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To cite this article: Felix Finster and José M Isidro 2023 *Class. Quantum Grav.* **40** 075017

View the article online for updates and enhancements.

A mechanism for dark matter and dark energy in the theory of causal fermion systems

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Abstract

It is shown that the theory of causal fermion systems gives rise to a novel mechanism for dark matter and dark energy. This mechanism is first worked out for cubical subsets of Minkowski space with periodic boundary conditions. Then it is studied in Friedmann–Lemaître–Roberson–Walker spacetimes. The magnitude of the effect scales like one over the lifetime of the Universe squared. In contrast to most models of dark matter and dark energy, our mechanism does not postulate any new particles. Instead, it is a result of the collective behavior of all the wave functions which form the Dirac sea, needed in order to arrange correlated initial and end quantum states of the Universe.

Keywords: dark energy, dark matter, causal fermion system

1. Introduction

The theory of *causal fermion systems* is a recent approach to fundamental physics (see the basics in section 2, the reviews [12, 13, 19], the textbooks [11, 18] or the website [1]). In this approach, spacetime and all objects therein are described by a measure ρ on a set $\mathcal{F}$ of linear operators on a Hilbert space $(\mathcal{H}, \langle.|.\rangle_{\mathcal{H}})$. The physical equations are formulated by means of the so-called *causal action principle*, a nonlinear variational principle where an action $\mathcal{S}$ is minimized under variations of the measure ρ.

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In the present paper we show that the theory of causal fermion systems gives rise to a novel mechanism for dark matter and dark energy with the scaling behavior

$$\text{dark energy:} \quad \Lambda \sim \frac{1}{T^2} \quad\quad\quad (1.1)$$

$$\text{dark matter:} \quad T_0^0 \sim \frac{1}{GT^2} \, , \quad\quad\quad (1.2)$$

where G is the gravitational constant, and T is the lifetime of the Universe. Here by 'lifetime of the Universe' we mean the time from the 'big bang' until reaching the final state of the Universe. For this 'final state' there are various scenarios: big crunch, big bounce, big rip or the heat death (for an overview see the textbook [8]). The physical value of T is unknown and depends on the specific cosmological model (for a few references on this question see [5, 33]). In order to get a first idea of the scalings, one can choose T as the current age of the Universe,

$$T \approx 14 \, \text{billion light years} \, . \quad\quad\quad (1.3)$$

In this case, one obtains in Planck units

$$\frac{1}{T^2} \approx 1.6 \times 10^{-122} \, ,$$

reproducing the correct orders of magnitude of the experimental values for dark energy and dark matter. This means that, at least for cosmological scenarios in which the lifetime of the Universe has the same order of magnitude as the present age of the Universe, our mechanism gives a possible explanation for the appearance of dark matter and dark energy, including the correct order of magnitude of its experimental values. In particular, one gets a simple understanding of why the cosmological constant is so small.

In order to put our results into context, we begin with a few general words on dark matter and dark energy (for a general account see [34]). There are an estimated 10 billion galaxies (1 billion $= 10^9$) in the observable portion of the Universe. Most of these galaxies are in recessional motion away from each other according to Hubble's linear law [44]. Moreover, the farthest-lying galaxies appear to recede from us at speeds higher than predicted by Hubble's simple law. Indeed very precise experiments [35, 36] have shown that this recession is accelerating exponentially in cosmic time so that, at large distances, gravity appears to develop a repulsive character. There appears to be an energy component acting as an accelerating agent in the Cosmos; this energy component has been called *dark energy*. One possible explanation for it is a non-vanishing cosmological constant Λ in Einstein's field equations; current data [7] suggest the tiny value $\Lambda = 2.9 \times 10^{-122}$ in natural units. For a detailed exposition of the cosmological constant and related theories the reader may for example consult the [2, 6, 9, 32, 37, 43]. Ultimately, any viable theory of quantum gravity (see for example [28]) must be in a position to explain these conundrums. A first ansatz towards this goal is provided by the holographic principle [4, 39, 40].

Moreover, there is a large fraction of unknown material within the observed Universe that has been dubbed *dark matter* (for general references see [3, 29]). This dark matter is non-luminous and interacts with baryonic matter essentially via the gravitational interaction only. The amount of evidence supporting this conclusion is impressive; for details see for example [41]. Here we will just mention a few key pieces of data suggesting that there is more matter in the Universe than meets the eye: the rotation curves of galaxies; velocities in galaxy clusters; x-ray emission by galaxy clusters; gravitational lensing by galaxy clusters. For example, the rotation velocities of stars around a center of attraction do not follow the usual Kepler law; instead they exhibit a constant rotational velocity, even when the stars are very far

out from the center of the galaxy. All this implies that there is an enormous amount of matter of non-baryonic origin still to be accounted for.

Altogether, our Universe is approximately flat, its mass/energy budget being roughly given by $b = 4.9\%$ (baryonic matter), $d_\text{m} = 26.8\%$ (dark matter) and $d_\text{e} = 68.3\%$ (dark energy). Thus our knowledge of matter within the Universe hardly accounts for 5% of its total, which makes the challenge to understand the remaining 95% all the more pressing.

Numerous theories of the dark Universe have been put forward, too many to be quoted here. In the present paper we propose a novel explanation for the dark Universe (both dark matter and dark energy). Our mechanism is based on the specific structures of a causal fermion system and the specific form of the causal action principle. The causal action principle can be regarded as a variational principle for a family of spinorial wave functions in spacetime (for a basic introduction to causal fermion systems and the causal action principle see the preliminaries in section 2). In the limiting case of Dirac wave functions, this family includes all the solutions of negative energy which form the Dirac sea. The resulting Dirac sea configuration is a minimizer of the causal action principle. This entails that the states forming the Dirac sea are 'invisible' in the sense that they do not enter the Euler—Lagrange (EL) equations of the causal action principle. In the so-called continuum limit as worked out in [11], the EL equations of the causal action principle give rise to classical field equations (like the Maxwell, Yang–Mills and Einstein equations). In this way, the causal action principle gives an explanation for why the Dirac sea does not enter the classical field equations (in particular, the naively obtained 'infinite negative energy density' of the Dirac sea drops out). However, *deviations from the vacuum Dirac sea structure* do *not* drop out of the field equations. This fact is crucial for the derivation of the Einstein equation from the causal action principle, because the deviation of the Dirac sea structure due to the curvature of spacetime gives rise to the Ricci tensor in the Einstein equations (for details see [11, sections 4.5 and 4.9]).

Our mechanism for dark matter and dark energy can also be understood along these lines, because it is a consequence of a deviation from the Dirac sea structure. But now the deviation is a consequence of the fact that we demand the physical system to be in a correlated quantum state both at its beginning ('big bang') and at its end (big crunch, big rip, big bounce or thermal death). In simple terms, this means that all the wave functions forming the Dirac sea should be 'in phase' at both the initial and the final state. The resulting boundary conditions make it necessary to slightly modify each wave function. The collective effect of all these small adjustments of Dirac wave functions amounts to a modification of the Dirac sea configuration which shows up in the Einstein equations as dark matter and dark energy with the scalings (1.1) and (1.2).

Before explaining this mechanism in some more detail, we point out that for our mechanism we do not need to postulate any new particles (like axions, WIMPs or supersymmetric partners). In fact, what appears as dark energy and dark matter is not to be considered as 'matter' or 'energy' in the usual sense. In this way, our approach bears some similarity with proposals made in [27, 42]. However, in contrast to these proposals, our mechanism is a result of a collective behavior of all the wave functions of the Dirac sea, needed in order to obtain correlated initial and end states of the Universe.

We conclude the introduction by describing our mechanism in some more detail in the simplest possible setting: a *time-periodic, flat Universe* obtained by identifying $t = 0$ with $t = T$ in Minkowski space. Our starting point is the measure ρ describing the vacuum in Minkowski space (for details see section 3). The resulting causal fermion system can be characterized equivalently by the family of physical wave functions, which can be understood in simple terms as describing the 'occupied one-particle states' of the system. An important feature of a causal fermion system is that the physical wave functions involve a regularization. We assume

that a suitably regularized Dirac sea configuration is a minimizer of the causal action principle. For the purpose of the present paper, we do not need to be specific about the detailed form of the regularization (see also the discussion of this point in section 3). For this reason, it suffices to consider the simplest possible regularization by a convergence-generating exponential factor. The resulting family of regularized physical wave functions is described by the *regularized kernel of the fermionic projector*

$$P^\varepsilon(x,y) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \, (p\!\!\!/ + m) \, \delta(p^2 - m^2) \, \theta(-p^0) \, e^{\varepsilon p^0} e^{-ip(x-y)} \tag{1.4}$$

(where $p(x-y)$ denotes the Minkowski inner product and $\varepsilon > 0$ is the regularization length).

