ADDITIVE COMBINATORICS USING EQUIVARIANT COHOMOLOGY

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ABSTRACT. We introduce a geometric method to study additive combinatorial problems. Using equivariant cohomology we reprove the Dias da Silva–Hamidoune theorem. We improve a result of Sun on the linear extension of the Erdős–Heilbronn conjecture. We generalize a theorem of G. Kós (the Grashopper problem) which in some sense is a simultaneous generalization of the Erdős–Heilbronn conjecture. We also prove a signed version of the Erdős–Heilbronn conjecture and the Grashopper problem. Most identities used are based on calculating the projective degree of an algebraic variety in two different ways.

CONTENTS

1. INTRODUCTION

In the present paper we would like to show a connection between additive combinatorics and equivariant cohomology and that this connection leads to new results.

1.1. Known results using equivariant cohomology.

A prototipical theorem in additive combinatorics is the Dias da Silva–Hamidoune theorem in [DdSH94], conjectured by Erdős and Heilbronn:

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Theorem. *Let $A \subset \mathbb{F}_p$ be a set, such that $|A| = n$ and $1 \le k \le n$, and let us use the notation*

$$\bigwedge^k A := \left\{ \sum_{i=1}^k a_i \mid a_i \in A,\ a_i \ne a_j \text{ if } i \ne j \right\}.$$

Then $|\bigwedge^k A| \ge \min\{(n-k)k+1, p\}$ holds.

Most proofs of the Dias da Silva–Hamidoune theorem and similar results in additive combinatorics uses the so called polynomial method of [ANR96], which uses polynomial interpolation formulas, like the Coefficient Formula of [KP12] and [Las10] (See also [KNPV15, Lemma 1.1]):

Lemma. *Let $f \in \mathbb{F}[x_1, \dots, x_n]$ be a polynomial of degree $\deg(f) \le d_1 + d_2 + \dots + d_n$. For arbitrary subsets $C_1, \dots, C_n$ of $\mathbb{F}$ with $|C_i| = d_i + 1$, the coefficient of $\prod x_i^{d_i}$ in f is*

$$\sum_{c_1 \in C_1} \sum_{c_2 \in C_2} \cdots \sum_{c_n \in C_n} \frac{f(c_1, c_2, \dots, c_n)}{\phi_1'(c_1)\phi_2'(c_2)\cdots\phi_n'(c_n)},$$

where $\phi_i(z) = \prod_{c \in C_i}(z - c)$.

We will review the polynomial method in Section 2.1 and show that the Coefficient Formula can be interpreted quite naturally as calculating the integral of an equivariant cohomology class over a product of projective spaces in two different ways. This way we can translate the existing polynomial methods to equivariant cohomology.

But we can do better: Instead of a product of projective spaces we can choose an other compact complex homogeneous space which fits better to the additive combinatorics problem we want to solve. For example the Dias da Silva–Hamidoune theorem leads naturally to an integral over the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$. In this geometric context the number $(n-k)k$ appearing in the estimate of the Dias da Silva–Hamidoune theorem can be interpreted as the dimension of the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$, as it can be seen from (3.3). We present a new short proof of the Dias da Silva–Hamidoune theorem based on these ideas in Section 3.

1.2. **New results.** For other problems we can explore the geometry of other homogeneous spaces. For example using partial flag manifolds we prove two theorems overlapping with a conjecture of Sun on a linear extension of the Erdős–Heilbronn conjecture:

Theorem. 4.2 *Let A be a subset of cardinality n of a field $\mathbb{F}$ with characteristic $p(\mathbb{F})$ and let $u_1, \dots, u_k \in \mathbb{F} \setminus 0$. If $p(\mathbb{F}) > d$ then the following holds:*

$$\left| \left\{ \sum_{i=1}^k u_i a_i : a_1, \dots, a_k \in A, \text{ and } a_i \ne a_j \text{ if } i \ne j \right\} \right| \ge d + 1,$$

for

$$d := k(n-k) + \sum_{1 \le i < j \le t} k_i k_j,$$

where $\{k_1, \dots, k_t\}$ denote the multiplicities of the u_i's.

Theorem. 4.3 *Let $u_1, \dots, u_n \in \mathbb{F}$ be different numbers of a field $\mathbb{F}$ with characteristic p and let $A = \{a_1, \dots, a_n\}$ be a subset of $\mathbb{F}$. If $n > 3$ and $p \le \binom{n}{2}$ then the following holds:*

$$\left| \left\{ \sum_{i=1}^n u_i a_{\pi(i)} : \pi \in S_n \right\} \right| = p.$$

Studying the full flag manifold leads us to a generalization of the Erdős–Heilbronn problem in characteristics 0:

Definition. 5.1 *A sequence of non negative integers* $\mathbf{b} = (b_1, \dots, b_{k-1})$ *is admissible if for all* $M_i \subset \mathbb{Z}$ *and* $|M_i| = b_i$*, and for all* $a_1, a_2, \dots, a_k$ *distinct integers there is a permutation* $\pi \in S_k$ *such that for all* $j = 1, \dots k-1$

$$\sum_{i=1}^{j} a_{\pi(i)} \notin M_i.$$

Theorem. 5.2 *A sequence* $\mathbf{b}$ *is admissible if and only if it satisfies the following system of linear inequalities: for any subset* $P \subset \{1, \dots, k-1\}$ *the condition*

$$\sum_{p \in P} b_p \quad \le \quad \Big|\{(i,j) : \exists p \in P, \text{ such that } 1 \le i \le p \le j \le k-1\}\Big|$$

holds.

We refer to this theorem as the Grasshopper problem for historical reasons explained later. Notice that for sequences $\mathbf{b}$ with only one non zero b_i term, Theorem 5.2 specializes to the Erdős–Heilbronn theorem in 0 characteristics.

In Theorem 5.14 we generalize Theorem 5.2 by showing that for small admissible sequences we need only permutations small in the Bruhat order to avoid the sets M_i by studying the degrees of Schubert varieties in the flag manifold.

All these examples are related to homogeneous spaces of the general linear Lie group $\mathrm{GL}(n) := \mathrm{GL}(n, \mathbb{C})$). However the techniques can be easily modified for other Lie groups.

In Section 6 we study the the complex symplectic group $\mathrm{Sp}(2n)$. This leads us to signed versions of the previous problems. For example we prove the signed Erdős–Heilbronn theorem:

Theorem 6.1. *If* A *is a set of distinct non zero residue classes* $a_1, \dots, a_n$ *modulo* p*,* $a_i + a_j \neq 0$ *and* $p > 2k(n-k) + \binom{k+1}{2}$*, then:*

$$\left|\left\{\sum_{i \in I} \pm a_i \mid I \subset (1, \dots, n), |I| = k\right\}\right| > 2k(n-k) + \binom{k+1}{2}.$$

The main tool of the paper is that we calculate the degree of a torus-invariant projective variety in two different ways: using the Atiyah–Bott–Berline–Vergne integration formula (ABBV formula for short) and the Borel–Hirzebruch formula. We included an appendix with a very short introduction to these formulas.

An interesting aspect of the method is that every lower bound obtained is the dimension of the homogeneous manifold used for the particular problem.

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2. Warmup: The Cauchy–Davenport theorem, connections with Lagrange interpolation and the Combinatorial Nullstellensatz

All the examples in this section can easily be proved by elementary or classical arguments. The section serves as an introduction to the cohomological method we are going to use in the paper. First we reprove the Cauchy–Davenport Theorem using the ABBV formula:

Theorem 2.1. *Let* A *and* B *be two subsets of the field* $\mathbb{F}_p$ *with* p *elements, where* p *is a prime. Assume that* $|A| = r$*,* $|B| = s$ *and* $r + s - 1 \le p$*, and let* $A + B$ *denote the sumset*

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Then $|A + B| \ge r + s - 1$*.*

Proof. Let V be an r-dimensional complex vector space with basis $e_1, \dots, e_r$ and U be an s-dimensional complex vector space with basis $f_1, \dots, f_s$. The group $G_1 = \mathrm{GL}(r)$ acts on V. Let $T(r)$ denote the compact maximal torus of diagonal matrices corresponding to the basis $e_1, \dots, e_r$. Similarly $G_2 = \mathrm{GL}(s)$ acts on U and $T(s)$ is the compact maximal torus corresponding to the basis $f_1, \dots, f_s$.

Consider the map $U \times V \to U \otimes V$ defined by the tensor product. The projectivization of this map is the Segre embedding $\mathbb{P}(U) \times \mathbb{P}(V) \to \mathbb{P}(U \otimes V)$.

The Segre variety $S_{r,s}$ is the image of the Segre embedding. It is an invariant subvariety of the action of the group $\mathrm{GL}(r) \times \mathrm{GL}(s)$ on the space $\mathbb{P}(U \otimes V)$, in fact it is the unique closed orbit.

By a classical formula the degree of the Segre variety is $\binom{r+s-2}{r-1}$. It is classically calculated as an integral:

$$\deg\big(\mathbb{P}(U) \times \mathbb{P}(V) \subset \mathbb{P}(U \otimes V)\big) = \binom{r+s-2}{r-1} = \int\limits_{S_{r,s}} c_1^{r+s-2}, \tag{2.1}$$

where c_1 is the restriction of the first Chern class of the dual of the tautological line bundle $L = \mathcal{O}(1) \to \mathbb{P}(U \otimes V)$.

We can also compute this degree using equivariant cohomology and the Atiyah–Bott–Berline–Vergne (ABBV for short) integration formula.

Proposition 2.2. [AB84] *Suppose that X is a compact manifold and T is a torus acting smoothly on X, and the fixed point set $F(X)$ of the T-action on X is finite. Then for any cohomology class $x \in H_T^*(X) := H_T^*(X;\mathbb{C})$*

$$\int\limits_X x = \sum_{f \in F(X)} \frac{x|_f}{e(T_fX)}. \tag{2.2}$$

Here $e(T_fX)$ is the T-equivariant Euler class of the tangent space T_fX. The right side is considered in the fraction field of the polynomial ring of $H_T^* = H^*(BT)$ (see more details in [AB84]*): part of the statement is that the denominators cancel when the sum is simplified.*

First notice that L is a $T(r) \times T(s)$ vector bundle—i.e. the $T(r) \times T(s)$-action lifts to L acting linearly on the fibers—so it has an equivariant first Chern class $\hat{c}_1$. Because the degree of the integrand is the dimension, any equivariant lift will give the same integral:

$$\deg\big(\mathbb{P}(U) \times \mathbb{P}(V) \subset \mathbb{P}(U \otimes V)\big) = \int\limits_{S_{r,s}} \hat{c}_1^{r+s-2}, \tag{2.3}$$

The fixed points of the action of the maximal torus $T(r) \times T(s)$ are the lines spanned by the vectors $e_i \otimes f_j$, and applying the ABBV formula for $X = S_{r,s} \cong \mathbb{P}^{r-1} \times \mathbb{P}^{s-1}$ we get

$$\binom{r+s-2}{r-1} = \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{(x_i + y_j)^{r+s-2}}{\prod\limits_{k \neq i}(x_i - x_k) \cdot \prod\limits_{l \neq j}(y_i - y_k)}, \tag{2.4}$$

where $H^*_{T(r)\times T(s)} = \mathbb{C}[x_1, \dots, x_r, y_1, \dots, y_s]$. We used that the tangent bundle of a projective space is of the form $\mathrm{Hom}(S, Q)$, where S and Q denote the tautological sub- and quotient bundles, respectively.

