Then for every set $Z \in \mathcal{M}$ of non-empty sets there is a function f with domain Z such that for all $y \in Z$, $\emptyset \subsetneq f(y) \subseteq y$ and every prime divisor of $|f(y)|$ is in P.

2.3. Loeb's theorem.

A topological space (X, τ) is called *compact* if for every $U \subseteq \tau$ such that $\bigcup U = X$ there is a finite subset $V \subseteq U$ such that $\bigcup V = X$.

Lemma 2.7. *(c.f. [*[Loeb65], ***Theorem 1****]). Let $\{X_i\}_{i \in I}$ be a family of compact spaces which is indexed by a set I on which there is a well-ordering $\leq$. If I is an infinite set and there is a choice function F on the collection $\{C : C \text{ is closed}, C \neq \emptyset, C \subset X_i \text{ for some } i \in I\}$, then the product space $\prod_{i \in I} X_i$ is compact in the product topology.*

2.4. A theorem of Fulkerson and Gross.

Fulkerson–Gross [FG65] proved the following lemma.

Lemma 2.8. *(c.f. [*[Kom15], ***Lemma 1****],* [FG65]*).* A finite graph (V, X) is chordal if and only if there is an ordering $<$ of V such that $\{w < v : \{w, v\} \in X\}$ is a clique for each $v \in V$.

3. GRAPH THEORETICAL OBSERVATIONS

3.1. Maximal independent set.

Observation 3.1. (ZF) For every $n \in \omega \backslash \{0, 1\}$, $\mathcal{P}_n$ is equivalent to AC_n.

Proof. ($\Leftarrow$) Fix $n \in \omega \backslash \{0, 1\}$, and let us assume AC_n. Let $G = (V_G, E_G)$ be a graph from the class P_{K_n} (c.f. §**1.1**, for definition of P_{K_n}). Let $\{G_i\}_{i \in I} = \{(V_{G_i}, E_{G_i})\}_{i \in I}$ be the components of G. By AC_n select $g_i \in V_{G_i}$ for each $i \in I$. We can see that $J = \{g_i : i \in I\}$ is a maximal independent set of G. For any $g_i, g_j \in J$ such that $g_i \neq g_j$, we have $\{g_i, g_j\} \notin E_G$. Consequently, J is an independent set. For the sake of contradiction, suppose J is not a maximal independent set. Then there is an independent set L which must contain two vertices x and y from V_{G_i} for some $i \in I$. Since $\{x, y\} \in E_G$, we obtain a contradiction.

($\Rightarrow$) Fix $n \in \omega \backslash \{0,1\}$, and let us assume $\mathcal{P}_n$. Consider a system of n-element sets $\mathcal{A} = \{A_i\}_{i \in I}$. We construct a graph $G = (V_G, E_G)$.

Constructing G: Let V_G consists of all the pairs (Y, y) such that $Y \in \mathcal{A}$ and $y \in Y$, and the edge set is defined as follows $\{(Y_1, y_1), (Y_2, y_2)\} \in E_G$ if and only if $Y_1 = Y_2$ and $y_1 \neq y_2$.

Clearly, the components of G are K_n. By $\mathcal{P}_n$, G has a maximal independent set M. Since M is an independent set, for each $Y \in \mathcal{A}$ there is at most one $y \in Y$ such that $(Y, y) \in M$. Since M is a maximal independent set, there is at least one $y \in Y$ such that $(Y, y) \in M$. Consequently, M determines a choice function for $\mathcal{A}$. $\square$

Observation 3.2. (ZF) AC^{ω}_{fin} is equivalent to $\mathcal{P}_{lf,c}$.

Proof. ($\Rightarrow$) We assume AC^{ω}_{fin}. Let $G = (V_G, E_G)$ be some non-empty locally finite, connected graph. Consider some $r \in V_G$. Let $V_0 = \{r\}$. For each integer $n \geq 1$, define $V_n = \{v \in V_G : d_G(r, v) = n\}$ where '$d_G(r, v) = n$' means there are n edges in the shortest path joining r and v. Each V_n is finite by locally finiteness of G, and $V_G = \bigcup_{n \in \omega} V_n$ by connectedness of G. By $UT(\aleph_0, fin, \aleph_0)$ (which is equivalent to AC^{ω}_{fin}(c.f. **Definition 2.4**)), V_G is countable. Consequently, V_G is well-ordered. We prove that every graph based on a well-ordered set of vertices has a maximal independent set in ZF. Let $G = (V_G, E_G)$ be a graph on a well-ordered set of vertices $V_G = \{v_\alpha : \alpha < \lambda\}$. Thus we can use transfinite recursion, without using any form of choice, to construct a maximal independent set. Let $M_0 = \emptyset$. Clearly, M_0 is an independent set. For any ordinal α, if M_α is a maximal independent set, then we are done. Otherwise, there is some $v \in V_G \backslash M_\alpha$, where $M_\alpha \cup \{v\}$ is an independent set of vertices. In that case, let $M_{\alpha+1} = M_\alpha \cup \{v\}$. For limit ordinals α, we use $M_\alpha = \bigcup_{i \in \alpha} M_i$. Clearly, $M = \bigcup_{i \in \lambda} M_i$ is a maximal independent set.

($\Leftarrow$) We assume $\mathcal{P}_{lf,c}$. Since AC^{ω}_{fin} is equivalent to its partial version PAC^{ω}_{fin} (c.f. **Definition 2.4** or [HR98]), it suffices to show PAC^{ω}_{fin}. Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a denumerable set of non-empty finite sets. Without loss of generality, we assume that $\mathcal{A}$ is disjoint. Consider a denumerable sequence $T = \{t_n : n \in \omega\}$ disjoint from $\mathcal{A}$. We construct a graph $G = (V_G, E_G)$.