   Taking this kernel as the starting point, we analyze effects which arise because spacetime has a finite lifetime. As mentioned before, by 'lifetime of the Universe' we mean the time from the 'big bang' until the final state of the Universe is reached. Here, for technical simplicity, we consider the simplest possible realization of a finite lifetime, namely a *time-periodic* Universe. In preparation, we consider the effects of *finite spatial volume*. Indeed, as we shall see later in this paper, finite spatial volume will have no measurable consequences, and for the analysis and results of the present paper it makes no difference whether the spatial volume is finite or infinite. Nevertheless, considering a system in finite spatial volume is useful for introducing our concepts. Suppose we consider a system in a three-dimensional box of length $2L$. Thus we replace Minkowski space by the spacetime cylinder

$$\mathcal{M} = \mathbb{R} \times [-L, L]^3.$$

   Choosing periodic boundary conditions, this means that the spatial momenta $\vec{k}$ must be on a three-dimensional lattice of spacing L/π, i.e.

$$\vec{k} \in \mathfrak{L} := \left(\frac{\pi}{L}\,\mathbb{Z}\right)^3.$$

   Likewise, in (1.4) one replaces the integral over the momenta by a sum,

$$P^\varepsilon_L(x,y) := \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\omega \left(\frac{\pi}{L}\right)^3 \sum_{\vec{k} \in \mathfrak{L}} (p\!\!\!/ + m) \, \delta(p^2 - m^2) \, \theta(-p^0) \, e^{\varepsilon \omega} e^{-ip(x-y)},$$

where $p := (\omega, \vec{k})$. These discrete momenta can also be realized by summing over the dual lattice in position space, i.e.

$$P^\varepsilon_L(x,y) = \sum_{\vec{\ell} \in (2L\mathbb{Z})^3} P^\varepsilon\left(x, y + (0, \vec{\ell})\right).$$

   By summing over the lattice, we arrange the periodic boundary conditions, because, shifting the summation index,

$$P^\varepsilon_L\left(x, y + (0, \vec{\ell'})\right) = \sum_{\vec{\ell} \in (2L\mathbb{Z})^3} P^\varepsilon\left(x, y + (0, \vec{\ell} + \vec{\ell'})\right)$$

$$= \sum_{\vec{\ell} \in (2L\mathbb{Z})^3} P^\varepsilon\left(x, y + (0, \vec{\ell})\right) = P^\varepsilon_L(x,y). \tag{1.5}$$

   We refer to this procedure as *periodization*. Since P^ε decays exponentially for large spatial distances like $\sim \exp(-m|\vec{x} - \vec{y}|)$ (for details see section 4.2 below), spatial periodization gives rise to contributions which decay exponentially in the size L of the box; this is why spatial periodization does not give rise to any measurable physical effects.
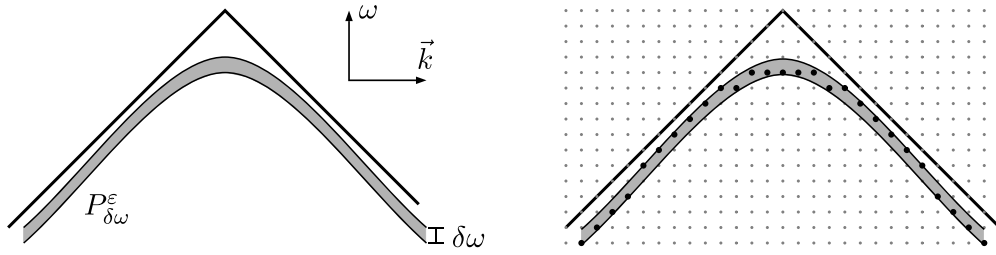
Figure 1. Periodizing a smeared-out Dirac sea configuration.

The situation becomes more interesting if the system should also be *periodic in time* with period T (which is to be thought of as the total life time of the Universe; for practical purposes we can only take T as the current age of the Universe as given and discussed in (1.3)). A simple periodization in time does *not* work, because the series

$$\sum_{\tau \in T\mathbb{Z}} P_L^\varepsilon \big(x, y + (\tau, 0)\big) \tag{1.6}$$

diverges. One way of understanding how this comes about is to note that $P(x, y)$ decays for large times only polynomially, at a rate which is not fast enough for the series to converge (for details see again section 4.2 below). Alternatively, the problem of (1.6) can be understood from the fact that making the system also periodic in time means that the momenta p must be on the four-dimensional lattice

$$p \in \frac{2\pi}{T}\, \mathbb{Z} \,\times\, \left(\frac{\pi}{L}\, \mathbb{Z}\right)^3 .$$

However, for generic values of m, this lattice does not intersect the mass hyperbola $p^2 = m^2$. In order to improve the situation, one needs to *smear out* the Dirac sea configuration, for example by replacing the δ-distribution in (1.4) by a multiple times a characteristic function supported in a 'strip of width $\delta\omega$' around the mass shell (see the left of figure 1; the detailed form of this smearing will be introduced below). Denoting the resulting distribution by $P_{\delta\omega}^\varepsilon$, periodizing in space and time becomes well-defined,

$$P_{\delta\omega, T, L}^\varepsilon(x, y) := \sum_{\tau \in T\mathbb{Z}} \sum_{\vec{\ell} \in (2L\mathbb{Z})^3} P_{\delta\omega}^\varepsilon \big(x, y + (\tau, \vec{\ell})\big) . \tag{1.7}$$

It corresponds in momentum space to a multiplication by δ-distributions supported on a four-dimensional lattice (see the right of figure 1).

Before discussing the role of the smearing in detail, we point out that the above construction leads to a violation of the Dirac equation, because the mass parameter is no longer fixed, but has been 'smeared out.' In other words, the time periodization makes it necessary to go beyond the standard physical description. In the theory of causal fermion systems, where the Dirac and Einstein equations emerge only in a limiting case (the so-called continuum limit as worked out in [11]), modifications of the Dirac equation arise naturally and do not cause any conceptual difficulties. One should also note that the energy scale of the smearing is of the order

$$\delta\omega \sim \frac{1}{T} .$$

Compared to the Compton scale, this energy scale is extremely small. Therefore, the smearing is a tiny effect. Nevertheless, as we shall work out in detail in what follows, it will lead to

corrections to the Einstein equations which can be interpreted as dark energy and dark matter with the scalings (1.1) and (1.2).

We next explain how the smearing is performed. Following the above assumption that the regularized kernel $P^\varepsilon(x,y)$ in Minkowski space must correspond to a minimizer of the causal action principle, our guiding principle is that the smearing and periodization in time should not leave the causal Lagrangian as far as possible unchanged. Moreover, we want to preserve

(a) the number of occupied states
(b) the normalization of the states.

These two requirements can be understood from the fact that the corresponding objects and quantities have a direct relation to inherent structures of the causal fermion system. Indeed, the number of occupied states corresponds to the dimension of the Hilbert space $\mathcal{H}$, whereas the normalization of the states and the corresponding unitary time evolution are related to a conservation law resulting from a Noether-like theorem for the causal action principle (more precisely, the conservation law for the commutator inner product; for details see [17, 20]). Keeping the number of occupied states fixed leads us to choose the width $\delta\omega$ independent of $\vec{k}$ as

$$\delta\omega = \frac{2\pi}{T}\,, \tag{1.8}$$

because with this choice, for every $\vec{k} \in \mathfrak{L}$ there is exactly one four-dimensional lattice point inside the strip (see the right of figure 1). In order to keep the normalization unchanged, we choose the weight function inside the strip to be constant. Moreover, we describe the states inside the strip again by solutions of the Dirac equation, but with a 'smeared' mass. Apart from being conceptually simple, this procedure also has the advantage of carrying over to curved spacetime (as will be explained in section 5 in the FLRW spacetime). This leads us to the ansatz

$$\hat{P}^\varepsilon_{\delta\omega}(p) = \frac{1}{\delta m(\vec{k})} \int_{m_0+\delta n(\vec{k})}^{m_0+\delta m(\vec{k})+\delta n(\vec{k})} (\slashed{p}+m)\,\delta(p^2-m^2)\,\theta(-p^0)\,e^{\varepsilon p^0}\,dm\,, \tag{1.9}$$

where m_0 is the unsmeared mass, and $\delta m(\vec{k})$ is chosen in such a way that the corresponding frequency shift $\delta\omega$ as given by

$$\delta\omega = \sqrt{|\vec{k}|^2 + \left(m+\delta m(\vec{k})\right)^2} - \sqrt{|\vec{k}|^2 + m^2} \tag{1.10}$$

satisfies (1.8) (for details see the computation after (4.3)). The function $\delta n(\vec{k})$ gives us the freedom to slightly modify the shape of the mass hyperbola with the goal of compensating the leading effect of the smearing (for details see remark 4.1 below). Working out all the relevant effects in detail, we find contributions to the Einstein equations of the form (1.1) and (1.2).

The paper is organized as follows. After the necessary preliminaries (section 2) we introduce causal fermion systems in Minkowski space (section 3). In section 4 we compute the effect of the smearing and periodization in detail and derive (1.1) and (1.2). In section 5 it is explained how the methods and results carry over to a Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime. In section 6 we conclude the paper with a brief discussion and outlook.

2. Preliminaries

This section provides the necessary background on causal fermion systems.

2.1. Causal fermion systems and the causal action principle

We begin with the general definitions.

Definition 2.1 (Causal fermion systems). Given a separable complex Hilbert space $\mathcal{H}$ with scalar product $\langle.|.\rangle_{\mathcal{H}}$ and a parameter $n \in \mathbb{N}$ (*the spin dimension*), we let $\mathcal{F} \subset L(\mathcal{H})$ be the set of all symmetric[3] operators on $\mathcal{H}$ of finite rank, which (counting multiplicities) have at most n positive and at most n negative eigenvalues. On $\mathcal{F}$ we are given a positive measure ρ (defined on a σ-algebra of subsets of $\mathcal{F}$). We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a *causal fermion system*.

A causal fermion system describes a spacetime together with all structures and objects therein. The physical equations are formulated for a causal fermion system by demanding that the measure ρ should be a minimizer of the causal action principle defined as follows. For any $x, y \in \mathcal{F}$, the product xy is an operator of rank at most $2n$. However, in general it is no longer symmetric because $(xy)^* = yx$, and this is different from xy unless x and y commute. As a consequence, the eigenvalues of the operator xy are in general complex. We denote the non-trivial eigenvalues counting algebraic multiplicities by $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$ (more specifically, denoting the rank of xy by $k \leqslant 2n$, we choose $\lambda_1^{xy}, \ldots, \lambda_k^{xy}$ as all the non-zero eigenvalues and set $\lambda_{k+1}^{xy}, \ldots, \lambda_{2n}^{xy} = 0$). We introduce the Lagrangian and the causal action by

$$\text{Lagrangian}: \mathcal{L}(x,y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(\left|\lambda_i^{xy}\right| - \left|\lambda_j^{xy}\right| \right)^2 \tag{2.1}$$

$$\text{causalaction}: \mathcal{S}(\rho) = \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x,y)\, d\rho(x)\, d\rho(y). \tag{2.2}$$

The *causal action principle* is to minimize $\mathcal{S}$ by varying the measure ρ under the following constraints,

$$\text{volume constraint}: \rho(\mathcal{F}) = \text{const} \tag{2.3}$$

$$\text{trace constraint}: \int_{\mathcal{F}} \text{tr}(x)\, d\rho(x) = \text{const} \tag{2.4}$$

$$\text{boundedness constraint}: \iint_{\mathcal{F} \times \mathcal{F}} |xy|^2\, d\rho(x)\, d\rho(y) \leqslant C, \tag{2.5}$$

where C is a given parameter, tr denotes the trace of a linear operator on $\mathcal{H}$, and the absolute value of xy is the so-called spectral weight,

$$|xy| := \sum_{j=1}^{2n} \left|\lambda_j^{xy}\right|.$$

This variational principle is mathematically well-posed if $\mathcal{H}$ is finite-dimensional (for details see [10] or [18, Chapter 12]).

