

# Turán and Ramsey numbers in linear triple systems II

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## Abstract

In this paper we continue our studies of Turán and Ramsey numbers in linear triple systems, defined as 3-uniform hypergraphs in which any two triples intersect in at most one vertex. In [7] the two main problems left open were the Turán number of the wicket and the Ramsey property of the sail. In this paper we present some progress towards both of these problems.

## 1 Introduction

### 1.1 The wicket and the sail

In this paper we continue our studies of *linear triple systems*, defined as 3-uniform hypergraphs in which any two triples intersect in at most one vertex. The  $(k, \ell)$ -family is the family of all linear triple systems with  $\ell$  triples on at most  $k$  vertices. For linear triple systems  $H, F$  we say that  $H$  is  $F$ -free if  $H$  does not contain any subsystem isomorphic to  $F$ . We consider  $F$  fixed and call it a *configuration*. The (linear) Turán number  $\text{ex}_L(n, F)$  (or simply just  $\text{ex}(n, F)$ ) of a configuration  $F$  is the maximum number of edges in  $F$ -free linear triple systems with  $n$  vertices.

A *Steiner triple system of order  $n$* ,  $\text{STS}(n)$ , is a linear triple system on  $n$  vertices, such that each pair of vertices appear in exactly one triple (sometimes the vertices are called *points* and the triples *blocks*). It is well-known that  $\text{STS}(n)$  exists if and only if  $n \equiv 1$  or  $n \equiv 3 \pmod{6}$ , such values of  $n$  are called *admissible*. A configuration is called  *$t$ -Ramsey* (introduced in [6]) if for all large enough admissible  $n$  ( $n \geq n_0(C, t)$ ), in every  $t$ -coloring of the blocks of any  $\text{STS}(n)$  there is a monochromatic copy of  $C$ .

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Here we continue our studies from [7] of Turán and Ramsey numbers in linear triple systems. While the paper is self-contained, familiarity with the concepts introduced in [7] would help the reader.

A famous conjecture of Brown, Erdős and T. Sós [2] (BES-conjecture) claims the following Turán-type property. (It is well-known that here we can restrict our attention to linear triple systems.)

**Conjecture 1.1** ([2]). *If a linear triple system on  $n$  vertices does not contain any member of the  $(k+3, k)$ -family then it has  $o(n^2)$  triples.*

For  $k=3$  there is only one member in the  $(6, 3)$ -family, the *triangle*, three pairwise intersecting triples without a common vertex. Conjecture 1.1 in this case was famously proved by Ruzsa and Szemerédi [10], in addition with the surprising lower bound: there are triangle-free linear triple systems with  $n^{2-o(1)}$  triples. This became known as the  $(6, 3)$ -theorem. The  $(6, 3)$ -theorem had a huge influence. For example the celebrated Triangle Removal Lemma (see [4] for a survey) was devised in order to find another proof for the  $(6, 3)$ -theorem. For some developments on the BES-conjecture see e.g. [3], [9], [11], [12], [13] or [14]. For additional background see [7].

Related to the BES-conjecture, in [7] we studied the Turán and Ramsey properties of certain specific configurations. It turned out that among small configurations (at most five edges) the most interesting were the wicket  $W$  (formed by three rows and two columns of a  $3 \times 3$  point matrix, see Figure 2) and the sail  $S$  (see Figure 3).

Indeed, the two main problems left open in [7] (see also [8]) for small configurations were the Turán number of the wicket and the Ramsey property of the sail.

**Problem 1.2** ([7]). *Is it true that  $ex(n, W) = o(n^2)$ ?*

In [7] we were able to prove only a weaker, Ramsey-type result, namely that  $W$  is  $t$ -Ramsey (see Theorem 1.5 below).

**Problem 1.3** ([6],[7]). *Is the sail  $t$ -Ramsey?*

Note that the sail cannot be forced by density: for  $n \equiv 0 \pmod{3}$  there are sail-free linear triple systems with  $\frac{n^2}{9}$  blocks. This is best possible as proved in [5].

In this paper we will present some progress towards both of these problems (see Theorem 1.4 and Corollary 1.7 below). First we introduce asymmetric Ramsey numbers. Given configurations  $C_1, C_2, \dots, C_t$ , we say that  $(C_1, C_2, \dots, C_t)$  is  $t$ -Ramsey if for all large enough admissible  $n$  ( $n \geq n_0(C_1, \dots, C_t)$ ), in every  $t$ -coloring of the blocks of any STS( $n$ ) there is a color  $i, 1 \leq i \leq t$  such that color  $i$  contains a copy of  $C_i$ .

Our first result is the following.

**Theorem 1.4.** *Let  $t$  be an arbitrary positive integer. Then  $(S, W, \dots, W)$  is  $t$ -Ramsey.*

In other words, at least one of the  $t$  configurations can be replaced with the sail; a first step in the direction of Problem 1.3. However, Problem 1.3 remains open. We note that Theorem 1.4 is proved (see Theorem 1.8 below) in a more general form, where the wicket is replaced by any extended  $s$ -configuration (see the definition in the next subsection).

