VAN KAMPEN DIAGRAMS AND SMALL CANCELLATION THEORY

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ABSTRACT

Van Kampen Diagrams and Small Cancellation Theory

Kelsey Lowrey

Given a presentation $\langle A \mid R \rangle$ of G, the word problem asks whether there exists an algorithm to determine which words in the free group, $F(A)$, represent the identity in G. In this thesis, we study small cancellation theory, developed by Lyndon, Schupp, and Greendlinger in the mid-1960s, which contributed to the resurgence of geometric group theory. We investigate the connection between Van Kampen diagrams and the small cancellation hypotheses. Groups that have a presentation satisfying the small cancellation hypotheses $C'\left(\frac{1}{6}\right)$, or $C'\left(\frac{1}{4}\right)$ and $T(4)$ have a nice solution to the word problem known as Dehn's Algorithm.

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- Additionally, it is worth mentioning that mathematician Paul Schupp passed away earlier this year. We offer condolences and also thank him for his contributions and groundbreaking work in the field of geometric group theory.

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GROUP THEORY PRELIMINARIES

1.1 Introduction

The study of algebraic structures and their properties in addition to the relationships that exist between structures represents a central area of mathematics. One of the main goals in studying algebraic structures is to extract information from objects or sets in a convenient and efficient manner. Group theory plays a crucial role in mathematics. The origin of group theory is traced back to studying the following three areas of mathematics: the theory of algebraic equations, number theory, and geometry. Group theory involves studying sets that are associated with a single binary operation that follow certain axioms – we refer to these sets as groups. Groups occupy a fundamental position within branches of study such as geometry and topology. Many concepts of group theory arose through the study of geometry, but beginning in the 1920s the focus shifted to more abstract, combinatorial ways of thinking about groups. The geometric approach to groups reemerged during the 1960's when Greendlinger, Lyndon and Schupp began to study small cancellation theory.

Combinatorial group theory focuses on studying groups through the lens of group presentations. Geometric group theory contributes to the study of finitely generated groups by investigating the relationship between algebraic, topological, and geometric properties of spaces on which these groups act, utilizing diagrams as visual tools.

The concept of a Van Kampen diagram was introduced by Egbert van Kampen in 1933. It was not until the 1960s when small cancellation theory was established by

Greendlinger, Lyndon, and Schupp [4] that Van Kampen diagrams became a standard tool in group theory, becoming especially popular in geometric group theory. Small cancellation theory involves group presentations, defining relations, and Van Kampen diagrams. Small cancellation theory illuminates the connection between the geometry of surfaces and certain group theoretical properties, particularly the word which we will explore in this thesis. A primary question that arises from a group and its presentation is the *word problem* which asks whether two words represent the same group element.

1.2 Words

The following definitions will aid us in our understanding of the word problem.

Definition. An **alphabet** is a finite set A where its elements are denoted as

$$A = \{a_1, ..., a_n\}.$$

The set of formal inverses of elements of A is $A^{-1} = \{a_1^{-1}, ..., a_n^{-1}\}$.

From now on we let $A = \{a_1, ..., a_n\}$.

Definition. A **letter** in A is any symbol element of $A \cup A^{-1}$.

Definition. A **word** in A is a finite string of letters from $A \cup A^{-1}$. We allow the empty word ϵ as well.

Definition. If $\omega = a_1 \cdots a_n$, then the length of ω, denoted $|\omega|$ is n, the number of letters in ω.

The free monoid on A, denoted A^*, is the collection of all words in $A \cup A^{-1}$ under the operation of concatenation. For example, if $A = \{x, y\}$, then typical elements of A^* include ϵ, $x^2y^{-1}x, ..., y^5x^{-1}yx^3, ...$

Definition. A word ω is **reduced** if no subword of the form $a_i a_i^{-1}$ or $a_i^{-1}a_i$ occur. Such subwords are referred to as **cancelling pairs**.

Definition. A reduced word $\omega = a_1 \cdots a_n$ is called **cyclically reduced** if a_n is not the inverse of a_1.

Definition. If $u \in A^*$, then the insertion or deletion of a cancelling pair is an **elementary transformation** of u.

We say that two words u and v are equivalent and write $u \sim v$ if there exists a finite sequence $\omega_1, \omega_2, ..., \omega_n$ where $u = \omega_1$, $v = \omega_n$ and each ω_{i+1} is obtained from ω_i by an elementary transformation. This forms an equivalence relation on A^*.

1.3 Free Groups

Free groups are essential to the study of infinite groups. The word obtained by concatenating ω_1 and ω_2 is denoted $\omega_1 \cdot \omega_2$. For example, if $\omega_1 = ab$ and $\omega_2 = ba$, then $\omega_1 \cdot \omega_2 = abba$. Letting $[\omega]$ denote the equivalence class of ω under $\sim$, we define a binary operation on $A^*/\sim$ by $[\omega_1][\omega_2] = [\omega_1 \cdot \omega_2]$. It can be shown that this operation is well-defined and associative. Our identity element is $[\epsilon]$ where ϵ is the empty word. If $\omega = a_1 \cdots a_n$ is a word, then write $\omega^{-1} = a_n^{-1} \cdots a_1^{-1}$ (where $(a_i^{-1})^{-1} = a_i$). The inverse of $[\omega]$ is $[\omega^{-1}]$. It follows that $A^*/\sim$ is a group under this operation.

Definition. The group $F(A) = A^*/\sim$ is the **free group** with basis A.

Every element of $F(A)$ has a unique reduced representative [5] (Theorem 1.2). This allows us to suppress the "bracket" notation and view $F(A)$ as the set of reduced words of A.

1.4 Normal Closure

Definition. Given the free group $F(A)$, let $R \subseteq F(A)$. The **normal closure** of R is the smallest normal subgroup of $F(A)$ that contains R and is denoted $\langle\langle R\rangle\rangle$. In other words, it is the intersection of all normal subgroups of $F(A)$ containing R:

$$\langle\langle R\rangle\rangle = \bigcap_{N \trianglelefteq F(A)} \{N \mid R \subseteq N\}.$$

Theorem 1.1. $\langle\langle R\rangle\rangle = \{\prod w_i s_i w_i^{-1} : w_i \in F(A), s_i \in R \cup R^{-1}, 1 \leq i \leq n\}$.

Proof. Let $K = \{\prod w_i s_i w_i^{-1} : w_i \in F(A), s_i \in R \cup R^{-1}, 1 \leq i \leq n\}$.

First, we show that $\langle\langle R\rangle\rangle \subseteq K$.

Let $x, y \in K$. We write

$$x = (w_1 s_1 w_1^{-1})(w_2 s_2 w_2^{-1}) \cdots (w_m s_m w_m^{-1})$$

and

$$y = (v_1 t_1 v_1^{-1})(v_2 t_2 v_2^{-1}) \cdots (v_p t_p v_p^{-1})$$

where $w_i, v_i \in F(A)$ and $s_i, t_i \in R \cup R^{-1}$. Then,

$$xy = (w_1 s_1 w_1^{-1})(w_2 s_2 w_2^{-1})...(w_m s_m w_m^{-1})(v_1 t_1 v_1^{-1})(v_2 t_2 v_2^{-1})...(v_p t_p v_p^{-1}) \in K$$

and

$$x^{-1} = (w_m^{-1} s_m w_m)...(w_2^{-1} s_2 w_2)(w_1^{-1} s_1 w_1) \in K.$$

Thus, $K \leq F(A)$.

Given $u \in F(A)$, observe that

$$\begin{aligned} uxu^{-1} &= u(w_1 s_1 w_1^{-1})(w_2 s_2 w_2^{-1}) \cdots (w_m s_m w_m^{-1})u^{-1} \\ &= (uw_1 s_1 w_1^{-1} u^{-1})(uw_2 s_2 w_2^{-1} u^{-1}) \cdots (uw_m s_m w_m^{-1} u^{-1}) \in K. \end{aligned}$$

Therefore, $K \trianglelefteq F(A)$. Since $R \subseteq K$ and $\langle\langle R \rangle\rangle$ is the smallest normal subgroup containing R, we conclude that $\langle\langle R \rangle\rangle \subseteq K$.

Now, we show that $K \subseteq \langle\langle R \rangle\rangle$.
Let $s \in R \cup R^{-1}$. As $\langle\langle R \rangle\rangle$ is normal in $F(A)$, $wsw^{-1} \in \langle\langle R \rangle\rangle$ for all $w \in F(A)$.

If $x \in K$, then by the observation above,

$$x = (w_1 s_1 w_1^{-1})(w_2 s_2 w_2^{-1})...(w_m s_m w_m^{-1}) \in \langle\langle R \rangle\rangle.$$

Therefore, $K \subseteq \langle\langle R \rangle\rangle$. □

1.5 Presentations

For a proof of the following see [3].

Theorem 1.2 (Universal Mapping Property for Free Groups)**.** *Let $F = F(A)$ be free with basis A, and let G be any group. Given a function $f : A \to G$, there exists a unique homomorphism $h : F(A) \to G$ such that $h|_A = f$.*

As a consequence, we have:

Theorem 1.3. *Every group G is isomorphic to a quotient of the free group.*

Proof. Let A be a set of generators for G, and let $f : A \to G$ be the inclusion mapping. By Theorem 1.2, there exists a unique homomorphism $h : F(A) \to G$ such that $h(a) = f(a)$ for all $a \in A$. We show that h is onto. Note that $A \subseteq im(h)$. Let $g \in G$. Then there exist $a_1, ..., a_k \in A$ and $m_1, ..., m_k$ such that $g = a_1^{m_1} \cdots a_k^{m_k}$. Let $x_i \in F(A)$ be such that $h(x_i) = a_i$. Thus, we have that:

$$\begin{aligned} h\Big(\prod_{i=1}^{k} x_i^{m_i}\Big) &= \Big(\prod_{i=1}^{k} h_i(x_i)^{m_i}\Big) \\ &= \Big(\prod_{i=1}^{k} a_i^{m_i}\Big) \\ &= g \end{aligned}$$

As h is onto, we conclude that $G \cong {F(A)}/{ker(h)}$ $\square$

Definition. Let G be a group and let $h : F(A) \to G$ be the surjective map defined in the proof of Theorem 1.3. A subset $R \subseteq ker(h)$ is a set of **defining relations** if $\langle\langle R\rangle\rangle = ker(h)$.

By Theorem 1.1, if R is a set of defining relations then every element in the kernel can be expressed as a finite product of conjugates of elements of $R \cup R^{-1}$. In some sense, elements of R are the building blocks of elements of $F(A)$ that represent the identity in G.

A concise method of defining a group is by utilizing *generators* and *relations* which we refer to as the presentation of a group.

Definition. Let A be a set and $R \subseteq F(A)$. Then $\langle A \mid R\rangle$ is a **presentation** for G if $G \simeq {F(A)}/{\langle\langle R\rangle\rangle}$. Elements of A are **generators** and elements of R are **relators**.

