University of Massachusetts Amherst

# Fourth Order Dispersion in Nonlinear Media 

Georgios Tsolias<br>University of Massachusetts Amherst

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# FOURTH ORDER DISPERSION IN NONLINEAR MEDIA 

A Dissertation Presented

by

GEORGIOS A. TSOLIAS

Submitted to the Graduate School of the<br>University of Massachusetts Amherst in partial fulfillment<br>of the requirements for the degree of

## DOCTOR OF PHILOSOPHY

February 2023

Department of Mathematics and Statistics
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# FOURTH ORDER DISPERSION IN NONLINEAR MEDIA 

A Dissertation Presented

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To Vicky and Alexia

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## ABSTRACT

## FOURTH ORDER DISPERSION IN NONLINEAR MEDIA

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In recent years, there has been an explosion of interest in media bearing quartic dispersion. After the experimental realization of so-called pure-quartic solitons, a significant number of studies followed both for bright and for dark solitonic structures exploring the properties of not only quartic, but also setic, octic, decic etc. dispersion, but also examining the competition between, e.g., quadratic and quartic dispersion among others.

In the first chapter of this Thesis, we consider the interaction of solitary waves in a model involving the well-known $\phi^{4}$ Klein-Gordon theory, bearing both Laplacian
and biharmonic terms with different prefactors. As a result of the competition of the respective linear operators, we obtain three distinct cases as we vary the model parameters. In the first the biharmonic effect dominates, yielding an oscillatory inter-wave interaction; in the third the harmonic effect prevails yielding exponential interactions, while we find an intriguing linearly modulated exponential effect in the critical second case, separating the above two regimes. For each case, we calculate the force between the kink and antikink when initially separated with sufficient distance. Being able to write the acceleration as a function of the separation distance, and its corresponding ordinary differential equation, we test the corresponding predictions, finding very good agreement, where appropriate, with the corresponding partial differential equation results. Where the two findings differ, we explain the source of disparities. Finally, we offer a first glimpse of the interplay of harmonic and biharmonic effects on the results of kink-antikink collisions and the corresponding single- and multi-bounce windows.

In the next two Chapters, we explore the competition of quadratic and quartic dispersion in producing kink-like solitary waves in a model of the nonlinear Schrödinger type bearing cubic nonlinearity. We present 6 families of multikink solutions and explore their bifurcations as a prototypical parameter is varied, namely the strength of the quadratic dispersion. We reveal a rich bifurcation structure for the system, connecting two-kink states with ones involving 4-, as well as 6 -kinks. The stability of all of these states is explored. For each family, we discuss a "lower branch" adhering to the energy landscape of the 2-kink states (also discussed in the previous Chapter). We also, however, study in detail the "upper branches" bearing higher numbers of kinks. In addition to computing the stationary states and analyzing their stability at the PDE level, we develop an effective particle theory that is shown to be surprisingly efficient in capturing the kink equilibria and
normal (as well as unstable) modes. Finally, the results of the bifurcation analysis are corroborated with direct numerical simulations involving the excitation of the states in a targeted way in order to explore their instability-induced dynamics.

While the previous two studies were focused on the one-dimensional problem, in the fourth and final chapter, we explore a two-dimensional realm. More specifically, we provide a characterization of the ground states of a higher-dimensional quadratic-quartic model of the nonlinear Schrödinger class with a combination of a focusing biharmonic operator with either an isotropic or an anisotropic defocusing Laplacian operator (at the linear level) and power-law nonlinearity. Examining principally the prototypical example of dimension $d=2$, we find that instability arises beyond a certain threshold coefficient of the Laplacian between the cubic and quintic cases, while all solutions are stable for powers below the cubic. Above the quintic, and up to a critical nonlinearity exponent $p$, there exists a progressively narrowing range of stable frequencies. Finally, above the critical p all solutions are unstable. The picture is rather similar in the anisotropic case, with the difference that even before the cubic case, the numerical computations suggest an interval of unstable frequencies. Our analysis generalizes the relevant observations for arbitrary combinations of Laplacian prefactor b and nonlinearity power p .

We conclude the thesis with a summary of its main findings, as well as with an outlook towards interesting future problems.

## TABLE OF CONTENTS

Page
ACKNOWLEDGMENTS ..... v
ABSTRACT ..... vii
LIST OF FIGURES ..... xii
INTRODUCTION ..... 1
CHAPTER

1. KINK-ANTIKINK INTERACTION FORCES AND BOUND STATES IN A $\phi^{4}$ MODEL WITH QUADRATIC AND QUATRIC DISPER- SION ..... 10
1.1 Model Setup \& Kink-Antikink Tail Behavior ..... 11
1.2 The force between kink and antikink ..... 16
1.3 Comparison of ODE and PDE models ..... 20
1.4 Velocity in versus velocity out curves and soliton collisions ..... 28
2. DARK SOLITONS UNDER HIGHER ORDER DISPERSION ..... 37
2.1 Model \& Theoretical Background. ..... 37
2.2 Numerical Findings. ..... 40
3. KINK-ANTIKINK INTERACTION FORCES AND BOUND STATES IN A NONLINEAR SCHRODDINGER MODEL WITH QUADRATIC AND QUATRIC DISPERSION ..... 47
3.1 Model Setup and Analysis ..... 48
3.1.1 Stationary States and Spectral Stability ..... 48
3.1.2 Effective Particle Model ..... 51
3.2 Numerical Results and Comparison ..... 55
3.2.1 Family 0 ..... 58
3.2.2 Families 1, 3, 5 ..... 62
3.2.3 Families 2 and 4 ..... 72
4. MIXED DISPERSION NONLINEAR SCHRÖDINGER EQUATION IN HIGHER DIMENSIONS: THEORETICAL ANALYSIS AND NU- MERICAL COMPUTATIONS ..... 82
4.1 Mathematical Setup and main results ..... 83
4.2 Construction of the waves: Preliminaries ..... 85
4.2.1 Well-posedness of the constrained minimization prob- lems ..... 85
4.2.2 The Gagliardo-Nirenberg-Sobolev inequalities with mixed dispersion ..... 87
4.2.3 Proof of Proposition 4.4 ..... 89
4.2.4 The anisotropic case: Proof of Proposition 4.5 ..... 96
4.3 Completion of the proofs of Theorems 4.1 and 4.2 ..... 98
4.3.1 Existence of the waves - isotropic case ..... 98
4.3.2 Existence of the waves - anisotropic case ..... 103
4.3.3 Spectral stability of the normalized waves ..... 104
4.4 Numerical Computations ..... 106
5. CONCLUSIONS ..... 114
BIBLIOGRAPHY ..... 119

# LIST OF FIGURES 

Figure
Page
1.1. The panels show the behavior of the right-side tails of the single kink, $\phi_{K}(x)$, for the cases $\alpha=1, \alpha=4, \alpha=5$ and $\beta=1$ (respectively, left to right) by graphing $1-\phi_{K}(x)$ multiplied be $e^{k x}$, where $k$ (positive) is $\operatorname{Re}(\lambda)$ in the complex case, $\lambda$ in the critical case, and the smaller (in absolute value) $\lambda$ in the real case. Superimposed are the fitted curves, also multiplied by $e^{k x}$. In all cases the red solid curve is $e^{k x}\left(1-\phi_{K}\right)$. The blue dash-dot curve is the fitted equation multiplied by $e^{k x}$; the specific equations for each case are $y=1.205 \cos (0.8660(x-0.9890)), y=3.274(x-0.8606)$ and $y=$ $3.308-14.9 e^{-x}$ (right to left respectively)
1.2. Top left panel shows the steady kink solutions and top right panel shows the spectral plane $\left(\lambda_{r}, \lambda_{i}\right)$ of eigenvalues $\lambda=\lambda_{r}+i \lambda_{i}$ of the linearized operator about the corresponding steady kink solution for fixed $\beta=1$ and $\alpha=0.5$ (blue circles), $\alpha=1$ (red x's), $\alpha=5$ (green diamonds), $\alpha=10$ (magenta stars). The most right of the top right panel is the zoomed in version that shows the internal modes for all cases. The bottom left panel shows the internal mode $\omega_{0}$ versus $\alpha$ for fixed $\beta=1$. The bottom right panel shows the internal mode $\omega_{0}$ versus $\beta$ for fixed $\alpha=1$.
1.3. $\alpha=1, \beta=1$, Phase portrait of the ODE Equation (ODE) in comparison with Eq. (1.3) (PDE). The blue solid curve corresponds to $X(0)=8, \dot{X}(0)=-0.02$. The red dash-dotted curve: $X(0)=8$, $\dot{X}(0)=-0.00555$. The light blue closed orbit: $X(0)=3.3, \dot{X}(0)=$ 0 . The green curve corresponds to $X(0)=8, \dot{X}(0)=-0.00356$. The pink solid closed orbit: $X(0)=7.4, \dot{X}(0)=-0.0002$.
1.4. $\alpha=1, \beta=1$. Comparisons of the PDE contour plot of the displacement field $u(x ; t)$ and the ODE trajectory. (blue solid curve). Left: $x_{0}=3.3, v_{\text {in }}=0$. This corresponds to the closed orbit (light blue) in Fig. 1.3. Right: $x_{0}=8, v_{i n}=-0.02$. This corresponds to blue solid curve in Fig. 1.3.
1.5. The left panel shows the energy vs $x_{0}$ for $\alpha=1, \beta=1$. Blue curve is twice the potential function of the ODE for the complex case (given in Eq. (1.16)). The data points are the (normalized) potential energies of the steady states of the PDE at $x_{0}=1.825,3.56,5.38,7.19,9.01$ which are shown in the right panel. The need to multiply the potential function of the ODE by two when comparing ODE and PDE stems from the fact that the energy calculation using a steady state of the PDE involves two solitons - kink and antikink. The right panel presents the static, equilibrium solutions corresponding to $x_{0} \approx 1.825$ (orange dashed-dot curve), $x_{0} \approx 3.56$ (blue solid curve), $x_{0}=5.38$ (red dashed curve), $x_{0} \approx 7.19$ (green dashed-dot curve) and $x_{0} \approx 9.01$ (purple dotted curve). Note that the steady state for the PDE occurs at $x_{0} \approx 1.825$ but the fixed point of the ODE is at $x_{0} \approx 1.75$
1.6. The spectral plane $\left(\lambda_{r}, \lambda_{i}\right)$ of eigenvalues $\lambda=\lambda_{r}+i \lambda_{i}$ of oscillations around the equilibria at $x_{0}=1.825$ (1st row left), $x_{0}=3.56$ (1st row right), $x_{0}=5.38$ ( 2 nd row left), $x_{0}=7.19$ (2nd row right), $x_{0}=9.01$ (bottom), for $\alpha=1$ and $\beta=1$.
1.7. The left panel shows phase plots for the real $\lambda$ case of $\alpha=5$ and $\beta=1$ using initial conditions $X(0)=8$ and $\dot{X}(0)=-0.003593$. The right panel illustrates phase plots for the complex $\lambda$ case of $\alpha=1$ and $\beta=1$ using initial conditions $X(0)=8$ and $\dot{X}(0)=-0.35$. In both cases, the ODE is shown by the blue solid curve and the PDE by the green dash-dotted curve. Insets show at what points the ODE model diverges from the PDE model. The ODE trajectory in the right panel is stopped at the point when $X=0$ because it becomes physically unrealistic beyond that point.
1.8. Contour plots of the PDE corresponding to the same parameter values and initial conditions as in the corresponding panels of Fig. 1.7. The ODE trajectory is superimposed in blue.
1.9. The blue circles on both panels are obtained by the numerical simulation of Eq. (1.3) where left panel represents $v_{1, \text { crit }}$ vs $\alpha$ when $\beta=1$ and the right panel represents $v_{1, \text { crit }}$ vs $\beta$ when $\alpha=1$. The red solid curve on the left panel is obtained by applying the formula $v_{1, \text { crit }}^{\alpha, 1}=\sqrt{\alpha} v_{1, \text { crit }}^{1, \beta}$ where $\beta=\frac{1}{\alpha^{2}}$ to the numerically obtained data (blue circles) on the right. The red solid curve on the right panel is obtained by applying the formula $v_{1, \text { crit }}^{1, \beta}=\beta^{1 / 4} v_{1, \text { crit }}^{\alpha, 1}$ to the numerically obtained data (blue circles) on the left.
1.10. The left panel shows $v_{2, \text { crit }}$ vs $\alpha$ when $\beta=1$ and the right panel shows $v_{2, \text { crit }}$ vs $\beta$ when $\alpha=1$. The blue circles and the red solid curves were obtained as described in Fig. 1.9.
1.11. The top left panel shows $v_{\text {out }}$ vs $v_{\text {in }}$ when $\alpha=1$ and $\beta=1$ with $v_{2, \text { crit }} \approx 0.5902$. The top right panel is the zoom-in about the first two-bounce curve. The bottom left panel is the zoom-in about the two three-bounce windows right before the critical velocity $v_{2, \text { crit }}$. The bottom right panel is the zoom in about the leftmost threebounce window on the bottom left panel. In both top right and bottom panels, the tails and their oscillatory behaviors are shown. One-bounce windows in the figures are in solid black. Two-bounce windows are in blue and three bounce windows are in green. The gray solid line on the top left panel is when the kink-antikink repel each other elastically.
1.12. $v_{\text {out }}$ vs $v_{\text {in }}$ when $\alpha=5$ and $\beta=1$ with $v_{c} \approx 0.7295$. The one-bounce window is in solid black. Two-bounce windows are in blue and three-bounce windows are in green.
1.13. Transition from dominant quartic progressively closer to dominant harmonic behavior, by changing $\alpha$ from 2 (left) to 3 (middle) and finally the critical case of $\alpha=4$ (right panel).
1.14. Contour plots when $\alpha=2.05$ and $\beta=1$ with $v_{c} \approx 0.6222$ for $X(0)=$ 10. Left panel is when $v_{\text {in }}=0.622$ and right panel is when $v_{\text {in }}=$ 0.621 .
2.1. (a) Classification of CW solution with $\beta_{2}$ and $\mu$; (b) modulational instability spectrum of CW for $\beta_{2}=-0.2$, with most unstable wavenumber $k_{m}=1.1$ corresponding to $\Omega=0.77 i$ (dot); (c) instability dynamics of CW for $\beta_{2}=-0.2$; and (d) dependence of spectral intensity $|\tilde{\psi}|^{2}$ on the evolution variable $z$ at wavenumber $k_{m}$ for results in panel (c), with slope (dashed line) giving $\Omega=0.77$ i, agreeing very well with the prediction.
2.2. Pure quartic dark soliton stationary solutions $\left(\beta_{2}=0, \mu=5\right)$ : (a) a well separated pair of dark solitons; (b) a mixed solution connecting dark solitons to an oscillation about 0 .
2.3. (a) Bifurcation diagram for the lowest-order dark soliton families with the labels corresponding to solutions shown in the lower panels. (b) and (c) Solutions corresponding to family 0 , ultimately connecting to a plane wave as $\beta_{2}$ decreases; (d)-(g) solutions from family 1 ; (h)-(k) solutions from family 2 . Left and right panels correspond to $\beta_{2}=-1, \beta_{2}=1$, respectively. Unstable/stable solutions shown with dashed/solid lines respectively. In all cases $\mu=5$.
2.4. Numerical propagation of dark soliton solutions from different families at $\beta_{2}=0$ : (a) family 1 is unstable to fission; (b) family 2 is stable for $\beta_{2}>0$; (c) family 2 upper branch, all upper branch composite solutions are unstable; (d) solution shown in Fig. 2.2(b), it has a very weak oscillatory instability.
2.5. Dark soliton generation from intensity notches in a CW background: (a) pure quartic dark soliton bound state with $\beta_{2}=0$; (b) repulsive dark soliton pair ( $\beta_{2}=0.5$ ); (c) dark soliton pair in the presence of modulational instability ( $\beta_{2}=-0.2$ ); (d) dark soliton complex at $\beta_{2}=0$. (a)-(c) Initial condition $\psi(t, 0)=\sqrt{\mu}\left(1-\exp \left(-t^{2} / 2\right)\right)$; (d) initial condition $\psi(t, 0)=\sqrt{\mu}(1+(\tanh (t-5)-\tanh (t+5)) / 2)$, all with $\mu=5$.
3.1. Bifurcation diagrams of six families of solutions: family 0 (black), family 1 (red), family 2 (blue), family 3 (green), family 4 (purple), family 5 (gray). In all cases $\mu=5, \gamma=1$, and $\beta_{4}=1$.
3.2. (a) Bifurcation diagram ( $Q$ vs $\beta_{2}$ ), the corresponding steady state solutions and spectra for Family 0, presented for a sequence of values of the quadratic dispersion parameter $\beta_{2}$. (b) the same bifurcation diagram as (a) but zoomed in about the intersection of the upper and the lower curves.
3.3. PDE and ODE initial conditions and dynamics for family 0 (upper branch only for PDE). The upper figure of (a) shows the plots of $\|u\|^{2}$ (black), $\left\|u+v_{1}\right\|^{2}$ (red), and $\left\|u-v_{1}\right\|^{2}$ (blue) for $\beta_{2}=0.5$, where $v_{1}$ is the eigenfunction corresponding to the only real PDE eigenvalue of 0.2231 . The lower left panel of (a) is the contour plot that results from using $\left\|u+0.01 v_{1}\right\|^{2}$ as the initial condition, and the lower right panel of $(\mathrm{a})$ is the contour plot that results from using $\left\|u-0.01 v_{1}\right\|^{2}$ as the initial condition. (b) gives the ODE values for the soliton positions (left of vertical line is the lower branch, right of the vertical line is the upper branch) and the ODE eigenvalues along the top (again lower branch left and upper branch right). Arrows on the points indicate the initial directions of the solitons (all directions would be reversed if $v_{1}$ is replaced by $-v_{1}$.)
3.4. Bifurcation diagram and the corresponding steady state solutions and spectrums for a) family 1, b) family 3 , c) family 5 .62
3.5. (a) Phase potraits for family 0 (dashed line) and family 1 (solid line) in the plane of $u-u_{x}$. Panel (a) shows a larger scale, while panels (b) and (c) manifest zooms near the right fixed point. One can see the resulting loops that are associated with the exponentially decaying in amplitude oscillatory tails connected with the saddle-spiral fixed point.
3.6. PDE and ODE initial conditions and dynamics for family 1 (upper branch only for PDE). The upper figures of (a), (b), (c) show the plots of $\|u\|^{2}$ (black), $\left\|u+v_{j}\right\|^{2}$ (red), and $\left\|u-v_{j}\right\|^{2}$ (blue) for $\beta_{2}=$ 0.5. $v_{j}$ is the eigenfunction corresponding to the PDE real eigenvalue $\lambda_{j}$, with (a) $\lambda_{1}: 0.3808$ (b) $\lambda_{2}: 0.2638$ and (c) $\lambda_{3}: 0.1263$. For each of (a), (b), (c), the lower left figure is the contour plot that results from using $\left\|u+0.01 v_{1}\right\|^{2}$ as the initial condition, and the lower right figure is the contour plot that results from using $\left\|u-0.01 v_{1}\right\|^{2}$ as the initial condition. (d) gives the ODE values for the soliton positions (left of the vertical line is the lower branch, right of the vertical line is the upper branch) and the ODE eigenvalues along the top (again lower branch shown left, and upper branch shown right). Arrows on the points indicate the initial directions of the solitons (all directions would be reversed if $v_{1}$ is replaced by $-v_{1}$.)
3.7. PDE and ODE initial conditions and dynamics for Family 3 (upper branch only for PDE). Similar to Figure 3.6, except for Family 3 instead of Family 1, with $\beta_{2}=0.5$ and PDE eigenvalues (a) $\lambda_{1}$ : 0.3083 and (b) $\lambda_{2}: 0.2104$.
3.8. PDE and ODE initial conditions and dynamics for Family 5 (upper branch only for PDE). Similar to Figure 3.6, except for Family 5 instead of Family 1, with $\beta_{2}=0.5$ and PDE eigenvalues (a) $\lambda_{1}$ : 0.3105 , (b) $\lambda_{2}: 0.2640$, and (c) $\lambda_{3}: 0.1509$.
3.9. PDE initial conditions and dynamics for lower branches of Families 1, 3, 5. Similar to Figure 3.6, except for lower branches instead of upper branches with (a) Family $1, \lambda_{1}=0.3012$, (b) Family 3, $\lambda_{1}=$ 0.0066 , (c) Family $5, \lambda_{1}=0.00014$. The only exception is the bottom right panel of (c). For the corresponding ODE initial conditions and dynamics see the bottom panels (left of the vertical line) in each of Figures 3.6, 3.7, and 3.8.
3.10. Bifurcation diagrams and the corresponding steady state solutions and spectra for (a) Family 2 (b) Family 4. The former are presented in the format shown previously of $Q$ vs. $\beta_{2}$. The latter indicate the eigenvalues and prescribe the motion of the solitary waves, in line with the desired eigenmode.
3.11. Inverse Participation Ratio plot for Family 1, upper branch. Numerical values shown are eigenvalues corresponding to the eigenvector whose IPR is calculated and plotted. Eigenvectors with index values smaller than shown do not contribute significant IPR values.
3.12. Dynamics corresponding to imaginary eigenvalues that are embedded in the continuous spectrum. In each case a small amount of an eigenvector with imaginary eigenvalue is added to the steady state, inducing an out-of-phase oscillation for a pair of solitons. The two panels in (a) represent out-of-phase oscillations for Families 0 (top figure, eigenvalue 2.0263) and 2 (bottom figure, eigenvalue 0.0446), both for the bottom branch. We show only the curve that represents the center of the soliton that appears on the positive side of the $x$ axis (and hence on top in the contour plots). All figures in (b) represent Family 1, top branch, with eigenvalue 2.0334. The three blue curves on the bottom again represent the motion of the center of each of the three solitons that appear on the positive side of the $x$-axis (corresponding to the top three solitons in the contour plot shown). These blue curves also appear superimposed on the contour plot, where due to scaling, the oscillations are not apparent.
3.13. Projection plots for $\beta_{2}=0.5$. In the top two rows, for a few selected solutions $u(x, t)$, we plot the scalar projection of $u(x, t)-u_{0}(x)$ in the direction of an eigenvector (as a function of time) using a semilog scale. For these solutions, the initial steady state $u_{0}(x)$ was slightly perturbed in the direction of said eigenvector. In each case the blue lines represent the projections, with the dotted blue lines representing positive perturbations, the dash-dot blue lines representing a negative perturbation; the red lines represent a least-squares straight line fit to the linear part of the blue curves. We observe a linear portion near the beginning of each plot, whose slope matches very closely with what is predicted by the corresponding eigenvalue. In all cases the slope of the projection curve matches the eigenvalue to two (for the smallest eigenvalues) or three decimal places. The cases are as follows. First row - steady state 3 (left) and steady state 1 (right), both lower branch (note the different time scales). Second row - steady state 4 , largest real eigenvalue (left - even eigenvector) and steady state 4 , second largest real eigenvalue (right - odd eigenvector - projections coincide). The figure in the third row shows how an initial (small) growth rate can transition to a larger growth rate (projection in blue). This figure corresponds to family 1, upper branch where the initial growth rate of 0.124 (fitted line in red) transitions to a growth rate of 0.381 (fitted line in black). Here $u_{0}(x)$ was perturbed in the direction of the eigenvector with eigenvalue 0.12634 and then projected onto the eigenvector with eigenvalue 0.38075 .
4.1. Two-parameter plane of the nonlinearity exponent parameter $p$ vs. the Laplacian prefactor $b$ (varying between 0 and 2 ); recall that the frequency $\omega$ is fixed to unity, while our computations are for dimension $d=2$. The figure shows the bifurcation loci separating spectrally stable solitons (under the curve) from unstable ones (above the curve). The right panel shows a blowup of the left one close to the edge point of $p=3$ and $b=2$.
4.2. Dependence of the squared $L^{2}$ norm, denoted by $P$, i.e., $P=\int_{\mathbf{R}^{2}}|u|^{2}$ for our computations, with respect to the frequency $\omega$ for different values of the nonlinearity exponent $p$, in the isotropic case for dimension $d=2$. These plots showcase the different stability regimes that can be found herein (see text for more details). The insets show the same graph over an expanded interval of frequencies, using a semi-logarithmic scale for the latter.
4.3. Several examples of the waveform of the solitary waves with $p=3$ in the isotropic case for different frequencies. We can observe how the solution profile changes from high $\omega$ to the linear limit of $\omega \rightarrow 0.25$. Notice the logarithmic scale of the colormap, and the (clear within that scale) zero-crossings of the solution. Figures for other values of $p$ are qualitatively similar.
4.4. Same as Fig. 4.1 but for the anisotropic case. . . . . . . . . . . . . . . 110
4.5. Same as Fig. 4.2 but for the anisotropic case and for different values of p. Again, a semi-logarithmic scale has been used for the frequencies. 112
4.6. Same as Fig. 4.3 but for the anisotropic case with $b=1$. Contrary to the isotropic case, the anisotropy reflects in the solution as it acquires, when approaching the linear limit $\omega \rightarrow 0.25$ a separable form in the $x$ and $y$ dependence with the nodal lines being uniform along direction $y$.

## INTRODUCTION

In this thesis we study forth order dispersion in nonlinear media. In $\phi^{4}$ KleinGordon models, we explore the existence problem of a single kink, the interaction of two such kinks, and collision simulations. In the nonlinear Schrödinger variant of the problem, we examine the existence problem of a single kink and of multikink states, and offer a systematic exploration of families of multi-kink states and their bifurcation diagrams, stability and effective theory. In higher dimensions, we investigate the dependence of the stability of the solutions on the interplay between the dimensionality and the nonlinearity of this model.

The study of nonlinear Klein-Gordon models is a topic that has a rich history. Many of the early developments on the subject have focused on the mathematically appealing theory of the inverse scattering transform and integrable systems $[1,2$, $3]$, such as the famous sine-Gordon equation [4, 5]. However, more recently, the intriguing features stemming from non-integrable dynamics have been at the center of numerous studies centered around, e.g., the $\phi^{4}$ model [6]. The latter has often been considered to be a prototypical system for phase transitions, ferroelectrics, and high-energy physics among other themes [4, 6]. Moreover, it has been a central point of both analytical and numerical explorations, involving kink interactions, collective coordinates, resonant dynamics (including with impurities) starting from the 1970's and extending over nearly 5 decades $[7,8,9,10,11,12,13,14,15,16,17,18]$ and even reaching to this day $[19,20,21]$; see also the recent recap of [22].

On the other hand, more recently, a diverse set of variants of the so-called nonlinear beam (or biharmonic) wave equation have been considered also; a collection of relevant examples includes, e.g., $[23,24,25,26]$. The corresponding models also span a diverse array of contexts, including, e.g., suspension bridges and the propagation of traveling waves therein. In nonlinear optics the engineering and realization of the so-called "pure-quartic solitons" (see more in the next section) in the lab has also substantially promoted the relevance of biharmonic models or, more generally, models that involve both regular quadratic and quartic dispersion.

While in the realm of nonlinear optics the most canonical setting to consider is a generalized nonlinear Schrödinger (NLS), in the present work we opt to consider the slightly simpler, yet highly informative, setting of a corresponding Klein-Gordon model. The rationale behind the latter choice involves the fact that the two models share the same existence properties, at least in one spatial dimension, yet the nature of the real field-theory renders the analytical calculations somewhat simpler, especially as regards the stability and dynamical implications of the inter-wave interactions. Given the strong connection between the two models, including via multiple scale expansions [33] (and the customary emergence in nonlinear optics of the NLS model as a paraxial approximation of the Klein-Gordon one [33]), the analysis of the simpler and prototypical nonlinear Klein-Gordon model can be insightful towards the existence, asymptotic and interaction properties of the solitary waves of the generalized NLS setting.

In the present work we formulate the existence problem of a single kink in a model incorporating quadratic and quartic dispersion in the presence of a $\phi^{4}$ potential (this part will be entirely analogous to the corresponding generalized NLS case). Subsequently, we explore the interaction of two such waves and identify their pairwise interaction force and how it depends on the model parameters. Subsequently,
conclusions of the analytical theory are tested against full numerical computations of the interaction dynamics. Lastly, collisions between two coherent structures are simulated, and the possible scenarios thereof are considered. Our aim is to reveal the possibility that either the biharmonic effect may dominate (yielding oscillatory tails and forces, equilibrium steady states of alternating stability etc.) or the harmonic effect will prevail (featuring exponential interactions and forces). The critical case between the two and its own intriguing behavior is revealed as well. In our study of collisions and, in particular, in the case kinks and antikinks interact and eventually separate, we create velocity-out versus velocity-in curves. These curves show windows of velocity-in values for which we see different numbers of bounces before the coherent structures separate. We compare this behavior to both the "pure $\phi^{4 "}$ case and the "pure biharmonic" special-case limits of the present model interpolating between them.

In the study of nonlinear dispersive waves, arguably one of the most wellestablished models with a wide range of possible applications is the nonlinear Schrödinger equation [40, 41, 42]. Its relevance has extended from mean-field limits of atomic gases [44, 43, 45], to the propagation of the envelope of the electric field in optical fibers [46, 33] and from water waves [42] to plasmas [47, 77] and beyond. Nevertheless, recent studies have recognized the experimental relevance and theoretical interest in exploring realms beyond those of purely quadratic dispersion, as accompanying the prototypical cubic nonlinearity (stemming from the Kerr effect [33] or the s-wave scattering of bosons [44, 43]).