## 1.2 $s$ -patterns, $s$ -configurations and projections

In order to state our other results we need some concepts introduced in [7]. Consider those properly edge-colored forests (acyclic graphs) that can be obtained as the union of  $s$  monochromatic matchings  $M^1, \dots, M^s$  with the following property: for any  $1 \leq i \leq s$  every edge in  $M_i$  has a vertex that is not covered by any edge of any  $M_j, i < j \leq s$ . We call a forest obtained this way an  $s$ -pattern. For example, the path  $aba$  is a 2-pattern but the path  $abab$  is not. Next we go one step further: we call a properly edge-colored forest an *extended  $s$ -pattern* if it is obtained from a disconnected  $s$ -pattern by joining two of its connected components with a single edge matching  $M^*$  of a new color. Note that an extended  $s$ -pattern may or may not be a  $(s + 1)$ -pattern. (The  $(s + 1)$ -pattern is preferable because it leads to stronger results.) For example, the path  $abcab$  is an extended 2-pattern but not a 3-pattern (see Figure 1).

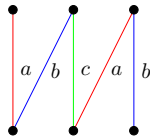


Figure 1: The  $abcab$  extended 2-pattern

A linear triple system  $\mathcal{H}$  is called an  $s$ -configuration (or an *extended  $s$ -configuration*) if it comes from an  $s$ -pattern (extended  $s$ -pattern) by augmenting all edges  $e \in M^i$  with a new *augmenting point*  $v_i$  to a triple  $e \cup v_i$  in such a way that the  $v_i$ 's are all distinct and disjoint from the vertices of the  $s$ -pattern as well (for extended  $s$ -patterns  $M^*$  is also augmented with a point that is distinct from all other augmenting points). For example Figure 2 shows how the wicket can be obtained by augmenting the extended 2-pattern  $abcab$ . (Augmenting points are shown on the figures by capitalizing the letters of the corresponding patterns.)

In the other direction we call it a *projection*. It is worth mentioning that different projection patterns may correspond to the same configuration.

In [7] we proved the following.

**Theorem 1.5** ([7]). *Any extended  $s$ -configuration  $C$  is  $t$ -Ramsey for all  $s, t \geq 1$ .*

Since the wicket  $W$  is an extended 2-configuration (see Figure 2), it is  $t$ -Ramsey. Our first result is a generalization of this result. But first we need the

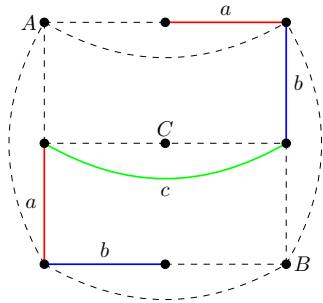


Figure 2: Augmenting the  $abcab$  extended 2-pattern to a wicket

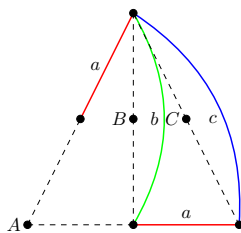


Figure 3: The sail  $S$  together with its projection

following concept: a linear triple system  $\mathcal{H}$  is  $(\eta, c)$ -dense (respectively  $(\eta, c)$ -sparse) with respect to disjoint vertex subsets  $U_1$  and  $U_2$  (shortly we say in  $(U_1, U_2)$ ), if for all subsets  $U' \subseteq U_1$ ,  $U'' \subseteq U_2$ ,  $|U'| \geq \eta|U_1|$ ,  $|U''| \geq \eta|U_2|$  at least (respectively less than)  $c$ -fraction of the pairs  $(u, v)$ ,  $u \in U'$ ,  $v \in U''$  are covered by a block of  $\mathcal{H}$ . In other words  $\mathcal{H}$  is  $c$ -dense in all large bipartite subgraphs of  $(U_1, U_2)$ . Equivalently, we can say that the bipartite shadow of  $\mathcal{H}$  is  $(\eta, c)$ -dense (respectively sparse) in  $(U_1, U_2)$  ( $(x, y)$ ,  $x \in U_1, y \in U_2$  is in the shadow of  $\mathcal{H}$  if it is covered by a block of  $\mathcal{H}$ .)

Our next result claims that Theorem 1.5 remains true if we  $t$ -color the blocks of an  $(\eta, c)$ -dense linear triple system instead of a complete STS( $n$ ).

**Theorem 1.6.** *For every  $c_1, c_2 > 0$ , integers  $s, t \geq 1$  and extended  $s$ -configuration  $C$ , there are positive constants  $\eta, n_0$  with the following properties. Let  $\mathcal{H}$  be a linear triple system on  $n \geq n_0$  vertices that is  $(\eta, c_1)$ -dense in  $(U_1, U_2)$  for some disjoint vertex subsets  $|U_1| = |U_2| \geq c_2 n$ . Then for any  $t$ -coloring of the blocks of  $\mathcal{H}$ , there is a monochromatic copy of  $C$ .*

Applying this with  $t = 1$  and  $C = W$ , we get the following corollary.