Example 1. If $R = \{aba^{-1}b^{-1}\} \subseteq F(a,b)$, then it can be shown that

$$F(a,b)\big/\langle\langle R\rangle\rangle \cong \mathbb{Z} \times \mathbb{Z}.$$

Therefore, $\mathbb{Z} \times \mathbb{Z}$ has presentation $\langle a, b \mid aba^{-1}b^{-1}\rangle$.

Example 2. If $R = \{a^2, b^2, (ab)^4\} \subseteq F(a,b)$, then it can be shown that

$$F(a,b)\big/\langle\langle R\rangle\rangle \cong D_4.$$

Therefore, D_4 has presentation $\langle a, b \mid a^2, b^2, (ab)^4\rangle$.

Example 3. If $R = \{a^5\} \subseteq F(a)$, then it can be shown that

$$F(A)\big/\langle a^5\rangle \cong \mathbb{Z}\big/5\mathbb{Z} = \mathbb{Z}_5.$$

Therefore, $\mathbb{Z}_5$ has presentation $\langle a \mid a^5\rangle$.

Example 3 illustrates a general result provided in Theorem 1.4.

Theorem 1.4. *Let $F(a)$ be the free group with basis a. Then $F(a)\big/\langle\langle a^n\rangle\rangle \simeq \mathbb{Z}_n$ and so $\mathbb{Z}_n$ has presentation $\langle a \mid a^n\rangle$.*

Definition. A group G is said to be **finitely generated** if G has a presentation $\langle A \mid R\rangle$ where A is finite.

Definition. A group G is said to be **finitely presented** if there is a finite set of defining relations, R, and a finite set of generators, A.

Note that every group has a presentation; however, the list of generators or relations may not be finite.

If G has presentation $\langle A \mid R \rangle$ and $G = {}^{F(A)}\!/_{\langle\langle R \rangle\rangle}$, then the elements of G are the cosets $\omega\langle\langle R \rangle\rangle$ where $\omega \in F(A)$. Given $w \in F(A)$, let $\overline{w} = w\langle\langle R \rangle\rangle$. Therefore, while working in $\langle A \mid R \rangle$ we write $\overline{\omega} = \overline{v}$ when $\omega^{-1}v \in \langle\langle R \rangle\rangle$.

Example 4. There are often instances when the set of defining relations in a presentation can be condensed. Consider the presentation

$$\langle x, y, z \mid x^2y^2,\ xz^{-1}x,\ yz^2x^2 \rangle.$$

This set of defining relations conveys that

$$\begin{aligned} xz^{-1}x &= 1 \\ z^{-1} &= x^{-2} \\ z &= x^2. \end{aligned}$$

Thus, we can rewrite our presentation as

$$\langle x, y \mid x^2y^2, yx^6 \rangle.$$

Additionally,

$$\begin{aligned} yx^6 &= 1 \\ y &= x^{-6}. \end{aligned}$$

Rewriting our presentation again yields

$$\langle x \mid x^{-10} \rangle.$$

By Theorem 1.4, this is a presentation for $\mathbb{Z}_{10}$.

In general, if G has presentation $\langle A \mid R\rangle$, it can be very difficult to determine the structure of G. For example, if G has presentation $\langle a, b, c \mid a^2b^3, abc^2\rangle$, it is difficult to conclude anything about the structure of G.

1.6 Symmetrized Presentations

Let $G = \langle A \mid R\rangle$ be a group presentation where $R \subseteq F(A)$ is a set of reduced and cyclically reduced words in the free group $F(A)$.

Definition. The **symmetrization** of $R \subseteq F(A)$ is the set of all distinct cyclic permutations of the defining relators r and of their inverses r^{-1}. We will denote the symmetrized set of R by R_*.

Example 1. Suppose a group G is given by the presentation

$$\langle a, b \mid a^2b^3\rangle.$$

R_* is $\{a^2b^3,\ ab^3a,\ b^3a^2,\ b^2a^2b,\ ba^2b^2,\ b^{-3}a^{-2},\ b^{-2}a^{-2}b^{-1},\ b^{-1}a^{-2}b^{-2},\ a^{-2}b^{-3},\ a^{-1}b^{-3}a^{-1}\}$.

1.7 Word Problem

Around 1912 mathematician Max Dehn studied and worked on three foundational decision problems in combinatorial group theory [2]. The first is known as the *word problem* and was motivated by topological considerations. Given a presentation $\langle A \mid R\rangle$ for a group G, this problem asks whether there exists an algorithm to determine which words in $F(A)$ represent the identity in G. A variation of this problem is to determine whether two elements ω_1, $\omega_2 \in F(A)$ are equal in G. If such an algorithm exists, then we say that G has a solvable word problem. Our goal is to understand the *word problem* in G which means that given $w \in F(A)$ we want to decide if $\overline{w} = 1$ in G. From the discussion above, $\overline{w} = 1$ in G if and only if $w \in \langle\langle R \rangle\rangle$ if and only if $w = \prod w_i s_i w_i^{-1}$ for some $w_i \in F(A), s_i \in R_*$ where $1 \leq i \leq n$.

Novikov [7] and Boone [1] independently proved that: there exist finitely presented groups with unsolvable word problem.

1.8 Dehn's Algorithm

In addition to Max Dehn posing the *word problem* for groups in general, he provided an algorithm which solved the word problem for fundamental groups of closed orientable two-dimensional manifolds. We outline Dehn's Algorithm below:

Begin with a symmetrized group presentation $\langle A \mid R \rangle$ for G. Given a reduced word $\omega \in F(A)$, we construct a sequence of reduced words $\omega_0, \omega_1, \omega_2, ..., \omega_n$ where $\omega = \omega_0$ so that $\omega_n = \epsilon$ if and only if $\overline{\omega} = 1$ in G.

Start:

We are given $\omega_0 = \omega$. Suppose ω_j is constructed.

- If ω_j is the empty word

  then terminate the algorithm.

- Else check if w_j contains a subword v that is a subword of a defining relator $r = uv \in R$ such that $\mid v \mid > \dfrac{\mid r \mid}{2}$.

  - Else if false

    then terminate algorithm with output w_j.

– Else if true

then replace v by u^{-1} in w_j and reduce. Denote the resulting reduced word by ω_{j+1}. Continue to first bullet point of algorithm.

End

Note that we always have $\mid \omega_0 \mid > \mid \omega_1 \mid > \mid \omega_2 \mid >$ ... implying that this process must terminate in at most $\mid \omega \mid$ steps. Furthermore, all the words ω_j represent the same element of G as ω. Thus, if the process terminates with the empty word, then ω represents the identity element of G.

For a symmetrized presentation $\langle A \mid R \rangle$, we say that Dehn's algorithm solves the word problem in G when the following is true: for any reduced word ω in $F(A)$, we have $\overline{\omega} = 1$ if and only if applying Dehn's Algorithm to ω terminates in the empty word.

Example 1. Consider the presentation of $\langle a, b \mid aba^{-1}b^{-1} \rangle$ for $\mathbb{Z} \times \mathbb{Z}$.

Note that the corresponding symmetrized presentation is

$$\langle a, b \mid aba^{-1}b^{-1},\ ba^{-1}b^{-1}a,\ a^{-1}b^{-1}ab,\ b^{-1}aba^{-1},\ bab^{-1}a^{-1},\ ab^{-1}a^{-1}b,\ b^{-1}a^{-1}ba,\ a^{-1}bab^{-1} \rangle.$$

Let $\omega = \omega_0 = a^2b^{-1}a^{-1}ba^{-1}bab^{-1}aba^{-1}b^{-2}a^{-1}b$.

Now, we can run Dehn's Algorithm.

Note that ω_0 contains the subword $v = b^{-1}a^{-1}b$ where v is a subword of the defining relator $r = uv = b^{-1}a^{-1}ba$. Thus we replace v with $u^{-1} = a^{-1}$ in ω_0 and reduce.

This yields:

$$\begin{aligned} \omega_1 &= a^2a^{-1}a^{-1}bab^{-1}aba^{-1}b^{-2}a^{-1}b \\ &= bab^{-1}aba^{-1}b^{-2}a^{-1}b \qquad \text{by reducing.} \end{aligned}$$

Returning to the first step of the algorithm, we see that ω_1 is not the empty word so we continue to the next step. Note that ω_1 contains the subword $v = bab^{-1}$ where v is a subword of the defining relator $r = uv = bab^{-1}a^{-1}$. Thus we replace v with $u^{-1} = a$ in ω_1 and reduce.

This yields:

$$\begin{aligned} \omega_2 &= aaba^{-1}b^{-2}a^{-1}b \\ &= a^2ba^{-1}b^{-2}a^{-1}b \qquad \text{by reducing.} \end{aligned}$$

Returning to the first step of the algorithm, we see that ω_2 is not the empty word so we continue to the next step. Note that ω_2 contains the subword $v = ba^{-1}b^{-1}$ where v is a subword of the defining relator $r = uv = ba^{-1}b^{-1}a$. Thus we replace v with $u^{-1} = a^{-1}$ in ω_2 and reduce.

This yields:

$$\begin{aligned} \omega_3 &= a^2a^{-1}b^{-1}a^{-1}b \\ &= ab^{-1}a^{-1}b \qquad \text{by reducing.} \end{aligned}$$

Returning to the first step of the algorithm, we see that ω_3 is not the empty word so we continue to the next step. Note that ω_3 contains the subword $v = b^{-1}a^{-1}b$ where

v is a subword of the defining relator $r = uv = b^{-1}a^{-1}ba$. Thus we replace v with $u^{-1} = a^{-1}$ in ω_3 and reduce.

This yields:

$$\begin{aligned}\omega_4 &= aa^{-1} \\ &= \epsilon \qquad \text{by reducing.}\end{aligned}$$

Returning to the beginning of the algorithm, we see that ω_4 returned the empty word and so we terminate the algorithm. Hence, we conclude that ω does indeed represent the identity element in $\mathbb{Z} \times \mathbb{Z}$.

Although Dehn's Algorithm is quite efficient in general, there may be alternate algorithms for solving the word problem for a given group. For example, observe that since $a = (1, 0)$ and $b = (0, 1)$ generate $\mathbb{Z} \times \mathbb{Z}$ then the words built using $\{a, b, a^{-1}, b^{-1}\}$ that represent the identity are the words where the sum of the a-exponents and b-exponents are both zero.

Chapter 2

VAN KAMPEN DIAGRAMS

2.1 Constructing Diagrams

Motivated by the work of Dehn, Lyndon, Schupp and Greendlinger developed techniques of encoding algebraic information about elements of a group in a diagram. We will see that these diagrams play a vital role in studying the word problem in a group.

We denote the Euclidean plane as $\mathbb{E}^2$. If $S \subseteq \mathbb{E}^2$, then ∂S denotes the boundary of S; the topological closure of S is denoted $\overline{S}$.

Definition. A **vertex** is a point of $\mathbb{E}^2$.

Definition. An **edge** is a bounded subset of $\mathbb{E}^2$ homeomorphic to the open unit interval.

Definition. A **region** is a bounded set homeomorphic to the open unit disk.

