Extremal graphs without long paths and large cliques

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Abstract Let $\mathcal{F}$ be a family of graphs. A graph is called $\mathcal{F}$-free if it does not contain any member of $\mathcal{F}$ as a subgraph. The Turán number of $\mathcal{F}$ is the maximum number of edges in an n-vertex $\mathcal{F}$-free graph and is denoted by $\mathrm{ex}\left(n, \mathcal{F}\right)$. The same maximum under the additional condition that the graphs are connected is $\mathrm{ex}_{\mathrm{conn}}\left(n, \mathcal{F}\right)$. Let P_k be the path on k vertices, K_m be the clique on m vertices. We determine $\mathrm{ex}(n, \{P_k, K_m\})$ if $k > 2m-1$ and $\mathrm{ex}_{\mathrm{conn}}(n, \{P_k, K_m\})$ if $k > m$ for sufficiently large n.

1 Introduction

In the present paper, all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph, the vertex and edge sets of G are denoted by $V(G)$ and $E(G)$, the numbers of vertices and edges in G by $v(G)$ and $e(G)$, respectively. We denote the degree of a vertex v in G by $d_G(v)$, the neighborhood of the vertex set V in G by $N_G(V)$. Let U_1, U_2 be vertex sets, denote by $e_G(U_1, U_2)$ the number of edges between U_1 and U_2 in G. We write $d(v)$ instead of $d_G(v)$, $N(V)$ instead of $N_G(V)$ and $e(U_1, U_2)$ instead of $e_G(U_1, U_2)$ if the underlying graph G is unambiguous. Denote by I_n the independent set on n vertices, by $G[B]$ the subgraph of G induced by the vertex set B and by $\overline{G}$ the edge complement of the graph G. A component of an undirected graph is an induced subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the rest of the graph. A vertex v in a graph G is called a cut vertex

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if deleting v from G increases the number of components of G.

Let $\mathcal{F}$ be a family of graphs. A graph is called $\mathcal{F}$-free if it does not contain any member of $\mathcal{F}$ as a subgraph. The Turán number of $\mathcal{F}$ is the maximum number of edges in an n-vertex $\mathcal{F}$-free graph and is denoted by $\mathrm{ex}\Big(n, \mathcal{F}\Big)$. Denote by $\mathrm{EX}(n, \mathcal{F})$ the set of $\mathcal{F}$-free graphs on n vertices with $\mathrm{ex}(n, \mathcal{F})$ edges and call a graph in $\mathrm{EX}(n, \mathcal{F})$ an extremal graph for $\mathcal{F}$. Let P_k be the path on k vertices, K_m be the clique on m vertices.

Vertices u and v are connected if there exists a path from u to v. Two disjoint vertex sets U and W are completely joined in G if $uw \in E(G)$ for all $u \in U$ and $w \in W$. Denote by $G_1 \bigotimes G_2$ the graph obtained from $G_1 \cup G_2$, the vertex disjoint union of graphs G_1 and G_2, and completely join $V(G_1)$ and $V(G_2)$. The Turán graph $T(n, p)$ is a complete multipartite graph formed by partitioning a set of n vertices into p subsets, with sizes as equal as possible, and connecting two vertices by an edge if and only if they belong to different subsets. Denote its size by $t(n, p)$.

In 1941, Turán [5] determined the Turán number for p-clique.

Theorem 1 (**Turán** [5]). *The number of edges in an n-vertex K_p-free ($p \geq 3$) graph is at most $t(n, p-1)$. Furthermore, $T(n, p-1)$ is the unique extremal graph.*

In 1959, Erdős and Gallai [2] determined the Turán number for P_k.

Theorem 2 (**Erdős and Gallai** [2]). *Let G be an n-vertex graph with more than $\frac{(k-2)n}{2}$ edges, $k \geq 2$. Then G contains a copy of P_k.*

Faudree and Schelp [3] and independently Kopylov [4] improved this result determining $\mathrm{ex}(n, P_k)$ for every $n > k > 0$ as well as the corresponding extremal graphs.

Theorem 3 (**Faudree and Schelp** [3] **and independently Kopylov** [4]). *Let $n \equiv r \pmod{k-1}$, $0 \leq r \leq k-1$, $k \geq 2$. Then*

$$\mathrm{ex}(n, P_k) = \frac{1}{2}(k-2)n - \frac{1}{2}r(k-1-r).$$

Faudree and Schelp also described the extremal graphs which are either

(a) vertex disjoint union of m ($n = m(k-1) + r$) complete graphs K_{k-1} and a K_r or

(b) k is even and $r = \frac{k}{2}$ or $\frac{k}{2} - 1$ then another extremal graph can be obtained by taking a vertex disjoint union if t copies of K_{k-1} $(0 \le t \le m)$ and a copy of $K_{\frac{k}{2}-1} \otimes \overline{K}_{n-(t+\frac{1}{2})(k-1)+\frac{1}{2}}$.

Kopylov [4] considered the extremal problem for P_k taken over all connected graphs. He determined the extremal values, 30 years later Balister, Győri, Lehel and Schelp found all the extremal graphs, too.

Theorem 4 (**Balister, Győri, Lehel and Schelp** [1])**.** *Let G be a connected graph on n vertices containing no path on k vertices, $n > k \ge 4$. Then $e(G)$ is bounded above by the maximum of $\binom{k-2}{2} + (n-k+2)$ and $\binom{\lceil \frac{k}{2} \rceil}{2} + \lfloor \frac{k-2}{2} \rfloor (n - \lceil \frac{k}{2} \rceil)$. If equality occurs then G is either $\left(K_{k-3} \cup \overline{K}_{n-k+2}\right) \otimes K_1$ or $\left(K_{k-2\lfloor \frac{k}{2} \rfloor+1} \cup \overline{K}_{n-\lceil \frac{k}{2} \rceil}\right) \otimes K_{\lfloor \frac{k}{2} \rfloor-1}$.*

Now let us turn to the problem of the present paper: try to determine $\mathrm{ex}(n, \{P_k, K_m\})$. If $k \le m$ then this is simply $\mathrm{ex}(n, P_k)$, therefore we can suppose $k > m$ for the rest of the paper.