**Corollary 1.7.** *For every  $c_1, c_2 > 0$ , there are positive constants  $\eta, n_0$  with the following properties. Let  $\mathcal{H}$  be a linear triple system on  $n \geq n_0$  vertices that*

is  $(\eta, c_1)$ -dense in  $(U_1, U_2)$  for some disjoint vertex subsets  $|U_1| = |U_2| \geq c_2 n$ . Then  $\mathcal{H}$  contains a copy of  $W$ .

In other words, if  $\mathcal{H}$  has a positive density not just overall but in all large bipartite subgraphs in  $(U_1, U_2)$  then it contains a wicket. This is further evidence in the direction of Problem 1.2. However, Problem 1.2 remains open.

Returning to Ramsey properties of small configurations, we will use Theorem 1.6 to prove the following more general theorem instead of Theorem 1.4.

**Theorem 1.8.** *Let  $t, s$  be arbitrary positive integers and let  $C$  be an extended  $s$ -configuration. Then  $(S, C, \dots, C)$  is  $t$ -Ramsey.*

In the special case when  $C = W$ , we get indeed Theorem 1.4.

In the next section we provide the tools. Then in Section 3 we prove Theorem 1.6 and in Section 4 Theorem 1.8.

## 2 Tools

For basic graph concepts see the monograph of Bollobás [1].

$V(G)$  and  $E(G)$  denote the vertex-set and the edge-set of the graph  $G$ . For a graph  $G$  and a subset  $U$  of its vertices,  $G|_U$  is the restriction of  $G$  to  $U$ .

If  $H$  is an  $s$ -pattern, then it is a bipartite graph (since it is acyclic), so we may assume that it has a bipartition  $V(H) = V_1 \cup V_2$  with all edges going between  $V_1$  and  $V_2$ . We will need the following lemma from [7] on the existence of  $s$ -patterns in dense bipartite graphs.

**Lemma 2.1** (Lemma 3.1 in [7]). *For every  $\delta, \kappa > 0$ , integer  $s \geq 1$  and  $s$ -pattern  $H$  with bipartition  $V(H) = V_1 \cup V_2$ , there are positive constants  $\gamma, n_0$  with the following property. Let  $G$  be a bipartite graph on  $n \geq n_0$  vertices with bipartition  $V(G) = U_1 \cup U_2$  and with at least  $\delta n^2$  edges between  $U_1$  and  $U_2$  that is properly colored by at most  $\kappa n$  colors. Then  $G$  contains at least  $\gamma n$  vertex disjoint copies of  $H$ , where in the different copies of  $H$  the same matching always gets the same color in  $G$  and  $V_i$  is always embedded into  $U_i$ ,  $i = 1, 2$ .*

This lemma in turn used the Regularity Lemma of Szemerédi [15].

## 3 Proof of Theorem 1.6

We proceed similarly to the proof of Theorem 1.5 in [7] but we replace complete bipartite graphs with dense bipartite graphs at the appropriate places. For the sake of completeness we present the details.

Assume that  $C$  is an extended  $s$ -configuration defined by the extended  $s$ -pattern  $H$ . Let the bipartition of  $H$  be  $V(H) = V_1 \cup V_2$  and denote the matchings in the definition of the extended  $s$ -pattern  $H$  by  $M^1, \dots, M^s, M^*$ . Let  $(u, v)$  be the single edge  $M^*$  with  $u \in V_1$  and  $v \in V_2$  and let  $H'$  be the disconnected  $s$ -pattern resulting from  $H$  after removing  $(u, v)$ . In  $H'$  we have two disconnected subgraphs  $H'_1$  and  $H'_2$  such that  $u \in V(H'_1)$  and  $v \in V(H'_2)$ .

Assume that  $n$  is a sufficiently large. Let  $\mathcal{H}$  be a linear triple system on  $n$  vertices that is  $(\eta, c_1)$ -dense in  $(U_1, U_2)$  for some disjoint vertex subsets  $|U_1| = |U_2| \geq c_2 n$ , where  $\eta$  is sufficiently small. Consider an arbitrary  $t$ -coloring of the blocks of  $\mathcal{H}$ . We must show that there is a monochromatic copy of  $C$ .

Consider the bipartite graph  $G$  between  $U_1$  and  $U_2$  that is the shadow of  $\mathcal{H}$ , i.e. we have the edge  $(x, y)$ ,  $x \in U_1, y \in U_2$  if and only if  $(x, y)$  is covered by a block of  $\mathcal{H}$ . The edges of  $G$  can be naturally colored with  $t$  colors by assigning to  $(x, y) \in E(G)$  the color of the unique block  $(x, y, z)$ . We refer to this coloring as the *primary* coloring. On the other hand, there is also a natural proper coloring of the edges of  $G$  with at most  $n$  colors by assigning to  $(x, y) \in E(G)$  the vertex  $z = f(x, y)$  of the unique block  $(x, y, z)$ . This coloring is called the  $M$ -coloring.

