On Helly numbers of exponential lattices*

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Abstract

Given a set $S \subseteq \mathbb{R}^2$, define the *Helly number of* S, denoted by $H(S)$, as the smallest positive integer N, if it exists, for which the following statement is true: for any finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^2$ such that the intersection of any N or fewer members of $\mathcal{F}$ contains at least one point of S, there is a point of S common to all members of $\mathcal{F}$.

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We prove that the Helly numbers of *exponential lattices* $\{\alpha^n \colon n \in \mathbb{N}_0\}^2$ are finite for every $\alpha > 1$ and we determine their exact values in some instances. In particular, we obtain $H(\{2^n \colon n \in \mathbb{N}_0\}^2) = 5$, solving a problem posed by Dillon (2021).

For real numbers $\alpha, \beta > 1$, we also fully characterize exponential lattices $L(\alpha, \beta) = \{\alpha^n \colon n \in \mathbb{N}_0\} \times \{\beta^n \colon n \in \mathbb{N}_0\}$ with finite Helly numbers by showing that $H(L(\alpha, \beta))$ is finite if and only if $\log_\alpha(\beta)$ is rational.

1 Introduction

Helly's theorem [11] is one of the most classical results in combinatorial geometry. It states that, for each $d \in \mathbb{N}$, if the intersection of any $d+1$ or fewer members of a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ is nonempty, then the entire family $\mathcal{F}$ has nonempty intersection. There have been numerous variants and generalizations of this famous result; see [1, 13] for example. One active direction of this research with rich connections to the theory of optimization, in particular to integer programming and LP-type problems [1, 4], is the study of variants of Helly's theorem with coordinate restrictions, which is captured by the following definition.

Let d be a positive integer. The *Helly number* of a set $S \subseteq \mathbb{R}^d$, denoted by $H(S)$, is the smallest positive integer N, if it exists, such that the following statement is true for every finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$: if the intersection of any N or fewer members of $\mathcal{F}$ contains at least one point of S, then $\bigcap \mathcal{F}$ contains at least one point of S. If no such number N exists, then we write $H(S) = \infty$. Helly's theorem in this language can be restated as $H(\mathbb{R}^d) = d + 1$.

A classical result of this sort is *Doignon's theorem* [8] where the set S is the integer lattice $\mathbb{Z}^d$. This result, which was also independently discovered by Bell [3] and by Scarf [15], states that $H(\mathbb{Z}^d) \leq 2^d$. This is tight as for $Q = \{0,1\}^d$ the intersection of any $2^d - 1$ sets in the family $\{\operatorname{conv}(Q \setminus \{x\}) \colon x \in Q\}$ contains a lattice point, but the intersection of all 2^d sets does not.

The theory of Helly numbers of general sets is developing quickly and there are many result of this kind [1, 13]. For example, De Loera, La Haye, Oliveros, and Roldán-Pensado [5] and De Loera, La Haye, Rolnick, and Soberón [6] studied the Helly numbers of differences of lattices and Garber [9] considered Hely numbers of crystals or cut-and-project sets.

The Helly number of a set S is closely related to the maximum size of a set that is empty in S. A subset $X \subseteq S$ is *intersect-empty* if $\left(\bigcap_{x \in X} \operatorname{conv}(X \setminus \{x\})\right) \cap S = \emptyset$. A convex polytope P with vertices in S is *empty in S* if P does not contain any points of S other than its vertices. In particular, an empty polytope does not contain points of S in the interior of its edges. For a discrete set S, we use $h(S)$ to denote the maximum number of vertices of an empty polytope in S. If there are empty polytopes in S with arbitrarily large number of vertices, then we write $h(S) = \infty$.

The following result by Hoffman [12] (which was essentially already proved by Doignon [8]) shows the close connection between intersect-empty sets and empty polygons in S and the S-Helly numbers; see also [2].

Proposition 1 ([12])**.** *If $S \subseteq \mathbb{R}^d$, then $H(S)$ is equal to the maximum cardinality of an intersect-empty set in S. If S is discrete, then $H(S) = h(S)$.*

Since all the sets S studied in this paper are discrete, we state all of our results using $h(\alpha)$ but, due to Proposition 1, our results apply to $H(\alpha)$ as well.

Very recently, Dillon [7] proved that the Helly number of a set S is infinite if S belongs to a certain collection of *product sets*, which are sets of the form $S = A^d$ with a certain kind of discrete set $A \subseteq \mathbb{R}$. His result shows, for example, that whenever p is a polynomial of degree at least 2 and $d \geq 2$, then $h(\{p(n) \colon n \in \mathbb{N}_0\}^d) = \infty$. However, there are sets for which Dillon's method gives no information, for example $\{2^n \colon n \in \mathbb{N}_0\}^2$. Thus, Dillon [7] posed the following question, which motivated our research.

Problem 1 (Dillon, [7])**.** *What is $h(\{2^n \colon n \in \mathbb{N}_0\}^2)$?*

In this paper, we study the Helly numbers of *exponential lattices* $L(\alpha)$ and $L(\alpha, \beta)$ in the plane where $L(\alpha) = \{\alpha^n \colon n \in \mathbb{N}_0\}^2$ and $L(\alpha, \beta) = \{\alpha^n \colon n \in \mathbb{N}_0\} \times \{\beta^n \colon n \in \mathbb{N}_0\}$ for real numbers $\alpha, \beta > 1$. In particular, we prove that Helly numbers of exponential lattices $L(\alpha)$ are finite and we provide several estimates that give exact values for α sufficiently large, solving Problem 1. We also show that Helly numbers of exponential lattices $L(\alpha, \beta)$ are finite if and only if $\log_\alpha(\beta)$ is rational.

2 Our results

For a real number $\alpha > 1$ and the exponential lattice $L(\alpha) = \{\alpha^n \colon n \in \mathbb{N}_0\}^2$, we abbreviate $h(L(\alpha))$ by $h(\alpha)$.

As our first result, we provide finite bounds on the numbers $h(\alpha)$ for any $\alpha > 1$. The upper bounds are getting smaller as α increases and reach their minimum at $\alpha = 2$.

Theorem 2. *For every real $\alpha > 1$, the maximum number of vertices of an empty polygon in $L(\alpha)$ is finite. More precisely, we have $h(\alpha) \leq 5$ for every $\alpha \geq 2$, $h(\alpha) \leq 7$ for every $\alpha \in [\frac{1+\sqrt{5}}{2}, 2)$, and*

$$h(\alpha) \leq 3 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 3$$

for every $\alpha \in (1, \frac{1+\sqrt{5}}{2})$*.*

We note that if $\alpha = 1 + \frac{1}{x}$ for $x \in (0, \infty)$, then the bound from Theorem 2 becomes $h(1 + \frac{1}{x}) \leq O(x \log_2(x))$. Moreover, we show that the breaking points of α for our upper bounds are determined by certain polynomial equations; see Section 3.

We also consider the lower bounds on $h(\alpha)$ and provide the following estimate.

Theorem 3. *We have* $h(\alpha) \geq 5$ *for every* $\alpha \geq 2$ *and* $h(\alpha) \geq 7$ *for every* $\alpha \in \left[\frac{1+\sqrt{5}}{2}, 2\right)$*. For every* $\alpha \in \left(1, \frac{1+\sqrt{5}}{2}\right)$*, we have*

$$h(\alpha) \geq \left\lfloor \sqrt{\frac{1}{\alpha - 1}} \right\rfloor .$$

If $\alpha = 1 + \frac{1}{x}$ where $x \in (0, \infty)$, then the lower bound from Theorem 3 becomes $h(1 + \frac{1}{x}) \geq \lfloor \sqrt{x} \rfloor$. So with decreasing α, the parameter $h(\alpha)$ indeed grows to infinity.

By combining Theorems 2 and 3, we get the precise value of the Helly numbers of $L(\alpha)$ with $\alpha \geq (1 + \sqrt{5})/2$. In particular, for $\alpha = 2$, we obtain a solution to Problem 1.

Corollary 4. *We have* $h(\alpha) = 5$ *for every* $\alpha \geq 2$ *and* $h(\alpha) = 7$ *for every* $\alpha \in [\frac{1+\sqrt{5}}{2}, 2)$*.*

We prove the following result which shows that even a slight perturbation of S can affect the value $h(S)$ drastically (note that this also follows by adding large empty polygons to S without changing its asymptotic density). We use the *Fibonacci numbers* $(F_n)_{n \in \mathbb{N}_0}$, which are defined as $F_0 = 1, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for every integer $n \geq 2$.