Construction 1: Suppose $\lfloor \frac{k}{2} \rfloor - 1 \le n$. $G_1 = T(\lfloor \frac{k}{2} \rfloor - 1, m-2) \otimes \overline{K_{n-\lfloor \frac{k}{2} \rfloor+1}}$.
The number of the edges in this graph is

$$f_n(m,k) = \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right)\left(n - \left\lfloor \frac{k}{2} \right\rfloor + 1 \right) + t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2 \right).$$

Construction 2: Suppose $k-1|n$, let $G_2 = \frac{n}{k-1} T(k-1, m-1)$ denote the graph obtained by taking $\frac{n}{k-1}$ vertex-disjoint copies of $T(k-1, m-1)$.
Clearly, the graphs $T(\lfloor \frac{k}{2} \rfloor - 1, m-2) \otimes \overline{K_{n-\lfloor \frac{k}{2} \rfloor+1}}$ and $\frac{n}{k-1} T(k-1, m-1)$ are $\{K_m, P_k\}$-free.

We believe that for large n (number of vertices) one of these constructions maximize the number of edges under the assumption that the graph contains neither a K_m nor a P_k. More precisely we guess that either Construction 1 gives the largest number of edges or the maximum is between $\frac{n}{k-1} t(k-1, m-1) - c(k,m)$ and $\frac{n}{k-1} t(k-1, m-1)$ where $c(k,m)$ does not depend on n.

But we are able to prove only the following two theorems.

Theorem 5. *Let G be a connected n-vertex $\{K_m, P_k\}$-free graph $m < k$. For sufficiently large n $(> N(k))$,*

$$\mathrm{ex}_{\mathrm{conn}}(n,\{K_m,P_k\}) = \left(\left\lfloor \frac{k}{2}\right\rfloor - 1\right)n + t\left(\left\lfloor \frac{k}{2}\right\rfloor - 1, m-2\right) - \left(\left\lfloor \frac{k}{2}\right\rfloor - 1\right)^2,$$

that is, Construction 1 is an extremal graph.

Theorem 6. *Let G be an n-vertex $\{K_m, P_k\}$-free graph, $2m-1<k$. For sufficiently large n ($> N'(k)$),*

$$\mathrm{ex}(n,\{K_m,P_k\}) = \left(\left\lfloor \frac{k}{2}\right\rfloor - 1\right)n + t\left(\left\lfloor \frac{k}{2}\right\rfloor - 1, m-2\right) - \left(\left\lfloor \frac{k}{2}\right\rfloor - 1\right)^2,$$

that is, Construction 1 is an extremal graph.

In Section 5 we will pose our conjecture for the remaining cases.

2 Lemmas for Theorem 5

Lemma 7. *Let G be an n-vertex graph with vertex partition $V(G) = X \cup Y$, where $|X| \geq 1$. Suppose that for each vertex $v \in X$, there is no path in G starting from v on $j+1$ ($j \geq 2$) vertices and each vertex $u \in Y$ is connected by a path to at least one vertex in X. Then $j = 2$ implies $e(G) \leq \frac{n+|Y|-|N(Y)|}{2}$ and $j \geq 3$ implies $e(G) \leq n(j-\frac{3}{2})$.*

Proof of Lemma 7. When $j=2$, all paths starting from $v \in X$ contain at most 2 vertices. Therefore each vertex $u \in Y$ is adjacent to at least one vertex in X.

(a) $G[Y]$ is empty. Otherwise, we find a path starting from $v \in X$ contain at least 3 vertices. Hence, each vertex in Y is adjacent to at least one vertex in X and $N(Y) \subset X$. We have the following easy steps.

(b) For each $u \in N(Y)$, $e(u, X-u) = 0$. Suppose the contrary, there exists an edge $uv \in E(G)$, where $v \in X-u$. Then we immediately find a path vuw, where $w \in Y$ and $uw \in E(G)$, which contradicts the fact that there is no path in G starting from $v \in X$ on 3 vertices.

(c) $e(G[X-N(Y)]) \leq \frac{n-|Y|-|N(Y)|}{2}$. Clearly, $|X-N(Y)| = n-|Y|-|N(Y)|$, and for each vertex $v \in X-N(Y)$, $N(v) \in X-N(Y)$. Since there is no path in G starting from

$v \in X$ on 3 vertices, we have $e(G[X - N(Y)]) \leq \frac{n-|Y|-|N(Y)|}{2}$.

Therefore, when $j = 2$ we have $e(G) \leq |Y| + \frac{n-|Y|-|N(Y)|}{2} = \frac{n+|Y|-|N(Y)|}{2}$.

When $j \geq 3$, we apply induction on $|Y|$ and separate the proof into two subcases. The base case is $|Y| = 0$. Clearly, in this case the theorem of Erdős and Gallai can be applied for X: $e(G) \leq |X|\frac{(j-1)}{2} \leq |X|(j - \frac{3}{2})$, which holds when $j \geq 2$.

Case 1. There exists a vertex $y \in Y$ which is not a cut vertex, such that $d(y) \leq j - 2$. By the inductional hypothesis we know that $e(G \setminus y) \leq (n-1)(j - \frac{3}{2})$. Thus, $e(G) \leq (n-1)(j - \frac{3}{2}) + j - 2 \leq n(j - \frac{3}{2})$.

Case 2. $d(y) \geq j - 1$ for each vertex $y \in Y$ which is not a cut vertex.

First, we show the following claim.

Claim 1. $P_{2j-2} \not\subseteq G[Y]$.

Proof of Claim 1. Suppose otherwise, let $P = (v_1, v_2, \ldots, v_{2j-2})$, $v_i \in Y (1 \leq i \leq 2j-2)$ be a path on $2j-2$ vertices in $G[Y]$. Since each vertex in Y is connected by a path to at least one vertex in X, there exists a $u \in X$, such that a path leads from u to P. We obtain the shortest path in this way when u is adjacent to v_{j-1} or v_j. In both cases, we find a path starting from u on $j+1$ vertices: either $(u, v_{j-1}, v_j, v_{j+1}, \ldots, v_{2j-2})$ or $(u, v_j, v_{j-1}, v_{j-2}, \ldots, v_1)$, a contradiction. □

It is also easy to check the following claim.

Claim 2. *If there exists a path P_{2j-1} in G, then X contains the middle vertex of this path only.*

Now we will finish the proof distinguishing two cases.