Let  $G_i$  denote the subgraph of  $G$  defined by the primary color  $i$ , where wlog assume that  $G_1$  is the most frequent primary color, say red. Then, since  $\mathcal{H}$  is  $(\eta, c_1)$ -dense in  $(U_1, U_2)$ , we have

$$|E(G_1)| \geq \frac{c_1 |U_1| |U_2|}{t}.$$

Applying Lemma 2.1 for  $H'$  and  $G_1$  with

$$\delta_1 = \frac{c_1}{4t} \text{ and } \kappa_1 = \frac{1}{2c_2},$$

we find at least

$$\gamma_1 |V(G_1)| \geq 2\gamma_1 c_2 n \tag{1}$$

vertex disjoint red copies of  $H'$  in  $G_1$ , where in the different copies of  $H'$  the same matching always gets the same  $M$ -color in  $G_1$  and  $V_i$  is always embedded into  $U_i$ ,  $i = 1, 2$ . Let  $A_1 \subset U_1$  denote the set of embedded images of  $u$  in these copies of  $H'$ , similarly let  $B_1 \subset U_2$  denote the set of embedded images of  $v$  in these copies of  $H'$ . Let  $L_1, \dots, L_m$  denote the copies of  $H'_1$  and  $u_i, 1 \leq i \leq m$  the image of  $u$  in these copies. Similarly,  $R_1, \dots, R_m$  denote the copies of  $H'_2$  and  $v_i, 1 \leq i \leq m$  the image of  $v$  in these copies. So  $A_1 = \{u_1, \dots, u_m\}$  and  $B_1 = \{v_1, \dots, v_m\}$ , and the  $i$ -th copy of  $H'$  consists of  $L_i$  and  $R_i$ .

A copy of  $H'$ ,  $L_i \cup R_i$ , is called *bad* if for some edge  $(x, y)$  in any of the copies  $L_j \cup R_j$  of  $H'$ ,  $f(x, y)$  is a vertex in  $L_i \cup R_i$ , otherwise  $L_i \cup R_i$  is good. Note that there are at most  $s$  bad copies since there are at most  $s$   $M$ -colors on the edges of  $H'$ . Let us remove the bad copies and denote by  $A'_1$  and  $B'_1$  the set of remaining vertices in  $A_1$  and  $B_1$ . Then using (1)

$$|A'_1| = |B'_1| \geq |A_1| - s \geq \gamma_1 |V(G_1)| - s \geq \frac{\gamma_1}{2} |V(G_1)| \geq \gamma_1 c_2 n. \tag{2}$$

Assume that  $L_i, R_j$  are good copies,  $(u_i, v_j)$  is an edge in  $G$  and its primary color is red (i.e. it is an edge in  $G_1$ ). In this case we try to extend the  $s$ -pattern with the edge  $(u_i, v_j)$  and find a red copy of  $C$ . We have to avoid the following two exceptional situations for success.

- $f(u_i, v_j) = f(a, b)$  for some edge  $(a, b)$  in  $L_i$  or in  $R_j$ . In this case we cannot extend the  $s$ -pattern with the edge  $(u_i, v_j)$  because its  $M$ -color is

not a new  $M$ -color. There are at most  $s|A'_1|$  possibilities for this situation. Indeed, the number of  $M$ -colors in  $H'$  is  $s$  and in each of these  $M$ -colors there can be at most  $|A'_1|$  edges between  $A'_1$  and  $B'_1$ .

- $f(u_i, v_j)$  is covered by  $V(L_i) \cup V(R_j)$ . In this case we can extend the  $s$ -pattern with the edge  $(u_i, v_j)$  to the required extended  $s$ -pattern but the point  $f(u_i, v_j)$  of the red block  $(u_i, v_j, f(u_i, v_j))$  may not be well placed, namely, it will not be a new vertex. However, each fixed  $M$ -color can appear at most once (since the copies  $L_i, R_j$  are disjoint), thus at most  $n$  pairs  $(u_i, v_j)$  can be in this situation. Indeed, for each  $z$  in  $L_i$ , the pair  $(z, u_i)$  is only in one block, hence there can be at most one  $j$  such that  $z = f(u_i, v_j)$ , and similarly for  $z$  in  $R_j$ . So each of the at most  $n$  vertices in  $L_1, \dots, L_m, R_1, \dots, R_m$  “ruins” at most one pair  $(u_i, v_j)$ .

Then if we have a non-exceptional red  $(u_i, v_j)$ , this will be the image of  $(u, v)$ . To get a red copy of  $H$  in  $G_1$ , we take  $L_i$ , the copy of  $H'_1$  containing  $u_i$  and  $R_j$ , the copy of  $H'_2$  containing  $v_j$ . Finally, adding back the corresponding 3rd vertices of the blocks (which are disjoint from this red copy of  $H$  by construction) we get a red copy of  $C$ .