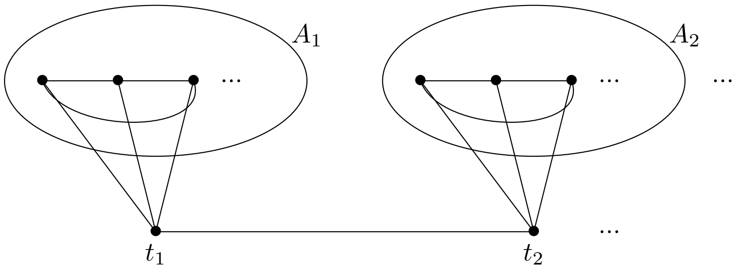

FIGURE 1. *The graph G.*

Constructing G: Let $V_G = (\bigcup_{n \in \omega} A_n) \cup T$. For each $n \in \omega$, let $\{t_n, t_{n+1}\} \in E_G$ and $\{t_n, x\} \in E_G$ for every element $x \in A_n$. Also for each $n \in \omega$, and any two $x, y \in A_n$ such that $x \neq y$, let $\{x, y\} \in E_G$ (see Figure 1).

Clearly, the graph G is connected and locally finite. By assumption, G has a maximal independent set of vertices, say M. Since M is maximal, M has to be infinite. Moreover, for each $i \in \omega$, either $t_i \in M$ or some $v \in A_i$ is in M. Since M is an independent set, for each $i \in \omega$ there is at most one $v \in A_i$ such that $v \in M$. Define $M' = \{v \in M : v \in A_i \text{ for some } i \in \omega\}$. If M' is infinite, then M' determines a partial choice function for $\mathcal{A}$.

Case (1). Suppose $M \backslash M'$ is finite. Then M' is infinite.

Case (2). Suppose $M \backslash M'$ is infinite. Since $\{t_n, t_{n+1}\} \in E_G$ for any $n \in \omega$, if $t_n \in M \backslash M'$, then $t_{n+1} \in M'$. Consequently, M' must be infinite as well. $\square$

Observation 3.3. (ZF) *$UT(\aleph_0,\aleph_0,\aleph_0)$ implies $\mathcal{P}_{lc,c}$, and $\mathcal{P}_{lc,c}$ implies $AC^{\aleph_0}_{\aleph_0}$.*

Proof. In order to prove the first implication, let $G = (V_G, E_G)$ be some non-empty locally countable connected graph. Consider some $r \in V_G$. Let $V_0 = \{r\}$. For each integer $n \geq 1$, define $V_n = \{v \in V_G : d_G(r, v) = n\}$. Since G is locally countable, each V_n is countable by $UT(\aleph_0,\aleph_0,\aleph_0)$. Also $V_G = \bigcup_{n\in\omega} V_n$ since G is connected. By $UT(\aleph_0,\aleph_0,\aleph_0)$, V_G is countable. Rest follows from the fact that every graph based on a well-ordered set of vertices has a maximal independent set in ZF (c.f. the proof of **Observation 3.2**). The second assertion follows from the arguments of **Observation 3.2**, since $AC^{\aleph_0}_{\aleph_0}$ is equivalent to $PAC^{\aleph_0}_{\aleph_0}$ in ZF (c.f. **Definition 2.4** or [HR98]). □

Remark 3.4. Fix $n \in \omega\backslash\{0, 1\}$. We denote by C_n the cycle graph with n-vertices. We denote by P_{C_n}, the class of those graphs whose only components are C_n. We denote by $\mathcal{P}'_n$ the statement *'Every graph from the class P_{C_n}, has a maximal independent set'*. We remark that AC_{P_n} implies $\mathcal{P}'_n$ in ZF where P_n is the **Perrin number** of n. Let $G = (V_G, E_G)$ be a graph from the class P_{C_n}. Let $\{G_i\}_{i\in I} = \{(V_{G_i}, E_{G_i})\}_{i\in I}$ be the components of P_{C_n}. Let M_i be the collection of different maximal independent sets of G_i for each $i \in I$. Since the number of different maximal independent sets in each component is P_n[1], by AC_{P_n} we can choose a $m_i \in M_i$ for each $i \in I$. Clearly, $\bigcup_{i\in I} m_i$ is a maximal independent set of G.

3.2. **Locally finite connected graphs.**

Observation 3.5. (ZF)AC^{ω}_{fin} implies the statement *'If (V, X) is a connected locally finite chordal graph, then there is an ordering $<$ of V such that $\{w < v : \{w, v\} \in X\}$ is a clique for each $v \in V$'.*

Proof. We note that by arguments in the proof of **Observation 3.2**, it is enough to see that the statement *'If (V, X) is a chordal graph based on a well orderable set of vertices, then there is an ordering $<$ of V such that $\{w < v : \{w, v\} \in X\}$ is a clique for each $v \in V$'* is provable in ZF. By **Lemma 2.8**, each finite subgraph $(W, X|W)$ has an ordering such that $\{w < v : \{w, v\} \in X \restriction W\}$ is a clique for every $v \in W$. We can encode every total ordering of a set W by a choice of one of $<, =, >$ for each pair $(x, y) \in W \times W$. Endow $\{<, =, >\}$ with the discrete topology and $T = \{<, =, >\}^{V\times V}$ with the product topology. Since V is well-ordered, $V \times V$ is well-ordered in ZF. Consequently, $\{<, =, >\} \times \{V \times V\}$ is well-ordered in ZF. By **Lemma 2.7**, T is compact. We use the compactness of T to prove the existence of the desired ordering. □

Remark 3.6. We list some other graph-theoretical statements from different papers, restricted to locally finite connected graphs, which are related to AC^{ω}_{fin}.