More concretely, over the past few years, a new direction within nonlinear optics has stemmed from the ability to engineer dispersion in optical systems in the laboratory, potentially completely eliminating quadratic dispersion and enabling quartic dispersion to be dominant [27]. This has led to the experimental obser-
vation of the briefly mentioned previously pure-quartic solitons (PQS) [27] and subsequently the realization of the pure-quartic soliton laser [28]. Apart from the study of the stationary and dynamical properties of those solitons, numerous other possibilities have emerged from this research thread, including, but not limited to the ability to program dispersion of higher order in fiber lasers [48], the possibility to explore the competing interaction of quadratic and quartic dispersion for bright solitary waves $[32,49][50,51]$, to examine the self-similar propagation of pulses in the presence of gain [52], their nature in the absence of Galilean invariance [53] or the possibility of multi-pulse solitary waves [63]. It is noteworthy that a number of studies have explored the existence and stability of solutions in related models bearing 4th order dispersion (or competing dispersions), as well as their potential for collapse [29, 31].

These recent developments in the area of higher-order dispersion solitons have so far focused in the study of bright soliton solutions, the possibility of which, in the presence of fourth order dispersion, has been known since the 1990s [60, 61, 54, 55]. Dark solitons, on the other hand, supported by higher-order dispersion remain largely unstudied.

Dark/kink-like solitary waves appear as dark intensity dips on a continuous wave (CW) background [64, 65]. Associated with the intensity minimum is a phase change of $\pi$, and this "kink" in the phase provides the dark soliton with added (topological) robustness against noise [66]. The possibility of dark solitons in the presence of fourth-order dispersion was alluded to in [54], and examples of dark solitons in the presence of both higher-order dispersion and quintic nonlinearity were found in [67]. In this work, we explore dark soliton under second- and forthorder dispersion an a pure Kerr (i.e., third-order) nonlinearity.

The experimental investigation of dark solitons in the presence of higher-order
dispersion is a significant challenge. Recently, methods using self-induced modulational instability with normal dispersion have allowed the generation of dark soliton trains in optical fiber cavities [68], with evidence that dark solitons are ubiquitous in these conditions [69]. Fiber laser cavities with conveniently programmable dispersion have proven to be ideal for the generation of higher-order dispersion bright solitons [28], and similar results have been found numerically in Kerr nonlinear microresonators [70, 71]. Encouragingly, dark solitons have also been found in normal dispersion Kerr nonlinear microresonators [72, 73] suggesting a cavity configuration is an ideal path for dark soliton experiments using higher-order normal dispersion. To disentangle the role of higher-order dispersion in dark soliton dynamics, the natural starting place is the conservative case (i.e., in the absence of cavity gain and loss). As such, in this work we consider a cavity-like configuration, with periodic boundary conditions, but constant energy.

The first steps towards a theory of higher-order dispersion dark solitons were taken using the $\phi^{4}$ Klein-Gordon model, in both the pure quartic form [38, 30] and in the presence of both quadratic and quartic terms, as presented in more detail in Chapter 1 in the present work. The so-called kink/antikink solutions in these real fields point to intricate dark soliton interactions in the optical case considered here.

Based on these results, here, we consider the fourth-order normal dispersion regime, in the presence of both normal and anomalous second-order dispersion and cubic Kerr nonlinearity. In the pure quadratic case there is a single dark soliton family of stationary kink solutions, with multiple dark solitons repelling one another. We find many more possibilities in the mixed quartic-quadratic dispersion case, opening up new directions for possible dark soliton experiments. We identify families of (multi-)dark solitons, examine their stability and instability numeri-
cally and dynamically, and conclude with a demonstration of possible dark soliton generation.

We systematically expand upon the branches of kink-like solutions (dark solitons [57]) and examine the bifurcation of these solutions in detail. Indeed, we examine the first 6 families of states, classified on the basis of the separation between the kink and the antikink. Our emphasis is not on the simpler single branch of kink solutions, but rather on the considerably more elaborate feature of the quadratic-quartic model, namely the possibility of existence of multi-kink bound states. We start from the simpler 2-kink states, which form the so-called "lower branches" of our bifurcation diagrams and continue the solutions in one of the key parameters of the system, namely the strength of the quadratic dispersion. For all of the relevant families (except for the "exceptional" 0th family which seems to emanate from the small amplitude limit), the branches feature a characteristic turning point which leads to an "upper branch" of states. The latter nucleates either one or two pairs of additional kinks, leading to states involving 4 -kink and 6 -kink solutions. We identify all of these states systematically and present a comprehensive overview of their stability properties. Equally importantly, in the limit of large $\beta_{2}$ (the quadratic dispersion parameter), we develop a theoretical formulation of the interacting kinks as "effective particles" (see, e.g., also our earlier considerations in [51]). This, in turn, allows us to identify the equilibrium configurations (and their kink locations) in the resulting interacting particle system, and examine the linear properties of these particles around the equilibria. We find that this particle picture is remarkably accurate at capturing the unstable and stable modes of the multi-kink states. Whenever relevant, we also complement the existence and stability studies with dynamical computations exploring the fate of the unstable states.

In the last part of the present work we revisit the topic of competing Laplacian and biharmonic dispersion terms, especially with a view to higher-dimensional considerations and the interplay of the power (exponent) $p$ of the nonlinearity and the dimensionality $d$ of the linear operator. More precisely, we consider the following mathematical models:

$$
\begin{align*}
& i u_{t}+\Delta^{2} u+b \Delta u-|u|^{p-1} u=0, \quad x \in \mathbf{R}^{d}  \tag{1}\\
& i u_{t}+\Delta^{2} u+b \partial_{x_{1}}^{2} u-|u|^{p-1} u=0, \quad x \in \mathbf{R}^{d} \tag{2}
\end{align*}
$$

Our work here focuses on the study of solitary waves of such models and their stability properties. In fact, we consider standing waves in the form $u=e^{-i \omega t} \Phi$, which results in the elliptic profile equations:

$$
\begin{align*}
& \Delta^{2} \Phi+b \Delta \Phi+\omega \Phi-|\Phi|^{p-1} \Phi=0, \quad x \in \mathbf{R}^{d}  \tag{3}\\
& \Delta^{2} \Phi+b \partial_{x_{1}}^{2} \Phi+\omega \Phi-|\Phi|^{p-1} \Phi=0, \quad x \in \mathbf{R}^{d} \tag{4}
\end{align*}
$$

We will refer to the model (1) as the isotropic case, while the model (2) as the anisotropic case (due to its different dispersion along the direction $x_{1}$ ).

Next, we set up the linear stability framework for these models. Namely, taking $u=e^{-i \omega t}(\Phi+v)$, plugging this in (1) (or (2) respectively) and ignoring the higher order terms (i.e. super-linear ones of the form $O\left(v^{2}\right)$ ), we obtain for $\vec{v}=(\operatorname{Re} v, \operatorname{Im} v)$,

$$
\begin{align*}
\vec{v}_{t} & =\mathcal{J} \mathcal{L} \vec{v}, \mathcal{J}=\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right), \mathcal{L}=\left(\begin{array}{cc}
\mathcal{L}_{+} & 0 \\
0 & \mathcal{L}_{-}
\end{array}\right)  \tag{5}\\
\mathcal{L}_{+} & =\Delta^{2}+b \Delta+\omega-p|\Phi|^{p-1},  \tag{6}\\
\mathcal{L}_{-} & =\Delta^{2}+b \Delta+\omega-|\Phi|^{p-1} . \tag{7}
\end{align*}
$$

Similarly, the eigenvalue problem for the anisotropic model (2) is also in the form
(5), with $\mathcal{L}_{ \pm}$given by

$$
\left\{\begin{array}{l}
\mathcal{L}_{+}=\Delta^{2}+b \partial_{x_{1}}^{2}+\omega-p|\phi|^{p-1} \\
\mathcal{L}_{-}=\Delta^{2}+b \partial_{x_{1}}^{2}+\omega-|\phi|^{p-1}
\end{array}\right.
$$

We now give a formal definition of spectral stability, which, in the context of the standing waves of the model of interest, is the central focus of the present work.

Definition 0.1 We say that the corresponding standing wave solution $e^{-i \omega t} \Phi$ is spectrally stable, provided the eigenvalue problem $\mathcal{J} \mathcal{L} v=\mu v$ does not have nontrivial solution $(v, \mu): v \in H^{4}\left(\mathbf{R}^{d}\right), \mu: \operatorname{Re} \mu>0$.

The closest in spirit work to the present one is that of [55]. In it, however, the author considers a different model, namely

$$
\begin{equation*}
i u_{t}+\gamma \Delta^{2} u+\Delta u+|u|^{p-1} u=0, \quad x \in \mathbf{R}^{d} \tag{8}
\end{equation*}
$$

The authors obtains a number of useful (and mostly rigorous) results for the standing waves for these models, especially in the regime ${ }^{1} \gamma<0,|\gamma| \ll 1$. Note however that this case, after some rescaling is equivalent to the case $b<0$ in (1), whereas our main interest is in the case $b>0$. The latter involves a competition (rather than a cooperation) of the linear contributions and, hence, represents a case of particular interest.

In the present setting, we examine systematically the isotropic case, but also compare it with the anisotropic one whereby the Laplacian operator is replaced by a second partial derivative along only a single spatial direction. We present theoretical results in both cases for the ground states of the system and their stability as a function of the nonlinearity power $p$ and the coefficient of the Laplacian (or

[^0]of the one-dimensional second partial derivative) $b$. Our principal theorems are, accordingly, stated in the next section.

We corroborate our theoretical analysis with detailed numerical computations that illustrate systematically both the isotropic and the anisotropic case with $d=2$, as a function of $b$ and also as a function of $p$. Starting with the isotropic case, we find that up to the cubic case of $p=3$, the relevant ground states are generically stable, irrespectively of the value of $b$. Beyond $p=3$ and for $3<p<5$, a critical threshold of $b$ exists such that below the relevant threshold, the wave is spectrally stable, while above, it destabilizes. Further, above $p=5$ and below a critical $p$, the waves will only be spectrally stable for an interval of $b$ 's, while upon crossing this critical threshold, a saddle-center bifurcation leads to the disappearance of all stable solutions of the isotropic setting. Interestingly, the anisotropic example bears numerous similarities with the above described isotropic case. The most notable difference that is worth highlighting is that even below $p=3$, the anisotropic case may bear instabilities for a narrow interval of $b$-values; more details are shown in our numerical computations that follow.

## C H A P TER 1

# KINK-ANTIKINK INTERACTION FORCES AND BOUND STATES IN A $\phi^{4}$ MODEL WITH QUADRATIC AND QUATRIC DISPERSION 

In this chapter ${ }^{1}$, we consider the interaction of solitary waves in a model involving the well-known $\phi^{4}$ Klein-Gordon theory, but now bearing both Laplacian and biharmonic terms with different prefactors. As a result of the competition of the respective linear operators, we obtain three distinct cases as we vary the model parameters. In the first the biharmonic effect dominates, yielding an oscillatory inter-wave interaction; in the third the harmonic effect prevails yielding exponential interactions, while we find an intriguing linearly modulated exponential effect in the critical second case, separating the above two regimes. For each case, we calculate the force between the kink and antikink when initially separated with sufficient distance. Being able to write the acceleration as a function of the separation distance, and its corresponding ordinary differential equation, we test the corresponding predictions, finding very good agreement, where appropriate, with the corresponding partial differential equation results. Where the two findings differ,

[^1]we explain the source of disparities. Finally, we offer a first glimpse of the interplay of harmonic and biharmonic effects on the results of kink-antikink collisions and the corresponding single- and multi-bounce windows.

### 1.1 Model Setup \& Kink-Antikink Tail Behavior

The standard $\phi^{4}$ Klein-Gordon theory yields the field equation

$$
\begin{equation*}
u_{t t}=u_{x x}-V^{\prime}(u) \tag{1.1}
\end{equation*}
$$

where $V(u)=\frac{1}{2}\left(u^{2}-1\right)^{2}$. In [30, 38], a variant of this equation was explored where the harmonic spatial derivative term was replaced by a biharmonic term of the form:

$$
\begin{equation*}
u_{t t}=-u_{x x x x}-V^{\prime}(u) \tag{1.2}
\end{equation*}
$$

Here, as indicated in the above section, motivated by the corresponding generalized NLS of [32], we explore a model incorporating the competition of the features of the two models:

$$
\begin{equation*}
u_{t t}=\alpha u_{x x}-\beta u_{x x x x}-V^{\prime}(u) \tag{1.3}
\end{equation*}
$$

where $\alpha$ and $\beta$ are assumed positive (to ensure the competition referred to above) and the potential function $V(u)$ is taken as before. When we pick $\alpha=1$ and $\beta=0$, we get Eq. (1.1) and when we pick $\alpha=0$ and $\beta=1$, we get Eq. (1.2). Notice that while one of the coefficients could be scaled out via a rescaling of space, we maintain both coefficients, in order to maintain the tractability of the special case limits of $(0,1)$ and $(1,0)$, i.e., biharmonic and harmonic respectively.

A central consideration of the present work is to explore both the features of a single solitary wave, but also to examine the interaction between two such waves, a
kink and an antikink. We will use a method developed by Manton (as in $[38,39]$ ) to find the force between a separated kink and antikink as a function of the separation distance. To do this we must first determine the tail behavior for a single kink or antikink. Once the force is determined, we can use the corresponding acceleration to generate an ODE, whose behavior can then be compared to the soliton trajectories of Eq. (1.3), i.e., the corresponding partial differential equation (PDE). As long as the separation distance between kink and antikink remains sufficiently large, the agreement between ODE and PDE should be quite good. However, in cases where the kink and antikink approach each other at distances comparable to their respective widths, then it is no longer obvious that the ODE model should be an adequate description of the full PDE dynamics and the exchanges of energy between the different modes present in the latter [6]. We will explore both the former agreement (at large distances) and the latter deviations (at short ones) in the numerical results below.

In order to determine the tail behavior of a single kink we proceed as follows. Substituting $\phi(x)=u(t, x)$ into Eq. (1.3) we get the steady-state equation

$$
\begin{equation*}
\alpha \phi^{\prime \prime}-\beta \phi^{\prime \prime \prime \prime}-V^{\prime}(\phi)=0 \tag{1.4}
\end{equation*}
$$

where ' denotes derivative with respect to the argument. To examine the relevant asymptotics, we substitute $\phi=1-\varepsilon e^{\lambda x}$ into Eq. (1.4). Neglecting terms of $\varepsilon^{2}$ and higher (i.e., linearizing), for the above mentioned $\phi^{4}$ potential, we get

$$
\begin{equation*}
-\alpha \lambda^{2}+\beta \lambda^{4}+4=0 \tag{1.5}
\end{equation*}
$$

It is easy to show that the roots of this equation are real for $\alpha \geq 4 \sqrt{\beta}$ and complex for $\alpha<4 \sqrt{\beta}$. In particular, for $\alpha<4 \sqrt{\beta}$ :

$$
\lambda_{1}=\frac{1}{2} \sqrt{\frac{4 \sqrt{\beta}+\alpha}{\beta}}+i \frac{1}{2} \sqrt{\frac{4 \sqrt{\beta}-\alpha}{\beta}}, \quad \lambda_{2}=\frac{1}{2} \sqrt{\frac{4 \sqrt{\beta}+\alpha}{\beta}}-i \frac{1}{2} \sqrt{\frac{4 \sqrt{\beta}-\alpha}{\beta}}
$$

$$
\lambda_{3}=-\frac{1}{2} \sqrt{\frac{4 \sqrt{\beta}+\alpha}{\beta}}+i \frac{1}{2} \sqrt{\frac{4 \sqrt{\beta}-\alpha}{\beta}}, \quad \lambda_{4}=-\frac{1}{2} \sqrt{\frac{4 \sqrt{\beta}+\alpha}{\beta}}-i \frac{1}{2} \sqrt{\frac{4 \sqrt{\beta}-\alpha}{\beta}}
$$

for $\alpha=4 \sqrt{\beta}$ (critical case), the degenerate roots are:

$$
\lambda_{1,2}=\sqrt{\frac{\alpha}{2 \beta}}, \quad \lambda_{3,4}=-\sqrt{\frac{\alpha}{2 \beta}}
$$

and for $\alpha>4 \sqrt{\beta}$ :

$$
\begin{gathered}
\lambda_{1}=\sqrt{\frac{\alpha-\sqrt{\alpha^{2}-16 \beta}}{2 \beta}}, \quad \lambda_{2}=\sqrt{\frac{\alpha+\sqrt{\alpha^{2}-16 \beta}}{2 \beta}} \\
\lambda_{3}=-\sqrt{\frac{\alpha-\sqrt{\alpha^{2}-16 \beta}}{2 \beta}}, \quad \lambda_{3}=-\sqrt{\frac{\alpha+\sqrt{\alpha^{2}-16 \beta}}{2 \beta}}
\end{gathered}
$$

For real $\lambda$, similarly to the pure $\phi^{4}$ case, our model for the tail behavior is

$$
\begin{equation*}
b e^{-a x}+d e^{-c x} \tag{1.6}
\end{equation*}
$$

for the critical case with the double roots the model is

$$
\begin{equation*}
b e^{-a x}(x-d) \tag{1.7}
\end{equation*}
$$

(accounting for the relevant generalized eigenvector) and for the complex $\lambda$ case (similarly to the pure biharmonic one), the model is

$$
\begin{equation*}
b e^{-a x} \cos (c(x-d)) \tag{1.8}
\end{equation*}
$$

We also know that in the real case $a=\lambda_{1}=\sqrt{\frac{\alpha-\sqrt{\alpha^{2}-16 \beta}}{2 \beta}}$ and $c=\lambda_{2}=\sqrt{\frac{\alpha+\sqrt{\alpha^{2}-16 \beta}}{2 \beta}}$ and in the complex case $a=\operatorname{Re}(\lambda)=\frac{1}{2} \sqrt{\frac{4 \sqrt{\beta}+\alpha}{\beta}}$ and $c=\operatorname{Im}(\lambda)=\frac{1}{2} \sqrt{\frac{4 \sqrt{\beta}-\alpha}{\beta}}$, while in the critical case $a=\sqrt{\frac{\alpha}{2 \beta}}$. We use curve fitting to get the other parameters $b$ and $d$. The results are in Table 1.1 for a sequence of prototypical case examples that we have considered.

In Fig. 1.1 we show the tail-behavior of a single kink, $\phi_{K}(x)$, for the cases $\alpha=1, \alpha=4, \alpha=5$ with $\beta=1$ (respectively, left to right). We graph the right

| $\alpha$ | $\beta$ | $\lambda$ | Tail behavior |
| :---: | :---: | :---: | :---: |
| 0 | 1 | $1-1 \mathrm{i}$ | $0.9700 e^{-x} \cos (x-0.4083)$ |
| 1 | 1 | $1.1180-0.8660 \mathrm{i}$ | $1.205 e^{-1.118 x} \cos (0.8660(x-0.9909))$ |
| 2 | 1 | $1.2247-0.7071 \mathrm{i}$ | $1.793 e^{-1.225 x} \cos (0.7071(x-1.824))$ |
| 3 | 1 | $1.3229-0.5000 \mathrm{i}$ | $3.614 e^{-1.323 x} \cos (0.5(x-3.299))$ |
| 3.5 | 1 | $1.3693-0.3536 \mathrm{i}$ | $6.662 e^{-1.369 x} \cos (0.3536(x-4.942))$ |
| 4 | 1 | 1.4142 | $3.363 e^{-1.414 x}(x-0.9786)$ |
| 4.5 | 1 | $1.1042,1.8113$ | $4.451 e^{-1.104 x}-12.92 e^{-1.811 x}$ |
| 5 | 1 | 1,2 | $3.354 e^{-x}-25.64 e^{-2 x}$ |
| 6 | 1 | $0.8740,2.2882$ | $2.679 e^{-0.8740 x}-157.4 e^{-2.288 x}$ |
| 1 | 0 | 2 | $2 e^{-2 x}$ |

Table 1.1. Single Kink Tail Behavior for different model parameter $(\alpha, \beta)$ in columns 1 and 2. Column 3 yields the corresponding (spatial) eigenvalues and column 4 the functional form providing the optimal fit to the tail behavior. One can read off the values of $a, b, c$, $d$ in column 4 by referring to Equations (1.6), (1.7), (1.8).
tail of $1-\phi_{K}(x)$ multiplied by $e^{k x}$ as well as a model fitted to the tail of $1-\phi_{K}(x)$, also multiplied by $e^{k x}$ (appropriate for $x$ sufficiently large). The value of $k$ is equal to the real part $\operatorname{Re}(\lambda)$ in the complex case, $\lambda$ in the critical case, and the smaller (in absolute value) $\lambda$ in the real case (corresponding to the slow decay). We can observe an excellent agreement in the oscillatory case (especially factoring in that we have multiplied the expression by an exponential, hence any deviation in the exponent would lead to an exponential growth). Similarly, also a remarkable fit can be discerned even in the critical case, revealing the underlying linear dependence modulating the exponential decay of the generalized eigenvector in this setting. The exponential case (originally doubly exponential turned into a single exponential upon multiplication by $e^{k x}$ ) is found to be less accurate. In the latter case of two real $\lambda$ an improved fit can be obtained to the model $b e^{-a x}+d e^{-c x}$ if $c$ is left as a free parameter in the curve-fitting process (rather than using the value specified above which results from Eq. (1.5)). It is an interesting question for future work, whether a weakly nonlinear theory can capture more accurately the correction to
the leading exponential dependence; however, for our present purposes, the current prediction capturing adequately the leading order exponential tail behavior will suffice.


Figure 1.1. The panels show the behavior of the right-side tails of the single kink, $\phi_{K}(x)$, for the cases $\alpha=1, \alpha=4, \alpha=5$ and $\beta=1$ (respectively, left to right) by graphing $1-\phi_{K}(x)$ multiplied be $e^{k x}$, where $k$ (positive) is $\operatorname{Re}(\lambda)$ in the complex case, $\lambda$ in the critical case, and the smaller (in absolute value) $\lambda$ in the real case. Superimposed are the fitted curves, also multiplied by $e^{k x}$. In all cases the red solid curve is $e^{k x}\left(1-\phi_{K}\right)$. The blue dashdot curve is the fitted equation multiplied by $e^{k x}$; the specific equations for each case are $y=1.205 \cos (0.8660(x-0.9890))$, $y=3.274(x-0.8606)$ and $y=3.308-14.9 e^{-x}$ (right to left respectively).

We now numerically calculate and illustrate several steady-state solutions, and investigate the corresponding spectra, for a single kink and different combinations of $\alpha$ and $\beta$ values. See Fig. 1.2, upper left panel, for the single-kink shapes corresponding to $\beta=1$ combined with several values of $\alpha$. We numerically calculate the spectrum for the single-kink case (which would also apply to the single antikink). As occurs for the standard $\phi^{4}$ model, an isolated point spectrum mode appears for each case considered, suggesting the possibility of internal vibrations (that, in turn, are well-known to play a role the outcome of kink-antikink collisions [6]). In the upper right panel of Fig. 1.2 we see that the spectrum for these cases consists of completely imaginary values, indicating stability. The bottom two panels of Fig.
1.2 illustrate the eigenvalues of the isolated internal mode. The cases shown are for varying $\alpha$ when $\beta=1$ (left panel; notice how the $\phi^{4}$ internal mode frequency limit of $\sqrt{3}$ is asymptotically approached as $\alpha$ becomes large) and for varying $\beta$ when $\alpha=1$ (right panel). When we investigate steady states for kink-antikink combinations (below) we will also find similar modes.

### 1.2 The force between kink and antikink

In order to find the force or acceleration between kink and antikink, we use the approach of Manton as in [38, 39]; see also [38] for details of the calculation in the case where $\alpha=0$ and $\beta=1$. We now briefly review some of the details of this force calculation. Consider the momentum $P=-\int_{x_{1}}^{x_{2}} u_{t} u_{x} d x$ on the interval $\left[x_{1}, x_{2}\right]$ ( $P$ is conserved when the integral is over the entire real line). Differentiating under the integral and using Eq. (1.3) we find that the force is given by

$$
\begin{equation*}
\frac{d P}{d t}=F=\left[-\frac{1}{2} u_{t}^{2}-\alpha \frac{1}{2} u_{x}^{2}+\beta u_{x} u_{x x x}-\beta \frac{1}{2} u_{x x}^{2}+V(u)\right]_{x_{1}}^{x_{2}} \tag{1.9}
\end{equation*}
$$

For a field configuration that is static or almost so, we can ignore the first term in the right-hand side bracket of Eq. (1.9). We consider a configuration

$$
\begin{equation*}
u(t, x)=\phi(x)=\phi_{K}(x+X(t))+\phi_{A K}(x-X(t))-1 \tag{1.10}
\end{equation*}
$$

where $-X(t)$ is the position of the kink and $X(t)$ is the position of the antikink, which represents a kink-antikink pair approaching each other (as $t$ gets larger). The kink and antikink positions are defined as the $x$-value of the intersection of each with the horizontal axis. Then, set $\eta=1-\phi, \eta_{K}=1-\phi_{K}$ and $\eta_{A K}=1-\phi_{A K}$ (where $\phi_{A K}$ is an antikink solution to Eq. (1.4)). Evaluating Eq. (1.9) from $x_{1}=x$


Figure 1.2. Top left panel shows the steady kink solutions and top right panel shows the spectral plane $\left(\lambda_{r}, \lambda_{i}\right)$ of eigenvalues $\lambda=\lambda_{r}+i \lambda_{i}$ of the linearized operator about the corresponding steady kink solution for fixed $\beta=1$ and $\alpha=0.5$ (blue circles), $\alpha=1$ (red x's), $\alpha=5$ (green diamonds), $\alpha=10$ (magenta stars). The most right of the top right panel is the zoomed in version that shows the internal modes for all cases. The bottom left panel shows the internal mode $\omega_{0}$ versus $\alpha$ for fixed $\beta=1$. The bottom right panel shows the internal mode $\omega_{0}$ versus $\beta$ for fixed $\alpha=1$.
to $x_{2} \rightarrow \infty$ results in

$$
\begin{equation*}
F=\alpha\left(\eta_{K}\right)_{x}\left(\eta_{A K}\right)_{x}-\beta\left(\eta_{K}\right)_{x}\left(\eta_{A K}\right)_{x x x}-\beta\left(\eta_{K}\right)_{x x x}\left(\eta_{A K}\right)_{x}+\beta\left(\eta_{K}\right)_{x x}\left(\eta_{A K}\right)_{x x}-4\left(\eta_{K}\right)\left(\eta_{A K}\right) \tag{1.11}
\end{equation*}
$$

where we assume that the kink-antikink separation $2 X$ is large. We have also assumed that $|x| \ll X$ and have used the approximation $V(\phi)=V(1-\eta) \approx 2 \eta^{2}$. Note that only the cross terms are left at this point.

For the complex case the model for the tail behavior is $\eta_{K}=b e^{-a x} \cos (c(x-d)$, appropriate for $x$ sufficiently large.

Carrying out the derivatives in Eq. (1.11) and using $a=\operatorname{Re}(\lambda)=\frac{1}{2} \sqrt{\frac{4 \sqrt{\beta}+\alpha}{\beta}}$ and $c=\operatorname{Im}(\lambda)=\frac{1}{2} \sqrt{\frac{4 \sqrt{\beta}-\alpha}{\beta}}$ we get the following expression for the force:

$$
\begin{equation*}
F=-2 \sqrt{\frac{16 \beta-\alpha^{2}}{\beta}} b^{2} e^{-X \sqrt{\frac{4 \sqrt{\beta}+\alpha}{\beta}}} \cos \left(\sqrt{\frac{4 \sqrt{\beta}-\alpha}{\beta}}(X-d)+\theta\right) . \tag{1.12}
\end{equation*}
$$

Here, we have that $\theta \in\left[0, \frac{\pi}{2}\right]$ such that $\tan \theta=\frac{\alpha}{\sqrt{16 \beta-\alpha^{2}}}$.
For the critical case the model for the tail behavior is $b e^{-a x}(x-d)$. This, upon substituting $a=\sqrt{\frac{\alpha}{2 \beta}}$ and $\alpha=4 \sqrt{\beta}$ results in the force formula:

$$
\begin{equation*}
F=-8 b^{2} \sqrt{2 \alpha} e^{-2 a X}\left(X-\frac{\sqrt{2 \alpha}}{4}-d\right) \tag{1.13}
\end{equation*}
$$

once again featuring a functional form reminiscent of that of the kink tail.
For the real case the tail behavior is $b e^{-a x}+d e^{-c x}$ and the force becomes

$$
F=-b^{2} e^{-2 X a}\left(a^{2} \alpha-3 a^{4} \beta+4\right)-d^{2} e^{-2 X c}\left(c^{2} \alpha-3 c^{4} \beta+4\right),
$$

where:

$$
\begin{equation*}
a=\sqrt{\frac{\alpha-\sqrt{\alpha^{2}-16 \beta}}{2 \beta}} \text { and } c=\sqrt{\frac{\alpha+\sqrt{\alpha^{2}-16 \beta}}{2 \beta}} . \tag{1.14}
\end{equation*}
$$

Using these values the force formula can be written as

$$
\begin{equation*}
F=\frac{\alpha^{2}-16 \beta-\alpha \sqrt{\alpha^{2}-16 \beta}}{\beta} b^{2} e^{-2 X a}+\frac{\alpha^{2}-16 \beta+\alpha \sqrt{\alpha^{2}-16 \beta}}{\beta} d^{2} e^{-2 X c} . \tag{1.15}
\end{equation*}
$$

Notice that the coefficient of the slow term is always negative while the coefficient of the fast term is always positive.

| $\alpha$ | $\beta$ | Mass | Acceleration |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 1.1852 | $6.351 e^{-2.0000 x} \cos (2.0000 x-0.8166)$ |
| 1 | 1 | 0.9540 | $11.79 e^{-2.2360 x} \cos (1.7320 x-1.4640)$ |
| 2 | 1 | 0.8052 | $27.66 e^{-2.4494 x} \cos (1.4142 x-2.0559)$ |
| 3 | 1 | 0.7031 | $98.30 e^{-2.6458 x} \cos (1.0000 x-2.4510)$ |
| 3.5 | 1 | 0.6633 | $259.1 e^{-2.7386 x} \cos (0.7072 x-2.429)$ |
| 4 | 1 | 0.6290 | $407.0 e^{-2.828 x}(x-1.686)$ |
| 4.5 | 1 | 0.5991 | $166.2 e^{-2.208 x}-3769 e^{-3.623 x}$ |
| 5 | 1 | 0.5728 | $117.8 e^{-2.000 x}-27545 e^{-4.000 x}$ |
| 6 | 1 | 0.5287 | $92.75 e^{-1.748 x}-2194577 e^{-4.576 x}$ |
| 1 | 0 | $4 / 3$ | $24 e^{-4 x}$ |

Table 1.2. Mass and acceleration as a function of the half-separation distance $x$ of the kink and antikink.