Proposition 5. *We have* $h(\{F_n \colon n \in \mathbb{N}_0\}^2) = \infty$*.*

We recall that $F_n = \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}}$ for every $n \in \mathbb{N}_0$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the *golden ratio* and $\psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi$ is its conjugate. Since $\psi < 1$, this formula shows that the points of $\{F_n \colon n \in \mathbb{N}_0\}^2$ are approaching the points of the scaled exponential lattice $\frac{\varphi}{\sqrt{5}} \cdot L(\varphi) = \{\frac{\varphi}{\sqrt{5}} \cdot \varphi^n \colon n \in \mathbb{N}_0\}^2$. Thus, Proposition 5 is in sharp contrast with the fact that $h(\frac{\varphi}{\sqrt{5}} \cdot L(\varphi)) = h(\varphi) \leq 7$, which follows from Theorem 2 and from the fact that affine transformations of any set $S \subseteq \mathbb{R}^d$ do not change $h(S)$. We also note Dillon's method [7] does not imply $h(\{F_n \colon n \in \mathbb{N}_0\}^2) = \infty$.

We also consider the more general case of exponential lattices where the rows and the columns might use different bases. For real numbers $\alpha > 1$ and $\beta > 1$, let

$L(\alpha, \beta)$ be the set $\{\alpha^n : n \in \mathbb{N}_0\} \times \{\beta^n : n \in \mathbb{N}_0\}$. Note that $L(\alpha) = L(\alpha, \alpha)$ for every $\alpha > 1$.

As our last main result, we fully characterize exponential lattices $L(\alpha, \beta)$ with finite Helly numbers $h(L(\alpha, \beta))$, settling the question of finiteness of Helly numbers of planar exponential lattices completely.

Theorem 6. *Let $\alpha > 1$ and $\beta > 1$ be real numbers. Then, $h(L(\alpha, \beta))$ is finite if and only if $\log_\alpha(\beta)$ is a rational number.*

Moreover, if $\log_\alpha(\beta) \in \mathbb{Q}$, that is, $\beta = \alpha^{p/q}$ for some $p, q \in \mathbb{N}$, then

$$\left\lfloor \frac{1}{pq} \left\lfloor \sqrt{\frac{1}{\alpha^{1/q} - 1}} \right\rfloor \right\rfloor \leq h(L(\alpha, \beta)) \leq pq \cdot h(\alpha^p).$$

The proof of Theorem 6 is based on Theorem 2 and on the theory of continued fractions and Diophantine approximations.

Open problems

First, it is natural to try to close the gap between the upper bound from Theorem 2 and the lower bound from Theorem 3 and potentially obtain new precise values of $h(\alpha)$.

Second, we considered only the exponential lattice in the plane, but it would be interesting to obtain some estimates on the Helly numbers of exponential lattices $\{\alpha^n : n \in \mathbb{N}_0\}^d$ in dimension $d > 2$.

We also mention the following conjecture of De Loera, La Haye, Oliveros, and Roldán-Pensado [5], which inspired the research of Dillon [7].

Conjecture 1 ([5])**.** *If $\mathcal{P}$ is the set of prime numbers, then $h(\mathcal{P}^2) = \infty$.*

Using computer search, Summers [16] showed that $h(\mathcal{P}^2) \geq 14$.

3 Proof of Theorem 2

Here, we prove Theorem 2 by showing that the number $h(\alpha)$ is finite for every $\alpha > 1$. This follows from the upper bounds $h(\alpha) \leq 5$ for $\alpha \geq 2$, $h(\alpha) \leq 7$ for every $\alpha \geq [\frac{1+\sqrt{5}}{2}, 2)$, and

$$h(\alpha) \leq 3 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 3$$

for any $\alpha \in (1, \frac{1+\sqrt{5}}{2})$.

We start by introducing some auxiliary definitions and notation. Let $\alpha > 1$ be a real number and consider the exponential lattice $L(\alpha)$. For $i \in \mathbb{N}_0$, the *ith* *column* of $L(\alpha)$ is the set $\{(\alpha^i, \alpha^n) \colon n \in \mathbb{N}_0\}$. Analogously, the *ith row* of $L(\alpha)$ is the set $\{(\alpha^n, \alpha^i) \colon n \in \mathbb{N}_0\}$.

For a point p in the plane, we write $x(p)$ and $y(p)$ for the x- and y-coordinates of p, respectively. Let P be an empty convex polygon in $L(\alpha)$. Let e be an edge of P connecting vertices u and v where $x(u) < x(v)$ or $y(u) < y(v)$ if $x(u) = x(v)$. We use $\overline{e}$ to denote the line determined by e and oriented from u to v. The *slope of* e is the slope of $\overline{e}$, that is, $\frac{y(v)-y(u)}{x(v)-x(u)}$.

We distinguish four types of edges of P; see part (a) of Figure 1. First, assume $x(u) \neq x(v)$ and $y(u) \neq y(v)$. We say that e is of *type I* if the slope of e is negative and P lies to the right of $\overline{e}$. Similarly, e is of *type II* if the slope of e is positive and P lies to the right of $\overline{e}$. An edge e has *type III* if the slope of e is negative and P lies to the left of $\overline{e}$. Finally, *type IV* is for e with positive slope and with P lying to the left of $\overline{e}$. It remains to deal with horizontal and vertical edges of P. A horizontal edge e is of type II if P lies below $\overline{e}$ and is of type III otherwise. Similarly, a vertical edge e is of type IV if P lies to the left of $\overline{e}$ and is of type III otherwise.

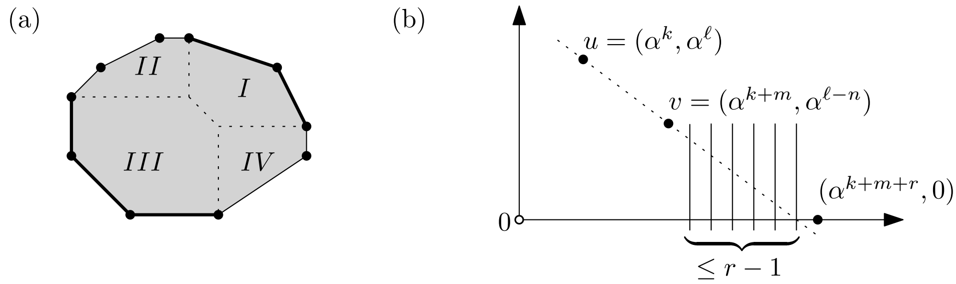

Figure 1: (a) The four types of edges of a convex polygon. (b) An illustration of the proof of Lemma 7.

Note that each edge of P has exactly one type and that the types partition the edges of P into four convex chains. We first provide an upper bound on the number of edges of those chains of P and then derive the bound on the total number of edges of P by summing the four bounds. We start by estimating the number of edges of P of type I.

Lemma 7. *The polygon P has at most $\lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right) \rceil$ edges of type I.*

Proof. First, let $r = \lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right) \rceil$ and note that $r \geq 1$ as $\alpha > 1$. Let e be the left-most edge of P of type I and let u and v be vertices of e. Since e is of type I, we have $u = (\alpha^k, \alpha^\ell)$ and $v = (\alpha^{k+m}, \alpha^{\ell-n})$ for some positive integers k, ℓ, m, and n.

We will show that the point $(\alpha^{k+m+r}, 0)$ lies above the line $\overline{e}$. Since there are at most $r-1$ columns of $L(\alpha)$ between the vertical line containing v and the vertical line containing $(\alpha^{k+m+r}, 0)$ and the point $(\alpha^{k+m+r}, 0)$ is below the lowest row of $L(\alpha)$, it then follows that there are at most r edges of P of type I; see part (b) of Figure 1.

Since the line $\overline{e}$ contains u and v, we see that

$$\overline{e} = \{(x, y) \in \mathbb{R}^2 \colon (\alpha^{\ell} - \alpha^{\ell-n})x + (\alpha^{k+m} - \alpha^{k})y = \alpha^{k+\ell+m} - \alpha^{k+\ell-n}\}.$$

It suffices to check that by substituting the coordinates of the point $(\alpha^{k+m+r}, 0)$ into the equation of the line $\overline{e}$ gives a left side that is at least $\alpha^{k+\ell+m} - \alpha^{k+\ell-n}$. The left side equals $\alpha^{k+\ell+m+r} - \alpha^{k+\ell+m-n+r}$ and thus we want

$$\alpha^{k+\ell+m+r} - \alpha^{k+\ell+m-n+r} \geq \alpha^{k+\ell+m} - \alpha^{k+\ell-n}.$$

By dividing both sides by $\alpha^{k+\ell}$ and by rearranging the terms, we can rewrite this expression as

$$\alpha^{-n}(1 - \alpha^{m+r}) \geq \alpha^{m} - \alpha^{m+r}.$$

Since $m, r > 0$ and $\alpha > 1$, we get $(1 - \alpha^{m+r}) < 0$ and thus the left side is increasing as n increases, so we can assume $n = 1$, leading to

$$\alpha^{-1} - \alpha^{m+r-1} \geq \alpha^{m} - \alpha^{m+r}.$$

We can again rearrange the inequality as

$$\alpha^{r} - \alpha^{r-1} - 1 \geq -\alpha^{-1-m},$$

where the right side is negative and approaches 0 as m tends to infinity, so we can replace it by 0, obtaining

$$\alpha^{r} - \alpha^{r-1} \geq 1.$$

This inequality is satisfied by our choice of r. □

We now estimate the number of edges of P that are of type III.