The integral of a cohomology class of degree less then the real dimension of the manifold on which we integrate is zero. Consequently, for any set of elements M in any field with

$|M| = r + s - 2$ the following identity holds:

$$\binom{r+s-2}{r-1} = \sum_{i=1}^{r}\sum_{j=1}^{s} \frac{\prod\limits_{m\in M}(x_i+y_j-m)}{\prod\limits_{k\neq i}(x_i-x_k)\cdot\prod\limits_{l\neq j}(y_j-y_l)} \tag{2.5}$$

Remark 2.3. The idea of adding constants to the linear factors of an identity is a very effective trick used by many authors in this field. We could trace it back to [ANR96], but it is also used in [KP12] and [PM09] and several other places. Geometrically this trick corresponds to tensoring the line bundle L with various other line bundles, but we will not explore this connection in this paper.

If $|A+B| \leq r+s-2$ then we can choose M, such that $A+B \subset M \subset \mathbb{F}_p$. Substitute now the elements of the set A into the x variables and the element of the set B into the y variables in the identity (2.5).

All terms of the right hand side are divisible by p, but the left hand side is not divisible by p, if $r+s-2 \leq p$.

This contradiction proves the theorem. □

2.1. **Cohomological proof of the Coefficient Formula.** The readers may notice that identity (2.4) is a special case of the Coefficient Formula below, proved and used e.g. in [KNPV15, Lemma 1.1]. For the history of the Coefficient Formula see loc. cit.

Lemma 2.4. ***(Coefficient Formula)*** *Let $f \in \mathbb{F}[x_1,\dots,x_n]$ be a polynomial of degree $\deg(f) \leq d_1+d_2+\cdots+d_n$. For arbitrary subsets $C_1,\dots,C_n$ of $\mathbb{F}$ with $|C_i| = d_i+1$, the coefficient of $\prod x_i^{d_i}$ in f is*

$$\sum_{c_1\in C_1}\sum_{c_2\in C_2}\cdots\sum_{c_n\in C_n}\frac{f(c_1,c_2,\dots,c_n)}{\phi_1'(c_1)\phi_2'(c_2)\cdots\phi_n'(c_n)},$$

where $\phi_i(z) = \prod_{c\in C_i}(z-c)$.

We claim that this formula can be deduced from the ABBV formula, at least for $\mathbb{F} = \mathbb{C}$. Let the manifold X be the product of projective spaces:

$$X = \bigtimes_{i=1}^{n} \mathbb{P}^{d_i}.$$

We have the torus $T(d_i+1)$ acting on $\mathbb{P}^{d_i}$ induced by its diagonal action on $\mathbb{C}^{d_i+1}$, so the product of these tori will act on X. Let x_i denote the equivariant first Chern class of the dual of the tautological line bundle of $\mathbb{P}^{d_i}$. Applying the ABBV formula for the integral of $f(\alpha_1,\dots,\alpha_n)$, where α_i is the first Chern class of the canonical bundle of $\mathbb{P}^{d_i}$ we obtain the Coefficient Formula by noticing that the integral of a polynomial $f(\alpha_1,\dots,\alpha_n)$ on $\bigtimes_{i=1}^{n}\mathbb{P}^{d_i}$ is nothing else, than the coefficient of $\prod \alpha_i^{d_i}$ in f.

You can also use the Combinatorial Nullstellensatz of [Alo99] to prove the Cauchy–Davenport theorem. As noted in [KNPV15], the Combinatorial Nullstellensatz is an immediate corollary of Lemma 2.4.

To prove the Cauchy–Davenport theorem you can also refer to the Alon–Nathanson–Ruzsa Lemma [ANR96, Thm 2.1], which is also an immediate corollary of Lemma 2.4.

Summing it up, the Coefficient Formula, the Combinatorial Nullstellensatz and the Alon–Nathanson–Ruzsa Lemma can be quickly proved using the ABBV formula for the product of projective spaces. Also, in some cases we know that the coefficient in question is positive because of geometric reasons: it is the degree of a projective variety.

In the following sections we show that using other manifolds with torus actions we can arrive to less familiar identities.

3. Dias da Silva–Hamidoune theorem revisited

In this section first we reprove the conjecture of Erdős and Heilbronn (published in [EG80]) first proved by Dias da Silva and Hamidoune in [DdSH94]:

Theorem 3.1. *Let $A \subset \mathbb{F}_p$ be a set, such that $|A| = n$ and $1 \le k \le n$, and let us use the notation*

$$\bigwedge^k A := \left\{ \sum_{i=1}^k a_i \mid a_i \in A,\ a_i \ne a_j \text{ if } i \ne j \right\}.$$

Then $|\bigwedge^k A| \ge \min\{(n-k)k+1, p\}$ holds.

Proof. Consider the action of $\mathrm{GL}(n)$ on the alternating k-forms $V = \bigwedge^k \mathbb{C}^n$. Then the minimal orbit in $\mathbb{P}V$ is the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ embedded into $\mathbb{P}V$ via the Plücker embedding. There are many ways to calculate the degree of this embedding, for example using Schubert calculus and the Pieri rule, as Schubert calculated originally (see also [Ful98]) or using the Borel–Hirzebruch formula [BH59] (see also [GW11, sec. 6] for details on the degree of the Grassmannian), the answer is the following classical formula:

$$\deg\big(\mathrm{Gr}_k(\mathbb{C}^n)\big) = \int\limits_{\mathrm{Gr}_k(\mathbb{C}^n)} c_1^{k(n-k)} = \big(k(n-k)\big)! \cdot \prod_{i=1}^k \frac{(i-1)!}{(n-i)!}, \tag{3.1}$$

where c_1 is the first Chern class of the dual of the tautological bundle.

The result is in the 0-th cohomology $H^0(\mathrm{Gr}_k(\mathbb{C}^n)) \cong \mathbb{Z}$. Since we also have $H^0_{T(n)}(\mathrm{Gr}_k(\mathbb{C}^n)) \cong \mathbb{C}$, and forgetting the action induces isomorphism between these two $\mathbb{C}$'s, we can simply replace the first Chern class with the $T(n)$-equivariant one:

$$\deg\big(\mathrm{Gr}_k(\mathbb{C}^n)\big) = \int\limits_{\mathrm{Gr}_k(\mathbb{C}^n)} \hat{c}_1^{k(n-k)}. \tag{3.2}$$

Recall that the torus $T(n)$ acts on the Grassmannian as restriction of the $\mathrm{GL}(n)$-action to the diagonal matrices, and $\hat{c}_1$ is the $T(n)$-equivariant first Chern class of the dual of the tautological subbundle over $\mathrm{Gr}_k(\mathbb{C}^n)$. The fixed points of the torus action are the coordinate subspaces $V_I := \langle e_{I_1}, \dots, e_{I_k} \rangle$, where $I = (I_1 < \cdots < I_k)$ is a k-element subset of $\{1, 2, \dots, n\}$. We use the notation $I \in \binom{n}{k}$. We apply the ABBV integration formula:

$$\int\limits_{\mathrm{Gr}_k(\mathbb{C}^n)} \hat{c}_1^{k(n-k)} = \sum_{J \in \binom{n}{k}} \frac{x_J^{k(n-k)}}{\prod\limits_{i\in J}\prod\limits_{j\notin J}(x_i - x_j)}, \tag{3.3}$$

where $H^*_{T(n)} = \mathbb{C}[x_1, \dots, x_n]$ for the x_i's being the „positive" generators and $x_J = \sum_{i\in J} x_i$.

We use again that if the degree of the equivariant cohomology class ω is smaller than $k(n-k)$—the dimension of $\mathrm{Gr}_k(\mathbb{C}^n)$—then $\int\limits_{\mathrm{Gr}_k(\mathbb{C}^n)} \omega = 0$. Consequently for any set of elements M with $|M| = k(n-k)$ in any field the following identity holds:

$$\big(k(n-k)\big)! \cdot \prod_{i=1}^k \frac{(i-1)!}{(n-i)!} = \sum_{J \in \binom{n}{k}} \frac{\prod\limits_{m\in M}(x_J - m)}{\prod\limits_{i\in J}\prod\limits_{j\notin J}(x_i - x_j)}. \tag{3.4}$$

Assume first that $p > (n-k)k$, and $|\bigwedge^k A| \le (n-k)k$, and choose an $M \subset \mathbb{F}_p$ such that $|M| = (n-k)k$ and $\bigwedge^k A \subset M$.

If we use the set M in the formula above and substitute the elements of A in the variables x_i, then the right hand side is divisible by p, because the numbers x_J are all in $\bigwedge^k A \subset M$.

On the other hand by (3.1) the left hand side is not divisible by p, if $p > (n-k)k$.

Remark 3.2. Using an idea we learned from Gyula Károlyi we can calculate the degree of the Grasmannian using the localization formula (3.4). Applying the substitution $x_i = i$ and choosing

$$M = \left\{ \binom{k+1}{2} + 1, \binom{k+1}{2} + 2, \dots, \binom{k+1}{2} + k(n-k) \right\},$$

we can see that all terms in the sum become zero except the one corresponding to $J = \{1, \dots, k\}$, and that term evaluates to

$$\frac{\big(k(n-k)\big)!}{\prod_{i=1}^{k} \prod_{j=k+1}^{n} (j-i)},$$

which is easily seen to be equal to the expression for the degree given above.

Now assume that $p \leq (n-k)k$. Let $B \subset \mathrm{GL}(n)$ be the Borel subgroup consisting of the invertible upper triangular matrices. The Schubert variety σ_I is the closure of the Borel orbit BV_I of the coordinate subspace V_I. The Schubert variety σ_I has dimension $\sum_{i=1}^k I_i - \frac{k(k+1)}{2}$. Note that there are other conventions to encode Schubert varieties e.g with partitions λ such that its Young diagram fits into a $k \times (n-k)$ rectangle. The translation is given by the *conversion formula*

$$I_j = n - k + j - \lambda_j. \tag{3.5}$$

We choose now a subset I such that the Schubert variety σ_I has dimension $p-1$.

By the classical formula of Schubert (see also [Ful98, Ex. 14.7.11]) the degree of σ_I via the Plücker embedding is

Theorem 3.3 (Schubert)**.**

$$\deg(\sigma_I) = \int_{\mathrm{Gr}_k(\mathbb{C}^n)} c_1^{p-1} \cdot [\sigma_I] = \frac{(p-1)!}{(I_1-1)! \cdots (I_k-1)!} \cdot \prod_{i<j} (I_j - I_i).$$

On the other hand we can calculate the degree of the Schubert variety by the ABBV integration formula. The Schubert variety is T-invariant, so it has a T-equivariant cohomology class $[\sigma_I]_T$ in the Grassmannian manifold. By the same argument as before the integral of an equivariant lift of the integrand will give the same result, so we get the following

$$\deg(\sigma_I) = \int_{\mathrm{Gr}_k(\mathbb{C}^n)} \hat{c}_1^{p-1} \cdot [\sigma_I] = \sum_{J \in \binom{n}{k}} \frac{x_J^{p-1} \cdot [\sigma_I]_T|_J}{\prod_{i \in J} \prod_{j \notin J} (x_i - x_j)}, \tag{3.6}$$

where $[\sigma_I]_T|_J$ is the restriction of the equivariant cohomology class $[\sigma_I]_T$ to the fixed point corresponding to J.