If G contains no P_{2j-1}, by Theorem 2, we can conclude $e(G) \leq \frac{2j-3}{2}n = n(j - \frac{3}{2})$. Otherwise, let $P = (v_1, v_2, \ldots, v_{2j-1}) \subseteq G$. By Claim 2, we know that $v_i \in Y$ $(1 \leq i \leq 2j - 1, i \neq j)$ and $v_j \in X$. Clearly, $N(v_1) \subseteq \{v_2, v_3, \ldots, v_j\}$ and $N(v_{2j-1}) \subseteq \{v_j, v_{j+1}, \ldots, v_{2j-2}\}$, otherwise, we find a copy of P_{j+1} starting from v_j, a contradiction. It is easy to see that v_1 and v_{2j-1} are not cut vertices, therefore, $d_{v_1} \geq j-1$ and $d(v_{2j-1}) \geq j-1$. If $N(v_1)$ contains a vertex not in $\{v_2, v_3, \ldots, v_j\}$ then there is a path of length $j+1$ starting at $v_j \in X$ and this is a contradiction. Hence, we have $N(v_1) = \{v_2, v_3, \ldots, v_j\}$ and $N(v_{2j-1}) = \{v_j, v_{j+1}, \ldots, v_{2j-2}\}$. Now, we will show that $\bigcup\limits_{i=1}^{j-1} N(v_i) = \{v_1, v_2, \ldots, v_j\}$ holds. If not, then there exits v_i $(2 \le i \le j-1)$, such that $v_i u \in E(G)$ and $u \notin \{v_1, v_2, \ldots, v_j\}$. In this case, we can easily find the path $(v_j, v_{j-1}, \ldots, v_{i+1}, v_1, v_2, \ldots, v_i, u)$ on $j+1$ vertices which is starting at v_j, a contradiction. Hence, $\bigcup\limits_{i=1}^{j-1} N(v_i) = \{v_1, v_2, \ldots, v_j\}$ and the number of edges which are incident to the vertex set $\{v_1, v_2, \ldots, v_{j-1}\}$ is at most $\binom{j}{2}$. After deleting the vertex set $\{v_1, v_2, \ldots, v_{j-1}\}$ from G and apply the induction hypothesis, we get $e(G) \le (n-j+1)(j-\frac{3}{2}) + \binom{j}{2} \le n(j-\frac{3}{2})$ when $j \ge 3$. The proof is completed. □

Lemma 8. *Let G be a connected graph on n vertices with a path P on $k-1$ vertices but no path on k vertices. Let $u \in V(G \setminus P)$ be a vertex adjacent to $s \ge 1$ vertices of P and assume a longest path Q in $G \setminus P$ starting at u has j $(j \ge 0)$ vertices. Then, $s + j \le \left\lfloor \frac{k}{2} \right\rfloor$.*

Proof of Lemma 8. Let $P = (v_1, v_2, \ldots, v_{k-1})$ and $v_{i_1}, v_{i_2}, \ldots, v_{i_s}$ $(1 < i_1 < i_2 < \cdots < i_s < k-1)$ be the vertices in P adjacent to u. Since $P_k \nsubseteq G$, the paths $(v_{k-1}, v_{k-2}, \ldots, v_{i_1}, u) \cup Q$ and $(v_1, v_2, \ldots, v_{i_s}, u) \cup Q$ contain at most $k-1$ vertices. Hence, $k-1-i_1+1+j \le k-1$ and $i_s + j \le k-1$, that is $i_1 \ge j+1$ and $i_s \le k-1-j$. These imply $i_s - i_1 \le k-2j-2$. It is easy to check that u cannot be joined to two consecutive vertices in P, that is $i_{j+1} \ne i_j + 1$ $(1 \le j \le s-1)$. Otherwise, G contains a path on k vertices: $(v_1, v_2, \ldots, v_{i_j}, u, v_{i_{j+1}}, \ldots, v_{k-1})$. Therefore, $i_{j+1} \ge i_j + 2$, and hence $2s-2 \le i_s - i_1$. We obtain $2s-2 \le i_s - i_1 \le k-2j-2$. Therefore, $s + j \le \left\lfloor \frac{k}{2} \right\rfloor$. □

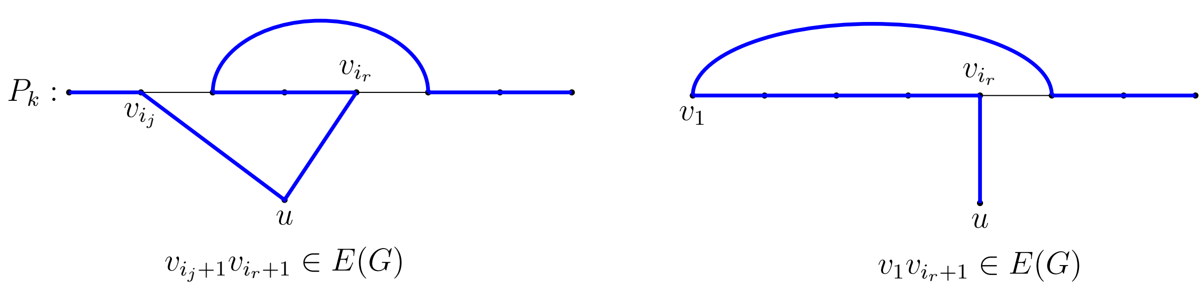

Figure 1: Cases for which G contains a P_{k+1}.

Lemma 9. *Let G be a connected graph on k vertices with no Hamiltonian path, but with a path $P = (v_1, v_2, \ldots, v_{k-1})$ on $k-1$ vertices. Suppose the vertex $u \in V(G \setminus P)$ has degree s, that is $N_P(u) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\}$, $i_1 < i_2 < \cdots < i_s$. Then*
(i) $s \leq \left\lfloor \frac{k}{2} \right\rfloor - 1$;
(ii) there are no edges of the form $v_{i_j+1}v_{i_r+1}$, $v_{i_j-1}v_{i_r-1}$, $v_1 v_{i_r+1}$ or $v_{i_r-1}v_{k-1}$, $(1 \leq j < r \leq s)$.

Proof of Lemma 9. (i) is obtained from Lemma 8, since $j = 1$ in this case.