Lemma 8. *The polygon P has at most $2\lceil \log_{\alpha}\left(\frac{\alpha+1}{\alpha}\right)\rceil + 1$ edges of type III for $1 < \alpha < 2$ and at most 2 such edges for $\alpha \geq 2$.*

Proof. Let $t = \lceil \log_{\alpha}\left(\frac{\alpha+1}{\alpha}\right)\rceil$ and $s = t + 1$ for $\alpha \in (1, 2)$ and $t = 1 = s$ for $\alpha \geq 2$. Suppose for contradiction that there are $s + t + 1$ edges of P of type III. Let $v_1, \ldots, v_{s+t+2}$ be the vertices of the convex chain that is formed by edges of P of type III. We use Q to denote the convex polygon with vertices $v_1, \ldots, v_{s+t+2}$. Note that Q is empty in $L(\alpha)$ as P is empty and $Q \subseteq P$.

Let v' be the point $(x(v_{s+2}), \alpha \cdot y(v_{s+2}))$, that is, v' is the point of $L(\alpha)$ that lies just above v_{s+2}; see part (a) of Figure 2. We will show that the point v' lies below the line $\overline{v_1 v_{s+t+2}}$. Since v' lies in the same column of $L(\alpha)$ as v_{s+2}, this then

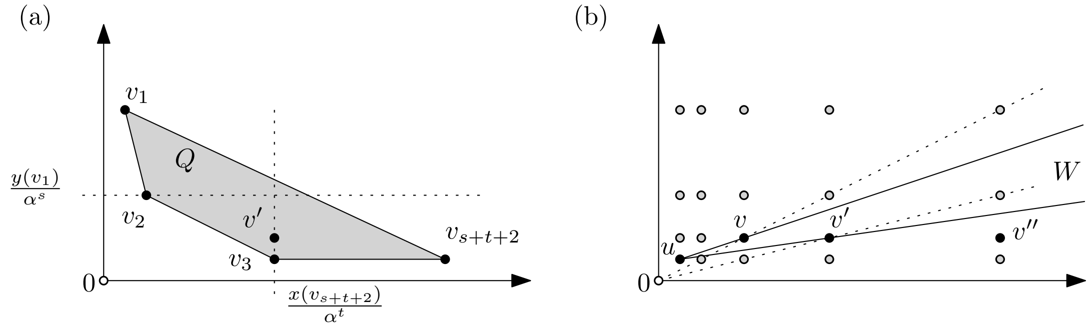

Figure 2: (a) An illustration of the proof of Lemma 8 for $s = 1 = t$. (b) An illustration of Lemma 9.

implies that v' lies in the interior of Q, contradicting the fact that Q is empty in $L(\alpha)$.

Note that $x(v') \leq \frac{x(v_{s+t+2})}{\alpha^t}$ and $y(v') \leq \frac{y(v_1)}{\alpha^s}$ as all edges $v_i v_{i+1}$ are of type III and thus the x- and y-coordinates decrease by a multiplicative factor at least α for each such edge. Since the only vertical edge might be v_1v_2 and the only horizontal edge might be $v_{s+t+1}v_{s+t+2}$, the x- or y-coordinates indeed decrease by the factor α at each step.

Let $v_1 = (\alpha^k, \alpha^\ell)$ and $v_{s+t+2} = (\alpha^{k+m}, \alpha^{\ell-n})$ for some positive integers k, ℓ, m, n. Note that $m, n \geq s + t$. The line determined by v_1 and v_{s+t+2} is then

$$\{(x, y) \in \mathbb{R}^2 \colon (\alpha^\ell - \alpha^{\ell-n})x + (\alpha^{k+m} - \alpha^k)y = \alpha^{k+\ell+m} - \alpha^{k+\ell-n}\}.$$

Since $x(v') \leq \frac{x(v_{s+t+2})}{\alpha^t}$ and $y(v') \leq \frac{y(v_1)}{\alpha^s}$, it suffices to check

$$(\alpha^\ell - \alpha^{\ell-n})\frac{\alpha^{k+m}}{\alpha^t} + (\alpha^{k+m} - \alpha^k)\frac{\alpha^\ell}{\alpha^s} < \alpha^{k+\ell+m} - \alpha^{k+\ell-n}.$$

After dividing by $\alpha^{k+\ell+m}$, this can be rewritten as

$$\alpha^{-t} + \alpha^{-s} < 1 - \alpha^{-m-n} + \alpha^{-t-n} + \alpha^{-s-m}.$$

Since $m, n \geq s + t$, the right hand side is decreasing with increasing m and n and thus we only need to prove

$$\alpha^{-s} + \alpha^{-t} \leq 1.$$

If $\alpha \geq 2$, then $s = 1 = t$ and this inequality becomes $2/\alpha \leq 1$, which is true. If $\alpha \in (1, 2)$, then $s = t + 1$ and the inequality becomes $1 + 1/\alpha \leq \alpha^t$, which is also true by our choice of t. □

It remains to bound the number of edges of P that are of types II and IV. Observe that if we switch the x- and y- coordinates of P, then edges of type II become edges of type IV and vice versa. Since the exponential lattice $L(\alpha)$ is symmetric with respect to the line $x = y$, we see that it suffices to estimate the number of edges of type II. To do so, we use the following auxiliary result.

Lemma 9. *Let u be a point of $L(\alpha)$ and let v and v' be two points of $L(\alpha)$ that are consecutive in a row R of $L(\alpha)$ that lies above the row containing u; see part (b) of Figure 2. Then, all points of $L(\alpha)$ that lie above R in the interior of the wedge spanned by the lines $\overline{uv}$ and $\overline{uv'}$ lie on at most $\lceil \log_\alpha(\frac{\alpha}{\alpha-1}) \rceil$ lines containing the origin.*

Proof. Similarly as in Lemma 7, we set $r = \lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right) \rceil$ and note that $r \geq 1$. We can assume without loss of generality that $u = (1, 1)$ as otherwise it suffices to renumber the points of $L(\alpha)$. We can also assume without loss of generality that neither of the points v and v' lies above the line $x = y$ as v and v' are consecutive on R and thus both cannot lie in opposite open halfplanes determined by this line.

Let o be the origin and consider the lines $\overline{ov}$ and $\overline{ov'}$. Then, the part of the line $\overline{ov}$ above the row R is above $\overline{uv}$; see part (b) of Figure 2. Similarly, the part of the line $\overline{ov'}$ above R is above $\overline{uv'}$. It follows that only points of $L(\alpha)$ that lie on a line $\overline{ow}$, where w is a point of $L(\alpha)$ to the right of v on R, can lie in the interior of W.

Let v'' be the point $(\alpha^r \cdot x(v'), y(v'))$, that is, v'' is the point of $L(\alpha)$ that lies at distance r to the right of v' on R, We will show that the part of the line $\overline{ov''}$ above R lies below $\overline{uv'}$. This will conclude the proof as all points of $L(\alpha)$ that lie in the interior of W above R have to then lie on one of the r lines $\overline{ow}$ with w lying between v and v'' on R.

It suffices to compare the slopes of the lines $\overline{ov''}$ and $\overline{uv'}$. Let $v' = (\alpha^m, \alpha^n)$ for some positive integers m and n. Then, the slope of $\overline{ov''}$ is

$$\frac{y(v'') - y(o)}{x(v'') - x(o)} = \frac{y(v')}{\alpha^r \cdot x(v')} = \frac{\alpha^n}{\alpha^{m+r}}$$

and the slope of $\overline{uv'}$ equals

$$\frac{y(v') - y(u)}{x(v') - x(u)} = \frac{y(v') - 1}{x(v') - 1} = \frac{\alpha^n - 1}{\alpha^m - 1}.$$

Thus, we want

$$\frac{\alpha^n - 1}{\alpha^m - 1} \geq \frac{\alpha^n}{\alpha^{m+r}}.$$

We can rewrite this inequality as

$$\alpha^{m+n+r} - \alpha^{m+r} \geq \alpha^{n+m} - \alpha^n,$$

which can be further rewritten by dividing both sides with α^n as

$$\alpha^{m+r}(1 - \alpha^{-n}) \geq \alpha^m - 1.$$

The left side is increasing with increasing n, so we can assume $n = 1$ and, by dividing both sides with α^m, we obtain

$$\alpha^r(1-\alpha^{-1}) \geq 1-\alpha^{-m}.$$

Now, the term α^{-m} on the right side approaches 0 from below with increasing m, so we can replace it by 0 obtaining

$$\alpha^r - \alpha^{r-1} \geq 1.$$

This inequality is satisfied by our choice of r. □

Now, we can apply Lemma 9 to obtain an upper bound on the number of edges of P of type II.