Definition. A **map M** is a finite collection of vertices, edges, and regions which are pairwise disjoint and satisfy:

1. If e is an edge of M, there are vertices a and b (not necessarily distinct) in M such that $\overline{e} = e \cup \{a\} \cup \{b\}$.
2. The boundary, ∂D, of each region D of M is connected and there is a set of edges $e_1, ..., e_n$ in M such that $\partial D = \overline{e_1} \cup ... \cup \overline{e_n}$.

The boundary of M is denoted as ∂M. If e is an edge with $\overline{e} = e \cup \{a\} \cup \{b\}$, the vertices a and b are called the *endpoints* of e. We consider maps as oriented objects.

An edge can be traversed in either of two directions. If e is some oriented edge going from endpoint v_1 to endpoint v_2, then we call vertex v_1 the *initial vertex* and v_2 the *terminal vertex* of e. The oppositely oriented edge is the *inverse of* e, denoted e^{-1} and goes from v_2 to v_1.

Definition. A **path** is a sequence of oriented edges $e_1, ..., e_n$ such that the initial vertex of e_{i+1} is the terminal vertex of e_i where $1 \leq i \leq n-1$. We also allow for the empty path.

We will often denote the path that traverses ∂M by P.

Definition. A path is **reduced** if it does not contain a successive pair of edges of the form ee^{-1}.

Definition. If D is a region of M with some orientation, then any cycle of minimal length which includes all the oriented edges of ∂D is a **boundary cycle** of D.

Definition. A **diagram** over a group $F(A)$ is an oriented map M and a function ϕ assigning to each oriented edge e of M as a *label* an element $\phi(e)$ of $F(A)$ such that if e is an oriented edge of M and e^{-1} is the oppositely oriented edge, then $\phi(e^{-1}) = \phi(e)^{-1}$.

If α is a path in M, $\alpha = e_1 \cdots e_k$, we define $\phi(\alpha) = \phi(e_1) \cdots \phi(e_k)$. If D is a region of M, a *label* of D is an element $\phi(\alpha)$ for α a boundary cycle of D.

Definition. A diagram M is said to be **connected** if there is a path in M between every pair of vertices.

Definition. A diagram M is **simply connected** if any simple closed curve which lies entirely in M can be contracted to a single point in M (a curve is called simple if it has no self intersections).

Suppose that G has presentation $\langle A \mid R \rangle$. Recall that given $w \in F(A)$, then $\overline{w} = 1$ in G if and only if $w \in \langle\langle R \rangle\rangle$ if and only if $w = \prod w_i s_i w_i^{-1}$ for some $w_i \in F(A), s_i \in R_*$ where $1 \leq i \leq n$. For convenience sake, we use the following notation: $\omega = c_1 \cdots c_n$ where each $c_i = w_i s_i w_i^{-1}$.

Definition. Let $F(A)$ be a free group with a given basis A. For each $\omega = c_1 \cdots c_n$ with each c_i as described above, we shall associate a **Van Kampen diagram** M which will be an oriented map labeled by a function ϕ into $F(A)$ satisfying the following properties:

1. If e is an edge of M, $\phi(e) \neq 1$.
2. M is connected and simply connected, with a distinguished vertex O on ∂M. There is a boundary cycle $e_1 \cdots e_t$ of M beginning at O such that the product $\phi(e_1) \cdots \phi(e_t)$ is reduced and $\phi(e_1) \cdots \phi(e_t) = c_1 \cdots c_n$.
3. If D is any region of M and $e_1 \cdots e_j$ is any boundary cycle of D, then $\phi(e_1) \cdots \phi(e_j)$ is reduced and is a cyclically reduced conjugate of some c_i.

Suppose we have a diagram M where the boundary path P has label ω which is unreduced as in Figure 1. We reduce the label by "sewing-up" subpaths of P that are products of two consecutive oriented edges whose labels are inverses of each other. These sewing-up operations can be iterated. At some stages of the process an operation may transform a region D with some boundary path pq into a 2-sphere. This sphere is then discarded, along with the tail edge that connected the sphere to the rest of the diagram. Note that the diagram in Figure 1 is what we shall refer to as being a balloon diagram.

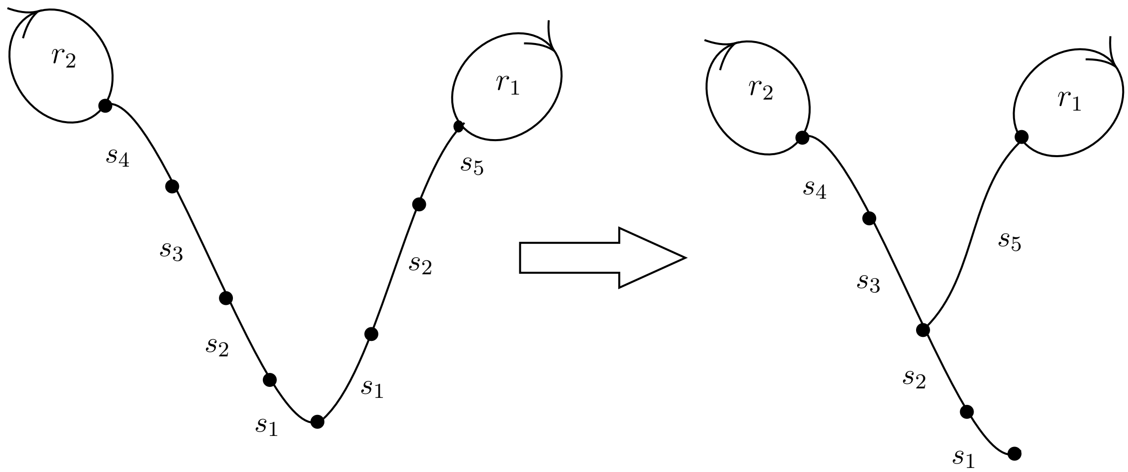


Figure 1: Left: Balloon diagram before reducing; Right: Balloon diagram after reducing

Definition. Let D be a region of M. We remove all vertices v of degree 2 (meaning the vertex has an an edge attached on both sides of it) on the boundary of D by **consolidating** v and the two edges incident with v into a new edge. Iterate this process as long as needed. If the boundaries of all regions are consolidated in this manner, a new region T results which we call a **tiling** of the region D.

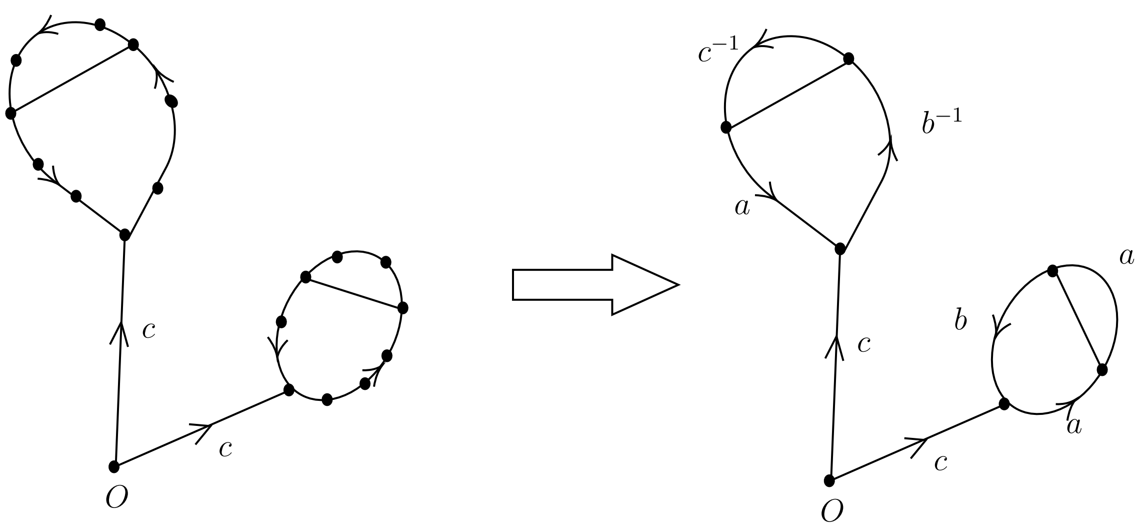


Figure 2: Right: Diagram M where the boundary of the regions are consolidated resulting in a tiling T.

2.2 Existence of a Van Kampen Diagram

We now show that the existence of a Van Kampen diagram for ω is equivalent to $\overline{\omega} = 1$ in G. Our proof is along the lines of Lyndon's *Combinatorial Group Theory* proofs of Theorem 1.1 and Lemma 1.2 [4].

Theorem 2.1. *A word $\omega \in F(A)$ is trivial in G if and only if there is a Van Kampen diagram for ω.*

Proof. Since $\overline{w} = 1$ in G, there exist $u_1, ..., u_n \in F(A)$ and $r_1, ..., r_n \in R_*$ such that

$$w = (u_1 r_2 u_1^{-1}) \cdots (u_n r_n u_n^{-1})$$

in $F(A)$. For each $1 \leq i \leq n$, write $c_i = u_i r_i u_i^{-1}$.

If $n = 0$, then M consists of a single vertex O. This satisfies properties (1)–(3) of the definition of a Van Kampen diagram. If $n = 1$, then $\omega = uru^{-1}$. We have two cases in this instance: $u = 1$ and $u \neq 1$. If $u = 1$, then M consists of the vertex O and a single edge labeled by r as indicated in Figure 3.

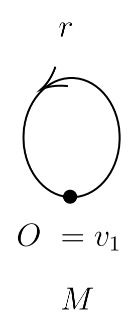


Figure 3. Base case when $u = 1$ consisting of a single edge and vertex.

If $u \neq 1$, then M consists of the balloon in Figure 4.

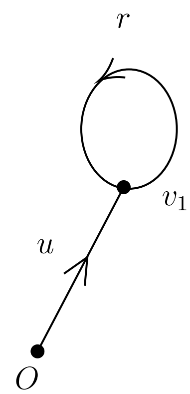


Figure 4. Base case when $u \neq 1$ consisting of a balloon.

One checks that the properties of Van Kampen diagrams are satisfied in both cases. For $n > 1$, for each $1 \leq i \leq n$ there is a corresponding balloon diagram M_i. We build M' by arranging $M_1, ..., M_n$ in order around a common base vertex O.

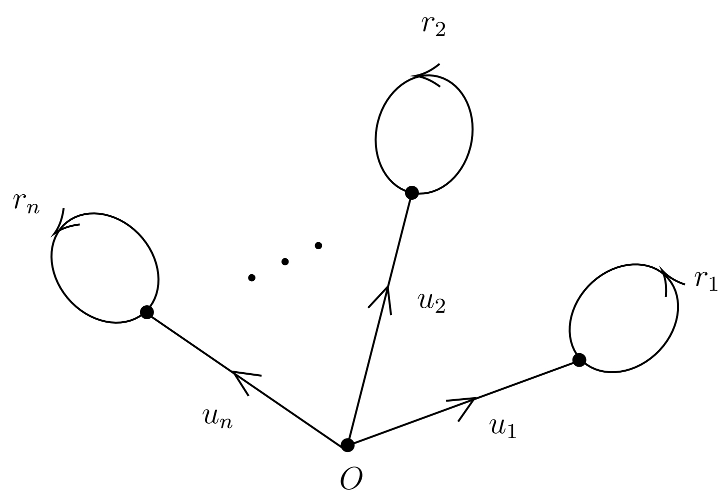


Figure 5. First stage of construction for $n > 1$.