Dividing the above formulae for the force by the mass gives the results in the acceleration column of Table 1.2. The values for $b$ and $d$ are determined by curve fitting the tail of a single kink, and are shown in Table 1.1.

Next, we integrate the expressions for the force on a kink to get the potential energy for each case, and also find all fixed points and their stability type. For the potential function in the complex case we have

$$
\begin{equation*}
U=-b^{2} \sqrt{\frac{16 \beta-\alpha^{2}}{2 \sqrt{\beta}}} e^{-X \sqrt{\frac{4 \sqrt{\beta}+\alpha}{\beta}}} \cos \left(\sqrt{\frac{4 \sqrt{\beta}-\alpha}{\beta}}(X-d)+\vartheta\right) \tag{1.16}
\end{equation*}
$$

for $\vartheta \in\left[0, \frac{\pi}{2}\right]$ such that $\tan \vartheta=\frac{\sqrt{4 \sqrt{\beta}+\alpha}}{\sqrt{4 \sqrt{\beta}-\alpha}}$. For saddle points we have

$$
X^{*}(k)=d+\left(\frac{\pi}{2}+2 k \pi-\theta\right) \sqrt{\frac{\beta}{4 \sqrt{\beta}-\alpha}}
$$

and for centers we get

$$
X^{*}(k)=d+\left(\frac{\pi}{2}+(2 k+1) \pi-\theta\right) \sqrt{\frac{\beta}{4 \sqrt{\beta}-\alpha}} .
$$

For the critical case (repeated $\lambda$ ) the potential function is

$$
\begin{equation*}
U=-2 b^{2} \alpha e^{\frac{-4 \sqrt{2}}{\sqrt{\alpha}} X}\left(X-d-\frac{\sqrt{2 \alpha}}{8}\right) \tag{1.17}
\end{equation*}
$$

and the center is given by

$$
X^{*}=d+\frac{\sqrt{2 \alpha}}{4} .
$$

For the real case we have the potential function

$$
\begin{equation*}
U=-b^{2} a \sqrt{\alpha^{2}-16 \beta} e^{-2 a X}+d^{2} c \sqrt{\alpha^{2}-16 \beta} e^{-2 c X}, \tag{1.18}
\end{equation*}
$$

and the center:

$$
\frac{1}{2(c-a)}\left(\log \frac{c^{2} d^{2}}{a^{2} b^{2}}\right)
$$

with $a$ and $c$ as given in Eq. (1.14).

### 1.3 Comparison of ODE and PDE models

Using the expressions for the force we can now write an ODE for the time evolution of the kink and antikink position for any given $(\alpha, \beta)$ combination. In Table 1.2 we have divided the force by the numerically calculated mass and used the curve-fitted values for $b$ and $d$ to get an acceleration expression for specific values of $\alpha$ and $\beta$. The corresponding ODE is then $\ddot{X}=-d U / d X$, where the acceleration of the right-hand side is provided in Table 1.2. This ODE for the position of the one coherent structure (while the other one is symmetrically located) is amenable to a phase portrait analysis, as shown in Fig. 1.3 and a comparison with the corresponding PDE results of Eq. (1.3) can be obtained both at that level and at the spatio-temporal evolution one as shown in Fig. 1.4.

In Fig. 1.3 we show trajectories for the case $\alpha=1$ and $\beta=1$ (complex case) that illustrate behavior near the steady states of the PDE (the fixed points of the ODE). For these cases, there is clearly excellent agreement between the ODE and PDE phase planes, which validates our force calculations. The calculated force laws


Figure 1.3. $\alpha=1, \beta=1$, Phase portrait of the ODE Equation (ODE) in comparison with Eq. (1.3) (PDE). The blue solid curve corresponds to $X(0)=8, \dot{X}(0)=-0.02$. The red dash-dotted curve: $X(0)=8, \dot{X}(0)=-0.00555$. The light blue closed orbit: $X(0)=3.3, \dot{X}(0)=0$. The green curve corresponds to $X(0)=8$, $\dot{X}(0)=-0.00356$. The pink solid closed orbit: $X(0)=7.4$, $\dot{X}(0)=-0.0002$.
work well as long as the separation between kink and antikink is sufficiently large. In the case of the PDE, we identify the motion of the coherent structure by using the intersection of the kink or antikink with the horizontal axis as the position, and also find the corresponding speed, and thus extract an effective phase portrait to be compared with the ODE results. In Fig. 1.4 we show two of the trajectories from Fig. 1.3 as contour plots of the PDE with the ODE trajectory superimposed on top (in blue). The left one among them is a robust oscillation around a stable fixed point in the form of a center (the light blue curve in Fig. 1.4). The other is a trajectory that is scattered from the innermost potential energy barrier due to the presence of the innermost saddle point, corresponding to a maximum of the effective energy landscape and is thus reflected. In this case, we see that the kinks do not make it to a collision but are rather reflected due to their interaction
landscape before the collision.


Figure 1.4. $\alpha=1, \beta=1$. Comparisons of the PDE contour plot of the displacement field $u(x ; t)$ and the ODE trajectory. (blue solid curve). Left: $x_{0}=3.3, v_{i n}=0$. This corresponds to the closed orbit (light blue) in Fig. 1.3. Right: $x_{0}=8, v_{i n}=-0.02$. This corresponds to blue solid curve in Fig. 1.3.

For another perspective on the quantification of the agreement between PDE and ODE results, see Fig. 1.5. In the left panel of this figure, we show the potential energy plot of the ODE, again for $\alpha=1, \beta=1$, in blue (using Eq. 1.16). The data points represent the potential energies of the steady states of the PDE calculated using the PDE energy $E=\int \frac{1}{2} \alpha u_{x}^{2}+\frac{1}{2} \beta u_{x x}^{2}+\frac{1}{2}\left(u^{2}-1\right)^{2} d x$ of the associated steady state configurations. The corresponding steady-states themselves are shown in the same figure, right panel. Note that since the calculated potential energy curve of the ODE approaches zero as the separation distance increases, the potential energies of the steady states of the PDE must also be normalized (i.e., calibrated) so that the limiting value is zero (by subtracting the potential energy of a steady state with very large separation). Again, clearly, the local maximum and minimum values of the ODE energy landscape coincide with the potential energies of the corresponding steady states of the PDE. The local minima correspond to stable steady states
of the PDE (centers of the ODE) and the local maxima correspond to unstable steady states of the PDE (saddle points of the ODE). Importantly, aside from the center-most potential energy maximum where the kink structures are so close that we cannot identify them as independent entities (and thus we do not expect the collective coordinate characterization to be as accurate), we observe that the agreement is very good. We remind the reader that the presence of this oscillatory energy landscape, its associated minima (centers) and maxima (saddles), and the respective stationary PDE configurations are distinctive features of the prevalence of the biharmonic term and are genuinely absent in the harmonic case (and more generally for $\alpha>4 \sqrt{\beta}$, when the harmonic contribution is dominant).

For the PDE, Eq. (1.3), we expect that the equilibrium solutions shown in Fig. 1.5 with $x_{0}=1.825, x_{0}=5.38$ and $x_{0}=9.01$ will be locally unstable and the ones with $x_{0}=3.56$ and $x_{0}=7.19$ will be locally stable. This is consistent with Fig. 1.6, where the spectral plots $\left(\lambda_{r}, \lambda_{i}\right)$ are shown for the eigenvalues $\lambda=$ $\lambda_{r}+i \lambda_{i}$ of the linearized field equation, for $\alpha=1$ and $\beta=1$. Using the expansion $u(x, t)=u_{0}(x)+\epsilon e^{\lambda t} w(x)$ around an equilibrium solution $u_{0}(x)$, we solve for the eigenvalues $\lambda$. For $x_{0}=3.56$ and $x_{0}=7.19$, we see that all the eigenvalues lie on the imaginary axis, so we conclude that the equilibrium for those cases are stable. However, for $x_{0}=1.825, x_{0}=5.38$ and $x_{0}=9.01$, we get one eigenvalue pair on the real (resulting in exponential instability) axis in addition to the ones on the imaginary. The eigenvalue that is the lowest imaginary one in the stable cases, as well as the single nonvanishing real pair in the unstable case are associated with the relative kink-antikink center motion. The nature of the mode is associated with stable oscillations in the former case and with unstable sliding in the latter setting. There also exists a vanishing pair of eigenvalues whose eigendirection leads to an energy-neutral rigid translation of the kink-antikink pair. The other


Figure 1.5. The left panel shows the energy vs $x_{0}$ for $\alpha=1, \beta=1$. Blue curve is twice the potential function of the ODE for the complex case (given in Eq. (1.16)). The data points are the (normalized) potential energies of the steady states of the PDE at $x_{0}=1.825,3.56,5.38,7.19,9.01$ which are shown in the right panel. The need to multiply the potential function of the ODE by two when comparing ODE and PDE stems from the fact that the energy calculation using a steady state of the PDE involves two solitons - kink and antikink. The right panel presents the static, equilibrium solutions corresponding to $x_{0} \approx 1.825$ (orange dashed-dot curve), $x_{0} \approx 3.56$ (blue solid curve), $x_{0}=5.38$ (red dashed curve), $x_{0} \approx 7.19$ (green dashed-dot curve) and $x_{0} \approx 9.01$ (purple dotted curve). Note that the steady state for the PDE occurs at $x_{0} \approx 1.825$ but the fixed point of the ODE is at $x_{0} \approx 1.75$.
nontrivial point spectrum pairs of 2 near-identical modes are analogous to the internal mode that was discussed earlier in the text for the single kink/antikink (for several combinations of $\alpha$ and $\beta$ ).

While in Fig. 1.3 we showed example phase portraits that resulted in very proximal correspondence between PDE and ODE (for the complex case), in Fig. 1.7 we show phase trajectories for both the real and complex cases that illustrate at what point the PDE and ODE solution curves may depart from each other (recall that the dash-dotted green line represents the position $X(t)$ of the antikink as measured by its intersection with the horizontal axis). In these cases, the kink


Figure 1.6. The spectral plane $\left(\lambda_{r}, \lambda_{i}\right)$ of eigenvalues $\lambda=\lambda_{r}+i \lambda_{i}$ of oscillations around the equilibria at $x_{0}=1.825$ (1st row left), $x_{0}=3.56$ (1st row right), $x_{0}=5.38$ ( 2 nd row left), $x_{0}=7.19$ (2nd row right), $x_{0}=9.01$ (bottom), for $\alpha=1$ and $\beta=1$.
and antikink get too close for the force law to remain valid. One can see that in the real case of the left panel this occurs at about $X=3$, while in the complex
case of the right panel at about $X=2$. Note also that the green curve indicates the formation of a bound state which is losing energy (in a way somewhat akin to a stable spiral but keeping in mind that in a bound state there are no longer an identifiable kink and antikink). Here, the important differences of the PDE dynamics from the conservative ODE of 1 degree-of-freedom (dof) become evident. The latter being energy conserving can only lead to reflection (or transmission) in such an example, while the former can transfer energy from the kink translational motion to other degrees of freedom (internal ones or radiation ones [6]), thus leading to the effective translational energy dispersion and thus the apparent trapping of the kinks into a so-called bion state. Successive 'breathings' of this bion state at the PDE level are mirrored in the progressively inward green curves (carrying less and less energy). At the ODE level, we make two more minor (in terms of the bigger picture of our story), yet technically relevant observations. Given the absence of ODE-PDE correspondence in the right panel we stop the ODE evolution once the kink-antikink pair directly collides (i.e., at $X=0$ ). On the other hand, the left panel has another intriguing but non-physical trait: the double exponential force (of opposite signs between the two exponentials) results in a landscape with a local minimum very close to $X=0$. We have found this feature to be an artifact of the theory and its lack of accuracy in the immediate vicinity of $X=0$. Let us reiterate, also in light of the above remarks, that the ODE models are based only on the behavior of the tails of the kinks and antikinks. When a kink and antikink are involved in an interaction, the model makes sense only when the structures are well-separated. Therefore the ODE model should not be expected to reflect the actual behavior of the system beyond the point where the waves are at a distance comparable to or smaller than their width, at which time they essentially forego their individual character.


Figure 1.7. The left panel shows phase plots for the real $\lambda$ case of $\alpha=5$ and $\beta=1$ using initial conditions $X(0)=8$ and $\dot{X}(0)=-0.003593$. The right panel illustrates phase plots for the complex $\lambda$ case of $\alpha=1$ and $\beta=1$ using initial conditions $X(0)=8$ and $\dot{X}(0)=$ -0.35 . In both cases, the ODE is shown by the blue solid curve and the PDE by the green dash-dotted curve. Insets show at what points the ODE model diverges from the PDE model. The ODE trajectory in the right panel is stopped at the point when $X=0$ because it becomes physically unrealistic beyond that point.

In Fig. 1.8 we show contour plots for the same values of $\alpha$ and $\beta$ and the same initial conditions as in Fig. 1.7, again with the ODE solution curves superimposed. I.e., these panels represent the spatio-temporal contour plot representation of the failure of the ODE theory to capture the PDE dynamics, as explained in the above discussion. As in Fig. 1.7 we can see that the ODE tracks the PDE simulation until around the time that the collision occurs. As a result of the latter, at the PDE level, a bound state emerges, while in the case of the ODE, the conservative nature of the 1 dof system does not allow any scenario other than reflection.


Figure 1.8. Contour plots of the PDE corresponding to the same parameter values and initial conditions as in the corresponding panels of Fig. 1.7. The ODE trajectory is superimposed in blue.

### 1.4 Velocity in versus velocity out curves and soliton collisions

We now investigate kink-antikink collisions in the context of escape velocity $\left(v_{\text {out }}\right)$ and multi-bounce windows as a function of incoming velocity $\left(v_{\text {in }}\right)$, in line with the extensive literature on the subject discussed in the Introduction (for a relatively recent summary in the $\phi^{4}$ case, see, e.g., [6]). Summarizing the kinkantikink collision dynamics in the $\phi^{4}$ model, we note the following. In this model, it has been shown that there exists a critical $v_{i n}$ value, which we label $v_{\text {crit }}$, such that for $v_{i n}>v_{\text {crit }}$, the kink and antikink interact once and then separate forever. For $v_{i n}<v_{\text {crit }}$ the kink and antikink can form a bound state, or can interact (bounce) any number of times, depending on $v_{i n}$, before separating forever. Furthermore, it is well-established since the work of [14] for the $\phi^{4}$ model, that the bounce windows corresponding to different numbers of bounces are nested in a fractal pattern. For example, three bounce windows occur at the edges of two-bounce windows, four bounce windows occur at the edges of three-bounce windows, and so on. Also, the
$v_{\text {in }}-v_{\text {out }}$ graph for a given window has the appearance of an inverted parabola, with the $v_{\text {out }}$ values going to zero at the edges of the window.

For a model with only a biharmonic term ( $\alpha=0$ in this paper) it was shown in [30] that two critical $v_{i n}$ values exist, with $v_{1, \text { crit }}<v_{2, \text { crit }} . v_{2, \text { crit }}$ is similar to $v_{\text {crit }}$ for the $\phi^{4}$ model in that for $v_{i n}>v_{2, \text { crit }}$ the kink and antikink interact once and then separate. For $v_{i n}<v_{1, \text { crit }}$ the kink and antikink repel elastically before interacting. For $v_{1, \text { crit }}<v_{i n}<v_{2, \text { crit }}$ the kink and antikink form a bound state. Near both critical values, we see oscillations in the $v_{\text {in }}-v_{\text {out }}$ graph, where the frequency of the oscillations rapidly increases as the critical values are approached. It is important to highlight how dramatically different this behavior is from the above behavior of the regular $\phi^{4}$ model, since essentially multi-bounce windows are fully absent in the biharmonic case, while the oscillations near the critical velocities are absent in the standard $\phi^{4}$ case.

Coming now to the case of the model considered herein, we note that it involves a mixture of the two cases just described. More specifically, we find that by fixing $\beta$ at $\beta=1$ and letting $\alpha$ increase from 0 to 6 , we see a transition from one case to the other. As $\alpha$ is increased (departing from the biharmonic case), more and more bounce windows begin to populate the region between $v_{1, \text { crit }}$ and $v_{2, \text { crit }}$. At first, these new bounce windows display oscillations in the $v_{i n}-v_{\text {out }}$ graphs near the edges of the windows, similar to what is seen in the bound-state region of the pure biharmonic case. With increasing $\alpha$ the oscillations diminish and the $v_{\text {out }}$ values at the edges of each window begin to approach zero as in the pure $\phi^{4}$ case. We will showcase these features qualitatively in the results that follow. Nevertheless, the delicate nature of the associated computations renders especially difficult the identification of effective 'critical points' where the behavior changes from the one reminiscent of the pure biharmonic problem to that reminiscent of the
pure harmonic one. In the case of the critical velocities, by rescaling we are able to relate the solutions to Eq. (1.3) for general $\alpha$ and $\beta=1$ to other combinations of $\alpha$ and $\beta$, as is now shown.

Let $u^{1, \beta}$ be a solution to

$$
u_{t t}=u_{x x}-\beta u_{x x x x}+2 u-2 u^{3} .
$$

and consider the coordinate transformation

$$
x \mapsto \xi=\frac{x}{a} .
$$

In the new coordinate system the solution can be rewritten as $u^{1, \beta}(x, t)=\tilde{u}(\xi, t)$. Of course, $u_{t t}^{1, \beta}=\tilde{u}_{t t}$ and $u_{x}^{1, \beta}=\frac{1}{a} \tilde{u}_{\xi}$, therefore $\tilde{u}$ obeys the equation

$$
\tilde{u}_{t t}=\frac{1}{a^{2}} \tilde{u}_{\xi \xi}-\frac{\beta}{a^{4}} \tilde{u}_{\xi \xi \xi \xi}+2 \tilde{u}-2 \tilde{u}^{3} .
$$

For $a^{4}=\beta$ we get

$$
\tilde{u}_{t t}=\frac{1}{\sqrt{\beta}} \tilde{u}_{\xi \xi}-\tilde{u}_{\xi \xi \xi \xi}+2 \tilde{u}-2 \tilde{u}^{3}
$$

so, $\tilde{u}(\xi, t) \equiv u^{\alpha, 1}(\xi, t)$ or,

$$
u^{1, \beta}(x, t)=u^{\alpha, 1}\left(\frac{x}{\beta^{1 / 4}}\right)
$$

for $\alpha=\frac{1}{\sqrt{\beta}}$.
Therefore we can obtain solutions to the model for the parameters $\alpha=1$ and $\beta$, using the solution for parameters $\alpha$ and $\beta=1$. Then, using this coordinate transformation, we find that the critical velocity $v_{1, c r i t}^{1, \beta}$ of the solution $u^{1, \beta}$ can be expressed in terms of the corresponding critical velocity $v_{1, \text { crit }}^{\alpha, 1}$ of the $u^{\alpha, 1}$ solution as

$$
\begin{equation*}
v_{1, c r i t}^{1, \beta}=\frac{d x}{d t}=a \frac{d \xi}{d t}=\beta^{1 / 4} v_{1, c r i t}^{\alpha, 1} \tag{1.19}
\end{equation*}
$$

where $\alpha=\frac{1}{\sqrt{\beta}}$.
Notice that when $\beta$ becomes large enough, we get $v_{1, c r i t}^{\alpha, 1} \approx v_{1, \text { crit }}^{0,1}$, so

$$
v_{1, \text { crit }}^{1, \beta} \sim \beta^{1 / 4} v_{1, \text { crit }}^{0,1} .
$$

Similarly

$$
v_{1, c r i t}^{\alpha, 1} \sim \sqrt{\alpha} v_{1, c r i t}^{1,0} .
$$

Furthermore, the above equations hold when $v_{1, \text { crit }}^{\alpha, 1}$ is replaced by $v_{2, \text { crit }}^{\alpha, 1}$.
In Fig. 1.9 we show graphs of $v_{1, \text { crit }}$ versus $\alpha$ for $\beta=1$ (left panel) and $v_{1, \text { crit }}$ versus $\beta$ for $\alpha=1$ (right panel). Fig. 1.10 is similar, but for $v_{2, \text { crit }}$. The blue circles on all panels are obtained by the numerical simulation of Eq. (1.3) where the left panels represent $v_{1, \text { crit }}$ vs $\alpha$ when $\beta=1$ and the right panels represent $v_{1, \text { crit }}$ vs $\beta$ when $\alpha=1$. In the panels of both figures, the red curves are obtained from the transformation given by Eq. (1.19) (plotted without markers for the transformed points and with connecting lines in order to make the graph more readable). The red curves are included to demonstrate the validity of Eq. (1.19) in comparison with direct PDE simulations.

Having identified the critical point scaling relations, we now turn to a direct examination of the collision features and associated multi-bounce windows. In Fig. 1.11 we show $v_{\text {in }}-v_{\text {out }}$ curves for $\alpha=1$ and $\beta=1$. For fixed $\beta=1$ we know that the force law changes from the complex $\lambda$ case to the real $\lambda$ case at $\alpha=4$, so we expect the case $\alpha=1$ to be somewhat similar to the pure biharmonic case of $\alpha=0$. In the upper left panel of Fig. 1.11 we see that the elastic collision region corresponds to $0<v_{\text {in }}<v_{1, \text { crit }} \approx 0.30805$ and the one-bounce region corresponds to $v_{2, \text { crit }} \approx 0.5902<v_{\text {in }}<1$. Note that we have chosen not to include $v_{i n}$ values greater than one. The region $v_{1, \text { crit }}<v_{i n}<v_{2, \text { crit }}$, which corresponds to a bound state when $\alpha=0$, is beginning to be populated by two and three bounce windows,


Figure 1.9. The blue circles on both panels are obtained by the numerical simulation of Eq. (1.3) where left panel represents $v_{1, \text { crit }}$ vs $\alpha$ when $\beta=1$ and the right panel represents $v_{1, \text { crit }}$ vs $\beta$ when $\alpha=1$. The red solid curve on the left panel is obtained by applying the formula $v_{1, \text { crit }}^{\alpha, 1}=\sqrt{\alpha} v_{1, \text { crit }}^{1, \beta}$ where $\beta=\frac{1}{\alpha^{2}}$ to the numerically obtained data (blue circles) on the right. The red solid curve on the right panel is obtained by applying the formula $v_{1, \text { crit }}^{1, \beta}=\beta^{1 / 4} v_{1, \text { crit }}^{\alpha, 1}$ to the numerically obtained data (blue circles) on the left.


Figure 1.10. The left panel shows $v_{2, \text { crit }}$ vs $\alpha$ when $\beta=1$ and the right panel shows $v_{2, \text { crit }}$ vs $\beta$ when $\alpha=1$. The blue circles and the red solid curves were obtained as described in Fig. 1.9.


Figure 1.11. The top left panel shows $v_{\text {out }}$ vs $v_{\text {in }}$ when $\alpha=1$ and $\beta=1$ with $v_{2, \text { crit }} \approx 0.5902$. The top right panel is the zoom-in about the first two-bounce curve. The bottom left panel is the zoom-in about the two three-bounce windows right before the critical velocity $v_{2, \text { crit }}$. The bottom right panel is the zoom in about the leftmost three-bounce window on the bottom left panel. In both top right and bottom panels, the tails and their oscillatory behaviors are shown. One-bounce windows in the figures are in solid black. Two-bounce windows are in blue and three bounce windows are in green. The gray solid line on the top left panel is when the kink-antikink repel each other elastically.
a byproduct of the inclusion of the quadratic dispersion. The top right panel shows the first two-bounce window. The bottom left panel shows the next two-bounce window, with three-bounce windows appearing just to the left. The bottom right panel shows the first three-bounce window in more detail. All windows display the
characteristic oscillations at the edges. Notice the important features of this case: on the one hand, the multi-bounce windows (which did not appear in the pure biharmonic case) are now present. On the other hand, they do not terminate as, e.g., in the case of the standard $\phi^{4}$ model [6, 22], but rather have the oscillatory terminations (with progressively shorter periodicity) encountered previously in [30] for the pure biharmonic case. Moreover, we have encountered a feature also absent in the standard (pure) $\phi^{4}$ case, namely higher-bounce windows appear only on one side (to the left) of the two bounce windows, while it is well-known [14] that they appear on both sides in the pure harmonic $\phi^{4}$ problem.


Figure 1.12. $v_{\text {out }}$ vs $v_{\text {in }}$ when $\alpha=5$ and $\beta=1$ with $v_{c} \approx 0.7295$. The onebounce window is in solid black. Two-bounce windows are in blue and three-bounce windows are in green.

In Fig. 1.12 we show $v_{\text {in }}-v_{\text {out }}$ curves for $\alpha=5$ and $\beta=1$. For this case, since $\alpha>4$ we have $\lambda$ real, and expect some similarity with the case of the pure $\phi^{4}$ model $(\beta=0)$. Indeed, the structure is similar to the fractal pattern we see in the $\phi^{4}$ case, with three-bounce windows at the edges of the two-bounce windows. However, we were not able to find three-bounce windows to the right of the twobounce windows. These should, presumably, emerge as $\alpha$ gets larger, or as $\beta$ gets
smaller. However, it is an open question requiring further systematic investigation how the self-similar (on both sides) picture of the pure $\phi^{4}$ model arises.


Figure 1.13. Transition from dominant quartic progressively closer to dominant harmonic behavior, by changing $\alpha$ from 2 (left) to 3 (middle) and finally the critical case of $\alpha=4$ (right panel).

We can begin to see how the system transitions from the $\alpha=0, \beta=1$ (pure quartic dispersion) case to the $\alpha=5, \beta=1$ (harmonic term dominant) case through some additional (less detailed) $v_{\text {in }}-v_{\text {out }}$ graphs; see, e.g., Fig. 1.13. The first two-bounce window for $\alpha=2$ (left panel, main figure, and see also Fig. 1.11 top left panel for $\alpha=1$ ) demonstrates a curious behavior. It appears (out of nowhere) at about $\alpha=0.711$, persisting to about $\alpha=2.8$ where it disappears. This is why it arises in the left panel, but not the middle one. Similarly, notice how the transition shrinks progressively the size of the gray line interval of 'no collision' for $0<v_{\text {in }}<v_{1, \text { crit }}$. It can be seen that this interval eventually disappears for $\alpha=4$ in the right panel of the figure, again showcasing how the transition between the two regimes (biharmonic vs. harmonic) emerges.

While in the discussion above, we have focused on features that transition the phenomenology between the two limits, it is important to realize that the wealth of the model considered here transcends that of solely the limit cases. For instance, Fig. 1.14 illustrates a phenomenon not seen in either the pure biharmonic or the


Figure 1.14. Contour plots when $\alpha=2.05$ and $\beta=1$ with $v_{c} \approx 0.6222$ for $X(0)=10$. Left panel is when $v_{\text {in }}=0.622$ and right panel is when $v_{\text {in }}=0.621$.
pure harmonic $\phi^{4}$ case, with $\alpha=2.05$ and $\beta=1$. What we see here is an initial interaction between kink and antikink, followed by separation of the solitons for a period of time, and then another approach of the pair. At this point one or more elastic collisions can occur, resulting in the appearance of multiple bounces. In the first panel of Fig. 1.14 we see a 'pseudo' two-bounce result, and in the second panel a pseudo three-bounce result. This can occur when the speed at which the kink and antikink approach each other for the second (or third) time is very small and therefore when we find ourselves in the small-speed reflection window of the complex eigenvalue case. In short, this is an unprecedented type of two-bounce since two-bounces cannot happen in the pure biharmonic case (where there is only bion formation and single bounce events [30]), but it can also not happen for pure harmonic $\phi^{4}$ where the small speed reflection scenario is absent. This is yet another manifestation of the rich phenomenology of the model combining harmonic and biharmonic dispersion.

## CHAPTER 2

## DARK SOLITONS UNDER HIGHER ORDER <br> DISPERSION

In this chapter ${ }^{1}$, we show theoretically that stable dark solitons can exist in the presence of pure quartic dispersion, and also in the presence of both quadratic and quartic dispersive effects, displaying a much greater variety of possible solutions and dynamics than for pure quadratic dispersion. The interplay of the two dispersion orders may lead to oscillatory non-vanishing tails, which enables the possibility of bound, potentially stable, multi-soliton states. Dark soliton-like states which connect to low amplitude oscillations are also shown to be possible. Dynamical evolution results corroborate the stability picture obtained, and possible avenues for dark soliton generation are explored.

