The Harris–Venkatesh conjecture for derived Hecke operators

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Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
under the Executive Committee
of the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2023

Abstract

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The Harris–Venkatesh conjecture posits a relationship between the action of derived Hecke operators on weight-one modular forms and Stark units. We prove the full Harris–Venkatesh conjecture for all CM weight-one modular forms. This reproves results of Darmon–Harris–Rotger–Venkatesh, extends their work to the adelic setting, and removes all assumptions on primality and ramification from the imaginary dihedral case of the Harris–Venkatesh conjecture. This is done by introducing the Harris–Venkatesh period on cuspidal one-forms on modular curves, introducing two-variable optimal modular forms, evaluating $\mathrm{GL}(2) \times \mathrm{GL}(2)$ Rankin–Selberg convolutions on optimal forms and newforms, and proving a modulo-ℓ^{t} comparison theorem between the Harris–Venkatesh and Rankin–Selberg periods. Furthermore, these methods explicitly describe local factors appearing in the constant of proportionality prescribed by the Harris–Venkatesh conjecture. We also look at the application of our methods to non-dihedral forms.

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Acknowledgements

There are many people who have supported my graduate school life at Columbia University, too many to completely enumerate in this space but for all of whom I am immensely grateful. I am fortunate to have had such inspiring company, both mathematically and personally.

I am deeply thankful to my advisor, Michael Harris, for introducing me to beautiful mathematical problems, for sharing his deep and broad insight, and for his mentorship throughout my doctoral studies. His continued guidance extends across the arts of mathematics, language, philosophy, and travel. For example, I cannot imagine another scenario in which I would have simultaneously and immersively studied the Spanish language and the theory of automorphic descent, but for his encouragement to join his project group at the 2019 BIRS–CMO workshop "Teoría de números en América" in Oaxaca, México. His unique perspective has been greatly enriching in many ways; it has been a privilege to learn from him.

There are many other mathematicians, both at Columbia and elsewhere, from whom I benefited greatly. I would like to express my gratitude to the members of my thesis committee, Michael Harris, Eric Urban, Will Sawin, Henri Darmon, and Akshay Venkatesh, for their support and time in examining this thesis. In writing this thesis itself, I am also grateful to Henri Darmon for discussions about quaternion algebras and periods on Shimura curves, to Wee-Teck Gan for discussions regarding explicit local theta liftings, and to Dick Gross for observations about Stark units and the Stark conjectures. I would like to furthermore thank Dorian Goldfeld, Hervé Jacquet, Johan de Jong, Chao Li, and Dipendra Prasad for sharing their wisdom and counsel over the years. My time at Columbia would not have been the same without my classmates, including Hugo, Chung

Hang, Noah, Yu-Sheng, Stan, Clara, Chi, Juan, and David, nor without our shared chats, laughs, and camaraderie. A special thanks as well to the staff who make the department run so smoothly, especially Nathan, Alenia, and Deniz.

Finally, my dearest family and friends have always been there for me in the periods of coursework and examinations, the genesis of this thesis, my struggles, my successes, everything in between, and everything beyond. To Mom, Dad, Andy, Angela, Amelia, Diana, Colin, Laura, Evan, Malina, and everyone else: thank you.

Dedication

To Michael Zhao (1995–2018).

Introduction

We study the conjecture of Harris–Venkatesh [HV19], which frames the general conjectures of Prasanna and Venkatesh [PV21, Ven19, GV18] on derived Hecke algebras and motivic cohomology groups in the coherent cohomology of the Hodge bundle on the modular curve (cf. [Ata22, Hor22, Oh22] for higher-dimensional coherent contexts). In this setting, Harris–Venkatesh gives a modular analogue of the Stark conjecture by relating Stark units to the predicted action of derived Hecke operators on weight-1 modular forms.