Lemma 10. *The polygon P has at most $\lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right) \rceil + 1$ edges of type II.*

Proof. Again, let $r = \lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right) \rceil$. Let u be the leftmost vertex of the convex chain C determined by the edges of P of type II. Similarly, let v be the second leftmost vertex of C. Note that since the edge uv is of type II, the vertex v lies in a row R of $L(\alpha)$ above the row containing u. Let v' be the point $(\alpha \cdot x(v), y(v))$, that is, point of $L(\alpha)$ that is to the right of v on R. Then, by Lemma 9, all points of $L(\alpha)$ that lie above R and in the interior of the wedge W spanned by the lines $\overline{uv}$ and $\overline{uv'}$ lie on at most r lines containing the origin.

Since P is empty in $L(\alpha)$, all vertices of C besides u, v, and possibly v' lie in W above R. Since all edges of C are of type II, every line determined by the origin and by a point of $L(\alpha)$ from the interior of W contains at most one vertex of C. Note that if v' is a vertex of C, then the only vertices of C are u, v, v'. Thus, in total C has at most $r + 2$ vertices and therefore at most $r + 1$ edges. □

We recall that, by symmetry, the same bound applies for edges of type IV and thus we get the following result.

Corollary 11. *The polygon P has at most $\lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right) \rceil + 1$ edges of type IV.* □

Since each edge of P is of one of the types I–IV, it immediately follows from Lemmas 7, 8, 10, and from Corollary 11 that the number of edges of P is at most

$$3\left\lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right)\right\rceil + 2 + 2\left\lceil \log_\alpha \left(\frac{\alpha+1}{\alpha}\right)\right\rceil + 1 \leq 5\left\lceil \log_\alpha \left(\frac{\alpha}{\alpha-1}\right)\right\rceil + 3,$$

as $\log_x \left(\frac{x}{x-1}\right) \geq \log_x \left(\frac{x+1}{x}\right)$ for every $x > 1$. In particular, this gives $h(2) \leq 8$ and $h\left(\frac{1+\sqrt{5}}{2}\right) \leq 13$. To obtain better bounds that are tight for $\alpha \geq \frac{1+\sqrt{5}}{2}$, we observe that not all types can appear simultaneously. To show this, we will use one last auxiliary result.

Let p and q be (not necessarily different) points lying on the same row R of $R(\alpha)$, each contained in an edge of P. Let L and L' be two lines containing p and q, respectively. If the slopes of L and L' are negative, then we call the part of the plane between L and L' below R a *slice of negative slope*; see part (a) of Figure 3 Analogously, *a slice of positive slope* is the part of the plane between L and L' above R if L and L' have positive slope.

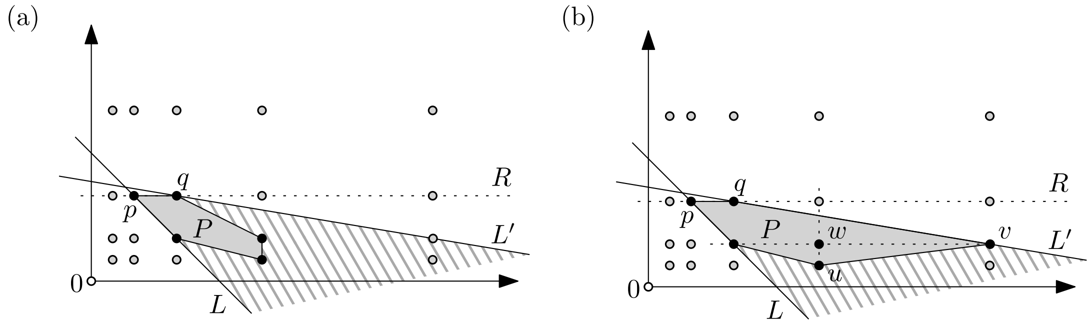

Figure 3: (a) An example of a slice of negative slope. The slice is denoted by dark gray stripes. (b) An illustration of the proof of Lemma 12 for negative slopes.

Lemma 12. *If the empty polygon P is contained in a slice of negative slope, then there is no non-vertical edge of P of type IV. Similarly, if P is contained in a slice of positive slope, then there is no edge of type I.*

Proof. By symmetry, it suffices to prove the statement for slices of negative slope. Suppose for contradiction that there is a non-vertical edge uv of type IV in a slice of negative slope determined by lines L and L' and points p and q as in the definition of a slice. Without loss of generality, we assume $x(u) < x(v)$.

Consider the point $w = (x(u), y(v))$ of $L(\alpha)$. Since uv is non-vertical, we have $w \notin \{u, v\}$. We claim that w is in the interior of P, contradicting the assumption that P is empty in $L(\alpha)$. Since uv is of type IV, the point u lies below the row containing w. However, since p is contained in an edge of P and P is in the slice, the boundary of P intersects this row to the left of w. Analogously, v is to the right of the column containing w and thus the boundary of P intersects this column above w. Then, however, w lies in the interior of P. $\square$

Finally, we can now finish the proof of Theorem 2.

Proof of Theorem 2. First, we observe that if all vertices of P lie on two columns of $L(\alpha)$, then P can have at most four vertices. So we assume that this is not the case. Let u be the leftmost vertex of P with the highest y-coordinate among all leftmost vertices of P. Let e_1 and e_2 be the edges of P incident to u. We denote the other edge of P incident to e_1 as e. We also use t_I, t_{II}, t_{III}, and t_{IV} to denote the number of edges of P of type I, II, III, and IV, respectively.

First, assume that e_1 is vertical. If e_2 is horizontal, then, since u is the top vertex of e_1 and P is not contained in two columns of $L(\alpha)$, the point $(\alpha \cdot x(u), y(u)/\alpha)$ of $L(\alpha)$ lies in the interior of P, which is impossible as P is empty in $L(\alpha)$.

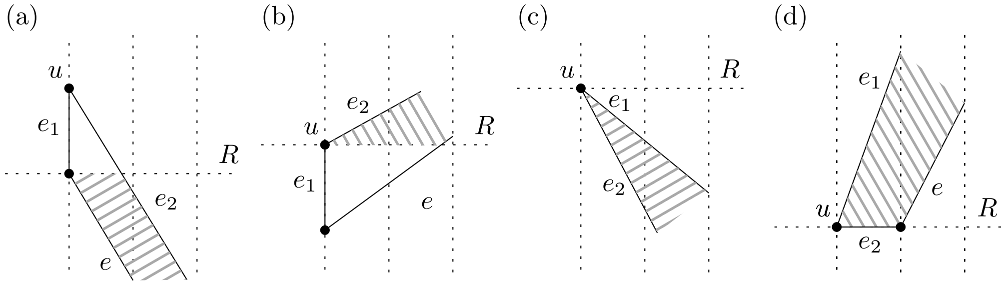

Figure 4: An illustration of the proof of Theorem 2.

If e_1 is vertical and the slope of e_2 is negative, then there is no edge of type II. Thus, the edge e intersects the row R of $L(\alpha)$ containing the other vertex of e_1 and $\overline{e}$ has negative slope. Then, the part of P below R is contained in the slice of negative slope determined by $\overline{e_2}$ and $\overline{e}$; see part (a) of Figure 4. By Lemma 12, there is no non-vertical edge of type IV in P. By Lemmas 7 and 8, the total number of edges of P is thus at most

$$t_I + t_{III} + 1 \leq \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 2 \left\lceil \log_\alpha \left(\frac{\alpha + 1}{\alpha} \right) \right\rceil + 2$$

for $\alpha \in (1, 2)$ and is by one smaller for $\alpha \geq 2$.

If e_1 is vertical and the slope of e_2 is positive, then, since P is empty, there is no edge of type III besides e_1 as otherwise the point $(\alpha \cdot x(u), y(u))$ of $L(\alpha)$ is in the interior of P. The edge e intersects the row R of $L(\alpha)$ containing u and $\overline{e}$ has positive slope. Thus, the part of P above R is contained in the slice of positive slope determined by $\overline{e_2}$ and $\overline{e}$; see part (b) of Figure 4. By Lemma 12, there is no edge of type I in P. By Lemma 10 and Corollary 11, the total number of edges of P is then at most

$$t_{II} + 1 + t_{IV} \leq 2 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 3.$$

In the rest of the proof, we can now assume that none of the edges e_1 and e_2 is vertical. We can label them so that the slope of e_1 is larger than the slope of e_2.

First, assume that the slope of e_1 is positive and the slope of e_2 is negative. Then, since the vertices of P do not lie on two columns of $L(\alpha)$, the point $(\alpha \cdot x(u), y(u))$ is contained in the interior of P, which is impossible as P is empty in $L(\alpha)$.