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[3] Here by a (bounded) symmetric operator A we mean that $\langle Au|v\rangle_{\mathcal{H}} = \langle u|Av\rangle_{\mathcal{H}}$ for all $u, v \in \mathcal{H}$. Representing the operator in an orthonormal basis, the resulting matrix is Hermitian. For bounded operators as considered here, the notions 'symmetric' and 'self-adjoint' coincide.

A minimizer of the causal action principle satisfies the following *EL equations*. For a suitable value of the parameter $\mathfrak{s} > 0$, the function $\ell : \mathcal{F} \to \mathbb{R}_0^+$ defined by

$$\ell(x) := \int_M \mathcal{L}_\kappa(x,y)\, d\rho(y) - \mathfrak{s} \tag{2.6}$$

is minimal and vanishes on the support[4] of ρ,

$$\ell|_{\mathrm{supp}\rho} \equiv \inf_{\mathcal{F}} \ell = 0\,. \tag{2.7}$$

Here the κ-*Lagrangian* $\mathcal{L}_\kappa$ is defined by

$$\mathcal{L}_\kappa \,:\, \mathcal{F} \times \mathcal{F} \to \mathbb{R}\,, \qquad \mathcal{L}_\kappa(x,y) := \mathcal{L}(x,y) + \kappa\,|xy|^2 \tag{2.8}$$

with a non-negative parameter κ, which can be thought of as the Lagrange parameter corresponding to the boundedness constraint. Likewise, the parameter $\mathfrak{s} \geqslant 0$ in (2.6) is the Lagrange parameter corresponding to the volume constraint. For the derivation and further details we refer to [21, section 2] or [18, chapter 7].

2.2. Spacetime and causal structure

Let ρ be a *minimizing* measure. *Spacetime* is defined as the support of this measure,

$$M := \mathrm{supp}\rho \,\subset\, \mathcal{F}\,,$$

where on M we consider the topology induced by $\mathcal{F}$ (generated by the operator norm on $L(\mathcal{H})$). Thus the spacetime points are symmetric linear operators on $\mathcal{H}$. The restriction of the measure $\rho|_M$ gives a volume measure on spacetime.

The operators in M contain a lot of information which, if interpreted correctly, gives rise to spacetime structures like causal and metric structures, spinors and interacting fields (for details see [11, chapter 1]). All the resulting objects are *inherent* in the sense that we only use information already encoded in the causal fermion system. Here we restrict attention to those structures needed in what follows. We begin with the following notion of causality:

Definition 2.2 (causal structure). For any $x,y \in \mathcal{F}$, we again denote the non-trivial eigenvalues of the operator product xy (again counting algebraic multiplicities) by $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$. The points x and y are called *spacelike* separated if all the λ_j^{xy} have the same absolute value. They are said to be *timelike* separated if the λ_j^{xy} are all real and do not all have the same absolute value. In all other cases (i.e. if the λ_j^{xy} are not all real and do not all have the same absolute value), the points x and y are said to be *lightlike* separated.

Restricting the causal structure of $\mathcal{F}$ to M, we get causal relations in spacetime.

The Lagrangian (2.1) is compatible with the above notion of causality in the following sense. Suppose that two points $x,y \in M$ are spacelike separated. Then the eigenvalues λ_i^{xy} all have the same absolute value. As a consequence, the Lagrangian (2.1) vanishes. Thus pairs of points with spacelike separation do not enter the action. This can be seen in analogy to the usual notion of causality where points with spacelike separation cannot influence each other. This is the reason for the notion 'causal' in *causal* fermion system and *causal* action principle.

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[4] The *support* of a measure is defined as the complement of the largest open set of measure zero, i.e.

$$\mathrm{supp}\,\rho := \mathcal{F} \setminus \bigcup \big\{\Omega \subset \mathcal{F}\,\big|\, \Omega \text{ is open and } \rho(\Omega) = 0\big\}\,.$$

It is by definition a closed set.

2.3. Spinors and physical wave functions

A causal fermion system also gives rise to spinorial wave functions in spacetime, as we now explain. For every $x \in \mathcal{F}$ we define the *spin space* S_x by $S_x = x(\mathcal{H})$; it is a subspace of $\mathcal{H}$ of dimension at most $2n$. It is endowed with the *spin inner product* $\prec .|. \succ_x$ defined by

$$\prec u|v \succ_x = -\langle u|xv \rangle_{\mathcal{H}} \qquad \text{(for all } u, v \in S_x \text{)}. \tag{2.9}$$

It is an important observation that every vector $u \in \mathcal{H}$ of the Hilbert space gives rise to a unique wave function denoted by ψ^u, which to every $x \in M$ associates a vector of the corresponding spin space $\psi^u(x) \in S_x$. It is obtained by orthogonal projection to the spin space,

$$\psi^u : M \to \mathcal{H} \qquad \text{with} \qquad \psi^u(x) := \pi_x u \in S_x M \quad \text{for all } x \in M. \tag{2.10}$$

We refer to ψ^u as the *physical wave function* of the vector $u \in \mathcal{H}$. Varying the vector $u \in \mathcal{H}$, we obtain a whole family of physical wave functions. This family is described most conveniently by the *wave evaluation operator* Ψ defined at every spacetime point $x \in M$ by

$$\Psi(x) : \mathcal{H} \to S_x, \qquad u \mapsto \psi^u(x). \tag{2.11}$$

It is a simple but important observation that every spacetime point operator can be recovered from its wave evaluation operator by (for the proof see for example [11, lemma 1.1.3]).

$$x = -\Psi(x)^* \Psi(x). \tag{2.12}$$

Having constructed the spacetime point operators, we also recover all the other inherent structures of a causal fermion system. Proceeding in this way, all spacetime structures can be regarded as being induced by the physical wave functions. Moreover, restricting attention to variations of Ψ, one can understand the causal action principle as a variational principle for the family of physical wave functions. Finally, one can construct concrete examples of causal fermion systems by choosing the physical wave functions more specifically as the quantum mechanical wave functions in a classical Lorentzian spacetime. In the next section we explain this construction in more detail.

2.4. Describing a Lorentzian spacetime by a causal fermion system

We now explain how a classical curved spacetime is described by a causal fermion system. Our starting point is Lorentzian spin geometry. Thus we let $(\mathcal{M}, g)$ be a smooth, globally hyperbolic, time-oriented Lorentzian spin manifold of dimension four. For the signature of the metric we use the convention $(+, -, -, -)$. We denote the corresponding spinor bundle by $S\mathcal{M}$. Its fibers $S_x\mathcal{M}$ (with $x \in \mathcal{M}$) are endowed with an inner product $\prec .|. \succ_x$ of signature $(2,2)$. Clifford multiplication is described by a mapping γ which satisfies the anti-commutation relations,

$$\gamma : T_x\mathcal{M} \to L(S_x\mathcal{M}) \qquad \text{with} \qquad \gamma(u)\gamma(v) + \gamma(v)\gamma(u) = 2g(u,v)\mathbb{1}_{S_x(\mathcal{M})}.$$

We also write Clifford multiplication in components with the Dirac matrices γ^j. The metric connections on the tangent bundle and the spinor bundle are denoted by ∇.

We denote the smooth sections of the spinor bundle by $C^\infty(\mathcal{M}, S\mathcal{M})$. The Dirac operator $\mathcal{D}$ is defined by

$$\mathcal{D} := i\gamma^j \nabla_j : C^\infty(\mathcal{M}, S\mathcal{M}) \to C^\infty(\mathcal{M}, S\mathcal{M}).$$

Given a real parameter $m \in \mathbb{R}$ (the *mass*), the Dirac equation reads

$$(\mathcal{D} - m)\psi = 0.$$

We mainly consider solutions in the class $C^\infty_{\mathrm{sc}}(\mathcal{M}, S\mathcal{M})$ of smooth sections with spatially compact support (i.e. wave functions whose restriction to any Cauchy surface is compact). On such solutions, one has the scalar product

$$(\psi \,|\, \phi)_m = \int_{\mathcal{N}} \prec \psi \,|\, \gamma(\nu)\, \phi \succ_x \, d\mu_{\mathcal{N}}(x)\,, \tag{2.13}$$

where $\mathcal{N}$ denotes any Cauchy surface and ν its future-directed normal (due to current conservation, the scalar product is in fact independent of the choice of $\mathcal{N}$; for details see [24, section 2]). Forming the completion gives the Hilbert space $(\mathcal{H}_m, (.|.)_m)$.

We next choose a closed subspace $\mathcal{H} \subset \mathcal{H}_m$ of the solution space of the Dirac equation. The induced scalar product on $\mathcal{H}$ is denoted by $\langle .|. \rangle_{\mathcal{H}}$. There is the technical difficulty that the wave functions in $\mathcal{H}$ are in general not continuous, making it impossible to evaluate them pointwise. For this reason, we need to introduce a *regularization* on the length scale ε, described mathematically by a linear

$$\text{regularization operator} \qquad \mathfrak{R}_\varepsilon \,:\, \mathcal{H} \to C^0(\mathcal{M}, S\mathcal{M})\,. \tag{2.14}$$

In the simplest case, the regularization can be realized by a convolution on a Cauchy surface or in spacetime (for details see [24, section 4] or [11, section § 1.1.2]). For us, the regularization is not merely a technical tool, but it realizes the concept that we want to change the geometric structures on the microscopic scale. With this in mind, we always consider the regularized quantities as those having mathematical and physical significance. Different choices of regularization operators realize different microscopic spacetime structures.