We use again that the integral of a class whose degree is less, than the dimension of the Grassmanian, is zero. This means, that for any set of elements M with $|M| = p-1$ in any field the following identity holds:

$$\frac{(p-1)!}{(I_1-1)! \cdots (I_k-1)!} \cdot \prod_{i<j} (I_j - I_i) = \sum_{J \in \binom{n}{k}} \frac{\prod_{m \in M} (x_J - m) \cdot [\sigma_I]_T|_J}{\prod_{i \in J} \prod_{j \notin J} (x_i - x_j)}. \tag{3.7}$$

Assuming that $|A_k| \leq p-1$ we can choose M, such that $A_k \subset M \subset F_p$ and $|M| = p-1$.

If we use this set M in the formula above and substitute the elements of A in the variables x_i, then we get that the right hand side is divisible by p, because the numbers x_I are all in $A_k \subset M$.

On the other hand, since $I_k \leq n < (n-k)k$, the left hand side is not divisible by p, which gives the desired contradiction, and we finished the proof. □

Remark 3.4. The degrees of the Schubert varieties tend to show up in additive combinatorics. A key step in the proof of the Alon–Nathanson–Ruzsa theorem [ANR96, Prop 1.2] is identifying these numbers as certain coefficients of a polynomial. The argument above gives a geometric reason for the appearance of these degrees.

4. The Sun conjecture

Z. W. Sun made the following conjecture in [Sun08] which can be viewed as a linear extension of the Erdős–Heilbronn conjecture.

Conjecture 4.1. *Let A be a subset of cardinality n of a field $\mathbb{F}$ with characteristic $p(\mathbb{F})$ and let $u_1, \ldots, u_k \in \mathbb{F} \setminus 0$. If $p(\mathbb{F}) \neq k+1$ then the following holds:*

$$\left| \left\{ \sum_{i=1}^{k} u_i a_i \ : \ a_1, \ldots, a_k \in A, \ \text{and } a_i \neq a_j \ \text{if } i \neq j \right\} \right| \geq \min\{p(F) - \delta, k(n-k)\} \tag{4.1}$$

In the above equation $\delta = 1$, if $n = 2$ and $u_1 + u_2 = 0$, and $\delta = 0$ otherwise.

Recently in [SZ12] Z. W. Sun and L. L. Zhao proved the conjecture if $p(\mathbb{F}) \geq \frac{k(3k-5)}{2}$.

We prove a stronger bound in the case $p(\mathbb{F})$ is big enough.

Notice that we do not assume that the u_i's are all different. For example if $u_i = 1$ for all $i \leq k$ we get back the Erdős–Heilbronn conjecture. Let $\{k_1, \ldots, k_t\}$ denote the multiplicities of the u_i's. In particular $\sum k_i = k$.

Using the notation

$$d := k(n-k) + \sum_{1 \leq i < j \leq t} k_i k_j,$$

we have the following:

Theorem 4.2. *Let A be a subset of cardinality n of a field $\mathbb{F}$ with characteristic $p(\mathbb{F})$ and let $u_1, \ldots, u_k \in \mathbb{F} \setminus 0$. If $p(\mathbb{F}) > d$ then the following holds:*

$$\left| \left\{ \sum_{i=1}^{k} u_i a_i : a_1, \ldots, a_k \in A, \ \text{and } a_i \neq a_j \ \text{if } i \neq j \right\} \right| \geq d+1. \tag{4.2}$$

Proof. We again create a polynomial identity using the ABBV formula to prove the theorem.

Consider now the $\mathrm{GL}(n)$-representation Γ_λ with highest weight

$$\lambda = (\mu_1, \ldots, \mu_1, \ldots, \mu_{t+1}, \ldots, \mu_{t+1}),$$

where the integer μ_i occurs k_i many times if $i \leq t$ and $\mu_{t+1} := 0$ occurs $n-k$ times. We also assume that $\mu_1 > \cdots > \mu_{t+1}$. The minimal orbit of $\mathbb{P}(\Gamma_\lambda)$ is the partial flag manifold Fl, consisting of the flags $0 \subset V_1 \subset \cdots \subset V_t \subset \mathbb{C}^n$, where $\dim(V_i) = \sum_{j=1}^{i} k_j$. Notice that $\dim(V_t) = k$. The partial flag manifold Fl is a complex manifold with dimension d and the group $\mathrm{GL}(n)$ acts on it smoothly and transitively. Let L be the restriction of the dual of the tautological bundle over $\mathbb{P}(\Gamma_\lambda)$ to the flag manifold. Let us denote the $T(n)$-equivariant first Chern class of L by $\hat{c}_1$. The fixed points of the torus action on the flag manifold are exactly the coordinate flags. If we have a fixed point $F = 0 \subset V_1 \subset \cdots \subset V_t \subset \mathbb{C}^n$, and a permutation $\pi \in S_n$, such that $V_i = \langle e_{\pi(1)}, \ldots, e_{\pi(\dim(V_i))} \rangle$, then the restriction of $\hat{c}_1$ to F is

$$\hat{c}_1|_F = \sum_{i=1}^{n} \lambda_i x_{\pi(i)},$$

where $H^*_{T(n)} = \mathbb{Z}[x_1, \ldots, x_n]$ and the x_i's are the "positive" generators. For a subset $I \subset \{1, 2, \ldots, n\}$ we use the notation $x_I = \sum_{i \in I} x_i$.

Now if we use the ABBV integration formula we get the following:

$$\deg(\mathrm{Fl}) = \int_{\mathrm{Fl}} \hat{c}_1^d = \sum_{I_1,\ldots,I_{t+1}} \frac{\left(\sum_{j=1}^{t+1} \mu_j x_{I_j} \right)^d}{\prod_{1 \le r < s \le t+1} \prod_{j \in I_s} \prod_{i \in I_r} (x_i - x_j)}, \tag{4.3}$$

where the summation goes along all partitions of the numbers $1, 2, \ldots, n$ into subsets $I_1, \ldots, I_{t+1}$, where $|I_j| = k_j$.

With the same argument as in the previous cases we can include a new set of variables $b_1, \ldots, b_d$:

$$\deg(\mathrm{Fl}) = \sum_{I_1,\ldots,I_{t+1}} \frac{\prod_{q=1}^{d} \left(\sum_{j=1}^{t+1} \mu_j x_{I_j} - b_q \right)}{\prod_{1 \le r < s \le t+1} \prod_{j \in I_s} \prod_{i \in I_r} (x_i - x_j)} \tag{4.4}$$

On the other hand by the Borel–Hirzebruch formula [BH59, 24.10 Thm] one can easily get that

$$\deg(\mathrm{Fl}) = d! \cdot \prod_{\lambda_i > \lambda_j} \frac{\lambda_i - \lambda_j}{j - i}. \tag{4.5}$$

Gross and Wallach gives a modern introduction to the Borel–Hirzebruch formula in [GW11], what we found very useful.

We can rewrite (4.5) in terms of the μ's:

$$\deg(\mathrm{Fl}) = d! \cdot \prod_{1 \le a < b \le t+1} \frac{(\mu_a - \mu_b)^{k_a k_b}}{\prod_{u=1}^{k_a} \prod_{v=1}^{k_b} (v - u + \sum_{i=a}^{b-1} k_i)}. \tag{4.6}$$

This means that we have the following polynomial identity in the variables μ_i, x_i, b_i:

$$d! \cdot \prod_{1 \le a < b \le t+1} \frac{(\mu_a - \mu_b)^{k_a k_b}}{\prod_{u=1}^{k_a} \prod_{v=1}^{k_b} (v - u + \sum_{i=a}^{b-1} k_i)} = \sum_{I_1,\ldots,I_{t+1}} \frac{\prod_{q=1}^{d} \left(\sum_{j=1}^{t+1} \mu_j x_{I_j} - b_q \right)}{\prod_{1 \le s < l \le t+1} \prod_{j \in I_s} \prod_{i \in I_r} (x_i - x_j)}. \tag{4.7}$$

The key observation here is that we can replace the μ_i's with variables.

Now suppose that our conjecture is false, $p(\mathbb{F}) > d$ and let $|B| = d$ be a subset of $\mathbb{F}$ containing the set

$$\left\{ \sum_{1 \le i \le k} u_i a_i \ : \ a_1, \ldots, a_k \in A, \text{ and } a_i \neq a_j \text{ if } i \neq j \right\}.$$

Now let us substitute the elements of B into the variables b_i, the elements of A into the variables x_i. Into the variables μ_i we substitute the u_i which had multiplicity k_i.

So now from the assumptions we get that the right hand side is zero, while the left hand side is non zero. This contradiction proves our theorem. □

Finally we prove our theorem for small primes in the special case $k = n$ and all the u_i are different:

Theorem 4.3. *Let $u_1, \dots, u_n \in \mathbb{F}$ be pairwise distinct elements of a field $\mathbb{F}$ with characteristic p and let $A = \{a_1, \dots, a_n\}$ be a subset of $\mathbb{F}$. If $n > 3$ and $p \le \binom{n}{2}$ then the following holds:*

$$\left|\left\{\sum_{i=1}^{n} u_i a_{\pi(i)} : \pi \in S_n\right\}\right| = p \tag{4.8}$$

Proof. Specializing (4.7) we get the identity:

$$V(\lambda_1, \dots, \lambda_n) \cdot \frac{\binom{n}{2}!}{V(1, 2, \dots, n)} = \sum_{\pi \in S_n} \frac{\left(\sum_{i=1}^{n} \lambda_i x_{\pi(i)}\right)^{\binom{n}{2}}}{V(x_{\pi(1)}, \dots, x_{\pi(n)})}, \tag{4.9}$$

where the λ_i's and the x_i's are variables. Here $V(x_1, \dots, x_n)$ denotes the Vandermonde determinant $\prod_{i<j} x_j - x_i$.

Formally you can think of the left hand side as the degree of the embedding of the full flag manifold to $\mathbb{P}(\Gamma_\lambda)$ for the highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i > \lambda_j$ for $i < j$.

We can assume that $p > \binom{n-1}{2}$, otherwise we know that $n \le p \le \binom{n-1}{2}$, so $n - 1 > 3$. In that case we can fix $\pi(n) = n$ and prove the statement for $n - 1$ rather than n.

Notice that if $n \ge 5$ then $n < \binom{n-1}{2} < p$, so $n < p$, and if $n = 4$, then again $p \ne 4$, so we have $n < p$.