Now we shall show that (ii) : $v_{i_j+1}v_{i_r+1}$, $v_{i_j-1}v_{i_r-1}$, $v_1 v_{i_r+1}$, $v_{i_r-1}v_{k-1} \notin E(G)$, $1 \leq j < r \leq s$. Without loss of generality, suppose otherwise there exists an edge of the form $v_{i_j+1}v_{i_r+1}$ $(1 \leq j < r \leq s)$ or $v_1 v_{i_r+1}$ then G would have a path on k vertices (see Figure 1), contradicting the assumption that G contains no Hamiltonian path. □

Lemma 10. *Let G be a graph on $\ell + 2$ vertices and $V(G) = M \cup v_1 \cup v_2$, $|M| = \ell$, such that there is no I_s in $G[M]$ and no I_{s+1} in G. Then $e(G) \geq \binom{\ell}{2} - t(\ell, s-1) + 1$.*

Proof of Lemma 10. By Turán's theorem, the maximum number of edges in an ℓ-vertex K_s-free graph is $t(\ell, s-1)$. Since there is no $I_s = \overline{K}_s$ in $G[M]$ and $v(G[M]) = \ell$, $e(G[M]) \geq \binom{\ell}{2} - t(\ell, s-1)$. Therefore, if $e(G) \leq \binom{\ell}{2} - t(\ell, s-1)$, then either (i) $e(G[M]) \leq \binom{\ell}{2} - t(\ell, s-1) - 1$ or (ii) $e(G[M]) = \binom{\ell}{2} - t(\ell, s-1)$, $v_1 v_2 \notin E(G)$ and no edges exist between M and $\{v_1, v_2\}$. For the case (i), clearly there exists an I_s in $G[M]$, so this is impossible. By Turán's theorem, we know that $T(\ell, s-1)$ is the unique extremal graph for K_s. Therefore, for the case (ii), when $G[M]$ is I_s-free and $e(G[M]) = \binom{\ell}{2} - t(\ell, s-1)$, we get that $G[M]$ is $\overline{T(\ell, s-1)}$. Choosing one vertex from each class of $\overline{T(\ell, s-1)}$, v_1 and v_2 form an I_{s+1} in G. This contradiction shows $e(G) \geq \binom{\ell}{2} - t(\ell, s-1) + 1$. □

Definition 1. *Let $T_r(n, \ell) = I_{n-r} \bigotimes T(r, \ell - 1)$ and let $t_r(n, \ell)$ denote the size of the graph $T_r(n, \ell)$.*

Observe that $f_n(m, k) = t_{\lfloor \frac{k}{2} \rfloor - 1}(n, m-1)$.

Lemma 11. *Let G be an n-vertex $\{K_m, P_k\}$-free connected graph containing a P_{k-1}. Suppose there exists a vertex $u \in V(G \setminus P_{k-1})$, such that u is adjacent to $\left\lfloor \frac{k}{2} \right\rfloor - 1$ vertices in $V(P_{k-1})$.*

Then, $e(G[V(P_{k-1})]) \leq t_{\lfloor \frac{k}{2} \rfloor -1}(k-1, m-1) = f_{k-1}(m,k)$.

Proof of Lemma 11. Let $P_{k-1} := (v_1, v_2, \ldots, v_{k-1})$, we separate the proof into two subcases.

Case 1. k *is even.*

Clearly u cannot be adjacent to v_1, v_{k-1} or consecutive vertices in P_{k-1}. Hence, when $|V(P_{k-1}) \cap N(u)| = \frac{k}{2} - 1$, the only possibility is $V(P_{k-1}) \cap N(u) = \{v_2, v_4, \ldots, v_{k-2}\}$. Since G is K_m-free, the graph induced by $V(P_{k-1}) \cap N(u)$, $G[V(P_{k-1}) \cap N(u)]$ is a K_{m-1}-free graph on $\frac{k}{2} - 1$ vertices. By Lemma 9, $G[v_1, v_3, v_5, \ldots, v_{k-1}]$ forms an empty graph on $\frac{k}{2}$ vertices.

Therefore, the number of edges of $G[V(P_{k-1})]$ is bounded by:

$$
\begin{aligned}
e\Big(G[V(P_{k-1})]\Big) &\leq \binom{k-1}{2} - \binom{\lfloor \frac{k}{2} \rfloor}{2} - \left\{ \binom{\lfloor \frac{k}{2} \rfloor - 1}{2} - t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2 \right) \right\} \\
&= \frac{(k-1)(k-2)}{2} - \frac{\frac{k}{2}\frac{k-2}{2}}{2} - \frac{\frac{k-2}{2}\frac{k-4}{2}}{2} + t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2 \right) \\
&= \frac{k}{2}\frac{k-2}{2} + t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2 \right) \\
&= t_{\lfloor \frac{k}{2} \rfloor -1}(k-1, m-1).
\end{aligned}
$$

Case 2. k *is odd.*

Since $e\Big(u, V(P_{k-1})\Big) = \left\lfloor \frac{k}{2} \right\rfloor - 1$ and u cannot be adjacent to v_1, v_{k-1} or consecutive vertices in P_{k-1}, then either

(*i*) $V(P_{k-1}) \cap N(u) = \{v_2, v_4, \ldots, v_{k-3}\}$ (or its symmetric version), or

(*ii*) $V(P_{k-1}) \cap N(u) = \{v_2, \ldots, v_{2f}, v_{2f+3}, v_{2f+5}, \ldots, v_{k-2}\}$, $(1 \leq f \leq \frac{k-5}{2})$ holds.

Suppose first (*i*): $V(P_{k-1}) \cap N(u) = \{v_2, v_4, \ldots, v_{k-3}\}$. Since G is K_m-free, $G[V(P_{k-1}) \cap N(u)]$ is a $(\left\lfloor \frac{k}{2} \right\rfloor - 1)$-vertex K_{m-1}-free graph while $G\Big[\{v_2, v_4, \ldots, v_{k-3}\} \cup \{v_{k-2}, v_{k-1}\}\Big]$ forms a K_m-free graph. By Lemma 10, at least $\binom{\lfloor \frac{k}{2} \rfloor - 1}{2} - t(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2) + 1$ more edges are missing in $G[V(P_{k-1})]$. Also, by Lemma 9, we see that $\{v_1, v_3, \ldots, v_{k-2}\}$ forms an independent set on $\left\lfloor \frac{k}{2} \right\rfloor$ vertices and edge set $\{v_1v_{k-1}, v_3v_{k-1}, \ldots, v_{k-4}v_{k-1}\} \nsubseteq E(G)$. Hence, $\binom{\lfloor \frac{k}{2} \rfloor}{2} + \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right)$ more edges are missing.