If the slopes of e_1 and e_2 are both non-positive, then there is no edge of type II besides the possibly horizontal edge e_1 as u is the leftmost vertex of P. By Lemma 12, there is also no non-vertical edge of type IV as P is contained in the slice of negative slopes determined by $\overline{e_1}$ and $\overline{e_2}$ or by $\overline{e}$ and $\overline{e_2}$ if e_1 is horizontal; see part (c) of Figure 4. Thus, by Lemmas 7 and 8, the number of edges of P is at most

$$t_I + 1 + t_{III} + 1 \leq \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 2 \left\lceil \log_\alpha \left(\frac{\alpha + 1}{\alpha} \right) \right\rceil + 3$$

for $\alpha \in (1, 2)$ and is by one smaller for $\alpha \geq 2$.

If the slopes of e_1 and e_2 are both non-negative, then there is no edge of type III besides the possibly horizontal edge e_2 (note that a vertical edge of type III would have u as its bottom vertex, which is impossible by the choice of u). Then, P is contained in the slice of positive slope determined by $\overline{e_1}$ and $\overline{e_2}$ or, if e_2 is horizontal, by $\overline{e_1}$ and $\overline{e}$; see part (d) of Figure 4. Lemma 12 then implies that there is also no edge of type I. We thus have at most

$$t_{II} + 1 + t_{IV} \leq 2 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 3$$

edges of P by Lemma 10 and Corollary 11.

Altogether, the upper bound on the number of edges of P is

$$\max \left\{ \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 2 \left\lceil \log_\alpha \left(\frac{\alpha + 1}{\alpha} \right) \right\rceil + 3, \; 2 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 3 \right\}$$

for $\alpha \in (1, 2)$ and the first term is smaller by 1 for $\alpha \geq 2$. This becomes 5 for $\alpha \geq 2$, $h(\alpha) \leq 7$ for $\alpha \geq [\frac{1+\sqrt{5}}{2}, 2)$, and at most $3 \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil + 3$ otherwise, since $\left\lceil \log_\alpha \left(\frac{\alpha+1}{\alpha} \right) \right\rceil \leq \left\lceil \log_\alpha \left(\frac{\alpha}{\alpha - 1} \right) \right\rceil$ for every $\alpha \in (1, \frac{1+\sqrt{5}}{2})$. □

4 Proof of Theorem 3

We prove the lower bounds on $h(\alpha)$ through the following three propositions.

Proposition 13. *For every $\alpha \geq 2$, we have $h(\alpha) \geq 5$.*

Proof. It is easy to check that $\text{conv}\{(1, \alpha^2), (\alpha, \alpha), (\alpha^2, 1), (\alpha^2, \alpha), (\alpha, \alpha^2)\}$ is an empty polygon in $L(\alpha)$ with 5 vertices for any α. □

Proposition 14. *For every $\alpha \in [\frac{1+\sqrt{5}}{2}, 2)$, we have $h(\alpha) \geq 7$.*

Proof. Let $k = k(\alpha)$ be a sufficiently large integer, and let

$$Q(\alpha) = \{(1, \alpha^k), (\alpha^{k-2}, \alpha^{k-1}), (\alpha^{k-1}, \alpha^{k-2}), (\alpha^k, 1), (\alpha^k, \alpha), (\alpha^{k-1}, \alpha^{k-1}), (\alpha, \alpha^k)\};$$

see Figure 5. We will show that $\mathrm{conv}(Q(\alpha))$ is an empty polygon in $L(\alpha)$ with 7 vertices.

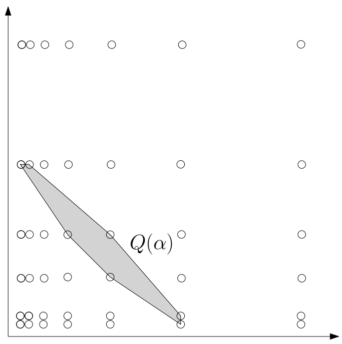

Figure 5: An illustration of the proof of Proposition 14.

First, we show that $Q(\alpha) \setminus \{(\alpha^{k-1}, \alpha^{k-1})\}$ is in convex position. For this, by symmetry, it is enough to check that the vector $(\alpha^{k-1}, \alpha^{k-2}) - (\alpha^k, 1)$ is to the left of $(1, \alpha^k) - (\alpha^k, 1)$. This is the case exactly if $\alpha^{k-1} - \alpha^k + \alpha^{k-2} - 1 < 0$. By rearranging we get $\alpha^{k-2}(\alpha+1-\alpha^2) < 1$, which holds for any k, since $\alpha+1-\alpha^2 \leq 0$ as $\alpha \geq (1+\sqrt{5})/2$.

Now, to show that the set $Q(\alpha)$ is in convex position, it is sufficient to check that $(\alpha^{k-1}, \alpha^{k-1}) - (\alpha^k, \alpha)$ is to the left of $(1, \alpha^k) - (\alpha^k, \alpha)$. This holds exactly if $\alpha^{k-1} - \alpha^k + \alpha^{k-1} - \alpha \geq 0$. By rearranging we get $2\alpha^{k-2}(2-\alpha) \geq 1$. Since $1 < \alpha < 2$, this holds if k is sufficiently large.

Thus, $\mathrm{conv}(Q(\alpha))$ has 7 vertices. To show that $\mathrm{conv}(Q(\alpha))$ is empty in $L(\alpha)$, we remark that points of the exponential lattice $L(\alpha)$ with at least one coordinate smaller than α^{k-1} are below the line through $(\alpha^{k-1}, \alpha^{k-2})$ and $(\alpha^{k-2}, \alpha^{k-1})$. Further, points with at least one coordinate larger than α^{k-1} are either above the line through $(1, \alpha^k)$ and (α, α^k) or to the right of the line through $(\alpha^k, 1)$ and (α^k, α). $\square$

Proposition 15. *For every $\alpha > 1$, we have $h(\alpha) \geq \left\lfloor \sqrt{\frac{1}{\alpha-1}} \right\rfloor$.*

Proof. For a positive integer k, let $P(k) = \{(\alpha^i, \alpha^{k-i}) : 1 \leq i \leq k\}$. Since $P(k)$ is contained in the hyperbola $h = \{(x,y) \in \mathbb{R}^2 \colon x, y > 0, xy = \alpha^k\}$, the points of $P(k)$ are in convex position, and $\mathrm{conv}(P(k))$ has k vertices. We will show that if $k \leq \sqrt{\frac{1}{\alpha-1}}$, then $\mathrm{conv}(P(k))$ is empty.

For points (x, y) of $L(\alpha)$ above h, we have $xy \geq \alpha^{k+1}$. Further, points (x, y) of $L(\alpha)$ with $xy \geq \alpha^{k+2}$ are separated from h by the hyperbola $h' = \{(x, y) \in$

$\mathbb{R}^2 \colon x, y > 0, xy = \alpha^{k+1}\}$. Thus, it is sufficient to check that h' is above the line ℓ connecting $(1, \alpha^k)$ with $(\alpha^k, 1)$. The closest point of h' to ℓ is $(\alpha^{(k+1)/2}, \alpha^{(k+1)/2})$, thus it is sufficient to check that this point is above ℓ. This holds if $2\alpha^{(k+1)/2} - \alpha^k - 1 \geq 0$ and we show that this inequality is satisfied for $k \leq \sqrt{\frac{1}{\alpha-1}}$.

Let $\alpha = 1 + s^2$ with some $s \in (0, 1)$. In this notation, $k \leq 1/s$ and we need to prove that $2(1 + s^2)^{(k+1)/2} \geq (1 + s^2)^k + 1$. Since $(1 + s^2)^{(k+1)/2} \geq 1 + s^2\frac{k+1}{2}$ by the Bernoulli inequality, and $(1 + s^2)^k \leq e^{s^2 k}$, it is sufficient to prove the stronger inequality $2(1 + s^2\frac{k+1}{2}) \geq e^{s^2 k} + 1$. The worst case, when $k = 1/s$, is equivalent to $1 + s + s^2 \geq e^s$, which holds for $s \in (0, 1)$ as can be seen by the Taylor expansion of e^s. □

5 Proof of Proposition 5

Let us denote $\mathcal{F} = \{F_n \colon n \in \mathbb{N}_0\}^2$. Suppose for contradiction that there is a positive integer k such that $h(\mathcal{F}) \leq k$. We will show that the points (F_{i+2}, F_i) with odd $i \in \{1, \ldots, 2k + 1\}$ are vertices of an empty convex polygon P in $\mathcal{F}$, contradicting the assumption $h(\mathcal{F}) \leq k$.