### 2.1 Model \& Theoretical Background.

We use the generalized nonlinear Schrödinger equation for the electric field envelope $\tilde{\Psi}$ with quadratic and quartic dispersion and a Kerr nonlinearity:

$$
\begin{equation*}
i \frac{\partial \tilde{\Psi}}{\partial \xi}+\frac{\tilde{\beta}_{4}}{24} \frac{\partial^{4} \tilde{\Psi}}{\partial \tau^{4}}-\frac{\tilde{\beta}_{2}}{2} \frac{\partial^{2} \tilde{\Psi}}{\partial \tau^{2}}+\gamma|\tilde{\Psi}|^{2} \tilde{\Psi}=0 \tag{2.1}
\end{equation*}
$$

[^2]where $\xi$ is the propagation distance, $\tau$ is the retarded time in the frame of the pulse and $\gamma$ is the nonlinear coefficient, which we shall take to be positive. The parameters $\tilde{\beta}_{2}=d v_{g}^{-1} / d \omega$ and $\tilde{\beta}_{4}=d^{3} v_{g}^{-1} / d \omega^{3}$, where $v_{g}$ is the group velocity and $\omega$ is the pulse carrier frequency, characterize the quadratic and quartic dispersion strengths respectively. We normalize the retarded time in units of $t_{0}=1 \mathrm{ps}, t=\tau / t_{0}$, and similarly the propagation length in terms of a characteristic propagation length $z_{0}=1 \mathrm{~mm}, z=\xi / z_{0}$. Finally, we rescale $\Psi=\sqrt{z_{0}} \gamma \tilde{\Psi}$. We restrict ourselves to positive (normal) quartic dispersion, and consequently fix $\tilde{\beta}_{4}=1 \mathrm{ps}^{4} \mathrm{~mm}^{-1}$, consistent with experiment [27], leading to $\beta_{4}=+1$. The resulting normalized model takes the form:
\[

$$
\begin{equation*}
i \frac{\partial \Psi}{\partial z}+\frac{1}{24} \frac{\partial^{4} \Psi}{\partial t^{4}}-\frac{\beta_{2}}{2} \frac{\partial^{2} \Psi}{\partial t^{2}}+|\Psi|^{2} \Psi=0 \tag{2.2}
\end{equation*}
$$

\]

where the normalized quadratic dispersion parameter is given by $\beta_{2}=\tilde{\beta}_{2} /\left(z_{0} t_{0}^{2}\right)$. Typical experimental values for associated optical settings would involve a nonlinearity coefficient of $\gamma=4.07 \mathrm{~W}^{-1} \mathrm{~mm}^{-1}[27]$, temporal units of 1 ps and propagation units of 1 mm . For the CW and numerical investigations considered below, this would correspond to a power of 1.2 W , a pulse length of 40 ps , and a propagation distance of 1000 mm .

In the conservative case, we can look for stationary solutions $\Psi(t, z)=\psi(t) \exp (i \mu z)$, where $\mu$ is a nonlinearity-induced phase shift characterizing the stationary solution, leading to the following equation for the real amplitude $\psi(t)$ :

$$
\begin{equation*}
-\mu \psi+\frac{1}{24} \frac{d^{4} \psi}{d t^{4}}-\frac{\beta_{2}}{2} \frac{d^{2} \psi}{d t^{2}}+\psi^{3}=0 \tag{2.3}
\end{equation*}
$$

We investigate dark soliton solutions to Eq. (2.3), connecting to the continuous wave background $\psi_{c w} \equiv \pm \sqrt{\mu}$ (which implies: $\mu>0$ ). In the four dimensional phase space characterizing solutions to Eq. (2.3) these dark soliton solutions may
either connect the positive CW solution back to itself (homoclinic solutions, for even numbers of solitons), or connect CW solutions of different signs (heteroclinic solutions, for odd numbers of solitons). We apply periodic boundary conditions so we only find homoclinic solutions. We characterize these using the complementary power, which for a cavity of length $2 L$ is $Q_{c}=\int_{-L}^{L}\left(\psi_{c w}^{2}-\psi^{2}\right) d t$.

As a starting point, we analyze the nature of the CW, on top of which the dark solitons are built. We consider purely real perturbations in the form $\psi=$ $\left[\psi_{0}+\epsilon \exp (\lambda t)\right]$, and upon substitution of this perturbation into Eq. (2.3) and keeping only linear terms in $\epsilon$ we obtain the following equation for $\lambda$ :

$$
\begin{equation*}
\frac{1}{24} \lambda^{4}-\frac{\beta_{2}}{2} \lambda^{2}-\mu+3 \psi_{0}^{2}=0 \tag{2.4}
\end{equation*}
$$

Setting $\psi_{0}=\psi_{c w}$ and solving for $\lambda$ we find that if $\mu \leq 3 \beta_{2}^{2} / 4$ we have four imaginary values for $\lambda$ if $\beta_{2}<0$, or four real values if $\beta_{2}>0$. The former implies that the CW is a center, so it is not possible to approach the CW and therefore no dark solitons connecting to the CW can exist. Conversely, when $\lambda$ is purely real, the CW is a saddle point and we have the more familiar dark soliton regime. If instead $\mu>3 \beta_{2}^{2} / 4$ the $\lambda$ form a complex quartet, so the CW is a saddle-spiral: any approach to the CW is accompanied by oscillations. This behaviour is analogous to that observed for the bright soliton (where instead $\beta_{4}<0$ ). The dependence of the CW solution on the 2D parameter plane is shown in Fig. 2.1(a); the quadratic dependence of $\mu$ on $\beta_{2}$ (solid curve), is as expected from the model's scaling properties.

As we shall see, the nature of the zero solution can also play a role in the form of the stationary states. Substituting $\psi_{0}=0$ into Eq. (2.4) the eigenvalues $\lambda$ are an imaginary pair and a real pair, which makes the zero solution a saddle-center, and therefore enables connections to oscillations about zero. This behaviour is distinct from the case of $\beta_{4}<0[62,32,49]$ where the zero solution is either approached
asymptotically, or not at all.
While the analysis of the eigenvalues (2.4) indicates where we can expect homoclinic (multi-)dark soliton solutions, a necessary condition for stability of these solutions is that the CW is modulationally stable. To this end, we examine the growth of linear waves by more generally perturbing the CW background:

$$
\begin{align*}
\Psi(t, z)=\left[\psi_{c w}\right. & +\epsilon_{1} \exp (i k t) \exp (i \Omega z)  \tag{2.5}\\
& \left.+\epsilon_{2}^{*} \exp (-i k t) \exp (-i \Omega z)\right] \exp (i \mu z)
\end{align*}
$$

where the $\epsilon_{i}$ denote small perturbations. Substitution of (2.5) into Eq. (2.2) and solving for linear $\epsilon_{i}$ shows how the perturbation eigenvalue $\Omega$ is related to the perturbation wavenumber $k$ :

$$
\begin{equation*}
\Omega^{2}=\left(-\mu+\frac{\beta_{2}}{2} k^{2}+\frac{1}{24} k^{4}+2 \psi_{c w}^{2}\right)^{2}-\psi_{c w}^{4} . \tag{2.6}
\end{equation*}
$$

If $\Omega^{2}<0$ then the perturbation undergoes exponential growth, with the boundary of modulational stability/instability occurring when $\Omega^{2}=0$. We see that we always have a low frequency ("acoustic") branch of modulational instability if $\beta_{2}<0$ (see Fig. 2.1(b)). Instability in this region proceeds with characteristic modulation of the background (Fig. 2.1(c)) and exponential growth of the most unstable wavenumber $k_{m}$ in the spectrum, $\tilde{\psi}$, of $\psi$. Note the almost perfect agreement between the growth rate predicted by Eq. (2.6) and the numerically observed value (Fig. 2.1(d)). The CW is modulationally stable if $\beta_{2} \geq 0$ and $\mu>0$, which is necessary for the stability of CW-based dark solitons.

### 2.2 Numerical Findings.

All propagation results (including the CW dynamics in Fig. 2.1) are obtained using a fourth-order split-step numerical scheme [74] applied to Eq. (2.2) with


Figure 2.1. (a) Classification of CW solution with $\beta_{2}$ and $\mu$; (b) modulational instability spectrum of CW for $\beta_{2}=-0.2$, with most unstable wavenumber $k_{m}=1.1$ corresponding to $\Omega=0.77 i$ (dot); (c) instability dynamics of CW for $\beta_{2}=-0.2$; and (d) dependence of spectral intensity $|\tilde{\psi}|^{2}$ on the evolution variable $z$ at wavenumber $k_{m}$ for results in panel (c), with slope (dashed line) giving $\Omega=$ $0.77 i$, agreeing very well with the prediction.


Figure 2.2. Pure quartic dark soliton stationary solutions $\left(\beta_{2}=0, \mu=5\right)$ : (a) a well separated pair of dark solitons; (b) a mixed solution connecting dark solitons to an oscillation about 0 .
$\Delta t=9.8 \times 10^{-3}$ and $\Delta z=7.6 \times 10^{-6}$. The robustness of the scheme is monitored by evaluating the Hamiltonian $H=\int(1 / 24)\left|d^{2} \psi / d t^{2}\right|^{2}+\left(\beta_{2} / 2\right)|d \psi / d t|^{2}+(1 / 2)|\psi|^{4} d t$ and verifying that this quantity is conserved during propagation. Stationary solutions are found by numerically solving Eq. (2.3) using a conjugate gradient method [74].

The linear analysis points to new possible features in dark soliton solutions. Figure 2.2 shows two extremes of possible pure-quartic solutions. Well separated dark solitons can be found (Fig. 2.2(a)), similar to those observed in the quadratic case, but with characteristic damped oscillations approaching the CW, consistent with earlier results for the real case [30]. Fig. 2.2(b) shows a new possibility enabled by the quartic dispersion, a connection between the CW and the saddle-centre at the origin. Such connections are impossible in the quadratic conservative case, but similar behavior has been observed in the presence of gain and loss [72].

Examining the possible solutions more systematically as a function of a system parameter, e.g. $\beta_{2}$ in Fig. 2.3(a), we find that many possible families exist, each with a characteristic spacing between the dark soliton pairs. The most closely spaced dark soliton pairs, which we call 'family 0' (Figs. 2.3(b) and (c)), connect the CW to the origin, and bifurcate from the CW solution at just less than $\beta_{2}=-2.5$. In contrast, all other families of increasingly widely spaced dark soliton pairs (e.g. 'family 1' and 'family 2' in Figs. 2.3(e) and (i) respectively), develop additional undulations as they proceed to negative $\beta_{2}$ (Figs. 2.3(d) and (h)), where they collide (in a saddle-center bifurcation) with an upper branch associated with large side oscillations (Fig. 2.3(f) and (j)). These upper branch solutions have additional dark soliton pairs on either side of the main pair, i.e., they are a triple pair family, which becomes increasingly evident as $\beta_{2}$ increases (Fig. 2.3(g) and (k)). These results are in line with the earlier real-field case analysis [38, 30], with the oscillatory


Figure 2.3. (a) Bifurcation diagram for the lowest-order dark soliton families with the labels corresponding to solutions shown in the lower panels. (b) and (c) Solutions corresponding to family 0, ultimately connecting to a plane wave as $\beta_{2}$ decreases; (d)-(g) solutions from family $1 ;(\mathrm{h})-(\mathrm{k})$ solutions from family 2 . Left and right panels correspond to $\beta_{2}=-1, \beta_{2}=1$, respectively. Unstable/stable solutions shown with dashed/solid lines respectively. In all cases $\mu=5$.


Figure 2.4. Numerical propagation of dark soliton solutions from different families at $\beta_{2}=0$ : (a) family 1 is unstable to fission; (b) family 2 is stable for $\beta_{2}>0$; (c) family 2 upper branch, all upper branch composite solutions are unstable; (d) solution shown in Fig. 2.2(b), it has a very weak oscillatory instability.
tails enabling isolated distances at which dark solitons become stationary.
We examine the stability of the solutions shown in Fig. 2.3 by perturbing the stationary solution and numerically computing the linearization eigenvalues [74]. A full presentation of the linear stability analysis is beyond the scope of this work; however, we find results consistent with the real case [30]. We find that family 0 and family 2 are stable for $\beta_{2} \geq 0$ (Fig. 2.3(c)), so stability alternates with spacing. This is topologically justified from the underlying alternating minima and maxima in the relevant effective landscape (c.f., e.g., Fig. 5 of [51] in the real-field case). All solutions are unstable for $\beta_{2}<0$ due to the modulational instability of the CW background. The continuous spectrum, as is typical for NLS dark solitons, spans the entire imaginary axis [45]. Stable and unstable families of solutions are shown respectively as solid and dashed lines in Fig. 2.3.

Examples of possible dynamics are shown in Fig. 2.4 for $\beta_{2}=0$. Figures 2.4(a)
and $2.4(\mathrm{~b})$ confirm respectively the instability of family 1 and the stability of family
2. While all upper branch solutions are unstable, the instability dynamics show long-lived non-stationary soliton dynamics which, as we shall see, appear to play a significant role in dark soliton generation processes. The instability dynamics shown in Fig. 2.4(c) are one example, with closely bound outer pairs appearing robust until disturbed by the instability products of the inner pair. The generalisation of family 0 to multiple oscillations about the zero solution, corresponding to Fig. 2.2(d), appears robust, as seen in Fig. 2.4(d), but linear stability analysis reveals a weak oscillatory instability.

We consider now the emergence of dark solitons from more general initial conditions. In our conservative system, the prototypical case is arguably a CW state with intensity notches, either of Gaussian or hyperbolic tangent form (domain-wall like).We note that such initial conditions are challenging to achieve in a cavity configuration; however, the dynamics is still instructive, and directly relevant to systems with appropriately engineered dispersion, including photonic crystal waveguides [27], Bose-Einstein condensates [57], and water waves [75].

With a Gaussian initial condition, we consider the dynamics in three different regimes: the pure quartic case $\left(\beta_{2}=0\right.$, Fig. 2.5(a)), positive $\beta_{2}\left(\beta_{2}=0.5\right.$, Fig. 2.5(b)), and negative $\beta_{2}\left(\beta_{2}=-0.2\right.$, Fig. 2.5(c)). In the pure-quartic case, a dark soliton pair appears, and exhibits a periodic cycle of collisions and separations, before escaping the local potential induced through the oscillatory tails, whereas for positive $\beta_{2}$, the dark solitons repel each other (which is the only possible dynamics in the pure positive quadratic dispersion case). For the negative $\beta_{2}$ case, the dark soliton pair is more tightly bound, but sits on a modulationally unstable CW background. A domain wall-like initial condition produces multiple dark soliton pairs, and even what appears to be a travelling complex of dark solitons (Fig. 2.5(d)),


Figure 2.5. Dark soliton generation from intensity notches in a CW background: (a) pure quartic dark soliton bound state with $\beta_{2}=0$; (b) repulsive dark soliton pair ( $\beta_{2}=0.5$ ); (c) dark soliton pair in the presence of modulational instability ( $\beta_{2}=-0.2$ ); (d) dark soliton complex at $\beta_{2}=0$. (a)-(c) Initial condition $\psi(t, 0)=\sqrt{\mu}\left(1-\exp \left(-t^{2} / 2\right)\right) ;(\mathrm{d})$ initial condition $\psi(t, 0)=$ $\sqrt{\mu}(1+(\tanh (t-5)-\tanh (t+5)) / 2)$, all with $\mu=5$.
similar to the instability dynamics observed earlier. We see that in the presence of instability (soliton or CW), solitons emerge, although they may be moving solitons. The nature of these moving solitons is an interesting direction for further research, particularly given the non-Galilean invariance of solitons in the presence of quartic dispersion [53].

## C H A P TER 3

# KINK-ANTIKINK INTERACTION FORCES AND BOUND STATES IN A NONLINEAR SCHRÖDINGER MODEL WITH QUADRATIC AND QUATRIC DISPERSION 

In this chapter ${ }^{1}$, we explore the competition of quadratic and quartic dispersion in producing kink-like solitary waves in a model of the nonlinear Schrödinger type bearing cubic nonlinearity. We present the first 6 families of multikink solutions and explore their bifurcations as the strength of the quadratic dispersion is varied. We reveal a rich bifurcation structure for the system, connecting two-kink states with states involving 4 -, as well as 6 -kinks. The stability of all these states is explored. For each family, we discuss a "lower branch" adhering to the energy landscape of the 2-kink states. We also, however, study in detail the "upper branches" bearing higher numbers of kinks. In addition to computing the stationary states and analyzing their stability within the partial differential equation model, we develop an effective particle ordinary differential equation theory that is shown to be surprisingly efficient in capturing the kink equilibria and normal (as well as unstable)

[^3]modes. Finally, the results of the bifurcation analysis are corroborated by means of direct numerical simulations involving the excitation of the states in a targeted way in order to explore their instability-induced dynamics.

### 3.1 Model Setup and Analysis

The generalized variant of the nonlinear Schrödinger (GNLS) equation that we study in the present work is [50]:

$$
\begin{equation*}
i u_{t}+\frac{\beta_{4}}{4!} u_{x x x x}-\frac{\beta_{2}}{2} u_{x x}+\gamma|u|^{2} u=0 \tag{3.1}
\end{equation*}
$$

where in an optical context $\beta_{4}$ characterizes the strength of the fourth-order dispersion, $\beta_{2}$ the strength of the second-order dispersion and $\gamma$ the strength of the cubic nonlinearity. We focus on the so-called quartic normal dispersion regime, where stable dark solitons have been found [50] in the presence of attractive nonlinearity, and so take $\beta_{4}>0$ and $\gamma>0$. Later we will restrict to $\beta_{4}=1$ and $\gamma=1$.

### 3.1.1 Stationary States and Spectral Stability

Eq. 3.1 is a Hamiltonian system, with conserved energy $\mathcal{E}$ given by

$$
\begin{equation*}
\mathcal{E}(u)=\frac{1}{2} \int_{-\infty}^{\infty}\left(\frac{\beta_{4}}{4!}\left|u_{x x}\right|^{2}+\frac{\beta_{2}}{2}\left|u_{x}\right|^{2}+\frac{\gamma}{2}|u|^{4}\right) d x \tag{3.2}
\end{equation*}
$$

Separating real and imaginary parts by taking $u=u_{R}+i u_{I}$, Eq. 3.1 can be written in standard Hamiltonian form as

$$
\begin{equation*}
\frac{\partial u}{\partial t}=J \mathcal{E}^{\prime}(u(t)), \tag{3.3}
\end{equation*}
$$

where $u=\left(u_{R}, u_{I}\right)^{T}, \mathcal{E}^{\prime}(u(t))$ is the functional derivative of $\mathcal{E}(u)$ evaluated at $u(t)$ and $J$ is the standard symplectic matrix

$$
J=\left[\begin{array}{cc}
0 & 1  \tag{3.4}\\
-1 & 0
\end{array}\right]
$$

Eq. 3.3 then becomes the pair of real-valued equations

$$
\begin{align*}
& \left(u_{R}\right)_{t}=-\frac{\beta_{4}}{24}\left(u_{I}\right)_{x x x x}+\frac{\beta_{2}}{2}\left(u_{I}\right)_{x x}-\left(u_{R}^{2}+u_{I}^{2}\right) u_{I}  \tag{3.5}\\
& \left(u_{I}\right)_{t}=\frac{\beta_{4}}{24}\left(u_{R}\right)_{x x x x}-\frac{\beta_{2}}{2}\left(u_{R}\right)_{x x}+\left(u_{R}^{2}+u_{I}^{2}\right) u_{R} . \tag{3.6}
\end{align*}
$$

In what follows, we are interested in stationary (time-independent amplitude) solutions of the form: $u(x, t)=e^{i \mu t} \phi(x)$. Substituting this ansatz into 3.1, we get the steady state model for the amplitude $\phi$

$$
\begin{equation*}
\frac{\beta_{4}}{4!} \phi^{\prime \prime \prime \prime}-\frac{\beta_{2}}{2} \phi^{\prime \prime}-\mu \phi+\gamma \phi^{3}=0 . \tag{3.7}
\end{equation*}
$$

In addition to the solution $\phi=0$, Eq. (3.7 has continuous wave (CW) solutions $\phi= \pm \sqrt{\mu / \gamma}$.

To facilitate our understanding of the nature of these fixed points, we rewrite 3.7 as a system of four first-order ordinary differential equations, using $U=\left(u_{1}, u_{2}, u_{3}, u_{4}\right)=$ $\left(\phi, \phi^{\prime}, \phi^{\prime \prime}, \frac{\beta_{4}}{24} \phi^{\prime \prime \prime}\right)$. Eq. 3.7 then becomes the first order system in $\mathbb{R}^{4}$

$$
U^{\prime}=F(U)=\left[\begin{array}{c}
u_{2}  \tag{3.8}\\
u_{3} \\
\frac{24}{\beta_{4}} u_{4} \\
\frac{\beta_{2}}{2} u_{3}+\mu u_{1}-\gamma u_{1}^{3}
\end{array}\right] .
$$

For $\beta_{4}>0, \mu>0$, and all $\beta_{2}$, the linearization about $\phi=0$ has a pair of real eigenvalues $\pm \alpha$ and a pair of imaginary eigenvalues $\pm \beta i$, thus the corresponding
equilibrium of 3.8 has a two-dimensional center subspace, and $\phi=0$ is a saddlecenter fixed point. The eigenvalues of the linearization about the CW states $\phi=$ $\pm \sqrt{\mu / \gamma}$ are instead

$$
\lambda= \pm \sqrt{\frac{6 \beta_{2} \pm 2 \sqrt{9 \beta_{2}^{2}-12 \beta_{4} \mu}}{\beta_{4}}}
$$

For fixed $\beta_{4}>0$ and $\mu>0$, corresponding equilibria $S^{ \pm}=( \pm \sqrt{\mu / \gamma}, 0,0,0)$ are saddle points of 3.8 when $\beta_{2}>-\beta_{2}^{*}$, where

$$
\begin{equation*}
\beta_{2}^{*}=2 \sqrt{\frac{\beta_{4} \mu}{3}} \tag{3.9}
\end{equation*}
$$

The stable and unstable manifolds of $S^{ \pm}$are both two-dimensional. When $\left|\beta_{2}\right|<$ $\beta_{2}^{*}$, the spatial eigenvalues are a complex quartet $\pm a \pm b i$, topologically corresponding to a saddle-spiral, and when $\left|\beta_{2}\right|>\beta_{2}^{*}$, they are two pairs of real eigenvalues $\pm b_{1}$ and $\pm b_{2}$, leading to a saddle point in the four dimensional space.

In our computations that will follow, the conditions $-\beta_{2}^{*}<\beta_{2}<\beta_{2}^{*}$ will play a pivotal role providing a set of bounds for $\beta_{2}$ under which the kink-antikink states of interest will exist. Later we will restrict our attention to the specific case of $\mu=5$.

A kink $\phi_{k}$ is a solution to 3.7 connecting the CW state at $-\sqrt{\mu / \gamma}\left(S^{-}\right)$to the one at $\sqrt{\mu / \gamma}\left(S^{+}\right)$. From a spatial dynamics perspective, this is a heteroclinic orbit connecting the saddle points $S^{-}$and $S^{+}$. If $\phi$ is a solution to Eq. (3.7), so is $-\phi$, thus for every kink solution $\phi_{k}$ there is a corresponding anti-kink $-\phi_{k}$. When $\beta_{4}=0$ and $\beta_{2}>0$, the exact formula for the stationary kink is given by

$$
\begin{equation*}
\phi_{k}(x)=\sqrt{\frac{\mu}{\gamma}} \tanh \left(\sqrt{\frac{\mu}{\beta_{2}}} x\right) . \tag{3.10}
\end{equation*}
$$

We take the existence of a primary kink solution to Eq. (3.7) as a hypothesis in what follows.

To study the stability of these solutions, we consider the linearization around $u(x, t)=e^{i \mu t} \phi(x)$, where $\phi(x)$ is a solution to Eq. (3.7). Adding the perturbation
as follows $u(x, t)=e^{i \mu t}[\phi(x)+v(x, t)]$, where $v(x, t)=v_{R}+i v_{I}$, substituting it into Eq. (3.1), we obtain two equations:

$$
\begin{aligned}
& \left(v_{R}\right)_{t}=-\frac{\beta_{4}}{4!} v_{I}^{\prime \prime \prime \prime}+\frac{\beta_{2}}{2} v_{I}^{\prime \prime}+\mu v_{I}-\gamma \phi^{2} v_{I} \\
& \left(v_{I}\right)_{t}=\frac{\beta_{4}}{4!} v_{R}^{\prime \prime \prime \prime}-\frac{\beta_{2}}{2} v_{R}^{\prime \prime}-\mu v_{R}+3 \gamma \phi^{2} v_{R}
\end{aligned}
$$

which can be written as

$$
\frac{\partial}{\partial t}\left[\begin{array}{c}
v_{R}  \tag{3.11}\\
v_{I}
\end{array}\right]=\left[\begin{array}{cc}
0 & -\mathcal{L}_{-}(\phi) \\
\mathcal{L}_{+}(\phi) & 0
\end{array}\right]\left[\begin{array}{l}
v_{R} \\
v_{I}
\end{array}\right]=-J \mathcal{L}(\phi)\left[\begin{array}{l}
v_{R} \\
v_{I}
\end{array}\right]
$$

where

$$
\begin{aligned}
\mathcal{L}_{+}(\phi) & =\frac{\beta_{4}}{4!} D^{4}-\frac{\beta_{2}}{2} D^{2}-\mu+3 \gamma \phi^{2} \\
\mathcal{L}_{-}(\phi) & =\frac{\beta_{4}}{4!} D^{4}-\frac{\beta_{2}}{2} D^{2}-\mu+\gamma \phi^{2} \\
\mathcal{L}(\phi) & =\left[\begin{array}{cc}
\mathcal{L}_{+}(\phi) & 0 \\
0 & \mathcal{L}_{-}(\phi)
\end{array}\right] .
\end{aligned}
$$

In what follows we find the lowest six families (as defined in Section 3.2) of stationary kink-antikink solutions numerically and examine their stability using Eq. (3.11).

### 3.1.2 Effective Particle Model

Using a method due to N. Manton [39] we now derive an ODE model of kink antikink interaction. For our model the Lagrangian is

$$
\begin{equation*}
L=\int_{-\infty}^{\infty} \mathcal{L} d x=\int_{-\infty}^{\infty}\left(\frac{i}{2}\left(u_{t}^{*} u-u^{*} u_{t}\right)-\frac{\beta_{2}}{2} u_{x}^{*} u_{x}-\frac{\beta_{4}}{4!} u_{x x}^{*} u_{x x}-\frac{1}{2} \gamma|u|^{4}\right) d x \tag{3.12}
\end{equation*}
$$

Invariance under translations gives rise to the conserved quantity

$$
\begin{equation*}
P=\int_{-\infty}^{\infty} \frac{i}{2}\left(u_{x}^{*} u-u^{*} u_{x}\right) d x \tag{3.13}
\end{equation*}
$$

which is the total momentum $P$ of the field $u$. In order to calculate the force $F$ between a kink and an antikink, we consider the momentum included in a finite interval $\left[x_{1}, x_{2}\right]$ and we differentiate with respect to time $t$.

$$
\begin{align*}
F=\frac{d P}{d t} & =\int_{x_{1}}^{x_{2}} \frac{i}{2}\left(u_{x t}^{*} u+u_{x}^{*} u_{t}-u_{t}^{*} u_{x}-u^{*} u_{x t}\right) d x \\
& =\int_{x_{1}}^{x_{2}} i\left(u_{x}^{*} u_{t}-u_{t}^{*} u_{x}\right) d x+\frac{i}{2}\left[u_{t}^{*} u-u^{*} u_{t}\right]_{x_{1}}^{x_{2}} \\
& =\left[\frac{\beta_{2}}{2} u_{x}^{*} u_{x}-\frac{\beta_{4}}{4!}\left(u_{x}^{*} u_{x x x}-u_{x x}^{*} u_{x x}+u_{x x x}^{*} u_{x}\right)-\frac{\gamma}{2}|u|^{4}+\frac{i}{2}\left(u_{t}^{*} u-u^{*} u_{t}\right)\right]_{x_{1}}^{x_{2}} \tag{3.14}
\end{align*}
$$

For $u(t, x)=e^{i \mu t} \phi(x)$, where $\phi(x)$ is a real static field this expression simplifies to

$$
\begin{equation*}
F=\left[\frac{\beta_{2}}{2} \phi^{\prime 2}+\frac{\beta_{4}}{4!}\left(\phi^{\prime \prime 2}-2 \phi^{\prime} \phi^{\prime \prime \prime}\right)-\frac{\gamma}{2} \phi^{4}+\mu \phi^{2}\right]_{x_{1}}^{x_{2}}=F_{x_{2}}-F_{x_{1}} \tag{3.15}
\end{equation*}
$$

which is zero, as expected for a static solution (the quantity inside the brackets is constant if $\phi$ satisfies Eq. (3.7).

Now, suppose we have a superposition of a kink centered at $x=-X$ and an antikink centered at $x=X$. Then the force on the antikink due to the kink is given by Eq. (3.15) for $x_{1}=0$ and $x_{2} \rightarrow \infty$, i.e., integrating across the antikink to find the force exerted on it due to the change of its momentum.