1. Komjáth–Galvin [KG91] proved that any graph based on a well-ordered set of vertices has a chromatic number and an irreducible good coloring in ZF. Consequently, the statements *'any locally finite connected graph has a chromatic number'* and *'any locally finite connected graph has an irreducible good coloring'* are provable under AC^{ω}_{fin} in ZF.
2. Hajnal [[Haj85], **Theorem 2**] proved that if the chromatic number of a graph G_1 is finite (say $k < \omega$), and the chromatic number of another graph G_2 is infinite, then the chromatic number of $G_1 \times G_2$ is k. In [BG20] we observed that if G_1 is based on a well-ordered set of vertices, then the following statement holds in ZF.
$$\text{`}\chi(E_{G_1}) = k < \omega \textit{ and } \chi(E_{G_2}) \geq \omega \textit{ implies } \chi(E_{G_1\times G_2}) = k.\text{'}$$
Consequently, under AC^{ω}_{fin} the above statement holds in ZF if G_1 is a locally finite connected graph.
3. Delhommé and Morillon [DM06] proved that AC^{ω}_{fin} is equivalent to the statement *'Every locally finite connected graph has a spanning tree'* in ZF.

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[1] We use the fact that the number of different maximal independent sets in an n-vertex cycle graph is the n-th Perrin number for $1 < n < \omega$.

4. A VARIANT OF CAC

Tachtsis communicated to us the following lemma.

Lemma 4.1. The following holds.

1. $UT(\aleph_0, \aleph_0, \aleph_0)$ implies the statement *'If* $(P, \leq)$ *is a poset such that* P *is well-ordered, and if all antichains in* P *are finite and all chains in* P *are countable, then* P *is countable'.*
2. $\aleph_1$ is regular implies the statement *'If* $(P, \leq)$ *is a poset such that* P *is well-ordered, and if all antichains in* P *are finite and all chains in* P *are countable, then* P *is countable'.*

Proof. We prove (1). Let $(P, \leq)$ be a poset such that P is well-ordered, all antichains in P are finite, and all chains are countable. Fix a well-ordering $\preceq$ of P. By way of contradiction, assume that P is uncountable. We construct an infinite antichain to obtain a contradiction. Since P is well-ordered by $\preceq$, we may construct (via transfinite induction) a maximal $\leq$-chain, V_0 say, without invoking any form of choice. Since V_0 is countable, it follows that $P - V_0$ is uncountable and every element of $P - V_0$ is incomparable to some element of V_0. Thus $P - V_0 = \bigcup\{W_p : p \in V_0\}$, where W_p is the set of all elements of $P - V_0$ which are incomparable to p. Since $P - V_0$ is uncountable and V_0 is countable, it follows by $UT(\aleph_0, \aleph_0, \aleph_0)$ that W_p is uncountable for some p in V_0. Let p_0 be the least (with respect to $\preceq$) such element of V_0. Now, construct a maximal $\leq$-chain in (the uncountable set) W_{p_0}, V_1 say, and let (similarly to the above argument) p_1 be the least (with respect to $\preceq$) element of V_1 such that the set W_{p_1} of all elements of W_{p_0} which are incomparable to p_1 is uncountable. Continuing in this fashion by induction (and noting that the process cannot stop at a finite stage), we obtain a countably infinite antichain $\{p_n : n \in \omega\}$, contradicting the assumption that all antichains are finite. Therefore, P is countable.

Similarly, we can prove (2). □

Modifying **Lemma 4.1**, we may observe that $UT(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ implies the statement *'If* $(P, \leq)$ *is a poset such that* P *is well-ordered, and if all antichains in* P *are finite and all chains in* P *have size* $\aleph_\alpha$*, then* P *has size* $\aleph_\alpha$*'* for any regular $\aleph_\alpha$ in ZF.

Corollary 4.2. The statement *'If* $(P, \leq)$ *is a poset such that* P *is well-ordered, and if all antichains in* P *are finite and all chains in* P *are countable, then* P *is countable'* holds in any Fraenkel-Mostowski model.

Proof. Follows from the fact that the statement $\aleph_1$ is a regular cardinal holds in every Fraenkel-Mostowski model (c.f. [[HKRST01], **Corollary 1**]). □

Theorem 4.3. (ZFA) *For any regular* $\aleph_\alpha$*, and* $n \in \omega\backslash\{0, 1\}$*,* $CAC_1^{\aleph_\alpha}$ *does not imply* AC_n^-*.*

Proof. Halbeisen–Tachtsis [[HT20], **Theorem 8**] constructed a permutation model (we donote by $\mathcal{N}_{HT}^1(n)$) where for arbitrary $n \geq 2$, AC_n^- fails but CAC holds. We fix an arbitrary integer $n \geq 2$ and recall the model constructed in the proof of [[HT20], **Theorem 8**] as follows.

Defining the ground model M: We start with a ground model M of $ZFA + AC$ where A is a countably infinite set of atoms written as a disjoint union $\bigcup\{A_i : i \in \omega\}$ where for each $i \in \omega$, $A_i = \{a_{i_1}, a_{i_2}, ...a_{i_n}\}$.

Defining the group $\mathcal{G}$ and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$:

- **Defining $\mathcal{G}$:** $\mathcal{G}$ is defined in [HT20] in a way so that **if $\eta \in \mathcal{G}$, then η only moves finitely many atoms** and for all $i \in \omega$, $\eta(A_i) = A_k$ for some $k \in \omega$. We recall the details from [HT20] as follows. For all $i \in \omega$, let τ_i be the n-cycle $a_{i_1} \mapsto a_{i_2} \mapsto ...a_{i_n} \mapsto a_{i_1}$. For every permutation ψ of ω, which moves only finitely many natural numbers, let ϕ_ψ be the permutation of A defined by $\phi_\psi(a_{i_j}) = a_{\psi(i)_j}$ for all $i \in \omega$ and $j = 1, 2, ..., n$. Let $\eta \in \mathcal{G}$ if and only if $\eta = \rho\phi_\psi$ where ψ is a permutation of ω which moves only finitely many natural numbers and ρ is a permutation of A for which there is a finite $F \subseteq \omega$

such that for every $k \in F$, $\rho \restriction A_k = \tau_k^j$ for some $j < n$, and ρ fixes A_m pointwise for every $m \in \omega \backslash F$.

- **Defining $\mathcal{F}$:** Let $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ generated by $\{\text{fix}_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$.