First, we show that the points (F_{i+2}, F_i) with odd $i \in \{1, \ldots, 2k + 1\}$ are in convex position. to show that, it suffices to show that the slopes of lines determined by three consecutive such points are decreasing. That is, we want to prove

$$\frac{F_i - F_{i-2}}{F_{i+2} - F_i} > \frac{F_{i+2} - F_i}{F_{i+4} - F_{i+2}}$$

for every odd $i \in \{1, \ldots, 2k - 3\}$. Since $F_k = F_{k-1} + F_{k-2}$ for every $k \geq 2$, this inequality can be rewritten as

$$\frac{F_{i-1}}{F_{i+1}} > \frac{F_{i+1}}{F_{i+3}}.$$

Thus, we want to show that $F_{i-1} \cdot F_{i+3} > F_{i+1}^2$. Again, using the Fibonacci recurrence, we can rewrite this expression as $F_{i-1}(3F_i + 2F_{i-1}) > F_i^2 + 2F_{i-1} \cdot F_i + F_{i-1}^2$, which can be simplified to $F_{i-1}(F_i + F_{i-1}) > F_i^2$ and further to $F_{i-1} \cdot F_{i+1} > F_i^2$. Using the Binet formula $F_k = \frac{\varphi^{k+1} - \psi^{k+1}}{\sqrt{5}}$, we see that this inequality is equivalent with

$$(\varphi^i - \psi^i)(\varphi^{i+2} - \psi^{i+2}) > (\varphi^{i+1} - \psi^{i+1})^2.$$

This can be expanded and rearranged to

$$2\varphi^{i+1} \cdot \psi^{i+1} > \varphi^{i+2} \cdot \psi^i + \varphi^i \cdot \psi^{i+2}.$$

Since i is odd, ψ^{i+1} is positive, and by diving both sides by $\varphi^{i+1} \cdot \psi^{i+1}$, we obtain

$$2 > \frac{\varphi}{\psi} + \frac{\psi}{\varphi} = -3,$$

as $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. Thus, the points are indeed in convex position.

To show that the polygon P is empty in $\mathcal{F}$, consider the line $L = \{(x, y) \in \mathbb{R}^2 \colon y = x/\varphi^2\}$. Any point (F_{i+2}, F_i) with odd i lies below L because

$$\frac{F_{i+2}}{\varphi^2} = \frac{1}{\varphi^2} \cdot \frac{\varphi^{i+3} - \psi^{i+3}}{\sqrt{5}} > \frac{\varphi^{i+1} - \psi^{i+1}}{\sqrt{5}} = F_i$$

since $\varphi^2 > \psi^2$ and $i+3$, $i+1$ are both even implying $\psi^{i+3}, \psi^{i+1} > 0$. Analogously, all points (F_{i+2}, F_i) with even i lie above L. For any i, every point (F_j, F_i) with $j \leq i+1$ lies above L, because $F_i \geq F_{j-1} > F_j/\varphi^2$. Each point (F_{i+2}, F_i) with $i > n$ lies at vertical distance less than $1/2$ from L as

$$\frac{F_{i+2}}{\varphi^2} = \frac{1}{\varphi^2} \cdot \frac{\varphi^{i+3} - \psi^{i+3}}{\sqrt{5}} = \frac{\varphi^{i+1} - \psi^{i+1}}{\sqrt{5}} + \frac{\varphi^2\psi^{i+1} - \psi^{i+3}}{\varphi^2\sqrt{5}} \leq F_i + \frac{\varphi^2\psi^2 - \psi^4}{\sqrt{5}}$$
$$< F_i + \frac{1}{2}.$$

Any point (F_{i+2}, F_j) with $j \leq i-1$ lies below L at vertical distance at least $1/2$ since the distance is either at least $F_i - F_j \geq 1$ if i is odd or it is at least $F_i - F_j - \frac{1}{2} \geq \frac{1}{2}$ if i is even. Thus, all points from $\mathcal{F} \setminus P$ are either above L or lie at vertical distance at least $1/2$ from L. It follows that P is empty convex polygon in $\mathcal{F}$ and $h(\mathcal{F}) \geq k+1$, a contradiction.

6 Proof of Theorem 6

Let $\alpha, \beta > 1$ be two real numbers. We prove that $h(L(\alpha, \beta))$ is finite if and only if $\log_\alpha(\beta)$ is a rational number.

6.1 Finite upper bound

First, assume that $\log_\alpha(\beta) \in \mathbb{Q}$. We will use Theorem 2 to show that the number $h(L(\alpha, \beta))$ is finite. Since $\log_\alpha(\beta) \in \mathbb{Q}$ and $\alpha, \beta > 1$, there are positive integers p and q such that $\beta = \alpha^{p/q}$. Suppose for contradiction that there is an empty polygon P in $L(\alpha, \beta)$ with at least $pq \cdot h(\alpha^p) + 1$ vertices. Note that this number of vertices is finite by Theorem 2. For $k \in \{0, \ldots, q-1\}$, we call a row of $L(\alpha, \beta)$ *congruent to* k if it is of the form $\{\alpha^n \colon n \in \mathbb{N}_0\} \times \beta^m$ for some integer m congruent to k modulo q. Analogously, a column of $L(\alpha, \beta)$ is *congruent to* $\ell \in \{0, \ldots, p-1\}$ if it is of the form $\alpha^m \times \{\beta^n : n \in \mathbb{N}_0\}$ for some m congruent to ℓ modulo p.

Now, since P contains at least $pq \cdot h(\alpha^p) + 1$ vertices, the pigeonhole principle implies that there are integers $k \in \{0, \ldots, q-1\}$ and $\ell \in \{0, \ldots, p-1\}$ such that at least $h(\alpha^p) + 1$ vertices of P that all lie in rows congruent to k and in columns congruent to ℓ. Let P' be the convex polygon that is spanned by these vertices. We claim that the polygon P' is not empty in $L(\alpha, \beta)$. Since $P' \subseteq P$, we get that P is also not empty in $L(\alpha, \beta)$, which contradicts our assumption about P.

To show that P' is not empty in $L(\alpha, \beta)$, consider the subset L of $L(\alpha, \beta)$ that contains only points of $L(\alpha, \beta)$ that lie in rows congruent to k and in columns congruent to ℓ. Clearly, vertices of P' lie in L and L is an affine image of $L(\alpha^p)$, which is scaled by the factors α^ℓ and $\beta^k = \alpha^{kp/q}$ in the x- and y-direction, respectively. Since affine mappings preserve incidences and P' has at least $h(\alpha^p) + 1$ vertices, it follows that P' is not empty in L. Since $L \subseteq L(\alpha, \beta)$, P' is not empty in $L(\alpha, \beta)$ either.

6.2 Finite lower bound

Let $\log_\alpha(\beta) \in \mathbb{Q}$ and $\beta = \alpha^{p/q}$ for some relative prime positive integers p and q. Observe that in this case $L(\alpha, \beta) \subset L(\alpha^{1/q})$. Thus, if an empty polygon in $L(\alpha^{1/q})$ is a subset of $L(\alpha, \beta)$, then it is an empty polygon in $L(\alpha, \beta)$.

Let $k = \left\lfloor \sqrt{1/(\alpha^{1/q} - 1)} \right\rfloor$ and consider the set $P = \{(\alpha^{i/q}, \alpha^{(k-i)/q}) : 1 \leq i \leq k\}$. It is an empty polygon in $L(\alpha^{1/q})$, as it is shown in the proof of Proposition 15. Since its subset $P' = \{(\alpha^{i/q}, \alpha^{(k-i)/q}) : 1 \leq i \leq k \text{ with } q|i \text{ and } p|k-i\}$ is a subset of $L(\alpha, \beta)$ and an empty polygon in $L(\alpha^{1/q})$, it is an empty polygon in $L(\alpha, \beta)$ with $\lfloor k/pq \rfloor$ vertices.

6.3 Infinite lower bound

Now, assume that $\log_\alpha(\beta) \notin \mathbb{Q}$. We will find a subset of $L(\alpha, \beta)$ forming empty convex polygon in $L(\alpha, \beta)$ with arbitrarily many vertices. To do so, we use a theory of continued fractions, so we first introduce some definitions and notation.

6.3.1 Continued fractions

Here, we recall mostly basic facts about so-called continued fractions, which we use in the proof. Most of the results that we state can be found, for example, in the book by Khinchin [14].

For a positive real number r, the *(simple) continued fraction of* r is an expression of the form

$$r = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

where $a_0 \in \mathbb{N}_0$ and $a_1, a_2, \ldots$ are positive integers. The simple continued fraction of r can be written in a compact notation as

$$[a_0; a_1, a_2, a_3, \ldots].$$

For every $n \in \mathbb{N}_0$, if we denote $\frac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$ and set $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, $q_0 = 1$, then the numbers p_n and q_n satisfy the recurrence

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2} \tag{1}$$

for each $n \in \mathbb{N}$. Observe that if r is irrational, then its continued fraction has infinitely many coefficients. Also, it follows from (1) that $\frac{p_n}{q_n} < r$ for n even and $\frac{p_n}{q_n} > r$ for n odd.

For example, if $r = \log_2(3)$, we get the continued fraction $[1; 1, 1, 2, 2, 3, 1, 5, 2, 23, \ldots]$ and the sequence $\left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}_0} = \left(\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{8}{5}, \frac{19}{12}, \frac{65}{41}, \frac{84}{53}, \frac{485}{306}, \ldots\right)$. For $r = \frac{1+\sqrt{5}}{2}$, we have $[1; 1, 1, 1, \ldots]$ and $\left(\frac{p_n}{q_n}\right)_{n \in \mathbb{N}_0} = \left(\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \ldots\right)$.