So assume that $p > \binom{n-1}{2}$ and let $k = \binom{n}{2} - p + 1$, where we have $k \le n - 1$. Now differentiate the identity k times in the variable λ_i:

$$\sum_{\pi \in S_n} \frac{x_{\pi(i)}^k \cdot \left(\sum_{i=1}^{n} \lambda_i x_{\pi(i)}\right)^{p-1}}{V(x_{\pi(1)}, \dots, x_{\pi(n)})} = \frac{\partial^k}{\partial \lambda_i^k} V(\lambda_1, \dots, \lambda_n) \cdot \frac{(p-1)!}{V(1, 2, \dots, n)}. \tag{4.10}$$

From these identities we get:

$$\sum_{\pi \in S_n} \frac{\left(\sum_{i=1}^{n} (\lambda_i x_{\pi(i)})^k\right) \cdot \left(\sum_{i=1}^{n} \lambda_i x_{\pi(i)}\right)^{p-1}}{V(x_{\pi(1)}, \dots, x_{\pi(n)})} = \sum_{i=1}^{n} \left(\lambda_i^k \cdot \frac{\partial^k}{\partial \lambda_i^k} V(\lambda_1, \dots, \lambda_n) \cdot \frac{(p-1)!}{V(1, 2, \dots, n)}\right). \tag{4.11}$$

Now we have the following easy identity for $k \le n - 1$ (use that the left hand side is also alternating):

$$\sum_{i=1}^{n} \left(\lambda_i^k \cdot \frac{\partial^k}{\partial \lambda_i^k} V(\lambda_1, \dots, \lambda_n)\right) = V(\lambda_1, \dots, \lambda_n) \cdot \binom{n}{k+1} k! \tag{4.12}$$

implying that

$$(4.13)\qquad \sum_{\pi\in S_n} \frac{\left(\sum_{i=1}^{n}(\lambda_i x_{\pi(i)})^k\right)\cdot\left(\sum_{i=1}^{n}\lambda_i x_{\pi(i)}\right)^{p-1}}{V(x_{\pi(1)},\dots,x_{\pi(n)})} = V(\lambda_1,\dots,\lambda_n)\cdot\binom{n}{k+1}k!\cdot\frac{(p-1)!}{V(1,2,\dots,n)}$$

Again we can include a set B of constants with $|B| = p-1$ without changing the value of the formula:

$$(4.14)\qquad \sum_{\pi\in S_n} \frac{\left(\sum_{i=1}^{n}(\lambda_i x_{\pi(i)})^k\right)\cdot\prod_{b\in B}\left(\sum_{i=1}^{n}\lambda_i x_{\pi(i)}-b\right)}{V(x_{\pi(1)},\dots,x_{\pi(n)})} = V(\lambda_1,\dots,\lambda_n)\cdot\binom{n}{k+1}k!\cdot\frac{(p-1)!}{V(1,2,\dots,n)}$$

Assume that we have a counterexample A and numbers $u_1,\dots,u_n\in\mathbb{F}$, then we have a set $B\subset\mathbb{F}$, such that $|B| = p-1$ and

$$\left\{\sum_{i=1}^{n} u_i a_{\pi(i)} : \pi\in S_n\right\}\subset B.$$

Now in the identity (4.14) we substitute the elements a_i into the variables x_i, the elements u_i into the variables λ_i. We have that the left hand side is 0 in $\mathbb{F}$, while the right hand side is not, which is a contradiction. □

5. The Grasshopper: a simultaneous generalization of the Erdős–Heilbronn problem

On the 50-th International Mathematics Olympiad for highschool students the following problem was given:

Imo 2009/6 [Khr]**:** *Let $a_1,\dots,a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = \sum_{i=1}^{n} a_i$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1,\dots,a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.*

The problem was said one of the hardest problems on IMO until that time, since only 3 people could solve it. The proof is an inductive combinatorial proof and deeply depends on the fact that the numbers $a_1,\dots,a_n$ are positive.

Géza Kós suggested the following form in the Mathematical and Physical Journal for Secondary Schools [kom09]:

Kömal: A. 496. *Let $a_1,a_2,\dots,a_k$ be distinct integers for $k=2n$ and let M be a set of n integers not containing 0 and $s = a_1+a_2+\cdots+a_{2n}$. A grasshopper is to jump along the real axis, starting at the point 0 and making k jumps with lengths $a_1,a_2,\dots,a_{2n}$ in some order. If $a_i>0$ then the grasshopper jumps to the right; while if $a_i<0$ then the grasshopper jumps to the left, to the point in the distance $|a_i|$ in the respective steps. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.*

This form was proved by the second author using the Combinatorial Nullstellensatz, however to prove that a certain coefficient of a polynomial is nonzero was cumbersome.

In [Kós11] the positivity of this coefficient is proved by an inductive tricky way. In this section we will identify this coefficient with an intersection number in a flag manifold.

The argument of [Kós11] in fact gives more: at every step we can give a different forbidden sets of integers M_i for $i=1,\dots,2n-1$ of cardinality n, and the grasshopper will be able to avoid the positions in M_i with his i-th jump. We would like to generalize this version: characterizing the sequences of non negative integers $\mathbf{b}=(b_1,\dots,b_{k-1})$ for which the grasshopper can avoid forbidden sets of integers M_i for $i=1,\dots,k-1$ of cardinality b_i. We will call these sequences $\mathbf{b}$ *admissible*:

Definition 5.1. A sequence of non negative integers $\mathbf{b}=(b_1,\dots,b_{k-1})$ is admissible if for all $M_i\subset\mathbb{Z}$ and $|M_i|=b_i$, and for all $a_1,a_2,\dots,a_k$ distinct integers there is a permutation $\pi\in S_k$ such that for all $j=1,\dots k-1$

$$\sum_{i=1}^{j} a_{\pi(i)}\notin M_i.$$

Then **A. 496.** is equivalent with the admissibility of $\mathbf{b}=(n,\dots,n)$ for $k=2n$.

Notice that we no longer assume that k is even. In this section we characterize the admissible sequences:

Theorem 5.2. *A sequence* $\mathbf{b}$ *is admissible if and only if it satisfies the following system of linear inequalities: for any subset* $P\subset\{1,\dots,k-1\}$ *the condition*

$$\sum_{p\in P} b_p \quad\le\quad \Big|\{(i,j):\exists p\in P,\ \text{such that } 1\le i\le p\le j\le k-1\}\Big| \tag{5.1}$$

holds.

You can interpret the right hand side as the number of 'intervals' of $\{1,\dots,k-1\}$ having nonempty intersection with P. This theorem tells us that it is enough to check 2^{k-1} simple inequalities to decide whether $\mathbf{b}$ is admissible.

Remark 5.3. An easy calculation shows that for sequences $\mathbf{b}$ with only one non zero b_i Theorem 5.2 specializes to the Erdős–Heilbronn theorem in 0 characteristics. So in the Erdős–Heilbronn problem there is only one forbidden set, as in the Grasshopper problem there are several. This way one can think of the Grasshopper problem as a simultaneous Erdős–Heilbronn problem.

First we create an identity by calculating the degree of the full flag manifold in two different ways similarly to the proof of Theorem 4.2.

Let Γ_λ be the irreducible representation of $\mathrm{GL}(k)$ with highest weight $\lambda=(\lambda_1>\lambda_2>\dots>\lambda_{k-1}>\lambda_k=0)$. This weight is in the open Weyl chamber, so the minimal orbit in $\mathbb{P}(\Gamma_\lambda)$ is the complete flag variety $\mathrm{Fl}=\mathrm{Fl}(k)$. Its degree in $\mathbb{P}(\Gamma_\lambda)$ is calculated by the integral

$$\deg\big(\mathrm{Fl}\subset\mathbb{P}(\Gamma_\lambda)\big)=\int_{\mathrm{Fl}}\hat{c}_1(L_\lambda)^d,$$

where $\hat{c}_1(L_\lambda)$ is the torus equivariant first Chern class of the canonical line bundle L_λ and $d=\binom{k}{2}=\dim(\mathrm{Fl})$. Using the ABBV formula to calculate the integral we get

$$\deg\big(\mathrm{Fl}\subset\mathbb{P}(\Gamma_\lambda)\big)=\sum_{\pi\in S_k}\frac{\left(\sum\limits_{i=1}^{k}\lambda_i x_{\pi(i)}\right)^d}{e(T_\pi\,\mathrm{Fl})}, \tag{5.2}$$

where $e(T_\pi\,\mathrm{Fl})=(-1)^{\binom{k}{2}}\operatorname{sgn}(\pi)V(\mathbf{x})$.

We can also use the Borel–Hirzebruch formula, which gives

$$\deg\big(\mathrm{Fl}\subset\mathbb{P}(\Gamma_\lambda)\big)=(-1)^{\binom{k}{2}}\binom{d}{\Delta}V(\lambda), \tag{5.3}$$

where $\Delta=(0,1,\dots k-1)$.

Now we arrive to the identity:

$$(5.4)\qquad (-1)^{\binom{k}{2}}\binom{d}{\Delta}V(\lambda)=\sum_{\pi\in S_k}\frac{\left(\sum_{i=1}^{k}\lambda_i x_{\pi(i)}\right)^d}{e(T_\pi \operatorname{Fl})}.$$

Notice that both sides are polynomials in the variables λ_i, and the identity is valid for integer substitutions in an open cone, therefore these polynomials agree.

Remark 5.4. The actual value $e(T_\pi \operatorname{Fl})=(-1)^{\binom{k}{2}}\operatorname{sgn}(\pi)V(\mathbf{x})$ of the denominator is not used in the following, however we can see that (5.4) is equivalent to

$$\binom{d}{\Delta}V(\lambda)V(\mathbf{x})=\operatorname{Alt}_{\mathbf{x}}\left((\lambda,\mathbf{x})^d\right),$$

where $(\lambda,\mathbf{x})$ denotes the scalar product of the vectors λ and $\mathbf{x}$ and $\operatorname{Alt}_{\mathbf{x}}$ refers to the antisymmetrization with respect to the $\mathbf{x}$ variables. This formula also can be proved directly, without using geometry.

We define the homogenous polynomial P in the variables v_i with degree d by substituting $\lambda_i=\sum_{j=i}^{k}v_j$ into (5.4), notice that the polynomial P does not depend on the variable v_k.

The coefficient of $v^{\mathbf{b}}$ for $\mathbf{b}=(b_1,\dots,b_{k-1})$ is:

$$(5.5)\qquad \operatorname{coef}(P,v^{\mathbf{b}})=\binom{d}{\mathbf{b}}\sum_{\pi\in S_k}\frac{\prod_{i=1}^{k-1}\left(\sum_{j=1}^{i}x_{\pi(j)}\right)^{b_i}}{e(T_\pi \operatorname{Fl})},$$

where $\binom{d}{\mathbf{b}}$ denotes the multinomial coefficient.

By the usual trick we can add lower order terms:

$$(5.6)\qquad \operatorname{coef}(P,v^{\mathbf{b}})=\binom{d}{\mathbf{b}}\sum_{\pi\in S_k}\frac{\prod_{m\in M_i}\left(\sum_{j=1}^{i}x_{\pi(j)}-m\right)}{e(T_\pi \operatorname{Fl})},$$

for given sets of numbers M_i with $|M_i|=b_i$. This implies that if the coefficient $\operatorname{coef}(P,v^{\mathbf{b}})$ is not zero, then $\mathbf{b}$ is admissible. Notice that the actual form of the denominators is not important, only the property that they are not zero if we substitute different numbers into the x_i's.

So we have the following expression:

$$(5.7)\qquad \frac{\binom{d}{\Delta}}{\binom{d}{\mathbf{b}}}\mu(\mathbf{b})=\sum_{\pi\in S_k}\frac{\prod_{m\in M_i}\left(\sum_{j=1}^{i}x_{\pi(j)}-m\right)}{e(T_\pi \operatorname{Fl})},$$

where $\mu(\mathbf{b})=(-1)^{\binom{k}{2}}\operatorname{coef}(V(\lambda),v^{\mathbf{b}})$.

Using a formula of Duan in [Dua03] for the degree of the flag manifold one can give a closed formula for $\mu(\mathbf{b})=\operatorname{coef}(V(\lambda),v^{\mathbf{b}})$, however this formula is a sum of terms with different signs, and it is not clear from it which coefficients are zero. So we try to decide which $\mu(\mathbf{b})$'s are non zero without calculating their value.