Defining the permutation model: Consider the FM-model $\mathcal{N}_{HT}^1(n)$ determined by M, $\mathcal{G}$ and $\mathcal{F}$.

Following **point 1** in the proof of [[HT20], **Theorem 8**], both A and $\mathcal{A} = \{A_i\}_{i\in\omega}$ are amorphous in $\mathcal{N}_{HT}^1(n)$ and no infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ has a Kinna–Wegner selection function. Consequently, AC_n^- fails. We follow the steps below to prove that for any regular $\aleph_\alpha$, $CAC_1^{\aleph_\alpha}$ holds in $\mathcal{N}_{HT}^1(n)$.

(1) Let $(P, \leq)$ be a poset in $\mathcal{N}_{HT}^1(n)$ such that all antichains in P are finite and all chains in P have size $\aleph_\alpha$. Let $E \in [A]^{<\omega}$ be a support of $(P, \leq)$. We can write P as a disjoint union of $\text{fix}_{\mathcal{G}}(E)$-orbits, i.e., $P = \bigcup\{Orb_E(p) : p \in P\}$, where $Orb_E(p) = \{\phi(p) : \phi \in \text{fix}_{\mathcal{G}}(E)\}$ for all $p \in P$. The family $\{Orb_E(p) : p \in P\}$ is well-orderable in $\mathcal{N}_{HT}^1(n)$ since $\text{fix}_{\mathcal{G}}(E) \subseteq Sym_{\mathcal{G}}(Orb_E(p))$ for all $p \in P$.

(2) Since **if $\eta \in \mathcal{G}$, then η only moves finitely many atoms**, $Orb_E(p)$ is an antichain in P for each $p \in P$. Otherwise there is a $p \in P$, such that $Orb_E(p)$ is not an antichain in $(P, \leq)$. Thus, for some $\phi, \psi \in \text{fix}_{\mathcal{G}}(E)$, $\phi(p)$ and $\psi(p)$ are comparable. Without loss of generality we may assume $\phi(p) < \psi(p)$. Since **if $\eta \in \mathcal{G}$, then η only moves finitely many atoms**, there exists some $k < \omega$ such that $\phi^k = 1_A$. Let $\pi = \psi^{-1}\phi$. Consequently, $\pi(p) < p$ and $\pi^k = 1_A$ for some $k \in \omega$. Thus, $p = \pi^k(p) < \pi^{k-1}(p) < ... < \pi(p) < p$. By transitivity of $<$, $p < p$, which is a contradiction.

(3) Since $Orb_E(p)$ is an antichain, it is finite. Consequently, $Orb_E(p)$ is well-orderable. Since $UT(WO, WO, WO)$ holds in $\mathcal{N}_{HT}^1(n)$, P is well-orderable by (1) and (2). Also we note that $UT(WO, WO, WO)$ implies UT($\aleph_\alpha$,$\aleph_\alpha$,$\aleph_\alpha$) in any FM-model (c.f. page 176 of [HR98]). So, we are done by **Lemma 4.1** and the point noted in the paragraph after **Lemma 4.1**.

□

Theorem 4.4. (ZFA) *For any regular $\aleph_\alpha$, $CAC_1^{\aleph_\alpha}$ does not imply 'There are no amorphous sets'.*

Proof. We consider the basic Fraenkel model (labeled as Model $\mathcal{N}_1$ in [HR98]) where 'there are no amorphous sets' is false, and $UT(WO, WO, WO)$ holds (c.f. [HR98]). Let $(P, \leq)$ be a poset in $\mathcal{N}_1$, and E be a finite support of $(P, \leq)$. By the arguments of the proof of **Theorem 4.3**, $\mathcal{O} = \{Orb_E(p) : p \in P\}$ is a well-ordered partition of P. Now for each $p \in P$, $Orb_E(p)$ is an antichain (c.f. the proof of [[Jec73], **Lemma 9.3**]). Thus, by methods from the proof of **Theorem 4.3**, $CAC_1^{\aleph_\alpha}$ holds in $\mathcal{N}_1$. □

Remark 4.5. Since $UT(WO, WO, WO)$ holds in $\mathcal{N}_{HT}^1(n)$ and $\mathcal{N}_1$, AC_{fin}^ω holds in $\mathcal{N}_{HT}^1(n)$ and $\mathcal{N}_1$. Consequently, by **Observation 3.2**, $\mathcal{P}_{lf,c}$ holds in $\mathcal{N}_{HT}^1(n)$ and $\mathcal{N}_1$.

Theorem 4.6. *(ZF) $CAC_1^{\aleph_0}$ implies $PAC_{fin}^{\aleph_1}$.*

Proof. Let $\mathcal{A} = \{A_n : n \in \aleph_1\}$ be a family of non-empty finite sets. Without loss of generality, we assume that $\mathcal{A}$ is disjoint. Define a binary relation $\leq$ on $A = \bigcup \mathcal{A}$ as follows: for all $a, b \in A$, let $a \leq b$ if and only if $a = b$ or $a \in A_n$ and $b \in A_m$ and $n < m$. Clearly, $\leq$ is a partial order on A. Also, A is uncountable. The only antichains of $(A, \leq)$ are the finite sets A_n where $n \in \aleph_1$. By $CAC_1^{\aleph_0}$, A has an uncountable chain, say C. Let $M = \{m \in \aleph_1 : C \cap A_m \neq \emptyset\}$. Since C is a chain and $\mathcal{A}$ is the family of all antichains of $(A, \leq)$, we have $M = \{m \in \aleph_1 : |C \cap A_m| = 1\}$. Clearly, $f = \{(m, c_m) : m \in M\}$, where for $m \in M$, c_m is the unique element of $C \cap A_m$, is a choice function of the uncountable subset $\mathcal{B} = \{A_m : m \in M\}$ of $\mathcal{A}$. Thus $\mathcal{B}$ is a $\aleph_1$-sized subfamily of $\mathcal{A}$ with a choice function. □

Theorem 4.7. *(ZFA) DC does not imply $CAC_1^{\aleph_0}$.*

Proof. We recall Jech's model (labeled as $\mathcal{N}_2(\aleph_\alpha)$ in [HR98]).