We will call the fractions $\frac{p_n}{q_n}$ the *convergents* of r. A *semi-convergent* of r is a number $\frac{p_{n-1}+ip_n}{q_{n-1}+iq_n}$ where $i \in \{0, 1, \ldots, a_{n+1}\}$. Note that each convergent of r is also a semi-convergent of r. The names are motivated by the use of convergents and semi-convergents as rational approximations of an irrational number r.

A rational number $\frac{p}{q}$ is a *best approximation* of an irrational number r, if any fraction $\frac{p'}{q'} \neq \frac{p}{q}$ with $q' < q$ satisfies

$$\left| q' \left(r - \frac{p'}{q'} \right) \right| > \left| q \left(r - \frac{p}{q} \right) \right|.$$

A rational number $\frac{p}{q}$ is a *best lower approximation of* r if

$$q' \left(r - \frac{p'}{q'} \right) > q \left(r - \frac{p}{q} \right) \geq 0$$

for all rational numbers $\frac{p'}{q'}$ with $\frac{p'}{q'} \leq r$, $\frac{p}{q} \neq \frac{p'}{q'}$, and $0 < q' \leq q$. Similarly, $\frac{p}{q}$ is a *best upper approximation of* r if

$$q' \left(r - \frac{p'}{q'} \right) < q \left(r - \frac{p}{q} \right) \leq 0$$

for all rational numbers $\frac{p'}{q'}$ with $\frac{p'}{q'} \geq r$, $\frac{p}{q} \neq \frac{p'}{q'}$, and $0 < q' \leq q$.

It is a well known fact that convergents are best approximations of r [14]. The following lemma about best lower and upper best approximations is a recent result of Hančl and Turek [10].

Lemma 16 ([10])**.** *Let r be a real number with $r = [a_0; a_1, a_2, \dots]$ and let $\frac{p_n}{q_n}$ be the nth convergent of r for each $n \in \mathbb{N}_0$. Then, the following three statements hold.*

1. *The set of best lower approximations of r consists of semi-convergents $\frac{p_{n-1}+ip_n}{q_{n-1}+iq_n}$ of r with n odd and $0 \leq i < a_{n+1}$.*
2. *The set of best upper approximations of r consists of semi-convergents $\frac{p_{n-1}+ip_n}{q_{n-1}+iq_n}$ of r with n even and $0 \leq i < a_{n+1}$, except for the pair $(n,i) = (0,0)$.*

Finally, a real number r is *restricted* if there is a positive integer M such that all the partial denominators a_i from the continued fraction of r are at most M. The restricted numbers are exactly those numbers r that are badly approximable by rationals [14], that is, there is a constant $c > 0$ such that for every $\frac{p}{q} \in \mathbb{Q}$ we have $\left|r - \frac{p}{q}\right| > \frac{c}{q^2}$.

We divide the rest of the proof of Theorem 6 into two cases, depending on whether $\log_\alpha(\beta)$ is restricted or not.

6.3.2 Unrestricted case

First, we assume that $\log_\alpha(\beta)$ is not restricted. Let $[a_0; a_1, a_2, a_3, \dots]$ be the continued fraction of $\log_\alpha(\beta)$ with $\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$ for every $n \in \mathbb{N}_0$. Then, for every positive integer m, there is a positive integer $n(m)$ such that $a_{n(m)+1} \geq m$. We use this assumption to construct, for every positive integer m, a convex polygon with at least m vertices from $L(\alpha,\beta)$ that is empty in $L(\alpha,\beta)$.

For a given m, consider the integer $n(m)$ and let W be the set of points

$$w_i = \left(\alpha^{p_{n(m)-1}+ip_{n(m)}}, \beta^{q_{n(m)-1}+iq_{n(m)}}\right)$$

where $i \in \{0, 1, \dots, a_{n(m)+1}\}$. That is, we consider points where the exponents form semi-convergents $\frac{p_{n(m)-1}+ip_{n(m)}}{q_{n(m)-1}+iq_{n(m)}}$ to $\log_\alpha(\beta)$. We abbreviate $p_{n,i} = p_{n(m)-1} + ip_{n(m)}$ and $q_{n,i} = q_{n(m)-1} + iq_{n(m)}$. Observe that $|W| \geq m$. We will show that W is the vertex set of an empty convex polygon in $L(\alpha,\beta)$. To do so, we assume without loss of generality that $n(m)$ is even so that $\frac{\beta^{q_{n(m)}}}{\alpha^{p_{n(m)}}} > 1$. The other case when $n(m)$ is odd is analogous.

First, we show that W is in convex position. In fact, we prove that all triples $(w_{i_1}, w_{i_2}, w_{i_3})$ with $i_1 < i_2 < i_3$ are oriented counterclockwise. It suffices to show this for every triple (w_i, w_{i+1}, w_{i+2}). To do so, we need to prove the inequality

$$\frac{y(w_{i+2}) - y(w_{i+1})}{x(w_{i+2}) - x(w_{i+1})} = \frac{\beta^{q_{n,i+2}} - \beta^{q_{n,i+1}}}{\alpha^{p_{n,i+2}} - \alpha^{p_{n,i+1}}} > \frac{\beta^{q_{n,i+1}} - \beta^{q_{n,i}}}{\alpha^{p_{n,i+1}} - \alpha^{p_{n,i}}} = \frac{y(w_{i+1}) - y(w_i)}{x(w_{i+1}) - x(w_i)}.$$

After dividing by $\frac{\beta^{q_{n(m)-1}}}{\alpha^{p_{n(m)-1}}}$, this can be written as

$$\frac{\beta^{(i+2)q_{n(m)}} - \beta^{(i+1)q_{n(m)}}}{\alpha^{(i+2)p_{n(m)}} - \alpha^{(i+1)p_{n(m)}}} > \frac{\beta^{(i+1)q_{n(m)}} - \beta^{iq^{n(m)}}}{\alpha^{(i+1)p_{n(m)}} - \alpha^{ip_{n(m)}}}.$$

If divide both sides by $\frac{\beta^{(i+1)q_{n(m)}} - \beta^{iq_{n(m)}}}{\alpha^{(i+1)p_{n(m)}} - \alpha^{ip_{n(m)}}}$, then the above inequality becomes

$$\frac{\beta^{q_{n(m)}}}{\alpha^{p_{n(m)}}} > 1.$$

This is true as $n(m)$ is even.

It remains to prove that the polygon Q with the vertex set W is empty in $L(\alpha, \beta)$. Suppose for contradiction that there is a point (α^p, β^q) of $L(\alpha, \beta)$ lying in the interior of Q. Let i be the minimum positive integer from $\{1, \ldots, a_{n(m)+1}\}$ such that $q < q_{n,i}$. Such an i exists as (α^p, β^q) is in the interior of Q. We then have $q_{n,i-1} < q < q_{n,i}$. Since (α^p, β^q) is in the interior of Q and W lies below the line $x = y$, we have $\frac{p}{q} > \log_\alpha(\beta)$. So it is enough to prove that (α^p, β^q) does not lie above the line $\overline{w_{i-1}w_i}$.

We have $p_{n,i} - \log_\alpha(\beta)q_{n,i} < p_{n,i-1} - \log_\alpha(\beta)q_{n,i-1}$ as $\frac{p_{n,i}}{q_{n,i}}$ is a best upper approximation of $\log_\alpha(\beta)$ and $q_{n,i-1} < q_{n,i}$. This implies $\frac{\beta^{q_{n,i-1}}}{\alpha^{p_{n,i-1}}} < \frac{\beta^{q_{n,i}}}{\alpha^{p_{n,i}}}$, or equivalently that w_i lies above the line determined by w_{i-1} and the origin.

Now if (α^p, β^q) lies above the line $\overline{w_{i-1}w_i}$, then it also lies above the line determined by w_{i-1} and the origin. Thus, $\frac{\beta^{q_{n,i-1}}}{\alpha^{p_{n,i-1}}} < \frac{\beta^q}{\alpha^p}$, implying

$$p - \log_\alpha(\beta)q < p_{n,i-1} - \log_\alpha(\beta)q_{n,i-1},$$

which means that $\frac{p}{q}$ is a better upper approximation of $\log_\alpha(\beta)$ than $\frac{p_{n,i-1}}{q_{n,i-1}}$. Thus, there exists a best upper approximation $\frac{p^*}{q^*}$ of $\log_\alpha(\beta)$ with $q_{n,i-1} < q^* < q_{n,i}$. This contradicts part (c) of Lemma 16 as $\frac{p^*}{q^*}$ is not a semi-convergent of $\log_\alpha(\beta)$.