For $x_{2} \rightarrow \infty$, let $\phi \rightarrow-\sqrt{\mu / \gamma}$. Then,

$$
\begin{equation*}
F_{x_{2}}=\frac{\mu^{2}}{2 \gamma} \tag{3.16}
\end{equation*}
$$

For $x$ in the region between the two kinks, let $\phi(x)=\sqrt{\mu / \gamma}-\eta(x)$, for $\eta$ small. Then keeping up to second order terms we get

$$
\begin{equation*}
F_{x_{1}} \approx\left[\frac{\beta_{2}}{2} \eta^{\prime 2}+\frac{\beta_{4}}{4!}\left(\eta^{\prime \prime 2}-2 \eta^{\prime} \eta^{\prime \prime \prime}\right)+\frac{\mu^{2}}{2 \gamma}-2 \mu \eta^{2}\right]_{x=0} \tag{3.17}
\end{equation*}
$$

Therefore the force acting on the antikink is given in terms of $\eta(x)$ by the following expression:

$$
\begin{equation*}
F \approx\left[-\frac{\beta_{2}}{2} \eta^{\prime 2}-\frac{\beta_{4}}{4!} \eta^{\prime \prime 2}+\frac{\beta_{4}}{4!} 2 \eta^{\prime} \eta^{\prime \prime \prime}+2 \mu \eta^{2}\right]_{x=0} \tag{3.18}
\end{equation*}
$$

If $X$ is large enough, so that the two kinks are well separated, then $\eta(x)$ can be very well approximated by the superposition of their tails. In particular, for large positive $x$, a single kink can be written as $\phi_{K}(x)=\sqrt{\frac{\mu}{\gamma}}-\chi(x)$ (and similarly, for large negative $x$ a single antikink can be written as $\left.\phi_{A K}(x)=\sqrt{\frac{\mu}{\gamma}}-\chi(-x)\right)$, where the tail $\chi(x)$ satisfies the linearized problem

$$
\begin{equation*}
2 \mu \chi-\frac{\beta_{2}}{2} \chi^{\prime \prime}+\frac{\beta_{4}}{4!} \chi^{\prime \prime \prime \prime}=0 \tag{3.19}
\end{equation*}
$$

For $\beta_{2}<2 \sqrt{\frac{\beta_{4 \mu}}{3}}$ the linearized equation has vanishing solutions of the form

$$
\begin{equation*}
\chi(x)=e^{-r x}(A \cos (k x)+B \cos (k x)) \tag{3.20}
\end{equation*}
$$

where

$$
\begin{equation*}
r=\sqrt{\frac{2 \sqrt{3 \beta_{4} \mu}+3 \beta_{2}}{\beta_{4}}} \quad \text { and } \quad k=\sqrt{\frac{2 \sqrt{3 \beta_{4} \mu}-3 \beta_{2}}{\beta_{4}}} . \tag{3.21}
\end{equation*}
$$

Then, the superposition of the tails gives

$$
\begin{equation*}
\sqrt{\frac{\mu}{\gamma}}-\eta(x)=\phi(x) \approx \phi_{K}(x+X)+\phi_{A K}(x-X)-\sqrt{\frac{\mu}{\gamma}} \tag{3.22}
\end{equation*}
$$

from which we obtain

$$
\begin{equation*}
\eta(x) \approx \chi(x+X)+\chi(-x+X) \tag{3.23}
\end{equation*}
$$

Substituting this into Eq. (3.18) and using Eq. (3.20), we finally get

$$
\begin{align*}
F & \approx\left(2\left(A^{2}-B^{2}\right) \frac{r^{2} k^{2} \beta_{4}}{3}+4 A B r k \beta_{2}\right) e^{-2 r X} \cos (2 k X)+  \tag{3.24}\\
& +\left(4 A B \frac{r^{2} k^{2} \beta_{4}}{3}-2\left(A^{2}-B^{2}\right) r k \beta_{2}\right) e^{-2 r X} \sin (2 k X)
\end{align*}
$$

For an effective ODE description we need to find the inertial mass of the kinks. For $c$ small enough, our numerical computations of traveling kinks suggest that the field $u$ configuration can be written as $u(x, t)=e^{i \mu t}\left(\phi_{K}(x-c t)-c^{2} v(x-c t)+i c w(x-c t)\right)$.

This corresponds to the profile of a kink $\phi_{K}$ moving to the right with constant speed $c$, since $|u|^{2}=\phi_{K}^{2}(x-c t)+\mathcal{O}\left(c^{2}\right)$, while $w$ denotes to the leading order imaginary (linear in $c$ ) and $v$ the leading order real (quadratic in $c$ ) correction.

Now, from Eq. (3.1) we get

$$
\begin{equation*}
i\left(u_{t}^{*} u+u^{*} u_{t}\right)=-\frac{\beta_{2}}{2}\left(u_{x x}^{*} u-u^{*} u_{x x}\right)+\frac{\beta_{4}}{4!}\left(u_{x x x x}^{*} u-u^{*} u_{x x x x}\right) \tag{3.25}
\end{equation*}
$$

and since $|u|^{2}$ is a function of $x-c t$, the time derivative can be expressed as a spatial derivative multiplied by $-c$.

$$
\begin{equation*}
i\left(u_{t}^{*} u+u^{*} u_{t}\right)=i\left(|u|^{2}\right)_{t}=-i c\left(|u|^{2}\right)_{x} \tag{3.26}
\end{equation*}
$$

Integrating over $x$ gives

$$
\begin{equation*}
-i c|u|^{2}=-\frac{\beta_{2}}{2}\left(u_{x}^{*} u-u^{*} u_{x}\right)+\frac{\beta_{4}}{4!}\left(u_{x x x}^{*} u-u_{x x}^{*} u_{x}+u_{x}^{*} u_{x x}-u^{*} u_{x x x}\right)-i c K \tag{3.27}
\end{equation*}
$$

where $K$ is an integrating real constant. Of course, for this equation to hold as $x \rightarrow \infty$, we need $K=\frac{\mu}{\gamma}$. Integrating one more time over the whole $x$-axis and rearranging the terms, we get:

$$
\begin{align*}
\int_{-\infty}^{\infty} \frac{\beta_{2}}{2}\left(u_{x}^{*} u-u^{*} u_{x}\right) d x= & -i c \int_{-\infty}^{\infty}\left(\frac{\mu}{\gamma}-|u|^{2}\right) d x d x  \tag{3.28}\\
& +\frac{\beta_{4}}{4!} \int_{-\infty}^{\infty}\left(u_{x x x}^{*} u-u_{x x}^{*} u_{x}+u_{x}^{*} u_{x x}-u^{*} u_{x x x}\right) d x
\end{align*}
$$

Then the total momentum of the traveling kink is given by
$P=\frac{c}{\beta_{2}} \int_{-\infty}^{\infty}\left(\frac{\mu}{\gamma}-|u|^{2}\right) d x+\frac{\beta_{4} c}{12 \beta_{2}} \int_{-\infty}^{\infty}\left(\phi_{K} w^{\prime \prime \prime}-\phi_{K}{ }^{\prime} w^{\prime \prime}+\phi_{K}{ }^{\prime \prime} w^{\prime}-\phi_{K}{ }^{\prime \prime \prime} w\right) d x+\mathcal{O}\left(c^{2}\right)$
and using the definition $P=M c$ for the inertial mass we find

$$
\begin{equation*}
M=\frac{1}{\beta_{2}} \int_{-\infty}^{\infty}\left(\frac{\mu}{\gamma}-\phi_{K}^{2}\right) d x+\frac{\beta_{4}}{12 \beta_{2}} \int_{-\infty}^{\infty}\left(\phi_{K} w^{\prime \prime \prime}-\phi_{K}^{\prime} w^{\prime \prime}+\phi_{K}^{\prime \prime} w^{\prime}-\phi_{K}^{\prime \prime \prime} w\right) d x \tag{3.30}
\end{equation*}
$$

where $w$ must satisfy

$$
\begin{equation*}
-\mu w-\frac{\beta_{2}}{2} w^{\prime \prime}+\frac{\beta_{4}}{4!} w^{\prime \prime \prime \prime}+\gamma \phi_{K}^{2} w=\phi_{K}^{\prime} \tag{3.31}
\end{equation*}
$$

In what follows, we will consider the mass that solely stems from the first term of Eq. (3.30), i.e., the standard renormalized mass of the defocusing NLS problem (that has also been used, e.g., towards the proof of the stability of the dark solitons thereof in, e.g., [76]. This will be justified a posteriori via the comparison of our results with the detailed numerical computations. A rigorous justification of this choice from first principles is an interesting topic for future study.

### 3.2 Numerical Results and Comparison

We now restrict our attention to numerical solutions of Eq. (3.7) with $\mu=5$, $\gamma=1$, and $\beta_{4}=1$. Letting $\beta_{2}$ vary, we get families of solutions to Eq. (3.7), where the members of each family are connected by numerical continuation with respect to $\beta_{2}$. We start by finding kink-antikink solutions at $\beta_{2}=0$ corresponding to larger and larger separation of the kink and antikink, as in [50]. We then use numerical continuation to create families of solutions which we refer to as family 0 (continuation of the smallest possible separation of kink-antikink at $\beta_{2}=0$ ), family 1 (continuation of the second smallest possible separation of kink-antikink at $\beta_{2}=0$ ), and so on for families $2,3,4$, and 5 . Additional families have been identified in our numerical computations, however to keep the presentation more succinct, we do not discuss them here.

An overarching summary of our results is shown in Fig. 3.1. Here we see that each family of solutions has both an upper and lower branch, which are connected by numerical continuation, with the exception of family 0 , where the upper branch
was not created as a numerical continuation of the lower branch. Before we delve into the details of individual branches, we identify the main features encompassing all the branches on the figure. Throughout our analysis we make use of the complementary power $Q$ to characterize the families of solutions,

$$
\begin{equation*}
Q=\int_{-L}^{L}\left(\frac{\mu}{\gamma}-\phi^{2}\right) d x \tag{3.32}
\end{equation*}
$$

where $L$ is the half-width of the stationary solution domain. This is effectively the same quantity as the one defined by $M$ in the previous section, however to more clearly distinguish between the two (bearing in mind that in the latter there is, in principle, also a contribution $\propto \beta_{4}$, we use a different symbolism here.

We can see on the right side of Fig. 3.1, as $\beta_{2} \rightarrow 1$, that the branches form 3 groups, with each group distinguished by the number of kinks present (2, 4 and 6 in increasing complementary power). The lowermost group involves what we will hereafter term "lower branches" for all the families considered below. This concerns the states involving 2 kinks that are well-separated in this large and positive $\beta_{2}$ limit. The next group, encompassing solely family 0 (black upper branch) and family 3 (green upper branch), involves solutions consisting of 4 kinks. Finally, after a similar "jump" in complementary power, we encounter all remaining upper branches (e.g., the red of family 1 , the blue of family 2 , the purple of family 4 and the gray of family 5), which are all solutions with 6 kinks. These results relate to the large and positive $\beta_{2}$ limit, but in the case where $\beta_{2}$ becomes negative, we encounter a similar partition between the branches. Namely, the black and green branches are somewhat "special". The former tends to a limit of progressively smaller complementary power, i.e., tending to a small-amplitude steady oscillation about the CW solution itself, while the latter has a turning point $\beta_{2}^{c r}$ which is distinct from that of all other branches. However the 4 remaining branches (1, 2, 4

(a)

Figure 3.1. Bifurcation diagrams of six families of solutions: family 0 (black), family 1 (red), family 2 (blue), family 3 (green), family 4 (purple), family 5 (gray). In all cases $\mu=5, \gamma=1$, and $\beta_{4}=1$.
and 5) seem quite similar at the level of this complementary power diagnostic. For these 4 branches all solutions starting on a lower branch feature a turning point for a negative value of $\beta_{2}$ (near -2$)$ and subsequently continue along an upper branch towards the 6-kink configuration discussed above. A final observation that we make based on earlier analysis [50] is that the CW is modulationally stable whenever $\beta_{2} \geq 0$, but when $\beta_{2}<0$ there is a continuous band of modulationally unstable wavenumbers with bounds $k= \pm \sqrt{-12 \beta_{2} / \beta_{4}}$. Indeed, the relevant branch(es) can never be stable for $\beta_{2}<0$. We now turn to details of each of the relevant families.


Figure 3.2. (a) Bifurcation diagram ( $Q$ vs $\beta_{2}$ ), the corresponding steady state solutions and spectra for Family 0, presented for a sequence of values of the quadratic dispersion parameter $\beta_{2}$. (b) the same bifurcation diagram as (a) but zoomed in about the intersection of the upper and the lower curves.

### 3.2.1 Family 0

We start by describing the family 0 presented in the bifurcation diagram of Fig. 3.2 and the dynamics plots of Fig. 3.3. Fig. 3.2(a) illustrates the profiles and spectral planes of the stability analysis associated with this branch for different values of $\beta_{2}$. If we look at the zoomed in bifurcation diagram on the right, Fig. 3.2(b), we see that the states appear to bifurcate from the CW background as localized wavepackets (with increasing $\beta_{2}$ ), for both the lower and upper branches. These branches tend to a small oscillation about the fixed point $\sqrt{\mu / \gamma}$ (for decreasing $\beta_{2}$ ), of approximately 0.04 , as $\beta_{2}$ tends to $\beta_{2}^{*} \approx-2.58$. This limiting amplitude gets smaller as the simulated $x$-domain gets larger (for double the size of the $x$-domain this amplitude is approximately 0.02 ) and so it is reasonable to assume that the limiting steady state flattens out as the size of the $x$-domain approaches infinity.

Following the lower branch to increasing $\beta_{2}$ (lower panels of Fig. 3.2(a)) we see


Figure 3.3. PDE and ODE initial conditions and dynamics for family 0 (upper branch only for PDE). The upper figure of (a) shows the plots of $\|u\|^{2}$ (black), $\left\|u+v_{1}\right\|^{2}$ (red), and $\left\|u-v_{1}\right\|^{2}$ (blue) for $\beta_{2}=0.5$, where $v_{1}$ is the eigenfunction corresponding to the only real PDE eigenvalue of 0.2231 . The lower left panel of (a) is the contour plot that results from using $\left\|u+0.01 v_{1}\right\|^{2}$ as the initial condition, and the lower right panel of (a) is the contour plot that results from using $\left\|u-0.01 v_{1}\right\|^{2}$ as the initial condition. (b) gives the ODE values for the soliton positions (left of vertical line is the lower branch, right of the vertical line is the upper branch) and the ODE eigenvalues along the top (again lower branch left and upper branch right). Arrows on the points indicate the initial directions of the solitons (all directions would be reversed if $v_{1}$ is replaced by $-v_{1}$.)
that the state can be described as a kink-anti-kink pair with an increasing separation distance. Indeed, the analysis of [51] which is generalized in Sec. 3.1.2, shows that the quartic dispersion induces an oscillatory tail in the kinks which (competing with the quadratic dispersion), in turn, enables the possibility of bound states between two kink-like structures. This landscape consists of an alternation of local energy minima (such as the present one), forming center points in the landscape of the soliton center dynamics, and local energy maxima, which, naturally, correspond to saddle points in the relevant landscape. For the centers like the one corresponding to the lower bifurcation branch, we expect stability, at least as far as the motion of the kink centers is concerned, and indeed we see in the bottom right of Fig. 3.2(a) that there are no instability eigenvalues (when $\beta_{2}>0$ ).

Turning our attention now to the upper branch in Fig. 3.2 we see that this corresponds to a state consisting of four kinks (i.e., four zeros in the amplitude at large $\beta_{2}$ ), but it also bifurcates from the nearly flat state at $\beta_{2}^{*} \approx-2.58$. Indeed, it appears (see the bottom insets of Fig. 3.2(b)) that the two and 4 kink states merge, in the sense of the complementary power Q , in the small amplitude limit (about the CW). In contrast to the lower branch however, the upper branch appears to always be unstable. Recall that for this family, the upper branch is not a numerical continuation of the lower branch (as is the case for the other families) but rather is calculated the same way that the lower branch is calculated, using a numerical continuation from a carefully chosen initial steady state at $\beta_{2}=0$.

While an oscillatory eigenvalue quartet seems to exist, we will not discuss such instabilities at length, as they appear in our computations to be strongly dependent on the computational domain size. Indeed, similarly to other such examples in the realm of dark solitons (starting with the work of [58]), the presence of so-called anomalous modes, pertaining to the motion of the solitary waves, inside the con-
tinuous spectrum gives rise to such resonances which are domain-size dependent as the latter determines the (finite-domain-induced) "quantization" of the continuous spectrum. On the other hand, we observe that the 4 kinks have 3 internal modes in their dynamics (in addition to their translational motion which is neutral and pertains to a so-called Nambu-Goldstone mode associated with the corresponding invariance). Indeed, for the 2 kinks, there is only one motion, in addition to their neutral translation, namely the out-of-phase relative motion thereof, while generally for $N$ kinks, we should expect $N-1$ such internal modes associated with the kink relative motions. In the case of the upper branch of family 0, Fig. 3.3 elucidates the situation. In particular, its right panel uses the approach pioneered by Manton [39] and analyzed above in Sec. 3.1.2 to identify the equilibrium kink configuration and performs a linearization analysis around it, in the form of a $4 \times 4$ system to obtain the effective particle normal modes Two of these are oscillatory (featuring one pair moving towards each other and one away from each other), while the third is an unstable real mode with the kinks moving in opposite outward directions. The kinks centered in the positive half-line move in unison and so do the ones in the negative half-line, but these two pairs move in opposite directions between them. This instability can lead to a splitting of the 4 -kink bound state into two 2-kink bound states as shown in Fig. 3.3(a), but it can also lead all 4 to collide at the center, featuring a long-lived breathing state before eventually separating. Notice that, as explained in the inset, in each of these dynamical evolution cases, we will present the case example where we have added the unstable eigenvector to the configuration, and also the one where we have subtracted it. These two possibilities have been used in order to seed the instability in two opposite directions, as seen in panel (a) of the figure and, similarly, in other examples of such seeding presented below.


Figure 3.4. Bifurcation diagram and the corresponding steady state solutions and spectrums for a) family $1, b$ ) family $3, \mathrm{c}$ ) family 5 .

### 3.2.2 Families 1, 3, 5

These families are grouped together as they concern unstable saddle configurations in their respective lower branches, as is clearly manifested in each of the bottom right insets in panels (a)-(c) in Fig. 3.4. Indeed, in each case the two-kink configurations pertain to the first, the second and the third local maxima of the energy landscape associated with the two 2-kink states which means that their out-of-phase motion should give rise to a dynamical instability. Consequently the two bottom right insets in each branch feature the associated real pair with a corresponding eigenmode that should dynamically destabilize the relevant state. As we move towards negative $\beta_{2}$, once again the modulational instability discussed previously takes place and all relevant configurations are unstable due to the continuous spectrum portion lying along the real axis.

Each of the relevant families features a turning point beyond which we move to the upper portion of the corresponding branches. These branches feature 6 kinks for the families 1 and 5 and 4 kinks for the family 3 . As discussed previously, family 3 , along with family 0 are special in this regard, while all other families feature 6 kink states in their upper portions.

To better understand this we need to turn our focus to the single kink bifurcation diagram. As we move to the left along the lower branch (and $\beta_{2}$ decreases), undulations on the oscillating tails on both sides of the kink increase in size. When the largest undulation (of which there are two, one on each side, due to symmetry) reaches a critical size, we reach the turning point. Then as we move on to the upper branch, (and now $\beta_{2}$ increases), the undulations decrease in size, except for the two largest ones. These continue to grow and eventually give rise to the two new kink pairs, one on each side of the kink.

Two-kink solutions behave in a similar manner, but the number and the location of the new kink pairs depend on the distance between the two original kinks. That distance can only be such that the oscillating tails between the two kinks interfere either constructively (odd families) or destructively (even families). In the latter case, the two largest undulations will be the ones outside of the kink pair and these will give rise to the two new kink pairs. In the former case, the two largest undulations will be the ones inside the kink pair, if there is enough space to do so, as in families $5,7,9 \ldots$, where two new pairs appear in the region inside the two kinks. In the case of family 3 , there is space for only one new pair to appear, while in the case of family 1 there is no space for any such pair at all, so the new pairs can appear from the undulations outside the two original kinks.

The interference between the oscillating tails of the two kinks can also explain the differences between the value of $\beta_{2}^{c r}$ of each family. In the constructive case, the largest undulations reach their critical size earlier, as we move to the left. So we reach the turning point for larger $\beta_{2}$, compared to the single kink case. In the destructive case, on the other hand, undulations are smaller so we need to move further to the left for them to reach their critical size. So the turning point corresponds to $\beta_{2}$ smaller than the one in the single kink case (with the notable


Figure 3.5. (a) Phase potraits for family 0 (dashed line) and family 1 (solid line) in the plane of $u-u_{x}$. Panel (a) shows a larger scale, while panels (b) and (c) manifest zooms near the right fixed point. One can see the resulting loops that are associated with the exponentially decaying in amplitude oscillatory tails connected with the saddle-spiral fixed point.
exception of family 1 ).
It should be added here that for the families 0 and 1 , we have also depicted the phase portrait of the plane $\left(u, u_{x}\right)$ of Fig. 3.5. The aim of the figure is to showcase how for family 0 , the kink-antikink profile only loops around 0 , but does not make it to loop around the (spatial) fixed point of $-\sqrt{\mu / \gamma}$, while in the case of family 1 , that looping does (as the first such example among the families) take place. The effectively self-similar pattern of the spatial configuration as it loops around the saddle-spiral fixed point at $u=\sqrt{\mu / \gamma}$ is further illustrated in the zooms of panels
(b) and (c).

Turning now to the upper branch of family 1 , the analysis of the relevant state is conveyed in Fig. 3.6. Panel (d) summarizes our theoretical predictions. More concretely, the 6 -kink state features 2 oscillatory modes and 3 real ones, in addition to the neutral translational one. The one with the largest growth rate $(\approx 0.391)$ features an out-of-phase motion of the two innermost kinks, while the other 4 remain essentially immobile. This instability is showcased in the dynamical evolution of panel (a) where we see that these inner kinks may either move outward colliding with the other two pairs (and forming breathing pairwise bound states, while the outermost kink is expelled) or they may move inward, collide and then move outward again, leading to the same fate as the previous example. We have also excited the two other unstable modes in panel (b) (for growth rate $\approx 0.267$ ) and panel (c) (with growth rate $\approx 0.12$ ), respectively. In the former case, the two inner kinks move in one direction, while the four outer ones move in the opposite direction. In both shown examples of panel (b), this leads to collisions and pairwise formations of one kink with a bound state pair. In each example where this happens, there is a "change of allegiance". The pair member closest to the single kink now forms a bound state with the formerly single kink, while the pair member furthest from the single kink is now "freed" and moves in the direction that the single kink used to move. In the case of the eigenmode excited in panel (c) the outer kinks move outward or inward, while the centermost pair stays put. However, what ends up being observed is more akin to the dynamics of panel (a), which appears to be the dominant instability, since the associated eigenmode growth rate is a factor of (nearly) 4 times larger than the eigenmode initially excited in panel (c). See also the discussion surrounding the case example shown later in Fig. 3.13(e).

In the case of family 3 , the corresponding dynamical picture is provided in


Figure 3.6. PDE and ODE initial conditions and dynamics for family 1 (upper branch only for PDE). The upper figures of (a), (b), (c) show the plots of $\|u\|^{2}$ (black), $\left\|u+v_{j}\right\|^{2}$ (red), and $\left\|u-v_{j}\right\|^{2}$ (blue) for $\beta_{2}=0.5 . v_{j}$ is the eigenfunction corresponding to the PDE real eigenvalue $\lambda_{j}$, with (a) $\lambda_{1}: 0.3808$ (b) $\lambda_{2}: 0.2638$ and (c) $\lambda_{3}$ : 0.1263 . For each of (a), (b), (c), the lower left figure is the contour plot that results from using $\left\|u+0.01 v_{1}\right\|^{2}$ as the initial condition, and the lower right figure is the contour plot that results from using $\left\|u-0.01 v_{1}\right\|^{2}$ as the initial condition. (d) gives the ODE values for the soliton positions (left of the vertical line is the lower branch, right of the vertical line is the upper branch) and the ODE eigenvalues along the top (again lower branch shown left, and upper branch shown right). Arrows on the points indicate the initial directions of the solitons (all directions would be reversed if $v_{1}$ is replaced by $-v_{1}$.)

(a)

(c)

Figure 3.7. PDE and ODE initial conditions and dynamics for Family 3 (upper branch only for PDE). Similar to Figure 3.6, except for Family 3 instead of Family 1, with $\beta_{2}=0.5$ and PDE eigenvalues (a) $\lambda_{1}$ : 0.3083 and (b) $\lambda_{2}: 0.2104$.

Fig. 3.7. The lower branch situation is again simple (with the out-of-phase motion of the two kinks predicted in panel (c) being responsible for the instability of this saddle-point configuration at a larger distance of $\approx 2.46$ at equilibrium. However, as indicated above, this is an example whereby the upper branch involves only 4 kinks. In this setting, our effective particle theory predicts the existence of 2 unstable modes with growth rates $\approx 0.305$ and $\approx 0.22$. The largest growth rate involves the inner kinks moving in one direction and the outer ones in the opposite, as shown in panel (a) of the figure. This leads, in line with what we saw before, to the collision of the inner pair with one of the outer kinks, and once again the same phenomenon of "change of allegiance" as discussed above. The less rapid growth is associated with a mode whereby the inner kinks stay put while the outer ones move either outward or inward (depending on the sign of the perturbation), as shown in panel (b). Among these cases, the outward motion is more "benign" as the outer kinks depart to (in principle) infinity, while the inner kinks remarkably are sitting at the equilibrium distance of the lower branch of family 0 and, hence, will stay at that distance indefinitely given the stability of the latter configuration. A far more elaborate scenario takes place when the outer kinks first move inward. In this case they collide with the central kinks leading to an expulsive event where, pairwise, two sets of kinks (the upper and lower ones, so to speak) are expelled outward in a breathing, propagating state. While this seems like a nearly bound state, the distance between the kinks appears to be increasing as they move suggesting that it does not pertain to a stable configuration. Nevertheless, an exploration of such breathing, propagating states could be an interesting topic for future study, as it is outside the scope of the present work.

We now turn to family 5 , which again like most families has 6 kinks in its upper portion (in addition to 2 substantially separated kinks at distance of $\approx 3.72$ in its


Figure 3.8. PDE and ODE initial conditions and dynamics for Family 5 (upper branch only for PDE). Similar to Figure 3.6, except for Family 5 instead of Family 1, with $\beta_{2}=0.5$ and PDE eigenvalues (a) $\lambda_{1}$ : 0.3105 , (b) $\lambda_{2}: 0.2640$, and (c) $\lambda_{3}: 0.1509$.
saddle-configuration lower portion). Here again, we encounter a situation involving 3 unstable modes of the upper branch, along with 2 oscillatory ones which have also been included for completeness in Fig. 3.8, in addition to the neutral translational mode. The unstable modes have growth rates of $\approx 0.304,0.267$ and 0.155 as indicated in panel (d) of the figure. The most unstable among these modes involves the out-of-phase motion of the two inner kink pairs of this configuration and the opposite to them, also out-of-phase motion of the outer kink pair. This can lead, as shown in panel (a) of the figure, e.g., to a collision of the kink pairs with the outer kinks, leading to a change of allegiance and then complex dynamics since the split innermost kinks collide between them and then again with the breathing pairs (leading to further change of allegiance etc.). In the case of the two pairs moving inward they collide with each other, while the outer kinks move outward. In this case, the complex dynamics of the 4 -kink collision near the center eventually leads, upon breathing, to two outer moving and breathing pairs, once again reminiscent of the ones we saw in family 3. Again, such dynamics as well as similar pair breathing and propagating, for instance, in panel (b) of the figure are motivating towards further study of such states. The case of panel (b) involves a weak in-phase motion of the 4 inner kinks and a stronger opposite direction motion of the 2 outer ones. This leads to a collision of one of the outer kinks with one of the inner pairs, and then a resulting cascade of two changes of allegiance as observed in both instances of panel (b) resulting eventually in two breathing pairs moving in one direction and two isolated kinks in the opposite direction. Finally, the weakest unstable mode of panel (c) involves all kinks for $x>0$ moving in the same direction and similarly all those for $x<0$ moving in the opposite direction. In the first example of panel (c) this leads to no collisions with the kinks continuing to move in their original direction. In the second example, all kinks move towards the center and the two


Figure 3.9. PDE initial conditions and dynamics for lower branches of Families 1, 3, 5. Similar to Figure 3.6, except for lower branches instead of upper branches with (a) Family 1, $\lambda_{1}=0.3012$, (b) Family 3 , $\lambda_{1}=0.0066$, (c) Family $5, \lambda_{1}=0.00014$. The only exception is the bottom right panel of (c). For the corresponding ODE initial conditions and dynamics see the bottom panels (left of the vertical line) in each of Figures 3.6, 3.7, and 3.8.
pairs collide there, leading to a breathing long-lived excitation, while the outer kinks initially moving inward are eventually led, through interaction (and perhaps the dominance of the most unstable mode of the highest growth rate) to move in the opposite direction, diverging away from the center.

We now turn to a description of the lower branches and their dynamics for these families for reasons of completeness. This is shown in Fig. 3.9. Panel (a) in the figure shows the 2 kink dynamics in family 1. The relevant unstable (saddle) configuration either destabilizes with the kinks moving outward, or does so with them moving inward (toward the center) colliding at $x=0$ and then subsequently moving outward. Similar examples are shown in panel (b) for the case of family 3 . The only difference in this case is that when the kinks move inward, they encounter a higher barrier (that imposed by the solution of family 1) and hence get trapped in the well between family 3 and family 1 . This is the well involving the stable solution of the family 2 around which the dynamics ends up orbiting in the bottom
right panel (b). Finally, a similar phenomenology is present in the case of panel (c). Interestingly, in this case, the oscillation is between family 5 and family 3 unstable saddle configurations, which means that the dynamics is orbiting around the center (stable) configuration of family 4.

### 3.2.3 Families 2 and 4

Lastly, we briefly refer to families 2 and 4, showcased in Fig. 3.10. Here, as explained at the level of the theoretical analysis of the energy landscape, but also corroborated by related numerics of the unstable families, the lower branches concern solutions that are stable. Indeed these are center configurations (around which the dynamics may orbit, as a result of the instability of the saddles above). This is reflected in the stable nature of the two bottom right subplots in panels (a) and (b) within Fig. 3.10. As before, crossing through negative $\beta_{2}$ in both families leads to modulationally unstable backgrounds with continuous spectrum crossing through to the real line. Past the turning point, we revert to the upper branches for each configuration which look fairly similar and essentially differ in the location of the resulting 6 kinks. Interestingly the inner kinks remain at the same distance as for the stable lower branch (for each of the families 2 and 4) and two outer pairs of kinks are added to the configuration at larger distances.

As regards the upper branches of each of these families, panels (c) and (d) of Fig. 3.10 reflect the theoretical predictions for their stability. In each case there are two unstable modes (rather than 3) which, in fact, have very proximal eigenvalues. It is for that reason that these two pairs of real modes cannot be distinguished in the two upper right insets of panels (a) and (b). Recall that the oscillatory instabilities are not systematically considered here given their size dependence. The other 3 pairs of nonzero modes of the 6 -kink system are imaginary and are


Figure 3.10. Bifurcation diagrams and the corresponding steady state solutions and spectra for (a) Family 2 (b) Family 4. The former are presented in the format shown previously of $Q$ vs. $\beta_{2}$. The latter indicate the eigenvalues and prescribe the motion of the solitary waves, in line with the desired eigenmode.
also given in panels (c) and (d). In either case, the destabilizing eigenmodes are similar and involve either an in-phase motion of the inner 2 kinks while the outer 4 ones are moving in the opposite direction or an out-of-phase motion of the inner kinks which on each "side" (i.e., for $x>0$ or $x<0$ ) is opposite to the motion of the outer kinks. For brevity, we do not present the dynamical implementation of these cases, although a similarly good agreement with the predictions of the theory has been found in this case.