Observe that M' satisfies properties (1) and (3) of Van Kampen diagrams. If the product $(u_1 r_1 u_1^{-1}) \cdots (u_n r_n u_n^{-1})$ is reduced, then M' satisfies properties (1)-(3) of Van Kampen diagrams as desired. However, suppose that $\phi(e_1) \cdots \phi(e_t)$ is not reduced. Let α be the boundary cycle of M' which begins at our base vertex O. We have

$\alpha = e_1 \cdots e_t$ and so it follows that

$$\phi(\alpha) = \phi(e_1) \cdots \phi(e_t) = c_1 \cdots c_n$$

in A^*.

Since we have assumed that property (2) does not hold for M', we have that α has two successive edges e and f such that $\phi(e)$ and $\phi(f)$ are inverses. Let e have initial and terminal vertices v_1 and v_2, respectively. Let f have initial and terminal vertices v_2 and v_3, respectively. Suppose at first that $v_1 \neq v_2$ and $v_1 \neq v_3$. We can fold the edge of e over onto the edge of f (whether or not $v_2 = v_3$) seen in Figure 6. This results in a diagram M'' with fewer edges.

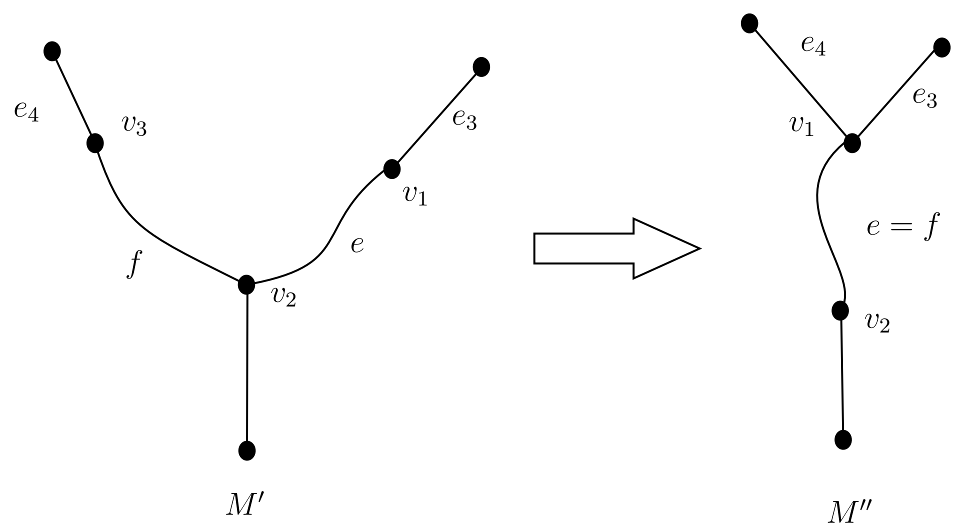


Figure 6. Folding edge of e over onto the edge of f.

We see that the boundary of our resulting M'' has fewer edges than α. If we started with v_3 being distinct from v_2 and v_1 we would proceed similarly as above.

If instead we have $v_1 = v_3$, then the closed edges of e and f would form a loop δ at the vertex v_1. We form M'' by deleting $\delta - v_1$ and M' interior to δ as seen in Figure 7.

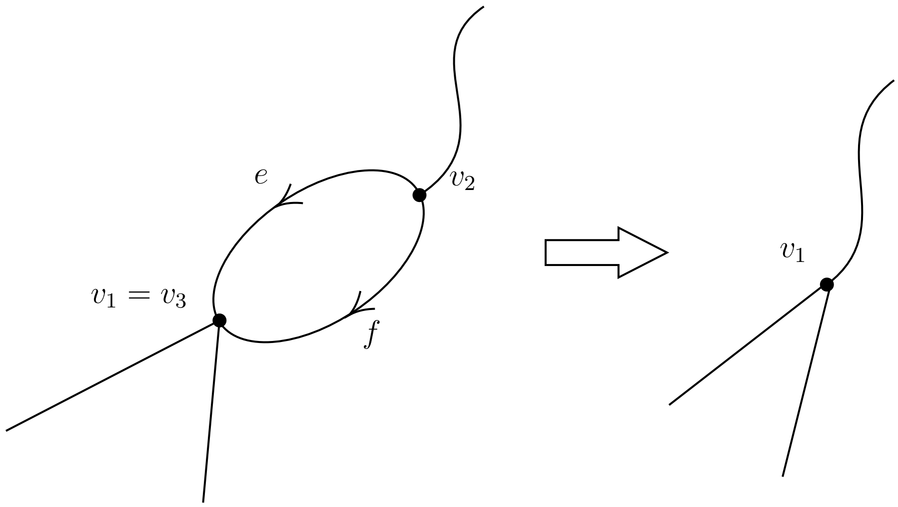
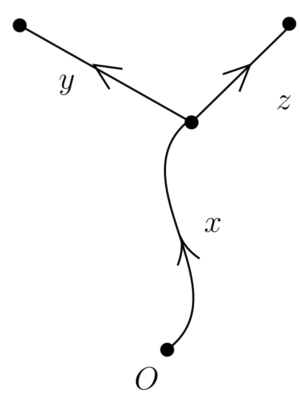


Figure 7. Deleting $\delta - v_1$ and M' interior to δ.

It follows that the boundary of M'' contains fewer edges than α. Further iteration of this process through the entire diagram would yield an M that satisfies all three properties of the Van Kampen diagram.

We proceed by induction on m where m denotes the number of regions we have in our Van Kampen diagram. If $m = 0$, there is nothing to prove as M is a tree with no loops. Therefore, we have $\phi(\alpha) = 1$ as cancellation reduces the word to the empty word.



Here we have that $xyy^{-1}zz^{-1}x^{-1} = 1$ and so $\overline{w} = 1$.

Figure 8. M is a tree with no loops.

If $m = 1$, then the boundary of M starting at O reads uru^{-1} where $u \in F(A)$ and $r \in R_*$ and so $\overline{w} = 1$.

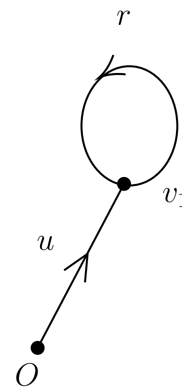


Figure 9. One region where M is a balloon.

We now assume the result holds for a Van Kampen diagram with k regions and M is a Van Kampen diagram with $k+1$ regions. Note that there must be some region D of M such that $\partial D \cap \partial M$ contains an edge. Form the Van Kampen diagram M' from M by deleting a single edge e in $\partial D \cap \partial M$ and the region D. Note that M' is still connected and simply connected.

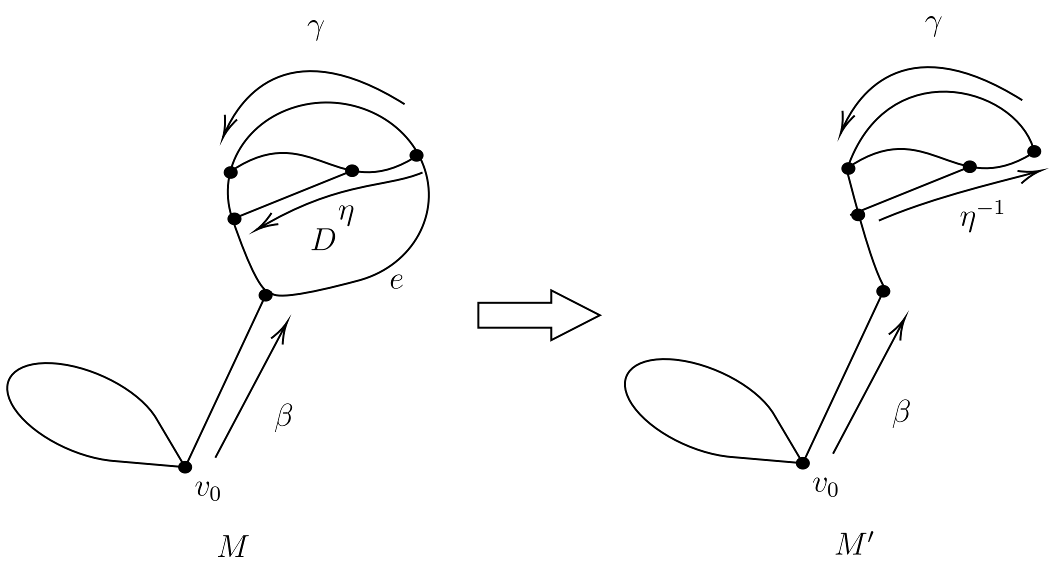


Figure 10. Deleting a single edge e in $\partial D \cap \partial M$ and the region D.

We can write $\alpha = \beta e \gamma$. Note that there is a boundary cycle $e\eta$ of D which begins at the edge e. In order to assign labels to our diagram let $\phi(\beta) = b$, $\phi(e) = z$, $\phi(\gamma) = c$, $\phi(\eta) = d$. Therefore, $\omega = \phi(\alpha) = bzc$. As the boundary cycle μ of M' beginning at v_0 is $\beta\eta^{-1}\gamma$, we have $\phi(\mu) = bd^{-1}c$. Since we now have k regions in M', we number them $D_1, ..., D_k$ and write $bd^{-1}c = (u_1 r_1 u_1^{-1}) \cdots (u_k r_k u_k^{-1})$ where r_i is the label of D_i. Observe that

$$\omega = bzc = (bd^{-1}c)(c^{-1}dzc)$$

where dz is the label of D. Letting $D_{k+1} = D$, $dz = r_{k+1}$, and $c^{-1} = u_{k+1}$ we have

$$\omega = bzc = (u_1 r_1 u_1^{-1}) \cdots (u_k r_k u_k^{-1})(u_{k+1} r_{k+1} u_{k+1}^{-1})$$

and so $\omega \in \langle\langle R \rangle\rangle$. Hence $\overline{\omega} = 1$. $\square$

2.3 Examples of Van Kampen Diagrams

We will now provide examples of (1) constructing a Van Kampen diagram for a word $\omega \in F(A)$ that is trivial in G and (2) showing that the existence of a Van Kampen diagram for ω implies that ω is trivial in G. We follow the process outlined in Theorem 2.1. Example 1 below can be found in Lyndon's text [4].

Example 1. We illustrate the construction of a Van Kampen diagram for

$$\omega = (ca^2bc^{-1})(cb^{-1}c^{-1}ac^{-1})(ca^{-1}c^2).$$

Note that ω is the identity in the group G with presentation $\langle a, b, c \mid ca^{-1}c^2,\ b^{-1}c^{-1}a,\ a^2b \rangle$.