- **Defining the ground model M.** We start with a ground model M of $ZFA+AC$ with an $\aleph_\alpha$-sized set A of atoms which is a disjoint union of $\aleph_\alpha$ pairs, so that $A = \bigcup\{A_\gamma : \gamma < \aleph_\alpha\}$, $A_\gamma = \{a_\gamma, b_\gamma\}$.
- **Defining the group $\mathcal{G}$ of permutations and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$.**
  - **Defining $\mathcal{G}$.** Let $\mathcal{G}$ be the group of all permutations of A which fix A_γ for all $\gamma < \aleph_\alpha$.
  - **Defining $\mathcal{F}$.** Let $\mathcal{F}$ be the normal filter on $\mathcal{G}$ which is generated by $\{\text{fix}_{\mathcal{G}}(E) : E \subset A, |E| < \aleph_\alpha\}$.
- **Defining the permutation model.** Consider the permutation model $\mathcal{N}_2(\aleph_\alpha)$ determined by M, $\mathcal{G}$ and $\mathcal{F}$.

Jech proved that $DC_{<\aleph_\alpha}$ is true in $\mathcal{N}_2(\aleph_\alpha)$. Let us consider the model $\mathcal{N}_2(\aleph_1)$. Clearly, $DC_{<\aleph_1}$ is true in $\mathcal{N}_2(\aleph_1)$. By **Theorem 4.6**, it is enough to show that $PAC^{\aleph_1}_{fin}$ fails in the model. We prove that the family $\mathcal{A} = \{A_\gamma : \gamma < \aleph_1\}$ of finite sets has no subfamily $\mathcal{B}$ of cardinality $\aleph_1$, such that $\mathcal{B}$ has a choice function. For the sake of contradiction, let $\mathcal{B}$ be a subfamily of cardinality $\aleph_1$ of $\mathcal{A}$ with a choice function $f \in \mathcal{N}_2(\aleph_1)$ and support $E \in [A]^{<\aleph_1}$. Since E is countable, there is a $\gamma < \aleph_1$ such that $A_\gamma \in \mathcal{B}$ and $A_\gamma \cap E = \emptyset$. Without loss of generality, let $f(A_\gamma) = a_\gamma$. Consider the permutation π which is the identity on A_η, for all $\eta \in \aleph_1 \backslash \{\gamma\}$, and let $(\pi \restriction A_\gamma)(a_\gamma) = b_\gamma \neq a_\gamma$. Then π fixes E pointwise, hence $\pi(f) = f$. So, $f(A_\gamma) = b_\gamma$ which contradicts the fact that f is a function. □

5. COFINAL WELL-FOUNDED SUBSETS IN ZFA

We modify the arguments from [[THS16],**Theorem 3.26**] and [[Tac18], **Theorem 10(ii)**] to observe the following.

Lemma 5.1. *Let A be a set of atoms. Let $\mathcal{G}$ be the group of permutations of A such that either each $\eta \in \mathcal{G}$ moves only finitely many atoms or there is a $n \in \omega \backslash \{0, 1\}$, such that for all $\eta \in \mathcal{G}$, $\eta^n = 1_A$. Let $\mathcal{F}$ be the normal filter of subgroups of $\mathcal{G}$ generated by $\{fix_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$. Then in the Fraenkel-Mostowski model $\mathcal{N}$ determined by A, $\mathcal{G}$, and $\mathcal{F}$, CS and CWF hold. Consequently, vDCP and LW hold.*

Proof. We follow the steps below.

1. Let $(P, \leq)$ be a poset in $\mathcal{N}$ and $E \in [A]^{<\omega}$ be a support of $(P, \leq)$. We can write P as a disjoint union of $\text{x}_{\mathcal{G}}(E)$-orbits, i.e., $P = \bigcup\{Orb_E(p) : p \in P\}$, where $Orb_E(p) = \{\phi(p) : \phi \in \text{x}_{\mathcal{G}}(E)\}$ for all $p \in P$. The family $\{Orb_E(p) : p \in P\}$ is well-orderable in $\mathcal{N}$ since $\text{x}_{\mathcal{G}}(E) \subseteq Sym_{\mathcal{G}}(Orb_E(p))$ for all $p \in P$.
2. We prove that $Orb_E(p)$ is an antichain in P for each $p \in P$. Otherwise there is a $p \in P$, such that $Orb_E(p)$ is not an antichain in $(P, \leq)$. Thus, for some $\phi, \psi \in \text{fix}_{\mathcal{G}}(E)$, $\phi(p)$ and $\psi(p)$ are comparable. Without loss of generality we may assume $\phi(p) < \psi(p)$. Let $\pi = \psi^{-1}\phi$. Consequently, $\pi(p) < p$.

   Case 1: Suppose there is a $n \in \omega \backslash \{0, 1\}$, such that for every $\eta \in \mathcal{G}$, $\eta^n = 1_A$. So $\pi^n = 1_A$. Thus, $p = \pi^n(p) < \pi^{n-1}(p) < ... < \pi(p) < p$. By transitivity of $<$, $p < p$, which is a contradiction.

   Case 2: Suppose each $\eta \in \mathcal{G}$, moves only finitely many atoms. Then for some $k < \omega$, $\pi^k = 1$. Rest follows from the arguments in **Case 1**.
3. We can follow [[THS16], **Theorem 3.26**] to see that CS holds in $\mathcal{N}$.
4. Although in every Fraenkel-Mostowski model, CS implies vDCP in ZFA (c.f. **Lemma 2.5**), we can recall the arguments from the 1^{st}-paragraph of [[THS16], **Page175**] to give a direct proof of vDCP in $\mathcal{N}$.
5. We can follow [[Tac18], **Theorem 10 (ii)**] to see that CWF holds in $\mathcal{N}$. By **Lemma 2.5**, LW holds in $\mathcal{N}$.