The key observation is that because of $\lambda_i-\lambda_j=v_i+\dots+v_{j-1}$, the positivity of the coefficient of $v^{\mathbf{b}}$ in the product is equivalent to the existence of a matching covering the lower class of the bipartite graph $\mathcal{B}_{\mathbf{b}}$ defined in 5.5:

Definition 5.5. We assign a bipartite graph $\mathcal{B}_{\mathbf{b}} = (U, D_{\mathbf{b}}, E_{\mathbf{b}})$ to a sequence of non negative integers $\mathbf{b} = (b_1, \dots, b_{k-1})$. The upper class U consists of the the pairs (j, l) with $1 \le j \le l \le k-1$ independently of $\mathbf{b}$. The lower class $D_{\mathbf{b}}$ consists of the pairs (i, t), where $1 \le i \le k-1$ and $1 \le t \le b_i$. $E_{\mathbf{b}}$ is the set of edges: there is an edge between (j, l) and (i, t) if $j \le i \le l$.

We say that the sequence $\mathbf{b}$ is a *matching sequence* if $\mathcal{B}_{\mathbf{b}}$ has a matching covering the lower class $D_{\mathbf{b}}$.

The elements of U correspond to the factors of $V(\lambda)$ and the elements of $D_{\mathbf{b}}$ correspond to the factors of $v^{\mathbf{b}}$. Note that $\mu(\mathbf{b}) > 0$ implies that $|\mathbf{b}| := \sum_i b_i = \binom{k}{2}$, so in these cases a matching of $D_{\mathbf{b}}$ is a perfect matching of the bipartite graph $\mathcal{B}_{\mathbf{b}}$. We will call these $\mathbf{b}$'s *perfect matching sequences.* However we also interested in sequences with $|\mathbf{b}| < \binom{k}{2}$, but these cases can be reduced to the $|\mathbf{b}| = \binom{k}{2}$ case (the choice of $a_i := i$ shows that for any admissible sequence $|\mathbf{b}| \le \binom{k}{2}$ holds): The structure of these bipartite graphs is quite simple, because the set of neighbours of a vertex $(i, t) \in D_{\mathbf{b}}$ is independent of t. Consequently a simple combinatorial argument gives the following.

Lemma 5.6. *The sequence* $\mathbf{b}$ *is matching if and only if there is a matching sequence* $\bar{\mathbf{b}} = (\bar{b}_1, \dots, \bar{b}_{k-1})$ *with* $|\bar{\mathbf{b}}| = \binom{k}{2}$ *dominating* $\mathbf{b}$, *i.e.* $\bar{b}_i \ge b_i$ *for* $i = 1, \dots, k-1$. □

The definition immediately implies that any sequence of non negative integers dominated by an admissible sequence is also admissible, therefore, we showed so far (by using the standard substitution trick into the right hand side of (5.7)) that if $\mathbf{b}$ is a matching sequence then it is also admissible. To finish one direction of the proof of Theorem 5.2 we need to prove that

Proposition 5.7. *A sequence of non negative integers* $\mathbf{b} = (b_1, \dots, b_{k-1})$ *is matching if and only if* $\mathbf{b}$ *satisfies the following system of linear inequalities: for any subset* $P \subset \{1, \dots, k-1\}$ *the condition*

$$b_P \le K(P) \tag{5.8}$$

holds, where $b_P = \sum_{p \in P} b_p$ *and* $K(P)$ *is the number of pairs* (i, j) *such that* $1 \le i \le j \le k-1$ *and there exists a* $p \in P$ *for which* $i \le p \le j$.

Proof. We use the Hall theorem [Hal35]. Using again the fact that the set of neighbors of a vertex $(i, t) \in D_{\mathbf{b}}$ is independent of t, we can see that it is enough to check the Hall condition only for the subsets H of $D_{\mathbf{b}}$ which have the property that if $(i, t) \in H$, then all vertices of the form (i, s) are also in H. For these subsets the Hall condition gives exactly the inequalities (5.8). □

For the other direction of Theorem 5.2 we need the following.

Proposition 5.8. *Admissible sequences are matching.*

Proof. In the proof we use the notation $\binom{k}{2} = [k]_2$ for typographical reasons. Let $\mathbf{b} = (b_1, \dots, b_{k-1})$ be an admissible sequence. We choose $a_i = i$ for $i = 1, \dots, k$. For a given $P = \{p_1, \dots, p_m\} \subset \{1, \dots, k-1\}$ of Proposition 5.7 we choose the forbidden sets M_{p_i} to be intervals of integers of length b_{p_i} in such a way that the grasshopper is forced to jump to the right of M_{p_i} in the p_i-th second. Admissibility of $\mathbf{b}$ implies the existence of a permutation $\pi \in S_k$ such that

$$s_j := \sum_{i=1}^{j} a_{\pi(i)} = \sum_{i=1}^{j} \pi(i) \notin M_j.$$

We call such a π an allowed permutation. For p_1 we have $s_{p_1} = \sum_{i=1}^{p_1} \pi(i) \ge \sum_{i=1}^{p_1} i = [p_1 + 1]_2$. Therefore for the choice

$$M_{p_1} = [[p_1 + 1]_2, [p_1 + 1]_2 + b_{p_1} - 1]$$

the grasshopper in the p_1-th second must be on the right of M_{p_1}, i.e.

$$s_{p_1} = \sum_{i=1}^{p_1} \pi(i) \ \geq \ [p_1+1]_2 + b_{p_1}$$

for any allowed permutation π.

In the p_2-th second the grasshopper must be further to the right by at least $\sum_{j=1}^{p_2-p_1} i = [p_2 - p_1 + 1]_2$:

$$s_{p_2} = \sum_{i=1}^{p_2} \pi(i) \ \geq \ [p_1+1]_2 + b_{p_1} + [p_2 - p_1 + 1]_2.$$

If we choose

$$M_{p_2} = \big[[p_1+1]_2 + b_{p_1} + [p_2-p_1+1]_2, [p_1+1]_2 + b_{p_1} + [p_2-p_1+1]_2 + b_{p_2} - 1\big],$$

then we assured that

$$s_{p_2} \geq [p_1+1]_2 + b_{p_1} + [p_2-p_1+1]_2 + b_{p_2}.$$

Continuing with the same strategy we choose $M_{p_i} = [x_i, x_i + b_{p_i} - 1]$ for

$$x_i = \sum_{j=0}^{i-1} [n_j+1]_2 + \sum_{j=1}^{i-1} b_{p_j},$$

where $n_0 = p_1$ and $n_j = p_{j+1} - p_j$ for $j = 1, \dots, m$. By induction we can see that

$$s_{p_m} \geq x_m + b_{p_m}.$$

For the last $k - p_m$ jumps the grasshopper moves again at least $\sum_{i=1}^{k-p_m} i = [k - p_m + 1]_2$ to the right:

$$s_k = [k+1]_2 \geq x_m + b_{p_m} + [k - p_m + 1]_2.$$

By definition

$$K(P) = [k]_2 - [k-p_m]_2 - \sum_{j=0}^{m-1} [n_j]_2,$$

and

$$[k]_2 - [k-p_m]_2 - \sum_{j=0}^{m-1} [n_j]_2 = [k+1]_2 - [k-p_m+1]_2 - \sum_{j=0}^{m-1} [n_j+1]_2,$$

since $(k - p_m) + \sum_{j=0}^{m-1} n_j = k$. This implies that $b_P \leq K(P)$, if there is a permutation $\pi \in S_k$ for which the grasshopper avoids the forbidden sets, and Proposition 5.7. implies that $\mathbf{b}$ is matching. □

And we finished the proof of Theorem 5.2. □

For perfect matching sequences the system of inequalities (5.8) can be replaced by a simpler one:

Proposition 5.9. *A sequence* $\mathbf{b}$ *with* $|\mathbf{b}| = \binom{k}{2}$ *is a perfect matching, i.e. the coefficient* $\mu(\mathbf{b}) = \mathrm{coef}(V(\lambda), v^{\mathbf{b}})$ *is non-zero if and only if*

$$\sum_{j=s}^{t} b_j \geq \binom{t-s+2}{2} \ \textit{for every } 1 \leq s \leq t \leq k-1. \tag{5.9}$$

Proof. Now the two partitions of the bipartite graph $\mathcal{B}_{\mathbf{b}}$ have the same size, so we can check the Hall condition in the other direction: Let B be a subset of U and let

$$\mathrm{Sh}(B) = \{a \mid 1 \le a \le k-1,\ \exists (j,l) \in B,\ j \le a \le l\}$$

be the "shadow" of B. Then the size of the neighborhood of B is $\sum_{a\in \mathrm{Sh}(B)} b_a$. Let us write $\mathrm{Sh}(B) = A_1 \cup \cdots \cup A_s$ as disjoint union of maximal intervals of integers.

The largest B with the same shadow has size $\sum_{i=1}^s \binom{|A_i|+1}{2}$ so the Hall condition is equivalent to the system of inequalities

$$\sum_{a\in \mathrm{Sh}(B)} b_a \ge \sum_{i=1}^{s} \binom{|A_i|+1}{2} \quad \text{for all possible shadows.} \tag{5.10}$$

(5.9) is a subsystem of this system of inequalities: the ones where the shadow is a single interval. Taking the sum of inequalities for all the intervals A_i we can see that (5.10) is equivalent to (5.9). □

Remark 5.10. Let E_i denote the rank i tautological subbundle over the full flag manifold and $|\mathbf{b}| = \binom{k}{2}$. Then the ABBV formula gives

$$\int_{\mathrm{Fl}} \prod_{i=1}^{k-1} c_1(E_i^*)^{b_i} = \sum_{\pi\in S_k} \frac{\prod_{i=1}^{k-1}\left(\sum_{j=1}^{i} x_{\pi(j)}\right)^{b_i}}{e(T_\pi \mathrm{Fl})}. \tag{5.11}$$

By the equations (5.4)–(5.7) we can identify this integral with the coefficient

$$\frac{\binom{d}{\Delta}}{\binom{d}{\mathbf{b}}}\mu(\mathbf{b}).$$

The cohomology ring $H^*(\mathrm{Fl})$ of the flag manifold is generated by the classes $c_1(E_i^*)$. Therefore the previous proof and Poincaré duality implies that $\mathbf{b}$ is a matching sequence if and only if $\prod_{i=1}^{k-1} c_1(E_i^*)^{b_i}$ is not zero in $H^*(\mathrm{Fl})$.

The bundles E_i^* are globally generated, which implies the non negativity of the coefficients $\mu(\mathbf{b})$.

Remark 5.11. It would be interesting to have a Grasshopper Theorem in characteristic p. Géza Kós mentions in [Kós11] that the coefficients $\mu(\mathbf{b})$ can have large prime factors. Therefore it is not clear how this mod p Grasshopper Theorem should look like.

5.1. **Schubert varieties in the flag manifold and Bruhat restrictions for the grasshopper.** It is intuitively clear that it is easier for the grasshopper to avoid the forbidden positions if there are less then $\binom{k}{2}$ of them. In this section we show that if $|\mathbf{b}| < \binom{k}{2}$ then the grasshopper does not have to use all the permutations in S_k.

The standard action of the linear group $\mathrm{GL}(k)$ on $\mathbb{C}^k$ induces an action on the flag manifold Fl. Let us denote the subgroup of $\mathrm{GL}(k)$ consisting of diagonal matrices by T and upper triangular matrices by B. For a permutation $w \in S_k$ let F_w be the coordinate flag, for which $V_i = \langle e_{w(1)}, \dots, e_{w(i)}\rangle$.