Let f be a modular form of weight 1 and level $\Gamma_1(N)$. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}(M)$ be its associated Artin representation realized on a free module M of rank 2 over $\mathbb{Z}[\chi_\rho]$ by Deligne–Serre, where χ_ρ is the character of ρ. Then ρ is realized on the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ of a finite Galois extension E of $\mathbb{Q}$.

If we fix an embedding $E \hookrightarrow \mathbb{C}$, then the Stark conjecture for the adjoint representation $\mathrm{Ad}(\rho)$ predicts the existence of a unit $\epsilon \in \mathcal{O}_E^\times$ and a positive integer W such that,

$$L'(\mathrm{Ad}(\rho), 0) = \frac{1}{W} \sum_{\sigma \in \mathrm{Gal}(E/\mathbb{Q})} \chi_{\mathrm{Ad}(\rho)}(\sigma) \log|\epsilon^\sigma|.$$

This formula is furthermore compatible with Galois conjugation of $\mathrm{Ad}(\rho)$ and conjugations of f under $\mathrm{Aut}(\mathbb{C})$, so it can be considered as being valued in $\mathbb{R} \otimes \mathbb{Z}[\chi_{\mathrm{Ad}(\rho)}]$. The right-hand side comes from the Stark regulator map evaluated on an element of the dual unit group $\mathcal{U}(\mathrm{Ad}(\rho)) := \mathrm{Hom}_{\mathrm{Gal}(E/\mathbb{Q})}(\mathrm{Ad}(\rho), \mathcal{O}_E^\times)$. With the convention that Frob_w is complex conjugation for real places w, there is a distinguished element $x_w := 2\rho(\mathrm{Frob}_w) - \mathrm{Tr}(\rho(\mathrm{Frob}_w)) \in \mathrm{Ad}(\rho)$ for each archimedean place w. Evaluation at x_w defines an injective map $\mathcal{U}(\mathrm{Ad}(\rho)) \hookrightarrow \mathcal{O}_E^\times \otimes \mathbb{Z}\big[\chi_{\mathrm{Ad}(\rho)}\big]$ with image

$e_{\mathrm{Ad}(\rho)}\mathcal{O}_E^\times$, where,

$$e_{\mathrm{Ad}(\rho)} = \frac{1}{|\mathrm{Gal}(E/\mathbb{Q})|}\sum_{\sigma} \chi_{\mathrm{Ad}(\rho)}(\sigma)\sigma^{-1}.$$

Composition with the usual logarithm of the absolute value on $E \hookrightarrow \mathbb{C}$ defines the Stark regulator map,

$$\mathrm{Reg}_{\mathbb{R}} : \mathcal{U}\big(\mathrm{Ad}(\rho)\big) \longrightarrow \mathbb{R} \otimes \mathbb{Z}\big[\chi_{\mathrm{Ad}(\rho)}\big].$$

The Stark conjecture then predicts that there exists a unique element $u_{\mathrm{Stark}} \in \mathcal{U}(\mathrm{Ad}(\rho))$ such that,

$$L'\big(\mathrm{Ad}(\rho), 0\big) = \mathrm{Reg}_{\mathbb{R}}(u_{\mathrm{Stark}}),$$

and again compatibly with Galois conjugation of $\mathrm{Ad}(\rho)$. These two formulations are related by the identity,

$$u_{\mathrm{Stark}}(x_w) = \frac{1}{W}\sum_{\sigma \in \mathrm{Gal}(E/\mathbb{Q})} \chi_{\mathrm{Ad}(\rho)}(\sigma)\epsilon^{\sigma}.$$

Up to a constant in $\mathbb{Q}(\chi_{\mathrm{Ad}(\rho)})$, the derivative $L'(\mathrm{Ad}(\rho), 0)$ can be replaced by the Petersson inner product $||f||^2 := \int_{\Gamma_1(N)\backslash\mathcal{H}} |f|^2 y \frac{dxdy}{\pi y^2}$, so the Stark conjecture for $\mathrm{Ad}(\rho)$ can be reformulated as an identity for some $c \in \mathbb{Q}(\chi_{\mathrm{Ad}(\rho)})^\times$:

$$c \cdot ||f||^2 = \mathrm{Reg}_{\mathbb{R}}(u_{\mathrm{Stark}}).$$

The Harris–Venkatesh conjecture is an analogue of the above identity, with $\mathbb{R}$ replaced by $\mathbb{F}_p^\times$ for each prime p.