□

5.1. **A model of ZFA.** Herrlich, Howard, and Tachtsis [[HHT12], **Theorem 11**, **Case 1**, **Case 2**] constructed two different classes of permutation models. Halbeisen–Tachtsis [[HT20], **Theorem 10(ii)**] proved that LOC_2^- does not imply $LOKW_4^-$ in ZFA. For the sake of convenience, we denote by $\mathcal{N}_{HT}^2$, the permutation model of [[HT20], **Theorem 10(ii)**]. The model $\mathcal{N}_{HT}^2$ is very similar to the model from [[HHT12],**Theorem 11**, **Case 2**] except the fact that in $\mathcal{N}_{HT}^2$ each permutation ϕ in the group $\mathcal{G}$ of permutations of the sets of atoms, can move only finitely many atoms. Fix a natural number n such that $n = 3$ or $n > 4$. We construct a model $\mathcal{M}_n$ of ZFA similar to the model constructed in [[HHT12],**Theorem 11**, **Case 1**], where each permutation ϕ in the group $\mathcal{G}$ of permutations of the sets of atoms, can move only finitely many atoms. Consequently, by **Lemma 5.1**, CS, vDCP, CWF, and LW hold in $\mathcal{M}_n$. In particular we prove that (LOC_2^- + CS + CWF) does not imply LOC_n^- in ZFA if $n \in \omega$ such that $n = 3$ or $n > 4$.

Theorem 5.2. *Let n be a natural number such that $n = 3$ or $n > 4$. Then there is a model $\mathcal{M}_n$ of ZFA where the following hold.*

(1) *If $X \in \{LOC_2^-, CS, vDCP, CWF, LW\}$, then X holds.*
(2) *LOC_n^- fails.*
(3) *If $X \in \{\mathcal{P}_n, DT, LT\}$, then X fails.*

Proof. Fix a natural number n such that $n = 3$ or $n > 4$.

Defining the ground model M: Let κ be any infinite well-ordered cardinal number. We start with a ground model M of $ZFA + AC$ where A is a κ-sized set of atoms written as a disjoint union $\bigcup\{A_\alpha : \alpha < \kappa\}$, where $A_\alpha = \{a_{\alpha,1}, a_{\alpha,2}, ..., a_{\alpha,n}\}$ such that $|A_\alpha| = n$ for all $\alpha < \kappa$.

Defining the group $\mathcal{G}$ and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$:

- **Defining $\mathcal{G}$:** Let $\mathcal{G}$ be the *weak direct product* of $\mathcal{G}_\alpha$'s where $\mathcal{G}_\alpha$ is the alternating group on $\mathcal{A}_\alpha$ for each $\alpha < \kappa$. Hence, a permutation η of A is an element of $\mathcal{G}$ if and only if for every $\alpha < \kappa$, $\eta \restriction A_\alpha \in \mathcal{G}_\alpha$, and $\eta \restriction A_\alpha = 1_{A_\alpha}$ for all but finitely many ordinals $\alpha < \kappa$. Consequently, *every element $\eta \in \mathcal{G}$ moves only finitely many atoms.*
- **Defining $\mathcal{F}$:** Let $\mathcal{F}$ be the normal filter of subgroups of $\mathcal{G}$ generated by $\{\text{fix}_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$.

Defining the permutation model: Consider the permutation model $\mathcal{M}_n$ determined by M, $\mathcal{G}$ and $\mathcal{F}$.

(1). If $X \in \{LOC_2^-, CS, vDCP, CWF, LW\}$, then X holds in $\mathcal{M}_n$: Since every permutation $\phi \in \mathcal{G}$ moves only finitely many atoms, CS, vDCP, CWF, and LW holds in $\mathcal{M}_n$ by **Lemma 5.1**. Applying the group-theoretic facts from [[HHT12], **Theorem 11**, **Case 1**] and following the arguments of the proof of [[HT20], **Theorem 10(ii)**] we may observe that LOC_2^- holds in $\mathcal{M}_n$.

(2). LOC_n^- fails in $\mathcal{M}_n$: We prove that in $\mathcal{M}_n$, the well-ordered family $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ of n element sets does not have a partial choice function. For the sake of contradiction, let $\mathcal{B}$ be an infinite subfamily of $\mathcal{A}$ with a choice function $f \in \mathcal{M}_n$ and support $E \in [A]^{<\omega}$. Since E is finite, there is an $i < \kappa$ such that $A_i \in \mathcal{B}$ and $A_i \cap E = \emptyset$. Without loss of generality, let $f(A_i) = a_{i_1}$. Consider the permutation π which is the identity on A_j, for all $j \in \kappa - i$, and let $(\pi \restriction A_i)(a_{i_1}) = a_{i_2} \neq a_{i_1}$. Then π fixes E pointwise, hence $\pi(f) = f$. So, $f(A_i) = a_{i_2}$ which contradicts the fact that f is a function. Thus LOC_n^- fails in $\mathcal{M}_n$.

(3). If $X \in \{\mathcal{P}_n$, DT, LT$\}$, then X fails in $\mathcal{M}_n$: Since AC_n fails in the model from the arguments of the previous paragraph, $\mathcal{P}_n$ fails in the model by **Observation 3.1**. Since in $\mathcal{M}_n$, the linearly-ordered family $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ of n element sets does not have a choice function, DT fails in $\mathcal{M}_n$ by [[Tac19], **Theorem 3.1(ii)**]. Since in every Fraenkel–Mostowski model of ZFA, LT implies AC^{WO} (c.f.[[Tac19a], **Theorem 4.6(i)**]), LT fails in $\mathcal{M}_n$ since the well-ordered family $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ does not have a choice function. □

Corollary 5.3. (ZFA) *(LOC_2^- + CS + CWF) does not imply $CAC_1^{\aleph_0}$.*

Proof. Consider the permutation model $\mathcal{M}_n$ constructed in **Theorem 5.2** by letting the infinite well-ordered cardinal number κ to be $\aleph_1$. Rest follows from **Theorem 4.6** and the arguments of **Theorem 5.2(2)**. ☐

Following the arguments in the proof of **Theorem 5.2(3)**, we can also observe that DT and LT fails in the model from [[HT20], **Theorem 10(ii)**].