Thus we may assume that there is no such non-exceptional red  $(u_i, v_j)$ . Then in  $G|_{A'_1 \times B'_1}$  apart from at most  $s|A'_1| + n$  exceptional edges, all edges are colored (in the primary coloring) with the remaining  $(t - 1)$  colors (other than red). Assume wlog that  $G_2$  is the most frequent color out of these  $(t - 1)$  colors among the non-exceptional edges in  $G|_{A'_1 \times B'_1}$ . Then using the definition of  $(\eta, c_1)$ -dense and the fact that  $\eta$  is sufficiently small (compared to  $\gamma_1 c_2$ , see (2)) we get

$$|E(G|_{A'_1 \times B'_1})| \geq c_1 |A'_1| |B'_1|,$$

and thus

$$|E(G_2|_{A'_1 \times B'_1})| \geq \frac{c_1 |A'_1| |B'_1| - s|A'_1| - n}{t - 1} \geq \frac{c_1}{2(t - 1)} |A'_1| |B'_1|. \quad (3)$$

Indeed, this follows from

$$s|A'_1| + n \leq \frac{c_1 |A'_1| |B'_1|}{2},$$

which in turn follows from

$$\frac{2s}{c_1} + \frac{2}{\gamma_1 c_1 c_2} \leq |B'_1|$$

(using (2) and the fact that  $n$  is sufficiently large).

We will apply Lemma 2.1 for  $H'$  and  $G_2|_{A'_1 \times B'_1}$  with

$$\delta_2 = \frac{c_1}{8(t - 1)} \quad \text{and} \quad \kappa_2 = \frac{\kappa_1}{\gamma_1}.$$

Then indeed, from (3)

$$\begin{aligned} |E(G_2|_{A'_1 \times B'_1})| &\geq \frac{c_1}{2(t-1)} |A'_1| |B'_1| = \frac{c_1}{8(t-1)} (2|A'_1|)(2|B'_1|) \\ &= \delta_2 |V(G_2|_{A'_1 \times B'_1})|^2, \end{aligned}$$

and from (2)

$$\kappa_1 |V(G_1)| = \frac{\kappa_1}{\gamma_1} \frac{2\gamma_1 |V(G_1)|}{2} \leq \frac{\kappa_1}{\gamma_1} 2|A'_1| = \kappa_2 |V(G_2|_{A'_1 \times B'_1})|.$$

Applying Lemma 2.1 we find at least  $\gamma_2 |V(G_2|_{A'_1 \times B'_1})|$  vertex disjoint copies of  $H'$  in  $G_2|_{A'_1 \times B'_1}$ , where in the different copies of  $H'$  the same matching always gets the same  $M$ -color in  $G_2|_{A'_1 \times B'_1}$  and  $V_i$  is always embedded into  $U_i$ ,  $i = 1, 2$ .

We continue in this fashion. We will apply Lemma 2.1 for  $G_i|_{A'_{i-1} \times B'_{i-1}}$  and  $H'$  with

$$\delta_i = \frac{c_1}{8(t-(i-1))} \quad \text{and} \quad \kappa_i = \frac{\kappa_{i-1}}{\gamma_{i-1}}.$$

Then indeed,

$$|E(G_i|_{A'_{i-1} \times B'_{i-1}})| \geq \delta_i |V(G_i|_{A'_{i-1} \times B'_{i-1}})|^2,$$

and

$$\begin{aligned} \kappa_{i-1} |V(G_{i-1}|_{A'_{i-2} \times B'_{i-2}})| &= \frac{\kappa_{i-1}}{\gamma_{i-1}} \frac{2\gamma_{i-1} |V(G_{i-1}|_{A'_{i-2} \times B'_{i-2}})|}{2} \\ &\leq \frac{\kappa_{i-1}}{\gamma_{i-1}} 2|A'_{i-1}| = \kappa_i |V(G_i|_{A'_{i-1} \times B'_{i-1}})|. \end{aligned}$$

Applying Lemma 2.1 we find at least  $\gamma_i |V(G_i|_{A'_{i-1} \times B'_{i-1}})|$  vertex disjoint copies of  $H'$  in  $G_i|_{A'_{i-1} \times B'_{i-1}}$ , where in the different copies of  $H'$  the same matching always gets the same  $M$ -color in  $G_i|_{A'_{i-1} \times B'_{i-1}}$  and  $V_i$  is always embedded into  $U_i$ ,  $i = 1, 2$ . Note that all edges of  $G_i$  have the same primary color, so these copies of  $H'$  we find are monochromatic in this color. Let  $A_i \subset A'_{i-1}$  denote the set of embedded images of  $u$  in these copies of  $H'$ , similarly let  $B_i \subset B'_{i-1}$  denote the set of embedded images of  $v$  in these copies of  $H'$ . We remove the bad copies of  $H'$  and denote by  $A'_i$  and  $B'_i$  the set of remaining vertices in  $A_i$  and  $B_i$ .