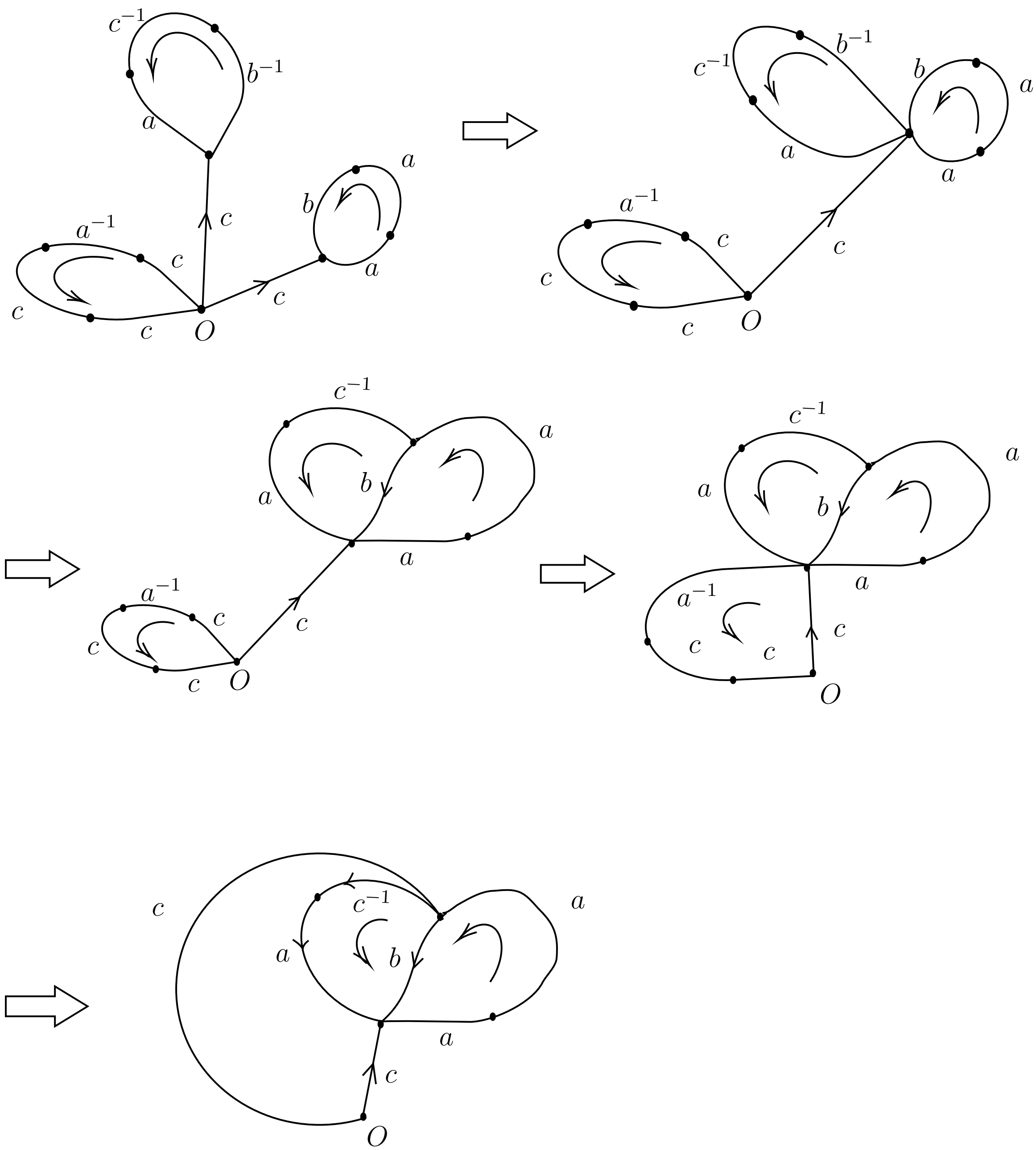


Figure 11: Region corresponding to $(bc^{-1}a)$ has no edges on the boundary of the final diagram.

We now provide an example following the process outlined in the proof of Theorem 2.1 to show the existence of a Van Kampen diagram implies that ω is trivial in G. We accomplish this by showing $\omega \in \langle\langle R \rangle\rangle$. This example can be found in Strebel's Appendix [8].

Example 2. The binary dihedral group Γ has the following presentation:

$$\langle a, b, c \mid a^2 = b^m = c^2 = abc \rangle.$$

Let $m = 6$.

We can eliminate c in this presentation:

$$\begin{aligned} abc &= c^2 && [\text{ apply } c^{-1} \text{ to the right of both sides }] \\ ab &= c. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} c^2 &= (ab)^2 && [\text{ plug in } ab \text{ for } c \text{ }] \\ (ab)^2 &= a^2 && [\text{ from the presentation as } c^2 = a^2 \text{ }] \\ abab &= a^2 \\ bab &= a && [\text{ apply } a^{-1} \text{ on the left of both sides }] \\ ab &= b^{-1}a && [\text{ apply } b^{-1} \text{ on the left of both sides }] \\ aba^{-1} &= b^{-1} && [\text{ apply } a^{-1} \text{ on the right of both sides }]. \end{aligned}$$

We now obtain the presentation:

$$\pi : \langle a, b \mid aba^{-1}b,\ b^6a^{-2} \rangle.$$

For Γ, write $r_1 = aba^{-1}b$ and $r_2 = b^6a^{-2}$. We will show that $b^{12} = 1$ in Γ.

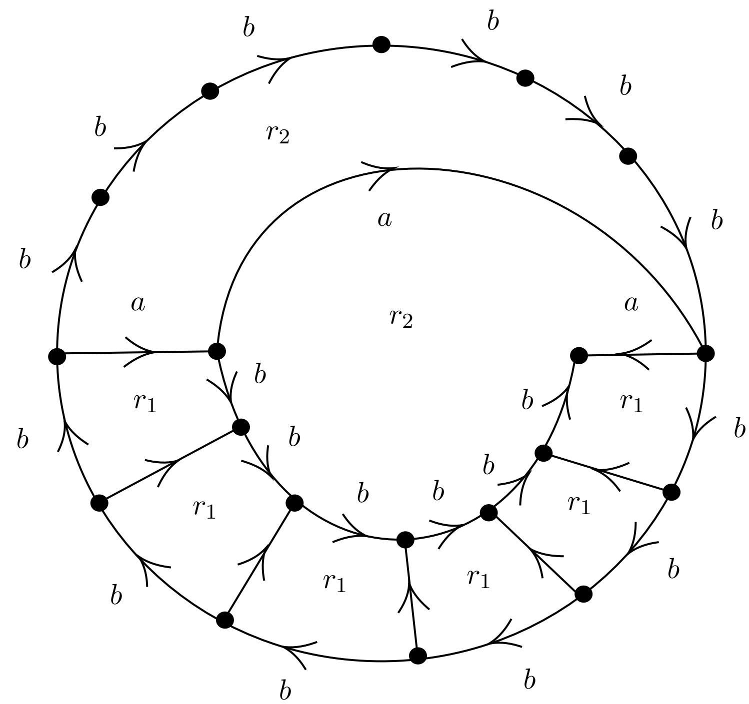


Figure 12. Van Kampen diagram, M.

Consider the Van Kampen diagram M in Figure 12. Starting at any exterior vertex we read b^{12} around the boundary path. Also note that the boundary labeled about any region reads r_1 or r_2. Let M' be the new diagram after we cut M along the portion of ∂D_1, labeled a^2 (see Figure 13).

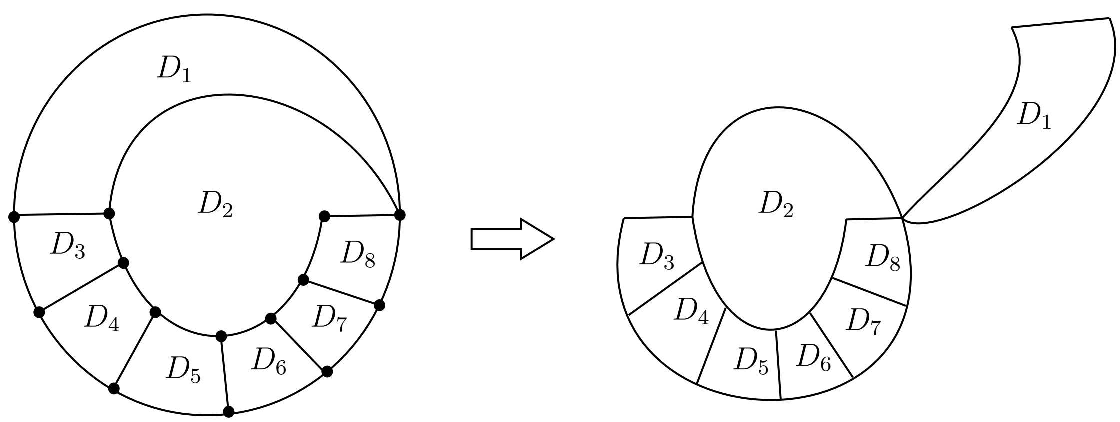


Figure 13. Cutting M along the portion of ∂D_1, labeled a^2.

- $\underline{D_1}$: Note that $b^{12} = b^6b^6$. We have that $\partial D_1 = a^{-2}b^6$ which is r_2. Therefore, $\partial M' = b^6a^2$. Thus, we have that

$$b^{12} = \underbrace{b^6a^2}_{\partial M'}[a^{-2}b^6].$$

- $\underline{D_2}$: Note that

$$\begin{aligned} b^6a^2 &= b^6aa \\ &= b^6a(b^6a^{-1})ab^{-6}a \qquad \text{where } \partial D_2 = ab^{-6}a = r_2, \quad \partial M' = b^6ab^6a^{-1}. \end{aligned}$$

Thus

$$b^{12} = \underbrace{(b^6ab^6a^{-1})}_{\partial M'}\underbrace{[ab^{-6}a]}_{\partial D_2}[a^{-2}b^6].$$

- $\underline{D_3}$: Note that

$$b^6ab^6a^{-1} = \underbrace{(b^5ab^5a^{-1})}_{\partial M'}\underbrace{ab^{-5}}_{\text{path to get to } D_3}\underbrace{(a^{-1}bab)}_{\partial D_3}\underbrace{b^5a^{-1}}_{\text{path back}}.$$

Note $\partial D_3 = r_1$. Thus,

$$b^{12} = (b^5ab^5a^{-1})(ab^{-5})[a^{-1}bab](b^5a^{-1})[ab^{-6}a][a^{-2}b^6].$$

- $\underline{D_4}$: Note that

$$b^5ab^5a^{-1} = \underbrace{(b^4ab^4a^{-1})}_{\partial M'}ab^{-4}\underbrace{(a^{-1}bab)}_{\partial D_4}b^4a^{-1}.$$

Note that $\partial D_4 = r_1$. Thus,

$$b^{12} = (b^4ab^4a^{-1})(ab^{-4})[a^{-1}bab](b^4a^{-1})(ab^{-5})[a^{-1}bab](b^5a^{-1})[ab^{-6}a][a^{-2}b^6].$$

- $\underline{D_5}$: Note that

$$b^4ab^4a^{-1} = \underbrace{[b^3ab^3a^{-1}]}_{\partial M'} ab^{-3} \underbrace{(a^{-1}bab)}_{\partial D_5} b^3a^{-1}.$$