5.2. **The model $\mathcal{N}_2^*(p)$.** We improve the choice strength of the result of [[CHHKR08], **Theorem 4.8**] if k is a prime, applying the methods of [HT13].

Theorem 5.4. *Fix a prime $p \in \omega \backslash \{0, 1, 2\}$. Then in $\mathcal{N}_2^*(p)$, the following hold.*

(1) *CWF holds.*
(2) *$MC(q)$ holds for all prime $q \neq p$.*
(3) *If p is not a divisor of n, then AC_n and $\mathcal{P}_n$ hold.*

Proof. **(1). CWF holds:** We note that $\mathcal{N}_2^*(p)$ was constructed via a group $\mathcal{G}$ such that $\mathcal{G}$ was abelian and for all $\phi \in \mathcal{G}$, $\phi^p = 1_A$ (c.f. [[CHHKR08], **Theorem 4.8**]). Also the normal filter $\mathcal{F}$ of subgroups of $\mathcal{G}$ was generated by $\{\text{fix}_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$. Thus, CWF holds in $\mathcal{N}_2^*(p)$ by **Lemma 5.1**.

(2). $MC(q)$ holds for all prime $q \neq p$: We prove that in $\mathcal{N}_2^*(p)$, $MC(q)$ holds for all prime $q \neq p$. To see this we observe that $\mathcal{N}_2^*(p)$ satisfies the hypotheses of **Lemma 2.6** with $P = \{p\}$.

- First, we note that $\mathcal{G}$ is Abelian (c.f. [[CHHKR08], **Theorem 4.8**]).
- We follow the arguments from the proof of [[HT13], **Theorem 4.6**] to see that for all $t \in \mathcal{N}_2^*(p)$, $Orb_{\mathcal{G}}(t)$ is finite. Fix a $t \in \mathcal{N}_2^*(p)$. By the Orbit-Stabilizer theorem, $|Orb_{\mathcal{G}}(t)| = [\mathcal{G}/Stab_{\mathcal{G}}(t)]$, where $Stab_{\mathcal{G}}(t)$ is the stabilizer subgroup of $\mathcal{G}$ with respect to t, i.e., $Stab_{\mathcal{G}}(t) = \{g \in \mathcal{G} : g(t) = t\}$. Let $E_t = \cup_{i=0}^{l} A_i$ be a support of t. Clearly, if $\phi, \psi \in \mathcal{G}$ which agree on E_t, then $\phi Stab_{\mathcal{G}}(t) = \psi Stab_{\mathcal{G}}(t)$. By the definition of $\mathcal{G}$, for all $\phi \in \mathcal{G}$, $\phi^p = 1_A$. So $[\mathcal{G}/Stab_{\mathcal{G}}(t)] \leq p^{l+1}$. Thus $Orb_{\mathcal{G}}(t)$ is finite.
- Since $\mathcal{G}$ is such that for all $\phi \in \mathcal{G}$, $\phi^p = 1_A$ (c.f. [[CHHKR08], **Theorem 4.8**]), we can see that part (3) of **Lemma 2.6** is also satisfied. Fix $\psi \in \mathcal{G}$. Let p_1 be a prime divisor of the order of ψ (i.e., p). Clearly, $p_1 = p \in P$.

By **Lemma 2.6**, for every family $\{X_i : i \in I\}$ of non-empty sets in $\mathcal{N}_2^*(p)$, there is a function F with domain I such that for all $i \in I$, we have that $F(i) \subseteq X_i$ and for all $i \in I$, every prime divisor of $|F(i)|$ is in P. Thus for every prime $q \neq p$, $MC(q)$ is true.

(3). If p is not a divisor of n, then AC_n and $\mathcal{P}_n$ hold: If p is not a divisor of n, then AC_n holds, by the arguments in the proof of [[HT13], **Theorem 4.7**(2)]. Consequently, if p is not a divisor of n, then $\mathcal{P}_n$ holds by **Observation 3.1**. ☐

Remark 5.5. We observe that CWF holds in the Second Fraenkel's model (labeled as Model $\mathcal{N}_2$ in [HR98]). Moreover, if $X \in \{\mathcal{P}_{lf,c}, \mathcal{P}_2\}$, then X fails in $\mathcal{N}_2$.

- We note that $\mathcal{N}_2$ was constructed via a group $\mathcal{G}$ such that for all $\phi \in \mathcal{G}$, $\phi^2 = 1_A$. By **Lemma 5.1**, CWF holds in $\mathcal{N}_2$.
- Since AC_2 fails in $\mathcal{N}_2$ (c.f. [HR98]), $\mathcal{P}_2$ fails in $\mathcal{N}_2$ by **Observation 3.1**.
- Since AC_{fin}^{ω} fails in $\mathcal{N}_2$ (c.f. [HR98]), $\mathcal{P}_{lf,c}$ fails in $\mathcal{N}_2$ by **Observation 3.2**.

Remark 5.6. Fix a prime $p_1 \in \omega$. Howard–Tachtsis [[HT13], **Theorem 4.7**] proved that $MC(q)$ holds in $\mathcal{N}_{22}(p_1)$ (c.f. the model from [[HT13], §**4.4**]) for every prime $q \neq p_1$. Fix a prime $p \in \omega \backslash \{0, 1, 2\}$.

- We note that $\mathcal{N}_{22}(p)$ was constructed via a group $\mathcal{G}$ such that for all $\phi \in \mathcal{G}$, $\phi^p = 1_A$. Consequently, by **Lemma 5.1**, CWF holds in $\mathcal{N}_{22}(p)$.