Indeed, we elaborate a bit further on the quantitative aspects of the comparison of the theory with our numerical computations now. This comparison can be seen as summarized in the two extensive tables 3.1 and 3.2. The former of these tables offers the comparison of the equilibrium configurations in the context of the ODE theoretical approach of section 3.1.2 and the full PDE results. In the latter, the zero crossings of configurations with 2 (all lower branches), 4 (upper branches of families 0 and 3) and 6 kinks (remaining upper branches) have been identified and listed. One can observe a very good agreement between the two. This only deteriorates a little in the cases of outermost kinks but is still qualitatively excellent and even quantitatively satisfactory.

An even more stringent test of the theory (in comparison to equilibrium positions of ODE vs. PDE) consisted of the examination of the relevant internal modes of vibration presented in Table 3.2. Here, we have included for completeness the motion induced by the mode (e.g., as we have already discussed, all lower branch non-vanishing modes should be out-of-phase), as well as the spatial parity of the mode. The former motion is useful towards understanding the unstable dynamics induced by the mode (this was also explained in the discussion of the different families above). The latter is in line with the expectations of Sturm-Liouville theory, wherever appropriate (given the 1d nature of our system). Remarkably, we see that

| Family | Branch | Soliton Position |  |
| :---: | :---: | :---: | :---: |
|  |  | ODE | PDE |
| 0 | lower | 0.5778 | 0.6084 |
|  | upper | 1.2063 | 1.2229 |
|  |  | 2.3620 | 2.4389 |
| 1 | lower | 1.2063 | 1.20534 |
|  | upper | 1.2063 | 1.20493 |
|  |  | 3.6190 | 3.63302 |
|  |  | 4.7747 | 4.84924 |
| 2 | lower | 1.8349 | 1.83539 |
|  | upper | 1.8349 | 1.83495 |
|  |  | 4.2476 | 4.26350 |
|  |  | 5.4032 | 5.47906 |
| 3 | lower | 2.4634 | 2.46336 |
|  | upper | 0.5778 | 0.60799 |
|  |  | 2.9905 | 3.03715 |
| 4 | lower | 3.0919 | 3.09138 |
|  | upper | 3.0919 | 3.27728 |
|  |  | 5.5046 | 5.70647 |
|  |  | 6.6602 | 6.92233 |
| 5 | lower | 3.7204 | 3.72945 |
|  | upper | 1.2063 | 1.22286 |
|  |  | 2.3620 | 2.43858 |
|  |  | 4.7747 | 4.86794 |

Table 3.1. Soliton positions, ODE versus PDE, for $\beta_{2}=0.5$


Table 3.2. ODE versus PDE eigenvalues and PDE eigenvector symmetry for $\beta_{2}=0.5$. The PDE eigenvalues listed are those that were identified using an inverse participation ratio (IPR) plot; * represents eigenvalues that were not apparent from the IPR plot, but were the closest PDE eigenvalues to the corresponding ODE eigenvalues that also had the same initial direction signature.
in this case as well, the effective particle method of Section 3.1.2 is fairly accurate in its prediction of both the oscillatory and the growing modes of the system. As the table shows, this turns out to be the case for both lower and upper branches, and for all the different families considered from 0 to 5 . It is important to note here that for the real modes, such a comparison is relatively straightforward as the modes are separated from the rest of the spectrum. However, such a comparison is far more involved when we are, in principle, seeking localized modes involving relative kink motions "buried" within the continuous spectrum. Nevertheless, we have developed a technique based on the inverse participation ratio (IPR) [59] which enables us to identify modes with high IPR, even when embedded in the continuous spectrum, and to compare them favorably in many cases with the theoretical predictions. We now briefly discuss the associated details.

The Inverse Participation Ratio can be defined for a function $u(x)$ as

$$
\begin{equation*}
\operatorname{IPR}=\frac{\int|u|^{4} d x}{\left(\int|u|^{2} d x\right)^{2}} \tag{3.33}
\end{equation*}
$$

When $u$ is an eigenvector (eigenfunction), this quantity can be used to find eigenvectors that are the most localized, even when there is a continuous background present. We create an IPR plot, which gives the IPR value for each eigenvector, listed in order of the corresponding eigenvalue (using Matlab's default method of ordering complex eigenvalues). An example, corresponding to Family 1, upper branch, is given in Figure 3.11. Since the eigenvalues come in pairs ( $\pm$ pairs for the real eigenvalues and complex conjugate pairs -in fact, quartets $\pm \lambda_{r} \pm i \lambda_{i}$ for the complex valued ones), only the first of each pair that "stands out" from the others is marked with an asterisk and labeled with its eigenvalue. From this plot we infer that three real and one purely imaginary eigenvalue correspond to the most localized eigenvectors. Note that the slightly elevated parts of the graph near
eigenvalue order number 1130 correspond to eigenvalues that have both non-zero real and imaginary parts (which are not considered) and the elevated part near 1200 corresponds to a zero eigenvalue (representing translational invariance). Thus, the four eigenvalues identified in the figure are the ones listed in Table 3.2. Also note that the eigenvalue 2.2871 i listed in Table 3.2 has an asterisk, indicating that it does not correspond to an elevated IPR value.


Figure 3.11. Inverse Participation Ratio plot for Family 1, upper branch. Numerical values shown are eigenvalues corresponding to the eigenvector whose IPR is calculated and plotted. Eigenvectors with index values smaller than shown do not contribute significant IPR values.

Figure 3.12 shows the dynamics for several embedded (purely imaginary) eigenvalues. The plots in the first (left) column verify that for typical lower branch cases, pure oscillations occur for long periods of time, with the frequency given by the corresponding eigenvalue (the top left plot is also, in fact, unchanged up to $t=300)$. The plots in the second column show that for typical upper branch
plots, the expected oscillations (with frequencies corresponding to the -imaginary part of the - respective eigenvalues) occur for short periods of time, after which nonlinearity takes over as the solitary wave paths start to interact. The blue curves track the centers of the kinks, and are needed as the contour plots do not have fine enough resolution to show the oscillations. The oscillations manifested in these graphs (and their localized nature around the kink equilibria) confirm that the modes selected by the high IPR are embedded ones within the continuous spectrum associated with the effective normal modes of the kink-antikink interacting particle system.

Lastly, we should mention that in addition to exploring the growth rate of unstable configurations via spectral stability analysis, we have also resorted to an alternative method to corroborate our numerical stability results through direct numerical simulations. Indeed, we have considered a method of perturbing the unstable eigenvectors and subsequently monitoring the instability growth rates. Typical case examples of the corresponding results are shown in Fig. 3.13. Here, we compare the findings of the linear stability computations (via red solid lines) with the PDE simulations (via blue lines). In each case the blue lines represent the projections arising from subtracting from $u(x, t)$ the equilibrium solution $u_{0}$ and then projecting the difference to the instability eigenvector. With the dotted blue lines, we represent the dynamical outcome of positive perturbations, while with the dash-dot blue lines the case of a negative perturbation. The the red lines represent a least-squares straight line fit to the linear part of the blue curves in these semilog plots. In this way, we can corroborate the growth rate observed in the spectral analysis via the instability dynamics observed in the full PDE model. In essentially all the cases considered the agreement is found to be very good with respect to our theoretical expectations.


Figure 3.12. Dynamics corresponding to imaginary eigenvalues that are embedded in the continuous spectrum. In each case a small amount of an eigenvector with imaginary eigenvalue is added to the steady state, inducing an out-of-phase oscillation for a pair of solitons. The two panels in (a) represent out-of-phase oscillations for Families 0 (top figure, eigenvalue 2.0263) and 2 (bottom figure, eigenvalue 0.0446), both for the bottom branch. We show only the curve that represents the center of the soliton that appears on the positive side of the $x$-axis (and hence on top in the contour plots). All figures in (b) represent Family 1 , top branch, with eigenvalue 2.0334 . The three blue curves on the bottom again represent the motion of the center of each of the three solitons that appear on the positive side of the $x$-axis (corresponding to the top three solitons in the contour plot shown). These blue curves also appear superimposed on the contour plot, where due to scaling, the oscillations are not apparent.


Figure 3.13. Projection plots for $\beta_{2}=0.5$. In the top two rows, for a few selected solutions $u(x, t)$, we plot the scalar projection of $u(x, t)-u_{0}(x)$ in the direction of an eigenvector (as a function of time) using a semilog scale. For these solutions, the initial steady state $u_{0}(x)$ was slightly perturbed in the direction of said eigenvector. In each case the blue lines represent the projections, with the dotted blue lines representing positive perturbations, the dash-dot blue lines representing a negative perturbation; the red lines represent a least-squares straight line fit to the linear part of the blue curves. We observe a linear portion near the beginning of each plot, whose slope matches very closely with what is predicted by the corresponding eigenvalue. In all cases the slope of the projection curve matches the eigenvalue to two (for the smallest eigenvalues) or three decimal places. The cases are as follows. First row - steady state 3 (left) and steady state 1 (right), both lower branch (note the different time scales). Second row - steady state 4, largest real eigenvalue (left - even eigenvector) and steady state 4, second largest real eigenvalue (right - odd eigenvector - projections coincide). The figure in the third row shows how an initial (small) growth rate can transition to a larger growth rate (projection in blue). This figure corresponds to family 1 , upper branch where the initial growth rate of 0.124 (fitted line in red) transitions to a growth rate of 0.381 (fitted line in black). Here $u_{0}(x)$ was perturbed in the direction of the eigenvector with eigenvalue 0.12634 and then projected onto the eigenvector with eigenvalue 0.38075 .

## C H A P TER 4

# MIXED DISPERSION NONLINEAR SCHRÖDINGER EQUATION IN HIGHER DIMENSIONS: THEORETICAL ANALYSIS AND NUMERICAL COMPUTATIONS 

In this chapter ${ }^{1}$, we provide a characterization of the ground states of a higherdimensional quadratic-quartic model of the nonlinear Schrödinger class with a combination of a focusing biharmonic operator with either an isotropic or an anisotropic defocusing Laplacian operator (at the linear level) and power-law nonlinearity. Examining principally the prototypical example of dimension $d=2$, we find that instability arises beyond a certain threshold coefficient of the Laplacian between the cubic and quintic cases, while all solutions are stable for powers below the cubic. Above the quintic, and up to a critical nonlinearity exponent $p$, there exists a progressively narrowing range of stable frequencies. Finally, above the critical $p$ all solutions are unstable. The picture is rather similar in the anisotropic case, with the difference that even before the cubic case, the numerical computations suggest an interval of unstable frequencies. Our analysis generalizes the relevant observations for arbitrary combinations of Laplacian prefactor $b$ and nonlinearity

[^4]power $p$.

### 4.1 Mathematical Setup and main results

We start by noting that the problem of interest possesses continuous spectrum, which effectively, per Weyl's theorem [78], amounts to the spectrum of the homogeneous background state: $\sigma\left(\mathcal{L}_{ \pm}\right)=\sigma\left(\Delta^{2}+b \Delta\right)=$ Range $\left[\xi \rightarrow|\xi|^{4}-b|\xi|^{2}+\omega\right]=$ $\left[\omega-\frac{b^{2}}{4},+\infty\right)$. If we do not expect embedded eigenvalues in the essential spectrum ${ }^{2}$, and since by a direct inspection $\mathcal{L}_{-}[\Phi]=0$, so $0 \in \sigma_{p . p .}\left(\mathcal{L}_{-}\right)$(where $\sigma_{p . p \text {. denotes }}$ the pure point spectrum), then we can clearly conclude $\omega \geq \frac{b^{2}}{4}$, which corresponds to the range of frequencies of the standing wave that we will be considering in what follows.

Our principal theme of study will consist of the standing wave solutions of (1) and (2) (that is, the solutions of (3) and (4)). Main interest is in the (spectral) stability of these waves. For future reference, we introduce the associated Hamiltonian functionals,

$$
\begin{aligned}
& I(u)=\left\{\frac{1}{2} \int_{\mathbf{R}^{d}}|\Delta u|^{2}-\frac{b}{2} \int_{\mathbf{R}^{d}}|\nabla u|^{2}-\frac{1}{p+1} \int_{\mathbf{R}^{d}}|u|^{p+1}\right\}, \\
& J(u)=\left\{\frac{1}{2} \int_{\mathbf{R}^{d}}|\Delta u|^{2}-\frac{b}{2} \int_{\mathbf{R}^{d}}\left|u_{x_{1}}\right|^{2}-\frac{1}{p+1} \int_{\mathbf{R}^{d}}|u|^{p+1}\right\}
\end{aligned}
$$

[^5]and the associated constrained minimization problems
\[

$$
\begin{align*}
& \left\{\begin{array}{l}
I(u) \rightarrow \min \\
\int_{\mathbf{R}^{d}}|u(x)|^{2} d x=\lambda
\end{array}\right.  \tag{4.1}\\
& \left\{\begin{array}{l}
J(u) \rightarrow \min \\
\int_{\mathbf{R}^{d}}|u(x)|^{2} d x=\lambda
\end{array}\right. \tag{4.2}
\end{align*}
$$
\]

The solutions of these problems, if they exist, are referred to as normalized waves for the corresponding variational problems.

For the rest of the paper, we consider the case $b>0$ only. We have the following result.

Theorem 4.1 (The isotropic case) Let $d \geq 2$ and $b>0$. Then, there exists $a$ unique $p_{*}(d) \in\left(1+\frac{4}{d}, 1+\frac{8}{d+1}\right)$, so that

- For $1<p<p_{*}(d)$, the constrained minimization problem (4.1) has a solution for every $\lambda>0$. Moreover, such solutions satisfy the Euler-Lagrange equation

$$
\begin{equation*}
\Delta^{2} \Phi+b \Delta \Phi+\omega \Phi-|\Phi|^{p-1} \Phi=0, \quad x \in \mathbf{R}^{d} \tag{4.3}
\end{equation*}
$$

for some $\omega=\omega(\lambda)>0$. In addition, all the functions $e^{-i \omega(\lambda) t} \Phi$ are spectrally stable in the context of the isotropic NLS (1).

- For $1+\frac{8}{d}>p>p_{*}(d)$, there exists $\lambda_{*}(p, d, b)>0$, so that the problem (4.1) has solutions for all $\lambda>\lambda_{*}(p, d)$. These solutions are spectrally stable.

Our numerical results suggest that $3.2<p_{*}(2)<3.4$, with the relevant value being in the vicinity of $p_{*}(2) \approx 3.3$, yet the subtle nature of the numerical considerations near the limit only affords us an approximate result in this context.

Next, we have the following result regarding the anisotropic case.

Theorem 4.2 (The anisotropic case) Let $d \geq 2$ and $b>0$. Then,

- For $1<p<1+\frac{4}{d}$, the constrained minimization problem (4.2) has a solution for every $\lambda>0$. Moreover, all of these solutions are spectrally stable.
- For $1+\frac{8}{d}>p>1+\frac{4}{d}$, there exists $\lambda_{*}(p, d, b)>0$, so that the problem (4.1) has solutions for all $\lambda>\lambda_{*}(p, d)$. These solutions are spectrally stable.

Remark: The statement of the Theorem 4.2 does not imply that all waves are spectrally stable, but rather only that the minimizers of the constrained minimization problem (4.2) are guaranteed to be spectrally stable. In fact, in later sections, we numerically explore waves (i.e. functions satisfying (4)), which are not necessarily spectrally stable. Interestingly they happen to co-exist with stable constrained minimizers in that for a range of $p$, there exist multiple solutions corresponding to different frequencies with some (2) of them being stable and one unstable. We now turn to the systematic construction of the waves of interest.

### 4.2 Construction of the waves: Preliminaries

We begin our considerations with an analysis of when the constrained minimization problem (4.1) is well-posed. That is, whether the quantity $I[u]$ is bounded from below.

### 4.2.1 Well-posedness of the constrained minimization problems

To this end, introduce the following functions

$$
\begin{aligned}
m(\lambda) & =\inf _{\|u\|_{L^{2}}^{2}=\lambda}\left\{\frac{1}{2} \int_{\mathbf{R}^{d}}|\Delta u|^{2}-\frac{b}{2} \int_{\mathbf{R}^{d}}|\nabla u|^{2}-\frac{1}{p+1} \int_{\mathbf{R}^{d}}|u|^{p+1}\right\} \\
n(\lambda) & =\inf _{\|u\|_{L^{2}}^{2}=\lambda}\left\{\frac{1}{2} \int_{\mathbf{R}^{d}}|\Delta u|^{2}-\frac{b}{2} \int_{\mathbf{R}^{d}}\left|u_{x_{1}}\right|^{2}-\frac{1}{p+1} \int_{\mathbf{R}^{d}}|u|^{p+1}\right\}
\end{aligned}
$$

It is not a priori clear that $m(\lambda), n(\lambda)$ are finite. We have the following result detailing that.

Lemma 4.3 Let $d \geq 1$. Then, for every $\lambda>0$ and $1<p<1+\frac{8}{d}$, we have that $-\infty<m(\lambda)<0$.

For $p>1+\frac{8}{d}, m(\lambda)=-\infty$.

Proof. Assume that $1<p<1+\frac{8}{d}$. By the Gagliardo-Nirenberg-Sobolev's inequalities

$$
\|u\|_{L^{p+1}\left(\mathbf{R}^{d}\right)} \leq C\|u\|_{\left.\dot{H}^{d\left(\frac{1}{2}-\frac{1}{p+1}\right.}\right)} \leq C_{d, p}\|\Delta u\|_{L^{2}}^{\frac{d}{2}\left(\frac{1}{2}-\frac{1}{p+1}\right)}\|u\|_{L^{2}}^{1-\frac{d}{2}\left(\frac{1}{2}-\frac{1}{p+1}\right)}
$$

Thus, for a function $u:\|u\|^{2}=\lambda$, we have

$$
\|u\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}^{p+1} \leq \lambda^{\frac{1}{2}\left(p+1-\frac{d}{4}(p-1)\right)}\|\Delta u\|_{L^{2}}^{\frac{d(p-1)}{4}}=: C_{\lambda}\|\Delta u\|_{L^{2}}^{\frac{d(p-1)}{4}} .
$$

Since $\frac{d(p-1)}{4}<2$, we conclude by Young's inequality that for every $\delta>0$,

$$
\|u\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}^{p+1} \leq\left(\frac{C_{\lambda}}{\delta^{\frac{d(p-1)}{8}}}\right)^{\frac{8}{8-d(p-1)}}+\delta\|\Delta u\|_{L^{2}}^{2} \leq D_{\lambda, \delta, p}+\delta\|\Delta u\|_{L^{2}}^{2}
$$

Trivially, $\|\nabla u\|^{2} \leq C_{d}\|\Delta u\|\|u\| \leq \delta\|\Delta u\|^{2}+\frac{C_{d}^{2} \lambda}{\delta}$, so by setting $\delta=\delta_{\lambda, p, b}$ appropriately small, we obtain that for a function $u:\|u\|^{2}=\lambda$,

$$
\begin{equation*}
I[u] \geq \frac{1}{4}\|\Delta u\|_{L^{2}}^{2}-C_{\lambda, p, b} \tag{4.4}
\end{equation*}
$$

whence the function $m$ is bounded from below.
Let $\phi$ be a test function, $\|\phi\|_{L^{2}}^{2}=\lambda$. We take the scaling transformation $\phi_{a}=$ $a^{d / 2} \phi(a x)$, so $\left\|\phi_{a}\right\|_{L^{2}}^{2}=\lambda$. We have

$$
I\left[\phi_{\epsilon}\right]=a^{4} \frac{\|\Delta \phi\|^{2}}{2}-b a^{2} \frac{\|\nabla \phi\|^{2}}{2}-\frac{a^{\frac{d(p+1)}{2}-d}}{p+1}\|\phi\|_{L^{p+1}}^{p+1} .
$$

Clearly, if $b>0$ and $0<a \ll 1$, the dominant term is $-b a^{2} \frac{\|\nabla \phi\|^{2}}{2}-\frac{a^{\frac{d(p+1)}{2}-d}}{p+1}\|\phi\|_{L^{p+1}}^{p+1}<$ 0 , whence $m(\lambda)<0$ for these values.

On the other hand, if $p>1+\frac{8}{d}$, we have $\frac{d(p+1)}{2}-d>4$, so that $\lim _{a \rightarrow+\infty} I\left[\phi_{a}\right]=$ $-\infty$.

The next lemmata are technical statements, which will however impact the restrictions one must impose on $p$ (and other parameters), in order to be able to construct the waves in Theorem 4.1. In fact, we shall need specific Gagliardo-Nirenberg-Sobolev type inequalities in order to resolve the existence requirements in Theorems 4.1 and 4.2.

### 4.2.2 The Gagliardo-Nirenberg-Sobolev inequalities with mixed dispersion

We start with the isotropic case.

Proposition 4.4 Let $b>0$. For every $d \geq 2$, there exists $p_{*}(d)$, so that: for all $1<p \leq p_{*}(d)$, the following estimate

$$
\begin{equation*}
\|\phi\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}^{p+1} \leq C\|\phi\|_{L^{2}}^{p-1}\left(\int_{\mathbf{R}^{d}}\left[|\Delta \phi|^{2}-b|\nabla \phi|^{2}+\frac{b^{2}}{4} \phi^{2}\right] d x\right) \tag{4.5}
\end{equation*}
$$

cannot hold for a given constant $C$ and all test functions $\phi$. In addition, $p_{*}(d)$ obeys the following

$$
\begin{equation*}
1+\frac{4}{d} \leq p_{*}(d) \leq 1+\frac{8}{d+1} \tag{4.6}
\end{equation*}
$$

On the other hand, for $1+\frac{8}{d}>p>p_{*}(d)$, there exists a constant $C=C_{p, d, b}$, so that

$$
\begin{equation*}
\|\phi\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}^{p+1} \leq C_{p, d, b}\|\phi\|_{L^{2}}^{p-1}\left(\int_{\mathbf{R}^{d}}\left[|\Delta \phi|^{2}-b|\nabla \phi|^{2}+\frac{b^{2}}{4} \phi^{2}\right] d x\right) \tag{4.7}
\end{equation*}
$$

Remark: The value of $p_{*}(1)=5$ was computed in [31]. Finding the exact value of $p_{*}(d), d \geq 2$ appears to be a hard problem in Fourier analysis, closely related to the restriction conjecture. Even in our proof of the upper bound in (4.6), we use the full strength of the Stein-Tomas restriction theorem in two spatial dimensions
(see for example p. 784, [79]) which does not appear to be enough to determine $p_{*}(d)$. Proposition 4.4 allows us to prove Theorem 4.1; see Section 4.3 below.

Next, we present the relevant GNS results (or lack thereof) in the anisotropic case. The result is much more definite than its counterpart Proposition 4.4.

Proposition 4.5 Let $b>0$. For every $d \geq 2$, and for all $1<p \leq 1+\frac{4}{d}$, the following estimate

$$
\begin{equation*}
\|\phi\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}^{p+1} \leq C\|\phi\|_{L^{2}}^{p-1}\left(\int_{\mathbf{R}^{d}}\left[|\Delta \phi|^{2}-b\left|\partial_{x_{1}} \phi\right|^{2}+\frac{b^{2}}{4} \phi^{2}\right] d x\right) \tag{4.8}
\end{equation*}
$$

cannot hold for a given constant $C$ and all test functions $\phi$.
On the other hand, for $1+\frac{8}{d}>p>1+\frac{4}{d}$, there exists a constant $C=C_{p, d, b}$, so that

$$
\begin{equation*}
\|\phi\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}^{p+1} \leq C_{p, d}\|\phi\|_{L^{2}}^{p-1}\left(\int_{\mathbf{R}^{d}}\left[|\Delta \phi|^{2}-b\left|\partial_{x_{1}} \phi\right|^{2}+\frac{b^{2}}{4} \phi^{2}\right] d x\right) \tag{4.9}
\end{equation*}
$$

In the remainder of this section, we present some preparatory material for the proofs of Propositions 4.4 and 4.5. To this end, we use the formula for the Fourier transform and its inverse as follows

$$
\hat{f}(\xi)=(2 \pi)^{-d / 2} \int_{\mathbf{R}^{d}} f(x) e^{-i x \cdot \xi} d x, \quad f(x)=(2 \pi)^{-d / 2} \int_{\mathbf{R}^{d}} \hat{f}(\xi) e^{i x \cdot \xi} d \xi
$$

The Plancherel's theorem states that $\|f\|_{L^{2}}=\|\hat{f}\|_{L^{2}}$. We will also make frequent use of the Bernstein inequality: for every $1 \leq p \leq q \leq \infty$ and every finite volume set $A \subset \mathbf{R}^{d}$, there exists $C=C_{d}$, so that

$$
\left\|P_{A} f\right\|_{L^{q}} \leq C|A|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{L^{p}}
$$

where $\widehat{P_{A} f}(\xi)=\chi_{A}(\xi) \hat{f}(\xi)$.
We are now ready to proceed to the specifics of the isotropic case.

### 4.2.3 Proof of Proposition 4.4

In consideration of the estimates (4.7), one can straightforwardly rescale to the case $b=2$, which we will henceforth use in our considerations for simplicity (although when completing the proof of our theorems in section IV below, we will present them for arbitrary b). Using Fourier transformation and Plancherel's theorem, we can rewrite

$$
\int_{\mathbf{R}^{d}}\left[|\Delta \phi|^{2}-2|\nabla \phi|^{2}+\phi^{2}\right] d x=\int_{\mathbf{R}^{d}}|\hat{f}(\xi)|^{2}\left(|\xi|^{2}-1\right)^{2} d \xi
$$

Further, one can use smooth decompositions near $|\xi|=1$ to study (4.7). More concretely, introduce a function $\psi \in C_{0}^{\infty}(\mathbf{R})$, so that $\psi(z)=1,|z|<1$ and $\psi(z)=0,|z|>2$. Then, let $\chi(z)=\psi(z)-\psi(2 z)$, so that supp $\chi \subset\left(\frac{1}{2}, 2\right)$ and $\sum_{j=-\infty}^{\infty} \chi\left(2^{j} z\right)=1, z \neq 0$. Now, introduce two multipliers

$$
\widehat{Q_{j} f}(\xi):=\chi\left(2^{-j}\left(|\xi|^{2}-1\right)\right) \hat{f}(\xi), \widehat{P_{m} f}(\xi):=\chi\left(2^{m}\left(|\xi|^{2}-1\right)\right) \hat{f}(\xi),
$$

and the corresponding versions $Q_{>j}:=\sum_{l>j} Q_{l}, Q_{\sim j}=Q_{j-1}+Q_{j}+Q_{j+1}$ and so on. Based on the relevant decomposition,

$$
I d=\sum_{j=0}^{\infty} Q_{j}+\sum_{m>0} P_{m}
$$

and $Q_{j}, j \geq 3$ Fourier restricts to a region $|\xi| \sim 2^{j / 2}$. We henceforth adopt the notation, $A \sim B$, for two positive quantities that satisfy if $\frac{1}{4} A \leq B \leq 4 A$.

It is actually not hard to come up with necessary and sufficient conditions on $p$ so that (4.7) holds, where $f$ is replaced by $Q_{>3} f$.

Estimates away from $|\xi|=1$

We can estimate by Sobolev embedding (or rather Bernstein inequality)

$$
\begin{equation*}
\left\|Q_{j} f\right\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}^{p+1} \leq C 2^{j \frac{(p-1) d}{4}}\left\|Q_{j} f\right\|_{L^{2}\left(\mathbf{R}^{d}\right)}^{p+1} \tag{4.10}
\end{equation*}
$$

Computing the right-hand side of (4.7) (with $b=2$ ), on the other hand, yields $2^{2 j}\left\|Q_{j} f\right\|_{L^{2}\left(\mathbf{R}^{d}\right)}^{p+1}$. One can now show (4.7) for $Q_{>3} f$, when $1<p<1+\frac{8}{d}$. Indeed, by the triangle inequality, (4.10)

$$
\begin{aligned}
\left\|Q_{>3} f\right\|_{L^{p+1}\left(\mathbf{R}^{d}\right)} & \leq C \sum_{j>3} 2^{j} \frac{(p-1) d}{(p(p+1)}
\end{aligned}\left\|Q_{j} f\right\|_{L^{2}\left(\mathbf{R}^{d}\right)} .
$$

where we have used $\frac{(p-1) d}{4}<2$ and $\left||\xi|^{2}-1\right| \sim 2^{j}$ on the support of the multiplier $Q_{j}$.

The situation is much more delicate for frequencies close to the sphere $|\xi|=1$, that is for the multipliers $P_{m}, m \gg 1$.

Estimates near $|\xi|=1$
Clearly, one has, by Bernstein inequalities, the estimates $\left\|P_{m} f\right\|_{L^{p+1}} \leq C_{m}\|f\|_{L^{2}}$, so the issue is the control of $P_{>m}$ for a fixed $m$. We claim that the central issue here is the exact bound in the estimate $\left\|P_{m} f\right\|_{L^{p+1}\left(\mathbf{R}^{d}\right)} \leq C\|f\|_{L^{2}}$. More precisely, define

$$
\begin{equation*}
\alpha(p, d)=\sup \left\{\alpha: \limsup _{m \rightarrow \infty} \sup _{\|f\|_{L^{2}}=1} 2^{\alpha m}\left\|P_{m} f\right\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}<\infty\right\} . \tag{4.11}
\end{equation*}
$$

Note that by the uniform boundedness principle and the definition of $\alpha(p, d)$, for every $\beta>\alpha(p, d)$, there is a $f_{\beta}:\left\|f_{\beta}\right\|_{L^{2}}=1$, so that

$$
\begin{equation*}
\limsup _{m \rightarrow \infty} 2^{\beta m}\left\|P_{m} f\right\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}=\infty \tag{4.12}
\end{equation*}
$$

For convenience, we drop the dependence on the dimension $d$ in $\alpha(p, d)$. Note that by the Bernstein's inequality $\alpha(p)>0$, in fact $\left\|P_{m} f\right\|_{L^{p+1}} \leq C 2^{-m\left(\frac{1}{2}-\frac{1}{p+1}\right)}\|f\|_{L^{2}}$, whence $\alpha(p, d) \geq\left(\frac{1}{2}-\frac{1}{p+1}\right)$. For the same reasons, it is clear that $p \rightarrow \alpha(p)$ is an increasing function. In addition $p \rightarrow \alpha(p)$ is a continuous function and $\alpha(1)=0$.