Note $\partial D_5 = r_1$. Thus,

$$\begin{aligned} b^{12} =&(b^3ab^3a^{-1})(ab^{-3})[a^{-1}bab](b^3a^{-1})(ab^{-4})[a^{-1}bab](b^4a^{-1})(ab^{-5})[a^{-1}bab](b^5a^{-1}) \\ &[ab^{-6}a][a^{-2}b^6]. \end{aligned}$$

- $\underline{D_6}$: Note that

$$b^3ab^3a^{-1} = \underbrace{b^2ab^2a^{-1}}_{\partial M'} ab^{-2} \underbrace{(a^{-1}bab)}_{\partial D_6} b^2a^{-1}.$$

Note $\partial D_6 = r_1$. Thus,

$$\begin{aligned} b^{12} =&[b^2ab^2a^{-1}](ab^{-2})[a^{-1}bab](b^2a^{-1})(ab^{-3})[a^{-1}bab](b^3a^{-1})(ab^{-4})[a^{-1}bab](b^4a^{-1}) \\ &(ab^{-5})[a^{-1}bab](b^5a^{-1})[ab^{-6}a][a^{-2}b^6]. \end{aligned}$$

- $\underline{D_7}$: Note that

$$b^2ab^2a^{-1} = \underbrace{baba^{-1}}_{\partial M'} ab^{-1} \underbrace{a^{-1}bab}_{\partial D_7} ba^{-1}.$$

Note $\partial D_7 = r_1$. Thus,

$$\begin{aligned} b^{12} =&[baba^{-1}](ab^{-1})[a^{-1}bab](ba^{-1})(ab^{-2})[a^{-1}bab](b^2a^{-1})(ab^{-3})[a^{-1}bab] \\ &(b^3a^{-1})(ab^{-4})[a^{-1}bab](b^4a^{-1})(ab^{-5})[a^{-1}bab](b^5a^{-1})[ab^{-6}a][a^{-2}b^6]. \end{aligned}$$

- Note that $\partial M' = baba^{-1} = r_1 = \partial D_8$.

Overall, we obtain

$$\begin{aligned} b^{12} = &[baba^{-1}](ab^{-1})[a^{-1}bab](ba^{-1})(ab^{-2})[a^{-1}bab](b^2a^{-1})(ab^{-3})[a^{-1}bab] \\ &(b^3a^{-1})(ab^{-4})[a^{-1}bab](b^4a^{-1})(ab^{-5})[a^{-1}bab](b^5a^{-1})[ab^{-6}a][a^{-2}b^6] \end{aligned}$$

which is a product of conjugates; therefore, $\omega = b^{12} \in \langle\langle R \rangle\rangle$. Hence $b^{12} = 1$ in Γ with $m = 6$. Note that this product is depicted in the balloon diagram of Figure 14.

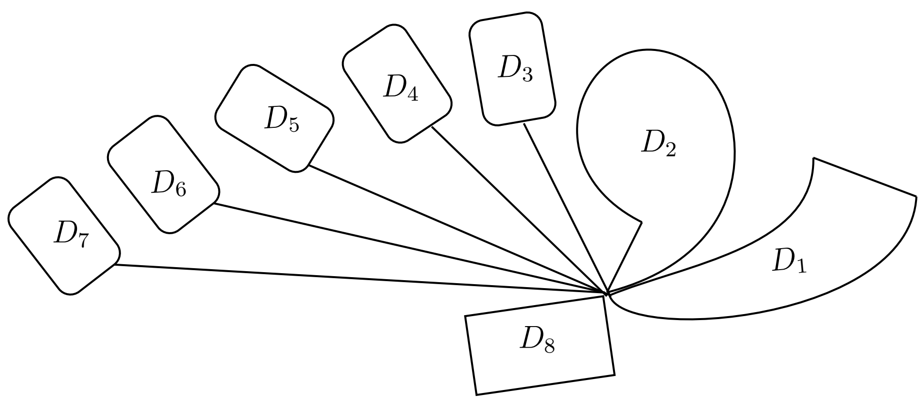


Figure 14. Iteration of forming M' by making each region into a balloon.

Theorem 2.1 is very useful as we are able to avoid the calculations above since b^{12} can be expressed as a Van Kampen diagram. We are then able to conclude that $b^{12} = 1$ in Γ.

Chapter 3

SMALL CANCELLATION THEORY

3.1 Small Cancellation Hypotheses

The methods involved in Dehn's Algorithm (Section 1.8) are geometric in nature. The first applications of those results to larger classes of groups utilized cancellation arguments of combinatorial group theory. These applications were separate from any geometric examinations of these groups. However, as previously mentioned, the geometric nature of Dehn's work resurfaced when Lyndon, Schupp, and Greendlinger initiated the geometric approach to small cancellation theory. Small cancellation theory studies groups with presentations satisfying some "small cancellation" conditions or hypotheses. These hypotheses provide us with a method to solve the world problem by looking at defining relations that have minimal overlap with each other. Groups satisfying sufficiently strong small cancellation hypotheses have a word problem which is solvable by Dehn's Algorithm. Throughout this section, let $R \subseteq F(A)$; and R_* denotes the symmetrization of R. Let $\langle A \mid R \rangle$ be a finite presentation for the group G, and let ω_0 be a non-empty reduced word in $F(A)$ such that $\overline{\omega} = 1$ in G. We denote a Van Kampen diagram over $\langle A \mid R \rangle$ as (M, ϕ, P) where M represents the simply connected diagram, ϕ is the labeling function for our edges, and P is the boundary path with label ω_0.

Definition. A word u is a **piece** relative to R if u is a common prefix of two distinct words of R_* (i.e. R_* contains two distinct elements of the form $r_1 = uv'$ and $r_2 = uv''$).

Since we typically work with a single symmetrized set at a time, it is common to omit the phrase "relative to R" and instead just call u a *piece*. Note, that u is canceled in the product $r_1^{-1}r_2$. Since R is symmetrized, a piece is a subword of an element of R which can be canceled by the multiplication of two non-inverse elements of R. The hypotheses of small cancellation theory make use of the fact that pieces are relatively small parts of elements of R.

Definition. Let λ be a positive real number. Then R satisfies **condition $\boldsymbol{C'(\lambda)}$** if the inequality

$$|u| < \lambda \cdot |r|$$

holds for every $r \in R_*$ and every prefix u of r that is a piece.

Definition. Let $p \in \mathbb{N}$. Then R satisfies **condition $\boldsymbol{C(p)}$** if no element of R_* is a product of fewer than p pieces.

Definition. Let $q \in \mathbb{N}$ such that $3 \leq h < q$. Suppose that $r_1, ..., r_h$ are elements of R_* with no successive elements r_i, r_{i+1} as an inverse pair. Then R is said to satisfy **condition $\boldsymbol{T(q)}$** if at least one of the products

$$r_1 r_2,\ ...,\ r_{h-1} r_h,\ r_h r_1$$

is reduced.

Note that every R satisfies $T(3)$. Typically used values of λ include $\frac{1}{4}$ and $\frac{1}{6}$ whose importance shall become apparent in Section 3.2 when we analyze the equation $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

A group G is referred to as a *fourth-group*, *sixth-group*, or *eigth-group* if it has a presentation $\langle A \mid R \rangle$ where R satisfies $C'\left(\frac{1}{4}\right)$, $C'\left(\frac{1}{6}\right)$, or $C'\left(\frac{1}{8}\right)$, respectively.

Lemma 3.1. $C'\left(\frac{1}{p}\right)$ implies $C(p+1)$.

Proof. Suppose R satisfies $C'\left(\frac{1}{p}\right)$ where $\lambda = \frac{1}{p}$. We assume without loss of generality that r is built of pieces. If $r \in R_*$, then $r = bc$ where b is a piece, and so $|b| < \frac{1}{p}|r|$. If $r = u_1 \cdots u_k$, assuming each u_i is a piece, then

$$|r| = |u_1| + ... + |u_k| < \frac{k}{p}|r|.$$

Therefore, $k > p$ as we need $\frac{k}{p} > 1$ in order to be strictly greater than $|r|$. Thus, $k \geq p+1$. It follows that no element of R_* is a product of fewer than $p+1$ pieces. $\square$

The geometric aspect of small cancellation theory is more beneficial than the combinatorial approach as the cancellation conditions $C(p)$ and $T(q)$ have specific geometric interpretations. In the geometric approach to small cancellation theory, the consequence of the cancellation hypotheses are studied in the context of Van Kampen diagrams. Let us look at some of the geometric outcomes of the small cancellation hypotheses.

Definition. An **extremal subdisc** D of M is a subdisc connected to the rest of M by an edge or by one vertex of D that is common to that of M.

Definition. A **boundary vertex** or **boundary edge** of M is a vertex or edge in ∂D.

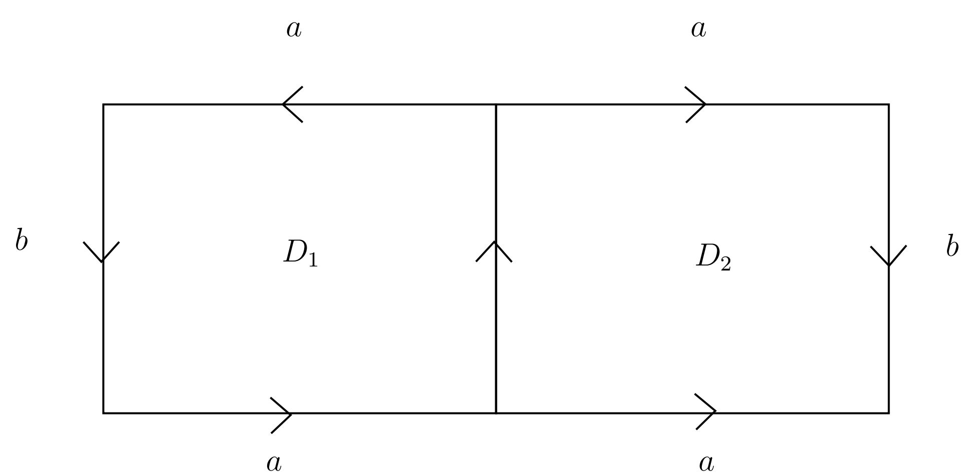
Definition. A **boundary face** of M is a face B of M such that $\partial B \cap \partial M \neq \emptyset$.

Note that if B is a boundary face of M, then $\partial B \cap \partial M$ does not need to contain an edge, as it may just include only one or more vertices. A boundary vertex, edge, or face may also be referred to as an exterior vertex, edge, or face.

Definition. A vertex, edge, or region of M which is not a boundary vertex, edge, or face is called **interior**.

Let (T, ϕ_T) be a labeled tiling of a subdisc of (M, ϕ, P) that has more than one face (see Definition 2.1). Each oriented interior edge of T corresponds to a particular edgepath P_{12} of M which runs on a common part of the boundaries of two faces B_1 and B_2. Let P_1 and P_2 be the boundary paths of B_1 and B_2 that start with P_{12}. If the labels of P_1 and P_2 are distinct, then the label of P_{12} is a piece relative to R. Subsequently, the label of each oriented interior edge of T is a piece relative to R, given that (M, ϕ, P) is reduced as defined below.