For a permutation w let $l(w)$ denote the number of inversions in it. If w_1 and w_2 are two permutations and $w_2 = s(i,j)w_1$ where $s(i,j)$ is a transposition and $l(w_2) = l(w_1) + 1$, then we say that w_2 covers w_1 and $w_2 > w_1$. The Bruhat order on S_k is the transitive and reflexive closure of this relation. The hierarchy of the B-orbits is governed by the Bruhat order (see e.g. [Bri05]):

Proposition 5.12.

a) The fixed points of T in Fl *are the coordinate flags F_w, where $w \in S_k$.*
b) Fl *is the disjoint union of the orbits $C_w := BF_w$, where $w \in S_k$.*
c) Let X_w be the Zariski closure of C_w in Fl*, then we have $\dim(X_w) = l(w)$ and:*

$$X_w = \bigcup_{v\in S_k,\ v\leq w} C_v, \tag{5.12}$$

where $v \leq w$ means the relation in the Bruhat order.

The varieties X_w are invariant for the T-action, their T-invariant cohomology classes in Fl will be denoted by $[X_w]$.

For $\lambda_1 > \lambda_2 > \cdots > \lambda_k = 0$ the flag manifold Fl is T-equivariantly embedded into $\mathbb{P}(\Gamma_\lambda)$, where the degree of the Schubert variety X_w is given by the following integral:

$$\deg\big(X_w \subset \mathbb{P}(\Gamma_\lambda)\big) = \int_{\mathrm{Fl}} \hat{c}_1(L)^{l(w)} \cdot [X_w]. \tag{5.13}$$

Notice that this degree is a polynomial $P_w(\lambda)$ in the variables $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k-1})$.

The ABBV integration formula then implies following:

$$P_w(\lambda) = \sum_{u\in S_k} \frac{[X_w]|_{F_u} \cdot \left(\sum_{i=1}^{k} x_{u(i)}\lambda_i\right)^{l(w)}}{\mathrm{sgn}(u)\cdot V(\mathbf{x})} \tag{5.14}$$

Proposition 5.12. implies that the variety X_w contains exactly the torus fixed points F_u, for which $u \leq w$ in the Bruhat order. The class $[X_w]$ is supported on X_w, implying that if $F_u \notin X_w$, then $[X_w]|_{F_u} = 0$. Consequently only the terms with $u \leq w$ contribute:

$$P_w(\lambda) = \sum_{u\leq w} \frac{[X_w]|_{F_u} \cdot \left(\sum_{i=1}^{k} x_{u(i)}\lambda_i\right)^{l(w)}}{\mathrm{sgn}(u)\cdot V(\mathbf{x})}. \tag{5.15}$$

Now we use again the change of variables:

$$\lambda_i = v_i + \cdots + v_{k-1}, \quad i = 1, \ldots, k-1.$$

With this notation let us denote $P_w(\lambda) = R_w(\mathbf{v})$ and let $\mathbf{b} = (b_1, \ldots, b_{k-1})$ be a sequence with $|\mathbf{b}| = l(w)$. By the very same calculation as before we get:

$$\mathrm{coef}(R_w, v^{\mathbf{b}}) = \sum_{u\leq w} \frac{[X_w]|_{F_u} \cdot \prod_{i=1}^{k-1}\left(\sum_{j=1}^{i} x_{u(j)}\right)^{b_i}}{\mathrm{sgn}(u)\cdot V(\mathbf{x})} \tag{5.16}$$

Notice that the number on the right hand side is an equivariant integral on the flag manifold of a class with degree equal to the dimension $\binom{k}{2}$, so it is a constant.

Remark 5.13. If we denote the ordinary cohomology class of X_w by $[X_w]$ too, then the right hand side is equal to $\int_{\mathrm{Fl}} [X_w] \cdot \prod_{i=1}^{k-1} c_1(E_i^*)^{b_i}$, which is non negative, since the bundles E_i^* are globally generated.

As before, for any sets M_i of integers with $|M_i| = b_i$, $i = 1, \dots, k-1$ we have the following identity:

$$\operatorname{coef}(R_w, v^{\mathbf{b}}) = \sum_{u \le w} \frac{[X_w]|_{F_u} \cdot \prod_{i=1}^{k-1} \prod_{m \in M_i} \left(\sum_{j=1}^{i} x_{u(j)} - m \right)}{\operatorname{sgn}(u) \cdot V(\mathbf{x})}, \tag{5.17}$$

implying the following theorem:

Theorem 5.14. *Let $a_1, a_2, \dots, a_k$ be distinct integers, $w \in S_k$ and let $M_1, \dots, M_{k-1}$ be sets of integers with $|M_i| = b_i$ not containing 0 and $s = a_1 + a_2 + \cdots + a_k$. Let $w \in S_k$ be a permutation with $|\mathbf{b}| = l(w)$. If $\operatorname{coef}(R_w, v^{\mathbf{b}}) \neq 0$, then there is a permutation $\pi \le w$, such that $\sum_{j=1}^{i} a_{\pi(j)} \notin M_i$ for all $1 \le i \le k-1$.* □

There are plenty of such $w, \mathbf{b}$ pairs:

Proposition 5.15.

(1) *If w is an arbitrary permutation, then there exists a $\mathbf{b} = (b_1, \dots, b_{k-1})$ with $|\mathbf{b}| = \sum_{i=1}^{k-1} b_i = l(w)$, such that $\operatorname{coef}(R_w, v^{\mathbf{b}}) \neq 0$.*
(2) *If $\mathbf{b} = (b_1, \dots, b_{k-1})$ is a matching sequence, then there exists a permutation w with $|\mathbf{b}| = \sum_{i=1}^{k-1} b_i = l(w)$ such that $\operatorname{coef}(R_w, v^{\mathbf{b}}) \neq 0$.*

Proof. Fix $m \le \binom{k}{2}$. The classes $[X_w]$ with $l(w) = m$ generate the cohomology group $H^{2m}(\mathrm{Fl})$. Also, by Remark 5.10, the classes $\prod_{i=1}^{k-1} c_1(E_i^*)^{b_i}$ with $\sum_{i=1}^{k-1} b_i = m$ generate the cohomology group $H^{2(\binom{k}{2}-m)}(\mathrm{Fl})$. Therefore Poincaré duality and Remark 5.13. implies the proposition. □

We can give a combinatorial description for the condition $\operatorname{coef}(R_w, v^{\mathbf{b}}) \neq 0$.

We say that $\mathrm{Id} = u_0 < u_1 < \cdots < u_{l(w)} = w$ is a Bruhat chain, if u_{i+1} covers u_i for $i = 0, 1, \dots, l(w)-1$, and we denote the set of Bruhat chains from Id to w by C.

For a cover $u_{i+1} = s(n,m)u_i$ let us define $A(u_i < u_{i+1}) = \sum_{t=n}^{m-1} v_t$ and say that v_t is compatible with the cover if v_t occurs in $A(u_i < u_{i+1})$. For a Bruhat chain $c = (\mathrm{Id} = u_0 < u_1 < \cdots < u_{l(w)} = w)$ we use the notation $A(c) = \prod_{i=0}^{l(w)-1} A(u_i < u_{i+1})$.

Postnikov and Stanley gives the following description for the polynomial R_w:

Proposition 5.16. [PS09] *With the notations above we have*

$$R_w = \sum_{c \in C} A(c).$$

This implies that

Corollary 5.17. $\operatorname{coef}(R_w, v^{\mathbf{b}}) \neq 0$ *if and only if there exists a Bruhat chain $c = \mathrm{Id} < u_1 < \cdots < u_{l(w)} = w$, and variables $v_{x(i)}$ for $0 \le i \le l(w)-1$, such that $v_{x(i)}$ is compatible with the cover $u_i < u_{i+1}$ and $\prod_{i=0}^{l(w)-1} v_{x(i)} = v^{\mathbf{b}}$.* □

We give another easier equivalent condition, and sketch the proof of the equivalence.

Corollary 5.18. *Let $w \in S_k$ be a permutation and let F be the set of inversions in w. For each inversion $f = (i,j) \in F$ define the linear polynomial $L(f) = \sum_{k=i}^{j-1} v_k$ and the polynomial $L_w = \prod_{f \in F} L(f)$. Assume that $l(w) = |\mathbf{b}|$.*

(1) With the notations above $\operatorname{coef}(R_w, v^{\mathbf{b}}) \neq 0$ *if and only if* $\operatorname{coef}(L_w, v^{\mathbf{b}}) \neq 0$.
(2) For $s \leq t$ let $K_{s,t}$ denote the number of inversions (i,j) of w such that $s \leq i < j \leq t+1$. Then $\operatorname{coef}(R_w, v^{\mathbf{b}}) \neq 0$ *if and only if*

$$\sum_{i=s}^{t} b_i \geq K_{s,t} \tag{5.18}$$

for all $1 \leq s \leq t \leq k-1$.

Proof. The second statement is just the usual Hall-theoretical reformulation of the first statement, so it is enough to prove the first. The direction that $\operatorname{coef}(R_w, v^{\mathbf{b}}) \neq 0$ implies $\operatorname{coef}(L_w, v^{\mathbf{b}}) \neq 0$ can be proved by induction on $l(w)$.

For the other direction it is enough to show that there exists a Bruhat chain $c = \mathrm{Id} < u_1 < \cdots < u_{l(w)} = w$, such that $A(c) = L_w$, which can also be proved by induction on $l(w)$. We can find a permutation w' and a transposition (i,j) with $w' \cdot (i,j) = w$, $w(i) = w(j) + 1$, and $l(w) = l(w') + 1$. With this choice F consists of the set of inversions in w' and (i,j). If we apply the induction hypothesis for w' we get a Bruhat chain c' with $A(c') = L(w')$, implying that $A(c) = L_w$ holds for $c = c' \cup (i,j)$. □

Notice that specializing (5.18) to the longest permutation we recover the system of inequalities (5.9). We conjecture that the condition (5.18) is sharp: the grasshopper can survive with permutations under w and forbidden sets M_i with size b_i if and only if (5.18) holds.

6. The Symplectic case

There are natural generalizations of the previous constructions. (We thank Richard Rimanyi for drawing our attention to this possibility.) So far we studied Grassmannians and flag manifolds which are homogeneous spaces for the simple groups $\mathrm{SL}(n)$. There are three other infinite series of complex Lie groups (more exactly simple Lie algebras). Since the corresponding Weyl groups are similar they lead to very similar variants of our previous constructions. It turns out that the case of the symplectic groups is the most convenient. We discuss these statements in somewhat less details. For example studying the degree of the symplectic (or sometimes called Lagrangian) Grassmannians leads to a *signed* Erdős–Heilbronn theorem.

Theorem 6.1. *If A is a set of distinct non zero residue classes $a_1, \ldots, a_n$ modulo p, $a_i + a_j \neq 0$ and $p > 2k(n-k) + \binom{k+1}{2}$, then:*

$$\left| \left\{ \sum_{i \in I} \pm a_i \;\middle|\; I \subset (1, \ldots, n), |I| = k \right\} \right| > 2k(n-k) + \binom{k+1}{2}$$

This can be proved analogously to our geometric proof for the Dias da Silva–Hamidoune theorem, by replacing the Grassmannian with the symplectic Grassmannian. Notice that the right hand side $d = 2k(n-k) + \binom{k+1}{2}$ is exactly the dimension of the symplectic Grassmannian $\mathrm{SGr}_k(\mathbb{C}^{2n})$.