6.3.3 Restricted case

Now, assume that the number $\log_\alpha(\beta)$ is restricted. Let $[a_0; a_1, a_2, a_3, \ldots]$ be the continued fraction of $\log_\alpha(\beta)$ with $\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$ for every $n \in \mathbb{N}_0$. Let $M = M(\alpha, \beta)$ be a number satisfying

$$a_n \leq M \tag{2}$$

for every $n \in \mathbb{N}_0$ and let $c = c(\alpha, \beta) > 0$ be a constant such that

$$\left|\log_\alpha(\beta) - \frac{p}{q}\right| > \frac{c}{q^2} \tag{3}$$

holds for every $\frac{p}{q} \in \mathbb{Q}$. Recall that $\frac{\alpha^{p_n}}{\beta^{q_n}} < 1$ for even n and $\frac{\alpha^{p_n}}{\beta^{q_n}} > 1$ for odd n. Note also that the sequence $\left(\frac{\alpha^{p_n}}{\beta^{q_n}}\right)_{n\in\mathbb{N}_0}$ converges to 1 as $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}_0}$ converges to $\log_\alpha(\beta)$. Moreover, the terms of $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}_0}$ with odd indices form a decreasing subsequence and the terms with even indices determine an increasing subsequence.

Let $n_0 = n_0(\alpha, \beta)$ be a sufficiently large positive integer and let V be the set of points $v_n = (\alpha^{p_n}, \beta^{q_n})$ for every odd $n \geq n_0$. Note that V is a subset of $L(\alpha, \beta)$.

We first show that V is in convex position. In fact, we prove a stronger claim by showing that the orientation of every triple $(v_{n_1}, v_{n_2}, v_{n_3})$ with $n_1 < n_2 < n_3$ is counterclockwise. It suffices to show this for every triple (v_{n-4}, v_{n-2}, v_n). To do so, we prove that the slopes of the lines determined by consecutive points of V are increasing, that is,

$$\frac{y(v_n) - y(v_{n-2})}{x(v_n) - x(v_{n-2})} = \frac{\beta^{q_n} - \beta^{q_{n-2}}}{\alpha^{p_n} - \alpha^{p_{n-2}}} > \frac{\beta^{q_{n-2}} - \beta^{q_{n-4}}}{\alpha^{p_{n-2}} - \alpha^{p_{n-4}}} = \frac{y(v_{n-2}) - y(v_{n-4})}{x(v_{n-2}) - x(v_{n-4})}$$

for every even $n \geq n_0$. By dividing both sides of the inequality with $\frac{\beta^{q_{n-2}}}{\alpha^{p_{n-2}}}$, we rewrite this expression as

$$\frac{\beta^{q_n - q_{n-2}} - 1}{\alpha^{p_n - p_{n-2}} - 1} > \frac{1 - \beta^{q_{n-4} - q_{n-2}}}{1 - \alpha^{p_{n-4} - p_{n-2}}}.$$

Using (1), this is the same as

$$\frac{\beta^{a_n q_{n-1}} - 1}{\alpha^{a_n p_{n-1}} - 1} > \frac{1 - \beta^{-a_{n-2} q_{n-3}}}{1 - \alpha^{-a_{n-2} p_{n-3}}}.$$

The above inequality can be rewritten as

$$(\beta^{a_n q_{n-1}} - 1)(1 - \alpha^{-a_{n-2} p_{n-3}}) > (\alpha^{a_n p_{n-1}} - 1)(1 - \beta^{-a_{n-2} q_{n-3}}),$$

where $\beta^{q_{n-1}} > \alpha^{p_{n-1}} > 1$ and $1 > \alpha^{-p_{n-3}} > \beta^{-q_{n-3}} > 0$ as $n-1$ and $n-3$ are even. Therefore, if the above inequality holds for $a_n = 1 = a_{n-2}$, then it holds for any a_n and a_{n-1} as both numbers are always at least 1. Thus, it suffices to show

$$(\beta^{q_{n-1}} - 1)(1 - \alpha^{-p_{n-3}}) > (\alpha^{p_{n-1}} - 1)(1 - \beta^{-q_{n-3}}). \tag{4}$$

We prove this using the following simple auxiliary lemma.

Lemma 17. *Consider the function* $f\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *given by* $f(x,y) = (x-1)(1-1/y)$. *Let* $x, y, x', y' > 1$ *be real numbers such that* $1 - \frac{1}{y} - \frac{x}{x'} > 0$. *Then,* $f(x',y) > f(x,y')$.

Proof. We have

$$f(x',y) - f(x,y') = (x'-1)\left(1-\frac{1}{y}\right) - (x-1)\left(1-\frac{1}{y'}\right)$$
$$= x' - \frac{x'-1}{y} - x + \frac{x-1}{y'} > x' - \frac{x'}{y} - x = x'\left(1 - \frac{1}{y} - \frac{x}{x'}\right) > 0,$$

where the last inequality follows from $1 - \frac{1}{y} - \frac{x}{x'} > 0$. □

Now, by choosing $x = \alpha^{p_{n-1}}$, $x' = \beta^{q_{n-1}}$, $y = \alpha^{p_{n-3}}$, and $y' = \beta^{q_{n-3}}$, the inequality (4) becomes $f(x',y) > f(x,y')$. In order to prove it, we just need to verify the assumptions of Lemma 17. We clearly have $x, x', y, y' > 1$. It now suffices to show $1 - \frac{1}{y} - \frac{x}{x'} > 0$. By (3), we obtain that $q_{n-1}\log_\alpha(\beta) - p_{n-1} \geq c/q_{n-1}$, thus

$$\frac{x}{x'} = \frac{\alpha^{p_{n-1}}}{\beta^{q_{n-1}}} \leq \alpha^{-c/q_{n-1}}.$$

Now, to bound q_{n-1} in terms of p_{n-3}, equation (1) gives

$$q_{n-1} = a_{n-1}q_{n-2} + q_{n-3} \leq (M+1)q_{n-2} = (M+1)(a_{n-2}q_{n-3} + q_{n-4})$$
$$\leq (M+1)^2 q_{n-3} \leq 2\log_\beta(\alpha)(M+1)^2 p_{n-3},$$

where we used (2) and $q_{n-4} \leq q_{n-3} \leq q_{n-2}$, $q_{n-3} \leq 2\log_\beta(\alpha)p_{n-3}$ for n large enough. It follows that $q_{n-1} \leq M'p_{n-3}$ for a suitable constant $M' = M'(\alpha,\beta) > 0$. Thus,

$$1 - \frac{1}{y} - \frac{x}{x'} \geq 1 - \alpha^{-p_{n-3}} - \alpha^{-c/q_{n-1}} \geq 1 - \alpha^{-p_{n-3}} - \alpha^{-c/(M'p_{n-3})},$$

which is at least

$$\frac{c\ln\alpha}{2M'p_{n-3}} - \frac{1}{\alpha^{p_{n-3}}}$$

as $1 - c\ln\alpha/(2M'p_{n-3}) \geq e^{-2c\ln\alpha/(2M'p_{n-3})} = \alpha^{-c/(M'p_{n-3})}$ if $0 < c\ln\alpha/(2M'p_{n-3}) < 1/2$. The last expression is positive if $n \geq n_0$ and n_0 is sufficiently so that p_{n-3} is large enough.

It remains to show that the convex polygon P with the vertex set V is empty in $L(\alpha,\beta)$. We proceed analogously as in the unrestricted case. Suppose for contradiction that there is a point (α^p, β^q) of $L(\alpha,\beta)$ lying in the interior of P. Then, let $v_n = (\alpha^{p_n}, \beta^{q_n})$ be the lowest vertex of P that has (α^p, β^q) below. Such a vertex v_n exists, as V contains points with arbitrarily large y-coordinate. By the choice of v_n, we obtain $q_{n-2} < q < q_n$. Since (α^p, β^q) is in the interior of P and V lies below the line $x = y$, we have $\frac{p}{q} > \log_\alpha(\beta) > \frac{p_{n-1}}{q_{n-1}}$. Moreover, since all triples from V are oriented counterclockwise, the point (α^p, β^q) lies above the line $\overline{v_{n-2}v_n}$.

Let

$$w_i = (\alpha^{p_{n-2}+ip_{n-1}}, \beta^{q_{n-2}+iq_{n-1}})$$

where $i \in \{0, 1, \ldots, a_n\}$ similarly as in the proof of the unrestricted case. There, it was shown that all the triples w_{i-1}, w_i, w_{i+1} are oriented counterclockwise, thus all the points w_i with $i \in \{1, \ldots, a_n - 1\}$ lie below the line $\overline{v_{n-2}v_n}$. Thus, if (α^p, β^q) lies above the segment connecting v_{n-2} and v_n, then there is an i such that (α^p, β^q) lies above the segment connecting w_{i-1} and w_i. As in the last two paragraphs of the proof of the unrestricted case, the position of (α^p, β^q) implies the inequality $p - \log_\alpha(\beta)q < p_{n,i-1} - \log_\alpha(\beta)q_{n,i-1}$, and the contradiction follows from part (c) of Lemma 16, as there can be no best upper approximation of $\log_\alpha(\beta)$ which is not a semi-convergent of $\log_\alpha(\beta)$.

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