Evaluating the regularization operator at a spacetime point $x \in \mathcal{M}$ gives the *regularized wave evaluation operator* $\Psi^\varepsilon(x)$,

$$\Psi^\varepsilon(x) = \mathfrak{R}_\varepsilon(x) \,:\, \mathcal{H} \to S_x\mathcal{M}\,. \tag{2.15}$$

We also take its adjoint (with respect to the Hilbert space scalar product $\langle .|. \rangle_{\mathcal{H}}$ and the spin inner product $\prec .|. \succ_x$),

$$\left(\Psi^\varepsilon(x)\right)^* \,:\, S_x\mathcal{M} \to \mathcal{H}\,.$$

Multiplying $\Psi^\varepsilon(x)$ by its adjoint gives the operator

$$F^\varepsilon(x) := -\left(\Psi^\varepsilon(x)\right)^* \Psi^\varepsilon(x) \,:\, \mathcal{H} \to \mathcal{H}\,, \tag{2.16}$$

referred to as the *local correlation operator* at the spacetime point x. The local correlation operator is also characterized by the relation

$$(\psi \,|\, F^\varepsilon(x)\, \phi) = - \prec (\mathfrak{R}_\varepsilon \psi)(x)|(\mathfrak{R}_\varepsilon \phi)(x) \succ_x \qquad \text{for all } \psi, \phi \in \mathcal{H}\,. \tag{2.17}$$

Taking into account that the inner product on the Dirac spinors at x has signature $(2,2)$, it is a symmetric operator on $\mathcal{H}$ of rank at most four, which (counting multiplicities) has at most two positive and at most two negative eigenvalues. Varying the spacetime point, we obtain a mapping

$$F^\varepsilon \,:\, \mathcal{M} \to \mathcal{F} \subset L(\mathcal{H})\,,$$

where $\mathcal{F}$ denotes all symmetric operators of rank at most four with at most two positive and at most two negative eigenvalues. Finally, we introduce the measure ρ on $\mathcal{F}$ by taking the push-forward of the volume measure on $\mathcal{M}$ under the mapping F^ε,

$$\rho := (F^\varepsilon)_* \mu_{\mathcal{M}} \tag{2.18}$$

(thus $\rho(\Omega) := \mu_{\mathcal{M}}((F^\varepsilon)^{-1}(\Omega))$). The resulting structure $(\mathcal{H}, \mathcal{F}, \rho)$ is a causal fermion system of spin dimension two.

2.5. Classical gravity and beyond

The construction of the causal fermion system in the previous section involves the Lorentzian metric g. But this Lorentzian metric does not need to satisfy the Einstein equation. Instead of postulating the Einstein equations, one can *derive* these equations from the causal action principle. To this end, one evaluates the EL equation (2.7) for the causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ constructed in the previous section in the limit $\varepsilon \searrow 0$ when the regularization is removed. This analysis, referred to as the *continuum limit*, is carried out in detail in [11, chapter 4]. The main result is that the EL equations are satisfied asymptotically for small $\varepsilon > 0$ only if the Lorentzian metric satisfies the Einstein equations, up to possible higher order corrections in curvature (which scale in powers of $(G \, \mathrm{Riem})$, where G is the gravitational coupling constant and Riem is the curvature tensor), i.e. (see [11, theorems 4.9.3 and 5.4.4])

$$R_{jk} - \frac{1}{2} R \, g_{jk} + \Lambda \, g_{jk} = G T_{jk} + \mathcal{O}\big(G^2 \, \mathrm{Riem}^2\big) \, . \tag{2.19}$$

We now briefly outline how this result is derived in [11, chapter 4]. Given a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ describing a globally hyperbolic spacetime $(\mathcal{M}, g)$ (as constructed in section 2.4), the task is to evaluate the EL equation (2.7). For the computations, it is very useful that the EL equations can be expressed in terms of the *kernel of the fermionic projector* $P(x, y)$. Here we do not need to enter the details (which can be found in [11, § 1.1.3 and § 1.4.1]), but it suffices to explain what the kernel of the fermionic projector is and how it encodes the gravitational field. Abstractly, the kernel of the fermionic projector is defined in terms of the wave evaluation operator (2.11) by

$$P(x, y) = -\Psi(x) \, \Psi(y)^* \, : \, S_y \to S_x \, .$$

In an orthonormal basis (e_i) of $\mathcal{H}$, it can be expressed as

$$P(x, y) = -\sum_i |\psi^{e_i}(x) \succ \prec \psi^{e_i}(y)| \, , \tag{2.20}$$

showing that it is composed of all the physical wave functions of the system (for details see also [11, § 1.2.4]). For a causal fermion system constructed in a Lorentzian spacetime, one can identify $P(x, y)$ with the *regularized kernel of the fermionic projector* $P^\varepsilon(x, y)$ defined by (for details see [11, section 1.2])

$$P^\varepsilon(x, y) = -\Psi^\varepsilon(x) \big(\Psi^\varepsilon(y)\big)^* \, : \, S_y \mathcal{M} \to S_x \mathcal{M} \, . \tag{2.21}$$

In the continuum limit, one studies the EL equations expressed in terms of $P^\varepsilon(x, y)$ asymptotically for small $\varepsilon > 0$. In this asymptotic regime, it suffices to analyze the behavior of $P^\varepsilon(x, y)$ near the light cone (i.e. if x and y have nearly lightlike separation). An explicit analysis is possible based on the method of integration along characteristics (see the *regularized light-cone expansion* in [11, section 2.2] or, more generally, the *regularized Hadamard expansion* [22]). Evaluating the resulting expressions in the EL equation (2.7), one gets equations involving the gravitational field and the Dirac wave functions (see [11, section 4.5]). In models containing neutrinos, these equations give rise to the Einstein equation (2.19) (see [11, section 4.9]).

Before explaining how to go beyond classical gravity, we point out that the above constructions only give a *special class of examples* of causal fermion systems describing classical spacetimes. In general, the measure ρ defined by (2.18) will *not* be a minimizer of the causal action principle. We only know that it is an *approximate* minimizer if the Einstein equations are satisfied. A good strategy for getting an *exact* minimizer is to take the causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ as constructed in section 2.4 from a Lorentzian spacetime as the starting

point, and to minimize the causal action principle further by varying the wave evaluation operator (2.11) of this causal fermion system. The resulting physical wave functions will no longer satisfy the Dirac equation. Even more, in general a Lorentzian metric ceases to exist. In other words, we are leaving the realm of classical gravity and differential geometry. The geometry of the resulting *quantum spacetime* has been explored in [15]. We also refer the reader interested in this direction of research to the survey articles [12, section 4] or [13, section 4].

A more modest approach and more computational approach for going beyond classical gravity is to work out *small corrections* to the Dirac and Einstein equations described by the EL equations of the causal action principle. This strategy was first used in [16], where modifications of the Dirac equations were derived from the EL equations and the corresponding conservation laws, giving rise to a mechanism of *baryogenesis*. In the present paper, we proceed in the same spirit to again derive modifications of the Dirac equation, but now giving a possible explanation for the dark Universe. We note that a systematic study of all corrections to the Dirac and Einstein equations (also including dynamical coupling constants as envisioned in the preliminary preprint [25]) is still lacking and seems an interesting field of future research.

3. The Minkowski vacuum as a minimizer of the causal action

We now specify the construction of section 2.4 in order to describe the Minkowski vacuum by a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$. In Minkowski space, the scalar product (2.13) simplifies to the usual spatial integral over the polarized probability density,

$$(\psi \,|\, \phi)_m = \int_{\mathbb{R}^3} \psi(t, \vec{x})^{\dagger} \phi(t, \vec{x}) \, d^3x \tag{3.1}$$

(where the dagger denotes the transposed and complex conjugated spinor). Moreover, we choose the subspace $\mathcal{H} \subset \mathcal{H}_m$ as the space spanned by all solutions of negative frequency. In this way, we implement the Dirac sea picture. This leaves us with the freedom to choose the regularization operator $\mathfrak{R}_\varepsilon$ in (2.14). The general strategy is to choose the regularization operator in such a way that the resulting causal fermion system becomes a minimizer of the causal action principle. At present, it is largely unknown how the resulting 'optimal regularization' should look like. A mathematically concise setting in which this question can be studied is the homogeneous causal action principle as analyzed in [14]. Here we shall not enter the results of this analysis. Instead, we merely *assume* that, by choosing a suitable regularization, we obtain an *exact minimizer* of the causal action principle.

This assumption has far-reaching consequences which are crucial for the arguments in our paper. Indeed, having an exact minimizer implies that the EL equations are satisfied without error terms. Taking the continuum limit, it follows in particular that the the Minkowski vacuum satisfies the Einstein equations. This means that, *in the Minkowski vacuum*, there is *no dark matter*, and the *cosmological constant vanishes*.

Taking these assumptions as our starting point, we shall compute how the situation changes when Minkowski space is replaced by a time-periodic Universe of a FLRW Universe with a correlated initial and end state. For these computation, it is irrelevant how the regularization of Minkowski space which realizes the minimum of the causal action principle actually looks like. Instead, it will suffice to work with simplest possible regularization obtained by the replacement $t \to t - i\varepsilon$. For this so-called $i\varepsilon$-regularization, the kernel of the fermionic projector takes the form (1.4) (for more details on the $i\varepsilon$-regularization see [11, § 1.2.5 and § 2.4.1]).

4. Dark matter and dark energy in a time-periodic universe

We now enter the detailed computations in flat spacetimes.