The estimate, unlike the unsigned version, is not sharp. Based on computer calculations we conjecture that the extremal cases for A are the arithmetic progressions

$$a, 3a, 5a, \ldots, (2n-1)a,$$

for which the number of signed sums is $k(2n-k)+\delta(k)$ where $\delta(k)$ is 1 if $k \neq 2$ and 0 if $k=2$. For $k>2$ and n small this can be proved by computer. For $k=2$ the estimate is sharp.

It turned out that there is a simpler proof which is a slight modification of the proof of the Dias da Silva–Hamidoune theorem.

Proof. Let's have a Schubert variety $\sigma_\lambda \subset \mathrm{Gr}_k(\mathbb{C}^{2n})$, by theorem3.3 we have:

$$\deg(\sigma_\lambda) = \frac{d!}{(I_1-1)!\cdots(I_k-1)!}\cdot\prod_{i<j}(I_j-I_i) = \int_{\mathrm{Gr}_k(\mathbb{C}^{2n})} c_1^d[\sigma_\lambda],$$

where $I_j := 2n-k+j-(k-j) = 2(n-k+j)$ and $d=\dim(\sigma_\lambda)$. The cohomology class $[\sigma_\lambda] \in H^*\big(\mathrm{Gr}_k(\mathbb{C}^{2n})\big)$ is given by the Schur polynomial $s_\lambda(\alpha) = \det(\alpha_j^{\lambda_i+k-i})/V(\alpha)$ where the α_i's are the Chern roots of S^*, the dual of the tautological subbundle. As s_λ is symmetric it can be also expressed as a polynomial of the elementary symmetric polynomials of the α_i's, i.e. the Chern classes $c_i(S^*)$. To apply the ABBV formula we need an equivariant lift of the integrand: we replace the Chern classes $c_i(S^*)$ with the torus-equivariant ones using that S^* is a $T(2n)$-bundle. Using the notation $\widehat{[\sigma_\lambda]}$ for the lift of $[\sigma_\lambda]$ we have that

$$\int_{\mathrm{Gr}_k(\mathbb{C}^{2n})} c_1^d[\sigma_\lambda] = \int_{\mathrm{Gr}_k(\mathbb{C}^{2n})} \hat{c}_1^d\widehat{[\sigma_\lambda]}.$$

For $\lambda=(k-1,\dots,0)$ we have

$$s_\lambda(\alpha) = \prod_{1\le i<j\le k}(\alpha_i+\alpha_j).$$

Applying now the ABBV formula and using the usual dehomogenizing trick with a d-element set M we get

$$\frac{d!}{(I_1-1)!\cdots(I_k-1)!}\cdot\prod_{i<j}(I_j-I_i) = \sum_{J\in\binom{2n}{k}} \frac{\prod\limits_{m\in M}(x_J-m)\cdot\prod\limits_{i,j\in J, i<j}(x_i+x_j)}{\prod\limits_{i\in J}\prod\limits_{j\notin J}(x_i-x_j)}.$$

Then substitute the elements of $A\cup -A$ into the x_i's. Notice that because of the factor $\prod\limits_{i,j\in J,i<j}(x_i+x_j)$ the only non zero terms correspond to subsets not containing both a and $-a$. Then we can use the argument of Theorem 3.1. by choosing M as the possible k-term signed sums to finish the proof. □

Remark 6.2. One can prove the signed Erdős–Heilbronn theorem using the Combinatorial Nullstellensatz to the polynomial

$$P(\mathbf{x}) = \prod_{1\le i<j\le k}(x_j^2-x_i^2)\prod_{b\in B}\left(\sum_{i=1}^k x_i - b\right).$$

where $B\subset\mathbb{F}_p$ and $|B|=d$. The coefficient of $\prod_{1\le i\le k} x_i^{2n-i}$ can be calculated using the standard theory of symmetric polynomials, namely with the hook rule. This coefficient is exactly the degree of the Schubert variety σ_λ, and the calculation is a proof for this special case of the Schubert degree formula.

6.1. **The small p case.** For $p \le d = 2k(n-k) + \binom{k+1}{2}$ we want to find a Schubert variety $\sigma_\lambda \subset \mathrm{Gr}_k(\mathbb{C}^{2n})$ with dimension $p-1$ and we also want its cohomology class to be divisible by $\prod_{1\le i<j\le k} (\alpha_i+\alpha_j)$. This can be achieved by substituting α_i^2's into a Schur polynomial s_λ. Then

$$s_\lambda(\alpha_1^2,\dots,\alpha_k^2) = \frac{\det(\alpha_j^{2\lambda_i+2k-2i})}{\prod\limits_{1\le i<j\le k}(\alpha_i+\alpha_j)(\alpha_i-\alpha_j)},$$

implying that

$$s_\mu(\alpha) = s_\lambda(\alpha_1^2,\dots,\alpha_k^2)\prod_{1\le i<j\le k}(\alpha_i+\alpha_j),$$

for $\mu_i = 2\lambda_i + k - i$. The only problem is that the degree of $s_\lambda(\alpha_1^2,\dots,\alpha_k^2)$ is even, so we can find a desired Schubert variety with dimension $p-1$ only if k is even (using that k is congruent to $\binom{k}{2}+\binom{k+1}{2}$ modulo 2). Therefore we arrived at the following:

Theorem 6.3. *If A is a set of distinct non zero residue classes $a_1,\dots,a_n$ modulo p, $a_i+a_j \ne 0$ and $p \le 2k(n-k)+\binom{k+1}{2}$, then:*

$$\left|\left\{\sum_{i\in I} \pm a_i \;\middle|\; I \subset (1,\dots,n), |I| = k\right\}\right| = p,$$

if k is even. If k is odd, then the zero residue class might be missing.

We conjecture however that even if k is odd, the zero residue class is not missing.

The grasshopper problem also has a symplectic version, we discuss it in the following:

6.2. **Grasshopper with signs.** Let $a_1,a_2,\dots,a_k$ be distinct non negative integers and let $|M_i| = b_i$ sets for $1\le i\le k$. A grasshopper is to jump along the real axis, starting at the point 0 and make k jumps with lengths $a_1,a_2,\dots,a_k$ in some order, and at each step the grasshopper can decide if he decides to jump to the left or to the right. How many forbidden positions can be given at each step, such that the grasshopper can avoid them?

Or more formally: what are the integer sequences $(b_1,\dots,b_k)$, for which for all non negative $a_1,a_2,\dots,a_k$ sequences there is a permutation $\pi\in S_k$ and a sign function $s:(1,\dots,k)\to\{-1,+1\}$, such that

$$\sum_{1\le j\le i} s(j)a_{\pi(j)} \notin M_i?$$

Let Γ_λ be the irreducible representation of $Sp(2k,\mathbb{C})$ with highest weight $\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_{k-1} > \lambda_k > 0)$. The minimal orbit in $\mathbb{P}(\Gamma_\lambda)$ is the symplectic flag variety $S\,\mathrm{Fl} = S\,\mathrm{Fl}(k)$. Its degree in $\mathbb{P}(\Gamma_\lambda)$ is calculated by the integral

$$\deg\big(S\,\mathrm{Fl}\subset\mathbb{P}(\Gamma_\lambda)\big) = \int_{\mathrm{Fl}} \hat c_1(L_\lambda)^d,$$

where L_λ is the canonical bundle—the pullback of the canonical bundle over $\mathbb{P}(\Gamma_\lambda)$, $\hat c_1(L_\lambda)$ is the first T-equivariant Chern class of L_λ and $d = \dim(S\,\mathrm{Fl}) = k^2$.

By the Borel–Hirzebruch formula we have:

$$\deg\big(S\,\mathrm{Fl}\subset\mathbb{P}(\Gamma_\lambda)\big) = \frac{d!}{k!}\prod_{i=1}^k \lambda_i \prod_{i<j\le k}\frac{\lambda_i^2-\lambda_j^2}{j^2-i^2}.$$

On the other hand the fixed points of the $T(k)$-action—where $T(k)$ is a maximal torus of $Sp(2k,\mathbb{C})$—are parametrized by the pairs (π, s) where $\pi \in S_k$ is a permutation and $s : (1,\dots,k) \to \{-1,+1\}$ is a sign function. Applying the ABBV integral formula we get:

$$\deg\left(S\,\mathrm{Fl} \subset \mathbb{P}(\Gamma_\lambda)\right) = \int_{\mathrm{Fl}} c_1^T(L)^d = \sum_{(\pi,s)} \frac{\left(\sum_{i=1}^{k} s(i)\lambda_i x_{\pi(i)}\right)^d}{e(T_{(\pi,s)}S\,\mathrm{Fl})}, \tag{6.1}$$

where

$$e(T_{(\pi,s)}S\,\mathrm{Fl}) = \prod_{1\le i\le j\le k} s(i)x_{\pi(i)} + s(j)x_{\pi(j)} \prod_{1\le i<j\le k} s(i)x_{\pi(i)} - s(j)x_{\pi(j)}.$$

The right hand side of (6.1) is a polynomial in the variables λ_i.

We define the homogenous polynomial P in the variables v_i with degree d by substituting $\lambda_i = \sum_{j=i}^{k} v_j$. The coefficient of $v^{\mathbf{b}}$ for $\mathbf{b} = (b_1,\dots,b_k)$ is:

$$\mathrm{coef}(P, v^{\mathbf{b}}) = \binom{d}{\mathbf{b}} \sum_{(\pi,s)} \frac{\prod_{i=1}^{k}\left(\sum_{j=1}^{i} s(j)x_{\pi(j)}\right)^{b_i}}{e(T_{(\pi,s)}S\,\mathrm{Fl})}. \tag{6.2}$$

By the usual trick we can add lower order terms:

$$\mathrm{coef}(P, v^{\mathbf{b}}) = \sum_{(\pi,s)} \frac{\prod_{i=1}^{k}\prod_{m\in M_i}\left(\sum_{j=1}^{i} s(j)x_{\pi(j)} - m\right)}{e(T_{(\pi,s)}S\,\mathrm{Fl})}. \tag{6.3}$$

We can substitute the numbers a_i into the variables x_i because $a_i \neq a_j$ if $i \neq j$ and $a_i \neq -a_j$, so the denominator is not zero. This implies that the grasshopper can always jump if $\sum_{1\le i\le k} b_i = k^2$ and $\mathrm{coef}(P, v^{\mathbf{b}}) \neq 0$.

The Borel–Hirzebruch formula gives:

$$P(\mathbf{v}) = K \cdot \prod_{1\le i\le j\le k}\left(\sum_{l=i}^{k} v_l + \sum_{l=j}^{k} v_l\right) \prod_{1\le i<j\le k}\left(\sum_{l=i}^{j-1} v_l\right), \tag{6.4}$$

where the K is a positive constant. We can see that $\mathrm{coef}(P, v^{\mathbf{b}}) \neq 0$ if and only if $\mathrm{coef}(Q, v^{\mathbf{b}}) \neq 0$, where:

$$Q(\mathbf{v}) = \prod_{1\le i\le j\le k}\left(\sum_{l=i}^{k} v_l\right) \prod_{1\le i<j\le k}\left(\sum_{l=i}^{j-1} v_l\right) \tag{6.5}$$

We can assign a bipartite graph to $\mathbf{b}$ similarly to Definition 5.5. such that the non vanishing of $\mathrm{coef}(Q, v^{\mathbf{b}})$ is equivalent to the existence of a perfect matching. Slightly modifying the argument of Proposition 5.9. we arrive at the following:

Theorem 6.4. *Let $a_1, a_2, \dots, a_k$ be distinct non negative integers, and let M_i for $1 \le i \le k$ be sets of integers with $\sum_{1\le i\le k} b_k = k^2$ for $|M_i| = b_i$. Assume moreover that the following conditions hold:*

(1) If $1 \le i \le j \le k-1$ *then* $\sum_{i \le l \le j} b_l \ge \binom{j-i+2}{2}$,

(2) $1 \le i \le k$, *then* $\sum_{i \le l \le k} b_l \ge (k-i+1)^2$.