Case (ii) is very similar. $G[V(P_{k-1}) \cap N(u)]$ is a $\left(\left\lfloor \frac{k}{2} \right\rfloor - 1\right)$-vertex K_{m-1}-free graph and $G\Big[\{v_2, \dots, v_{2f}, v_{2f+3}, v_{2f+5}, \dots, v_{k-2}\} \cup \{v_{2f+1}, v_{2f+2}\}\Big]$ forms a K_m-free graph. By Lemma 10, at least $\binom{\lfloor \frac{k}{2} \rfloor - 1}{2} - t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2\right) + 1$ edges are missing. Also, by Lemma 9, we see that $\{v_1, v_3, \dots, v_{2f+1}, v_{2f+4}, v_{2f+6}, \dots, v_{k-1}\}$ forms an independent set on $\left\lfloor \frac{k}{2} \right\rfloor$ vertices, edges $\{v_1v_{2f+1},\ v_3v_{2f+1},\ \dots,\ v_{2f-1}v_{2f+1}\} \notin E(G)$ and edges $\{v_{2f+2}v_{2f+4},\ v_{2f+2}v_{2f+6},\ \dots,\ v_{2f+2}v_{k-1}\} \notin E(G)$. Hence, $\binom{\lfloor \frac{k}{2} \rfloor}{2} + \left(\left\lfloor \frac{k}{2} \right\rfloor - 1\right)$ more edges are missing, again.

Therefore, we get the following upper bound on $e(G[V(P_{k-1})])$ in both cases (i) and (ii),

$$
\begin{aligned}
e\Big(G[V(P_{k-1})]\Big) &\le \binom{k-1}{2} - \binom{\lfloor \frac{k}{2} \rfloor}{2} - \left(\left\lfloor \frac{k}{2} \right\rfloor - 1\right) - \left\{\binom{\lfloor \frac{k}{2} \rfloor - 1}{2} - t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2\right) + 1\right\} \\
&= \frac{(k-1)(k-2)}{2} - \frac{\frac{k-1}{2}\frac{k-3}{2}}{2} - \frac{k-3}{2} - \frac{\frac{k-3}{2}\frac{k-5}{2}}{2} + t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2\right) - 1 \\
&= \left(\frac{k-1}{2}\right)^2 + t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2\right) - 1 \\
&= t_{\lfloor \frac{k}{2} \rfloor - 1}(k-1, m-1).
\end{aligned}
$$

□

Recall that when G is an n-vertex $\{K_m, P_k\}$-free graph and containing a path on $k-1$ vertices, then $e\Big(u, V(P_{k-1})\Big) \le \left\lfloor \frac{k}{2} \right\rfloor - 1$ holds for each $u \in V(G \setminus P_{k-1})$ (Lemma 9 (i)). Partition the vertices in $G \setminus P_{k-1}$ into mutually disjoint sets in two different ways.

(i) Let $u \in A_i$, if $e\Big(u, V(P_{k-1})\Big) = i$, $0 \le i \le \left\lfloor \frac{k}{2} \right\rfloor - 1$. Hence, $\bigcup_{i=0}^{\lfloor \frac{k}{2} \rfloor - 1} A_i = V(G \setminus P_{k-1})$ and $A_i \cap A_j = \emptyset$ $(i \ne j)$. Clearly, $A_{\lfloor \frac{k}{2} \rfloor - 1}$ is an independent set, otherwise, we find a copy of P_k in G.

(ii) Re-arrange the vertices in $G \setminus P_{k-1}$ progressively: when $u \in A_{i_1}$ and u is also connected by a path to vertices belonging to $A_{i_2}, A_{i_3}, \dots,$ and A_{i_r}, then we put u into set $B_{\max\{i_1, i_2, \dots, i_r\}}$. Clearly, B_i and B_j $(i \ne j)$ are disjoint and mutually independent. Since G is connected, we get $B_0 = \emptyset$. Observe that $A_i \cap B_j = \emptyset$ if $i > j$.

Lemma 12. *Let G be an n-vertex $\{K_m, P_k\}$-free connected graph with a path on $k-1$*

vertices. Let A_i and B_i are the sets of vertices defined above. Then,

$$e(G) - e\left(G[V(P_{k-1})]\right) \leq \begin{cases} (\left\lfloor \frac{k}{2} \right\rfloor - 1)(n-k+1), & A_{\lfloor \frac{k}{2} \rfloor -1} \neq \emptyset, \\ (\left\lfloor \frac{k}{2} \right\rfloor - \frac{3}{2})(n-k+1), & A_{\lfloor \frac{k}{2} \rfloor -1} = \emptyset. \end{cases}$$

Proof of Lemma 12. Since B_i and B_j $(1 \leq i \neq j \leq \left\lfloor \frac{k}{2} \right\rfloor - 1)$ are mutually independent, obviously,

$$e(G) - e\left(G[V(P_{k-1})]\right) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor -1} e(G[B_i]) + \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor -1} e(B_i, V(P_{k-1})). \tag{1}$$

Let $X_i = A_i \cap B_i$ and $Y_i = B_i - X_i$. By the definitions of A_i and B_i, each vertex in Y_i is connected (by a path) to at least one vertex in X_i and if $X_i = \emptyset$ then $Y_i = \emptyset$. By Lemma 8 we know that the number of vertices of the longest path in $G[B_i]$ starting at a vertex $v \in B_i \cap A_i$ is at most $\left\lfloor \frac{k}{2} \right\rfloor - i$.

Hence if $i = \left\lfloor \frac{k}{2} \right\rfloor - 1$ then there is no edge in $G \setminus P_{k-1}$ starting from the vertices of $A_{\lfloor \frac{k}{2} \rfloor -i}$, therefore $B_{\lfloor \frac{k}{2} \rfloor -1} = A_{\lfloor \frac{k}{2} \rfloor -1}$ and it contains no edge.

In the case of $B_{\lfloor \frac{k}{2} \rfloor -2}$ the maximum length of a path starting from $X_{\lfloor \frac{k}{2} \rfloor -2} = A_{\lfloor \frac{k}{2} \rfloor -2} \cap B_{\lfloor \frac{k}{2} \rfloor -2}$ is 2. Applying Lemma 7 for $X_{\lfloor \frac{k}{2} \rfloor -2}$ and $Y_{\lfloor \frac{k}{2} \rfloor -2}$ we obtain

$$e(G[B_{\lfloor \frac{k}{2} \rfloor -2}]) \leq \frac{1}{2}|X_{\lfloor \frac{k}{2} \rfloor -2}| + |Y_{\lfloor \frac{k}{2} \rfloor -2}| = \frac{1}{2}|A_{\lfloor \frac{k}{2} \rfloor -2} \cap B_{\lfloor \frac{k}{2} \rfloor -2}| + |B_{\lfloor \frac{k}{2} \rfloor -2} - A_{\lfloor \frac{k}{2} \rfloor -2}|$$

and

$$e(G[B_{\lfloor \frac{k}{2} \rfloor -2}]) \leq \frac{1}{2}|A_{\lfloor \frac{k}{2} \rfloor -2}| + |B_{\lfloor \frac{k}{2} \rfloor -2} - A_{\lfloor \frac{k}{2} \rfloor -2}|. \tag{2}$$