4.1. Smearing in the mass parameter

We set

$$
P_m^\varepsilon(p) := (p\!\!\!/ + m)\, \delta\big(p^2 - m^2\big)\, \theta(-p^0)\, e^{\varepsilon p^0}\,.
\tag{4.1}
$$

We now 'smear out' the mass by setting

$$
P^\varepsilon = \frac{1}{\delta m(\vec{k})} \int_{m_0 + \delta n(\vec{k})}^{m_0 + \delta m(\vec{k}) + \delta n(\vec{k})} P_m^\varepsilon\, dm\,,
\tag{4.2}
$$

where $p = (\omega, \vec{k})$. We choose δm such that the width in ω is given by

$$
\delta\omega = \frac{2\pi}{T}\,,
\tag{4.3}
$$

where T is the lifetime of our Universe. Taking the difference of the dispersion relations

$$
(\omega - \delta\omega)^2 - |\vec{k}|^2 = (m_0 + \delta m + \delta n)^2 \qquad \text{and} \qquad \omega^2 - |\vec{k}|^2 = (m_0 + \delta n)^2\,,
$$

we obtain for any momentum $\vec{k}$ the condition

$$
(\omega - \delta\omega)^2 - \omega^2 = (m_0 + \delta m + \delta n)^2 - (m_0 + \delta n)^2\,.
$$

Expanding linearly, substituting (4.3) and solving for δm gives

$$
\delta m = \frac{|\omega|}{m_0}\, \frac{2\pi}{T} + \mathcal{O}\big((m_0 T)^{-2}\big)\,.
\tag{4.4}
$$

The function $\delta n(\vec{k})$ gives us the freedom to 'slightly deform' the mass shell; this will be explained in remark 4.1 below.

In order to determine the resulting contribution to the Einstein equations as obtained in the continuum limit, it suffices to compute the vector component of P^ε. In view of (4.1) and (4.2), we may pull out the Dirac matrices,

$$
P^\varepsilon(p) = p\!\!\!/\, R(p)\, \theta(-p^0)\, e^{\varepsilon p^0} + (\text{scalar component})
\tag{4.5}
$$

with

$$
R(p) := \frac{1}{\delta m(\vec{k})} \int_{m_0 + \delta n(\vec{k})}^{m_0 + \delta m(\vec{k}) + \delta n(\vec{k})} \delta\big(p^2 - m^2\big)\, dm
\tag{4.6}
$$

(here by 'scalar component' we mean a multiple of the identity matrix). We next expand the δ distribution in the integrand in the mass parameter,

$$
\delta\big(p^2 - m^2\big) = \delta\big(p^2 - m_0^2\big) + \delta'\big(p^2 - m_0^2\big)\, (m^2 - m_0^2)
$$
$$
+ \frac{1}{2}\, \delta''\big(p^2 - m_0^2\big)\, (m^2 - m_0^2)^2 + \mathcal{O}\big((m^2 - m_0^2)^3\big)\,.
$$

Now we can carry out the integral in (4.6),

$$
\begin{aligned}
R &= \delta\left(p^2 - m_0^2\right) + \delta'\left(p^2 - m_0^2\right) \frac{1}{\delta m(\vec{k})} \int_{m_0 + \delta n(\vec{k})}^{m_0 + \delta m(\vec{k}) + \delta n(\vec{k})} \left(m^2 - m_0^2\right) dm \\
&\quad + \frac{1}{2}\, \delta''\left(p^2 - m_0^2\right) \frac{1}{\delta m(\vec{k})} \int_{m_0 + \delta n(\vec{k})}^{m_0 + \delta m(\vec{k}) + \delta n(\vec{k})} \left(m^2 - m_0^2\right)^2 dm + \mathcal{O}\left(\left(\delta m + |\delta n|\right)^3\right) \\
&= \delta\left(p^2 - m_0^2\right) + \delta'\left(p^2 - m_0^2\right) \left(m_0\left(\delta m + 2\delta n\right) + \frac{1}{3}\left(\delta m\right)^2 + \delta m\, \delta n + \left(\delta n\right)^2\right) \\
&\quad + \frac{1}{2}\, \delta''\left(p^2 - m_0^2\right) \left(\frac{4}{3}\, m_0^2 \left(\delta m^2 + 3(\delta m)(\delta n) + 3(\delta n)^2\right)\right) + \mathcal{O}\left(\left(\delta m + |\delta n|\right)^3\right).
\end{aligned}
$$

According to (4.5), the vector component of P is obtained from this formula by multiplying with p'.

We next want to choose δn in such a way that the leading contributions $\sim \delta'$ vanish. To this end, we choose

$$
\delta n = -\frac{\delta m}{2} - \frac{1}{24}\, \frac{(\delta m)^2}{m_0}\,, \tag{4.7}
$$

because then

$$
\begin{aligned}
m_0\left(\delta m + 2\delta n\right) &= -\frac{1}{12}\,(\delta m)^2 \\
\frac{1}{3}\,(\delta m)^2 + \delta m\, \delta n + (\delta n)^2 &= \frac{1}{3}\,(\delta m)^2 - \frac{1}{2}\,(\delta m)^2 + \frac{1}{4}\,(\delta m)^2 + \mathcal{O}\left((\delta m)^3\right) \\
&= \frac{1}{12}\,(\delta m)^2 + \mathcal{O}\left((\delta m)^3\right),
\end{aligned}
$$

as desired (this choice will be discussed in remark 4.1 below). We thus obtain

$$
R = \delta\left(p^2 - m_0^2\right) + \frac{1}{6}\, m_0^2\, (\delta m)^2\, \delta''\left(p^2 - m_0^2\right) + \mathcal{O}\left((\delta m)^3\right). \tag{4.8}
$$

Remark 4.1 (Choice of n). The ansatz (4.2) was already motivated in the introduction. It was also explained why $\delta\omega$ and δm are to be chosen according to (4.3) and (4.4) (see also (1.10) in the introduction). However, the choice of n according to (4.7) still requires a detailed explanation. The first summand in (4.7) can be understood immediately from the fact that we do not want to change the mean value of the mass. More precisely, it gives rise to a smearing

$$
\frac{1}{\delta m(\vec{k})} \int_{m_0 - \frac{\delta m(\vec{k})}{2}}^{m_0 + \frac{\delta m(\vec{k})}{2}} P_m\, dm + \mathcal{O}(\Delta m)
$$

which is symmetric about m_0. This seems the natural choice.

The quadratic term in (4.7)

$$
-\frac{1}{24}\, \frac{(\delta m)^2}{m_0}
$$

is less obvious. This choice can be understood from the requirement that the regularized kernel $P^\varepsilon(x, y)$ must be a minimizer of the causal action principle. The argument goes as follows: The corresponding contribution to the smeared δ-distribution is given by (here and in what follows, the symbol $\asymp$ denotes a selected contribution to the expression on the left)

$$R \asymp \delta'(p^2 - m_0^2) \, 2m_0 \, \delta n + \mathcal{O}\left((\delta m + |\delta n|)^3\right)$$

$$= -\frac{1}{12} \, \delta'(p^2 - m_0^2) \, (\delta m)^2 + \mathcal{O}\left((\delta m + |\delta n|)^3\right).$$

Using (4.4), we can write this contribution as

$$R \asymp -\frac{\pi^2}{3} \frac{1}{T^2} \, \delta'(p^2 - m_0^2) \, \frac{\omega^2}{m_0^2} = -\frac{\pi^2}{6} \frac{1}{m_0^2 T^2} \, \omega \, \frac{\partial}{\partial \omega} \delta(p^2 - m_0^2).$$

Taking the Fourier transform, the factor $\omega \partial_\omega$ becomes the operator $-(1 + t\partial_t)$, because

$$\int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \, \omega \left(\frac{\partial}{\partial \omega} \delta(p^2 - m_0^2)\right) \theta(-p^0) \, e^{ip\xi}$$

$$= -i \frac{\partial}{\partial t} \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \left(\frac{\partial}{\partial \omega} \delta(p^2 - m_0^2)\right) \theta(-p^0) \, e^{ip\xi}$$

$$= i \frac{\partial}{\partial t} \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \, \delta(p^2 - m_0^2) \, \theta(-p^0) \, \frac{\partial}{\partial \omega} e^{ip\xi}$$

$$= -\frac{\partial}{\partial t} \left(t \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \, \delta(p^2 - m_0^2) \, \theta(-p^0) \, e^{ip\xi} \right)$$

$$= -\left(1 + t \frac{\partial}{\partial t}\right) \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \, \delta(p^2 - m_0^2) \, \theta(-p^0) \, e^{ip\xi}$$

(here we take into account only the leading contribution in ε). Thus the corresponding contribution to the regularized kernel is

$$P_{\delta\omega}^\varepsilon(x,y) \asymp \frac{\pi^2}{6} \frac{1}{m_0^2 T^2} \, i\partial_x\left((1 + t\partial_t) \mathcal{T}_{m_0^2}^\varepsilon(\xi)\right), \tag{4.9}$$

where $\mathcal{T}_{m^2}^\varepsilon(\xi)$ is the regularized Fourier transform of the lower mass shell,

$$\mathcal{T}_{m^2}^\varepsilon(\xi) = \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \, \delta(p^2 - m^2) \, \theta(-p^0) \, e^{\varepsilon p^0} e^{ip\xi}. \tag{4.10}$$

The time derivative ∂_t in (4.9) changes the singularity structure on the light cone by making the order of the singularity stronger. This is inhibited by the causal action principle, because stronger singularities on the light cone give rise to extremely large contributions to the causal action. Therefore, we can argue that minimizing the causal action forces us to choose n according to (4.7). ◊

It remains to analyze the resulting formula (4.8) and to make the connection to dark matter and dark energy. To this end, we now compute the corresponding contributions to the kernel of the fermionic projector.