- In $\mathcal{N}_{22}(p)$, AC_n is true for all $n \in \omega\backslash\{0,1\}$ such that p is not a divisor of n (c.f. [[HT13], **Theorem 4.7(2)**]). Consequently, by **Observation 3.1**, $\mathcal{P}_n$ holds in $\mathcal{N}_{22}(p)$ if p is not a divisor of n.
- Since AC^{ω}_{fin} fails in $\mathcal{N}_{22}(p)$ (c.f. the proof of [[HT13], **Theorem 4.7(3)**]), $\mathcal{P}_{lf,c}$ fails in $\mathcal{N}_{22}(p)$ by **Observation 3.2**.

6. Summary

6.1. Synopsis of theorems, observations, and remarks.

- (ZF) $(\forall n \in \omega\backslash\{0,1\}) AC_n \leftrightarrow \mathcal{P}_n$ (c.f. [§3, **Observation 3.1**]).
- (ZF) $UT(\aleph_0,\aleph_0,\aleph_0) \rightarrow \mathcal{P}_{lc,c} \rightarrow AC^{\aleph_0}_{\aleph_0} \rightarrow AC^{\omega}_{fin} \longleftrightarrow \mathcal{P}_{lf,c}$ (c.f. [§3, **Observation 3.2**, **Observation 3.3**]).
- (ZF) AC^{ω}_{fin} implies the statement *'If $G = (V_G, E_G)$ is a connected locally finite chordal graph, then there is an ordering $<$ of V_G such that $\{w < v : \{w, v\} \in E_G\}$ is a clique for each $v \in V_G$'* (c.f. [§3, **Observation 3.5**]).
- (ZFA) For every $n \in \omega\backslash\{0,1\}$, for any regular $\aleph_\alpha$, $CAC^{\aleph_\alpha}_1 \not\rightarrow AC^-_n$ (c.f. [§4, **Theorem 4.3**]).
- (ZFA) For any regular $\aleph_\alpha$, $CAC^{\aleph_\alpha}_1 \not\rightarrow$ 'There are no amorphous sets' (c.f. [§4, **Theorem 4.4**]).
- In $\mathcal{N}^1_{HT}(n)$ and $\mathcal{N}_1$, $\mathcal{P}_{lf,c}$ holds (c.f. [§4, **Remark 4.5**]).
- (ZF) $CAC^{\aleph_0}_1 \rightarrow PAC^{\aleph_1}_{fin}$ (c.f. [§4, **Theorem 4.6**]).
- (ZFA) DC $\not\rightarrow CAC^{\aleph_0}_1$ (c.f. [§4, **Theorem 4.7**]).
- (ZFA) Let $n \in \omega$ such that $n = 3$ or $n > 4$. Then (LOC$^-_2$ + CS + CWF) $\not\rightarrow X$, if $X \in \{LOC^-_n, DT, LT\}$ (c.f. [§5, **Theorem 5.2**]).
- (ZFA) (LOC$^-_2$ + CS + CWF) $\not\rightarrow CAC^{\aleph_0}_1$ (c.f. [§5, **Corollary 5.3**]).
- For any prime $p \in \omega\backslash\{0,1,2\}$, CWF holds in $\mathcal{N}^*_2(p)$, $\mathcal{N}_2$, and, $\mathcal{N}_{22}(p)$.
- For any prime $p \in \omega\backslash\{0,1,2\}$, if p is not a divisor of n, then AC_n and $\mathcal{P}_n$ hold in $\mathcal{N}^*_2(p)$. (c.f. [§5, **Theorem 5.4**]).
- If $X \in \{\mathcal{P}_{lf,c}, \mathcal{P}_2\}$, then X fails in $\mathcal{N}_2$ (c.f. [§5, **Remark 5.5**]).
- For any prime $p \in \omega\backslash\{0,1,2\}$, $\mathcal{P}_n$ holds in $\mathcal{N}_{22}(p)$ if p is not a divisor of n, and $\mathcal{P}_{lf,c}$ fails in $\mathcal{N}_{22}(p)$ (c.f. [§5, **Remark 5.6**]).

6.2. Table of statements and models.

The following table depicts the truth/falsity of statements that we studied in different permutation models. The bold-letter entries '**T** (True) and '**F**' (False) denote the new results in this note. The normal-letter F and T denote the known results.

Table of statements depicting their truth/falsity in certain models				
Models	$CAC^{\aleph_\alpha}_1$	CWF	$\mathcal{P}_{lf,c}$	$\mathcal{P}_n$
$\mathcal{N}^*_2(p)$ $(p > 2)$ (Theorem 5.4)		**T**	**F** if $p = 3$	**T** if $p \nmid n$
$\mathcal{N}_2$ (Remark 5.5)		**T**	**F**	**F** if $n = 2$
$\mathcal{N}_{22}(p)$ $(p \in \omega\backslash\{0,1,2\})$ (Remark 5.6)		**T**	**F**	**T** if $p \nmid n$
$\mathcal{M}_n$ $(n = 3/n > 4)$ (Theorem 5.2)		**T**		**F**
$\mathcal{N}^1_{HT}(n)$ $(n \geq 2)$ (Theorem 4.3)	**T**	T	**T**	**F**
$\mathcal{N}_1$ (Theorem 4.4)	**T**	T	**T**	**F** if $n = 2$

In [BG20], we observed that CWF holds in $\mathcal{N}^1_{HT}(n)$. In $\mathcal{N}^1_{HT}(n)$, AC_n fails. Consequently, $\mathcal{P}_n$ fails in the model by **Observation 3.1**. Since AC^{ω}_{fin} fails in $\mathcal{N}^*_2(3)$ (see [HR98]), $\mathcal{P}_{lf,c}$ fails in $\mathcal{N}^*_2(3)$ by **Observation 3.3**. In $\mathcal{N}_1$, AC_2 fails (c.f. [HR98]). So $\mathcal{P}_2$ fails in $\mathcal{N}_1$.

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