For the Harris–Venkatesh conjecture, the analogue of the Petersson inner product is the action of the Shimura class $\mathfrak{S}_p$ as described by Harris–Venkatesh [HV19, Section 3.1] and Darmon–Harris–Rotger–Venkatesh [DHRV22, Section 1.1]. Let p be a prime not dividing $6N$. There is an étale Galois covering $X_1(p) \longrightarrow X_0(p)$ with group $\mathbb{F}_p^\times$. This defines the element,

$$\mathfrak{S}_p \in H^1_{\text{ét}}\big(X_0(p), \mathbb{F}_p^\times\big) = H^1_{\text{ét}}(X_0(p), \mathbb{Z}/(p-1)\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p^\times.$$

Now consider the base change of the modular curve $X_0(p) \otimes \mathbb{Z}/(p-1)\mathbb{Z}$; the push-forward of the étale sheaf $\mathbb{Z}/(p-1)\mathbb{Z} \longrightarrow \mathcal{O}_{X_0(p)\otimes\mathbb{Z}/(p-1)\mathbb{Z}}$ gives the Shimura class,

$$\mathfrak{S}_p \in H^1_{\text{ét}}(X_0(p) \otimes \mathbb{Z}/(p-1)\mathbb{Z}, \mathbb{G}_a) \otimes \mathbb{F}_p^\times,$$

which can also be viewed as an element of Zariski cohomology $H^1(X_0(p) \otimes \mathbb{Z}/(p-1)\mathbb{Z}, \mathcal{O}) \otimes \mathbb{F}_p^\times$. By Serre duality, $\mathfrak{S}_p$ is also an element of $\mathrm{Hom}\big(H^0\big(X_0(p), \Omega^1\big), \mathbb{F}_p^\times\big)$, i.e. as a map from weight 2 modular forms to $\mathbb{F}_p^\times$. By adding $\cup\mathfrak{S}_p$ to the usual Hecke operator defined by the pull-back and push-forward of $\pi_1, \pi_2 : X_{\Gamma_1(N)\cap\Gamma_0(p)} \to X_{\Gamma_1(N)}$, this defines a *derived Hecke operator* $T_{p,N}$ on the space of cusp forms of weight 1 and level N coprime to p,

$$\begin{tikzcd}
H^0\big(X_{\Gamma_1(N),\mathbb{Z}/(p-1)\mathbb{Z}}, \omega(\mathrm{Cusp})\big) \arrow[r, dashed, "T_{p,N}"] \arrow[d, "\pi_1^*"] & H^1\big(X_{\Gamma_1(N),\mathbb{Z}/(p-1)\mathbb{Z}}, \omega(\mathrm{Cusp})\big) \otimes \mathbb{F}_p^\times \\
H^0\big(X_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}/(p-1)\mathbb{Z}}, \omega(\mathrm{Cusp})\big) \arrow[r, "\cup\mathfrak{S}_p"] & H^1\big(X_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}/(p-1)\mathbb{Z}}, \omega(\mathrm{Cusp})\big) \otimes \mathbb{F}_p^\times. \arrow[u, "\pi_{2*}"']
\end{tikzcd}$$

The $\mathbb{F}_p^\times$-analogue of the Stark regulator $\mathrm{Reg}_\mathbb{R}$ is given by the following distinguished element for each place w of E over p,

$$x_w := 2\rho(\mathrm{Frob}_w) - \mathrm{Tr}(\rho(\mathrm{Frob}_w)) \in \mathrm{Ad}(\rho).$$