Proposition 4.2. *The 'smearing' of the mass parameter (4.2) with δm and δn according to (4.4) and (4.7) gives rise to corrections to the kernel of the fermionic projector of the form*

$$(P^\varepsilon - P_{m_0}^\varepsilon)(x,y) = \frac{\pi^2}{6} \frac{1}{T^2} \, t^2 \, P^\varepsilon(x,y) \tag{4.11}$$

$$-\frac{i\pi^2}{3} \frac{1}{T^2} \, \gamma^0 \, t \, \mathcal{T}_{m_0^2}^\varepsilon(\xi) \tag{4.12}$$

$$+\frac{i\pi^2}{12} \frac{1}{T^2} \int_{-\infty}^{\infty} \epsilon(\tau) \, \gamma^0 \, \mathcal{T}_{m_0^2}^\varepsilon(\xi + \tau e_0) \, d\tau \tag{4.13}$$

$$- \frac{\pi^2}{12} \frac{1}{T^2} \int_{-\infty}^{\infty} \epsilon(\tau)\,(t+\tau)\,P^\varepsilon(x, y + \tau e_0)\,d\tau$$

$$- (\text{scalar component}) + \mathcal{O}\big((m_0 T)^{-3}\big), \qquad (4.14)$$

where $\mathcal{T}_{m_0^2}^\varepsilon$ is the distributional kernel (4.10) (and ε is the sign function $\epsilon(\tau) = 1$ for $\tau \geqslant 0$ and $\epsilon(\tau) = -1$ otherwise).

Proof. Applying (4.4) to (4.8), we obtain

$$R = \delta\big(p^2 - m_0^2\big) + \frac{2\pi^2}{3} \frac{\omega^2}{T^2} \delta''\big(p^2 - m_0^2\big) + \mathcal{O}\big((m_0 T)^{-3}\big). \qquad (4.15)$$

We rewrite the correction term in this equation as

$$\frac{2\pi^2}{3} \frac{\omega^2}{T^2} \delta''\big(p^2 - m_0^2\big) = \frac{2\pi^2}{3} \frac{\omega^2}{T^2} \left(\frac{1}{2\omega} \frac{\partial}{\partial\omega} \left(\frac{1}{2\omega} \frac{\partial}{\partial\omega} \delta\big(p^2 - m_0^2\big) \right) \right)$$

$$= \frac{\pi^2}{6} \frac{1}{T^2} \left(\frac{\partial^2}{\partial\omega^2} \delta\big(p^2 - m_0^2\big) - \frac{1}{\omega} \frac{\partial}{\partial\omega} \delta\big(p^2 - m_0^2\big) \right). \qquad (4.16)$$

Now we can compute the Fourier transform of the resulting terms,

$$\int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi)^4} \left(\frac{\partial^2}{\partial\omega^2} \delta\big(p^2 - m_0^2\big) \right) \theta(-p^0)\, e^{ip\xi}$$

$$= -t^2 \int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi)^4} \delta\big(p^2 - m_0^2\big) \theta(-p^0)\, e^{ip\xi}$$

$$\int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi)^4} \frac{1}{\omega} \left(\frac{\partial}{\partial\omega} \delta\big(p^2 - m_0^2\big) \right) \theta(-p^0)\, e^{ip\xi}$$

$$= -\frac{i}{2} \int_{-\infty}^{\infty} \epsilon(\tau) \left(\int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi)^4} \left(\frac{\partial}{\partial\omega} \delta\big(p^2 - m_0^2\big) \right) \theta(-p^0)\, e^{ip\xi + i\omega\tau} \right) d\tau \qquad (4.17)$$

$$= -\frac{1}{2} \int_{-\infty}^{\infty} \epsilon(\tau)\,(t+\tau) \left(\int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi)^4} \delta\big(p^2 - m_0^2\big) \theta(-p^0)\, e^{ip\xi + i\omega\tau} \right) d\tau. \qquad (4.18)$$

For clarity, we remark that in (4.17) we used the distributional formula

$$\frac{1}{\omega} = -\frac{i}{2} \lim_{\delta \searrow 0} \int_{-\infty}^{\infty} \epsilon(\tau)\, e^{-\delta|\tau|}\, e^{i\omega\tau}\, d\tau.$$

After pulling the τ-integral outside, we can take the limit $\delta \searrow 0$, because the expression inside the brackets has suitable decay properties in τ. Using these formulas in (4.5), in analogy to (4.9) we obtain the corrections to $P_{\delta\omega}^\varepsilon(x, y)$ given by

$$\frac{\pi^2}{6} \frac{1}{T^2} i\slashed{\partial}_x \left(t^2\, \mathcal{T}_{m_0^2}^\varepsilon(\xi) \right) - \frac{\pi^2}{12} \frac{1}{T^2} i\slashed{\partial}_x \int_{-\infty}^{\infty} \epsilon(\tau)\,(t+\tau)\, \mathcal{T}_{m_0^2}^\varepsilon(\xi + \tau e_0)\, d\tau.$$

Carrying out the derivatives inside the integrand gives the result. $\qquad\square$

The corrections computed in this proposition give rise to dark matter and dark energy with the scalings (1.1) and (1.2), as we now explain. The energy-momentum tensor of a Dirac wave function ψ has the general form

$$T_{jk} = \frac{1}{2} \text{Re} \prec \psi \,|\, (i\gamma_j \nabla_k + i\gamma_k \nabla_j)\psi \succ$$

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(for a derivation see for example [26]). Noting that the kernel of the fermionic projector is composed of all physical wave functions (see (2.20)), the total energy-momentum tensor is given formally by

$$T_{jk}(x) = -\frac{1}{2}\mathrm{ReTr}_{S_x\mathscr{M}}\left(\left(i\gamma_j\nabla_k + i\gamma_k\nabla_j\right)P^{\varepsilon}(x,x)\right).\qquad(4.19)$$

The reason why this equation holds only formally is that it involves the energy-momentum of all the sea states, which clearly must not be included in the Einstein equations. The causal action principle takes care of this issue without the need of counter terms, simply because the vacuum Dirac sea is a minimizer (as explained in section 3). Consequently, the sea states do not show up in the EL equations (more technically, for the vacuum Dirac sea configuration the positive and negative terms in the Lagrangian (2.1) cancel each other). However, *changes* to the vacuum Dirac sea configurations in general do *not* drop out of the EL equations. More precisely, the term (4.11) changes the eigenvalues of the closed chain only by a prefactor. Therefore, it does drop out of the Lagrangian. The other terms (4.12)–(4.14), however, do not drop out, and we may compute the corresponding contributions to the energy-momentum tensor simply with the help of (4.19). Since the summands (4.12) and (4.13) involves the Dirac matrix γ^0, the contribute only to T_0^0 and can be understood as dark matter (1.2). The summand (4.14), on the other hand, also involves the spatial Dirac matrices, thereby contributing to both dark energy (1.1) and dark matter (1.2). In view of the factors $1/T^2$ in (4.11)–(4.14), all these contributions have the desired scaling behavior (1.1) and (1.2).

We finally point out that the contributions (4.11)–(4.14) are singular on the light cone. This means that, in order to compute how they contribute to the EL equations of the causal action principle, one needs to employ the formalism of the continuum limit (as outlined in section 2.5). In this formalism, the prefactors becomes empirical so-called regularization parameters. In particular, at present the signs of the prefactors are unknown.

4.2. Periodizing the spacetime

In this section, we compute the effect of the periodization (1.7). The result of our analysis will be that the resulting contributions to the Einstein equations are much smaller than (1.1) and (1.2), so that the periodization is of no relevance for the dark Universe. Our starting point is again the smeared Dirac sea (4.2) and (4.1) with δm and δn as given by (4.4) and (4.7). But now we need to expand $P(\xi)$ for large ξ. In preparation, we analyze the distribution $\mathcal{T}_{m^2}^{\varepsilon}$ introduced in (4.10). The Fourier integral over the lower mass shell can be carried out explicitly using Bessel functions to obtain (for details see [11, proof of lemma 1.2.9])

$$\mathcal{T}_{m^2}^{\varepsilon}(\xi) = \frac{m^2}{(2\pi)^3}\frac{K_1\left(m\sqrt{r^2 + (\varepsilon + it)^2}\right)}{m\sqrt{r^2 + (\varepsilon + it)^2}},\qquad(4.20)$$

where we again set $\xi = y - x$ as well as $t = \xi^0$ and $r = |\vec{\xi}|$. Expanding the modified Bessel function K_1 for large values of its arguments t of r, one finds that (see [30, equation (10.40.2)])

$$\mathcal{T}_{m^2}^{\varepsilon}(t,\vec{0}) \simeq \frac{\sqrt{m}}{|t|^{\frac{3}{2}}}e^{-imt}\left(1 + \mathcal{O}\left(|t|^{-1}\right)\right)\qquad(4.21)$$

$$\mathcal{T}_{m^2}^{\varepsilon}(0,\vec{\xi}) = \frac{\sqrt{m}}{r^{\frac{3}{2}}}e^{-mr}\left(1 + \mathcal{O}\left(r^{-1}\right)\right).\qquad(4.22)$$

Finally, the regularized kernel of the fermionic projector (1.4) is obtained by differentiation,

$$P^{\varepsilon}(x,y) = (i\partial\!\!\!/_x + m)\,\mathcal{T}_{m^2}^{\varepsilon}(\xi).\qquad(4.23)$$

The exponential decay in spacelike directions (4.22) explains why, as already mentioned in the introduction after (1.5), a periodization in space does not lead to any measurable physical effects. In timelike directions, however, one has only polynomial decay,

$$P^{\varepsilon}(x,y)\big|_{\xi=(t,\vec{0})} \simeq \frac{m^{\frac{3}{2}}}{|t|^{\frac{3}{2}}}\, e^{-imt}\Big(1 + \mathcal{O}\big(|t|^{-1}\big)\Big)\,.$$

In order to compute the resulting effect of the periodization in time, following the consideration after (1.6) we need to take into account the smearing in the mass parameter. Therefore, our next task is to compute the effect of the smearing in the mass parameter for large times. Our starting point is again the formula (4.6) for the smeared lower mass shell. But, instead of expanding in the mass parameter, we now rewrite the smearing by a convolution in the variable ω. In position space, this convolution corresponds to multiplication by a cutoff function in time, giving us the desired information for large times.