Similarly, again by Lemma 7, we have

$$e(G[B_{\lfloor \frac{k}{2} \rfloor -i}]) \leq |B_{\lfloor \frac{k}{2} \rfloor -i}| \left(i - \frac{3}{2}\right) \ (i \geq 3). \tag{3}$$

Adding up the inequalities (2) and (3) for $i = 2, 3, \dots$ we obtain an upper bound on the first term of the right hand side of (1).

$$\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor -1} e(G[B_j]) = \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor -2} e(G[B_j]) \leq \frac{1}{2}|A_{\lfloor \frac{k}{2} \rfloor -2}| + \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor -3} |B_{\lfloor \frac{k}{2} \rfloor -2} \cap A_\ell| + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor -3} |B_j| \left(\left\lfloor \frac{k}{2} \right\rfloor - j - \frac{3}{2} \right)$$

$$\leq \frac{1}{2}|A_{\lfloor \frac{k}{2} \rfloor -2}| + \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor -3} |B_{\lfloor \frac{k}{2} \rfloor -2} \cap A_\ell| + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor -3} \sum_{\ell=0}^{j} |B_j \cap A_\ell| \left(\left\lfloor \frac{k}{2} \right\rfloor - j - \frac{3}{2} \right).$$

Here j can be replaced in the last factor by ℓ. The coefficient 1 in the middle term can also be replaced by $\left(\left\lfloor \frac{k}{2} \right\rfloor - \ell - \frac{3}{2}\right)$ if $\ell \leq \left\lfloor \frac{k}{2} \right\rfloor - 3$. This leads to the following upper bound:

$$\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor -1} e(G[B_j]) \leq \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor -2} |A_\ell| \left(\left\lfloor \frac{k}{2} \right\rfloor - \ell - \frac{3}{2} \right). \tag{4}$$

It is easy to determine the last term in (1).

$$\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor -1} e(B_i, V(P_{k-1})) = \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor -1} e(A_\ell, V(P_{k-1})) = \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor -1} \ell |A_\ell|. \tag{5}$$

Take the sum of (4) and (5).

$$e(G) - e\left(G[P_{k-1}]\right) \leq \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right) |A_{\lfloor \frac{k}{2} \rfloor -1}| + \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor -2} |A_\ell| \left(\left\lfloor \frac{k}{2} \right\rfloor - \frac{3}{2} \right). \tag{6}$$

In the case when $A_{\lfloor \frac{k}{2} \rfloor -1} \neq \emptyset$ holds the following simpler upper bound will be sufficient:

$$\sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor -1} |A_\ell| \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right).$$

Here $\sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor -1} |A_\ell| = n - k + 1$ therefore

$$e(G) - e\left(G[P_{k-1}]\right) \leq (n - k + 1) \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right)$$

holds, proving the first row of our statement. However if $A_{\lfloor \frac{k}{2} \rfloor -1} = \emptyset$ then the stronger upper estimate

$$(n - k + 1) \left(\left\lfloor \frac{k}{2} \right\rfloor - \frac{3}{2} \right)$$

is obtained from (6). □

3 Proof of Theorem 5

Proof of Theorem 5. Several cases will be distinguished.

Case 1. G contains no path of length $k-1$: $P_{k-1} \nsubseteq G$.

Case 1.1. k is even.

By Theorem 2, we can see that $e(G) \leq \frac{(k-3)n}{2}$. Clearly, there exists an integer n_1, such that $n \geq n_1$, $\frac{(k-3)n}{2} \leq \frac{(k-2)n}{2} - (\left\lfloor \frac{k}{2} \right\rfloor - 1)^2 + t(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2) = f_n(m,k)$ holds for $n > n_1$.

Case 1.2 k is odd.

Case 1.2.1. G contains no path of length $k-2$: $P_{k-2} \nsubseteq G$.

By Theorem 2, we can see that $e(G) \leq \frac{(k-4)n}{2}$. Similarly, there exists an integer n_2, such that $n \geq n_2$, $\frac{(k-4)n}{2} \leq \frac{(k-3)n}{2} - \left(\left\lfloor \frac{k}{2} \right\rfloor - 1\right)^2 + t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2\right) = f_n(m,k)$ holds for $n > n_2$.

Case 1.2.2 G contains a path of length $k-2$: $P_{k-2} = (v_1, v_2, \ldots, v_{k-2})$.

Case 1.2.2.1 There exists a vertex $u \in V(G \setminus P_{k-2})$ having $\lfloor \frac{k-1}{2} \rfloor - 1$ neighbors in P_{k-2}.

By Lemma 11 we have $e\Big(G[V(P_{k-2})]\Big) \leq t_{\lfloor \frac{k-1}{2} \rfloor - 1}(k-2, m-1)$.

By Lemma 12 we see that $e\Big(G[V(G \setminus P_{k-2})]\Big) + e\Big(V(P_{k-2}), V(G \setminus P_{k-2})\Big) \leq \Big(\lfloor \frac{k-1}{2} \rfloor - 1\Big)(n-k+2)$.

Therefore, we get

$$\begin{aligned} e(G) &= e\Big(G[V(P_{k-2})]\Big) + e\Big(G[V(G \setminus P_{k-2})]\Big) + e\Big(V(P_{k-2}), V(G \setminus P_{k-2})\Big) \\ &\leq \frac{k-1}{2}\frac{k-3}{2} + t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2\right) + \frac{k-3}{2}(n-k+2) \\ &= \left(\left\lfloor \frac{k}{2} \right\rfloor - 1\right)n + t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2\right) - \left(\left\lfloor \frac{k}{2} \right\rfloor - 1\right)^2 = f_n(m,k). \end{aligned}$$

In this subcase, we are done.

Case 1.2.2.2 For each vertex $u \in V(G \setminus P_{k-2})$, $e\Big(u, V(P_{k-2})\Big) \leq \left\lfloor \frac{k}{2} \right\rfloor - 2$ holds.

By Turán's theorem, we have $e\Big(G[V(P_{k-2})]\Big) \leq t(k-2, m-1)$.