Again if there is a non-exceptional edge in  $G_i$  between  $A'_i$  and  $B'_i$ , then we are done. Otherwise in  $G|_{A'_i \times B'_i}$  apart from at most  $is|A'_i| + in$  exceptional edges, all edges are colored (in the primary coloring) with the remaining  $(t-i)$  colors. Assume wlog that  $G_{i+1}$  is the most frequent color out of these  $(t-i)$  colors among the non-exceptional edges in  $G|_{A'_i \times B'_i}$ . Then using the definition of  $(\eta, c_1)$ -dense and the fact that  $\eta$  is sufficiently small, we get

$$|E(G|_{A'_i \times B'_i})| \geq c_1 |A'_i| |B'_i|,$$

and thus

$$|E(G_{i+1}|_{A'_i \times B'_i})| \geq \frac{c_1 |A'_i| |B'_i| - is|A'_i| - in}{t-i} \geq \frac{c_1}{2(t-i)} |A'_i| |B'_i|,$$



(assuming that  $n$  is sufficiently large).

Finally we arrive at  $G_t$  between  $A'_{t-1}$  and  $B'_{t-1}$ , where all but  $O(n)$  edges are of the last primary color, and in particular the density of  $G_t|_{A'_{t-1} \times B'_{t-1}}$  is at least  $c_1/2$  (using again the definition of  $(\eta, c_1)$ -dense and the fact that  $\eta$  is sufficiently small). Applying Lemma 2.1 for  $H'$  and  $G_t|_{A'_{t-1} \times B'_{t-1}}$  with

$$\delta_t = \frac{c_1}{8} \quad \text{and} \quad \kappa_t = \frac{\kappa_{t-1}}{\gamma_{t-1}},$$

we find at least  $\gamma_t |V(G_t|_{A'_{t-1} \times B'_{t-1}})|$  vertex disjoint copies of  $H'$  in  $G_t|_{A'_{t-1} \times B'_{t-1}}$ , where in the different copies of  $H'$  the same matching always gets the same  $M$ -color in  $G_t|_{A'_{t-1} \times B'_{t-1}}$  and  $V_i$  is always embedded into  $U_i$ ,  $i = 1, 2$ . Let  $A_t \subset A'_{t-1}$  denote the set of embedded images of  $u$  in these copies of  $H'$ , similarly let  $B_t \subset B'_{t-1}$  denote the set of embedded images of  $v$  in these copies of  $H'$ . We remove the bad copies of  $H'$  and denote by  $A'_t$  and  $B'_t$  the set of remaining vertices in  $A_t$  and  $B_t$ . Now there must be a non-exceptional edge in  $G_t$  between  $A'_t$  and  $B'_t$  because this is the last color available. Indeed, other than at most  $ts|A'_t| + tn$  exceptional edges, all edges must have this color. Then we are done similarly as before.  $\square$

## 4 Proof of Theorem 1.8

We will use similar ideas as in the proof of Theorem 1.6, but we will also use Theorem 1.6 explicitly.

Assume that  $C$  is an extended  $s$ -configuration defined by the extended  $s$ -pattern  $H$ . Let the bipartition of  $H$  be  $V(H) = V_1 \cup V_2$  and denote the matchings in the definition of the extended  $s$ -pattern  $H$  by  $M^1, \dots, M^s, M^*$ . Let  $(u, v)$  be the single edge  $M^*$  with  $u \in V_1$  and  $v \in V_2$  and let  $H'$  be the disconnected  $s$ -pattern resulting from  $H$  after removing  $(u, v)$ . In  $H'$  we have two disconnected subgraphs  $H'_1$  and  $H'_2$  such that  $u \in V(H'_1)$  and  $v \in V(H'_2)$ .

Assume that  $n$  is a sufficiently large admissible integer and we have a  $t$ -coloring of the blocks of an STS( $n$ ) =  $(V, \mathcal{B})$ . Partition  $V$  into two almost equal parts  $U_1$  and  $U_2$ . Consider the complete bipartite graph  $G$  between  $U_1$  and  $U_2$  (or equivalently the shadow graph of  $\mathcal{B}$  between  $U_1$  and  $U_2$ ). We consider again the primary coloring and the  $M$ -coloring  $f(x, y)$  of the edges of  $G$  as in the proof of Theorem 1.6. Denote the subgraph of  $G$  induced by the  $i$ -th primary color by  $G_i$ . Assume wlog that the first primary color is red. Thus we either have to find a red sail in  $\mathcal{B}$  or a monochromatic  $C$  in a non-red color.

Let  $A'_0 = U_1$ ,  $B'_0 = U_2$  and  $\gamma'_0 = 1/2$  (to initialize the iteration described below). Let  $c > 0$  be sufficiently small compared to  $1/t$  and let  $\eta_1$  be sufficiently small, so that we could apply Theorem 1.6 with  $c_1 = 1/3$ ,  $c_2 = (\gamma'_0)^2/4 = 1/16$  and  $\eta = 4\eta_1/\gamma'_0$  (note that if Theorem 1.6 is true for a particular  $\eta$ , then it is also true for any smaller  $\eta$ ). Assume first that there is a non-red primary color, wlog  $G_2$ , that is not  $(\eta_1, c)$ -sparse in  $(U_1, U_2)$ , i.e. there are subsets  $U'_1 \subseteq U_1$ ,  $U'_2 \subseteq U_2$ ,  $|U'_1| \geq \eta_1|U_1|$ ,  $|U'_2| \geq \eta_1|U_2|$  and