We introduce the parameter τ by the condition

$$\omega^2 - m^2 = \big(\omega + \tau\big)^2 - m_0^2\,;$$

it tells us about the shift in frequency as caused by the mass change. Using that ω is negative, we can express τ and m in terms of each other by

$$m(\tau) = \sqrt{m_0^2 + \omega^2 - \big(\omega + \tau\big)^2} = \sqrt{m_0^2 - 2\omega\tau - \tau^2} \tag{4.24}$$

$$\tau(m) = -\omega - \sqrt{\omega^2 - m^2 + m_0^2}\,. \tag{4.25}$$

Moreover,

$$m\,dm = (-\omega - \tau)\,d\tau\,, \qquad dm = -\frac{\omega + \tau}{m(\tau)}\,d\tau\,.$$

This makes it possible to rewrite (4.6) as

$$R = \frac{1}{\delta m(\vec{k})} \int_{\tau(m_0+\delta n)}^{\tau(m_0+\delta m+\delta n)} \delta\big((-\omega+\tau)^2 - |\vec{k}|^2 - m_0^2\big)\left(-\frac{\omega+\tau}{m(\tau)}\right) d\tau\,. \tag{4.26}$$

A direct computation using (4.25), (4.4) and (4.7) yields

$$\tau(m_0 + \delta n) = -\frac{\pi}{T} + m_0\,\mathcal{O}\big((m_0 T)^{-2}\big)$$

$$\tau(m_0 + \delta m + \delta n) = \frac{\pi}{T} + m_0\,\mathcal{O}\big((m_0 T)^{-2}\big)$$

(apart from the error terms, these relations are also obvious from (4.3) and (4.7)). Again using (4.4), we obtain

$$\begin{aligned}
R &= \frac{m_0}{|\omega|}\,\frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \delta\big((-\omega+\tau)^2 - |\vec{k}|^2 - m_0^2\big)\left(-\frac{\omega+\tau}{m(\tau)}\right) d\tau\,\Big(1 + \mathcal{O}\big((m_0 T)^{-1}\big)\Big)\\
&= \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \delta\big((-\omega+\tau)^2 - |\vec{k}|^2 - m_0^2\big)\,d\tau\,\Big(1 + \mathcal{O}\big((m_0 T)^{-1}\big)\Big)\\
&= \int_{-\infty}^{\infty} \delta\big((-\omega+\tau)^2 - |\vec{k}|^2 - m_0^2\big)\,\hat{\eta}_T(\tau)\,d\tau + \mathcal{O}\big((m_0 T)^{-1}\big)\,,
\end{aligned} \tag{4.27}$$

where $\hat{\eta}_T$ is a multiple times the characteristic function,

$$\hat{\eta}_T := \frac{T}{2\pi}\,\chi_{\left[-\frac{\pi}{T},\frac{\pi}{T}\right]}\,.$$

In other words, the leading effect of the mass smearing for large $m_0 T$ can be described simply by taking the convolution in ω with the function $\hat{\eta}_T$. In position space, this simply corresponds to multiplication by the Fourier transform of $\hat{\eta}_T$ given by

$$\eta_T(t) = \int_{-\infty}^{\infty} \hat{\eta}_T(\omega)\, e^{-i\omega t}\, d\omega = \frac{T}{\pi t} \sin\left(\frac{\pi t}{T}\right).$$

Therefore, the leading effect of the mass smearing for large times amounts to multiplying the kernel by a function which is T-periodic and vanishes at $t = nT$ with $n \neq 0$.

Having clarified the effect of the mass smearing in position space for large times, we can now perform the periodization in time. Using (4.23), similar to (1.6) we consider the series

$$\sum_{\tau \in T\mathbb{Z}} (i\partial_x + m)\left(\eta_T(\tau)\, \mathcal{T}_{m^2}^{\varepsilon}(x, y + (\tau, 0))\right).$$

In order to determine the scaling behavior, it suffices to consider the summand with $\tau = T$. Since $\eta_T(t)$ vanishes at $t = T$, the leading contribution scales like

$$(i\partial_x + m)\left(\eta_T(T + \xi^0)\, \mathcal{T}_{m^2}^{\varepsilon}(x, y + (T, 0))\right)$$
$$\sim \gamma^0 \left(i\eta_T'(T + \xi^0) + m\,\eta(T + \xi^0)\right) \mathcal{T}_{m^2}^{\varepsilon}(x, y + (T, 0))$$
$$\sim \gamma^0\, \frac{\sqrt{m}}{T^{\frac{3}{2}}}\left(\frac{1}{T} + \frac{imt}{T}\right) = \gamma^0\, \frac{\sqrt{m}}{T^{\frac{5}{2}}}\left(1 + imt\right),$$

where in the last line we used (4.21). Compared to the contributions computed in proposition 4.2, these contributions are by a scaling factor $(m_0 T)^{-\frac{1}{2}}$ smaller and can therefore be neglected.

By direct computation, one finds that the contributions by the error terms in (4.27) are even smaller and can also be neglected. In order to explain how this comes about, we consider exemplarily the contributions by the quadratic term in (4.7). Expanding (4.26), the resulting contribution to R takes the form

$$-\frac{1}{24}\, \frac{\delta m(\vec{k})}{m_0} \int_{\tau(m_0 - \delta m/2)}^{\tau(m_0 + \delta m/2)} \frac{\partial}{\partial \omega} \delta\left((-\omega + \tau)^2 - |\vec{k}|^2 - m_0^2\right)\left(-\frac{\omega + \tau}{m(\tau)}\right) d\tau.$$

As explained after (4.27), the τ-integration gives rise to the multiplication by $\eta_T(t)$. Compared to (4.27), we now have the additional factor

$$\frac{(\delta m)^2}{m_0}\, \frac{\partial}{\partial \omega} \overset{(4.4)}{\simeq} \frac{\omega^2}{m_0^3 T^2}\, \frac{\partial}{\partial \omega}.$$

Taking the Fourier transform similar to the computation after (4.16), we obtain an additional scaling factor which is at least as small as $(m_0 T)^{-1}$, justifying that this contribution is negligible. For the other contributions one can argue similarly.

There is one point in the periodization which requires closer attention: if we periodize both in space and time (as shown in figure 1) we also get contributions when the vector $(\tau, \vec{\ell})$ in (1.7) is close to the light cone. The following consideration shows that the resulting contributions to the kernel of the fermionic projector are also negligible: The scaling behavior can be determined by considering the simple $i\varepsilon$ regularization obtained by the replacement $t \to t - i\varepsilon$. Then the Bessel function in (4.20) becomes (for details see [11, § 1.2.5] or [31, section 5.1 and appendix B])

$$K_1\left(m\sqrt{r^2 + (\varepsilon + it)^2}\right).$$

Evaluating this function on the light cone $t = \pm r$, one obtains exponential decay

$$K_1\big(m\sqrt{\varepsilon^2 + 2i\,\varepsilon t}\,\big) \lesssim K_1\big(m\sqrt{\varepsilon\,|t|}\,\big) \lesssim e^{-m\sqrt{\varepsilon\,|t|}}\,.$$

As a consequence, the functions $\mathcal{T}_{m^2}^{\varepsilon}$ and P^{ε} decay exponentially in lightlike directions as

$$e^{-m\sqrt{\varepsilon T}}\,. \tag{4.28}$$

This decay is very slow because the regularization length ε comes into play. Nevertheless, we get exponential decay on cosmological scales, as is verified as follows. Choosing m as the mass of the τ-lepton, ε as the Planck length and T according to (1.3), we obtain

$$m\varepsilon\,\sqrt{mT} \approx 10$$

and thus

$$m^2\,\varepsilon T \approx 10\,\sqrt{mT} \approx 4 \times 10^{21}\,.$$

As a consequence, the exponential factor (4.28) is extremely small and may be neglected. With this in mind, in the periodization we may disregard null directions. A more detailed estimate of the kernel of the fermionic projector in a strip around the light cone can be found in [31, appendix B].

5. Dark matter and dark energy in an FLRW universe

In the previous section we explained how smearing in the mass and periodizing flat Minkowski space gives rise to contributions to the Einstein equations which can be interpreted as dark matter and dark energy. We now explain how these findings can be extended to the more general and more realistic situation of a FRLW Universe. It turns out that, based on the formulation of the Dirac equation in FRLW spacetimes in [23], this generalization is rather straightforward. Following the conventions in this paper, we choose conformal coordinates and write the line element of a spatially homogeneous and isotropic space-time as

$$ds^2 = S(\tau)^2\Big(d\tau^2 - d\chi^2 - f(\chi)^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2)\Big). \tag{5.1}$$

Here τ is a time coordinate, $\varphi \in [0, 2\pi)$ and $\vartheta \in (0, \pi)$ are angular coordinates, and χ is a radial coordinate. In the three cases of a closed, open and flat Universe, the function f and the range of χ are given respectively by

$$\begin{cases} \text{closed universe:} & f(\chi) = \sin(\chi)\,, & \chi \in (0, \pi) \\ \text{open universe:} & f(\chi) = \sinh(\chi)\,, & \chi > 0 \\ \text{flat universe:} & f(\chi) = \chi\,, & \chi > 0\,. \end{cases}$$

In the Dirac equation, the spatial dependence can be separated, giving rise to a spectral parameter λ. The time dependence is described by the ordinary differential equation

$$\left[i\,\partial_\tau - Sm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 0\,. \tag{5.2}$$

Our mechanism for dark matter and dark energy is driven by the solutions whose kinetic energy is much larger than the rest mass. Therefore, the solutions can be described by the WKB approximation (for details see [23, section 3])

$$
\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}(\tau) = U(\tau)^{-1} \begin{pmatrix} c_1 \exp\left(-i \int^{\tau} \sqrt{m^2 S^2 + \lambda^2}\, d\tau\right) \\ c_2 \exp\left(i \int^{\tau} \sqrt{m^2 S^2 + \lambda^2}\, d\tau\right) \end{pmatrix},
\tag{5.3}
$$

where $U(\tau)$ is a unitary matrix which diagonalizes the matrix potential in (5.2), i.e.

$$
U \begin{pmatrix} Sm & -\lambda \\ -\lambda & -Sm \end{pmatrix} U^{-1} = \sqrt{m^2 S^2 + \lambda^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\tag{5.4}
$$