A convenient characterization of $\alpha(p, d)$ is the following: for every $\epsilon>0$, there is $C_{\epsilon}$, so that

$$
\begin{equation*}
\left\|P_{m} f\right\|_{L^{p+1}\left(\mathbf{R}^{d}\right)} \leq C_{\epsilon} 2^{(-\alpha(p, d)+\epsilon) m}\|f\|_{L^{2}\left(\mathbf{R}^{d}\right)} \tag{4.13}
\end{equation*}
$$

We claim that $\alpha(p, d)$ determines the value of $p_{*}(d)$ in Proposition 4.4.
In fact, we claim that $p_{*}(d)$ is the unique solution of the equation $\alpha(p, d)=\frac{2}{p+1}$. We now prove this claim. First, we show that this equation has a unique solution. To start with, the continuous function $h(p):=\alpha(p)-\frac{2}{p+1}$ is increasing, with $h(1)=$ $-1<0$, while $h(p) \geq \frac{1}{2}-\frac{3}{p+1}>0$ for $p \geq 5$, so there will be a solution $p \in(1,5)$. In fact, below we show better bounds on $\alpha(p, d)$, which imply existence of solutions in the interval of interest, namely $\left(1,1+\frac{8}{d}\right)$, but the existence of solutions anywhere in $(1, \infty)$ will suffice for now.

Next, we show that for $p<p_{*}(d),(4.5)$ holds. This means that $\alpha(p)-\frac{2}{p+1}<0$. Introduce $\beta>\alpha(p)$, so that $\beta<\frac{2}{p+1}$. According to the remarks made earlier, this allows us to find a function $f_{\beta}:\left\|f_{\beta}\right\|_{L^{2}}=1$, so that (4.12) holds true. Assume then, for a contradiction, that (4.5) holds for some constant $C$. This means that for all $\phi \neq 0$,

$$
\begin{equation*}
\frac{\|\phi\|_{L^{p+1}}}{\|\phi\|_{L^{2}}^{\frac{p-1}{p+1}}\left(\int\left[|\Delta \phi|^{2}-2|\nabla \phi|^{2}+\phi^{2}\right] d x\right)^{\frac{1}{p+1}}} \leq C . \tag{4.14}
\end{equation*}
$$

In particular, taking into account the properties of $P_{m}$ and our earlier calculations with regards to the quantity in the denominator of (4.14), we can take $\phi_{m}=P_{m} f_{\beta}$,

$$
\begin{equation*}
\sup _{m} \frac{\left\|P_{m} f_{\beta}\right\|_{L^{p+1}}}{\left\|P_{m} f_{\beta}\right\|_{L^{2}} 2^{-\frac{2 m}{p+1}}} \leq C . \tag{4.15}
\end{equation*}
$$

Now, since $\left\|P_{m} f_{\beta}\right\|_{L^{2}} \leq\left\|f_{\beta}\right\|_{L^{2}}=1$, it follows that $\sup _{m}\left\|P_{m} f_{\beta}\right\|_{L^{p+1}} 2^{\frac{2 m}{p+1}} \leq C$. But this is a contradiction with (4.12), since

$$
\left\|P_{m} f_{\beta}\right\|_{L^{p+1}} 2^{\beta m} 2^{m\left(\frac{2}{p+1}-\beta\right)}=\left\|P_{m} f_{\beta}\right\|_{L^{p+1}} 2^{\frac{2 m}{p+1}} \leq C
$$

whereas on the left hand side $\lim \sup _{m}\left\|P_{m} f_{\beta}\right\|_{L^{p+1}} 2^{\beta m}=\infty$, and $\lim _{m} 2^{m\left(\frac{2}{p+1}-\beta\right)}=$ $\infty$.

Assume now $p>p_{*}(d)$, so $\alpha(p)>\frac{2}{p+1}$. So, we can find $\beta: \alpha(p)>\beta>\frac{2}{p+1}$. Then, we have the estimate, (see (4.13), but applied to $P_{m}^{2} f$ )

$$
\begin{equation*}
\left\|P_{m} f\right\|_{L^{p+1}\left(\mathbf{R}^{d}\right)} \leq C_{\epsilon} 2^{-\beta m}\left\|P_{m} f\right\|_{L^{2}\left(\mathbf{R}^{d}\right)} \tag{4.16}
\end{equation*}
$$

We will show that (4.7) holds. In view of the estimates away from $|\xi|=1$, which establish (4.7) for $Q_{>3} f$, it suffices to consider $P_{>m_{0}} f$ for $m_{0}$ sufficiently large only. We take $m_{0}=10$ for concreteness. By (4.16), we have

$$
\begin{aligned}
\left\|P_{>10} f\right\|_{L^{p+1}} & \leq \sum_{m>10}\left\|P_{m} f\right\|_{L^{p+1}} \leq C \sum_{m>10} 2^{-\beta m}\left\|P_{m} f\right\|_{L^{2}} \\
& \leq C\|f\|_{L^{2}}^{\frac{p-1}{p+1}}\left(\sum_{m>10}\left\|P_{m} f\right\|_{L^{2}}^{2} 2^{-2 m}\right)^{\frac{1}{p+1}} \\
& \leq C\|f\|_{L^{2}}^{\frac{p-1}{p+1}}\left(\int\left[|\Delta f|^{2}-2|\nabla f|^{2}+f^{2}\right] d x\right)^{\frac{1}{p+1}}
\end{aligned}
$$

This establishes (4.7) and so the estimates (or lack thereof) in Proposition 4.4 are established in full.

We now focus our attention on appropriate estimates on $p_{*}(d)$.

Estimates on the value of $p_{*}(d)$

As we saw above, the value $p_{*}(d)$ is intimately related to the precise estimates of $P_{m}: L^{2}\left(\mathbf{R}^{d}\right) \rightarrow L^{p+1}\left(\mathbf{R}^{d}\right)$. Recall that there was the trivial bound based on the Bernstein's inequality, $\alpha(p, d) \geq\left(\frac{1}{2}-\frac{1}{p+1}\right)$, but we now aim for a much more
sophisticated one. Before we proceed, we need to introduce some quantities that will be helpful in our considerations. The surface measure on $\mathbf{S}^{d-1}$ is defined via $d \sigma(x)=\delta\left(|x|^{2}-1\right)$ and its Fourier transform is (see [79], Appendix B.4)

$$
\widehat{d \sigma}(\xi)=(2 \pi)^{-d / 2} \int_{\mathbf{S}^{d-1}} e^{-i \xi \cdot \theta} d \theta=c_{d} \frac{J_{\frac{d-2}{2}}(|\xi|)}{|\xi|^{\frac{d-2}{2}}}=: S(\xi)
$$

where $c_{d}$ is a constant and $J_{n}$ are the standard Bessel functions. Furthermore, see Appendix B.5, [79], for any radial function $f(x)=f_{0}(|x|)$, one can compute its Fourier transform as follows

$$
\hat{f}(\xi)=C_{d}|\xi|^{-\frac{d-2}{2}} \int_{0}^{\infty} f_{0}(r) J_{\frac{d-2}{2}}(r|\xi|) r^{\frac{d}{2}} d r
$$

In this way, when we take the multipliers associated to $P_{m}$, namely $f_{0}(r)=$ $\chi\left(2^{m}\left(r^{2}-1\right)\right)$, we see that its kernel $K_{m}$ (i.e., $\left.P_{m} f=f * K_{m}\right)$ can be expressed in terms of an averaging operator involving the kernel $S=\widehat{d \sigma}$ as follows
$K_{m}(x)=C_{d} \int_{0}^{\infty} \chi\left(2^{m}\left(r^{2}-1\right)\right) \frac{J_{\frac{d-2}{2}}(r|x|)}{|x|^{\frac{d-2}{2}}} r^{\frac{d}{2}} d r=C_{d} \int_{0}^{\infty} \chi\left(2^{m}\left(r^{2}-1\right)\right) r^{d-1} S(r|x|) d r$.
Let us now fix $q>2$. We wish to establish an estimate for the operator norm $P_{m}: L^{2} \rightarrow L^{q}$. Note that $P_{m}$ is trivially bounded, but the issue is to determine precise bounds on the norm, as a function of $m$. Due to the fact that $\chi$ is realvalued, $P_{m}: L^{q^{\prime}} \rightarrow L^{2}$, so in order to compute $\left\|P_{m}\right\|_{B\left(L^{2} \rightarrow L^{q}\right)}$, we might instead consider $P_{m}^{2}: L^{q^{\prime}} \rightarrow L^{q}$ and in addition

$$
\left\|P_{m}\right\|_{B\left(L^{2} \rightarrow L^{q}\right)}=\sqrt{\left\|P_{m}^{2}\right\|_{B\left(L^{q^{\prime}} \rightarrow L^{q}\right)}} .
$$

Now, $P_{m}^{2} f=\tilde{K}_{m} * f, \tilde{K}_{m}=C_{d} \int_{0}^{\infty} \chi^{2}\left(2^{m}\left(r^{2}-1\right)\right) r^{d-1} S(r|x|) d r$. The advantage in this formulation is that the mapping properties of the operator $f \rightarrow f * \widehat{d \sigma}=f * S$ are well-understood. In fact, this is the content of the celebrated Stein-Tomas theorem. To summarize, (see (10.4.7), p. 784, [79]), there is the estimate

$$
\begin{equation*}
\|f * S\|_{L^{q}\left(\mathbf{R}^{d}\right)} \leq C\|f\|_{L^{q^{\prime}}\left(\mathbf{R}^{d}\right)}, q_{d}=\frac{2(d+1)}{d-1} \tag{4.17}
\end{equation*}
$$

With this value of $q_{d}$, we then conclude that since $f * \tilde{K}_{m}=C_{d} \int_{0}^{\infty} \chi^{2}\left(2^{m}\left(r^{2}-\right.\right.$ 1)) $r^{d-1}[S(r|\cdot|) * f] d r$, we have the estimate, based on the Stein-Tomas bound (4.17),

$$
\left\|P_{m}^{2} f\right\|_{L^{q}}=\left\|f * \tilde{K}_{m}\right\|_{L^{q}} \leq C 2^{-m}\|f\|_{L^{q^{\prime}}}
$$

Note here that the factor $2^{-m}$ is gained through the integration in $r$, while the estimate for the term $\|[S(r|\cdot|) * f]\|_{L^{q}}$ comes from (4.17). Accordingly, this gives the estimate

$$
\begin{equation*}
\left\|P_{m} f\right\|_{L^{q_{d}}} \leq C 2^{-m / 2}\|f\|_{L^{2}}, q_{d}=\frac{2(d+1)}{d-1} \tag{4.18}
\end{equation*}
$$

Interpolating this estimate with the trivial one $\left\|P_{m} f\right\|_{L^{2}} \leq C\|f\|_{L^{2}}$, we conclude that for every $2 \leq q \leq \frac{2(d+1)}{d-1}$, there is

$$
\begin{equation*}
\left\|P_{m} f\right\|_{L^{q}} \leq C 2^{-m \frac{d+1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}\|f\|_{L^{2}} \tag{4.19}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\alpha(p, d) \geq \frac{d+1}{2}\left(\frac{1}{2}-\frac{1}{p+1}\right) . \tag{4.20}
\end{equation*}
$$

In particular, it is clear that $\alpha(p, d)-\frac{2}{p+1}>0$, if $p>1+\frac{8}{d+1}$, which means that we have established the upper bound $p_{*}(d)<1+\frac{8}{d+1}$.

In order to establish the lower bound for $p_{*}(d)$, we test the ratio $\frac{\left\|P_{m} f\right\|_{L^{r}\left(\mathbf{R}^{d}\right)}}{\|f\|_{L^{2}\left(\mathbf{R}^{d}\right)}}$ for $r>2$, with $f=K_{m}$, defined above. For the correct asymptotics, we need to recall (see Appendix B.6, [79]) that for every $r \gg 1$,

$$
J_{k}(r)=c \frac{\cos \left(r-\frac{\pi k}{2}-\frac{\pi}{4}\right)}{\sqrt{r}}+O\left(r^{-3 / 2}\right)
$$

Now,

$$
\begin{aligned}
\tilde{K}_{m}(x) & =C_{d} \int_{0}^{\infty} \chi^{2}\left(2^{m}\left(r^{2}-1\right)\right) r^{d-1} S(r|x|) d r= \\
& =\text { const. }|x|^{-\frac{d-1}{2}} \int_{0}^{\infty} \chi^{2}\left(2^{m}\left(r^{2}-1\right)\right) r^{d-1} \frac{\cos \left(r|x|-\frac{\pi(d-2)}{4}-\frac{\pi}{4}\right)}{r^{\frac{d-1}{2}}} d r+2^{-m} O\left(|x|^{-\frac{d+1}{2}}\right)
\end{aligned}
$$

It is then easy to see that for $2^{-m} \ll \delta \ll 1$ and $|x| \sim \delta 2^{m}, m \gg 1$, in the integral above there is the approximate formula

$$
\cos \left(r|x|-\frac{\pi(d-2)}{4}-\frac{\pi}{4}\right)=\cos \left(|x|-\frac{\pi(d-2)}{4}-\frac{\pi}{4}\right)+O(\delta) .
$$

This implies that for a fixed portion of the set $|x| \sim \delta 2^{m}, \cos \left(r|x|-\frac{\pi(d-2)}{4}-\frac{\pi}{4}\right) \geq$ $\frac{1}{2}$, whence we have that $\tilde{K}_{m}$ obeys, on this fixed portion of the set, the bound $\left|\tilde{K}_{m}(x)\right| \gtrsim 2^{-m \frac{d+1}{2}}$. Thus,

$$
\left\|P_{m} f\right\|_{L^{r}\left(\mathbf{R}^{d}\right)} \geq c 2^{-\frac{m}{2}} 2^{-m d\left(\frac{1}{2}-\frac{1}{r}\right)}
$$

while by Plancherel's theorem

$$
\|f\|_{L^{2}}=\left\|K_{m}\right\|_{L^{2}}=\left(\int_{\mathbf{R}^{d}}\left|\chi\left(2^{m}\left(|\xi|^{2}-1\right)\right)\right|^{2} d \xi\right)^{\frac{1}{2}} \sim 2^{-\frac{m}{2}} .
$$

Thus,

$$
\frac{\left\|P_{m} f\right\|_{L^{r}\left(\mathbf{R}^{d}\right)}}{\|f\|_{L^{2}\left(\mathbf{R}^{d}\right)}} \geq c 2^{-m d\left(\frac{1}{2}-\frac{1}{r}\right)} .
$$

It follows that one has the inequality complementary to (4.20),

$$
\begin{equation*}
\alpha(p, d) \leq d\left(\frac{1}{2}-\frac{1}{p+1}\right) . \tag{4.21}
\end{equation*}
$$

We can now derive an estimate for $p_{*}(d)$. Indeed,

$$
\alpha(p, d)-\frac{2}{p+1} \leq d\left(\frac{1}{2}-\frac{1}{p+1}\right)-\frac{2}{p+1}<0
$$

if $p<1+\frac{4}{d}$. Thus, we conclude that $p_{*}(d)>1+\frac{4}{d}$. This finishes the proof of Proposition 4.4.

Our next goal is to analyze the relevant GNS inequalities in the non-isotropic case.

### 4.2.4 The anisotropic case: Proof of Proposition 4.5

Again, a simple rescaling argument reduces matters to the case $b=2$, as in the proof of Proposition 4.4. The arguments for the anisotropic case are pretty similar, once we realize the important differences in the dispersion relations. More specifically, using Plancherel's theorem in this case:

$$
\int_{\mathbf{R}^{d}}\left[|\Delta \phi|^{2}-2\left|\partial_{x_{1}} \phi\right|^{2}+|\phi|^{2}\right] d x=\int_{\mathbf{R}^{d}}|\hat{\phi}(\xi)|^{2}\left[\left(\xi_{1}^{2}-1\right)^{2}+\left|\xi^{\prime}\right|^{4}+2 \xi_{1}^{2}\left|\xi^{\prime}\right|^{2}\right] d \xi
$$

where $\xi^{\prime}=\left(\xi_{2}, \ldots, \xi_{d}\right)$. For future reference, introduce the dispersion related function $h(\xi):=\left(\xi_{1}^{2}-1\right)^{2}+\left|\xi^{\prime}\right|^{4}+2 \xi_{1}^{2}\left|\xi^{\prime}\right|^{2}$. Based on this formula, we discuss the validity of (4.8).

We start our analysis by considering some easy regions. One such region, is when $\xi_{1}$ is away from $\pm 1$. Quantitatively, $\left|\xi_{1}^{2}-1\right| \geq \frac{1}{100}$ and say $f_{0}:=P_{\left|\xi_{1}^{2}-1\right| \geq \frac{1}{100}} f$. In this case, we clearly have $h(\xi) \sim 1+\xi_{1}^{4}+\left|\xi^{\prime}\right|^{4} \sim\langle\xi\rangle^{4}$. In such a scenario, it is easy to analyze $\left\|f_{0}\right\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}$, in particular what it takes for (4.9) to hold (and (4.8) to fail respectively).

More specifically, assuming $1<p<1+\frac{8}{d}$, we have by Bernstein's inequality and Plancherel's equality

$$
\begin{aligned}
\left\|f_{0}\right\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}^{p+1} & \leq C\left(\left\|f_{0}\right\|_{L^{2}}+\sum_{k=0}^{\infty} 2^{k d\left(\frac{1}{2}-\frac{1}{p+1}\right)}\left\|P_{k} f_{0}\right\|_{L^{2}}\right)^{p+1} \\
& \leq C\left\|f_{0}\right\|_{L^{2}}^{p-1} \sum_{k=0}^{\infty} 2^{4 k} \int_{|\xi| \sim 2^{k}}\left|\hat{f}_{0}(\xi)\right|^{2} d \xi \\
& \leq C\|f\|_{L^{2}}^{p-1} \int_{\mathbf{R}^{d}}\left|\hat{f}_{0}(\xi)\right|^{2} h(\xi) d \xi \\
& \leq C\|f\|_{L^{2}}^{p-1} \int_{\mathbf{R}^{d}}\left[|\Delta f|^{2}-2\left|\partial_{x_{1}} f\right|^{2}+|f|^{2}\right] d x
\end{aligned}
$$

This shows that $1<p<1+\frac{8}{d}$ is a sufficient condition for the validity of (4.9), in the case, where $\xi_{1}$ is away from $\pm 1$. On the other hand, testing (4.9) with a function
of the type $\hat{f}(\xi)=\varphi\left(2^{-k} \xi\right)$ for $k \gg 1$, shows that $1<p<1+\frac{8}{d}$ is necessary as well.

We now turn our attention to the more interesting cases, namely $\left|\xi_{1}^{2}-1\right| \sim 2^{-m}$, $m \gg 1$. In this case,

$$
h(\xi) \sim 2^{-2 m}+\left|\xi^{\prime}\right|^{4}+\left|\xi^{\prime}\right|^{2}
$$

The case $\left|\xi^{\prime}\right| \geq \frac{1}{100}$ reduces to $h(\xi) \sim\langle\xi\rangle^{4}$, which was just analyzed. So, it remains to consider the cases $\left|\xi^{\prime}\right|<\frac{1}{100}$. So, the dispersion relation will be exactly $h(\xi) \sim$ $2^{-2 m}+\left|\xi^{\prime}\right|^{2}$. Further, by changing the Fourier variables $\xi_{1} \rightarrow \xi_{1} \pm 1$ (which on the physical side means replacing $f$ with $f \rightarrow e^{\mp i x_{1}} f$, a harmless operation in terms of all $\|\cdot\|_{L^{q}}$ norms), we are reduced to studying the question: for which values of $p$ can the inequality hold

$$
\begin{equation*}
\|f\|_{L^{p+1}}^{p+1} \leq C\|f\|_{L^{2}}^{p-1} \int_{\mathbf{R}^{d}}|\hat{f}(\xi)|^{2}|\xi|^{2} d \xi \tag{4.22}
\end{equation*}
$$

where $f$ is a function supp $\hat{f} \subset\{\xi:|\xi| \ll 1\}$.
We will now show that $p \geq 1+\frac{4}{d}$ is a necessary and sufficient condition for (4.22) to hold. We have already established that $1+\frac{8}{d}>p$ is necessary and sufficient for the region away from $\xi_{1}= \pm 1$, which will of course need to be intersected with the necessary and sufficient condition for (4.22) to hold.

To this end, assume that $p \geq 1+\frac{4}{d}$. Consider $f=\sum_{k=0}^{\infty} P_{-k} f$ (recall supp $\hat{f} \subset$ $\{\xi:|\xi| \ll 1\}$ ), so by standard properties of the Littlewood-Paley decompositions and the Bernstein's inequality

$$
\|f\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}^{2} \leq C \sum_{k=0}^{\infty}\left\|P_{-k} f\right\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}^{2} \leq C \sum_{k=0}^{\infty} 2^{-2 k d\left(\frac{1}{2}-\frac{1}{p+1}\right)}\left\|P_{-k} f\right\|_{L^{2}\left(\mathbf{R}^{d}\right)}^{2}
$$

Further applying Cauchy-Schwartz

$$
\begin{aligned}
\|f\|_{L^{p+1}\left(\mathbf{R}^{d}\right)}^{2} & \leq C\left(\sum_{k=0}^{\infty}\left\|P_{-k} f\right\|_{L^{2}}^{2}\right)^{\frac{p-1}{p+1}}\left(\sum_{k} 2^{-2 k d\left(\frac{1}{2}-\frac{1}{p+1}\right) \frac{p+1}{2}}\left\|P_{-k} f\right\|_{L^{2}\left(\mathbf{R}^{d}\right)}^{2}\right)^{\frac{2}{p+1}} \\
& \leq C\|f\|_{L^{2}}^{\frac{p-1}{p+1}}\left(\sum_{k} 2^{-k d \frac{p-1}{2}}\left\|P_{-k} f\right\|_{L^{2}\left(\mathbf{R}^{d}\right)}^{2}\right)^{\frac{2}{p+1}}
\end{aligned}
$$

It follows that whenever $p \geq 1+\frac{4}{d}$, we have

$$
\|f\|_{L^{p+1}}^{p+1} \leq C\|f\|_{L^{2}}^{p-1} \sum_{k=0}^{\infty} 2^{-2 k}\left\|P_{-k} f\right\|_{L^{2}\left(\mathbf{R}^{d}\right)}^{2} \leq C\|f\|_{L^{2}}^{p-1} \int_{\mathbf{R}^{d}}|\hat{f}(\xi)|^{2}|\xi|^{2} d \xi .
$$

This establishes (4.22) under the assumption $p \geq 1+\frac{4}{d}$. Conversely, assuming that (4.7) holds, we test it with a function $f: \hat{f}(\xi)=\chi\left(2^{k}\left(\xi_{1}-1\right), 2^{k} \xi^{\prime}\right), k \gg 1$. This yields the inequality $p \geq 1+\frac{4}{d}$. Thus, we have finally established that the necessary and sufficient condition for (4.9) to hold is exactly $1+\frac{8}{d}>p \geq 1+\frac{4}{d}$.

### 4.3 Completion of the proofs of Theorems 4.1 and 4.2

We start our presentation with the proof for the existence of the waves. Along the way, we establish a few necessary spectral properties of the corresponding linearized operators, which will be instrumental in the spectral stability considerations.

### 4.3.1 Existence of the waves - isotropic case

In this section, we present the proofs for the existence (or at least a very detailed scheme of the proof) for the isotropic case. We start with a few words about strategy, even though our approach, in principle, is a quite natural one. It was established in Lemma 4.3 that the constrained minimization problem (4.1) is wellposed, and in fact $-\infty<m(\lambda)<0$. We would like to show that there is a minimizer
for this problem, which subsequently will be shown to satisfy the Euler-Lagrange equation (4.3). To this end, consider a minimizing sequence, say $\phi_{k} \in H^{2}\left(\mathbf{R}^{d}\right)$. Ultimately, we would like to show that a strongly convergent subsequence of $\phi_{k}$ will converge to a solution $\Phi$. The central issue that we need to address is the nontriviality of such a minimizing sequence. This is the subject of the next technical lemma.

Lemma 4.6 Let $b>0, d \geq 2$ and $1<p<1+\frac{8}{d}$. Let also

- $1<p \leq p^{*}(d)$ and $\lambda>0$
- $p^{*}(d)<p<1+\frac{8}{d}$ and $\lambda>\lambda_{b, p, d}$.

Then, there exists a subsequence of $\phi_{k}$ so that for some $L_{1}>0, L_{2}>0, L_{3}>0$,

$$
\begin{equation*}
\int_{\mathbf{R}^{d}}\left|\Delta \phi_{k}\right|^{2} d x \rightarrow L_{1} ; \quad \int_{\mathbf{R}^{d}}\left|\nabla \phi_{k}\right|^{2} d x \rightarrow L_{2} ; \quad \int_{\mathbf{R}^{d}}\left|\phi_{k}\right|^{p+1} d x \rightarrow L_{3} \tag{4.23}
\end{equation*}
$$

Informally, the claim is that for $1<p \leq p^{*}(d), \lambda>0$ and for $p^{*}(d)<p<1+\frac{8}{d}$, $\lambda>\lambda_{b, p, d}$ (where $\lambda_{b, p, d}$ is some threshold depending on the parameters $b, p, d$ ), one has non-trivial minimizing sequences. Note that this does not yet show the existence of a limit, for which we bring the full weight of the compensated compactness theory to bear. At the same time this rules out some of the main obstacles toward the strong convergence of, a subsequence of a translate of $\phi_{k}$, to a minimizer. Proof. By the estimate (4.4), it is clear that $\left\{\int_{\mathbf{R}^{d}}\left|\Delta \phi_{k}\right|^{2} d x\right\}_{k}$ is a bounded sequence. Since $\left\|\phi_{k}\right\|^{2}=\lambda$ is fixed, by Sobolev embedding it follows that $\int_{\mathbf{R}^{d}}\left|\nabla \phi_{k}\right|^{2} d x, \int_{\mathbf{R}^{d}}\left|\phi_{k}\right|^{p+1} d x$ are bounded as well. We can take subsequences to ensure that the convergences in (4.23) hold true.

Now, it remains to establish the non-trivial claim, namely that all $L_{1}, L_{2}, L_{3}$ are non-zero. Assume for a contradiction that $L_{3}=0$. Introduce

$$
\tilde{I}[u]:=\left\{\frac{1}{2} \int_{\mathbf{R}^{d}}|\Delta u|^{2}-\frac{b}{2} \int_{\mathbf{R}^{d}}|\nabla u|^{2}\right\} .
$$

Clearly, $\tilde{I}[u] \geq I[u]$, whereas $\lim _{k} \tilde{I}\left[\phi_{k}\right]=\lim _{k} I\left[\phi_{k}\right]=\inf _{\|u\|^{2}=\lambda} I[u] \leq \inf _{\|u\|^{2}=\lambda} \tilde{I}[u]$. It follows that $\phi_{k}$ is a minimizing sequence for the problem $\inf _{\|u\|^{2}=\lambda} \tilde{I}[u]$ and the minima coincide. On the other hand, by Plancherel's theorem

$$
\begin{equation*}
\int_{\mathbf{R}^{d}}|\Delta u|^{2}-b \int_{\mathbf{R}^{d}}|\nabla u|^{2}+\frac{b^{2}}{4}\|u\|^{2}=\int_{\mathbf{R}^{d}}|\hat{u}(\xi)|^{2}\left(|\xi|^{2}-\frac{b}{2}\right)^{2} d \xi \geq 0 \tag{4.24}
\end{equation*}
$$

whence $\inf _{\|u\|^{2}=\lambda} \tilde{I}[u] \geq-\frac{b^{2}}{8} \lambda$. In fact, there is equality, i.e., $\inf _{\|u\|^{2}=\lambda} \tilde{I}[u]=-\frac{b^{2}}{8} \lambda$ as the inequality in (4.24) may be saturated by choosing a function $u$, so that $\hat{u}$ is supported arbitrarily close to $|\xi|=\frac{\sqrt{b}}{2}$.

All in all, it follows that $\inf _{\|u\|^{2}=\lambda} I[u]=-\frac{b^{2}}{8} \lambda$. Applying this to arbitrary $f \neq 0$, and then $u=\sqrt{\lambda} \frac{f}{\|f\|}$, so that $\|u\|^{2}=\lambda$, we have

$$
\frac{2 \lambda^{\frac{p-1}{2}}}{p+1} \int_{\mathbf{R}^{d}}|f|^{p+1} \leq\|f\|^{p-1}\left\{\int_{\mathbf{R}^{d}}|\Delta f|^{2}-b|\nabla f|^{2}+\frac{b^{2}}{4}|f|^{2}\right\} .
$$

This last inequality is in contradiction with (4.5) for $1<p \leq p^{*}(d)$, and with (4.6) for all large enough $\lambda$. This completes the proof for $L_{3}>0$ under these assumptions.

Assuming that either $L_{1}=0$ or $L_{2}=0$ implies, by the standard GagliardoNirenberg inequality, the fact that $L_{3}=0$, which we have just shown to be impossible.