$$E(G_2|_{U'_1 \times U'_2}) \geq c|U'_1||U'_2|.$$

Applying Lemma 2.1 for  $H'$  and  $G_2|_{U'_1 \times U'_2}$  with

$$\delta = \frac{c}{4} \text{ and } \kappa_1 = \frac{1}{2\eta_1},$$

we find at least

$$\gamma_1 |V(G_2|_{U'_1 \times U'_2})| \geq \gamma_1 \eta_1 n \quad (4)$$

vertex disjoint red copies of  $H'$  in  $G_2|_{U'_1 \times U'_2}$ , where in the different copies of  $H'$  the same matching always gets the same  $M$ -color in  $G_2$  and  $V_i$  is always embedded into  $U_i$ ,  $i = 1, 2$ . Let  $A_1 \subset U'_1$  denote the set of embedded images of  $u$  in these copies of  $H'$ , similarly let  $B_1 \subset U'_2$  denote the set of embedded images of  $v$  in these copies of  $H'$ . Let  $L_1, \dots, L_m$  denote the copies of  $H'_1$  and  $u_i$ ,  $1 \leq i \leq m$  the image of  $u$  in these copies. Similarly,  $R_1, \dots, R_m$  denote the copies of  $H'_2$  and  $v_i$ ,  $1 \leq i \leq m$  the image of  $v$  in these copies. So  $A_1 = \{u_1, \dots, u_m\}$  and  $B_1 = \{v_1, \dots, v_m\}$ , and the  $i$ -th copy of  $H'$  consists of  $L_i$  and  $R_i$ .

As above in the proof of Theorem 1.6 we remove the at most  $s$  bad copies and denote by  $A'_1$  and  $B'_1$  the set of remaining vertices in  $A_1$  and  $B_1$ . Then using (4)

$$|A'_1| = |B'_1| \geq |A_1| - s \geq \gamma_1 \eta_1 n - s \geq \frac{\gamma_1 \eta_1}{2} n = \gamma'_1 n.$$

As above, we may assume that in  $G|_{A'_1 \times B'_1}$  apart from at most  $s|A'_1| + n$  exceptional edges, all edges are colored (in the primary coloring) with the remaining  $(t-1)$  colors (non- $G_2$ ). We continue the process in  $(A'_1, B'_1)$ .

We continue in this fashion as in the proof of Theorem 1.6. Suppose the process stops after  $i$  steps for some  $0 \leq i \leq t-1$ . This includes the two extreme cases:  $i = 0$ , when the process immediately stops in  $(A'_0, B'_0)$  and  $i = t-1$ , when the process goes through for all non-red colors. This is the main difference compared to the proof of Theorem 1.6, here the process might stop earlier if most of the edges are red in  $G|_{A'_i \times B'_i}$ . Thus if we stop after  $i$  steps for some  $0 \leq i \leq t-1$ , we have sets  $A'_i$  and  $B'_i$  with

$$|A'_i| = |B'_i| \geq \gamma'_i n, \quad (5)$$

and in  $G|_{A'_i \times B'_i}$  apart from at most  $O(n)$  exceptional edges, all edges are colored (in the primary coloring) with the remaining  $(t-i)$  colors. Let  $\eta_{i+1}$  be sufficiently small, so that we could apply Theorem 1.6 with  $c_1 = 1/3$ ,  $c_2 = (\gamma'_i)^2/4$  and  $\eta = 4\eta_{i+1}/\gamma'_i$ . Since the process stopped, all the remaining  $(t-i-1)$  non-red primary colors  $G_i$  are  $(\eta_{i+1}, c)$ -sparse in  $(A'_i, B'_i)$ , i.e. for all subsets  $U'_1 \subseteq A'_i$ ,  $U'_2 \subseteq B'_i$ ,  $|U'_1| \geq \eta_{i+1}|A'_i|$ ,  $|U'_2| \geq \eta_{i+1}|B'_i|$  we have

$$E(G_i|_{U'_1 \times U'_2}) < c|U'_1||U'_2|.$$

Note that this is also true for the other  $i$  non-red primary colors as well, since we can only have linearly many edges in these colors. Thus we have

$$E(G_1|_{U'_1 \times U'_2}) > (1 - (t-1)c)|U'_1||U'_2|. \quad (6)$$

Since  $c$  is small compared to  $1/t$ , (6) clearly implies that  $G_1$  is  $(\eta_{i+1}, 1/2)$ -dense in  $(A'_i, B'_i)$ .

Now it is time to look for a red sail. Recall that the projection of the sail is an  $abc$  triangle with an  $a$  edge hanging off from the vertex incident to  $b$  and  $c$  (see Figure 3), i.e. this is the pattern we have to look for in the red shadow graph of  $\mathcal{B}$ .