Next, it is convenient to transform from cosmological time τ to observer time t defined by

$$
dt = S(\tau)\, d\tau\,.
$$

Moreover, writing the phase in (5.3) as $\omega(t)\, dt$ we obtain the simple dispersion relation

$$
\omega(t)^2 - \frac{\lambda^2}{S(t)^2} = m^2\,.
$$

Therefore, we can interpret λ/S as spatial momentum. The only difference compared to the dispersion relation in Minkowski space is that the momentum is not a separation constant, but that it depends on the scale function. This also leads to a scale dependence of the frequency,

$$
\omega = \omega(t, \lambda, m) := -\sqrt{\frac{\lambda^2}{S(t)^2} + m^2}\,.
$$

Locally, the kernel of the fermionic projector in the FRLW spacetime can be written in analogy to (1.4) as

$$
P_m^{\varepsilon}(x,y) = \int_{-\infty}^{\infty} \frac{d^4 p}{(2\pi)^4}\, (p\!\!\!/ + m)\, \delta\left(\omega(t)^2 - \frac{\lambda(\vec{p})^2}{S(t)^2} - m^2\right) \theta(-p^0)\, e^{\varepsilon p^0} e^{-ip(x-y)}
\tag{5.5}
$$

(for details see [23, section 5]). Therefore, setting

$$
p^2 := \omega(t)^2 - \frac{\lambda^2}{S(t)^2}\,,
$$

the mass smearing can be used exactly as described in section 4.1. The only change is that the formula for δm (see (4.3) and (4.4)) needs to be modified to

$$
2\pi = \int_0^T \left(\omega\big(t, \lambda, m_0 + \delta n(\lambda) + \delta m(\lambda)\big) - \omega\big(t, \lambda, m_0 + \delta n(\lambda)\big) \right) dt
$$

and thus

$$
\delta m(\lambda) = \frac{2\pi}{\int_0^T \frac{m_0}{\sqrt{\frac{\lambda^2}{S(t)^2} + m_0^2}}\, dt} + \mathcal{O}\big((m_0 T)^{-2}\big)\,.
$$

Here $t = 0$ is the observer time of the big bang. The parameter T is the time from the big bang until it reaches its end state. For simplicity of presentation, we usually refer to T as the 'life time of our Universe,' although, strictly speaking, it could also be smaller. This would

mean that from time T on, the state of the Universe remains essentially the same. If the energy is much larger than the mass, this formula simplifies to

$$m_0 \, \delta m(\lambda) = \frac{2\pi}{\int_0^T \frac{S(\lambda)}{\lambda} \, dt} + \mathcal{O}\big((m_0 T)^{-2}\big) \,. \tag{5.6}$$

The resulting contributions to dark matter and dark energy can be computed locally in a given reference frame simply by using the formula for $\delta m(\lambda)$ in (4.8) and computing the corresponding contributions to the energy-momentum tensor exactly as explained at the end of section 4.1. From (4.8) we draw the following conclusions:

(1) Since the contributions to the energy-momentum tensor as computed in proposition 4.2 are a consequence of the mass smearing, which is performed globally in spacetime, dark matter and dark energy are evenly distributed in spacetime. Thus they do not depend on the scale factor $S(\tau)$.
(2) Via (5.6), the overall magnitude of the effect does depend on the global geometry.

6. Outlook

In this paper we saw that, arranging a Universe with a correlated quantum state beginning and end state by slightly modifying the frequencies of the physical wave functions, one gets contributions to the EL equations of the causal action principle which can be interpreted as dark matter and dark energy with the scalings (1.1) and (1.2). Our method for modifying the frequencies was to smear out the mass parameter in the Dirac equation and to select the frequencies compatible with the boundary conditions at the initial and final time. It is an open question whether our 'smearing' of the mass parameter can be understood in analogy to a *generalized Uncertainty Principle* like for example the Karolyhazy Uncertainty Relation [38]. We conclude by pointing out that, although this method seems a natural and canonical starting point, we expect that, ultimately, additional effects need to be taken into account. For instance, other modifications of the Dirac equation were derived in [16], giving rise to a mechanism of baryogenesis. Moreover, it is conceivable that the dynamical behavior of the regularization might give rise to changes of the masses and coupling constants on cosmological scales (see [25] for some preliminary considerations in this direction) A systematic study of all these corrections and their mutual interplay seems an interesting project for future research.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgment

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References

[1] Link to web platform on causal fermion systems (available at: www.causal-fermion-system.com)
[2] Ashtekar A 2017 Implications of a positive cosmological constant for general relativity *Rep. Prog. Phys.* **80** 102901
[3] Bahcall J, Piran T and Weinberg S 2008 Dark matter in the universe *4th Jerusalem Winter School for Theoretical Physics* 2nd edn (Singapore: World Scientific)
[4] Bousso R 2002 The holographic principle *Rev. Mod. Phys.* **74** 825–74
[5] Bousso R, Freivogel B, Leichenauer S and Rosenhaus V 2011 Eternal inflation predicts that time will end *Phys. Rev.* D **83** 023525
[6] Carroll S M 2001 The cosmological constant *Living Rev. Relativ.* **4** 2001–1
[7] Planck Collaboration 2020 Planck 2018 results. VI. cosmological parameters *Astron. Astrophys.* **641** A6
[8] Ellis G F R, Maartens R and MacCallum M A H 2013 *Relativistic Cosmology* (Cambridge: Cambridge University Press) 4th printing of the 2012 original
[9] Faraggi A E and Matone M 2020 The geometrical origin of dark energy *Eur. Phys. J.* C **80** 1094
[10] Finster F 2010 Causal variational principles on measure spaces *J. Reine Angew. Math.* **646** 141–94
[11] Finster F 2016 The continuum limit of causal fermion systems *Fundamental Theories of Physics* vol 186 (Berlin: Springer)
[12] Finster F 2018 Causal fermion systems: a primer for Lorentzian geometers *J. Phys.: Conf. Ser.* **968** 012004
[13] Finster F 2023 Causal fermion systems: classical gravity and beyond, *The Sixteenth Marcel Grossmann Meeting*, ed R Ruffini and G Vereshchagin (Singapore: World Scientific) pp 661–78
[14] Finster F, Frankl M and Langer C 2023 The homogeneous causal action principle on a compact domain in momentum space *Adv. Calc. Var.* (https://doi.org/10.1515/acv-2022-0038)
[15] Finster F and Grotz A 2012 A Lorentzian quantum geometry *Adv. Theor. Math. Phys.* **16** 1197–290
[16] Finster F, Jokel M and Paganini C F 2022 A mechanism of baryogenesis for causal fermion systems *Class. Quantum Grav.* **39** 165005
[17] Finster F, Kamran N and Oppio M 2021 The linear dynamics of wave functions in causal fermion systems *J. Differ. Equ.* **293** 115–87
[18] Finster F, Kindermann S and Treude J-H An introductory course on causal fermion systems (in preparation)
[19] Finster F and Kleiner J 2015 Causal fermion systems as a candidate for a unified physical theory *J. Phys.: Conf. Ser.* **626** 012020
[20] Finster F and Kleiner J 2016 Noether-like theorems for causal variational principles *Calc. Var.* **55** 41
[21] Finster F and Kleiner J 2017 A Hamiltonian formulation of causal variational principles *Calc. Var.* **56** 33
[22] Finster F and Kraus M 2020 The regularized Hadamard expansion *J. Math. Anal. Appl.* **491** 124340
[23] Finster F and Reintjes M 2009 The Dirac equation and the normalization of its solutions in a closed Friedmann-Robertson-Walker universe *Class. Quantum Grav.* **26** 105021
[24] Finster F and Reintjes M 2015 A non-perturbative construction of the fermionic projector on globally hyperbolic manifolds I – Space-times of finite lifetime *Adv. Theor. Math. Phys.* **19** 761–803
[25] Finster F and Röken C 2016 Dynamical gravitational coupling as a modified theory of general relativity (arXiv:1604.03872 [gr-qc])
[26] Finster F, Smoller J and Yau S-T 1999 Particlelike solutions of the Einstein-Dirac equations *Phys. Rev.* D **59** 104020
[27] Georgi H 2007 Unparticle physics *Phys. Rev. Lett.* **98** 221601
[28] Kiefer C 2012 Quantum Gravity *International Series of Monographs on Physics* vol 155 3rd edn (Oxford: Oxford University Press)
[29] Milgrom M 2015 Mond theory *Can. J. Phys.* **93** 107
[30] Olver F W J, Lozier D W, Boisvert R F and Clark C W 2010 *Digital Library of Mathematical Functions*, ed (Washington, DC: National Institute of Standards and Technology)
[31] Oppio M 2023 Hölder continuity of the integrated causal Lagrangian in Minkowski space *Adv. Theor. Math. Phys.* (to appear)
[32] Padmanabhan T 2008 Dark energy and gravity *Gen. Relativ. Gravit.* **40** 529–64
[33] Page D 2006 The lifetime of the universe *J. Korean Phys. Soc.* **49** 711–4
[34] Penrose R 2005 The road to reality *A Complete Guide to the Laws of the Universe* (New York: Alfred A. Knopf, Inc)

[35] Perlmutter S 1997 Cosmology from type Ia supernovae *Bull. Am. Astron. Soc.* **29** 1351
[36] Riess A *et al* 1998 Observational evidence from supernovae for an accelerating universe and a cosmological constant *Astron. J.* **116** 1009
[37] Singh T P 2020 Dark energy as a large scale quantum gravitational phenomenon *Mod. Phys. Lett. A* **35** 2050195
[38] Singh T P 2021 Quantum gravity, minimum length and holography *Pramana* **95** 1–7
[39] Susskind L 1995 The world as a hologram *J. Math. Phys.* **36** 6377–96
[40] Hooft G 't 1993 Dimensional reduction in quantum gravity *Conf. Proc.* C **930308** 284–96
[41] Trimble V 1987 Existence and nature of dark matter in the universe *Annu. Rev. Astron. Astrophys.* **25** 425
[42] Verlinde E 2017 Emergent gravity and the dark universe *SciPost Phys.* **2** 016
[43] Weinberg S 1989 The cosmological constant problem *Rev. Mod. Phys.* **61** 1–23
[44] Weinberg S 2008 *Cosmology* (Oxford: Oxford University Press)