By Lemma 12 we get

$$e\Big(G[V(G\setminus P_{k-2})]\Big)+e\Big(V(P_{k-2}),V(G\setminus P_{k-2})\Big)\le\left(\left\lfloor\frac{k-1}{2}\right\rfloor-\frac{3}{2}\right)(n-k+2).$$

Therefore, there exists an integer n_3, such that when $n\ge n_3$, the following result holds:

$$\begin{aligned}e(G)&=e\Big(G[V(P_{k-2})]\Big)+e\Big(G[V(G\setminus P_{k-2})]\Big)+e\Big(V(P_{k-2}),V(G\setminus P_{k-2})\Big)\\&\le t(k-2,m-1)+\frac{k-4}{2}(n-k+2)\\&\le\left(\frac{k-3}{2}\right)n-\left(\frac{k-3}{2}\right)^2+t\left(\frac{k-3}{2},m-2\right)\\&=f_n(m,k).\end{aligned}$$

Now we turn to the next case.

Case 2. G is connected and $P_{k-1}\subseteq G$.

Case 2.1. There exists a vertex $u\in G\setminus P_{k-1}$ such that u is adjacent to $\left\lfloor\frac{k}{2}\right\rfloor-1$ vertices of P_{k-1}.

By Lemma 11 we have $e\Big(G[V(P_{k-1})]\Big)\le t_{\lfloor\frac{k}{2}\rfloor-1}(k-1,m-1)$.

By Lemma 12 we can see that

$$e\Big(G[V(G\setminus P_{k-1})]\Big)+e\Big(V(P_{k-1}),V(G\setminus P_{k-1})\Big)\le\left(\left\lfloor\frac{k}{2}\right\rfloor-1\right)(n-k+1).$$

Therefore,

$$\begin{aligned}e(G)&=e\Big(G[V(P_{k-1})]\Big)+e\Big(G[V(G\setminus P_{k-1})]\Big)+e\Big(V(P_{k-1}),V(G\setminus P_{k-1})\Big)\\&\le t_{\lfloor\frac{k}{2}\rfloor-1}(k-1,m-1)+\left(\left\lfloor\frac{k}{2}\right\rfloor-1\right)(n-k+1)\\&=f_n(m,k).\end{aligned}$$

Case 2.2. Each vertex $u\in G\setminus P_{k-1}$ is adjacent to at most $\left\lfloor\frac{k}{2}\right\rfloor-2$ vertices of P_{k-1}.

By Turán's theorem, we see $e\Big(G[V(P_{k-1})]\Big)\le t(k-1,m-1)$.

By Lemma 12 we get

$$e\Big(G[V(G\setminus P_{k-1})]\Big)+e\Big(V(P_{k-1}),V(G\setminus P_{k-1})\Big)\leq\left(\left\lfloor\frac{k}{2}\right\rfloor-\frac{3}{2}\right)(n-k+1).$$

Therefore, there exists an integer n_4, such that when $n \geq n_4$, we get,

$$\begin{aligned}e(G) &= e\Big(G[V(P_{k-1})]\Big)+e\Big(G[V(G\setminus P_{k-1})]\Big)+e\Big(V(P_{k-1}),V(G\setminus P_{k-1})\Big)\\ &\leq t(k-1,m-1)+\left(\left\lfloor\frac{k}{2}\right\rfloor-\frac{3}{2}\right)(n-k+1)\\ &\leq t_{\lfloor\frac{k}{2}\rfloor-1}(k-1,m-1)+\left(\left\lfloor\frac{k}{2}\right\rfloor-1\right)(n-k+1)\\ &= f_n(m,k).\end{aligned}$$

Let $N(k) \geq n_1, n_2, n_3, n_4$. If G is connected and $n \geq N(k)$ then $e(G) \leq f_n(m,k)$ holds □

4 Proof of Theorem 6

First, we prove some lemmas.

Lemma 13. *If $n < k$ and $2m-1 < k$, then $t(n,m-1) < (\lfloor\frac{k}{2}\rfloor - 1)n$.*

Proof of Lemma 13. Since $t(n,m-1) \leq (\frac{n}{m-1})^2\binom{m-1}{2} = \frac{n^2(m-2)}{2(m-1)}$ it is sufficient to prove $\frac{n^2(m-2)}{2(m-1)} < \frac{k-3}{2}n$. Divide this inequality by n and prove the so obtained inequality for the largest possible value of n, namely $k-1$. But $\frac{(k-1)(m-2)}{(m-1)} < k-3$ is equivalent to our assumption $2m-1 < k$. □

It is also easy to check the following claim.

Lemma 14. *If $k \leq n$ and $2m-1 < k$, then*

$$t(k-2,m-1)+\left(\left\lfloor\frac{k-1}{2}\right\rfloor-\frac{3}{2}\right)(n-k+2)$$

$$\leq t(k-1,m-1)+\left(\left\lfloor\frac{k}{2}\right\rfloor-\frac{3}{2}\right)(n-k+1)<\left(\left\lfloor\frac{k}{2}\right\rfloor-1\right)n.$$

Proof of Lemma 14. In order to prove the first inequality observe that $T(k-2, m-1)$ is obtained from $T(k-1, m-1)$ by deleting a vertex in a class of size $\lceil \frac{k-1}{m-1} \rceil$. In other words, $k-1-\lceil \frac{k-1}{m-1} \rceil$ edges are deleted: $t(k-1, m-1) - t(k-2, m-1) = k-1-\lceil \frac{k-1}{m-1} \rceil$. We only need to see that this is at least $\lceil \frac{k-1}{2} \rceil - \frac{3}{2}$ that trivially holds for $m \geq 3$.

Start the proof of the second statement with the inequality $t(k-1, m-1) \leq (\frac{k-1}{m-1})^2 \binom{m-1}{2} = \frac{(k-1)^2(m-2)}{2(m-1)}$. Then the original inequality reduces to $\frac{(k-1)^2(m-2)}{2(m-1)} < \frac{n}{2} + (k-1)\left(\lfloor \frac{k}{2} \rfloor - \frac{3}{2}\right)$. Decrease the right hand side with the substitutions $k-1 < n$ and $\frac{k-1}{2} \leq \lfloor \frac{k}{2} \rfloor$. The so obtained inequality $\frac{(k-1)^2(m-2)}{2(m-1)} < \frac{k-1}{2} + (k-1)\frac{k-4}{2}$ is equivalent to $2m-1 < k$, again. □

Lemma 15. *There is an $\epsilon = \epsilon(k) > 0$ such that the upper estimate $(\left\lfloor \frac{k}{2} \right\rfloor - 1)n$ in both Lemmas 13 and 14 can be replaced by $\left(\left\lfloor \frac{k}{2} \right\rfloor - 1 - \epsilon\right)n$ if n is restricted to $n \leq N(k)$.*

Proof of Lemma 15. By Lemmas 13 and 14 we know that

$$\left\lfloor \frac{k}{2} \right\rfloor - 1 - \frac{t(n, m-1)}{n} > 0$$

and

$$\left\lfloor \frac{k}{2} \right\rfloor - 1 - \frac{t(k-1, m-1) + \left(\left\lfloor \frac{k}{2} \right\rfloor - \frac{3}{2}\right)(n-k+1)}{n} > 0$$

hold for every n. Of course the left hand sides can be arbitrarily small, but they have a positive minimum if $n \leq N(k)$ is supposed. Half of this minimum can be chosen as $\epsilon(k)$. □

Proof of Theorem 6. Let G_i $(1 \leq i \leq t)$ be the connected components of our graph G possessing the properties given in Theorem 6. ℓ_i denotes the number of vertices of G_i. We will give now different upper estimates on the number $e(G_i)$ of edges of the component G_i depending on its properties.