Consider the largest matching of the same  $M$ -color, denoted by  $M'$ , we can find in  $G_1|_{A'_i \times B'_i}$ . Using (5), the fact that  $G_1$  is  $(\eta_{i+1}, 1/2)$ -dense in  $(A'_i, B'_i)$  and that the number of  $M$ -colors is at most  $n$ , we have

$$|M'| \geq \frac{|A'_i||B'_i|}{2n} \geq \frac{\gamma'_i}{2}|A'_i|.$$

Let us divide  $M'$  into two equal halves (assume for simplicity that  $|M'|$  is even): we have the matching  $M'_1$  between sets  $A' \subseteq A'_i$  and  $B' \subseteq B'_i$  and we have the matching  $M'_2$  between sets  $A'' \subseteq A'_i$  and  $B'' \subseteq B'_i$ . Using (5) again we have

$$|M'_1| = |M'_2| \geq \frac{\gamma'_i}{4}|A'_i| \geq \frac{(\gamma'_i)^2}{4}n. \quad (7)$$

Consider the subgraph  $G_1|_{A'' \times B'}$ . We claim that  $G_1$  is  $(4\eta_{i+1}/\gamma'_i, 1/2)$ -dense in  $(A'', B')$ . Indeed, let us take  $U'_1 \subseteq A''$ ,  $U'_2 \subseteq B'$  such that

$$|U'_1|, |U'_2| \geq \frac{4\eta_{i+1}}{\gamma'_i}|A''| \geq \eta_{i+1}|A'_i|$$

(using (7)). But then by the fact that  $G_1$  is  $(\eta_{i+1}, 1/2)$ -dense in  $(A'_i, B'_i)$  we get that indeed

$$|E(G_1|_{U'_1 \times U'_2})| \geq \frac{|U'_1||U'_2|}{2}.$$

Consider an edge  $e$  in  $G_1|_{A'' \times B'}$ . Say  $e$  goes between edges  $(x, x') \in M'_1$  and  $(y, y') \in M'_2$ , so  $e = (y, x')$ . Similarly as in the proof Theorem 1.6, we say that  $e$  is *exceptional* if  $f(y, x')$  or  $f(x, y)$  is in  $\{x, x', y, y'\}$  (this cannot give a valid sail). Note that  $f(x, x') = f(y, y')$  cannot appear in  $\{x, x', y, y'\}$ . Clearly again we can only have at most  $O(n)$  exceptional edges.

Let us keep only the non-exceptional edges in  $G_1$ , denote this by  $G'_1$ , then clearly  $G'_1$  is  $(4\eta_{i+1}/\gamma'_i, 1/3)$ -dense in  $(A'', B')$ . Consider a non-exceptional  $G'_1$  edge  $e$  between edges  $(x, x') \in M'_1$  and  $(y, y') \in M'_2$ , so  $e = (y, x')$ . If  $(x, y)$  is also an edge in  $G_1$ , then we have the desired red pattern (where  $(x, x')$ ,  $(y, y')$  are the  $a$  edges and  $(x, y)$ ,  $(x', y)$  are the  $b$  and  $c$  edges), we can augment this to a red sail by adding the third vertices of the blocks. Thus we may assume that  $(x, y)$  is non-red; i.e. for all non-exceptional red edge  $(y, x')$  between  $A''$  and  $B'$  we get a non-red edge  $(x, y)$  between  $A'$  and  $A''$ . This implies that the non-red edges in the shadow graph of  $\mathcal{B}$  are  $(4\eta_{i+1}/\gamma'_i, 1/3)$ -dense in  $(A', A'')$ . Applying Theorem 1.6 with  $t - 1$  (i.e. the non-red colors),  $U_1 = A'$ ,  $U_2 = A''$ ,  $c_1 = 1/3$ ,  $c_2 = (\gamma'_i)^2/4$  and  $\eta = 4\eta_{i+1}/\gamma'_i$  (the conditions of the theorem are satisfied by the previous remark, (7) and our choice of  $\eta_{i+1}$ ) for these non-red edges in  $(A', A'')$ , we get a monochromatic non-red copy of  $C$ , as desired.  $\square$

## 5 Conclusion

We have shown that if a linear triple system  $\mathcal{H}$  is  $(\eta, c_1)$ -dense in  $(U_1, U_2)$  for some  $|U_1|, |U_2| \geq c_2 n$ , then it contains a wicket  $W$ . This is much better than the  $ex(n, W) \leq (1 - c)n^2$  we have shown in [8] in terms of the density (any  $c_1 > 0$  density is enough), but we need the strong structural condition of being  $(\eta, c_1)$ -dense. It is not clear how to find such  $(U_1, U_2)$  in general. This condition is somewhat similar to  $\eta$ -regularity. Unfortunately,  $\eta$  has to be sufficiently small compared to  $c_2$ , and thus the Regularity Lemma is not strong enough to find such  $(U_1, U_2)$ .

In our other result we have shown that the mixed  $(S, W, \dots, W)$  is  $t$ -Ramsey. The obvious next step would be to increase the number of sails until we have all sails as in Problem 1.3.

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