The rest of the proof for existence of a minimizer proceeds identically to the one presented in Section 3.2, [31]. Namely, first one establishes that the function $\lambda \rightarrow m(\lambda)$ is strictly sub-additive for $1<p<1+\frac{8}{d}$. That is, for all $\alpha \in(0, \lambda)$,

$$
\begin{equation*}
m(\lambda)<m(\alpha)+m(\lambda-\alpha) \tag{4.25}
\end{equation*}
$$

This is standard, and proceeds via the property that $\lambda \rightarrow \frac{m(\lambda)}{\lambda}$ is a non-increasing function, which can be obtained via elementary scaling arguments and the crucial property $\lim _{k} \int_{\mathbf{R}^{d}}\left|\phi_{k}\right|^{p+1}=L_{3}>0$, which was established in (4.23).

Next, taking a minimizing subsequence $\phi_{k}$, with the property (4.23), one applies the compensated compactness lemma to it. More specifically, by the P.L. Lions concentration compactness lemma (see Lemma 1.1, [80]), applied to $\rho_{k}:=\left|\phi_{k}\right|^{2} \in$ $L^{1}\left(\mathbf{R}^{d}\right),\left\|\rho_{k}\right\|_{L^{1}}=\lambda$ there is a subsequence (denoted again by $\rho_{k}$ ), so that at least one of the following is satisfied:

1. Tightness. There exists $y_{k} \in \mathbf{R}$ such that for any $\varepsilon>0$ there exists $R(\varepsilon)$ such that for all $k$

$$
\int_{B\left(y_{k}, R(\varepsilon)\right)} \rho_{k} d x \geq \int_{\mathbf{R}} \rho_{k} d x-\varepsilon
$$

2. Vanishing. For every $R>0$

$$
\lim _{k \rightarrow \infty} \sup _{y \in \mathbf{R}} \int_{B(y, R)} \rho_{k} d x=0
$$

3. Dichotomy. There exists $\alpha \in(0, \lambda)$, such that for any $\varepsilon>0$ there exist $R, R_{k} \rightarrow \infty, y_{k} \in \mathbf{R}^{d}$, such that

$$
\left\{\begin{array}{c}
\left|\int_{B\left(y_{k}, R\right)} \rho_{k} d x-\alpha\right|<\varepsilon,\left|\int_{R<\left|x-y_{k}\right|<R_{k}} \rho_{k} d x\right|<\varepsilon  \tag{4.26}\\
\left|\int_{R_{k}<\left|x-y_{k}\right|} \rho_{k} d x-(\lambda-\alpha)\right|<\varepsilon
\end{array}\right.
$$

Then, one shows that the dichotomy cannot occur. The proof proceeds via an argument that shows that dichotomy leads to a inequality of the form $m(\lambda) \geq$ $m(\alpha)+m(\lambda-\alpha)$, with $\alpha$ as in the dichotomy alternative. This of course contradicts the strict sub-additivity (4.25). Next, vanishing leads, via the standard Gagliardo-Nirenberg's, to $\lim _{k} \int_{\mathbf{R}^{d}}\left|\phi_{k}\right|^{p+1}=0$, in a contradiction with (4.23), namely $\lim _{k} \int_{\mathbf{R}^{d}}\left|\phi_{k}\right|^{p+1}=L_{3}>0$.

Hence, one concludes tightness. But tightness means that for some sequence $y_{k} \in \mathbf{R}^{d}$, there is strong $L^{2}$ convergence for $\left\{\phi_{k}\left(x-y_{k}\right)\right\}$. Denote $\Phi(\cdot):=\lim _{k} \phi_{k}(\cdot-$
$\left.y_{k}\right)$. By the lower semi-continuity of the $L^{2}$ norm with respect to weak convergence, we also conclude that $\lim _{k}\left\|\Phi(\cdot)-\phi_{k}\left(\cdot-y_{k}\right)\right\|_{H^{2}}=0$, whence $\Phi$ is a constrained minimizer of (4.1), all under the assumptions of Lemma 4.6.

Now, we take on the question for the Euler-Lagrange equation (4.3). To this end, fix a test function $h$ and consider the scalar function

$$
g(\epsilon):=I\left(\sqrt{\lambda} \frac{\Phi+\delta h}{\|\Phi+\delta h\|}\right) .
$$

Since $g$ is differentiable in a neighborhood of the origin, and achieves its minimum there, we have that $g^{\prime}(0)=0$. Since this is true for all test functions $h$, the resulting expression is that $\Phi$ is a distributional solution of (4.3). It is standard result in elliptic theory to conclude that such a solution $\Phi \in H^{4}\left(\mathbf{R}^{d}\right)$. In addition, one can establish asymptotics for such functions, but we will not do so herein.

Next, we consider the second derivative necessary condition for a minimum at zero, which states that $g^{\prime \prime}(0) \geq 0$. Assuming that the test function $h \perp \Phi$, we conclude $\left\langle\mathcal{L}_{+} h, h\right\rangle \geq 0$, which is exactly $\left.\mathcal{L}_{+}\right|_{\{\Phi\}^{\perp}} \geq 0$. In fact, this is sharp, because by a direct inspection ${ }^{3} \mathcal{L}_{+}[\nabla \Phi]=0$. Also, since $\left\langle\mathcal{L}_{+} \Phi, \Phi\right\rangle=-(p-$ 1) $\int|\Phi|^{p+1} d x<0$, we conclude that $\mathcal{L}_{+}$indeed has a negative eigenvalue. This coupled with $\left.\mathcal{L}_{+}\right|_{\{\Phi\}^{\perp}} \geq 0$ confirms that $\mathcal{L}_{+}$has exactly one negative eigenvalue.

Finally, we show that $\mathcal{L}_{-} \geq 0$. Assume not. Then, there is $\psi \perp \Phi,\|\psi\|=1$, $\mathcal{L}_{-} \psi=-\sigma^{2} \psi$. Note however that $\mathcal{L}_{-}>\mathcal{L}_{+}$, whence

$$
0 \leq\left\langle\mathcal{L}_{+} \psi, \psi\right\rangle<\left\langle\mathcal{L}_{-} \psi, \psi\right\rangle=-\sigma^{2}
$$

which is a contradiction. Looking closely, this also shows that 0 is a simple eigenvalue for $\mathcal{L}_{-}$, because then, we take $\psi \perp \Phi: \mathcal{L}_{\psi}=0$, and this still leads to a contradiction as above. Thus, we have shown the following proposition.

[^6]Proposition 4.7 Let $b>0, d \geq 2,1<p<1+\frac{8}{d}$ and one of the two assumptions below is verified

- $1<p \leq p^{*}(d)$ and $\lambda>0$
- $p^{*}(d)<p<1+\frac{8}{d}$ and $\lambda>\lambda_{b, p, d}$.

Then, there exists $\Phi=\Phi_{\lambda}$, a constrained minimizer of (4.1) and $\omega=\omega_{\lambda}>$ 0 . In addition, $\Phi$ satisfies the Euler-Lagrange equation (4.3), and the linearized Schrödinger operators $\mathcal{L}_{ \pm}$satisfy

1. $\mathcal{L}_{-}[\Phi]=0,0 \in \sigma_{\text {p.p. }}\left(\mathcal{L}_{-}\right)$is a simple eigenvalue, and $\left.\mathcal{L}_{-}\right|_{\{\Phi\}^{\perp}} \geq \delta>0$, for some $\delta>0$
2. $\left.\mathcal{L}_{+}\right|_{\{\Phi\}^{\perp}} \geq 0$. Moreover, $n\left(\mathcal{L}_{+}\right)=1$

This completes the existence part of Theorem 4.1.

### 4.3.2 Existence of the waves - anisotropic case

Following identical steps as in Section 4.3.1, we establish the following analog of Lemma 4.6.

Lemma 4.8 Let $b>0, d \geq 2$ and $1<p<1+\frac{8}{d}$. Let also

- $1<p \leq 1+\frac{4}{d}$ and $\lambda>0$
- $1+\frac{4}{d}<p<1+\frac{8}{d}$ and $\lambda>\lambda_{b, p, d}$.

Then, there exists a subsequence of $\phi_{k}$ so that for some $L_{1}>0, L_{2}>0, L_{3}>0$,

$$
\begin{equation*}
\int_{\mathbf{R}^{d}}\left|\Delta \phi_{k}\right|^{2} d x \rightarrow L_{1} ; \quad \int_{\mathbf{R}^{d}}\left|\nabla \phi_{k}\right|^{2} d x \rightarrow L_{2} ; \quad \int_{\mathbf{R}^{d}}\left|\phi_{k}\right|^{p+1} d x \rightarrow L_{3} \tag{4.27}
\end{equation*}
$$

The proof of Lemma 4.8 proceeds in an identical manner to the proof of Lemma 4.6 in Section 4.3.1, with the suitable replacement of isotropic Proposition 4.4 with its anisotropic analog Proposition 4.5. Once this step is completed, one establishes the strong sub-linearity of the cost function $n(\lambda)$, similar to the sub-linearity of $m(\lambda)$. The next step, again identical to the corresponding step for the isotropic case, is to show that once we take a minimizing sequence $\phi_{k}$, the method of compensated compactness goes through for the functions $\rho_{k}:=\phi_{k}^{2}$. This establishes the existence of the minimizer $\Phi$. Similarly, it satisfies the Euler-Lagrange equation and the spectral properties hold true. We collect the results in the next Proposition.

Proposition 4.9 Let $b>0, d \geq 2,1<p<1+\frac{8}{d}$ and one of the two assumptions below are verified

- $1<p \leq 1+\frac{4}{d}$ and $\lambda>0$
- $1+\frac{4}{d}<p<1+\frac{8}{d}$ and $\lambda>\lambda_{b, p, d}$.

Then, there exists $\Phi=\Phi_{\lambda}, \omega=\omega_{\lambda}>0$, a constrained minimizer of (4.2). In addition, $\Phi$ satisfies the Euler-Lagrange equation (4) and the linearized operators $\mathcal{L}_{ \pm}$obey

1. $\mathcal{L}_{-} \geq 0$. More specifically, $\mathcal{L}_{-}[\Phi]=0,0 \in \sigma_{\text {p.p. }}\left(\mathcal{L}_{-}\right)$is a simple eigenvalue, and $\left.\mathcal{L}_{-}\right|_{\{\Phi\}^{\perp}} \geq \delta>0$, for some $\delta>0$
2. $\left.\mathcal{L}_{+}\right|_{\{\Phi\}^{\perp}} \geq 0$. Moreover, $n\left(\mathcal{L}_{+}\right)=1$.

### 4.3.3 Spectral stability of the normalized waves

In this section, we show the spectral stability of the waves constructed as constrained minimizers as (4.1), (4.2) respectively. Starting with the eigenvalue prob-
lem (5), we have that instability is equivalent to the solvability of the system

$$
\left\{\begin{array}{l}
\mathcal{L}_{-} g=-\mu f  \tag{4.28}\\
\mathcal{L}_{+} f=\mu g
\end{array}\right.
$$

for some $\mu: \operatorname{Re} \mu>0$. So, applying $\mathcal{L}_{-}$to the second equation, we see that (4.28) reduces to the solvability of

$$
\begin{equation*}
\mathcal{L}_{-} \mathcal{L}_{+} f=-\mu^{2} f \tag{4.29}
\end{equation*}
$$

Conversely, if (4.29) has a non-trivial solution $\mu, f$, then $g:=\mu^{-1} \mathcal{L}_{+} f$ has a nontrivial solution $\mu, f, g$. So, (4.28) and (4.29) are equivalent and we concentrate on the eigenvalue problem $\mathcal{L}_{-} \mathcal{L}_{+} f=-\mu^{2} f$ henceforth.

It follows immediately that $f \perp \Phi$. Thus, as $\left.\mathcal{L}_{-}\right|_{\{\Phi\}^{\perp}} \geq \delta>0$, it follows that there exists unique $\eta \in\{\Phi\}^{\perp}$, so that $f=\sqrt{\mathcal{L}_{-}} \eta$. Writing the relation $\mathcal{L}_{-} \mathcal{L}_{+} f=-\mu^{2} f$ in terms of $\eta$ yields

$$
\sqrt{\mathcal{L}_{-}}\left(\sqrt{\mathcal{L}_{-}} \mathcal{L}_{+} \sqrt{\mathcal{L}_{-}} \eta+\mu^{2} \eta\right)=0
$$

As $\sqrt{\mathcal{L}_{-}} \mathcal{L}_{+} \sqrt{\mathcal{L}_{-}} \eta+\mu^{2} \eta \in\{\Phi\}^{\perp}=\operatorname{Ker}\left(\mathcal{L}_{-}\right)^{\perp}$, we conclude that $\sqrt{\mathcal{L}_{-}} \mathcal{L}_{+} \sqrt{\mathcal{L}_{-}} \eta+$ $\mu^{2} \eta=0$. Thus,

$$
\begin{equation*}
\sqrt{\mathcal{L}_{-}} \mathcal{L}_{+} \sqrt{\mathcal{L}_{-}} \eta=-\mu^{2} \eta \tag{4.30}
\end{equation*}
$$

Note however that the operator $\sqrt{\mathcal{L}_{-}} \mathcal{L}_{+} \sqrt{\mathcal{L}_{-}}$is symmetric now, whence $-\mu^{2} \in$ $\sigma_{p . p .}\left(\sqrt{\mathcal{L}_{-}} \mathcal{L}_{+} \sqrt{\mathcal{L}_{-}}\right)$, so $-\mu^{2} \in \mathbf{R}$. We have already shown that there could not be oscillatory instabilities. Furthermore, testing (4.30) with $\eta \in\{\Phi\}^{\perp}$, we obtain

$$
-\mu^{2}\|\eta\|^{2}=\left\langle\mathcal{L}_{+} \sqrt{\mathcal{L}_{-}} \eta, \sqrt{\mathcal{L}_{-}} \eta\right\rangle=\left\langle\mathcal{L}_{+} f, f\right\rangle .
$$

Since $f \in\{\Phi\}^{\perp}$ and $\left.\mathcal{L}_{+}\right|_{\{\Phi\}^{\perp}} \geq 0$, it follows that $\left\langle\mathcal{L}_{+} f, f\right\rangle \geq 0$, whence $-\mu^{2} \geq 0$. This implies that all spectrum is stable, hence the spectral stability of $\Phi$ follows.

### 4.4 Numerical Computations

In the present section, we show a number of numerical computations for $d=2$ which corroborate and complement our analytical results on the existence and stability of solitons for both the isotropic and anisotropic models with competing Laplacian and biharmonic operators.



Figure 4.1. Two-parameter plane of the nonlinearity exponent parameter $p$ vs. the Laplacian prefactor $b$ (varying between 0 and 2 ); recall that the frequency $\omega$ is fixed to unity, while our computations are for dimension $d=2$. The figure shows the bifurcation loci separating spectrally stable solitons (under the curve) from unstable ones (above the curve). The right panel shows a blowup of the left one close to the edge point of $p=3$ and $b=2$.

We start with the isotropic case. A summary of our results can be firstly found in Fig. 4.1 which contains a two-parameter ( $p$ vs $b$ ) diagram. Here, the depicted curve separates the regime of spectrally stable waves (under the curve) from spectrally and dynamically unstable ones (over the curve) for fixed frequency $\omega=1$. It is important to recall here that any pair $(b, \tilde{\omega}=1)$ for a given $b$ and fixed $\tilde{\omega}$ can be converted upon rescaling to a pair $\left(\tilde{b}=1, \omega=1 / b^{2}\right)$, i.e., results pertaining to $b$ variation for fixed $\tilde{\omega}$ are tantamount to ones with fixed $\tilde{b}$ and variable $\omega$. By using the latter representation, it is possible to connect to the well-known

Vakhitov-Kolokolov criterion for the spectral stability, based on the monotonicity of the $P(\omega)$ dependence [81]. Increasing dependence of $P(\omega)$ (or, analogously, decreasing dependence of $P(b)$ ) is necessary for spectral stability, while a decreasing dependence (or increasing dependence of $P(b)$ ) leads to spectral (and dynamical) instability for the single-humped states considered herein. Furthermore, it should be noted that the limit of $b=0$ is tantamount to $\omega \rightarrow \infty$, while $b \rightarrow 2$ corresponds to $\omega \rightarrow 1 / 4$ within the above scaling (the linear limit), setting the scales of variation of the respective parameters.

Representations of the dependence of $P$ with respect to $\omega$ for different values of $p$ can be found in Fig. 4.2. It can be clearly seen that in the case of $p=3$, similarly to what happens for all values with $1<p<3, P$ increases monotonically with $\omega$, pertaining to a regime of spectral stability. Our numerical results seem to suggest the presence of a $p^{*} \approx 3.3$ (see once again the right end of the curves in the panels of Fig. 4.1). For $1<p<p^{*}$, in line with Theorem 1, we find ground state minimizers for all values of $P \equiv \lambda$. By $P$ here, we denote the squared $L^{2}$ norm due to its being tantamount to the optical power in the corresponding physical problem. For values of the exponent $p$ that lie within $p^{*}<p<5$, the power $P$ features an interval of monotonic decrease with $\omega$ close to the linear limit (of dispersing waveforms). Indeed, the corresponding solutions near the linear limit are unstable, while for sufficiently large frequencies the solutions become spectrally stable, as seen in the top right panel which corresponds to $p=5$. This finding also corroborates the results of Theorem 1, since in the latter interval, it is not possible to reach powers $P(\lambda)$ below the minimum of the corresponding curve. For $p>5$ and below a critical, dimension-dependent threshold (which for our twodimensional case is $p_{c r}=6.565$ ), instabilities arise both for sufficiently small (near the linear limit) and sufficiently large (instabilities due to collapse) values of $\omega$,
as it is shown in the bottom left panel for $p=6$; in this case, the only stable frequencies are the intermediate ones, corresponding to the interval of growing $P$. Finally, when going above the relevant critical value of $p$ (see bottom right panel, corresponding to $p=7$ ), the soliton is spectrally and dynamically unstable for all the frequencies, given the monotonically decreasing dependence of $P$ with respect to $\omega$. Notice that these findings for $p>5 \equiv 1+\frac{8}{d}$ complement in a natural way the rigorous results of Theorem 1 .


Figure 4.2. Dependence of the squared $L^{2}$ norm, denoted by $P$, i.e., $P=$ $\int_{\mathbf{R}^{2}}|u|^{2}$ for our computations, with respect to the frequency $\omega$ for different values of the nonlinearity exponent $p$, in the isotropic case for dimension $d=2$. These plots showcase the different stability regimes that can be found herein (see text for more details). The insets show the same graph over an expanded interval of frequencies, using a semi-logarithmic scale for the latter.


Figure 4.3. Several examples of the waveform of the solitary waves with $p=3$ in the isotropic case for different frequencies. We can observe how the solution profile changes from high $\omega$ to the linear limit of $\omega \rightarrow 0.25$. Notice the logarithmic scale of the colormap, and the (clear within that scale) zero-crossings of the solution. Figures for other values of $p$ are qualitatively similar.

Figure 4.3 showcases the relevant isotropic (radially symmetric) waveforms and a variety of different frequencies, starting from the highly nonlinear limit of large $\omega$ (where the width of the solution shrinks, while its amplitude grows), to progressively lower frequencies, eventually approaching the linear limit of small amplitude as $\omega \rightarrow 1 / 4$. It is important to note the logarithmic scale of the relevant colorbar,
associated to continuously decreasing amplitudes as $\omega$ decreases. Noticeable also within this scale are the nodal lines of the solution, given the oscillatory nature of the linear tail as a result of the competition between the harmonic and biharmonic terms. Although this figure corresponds to $p=3$, it is qualitatively similar to the outcome for other values of $p$.


Figure 4.4. Same as Fig. 4.1 but for the anisotropic case.

We have made a similar analysis for the anisotropic case. The two-parameter diagram of $p$ versus $b$ is displayed in Fig. 4.4. Indeed, the phenomenology is quite similar to the isotropic case, although with some notable differences that can be observed not only near the right edge of the curve of Fig. 4.4 but also in the $P(\omega)$ plots for different values of $p$ in Fig. 4.5. For low enough $p$ (as, e.g., for $p=2$ ), the soliton is stable for every frequency, and a solution exists for all values of $P \equiv \lambda$, in line with Theorem 2. However, contrary to the isotropic case, this monotonic dependence of $P$ on $\omega$ does not persist up to $p=3$. Indeed, there exists an interval of $b$ 's (or, equivalently, of frequencies) for $p$ roughly between 2.481 and 3 in our two-dimensional setting, whereby $P(\omega)$ presents a maximum and a minimum and, as a consequence, the soliton becomes unstable in that interval (as shown, e.g., in
the plot for $p=2.8$ ); this suggests that the linear limit is not approached in the same way as in the isotropic case near the critical point of $p=3$. Incidentally, it is especially relevant to note that the linear limit itself bears nontrivial differences as now the second partial derivative only occurs along $x$ direction. This leads, near the linear limit, to an oscillatory pattern solely along the $x$ direction, while the solution becomes separable in the form $X(x) \times Y(y)$. This can be clearly observed in the relevant solution panels in Fig. 4.6.

It is also relevant to note here that our numerical results do not contradict Theorem 2, although in the very vicinity of the linear limit and for values of $p$ between 2.5 and 3 , we cannot fully confirm the relevant theory. In particular, a careful observation of Fig. 4.4, e.g., for $p=2.8$ (top right panel) suggests a nonmonotonic dependence of $P$ on $\omega$ but as the linear limit is approached, we are unable to resolve the question of whether all values of $P$ are accessible, as one approaches closer and closer to $\omega=1 / 4$, in line with the expectations of Theorem 2. While the Theorem prompts us to expect that to be the case (and the numerics are also suggestive in this vein), the highly computationally expensive, anisotropic 2D computations needed have not allowed us to fully confirm this limit, which remains an interesting, open computational question for future studies.

When $p$ is increased from $p=3$, we observe a similar phenomenology as in the isotropic case, i.e. the curve $P(\omega)$ is monotonically decreasing near the linear limit and becomes monotonically increasing (spectrally stable) after a local minimum (see the plot for $p=5$ in Fig. 4.5). This phenomenology changes again (resembling the isotropic case) for $p>5$, as shown in the plot for $p=5.2$; here, an interval of stability for intermediate frequencies can be seen to arise. Finally, for sufficiently large values of $p$, again similarly to the isotropic limit, the waves become generically unstable for all frequencies, as illustrated by the monotonically decreasing


Figure 4.5. Same as Fig. 4.2 but for the anisotropic case and for different values of $p$. Again, a semi-logarithmic scale has been used for the frequencies.
dependence of $P(\omega)$ in the plot for $p=6$. It is interesting to point out, however, that the relevant threshold is considerably lower in the anisotropic case where it is around $p=5.407$ for $d=2$, while in the isotropic one the threshold is around $p=6.565$ for $d=2$.


Figure 4.6. Same as Fig. 4.3 but for the anisotropic case with $b=1$. Contrary to the isotropic case, the anisotropy reflects in the solution as it acquires, when approaching the linear limit $\omega \rightarrow 0.25$ a separable form in the $x$ and $y$ dependence with the nodal lines being uniform along direction $y$.

## CHAPTER5

## CONCLUSIONS

In the first part of this work we have explored a model featuring the competition of a harmonic and biharmonic linear operator in a quadratic-quartic $\phi^{4}$ model. We have argued that this model is of intrinsic mathematical interest due to the distinct implications of the different linear operators and also the unique features created by their interplay that neither of the 'pure' (quadratic or quartic dispersion) models possesses. The harmonic part creates a saddle point in the spatial dynamics and hence leads to exponentially decaying waveforms. On the other hand, the biharmonic operator leads to complex eigenvalues and a spiral point in the corresponding spatial dynamics. Here, we have seen the interplay of these two possibilities creating an effective competition between the two tendencies. We have observed that this competition leads to a critical point (with an intriguing behavior in its own right, i.e., a linearly modulated exponential) and on the two sides of this criticality either the biharmonic oscillatory effect or the harmonic exponential decay effect prevail respectively. This crucially affects the interactions between the kinks which we have also explicitly identified and corroborated by means of detailed comparison of both the single wave tails and of the two-coherent-structure interactions. We have also elucidated the extent to which this collective coordinate approach can be reliably used and illustrated its failure when the kinks get too close to each other.

Additionally, we have examined the collisions, bounce and multi-bounce windows stemming from the kink interactions and have shown how the critical velocities and corresponding windows are modified as a function of the quadratic-quartic model parameters.

These findings had a significant bearing on the corresponding quadratic-quartic NLS model the study of which was the natural continuation of our work. In particular, we revisited the topic of media with competing quadratic and quartic dispersions in the context of nonlinear structures commonly considered in self-defocusing media, namely kink-like states in the form of dark solitary waves. We have focused our attention on the setting of multiple such structures and have proposed a systematic understanding of pairs of such kinks on the basis of an energetic landscape emanating from the kink-antikink interaction. In a similar manner to the $\phi^{4}$ model, the competition of the different dispersions, and indeed crucially the presence of the quartic effects enable the presence of oscillatory tails and of potential bound states for multi-kink states. We have analyzed the first few center- and saddleconfigurations of this type, indeed 3 center states (families $0,2,4$ ) and 3 saddle ones (families 1, 3, and 5). In addition to presenting a systematic continuation of the states in one of the most natural parametric variations of the system (the coefficient of the quadratic dispersion), we have followed the solutions past their (typical, aside from family 0) turning points, identifying their respective upper branches, unveiling, in turn, solutions associated with 4, as well as with 6 kinks.

We have provided a systematic particle picture that offers the possibility of a systematic classification of the obtained states, irrespective of the number of kinks based on their interactions, provided that the kinks are sufficiently well separated, i.e., for large enough positive quadratic dispersion $\beta_{2}$ in our system. This analysis was used to accurately capture the equilibrium distance of the kinks, as well as their
internal excitation modes. In a wide range of corresponding families and examples, stable and unstable, lower and upper branch ones, the method was found to provide systematic insights regarding the kink dynamics and their stability.

Our considerations offer a systematic view of the possible stationary multisoliton solution families and can be naturally extended to either higher-order families or heteroclinic ones involving an odd number of kinks. Both directions have been successfully attempted, although they are not detailed herein.

Finally, in an additional dimension of considerations, we turned our focus to the study of the competition between a focusing quartic and a defocusing quadratic dispersion term in an NLS model, with a numerical emphasis on the mor cmputationally tractable case of $d=2$. In particular, we have considered a setting in which there is a competition between a focusing quartic and a defocusing quadratic dispersion term. Our Theorems 1 and 2 have offered a rigorous perspective on the relevant phenomenology, providing bounds on the nonlinearity exponent (as a function of dimension) for which minimizers of the (squared) $L^{2}$ norm exist for all values of that quantity, as well as ones for which such minimizers do not exist for all powers. This has been done both for the isotropic case involving radial solutions, as well as for the anisotropic one where the second derivative term was only active along a particular direction. We have complemented these findings with detailed numerical results and corresponding multi-parameter diagrams detailing the stability of the single-humped states of the system. In both isotropic and anisotropic cases, we found that the when exponent $p$ of the nonlinear term is sufficiently small, the dependence of the power on the frequency is monotonic, while above a certain threshold more complex non-monotonic dependencies arise. Our numerical results for the case of $d=2$ go beyond the accessible limits to our Theorems, identifying possible stable solutions even beyond the exponent bound of $p=1+8 / d$ (where
$d$ is the dimension), as well as identifying exponents beyond which no spectrally stable solutions arise.

## Outlook

While we believe that these results offer numerous insights into systems with competing quadratic and quartic dispersions, there are also numerous open questions to consider. Following the analysis of the $\phi^{4}$ model, a natural next step (albeit a rather nontrivial one, given the complications that have recently arisen even in the standard $\phi^{4}$ case [6]) is to attempt to explore the role of the kink/antikink internal modes in their collisions and the interplay between the kinetic energy stored in the kink translational modes, the vibrational energy of the internal modes and the dispersive radiation of the extended modes.

A more demanding consideration suggested by our results on the NLS model involves the setting of traveling excitations. In addition to the loss of Galilean invariance (in the presence of quartic dispersion) [53] rendering interesting the existence and stability analysis of single kinks, we have found that bound, breathing states of two kinks are quite common and would be worth seeking as potentially exact solutions and to understand their stability.

As concerns the the higher-dimensional setting, an important question concerns the close proximity of the linear limit for the anisotropic case when $p$ is in the vicinity of $p=3 \equiv 1+4 / d$ (for our case of $d=2$ ). More generally, for the case considered herein, it would be interesting to also examine if higher-excited states, including multi-soliton ones, as well as vortical ones are feasible and potentially also spectrally stable (and under what conditions). Additionally, numerical studies of the more computationally demanding case of $d=3$ would also be worthwhile in
connection to the Theorems presented herein. Last but not least, exploring similar questions with the recently accessible experimentally, even orders of dispersion [48] would also be of particular interest.

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[^0]:    ${ }^{1}$ Note that the case $\gamma>0$ in (8) is not as relevant physically as it does not support (bright) localized waves. It basically corresponds to the de-focusing case in the standard NLS framework.

[^1]:    ${ }^{1}$ The contents of this chapter are published in Journal of Physics A: Mathematical and Theoretical [51] and appear here with permission.

[^2]:    ${ }^{1}$ The contents of this chapter are published in Optics Letters [50] and appear here with permission.

[^3]:    ${ }^{1}$ The contents of this chapter have been submitted for publication in Communications in Nonlinear Science and Numerical Simulation.

[^4]:    ${ }^{1}$ The contents of this chapter are published in Journal of Physics A: Mathematical and Theoretical [56] and appear here with permission.

[^5]:    ${ }^{2}$ However, there are fourth order differential operators with fast decaying potentials, which have embedded eigenvalues in their continuous spectrum. This is in sharp contrast with the second order operators, which may possess eigenvalues only at the edges of the continuous spectrum.

[^6]:    ${ }^{3}$ Note however that $\nabla \Phi \perp \Phi$