Then there is a permutation $\pi \in S_k$ *and a sign function* $s : (1, \dots, k) \to \{-1, +1\}$, *such that for all* $1 \le i \le k$

$$\sum_{1 \le j \le i} s(j) a_{\pi(j)} \notin M_i.$$

Remark 6.5. It is not difficult to see that the signed Erdős–Heilbronn theorem over the integers follows from the signed grasshopper theorem, exactly as for the unsigned versions.

7. Concluding Remarks

It turns out that all results in the paper can be reproved using the Combinatorial Nullstellensatz. Nevertheless we hope that the advantages of our geometric method is clear: the right estimate is suggested by the dimension of a homogeneous space, and the non-vanishing of the relevant coefficient is guaranteed since it is the degree of a variety. This identification also helps to use existing formulas like the Borel–Hirzebruch formula to calculate this coefficient and study its prime factors.

It is tempting to apply this geometric approach to problems where the coefficient formula was succesfully used like the Dyson identity [KP12] or the q-Dyson identity [KN14]. In fact if we specialize for $n = 2$ the proof Karasev and Petrov gave to the Dyson identity in [KP12], then it is essentially equivalent to the calculation of the degree of the Segre embedding using the identity (2.4). It also looks promising to approach the inverse Erdős–Heilbronn problem this way.

Appendix A. The Borel–Hirzebruch formula

The main references are [FH91] and [GW11]. Let G be a simple complex Lie group ($\mathrm{SL}(n)$ or $\mathrm{Sp}(2n)$ in the paper). Given a dominant weight λ we have an irreducible representation Γ_λ of G with highest weight λ we have an action of G on the projective space $\mathbb{P}(\Gamma_\lambda)$. Its minimal orbit $X_\lambda \subset \mathbb{P}(\Gamma_\lambda)$ is a homogeneous space for G. Let R^+ denote the positive roots of G. For a root $\alpha \in R^+$ let $\alpha^\vee$ denote the corresponding coroot. Unfortunately there are different conventions, we use the one that a coroot is an element of the Cartan subalgebra. This corresponds to the H_α notation of Fulton and Harris in [FH91]. The Borel–Hirzebruch formula calculates the projective degree of the minimal orbit:

Theorem A.1. *With the notation above*

$$\deg\big(X_\lambda \subset \mathbb{P}(\Gamma_\lambda)\big) = d! \prod_{\alpha \in T_\lambda} \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle},$$

where $T_\lambda = \{\alpha \in R^+ : \langle \lambda, \alpha^\vee \rangle \ne 0\}$, $d = |T_\lambda|$ *and* ρ *is half the sum of the positive roots.*

The pairing $\langle \cdot, \cdot \rangle$ is the dual pairing. d is also the dimension of X_λ.

A.1. $\mathrm{GL}(n)$. We switch to $\mathrm{SL}(n)$ since $\mathrm{GL}(n)$ is not a simple group. The Cartan subalgebra consists of the zero trace diagonal matrices. The weights are generated by the functionals L_i for $i = 1, \dots, n$ with value the i-th entry of the diagonal matrix. Notice that $\sum_{i=1}^n L_i = 0$. The positive roots are

$$R^+ = \{L_i - L_j : i < j\}.$$

Half the sum of the positive roots is

$$\rho = \sum_{i=1}^{n-1} (n-i) L_i.$$

Let H_i be the standard basis for the diagonal matrices, then

$$(L_i - L_j)^\vee = H_i - H_j.$$

The dominant weights are of the form $\sum_{i=1}^n \lambda_i L_i$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Because of the relation $\sum_{i=1}^n L_i = 0$ you can choose λ_n to be 0. The minimal orbit is a partial flag manifold corresponding to the multiplicities of the pairwise distinct λ_i's. In particular the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ corresponds to $\lambda = (1,1,\dots,1,0,\dots,0)$ with k copies of 1's.

A.2. $\mathrm{Sp}(2n)$**.** The Cartan subalgebra of the symplectic group $\mathrm{Sp}(2n)$ can be identified with the space of $2n \times 2n$ diagonal matrices with $a_{n+i,n+i} = -a_{i,i}$. The weights are generated by the functionals L_i for $i = 1,\dots,n$ with value $a_{i,i}$. The positive roots are

$$R^+ = \{L_i - L_j : i < j\} \cup \{L_i + L_j : i \leq j\}.$$

Half the sum of the positive roots is

$$\rho = \sum_{i=1}^{n} (n-i+1) L_i.$$

Let H_i be the diagonal matrix with $a_{i,i} = -a_{n+i,n+i} = 1$, and the other entries 0. Then

$$(L_i - L_j)^\vee = H_i - H_j, \quad (L_i + L_j)^\vee = H_i + H_j \text{ for } \ i < j, \quad \text{and } (2L_i)^\vee = H_i.$$

The dominant weights are of the form $\sum_{i=1}^n \lambda_i L_i$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimal orbit is the symplectic flag manifold for pairwise distinct λ_i's. The symplectic Grassmannian $S\,\mathrm{Gr}_k(\mathbb{C}^{2n})$ corresponds to $\lambda = (1,1,\dots,1,0,\dots,0)$ with k copies of 1's.

Appendix B. The ABBV formula for minimal orbits

A maximal torus $T(n)$ of G is acting on the minimal orbit X_λ. The fixed points can be identified with the orbit of λ under the action of the Weyl group W_G of G, i.e with the cosets W_G/W_λ, where W_λ is the stabilizer subgroup. The tangent space in the fixed point corresponding to λ has weights $\tau_\lambda := \{\alpha \in R^- : \langle \lambda, \alpha^\vee \rangle \neq 0\}$. To get the weights for the other fixed points you apply the Weyl group. Therefore the integral of a $T(n)$-equivariant cohomology class α is

$$\int_{X_\lambda} \alpha = \sum_{f \in W_G/W_\lambda} \frac{\alpha|_f}{\prod\limits_{\omega \in \tau_\lambda} f(\omega)}.$$

To calculate the degree we need the restrictions of the first $T(n)$-equivariant Chern class of the canonical bundle:

$$\hat{c}_1(L_\lambda)|_f = \sum_{i=1}^n \lambda_i f(x_i),$$

where $H^*_{T(n)} = \mathbb{Z}[x_1,\dots,x_n]$ and the x_i's are the „positive" generators. So there is a double twist in the notation. For $\mathrm{GL}(n)$ and $\mathrm{Sp}(2n)$ the x_i's correspond to the $-L_i$'s, and the L_i's correspond to the weights of the tautological subbundles (as opposed to the canonical subundles).

B.1. $\mathrm{GL}(n)$**.** The Weyl group is the symmetric group S_n acting on the weights by permuting the coefficients. For $\lambda = (1,0,\dots,0)$ the minimal orbit is the whole projective space $\mathbb{P}(\mathbb{C}^n)$ and, more generally for $\lambda = (1,\dots,1,0,\dots,0)$ with k copies of 1's the minimal orbit is the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ in the Plücker embedding.

B.2. Sp$(2n)$. The Weyl group is the semidirect product of the symmetric group S_n and $\mathbb{Z}_2^n$ acting on the weights by permuting the coefficients and multiplying the i-th coefficient by -1, respectively.

References

[AB84] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.

[Alo99] Noga Alon. Combinatorial Nullstellensatz. volume 8, pages 7–29. 1999. Recent trends in combinatorics (Mátraháza, 1995).

[ANR96] N. Alon, M. B. Nathanson., and I. Ruzsa. The polynomial method and restricted sums of congruence classes. *J. Number Theory*, 56(2):404–417, 1996.

[BH59] A. Borel and F. Hirzebruch. Characteristic classes and homogeneous spaces. II. *Amer. J. Math.*, 81:315–382, 1959.

[Bri05] M. Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, Trends Math., pages 33–85. Birkhäuser, Basel, 2005.

[DdSH94] J. A. Dias da Silva and Y. O. Hamidoune. Cyclic spaces for Grassmann derivatives and additive theory. *Bull. London Math. Soc.*, 26(2):140–146, 1994.

[Dua03] H. Duan. The degree of a Schubert variety. *Adv. Math.*, 180(1):112–133, 2003.

[EG80] P. Erdős and R. L. Graham. *Old and new problems and results in combinatorial number theory*, volume 28 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*. Université de Genève, L'Enseignement Mathématique, Geneva, 1980.

[FH91] W. Fulton and J. Harris. *Representation theory. A first course*. Number 125 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1991.

[Ful98] W. Fulton. *Intersection Theory*. Springer, 1984, 1998.

[GW11] B. H. Gross and N. R. Wallach. On the Hilbert polynomials and Hilbert series of homogeneous projective varieties. In *Arithmetic geometry and automorphic forms*, volume 19 of *Adv. Lect. Math. (ALM)*, pages 253–263. Int. Press, Somerville, MA, 2011.

[Hal35] P Hall. On representatives of subsets. *Journal of the London Mathematical Society*, 1(1):26–30, 1935.

[Khr] D. Khramtsov. Imo 2009, problem 6. available at http://www.artofproblemsolving.com/Forum/viewtopic.php?t=289051.

[KN14] Gy. Károlyi and Z. L. Nagy. A simple proof of the Zeilberger-Bressoud q-Dyson theorem. *Proc. Amer. Math. Soc.*, 142(9):3007–3011, 2014.

[KNPV15] Gy. Károlyi, Z. L. Nagy, F. V. Petrov, and V. Volkov. A new approach to constant term identities and Selberg-type integrals. *Advances in Mathematics*, 277:252 – 282, 2015.

[kom09] Középiskolai Matematikai és fizikai lapok: Kömal. page 542, december 2009.

[Kós11] G. Kós. On the grasshopper problem with signed jumps. *American Mathematical Monthly*, 118(10):877–886, 2011.

[KP12] R. N. Karasev and F. V. Petrov. Partitions of nonzero elements of a finite field into pairs. *Israel Journal of Mathematics*, 192(1):143–156, Nov 2012.

[Las10] MichałLasoń. A generalization of combinatorial Nullstellensatz. *Electron. J. Combin.*, 17(1):Note 32, 6, 2010.

[PM09] E. Preissmann and M. Mischler. Seating couples around the King's table and a new characterization of prime numbers. *Amer. Math. Monthly*, 116(3):268–272, 2009.

[PS09] A. Postnikov and R. P. Stanley. Chains in the Bruhat order. *J. Algebraic Combin.*, 29(2):133–174, 2009.

[Sun08] Z. W. Sun. On value sets of polynomials over a field. *Finite Fields Appl.*, 14(2):470–481, 2008.

[SZ12] Z. W. Sun and L. L. Zhao. Linear extension of the Erdős–Heilbronn conjecture. *J. Combin. Theory Ser. A*, 119(2):364–381, 2012.

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