Definition. Let M be an arbitary diagram of $F(A)$. A diagram (M, ϕ, P) is **reduced** if it does not contain two faces B_1, B_2 and an oriented edge e running on part of $\partial B_1 \cap \partial B_2$ in such a way that the boundary paths P_1 and P_2 of B_1 and B_2 having e as their first edge, carry the same label.



$P_1 = eaba^{-1}$ and $P_2 = eaba^{-1}$.

Figure 15. An example of a diagram M which is not reduced.

The following theorem is important to the study of small cancellation groups. A proof is found in Lyndon's text [4], Lemma 2.1.

Theorem 3.2 (Existence of reduced diagrams for reduced words of a group presentation)**.** *Assume $\langle A \mid R\rangle$ is a presentation and ω_0 is a reduced word of $\langle A \mid R\rangle$. Then* *there exists a reduced diagram* (M_0, ϕ_0, P_0) *over* $\langle A \mid R\rangle$ *having* ω_0 *as the label of its boundary path* P_0.

Note that ω_0 is a product of reduced relators that is reduced. That is, ω_0 is the identity on the boundary path, but is not a relator itself; $\omega_0 = 1$ in G but $\omega_0 \neq 1$ in $F(A)$.

Now we will cover some notational conventions that we will be using. Let T be the tiling of a subdisc D, which we obtain by consolidating the edges of a reduced diagram for G.

1. If v is a vertex of a map M then $d(v)$, *the degree of* v, will denote the number of oriented edges having v as an initial vertex. Note, that if an edge has both of its endpoints at v we count this edge twice.

2. The *degree of a face* B, denoted as $d(B)$, is the number of edges making up ∂B.

3. We denote the number of interior edges of ∂B as $i(B)$. Note, we count an edge twice if it appears twice in a boundary cycle of B.

4. We denote the number of exterior edges of ∂B as $e(B)$.

5. We denote the number of edges making up ∂D as $d(D)$.

Theorem 3.3. *Let* R_* *be a symmetrized set of elements of a free group* $F(A)$*, and let* M *be a reduced diagram.*

1. *If* R *satisfies* $C(k)$*, then each face* D *of* M *such that* $\partial D \cap \partial M$ *does not contain an edge has* $d(D) \geq k$.

2. *If* R *satisfies* $T(m)$*, then each interior vertex* v *of* M *has* $d(v) \geq m$.

A proof of this theorem is found in Lyndon's work [4], Lemma 2.2.

Our geometric evaluation of small cancellation hypotheses leads us in the direction of needing to study maps where the degrees of the vertices and regions satisfy certain inequalities.

3.2 Small Cancellation Inequality

Let p and q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. It is well-known that the only positive integer solutions (p, q) to this equation are $(3, 6), (4, 4)$, and $(6, 3)$. This equation is motivated by the three regular tessellations of the plane: by triangles, squares, or hexagons. As the angle sum of a polygon with p sides is $\pi(p-2)$, this means that each interior angle of a regular polygon measures $\frac{\pi(p-2)}{p}$. The number of polygons meeting at a point is q. The product is

$$\frac{\pi(p-2)}{p} q = 2\pi.$$

We have two types of maps we will want to consider.

Definition. If M is a non-empty map such that each interior vertex of M has degree at least p and *all* regions of M have degree at least q, then M will be called a $[\boldsymbol{p}, \boldsymbol{q}]$ **map**.

Definition. If M is a non-empty map such that each interior vertex of M has degree at least p and each *interior* region of M has degree at least q, then M will be called a $(\boldsymbol{p}, \boldsymbol{q})$ **map**.

Some additional notational conventions we will be needing in the forthcoming proofs are covered next. Let M be an arbitrary map.

- $\sum$: Summation signs denote summations over vertices or regions of M.
- $\sum_v d(v)$: is the sum of the degrees of all the vertices of M.
- $\sum_B d(B)$: is the sum of the degrees of all the faces of M.
- $\sum^\bullet$: summation restricted to boundary vertices or faces.
- $\sum^\circ$: summation over interior vertices or faces.
- $\sum^\bullet d(v)$: sum of degrees of boundary vertices of M.
- $\sum^\circ d(B)$: sum of the degrees of the interior faces.
- V = number of vertices of M.
- E = number of unoriented edges of M.
- F = number of faces of M.

Now, we will derive a key inequality.

Consider a tiling T of a disc D. Recall Euler's Formula for D is

$$1 = V - E + F. \tag{1}$$

Here, the unbounded region is not considered.

Additionally, we need the following equations:

$$2E = \sum_v d(v) \tag{2}$$

$$2E = \sum_B d(B) + d(D). \tag{3}$$

We get the first equation by observing that the sum of the degrees of the vertices counts each edge twice. We get the second equation by double counting the interior edges with d(B) and counting the boundary edges once. Adding $d(D)$ to double count boundary edges gives us $2E$.

We want to take linear combinations of the above equations to eliminate E. Note that the following equation holds for the pairs $(6,3)$ and $(4,4)$:

$$p = 2\left(\frac{p}{q}\right) + 2. \tag{4}$$

Multiplying (1) by p results in:

$$p = pV - pE + pF. \tag{1'}$$

Multiplying (2) by $-\frac{p}{q}$ gives us:

$$-\frac{p}{q}(2E) = -\frac{p}{q}\sum_{v} d(v). \tag{2'}$$

Multiplying (3) by -1 leads to:

$$-2E = -\sum_{B} d(B) - d(D). \tag{3'}$$

By adding the left and right-hand sides of $(1')+(2')+(3')$ we obtain:

$$p - \frac{p}{q}(2E) - 2E = pV - pE + pF - \frac{p}{q}\sum_{v} d(v) - \sum_{B} d(B) - d(D)$$

We now add pE to both sides of the equation above to get:

$$p - \frac{p}{q}(2E) - 2E + pE = pV + pF - \frac{p}{q}\sum_v d(v) - \sum_B d(B) - d(D).$$

Now we substitute (4) for pE:

$$p - \frac{p}{q}(2E) - 2E + \left(2\left(\frac{p}{q}\right) + 2\right)E = pV + pF - \frac{p}{q}\sum_v d(v) - \sum_B d(B) - d(D).$$

Distributing E to get:

$$p - \frac{p}{q}(2E) - 2E + \frac{p}{q}(2E) + 2E = pV + pF - \frac{p}{q}\sum_v d(v) - \sum_B d(B) - d(D).$$

Now by cancellation we have:

$$p = \left(pV - \frac{p}{q}\sum_v d(v)\right) + \left(pF - \sum_B d(B) - d(D)\right). \tag{5}$$

Note that as V is the number of vertices, we can pull the term pV into the summation $\sum_v d(v)$. However, since this sum is multiplied by $\frac{p}{q}$, we need to compensate for this term. Since $\frac{p}{q} \cdot q = p$, this yields

$$\frac{p}{q}\sum_v \Big[q - d(v)\Big].$$

Similarly, as F is the number of faces, we can pull the term pF into the summation $\sum_B d(B)$. This yields

$$\sum_B \Big[p - d(B)\Big].$$

We can rewrite Equation (5) as

$$p = \frac{p}{q}\sum_v \Big[q - d(v)\Big] - d(D) + \sum_B \Big[p - d(B)\Big]. \tag{6}$$

Now, we will decompose Equation (6) considering our 3 different (p, q) pairs.

- $\underline{(p, q) = (6, 3)}$: Decompose the sum ranging over faces B in (6) according to the value of $e(B)$.

  $$6 = 2\sum_v \Big[3-d(v)\Big]-d(D)+\sum_{e(B)=0} \Big[6-d(B)\Big]+\sum_{e(B)=1} \Big[6-d(B)\Big]+\sum_{\substack{e(B)=k \\ k\geq 2}} \Big[6-d(B)\Big].$$

  Note that $d(B) = e(B) + i(B)$. Therefore

  $$6 - d(B) = 6 - (e(B) + i(B)).$$

  So, we have

  $$6 = 2\sum_v \Big[3-d(v)\Big]-d(D)+\sum_{e(B)=0} \Big[6-i(B)\Big]+\sum_{e(B)=1} \Big[5-i(B)\Big]+\sum_{\substack{e(B)=k \\ k\geq 2}} \Big[(6-k)-i(B)\Big].$$

  Additionally, we have that $d(D) = \sum_B e(B)$. By considering the values of $e(B)$, this yields:

  $$\begin{aligned} 6 = 2\sum_v &\Big[3 - d(v)\Big] \\ &+ \sum_{e(B)=0} \Big[6 - i(B)\Big] + \sum_{e(B)=1} \Big[4 - i(B)\Big] + \sum_{\substack{e(B)=k \\ k\geq 2}} \Big[(6 - 2k) - i(B)\Big]. \end{aligned} \tag{7}$$

- $\underline{(p,q) = (4,4)}$: Decompose the sum ranging over faces B in (6) according to the value of $e(B)$.

$$4 = \sum_v \Big[4 - d(v)\Big] - d(D) + \sum_{e(B)=0} \Big[4 - d(B)\Big] + \sum_{e(B)=1} \Big[4 - d(B)\Big] + \sum_{\substack{e(B)=k \\ k\geq 2}} \Big[4 - d(B)\Big].$$

Note that $d(B) = e(B) + i(B)$. Therefore

$$4 - d(B) = 4 - (e(B) + i(B)).$$

So, we have

$$\begin{aligned} 4 = & \sum_v \Big[4 - d(v)\Big] - d(D) \\ & + \sum_{e(B)=0} \Big[4 - i(B)\Big] + \sum_{e(B)=1} \Big[3 - i(B)\Big] + \sum_{\substack{e(B)=k \\ k\geq 2}} \Big[(4-k) - i(B)\Big]. \end{aligned}$$

We decompose the first sum from Equation (6) into a summation $\sum^{\bullet}$ over the exterior vertices and a summation $\sum^{\circ}$ over the interior vertices which results in

$$\begin{aligned} 4 = & \sum\nolimits_v^{\bullet} \Big[4 - d(v)\Big] + \sum\nolimits_v^{\circ} \Big[4 - d(v)\Big] - d(D) \\ & + \sum_{e(B)=0} \Big[4 - i(B)\Big] + \sum_{e(B)=1} \Big[3 - i(B)\Big] + \sum_{\substack{e(B)=k \\ k\geq 2}} \Big[(4-k) - i(B)\Big]. \end{aligned}$$

Note that the number of boundary edges of D are in one-to-one correspondence with the number of exterior vertices of D. Therefore, we use the summand $-d(D)$ to replace $\sum^{\bullet} \Big[4 - d(v)\Big]$ by $\sum^{\bullet} \Big[3 - d(v)\Big]$ which results in:

$$4 = \sum\nolimits_v^{\bullet} \Big[3 - d(v)\Big] + \sum\nolimits_v^{\circ} \Big[4 - d(v)\Big]$$
$$+ \sum_{e(B)=0} \Big[4 - i(B)\Big] + \sum_{e(B)=1} \Big[3 - i(B)\Big] + \sum_{\substack{e(B)=k \\ k\geq 2}} \Big[(4-k) - i(B)\Big]. \quad (8)$$