1. $\ell_i > N(k)$. Then Theorem 5 implies $e(G_i) \leq f_{\ell_i}(m, k)$.

2. $k \leq \ell_i \leq N(k)$.

2.1. G_i contains a path P_{k-1} of length $k-1$.

2.1.1. There is a vertex $u \in G_i - P_{k-1}$ which is adjacent to P_{k-1} with $\lfloor \frac{k}{2} \rfloor - 1$ edges. Then one can repeat the reasoning of Case 2.1 of the proof of Theorem 5 to obtain $e(G_i) \leq$

$f_{\ell_i}(m,k)$.

2.1.2. There is no such vertex. Then Lemma 12 shows that the number of edges not in the path is at most $\left(\lfloor \frac{k}{2} \rfloor - \frac{3}{2}\right)(\ell_i - k + 1)$. The graph induced by P_{k-1} can contain at most $t(k-1, m-1)$ edges. Altogether $e(G_i) \le t(k-1,m-1) + \left(\lfloor \frac{k}{2} \rfloor - \frac{3}{2}\right)(\ell_i - k + 1)$ which is at most $\left(\lfloor \frac{k}{2} \rfloor - 1 - \epsilon\right)\ell_i$ by Lemma 15 (improvement of Lemma 14).

2.2. G_i contains no path of length $k-1$.

2.2.1. k is even. Then Theorem 2 gives $e(G_i) \le \ell_i \frac{k-3}{2} < \ell_i \left(\lfloor \frac{k}{2} \rfloor - 1 - \epsilon\right)$.

2.2.2. k is odd.

2.2.2.1. G_i contains no path of length $k-2$. Then Theorem 2 gives $e(G_i) \le \ell_i \frac{k-4}{2} < \ell_i \left(\lfloor \frac{k}{2} \rfloor - 1 - \epsilon\right)$.

2.2.2.2. It contains a P_{k-2}.

2.2.2.2.1. There is a vertex $u \in G_i - P_{k-2}$ which is adjacent to P_{k-2} with $\lfloor \frac{k-1}{2} \rfloor - 1$ edges. Then one can repeat the reasoning of Case 1.2.2.1 of the proof of Theorem 5 to obtain $e(G_i) \le f_{\ell_i}(m,k)$.

2.2.2.2.2. There is no such vertex. Then Lemma 12 shows that the number of edges not in the path is at most $\left(\lfloor \frac{k-1}{2} \rfloor - \frac{3}{2}\right)(\ell_i - k + 2)$. The graph induced by P_{k-2} can contain at most $t(k-2, m-1)$ edges. Altogether $e(G_i) \le t(k-2,m-1) + \left(\lfloor \frac{k-1}{2} \rfloor - \frac{3}{2}\right)(\ell_i - k + 2)$ which is at most $\left(\lfloor \frac{k}{2} \rfloor - 1 - \epsilon\right)\ell_i$ by Lemma 15 (improvement of Lemma 14).

3. $\ell_i < k$. Then the trivial estimate $t(\ell_i, m-1)$ and Lemma 13 results in $e(G_i) \le \left(\lfloor \frac{k}{2} \rfloor - 1 - \epsilon\right)\ell_i$ by Lemma 15 (improvement of Lemma 13).

In each case we have the upper bound either

$$e(G_i) \le f_{\ell_i}(m,k) = \left(\left\lfloor \frac{k}{2} \right\rfloor - 1\right)\ell_i - \left(\left\lfloor \frac{k}{2} \right\rfloor - 1\right)^2 + t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2\right), \tag{7}$$

where the last two terms form an additive constant independent of ℓ_i, or

$$e(G_i) \le \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 - \epsilon\right)\ell_i. \tag{8}$$

Suppose first that there is at least one i satisfying (7). Then (7) and (8) lead to

$$e(G) = \sum_i e(G_i) \le \sum_i \ell_i \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right) - \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right)^2 + t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2 \right)$$

$$= \sum_i e(G_i) \le n \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right) - \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right)^2 + t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2 \right) = f_n(m,k) \qquad (9).$$

Otherwise (8) can be used for every i:

$$e(G) = \sum_i e(G_i) \le \sum_i \ell_i \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 - \epsilon \right) = n \left(\left\lfloor \frac{k}{2} \right\rfloor - 1 - \epsilon \right).$$

This is smaller than $f_n(m,k)$ if

$$\left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right)^2 - t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2 \right) < \epsilon n$$

and the statement of the theorem holds with

$$N'(k) = \frac{1}{\epsilon} \left(\left(\left\lfloor \frac{k}{2} \right\rfloor - 1 \right)^2 - t\left(\left\lfloor \frac{k}{2} \right\rfloor - 1, m-2 \right) \right).$$

□

5 Remarks

We have only a conjecture for the case when G is not necessarily connected and $m+1 \le k \le 2m-1$.

Conjecture 1. *If $m+1 \le k \le 2m-1$ and k is odd and $k-1|n$ then Construction 2 gives the maximum while Construction 1 is the best for even k for large n.*

Let us consider now the more general case when H is a small given graph and try to determine $\mathrm{ex}(n, \{H, P_k\})$. Of course this question makes sense only when k is larger than the longest path in H.

Conjecture 2. *Suppose that the chromatic number of H is more than 2. Then*

$$\mathrm{ex}(n, \{H, P_k\}) = n \max \left\{ \left\lfloor \frac{k}{2} \right\rfloor - 1, \frac{\mathrm{ex}(k-1, H)}{k-1} \right\} + O_k(1).$$

The best constructions are 1. the generalization of our Construction 1 with $m = \chi(H)$ and 2. the vertex disjoint copies of $EX(k-1, H)$.

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