- $\underline{(p,q) = (3,6)}$: Decompose the sum ranging over faces B in (6) according to the value of $e(B)$.

$$3 = \frac{1}{2}\sum_v \Big[6 - d(v)\Big] - d(D)$$
$$+ \sum_{e(B)=0} \Big[3 - d(B)\Big] + \sum_{e(B)=1} \Big[3 - d(B)\Big] + \sum_{\substack{e(B)=k \\ k\geq 2}} \Big[3 - d(B)\Big].$$

Note that $d(B) = e(B) + i(B)$. Therefore

$$3 - d(B) = 3 - (e(B) + i(B)).$$

So, we have

$$3 = \frac{1}{2}\sum_v \Big[6 - d(v)\Big] - d(D)$$
$$+ \sum_{e(B)=0} \Big[3 - i(B)\Big] + \sum_{e(B)=1} \Big[2 - i(B)\Big] + \sum_{\substack{e(B)=k \\ k\geq 2}} \Big[(3-k) - i(B)\Big].$$

We decompose the first sum from Equation (6) into a summation $\sum^{\bullet}$ over the exterior vertices and a summation $\sum^{\circ}$ over the interior vertices.

$$\begin{aligned} 3 = \frac{1}{2}\sum\nolimits_v^{\bullet}\Big[6 - d(v)\Big] + \frac{1}{2}\sum\nolimits_v^{\circ}\Big[6 - d(v)\Big] - d(D) \\ + \sum_{e(B)=0}\Big[3 - i(B)\Big] + \sum_{e(B)=1}\Big[2 - i(B)\Big] + \sum_{\substack{e(B)=k \\ k\geq 2}}\Big[(3-k) - i(B)\Big]. \end{aligned}$$

Additionally, we have that $d(D) = -\frac{3}{2}d(D) + \frac{1}{2}d(D)$. Similar to the previous $(4,4)$ case, we use $-\frac{3}{2}d(D)$ to replace $\sum\nolimits_v^{\bullet}\Big[6 - d(v)\Big]$ by $\sum\nolimits_v^{\bullet}\Big[3 - d(v)\Big]$. Then we add $\frac{1}{2}d(D)$ to the sums over various values of $e(B)$. This yields:

$$\begin{aligned} 3 = \frac{1}{2}\sum\nolimits_v^{\bullet}\Big[3 - d(v)\Big] + \frac{1}{2}\sum\nolimits_v^{\circ}\Big[6 - d(v)\Big] \\ + \sum_{e(B)=0}\Big[3 - i(B)\Big] + \sum_{e(B)=1}\Big[\frac{5}{2} - i(B)\Big] + \sum_{\substack{e(B)=k \\ k\geq 2}}\Big[\Big(3 - \frac{1}{2}k\Big) - i(B)\Big]. \end{aligned} \tag{9}$$

Assuming that $d(v) \geq 3$ for every vertex v of T, we have that the summations over vertices in equations (7), (8), and (9) are not positive. Additionally, $i(B) \geq e(B)$ for every face T since T is a tiling and so the last sum in the equations above are not positive. By adding restrictions on the degrees of the interior vertices and faces, we obtain the following result.

Theorem 3.4 (Curvature Formula)**.** *Let (p,q) be one of the pairs $(6,3), (4,4)$ or $(3,6)$. Assume every vertex v of the tiling T of the region D has degree $d(v) \geq 3$, every interior vertex v has degree $d(v) \geq q$ and every interior face B of T has at least p edges. Then the following inequality is true:*

$$p \leq \sum_{e(B)=1}\Big[\frac{p}{q} + 2 - i(B)\Big]. \tag{1}$$

3.3 Applications of Small Cancellation Theory

The following result provides the vital link between Van Kampen diagrams and a solution to the word problem using Dehn's Algorithm.

Theorem 3.5. *Assume G has the presentation $\langle A \mid R\rangle$ satisfying either $C'\left(\frac{1}{6}\right)$, or $C'\left(\frac{1}{4}\right)$ and $T(4)$. If $\omega_0 \in F(A)$ is non-empty, reduced such that $\overline{\omega_0} = 1$ in G, then ω_0 contains a subword u which is a prefix of some $uv \in R_*$ with $|u| > |v|$.*

Proof. The hypothesis of Theorem 3.2 is satisfied and so there is a reduced Van Kampen diagram (M_0, ϕ_0, P_0) over $\langle A \mid R\rangle$ with ω_0 as the label of P_0. Note that, since the diagram is reduced, interior edges correspond to pieces. Since M_0 is simply connected, we consider several cases. It may be the case that M_0 is a disc with basepoint v_1 as seen below in Figure 16.

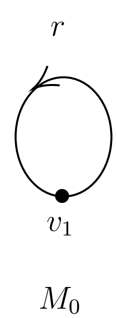


Figure 16. M_0 is a disc with basepoint v_1.

Otherwise, M_0 has an extremal subdisc D. If D is a single face by itself, then the subpath P_1 of P that starts and ends at v_1 has as its label a relator $u \in R_*$ as seen in Figure 17.

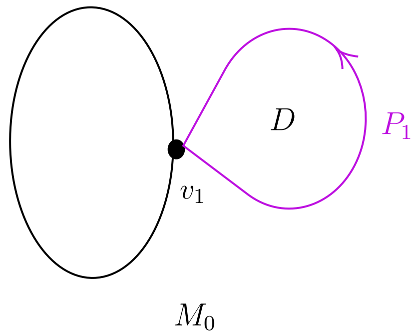


Figure 17. M_0 has an extremal subdisc D.

Otherwise, we remove the vertices of degree 2 in D by consolidating the edges to obtain a tiling T of D. By Lemma 3.1, $C'\left(\frac{1}{p}\right)$ implies $C(p+1)$. It follows immediately that $C(p)$ is also satisfied.

We specialize to the cases where $(p,q) = (6,3)$ or $(p,q) = (4,4)$. Observe that:

- $C'\left(\frac{1}{6}\right)$ implies $C(7)$ and $C(6)$. Every diagram satisfies $T(3)$.
  Thus, $(p,q) = (6,3)$.
- $C'\left(\frac{1}{4}\right)$ implies $C(5)$ and $C(4)$. We were given $T(4)$. Thus, $(p,q) = (4,4)$.

Therefore the tiling T satisfies inequality (1).

$$p \leq \sum_{e(B)=1} \left[\frac{p}{q} + 2 - i(B)\right].$$

Since $\frac{p}{q} = 2$ or $\frac{p}{q} = 1$ we have two cases to consider.

- For $(6,3)$, we have

$$\begin{aligned}\frac{p}{q} + 2 - i(B) &= 2 + 2 + i(B)\\ &= 4 - i(B).\end{aligned}$$

As every face of D has at least one interior edge, we have for all B that $i(B) \geq 1$.

- For $(4, 4)$, we have

$$\begin{aligned}\frac{p}{q} + 2 - i(B) &= 1 + 2 + i(B) \\ &= 3 - i(B).\end{aligned}$$

As every face of D has at least one interior edge, we have as well for all B that $i(B) \geq 1$.

As no summand on the right is larger than $\frac{p}{2}$, a face B contributes no more than $\frac{p}{2}$ to the sum. It follows that in order for our summation to be greater than or equal to p, there must exist *at least* two faces, which we denote as B and $\widehat{B}$ such that $e(B) = 1 = e(\widehat{B})$.

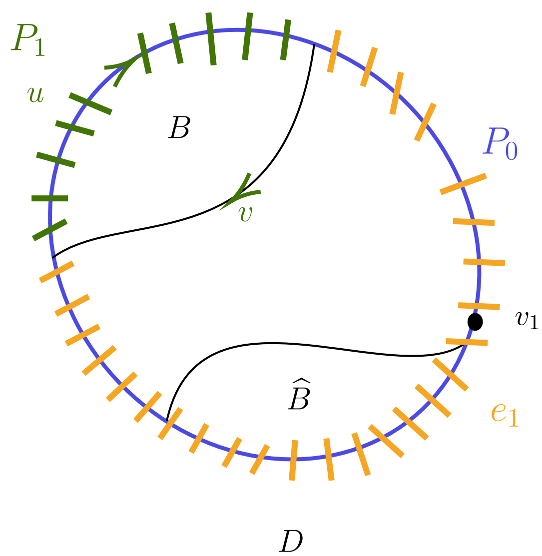


Figure 18. Existence of at least two faces B and $\widehat{B}$.

Note, v_1 can be an inner point of a sequence of edges that have been consolidated into a single edge e_1 of T. Assume without loss of generality that the exterior edge of B is distinct from e_1. This means the path P_0 (boundary path of M_0) has a connected subpath P_1 that traverses the exterior edge of B. The label u of the subpath P_1 is the prefix of the word $uv \in R_*$ that designates the boundary path of B starting with P_1. Consider the following two cases as we prepare to analyze v:

- (6,3):

$$\begin{aligned} 6 &\leq \sum_{e(B)=1} \left(\frac{6}{3} + 2 - i(B) \right) \\ &\leq \sum_{e(B)=1} (4 - i(B)). \end{aligned}$$

  For this inequality to hold, there must be *at least* two faces B and $\widehat{B}$ where $i(B) \leq 3$ and $i(\widehat{B}) \leq 3$.

- (4,4):

$$\begin{aligned} 4 &\leq \sum_{e(B)=1} \left(\frac{4}{4} + 2 - i(B) \right) \\ &\leq \sum_{e(B)=1} (3 - i(B)). \end{aligned}$$

  For this inequality to hold, there must be *at least* two faces B and $\widehat{B}$ where $i(B) \leq 2$ and $i(\widehat{B}) \leq 2$.

Note that $i(B)$ corresponds to the interior edge of B which is v. Therefore, as

$$\frac{p}{q} + 1 = \frac{6}{3} + 1 = 3$$

or

$$\frac{p}{q} + 1 = \frac{4}{4} + 1 = 2,$$

the suffix v is the product of at most $\dfrac{p}{q}+1$ pieces (where the pieces correspond to the labels of the interior edges of B). Suppose $b_1, ..., b_n$ are the pieces of v. Since R satisfies $C'\left(\dfrac{1}{p}\right)$, we know that for each piece b_i it holds that $|b_i| < \dfrac{1}{p}|uv|$.

Overall,

$$\begin{aligned}
|v| &= \sum_{i=1}^{n} |b_i| \\
&< \sum_{i=1}^{n} \frac{1}{p}|uv| \\
&\leq \left(\frac{p}{q}+1\right)\cdot\frac{1}{p}|uv| && \left[\text{as there are } \left(\frac{p}{q}+1\right) \text{ pieces}\right] \\
&= \frac{1}{2}|uv| && \left[\text{as } \frac{1}{p}+\frac{1}{q}=\frac{1}{2}\right].
\end{aligned}$$

Thus by the triangle inequality for the metric on groups,

$$2|v| < |uv| \leq |u| + |v|.$$

So, we have that $|u| > |v|$ as desired. $\square$

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