Evaluation at x_w defines an embedding into a space of units,

$$\mathcal{U}\big(\mathrm{Ad}(\rho)\big) \longrightarrow \big(\mathcal{O}_E^\times\big)^{\mathrm{Frob}_w} \otimes \mathbb{Z}\big[\chi_{\mathrm{Ad}(\rho)}\big],$$

whose image in $\mathcal{O}_{F_w}^\times \otimes \mathbb{Z}[\chi_{\mathrm{Ad}(\rho)}]$ is in $\mathbb{Z}_p^\times \otimes \mathbb{Z}[\chi_{\mathrm{Ad}(\rho)}]$. Thus, reduction modulo the ideal corresponding to w defines a regulator map (called "reduction of a Stark unit" in [HV19, DHRV22]),

$$\mathrm{Reg}_{\mathbb{F}_p^\times} : \mathcal{U}\big(\mathrm{Ad}(\rho)\big) \longrightarrow \mathbb{F}_p^\times \otimes \mathbb{Z}\big[\chi_{\mathrm{Ad}(\rho)}\big].$$

With the Shimura class and the $\mathbb{F}_p^\times$ regulator map, we present the conjecture of Harris–Venkatesh [HV19, Conjecture 3.1] away from primes 2 and 3. Let $\ell \geq 5$ be a prime that divides $p-1$ and is coprime to N, and let t be the largest exponent of ℓ such that ℓ^t divides $p-1$. We fix a *discrete logarithm*, which is a surjective homomorphism,

$$\log_\ell : \mathbb{F}_p^\times \twoheadrightarrow \mathbb{Z}/\ell^t\mathbb{Z},$$

Conjecture 1 (Harris–Venkatesh conjecture)**.** Let f be a Hecke new cusp form of weight 1 and level N. There is an element $u_f \in \mathcal{U}(\mathrm{Ad}(\rho))$ and a positive integer m such that for any primes $p, \ell \geq 5$ coprime to N,

$$m \cdot \log_\ell \mathfrak{S}_p\big(\mathrm{Tr}_p^{Np}(f(z)f^*(pz))\big) = \log_\ell \mathrm{Reg}_{\mathbb{F}_p^\times}(u_f),$$

where f^* is the dual newform of f.

Remark 2*.* Assuming the Stark conjecture for $\mathrm{Ad}(\rho)$, we can take $u_f = c \cdot u_{\mathrm{Stark}}$ with some nonzero $c \in \mathbb{Z}[\chi_{\mathrm{Ad}(\rho)}]$. We can also replace $\mathfrak{S}_p$ with the Harris–Venkatesh period $\mathcal{P}_{\mathrm{HV}}$, which we define later and which has the property,

$$\mathfrak{S}_p\big(\mathrm{Tr}_p^{Np}(f(z)f^*(pz))\big) = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] \cdot \mathcal{P}_{\mathrm{HV}}(f \otimes f^*).$$

For a modular form of weight 1, its associated 3-dimensional adjoint representation $\mathrm{Ad}(\rho)$ factors through $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times = \mathrm{PGL}_2(\mathbb{C}) = \mathrm{SO}_3(\mathbb{C})$ and has finite image; this image is therefore a finite subgroup of $\mathrm{SO}_3(\mathbb{C})$, which must either be cyclic, D_{2n}, A_4, S_4, or A_5. Eisenstein series are the forms with cyclic image, dihedral forms are those with image D_{2n}, and the remaining forms are called "exotic". The Stark conjecture is known in the Eisenstein and dihedral cases, but remains open in general for the three exotic cases.

The evaluation of $\mathfrak{S}_p$ at Eisenstein series was considered by Mazur [Maz77, p. 103] and computed by Merel [Mer96] (cf. the discussion in [HV19, Section 5.2]). The first theoretical steps toward the Harris–Venkatesh conjecture in the dihedral setting were done by Darmon–Harris–Rotger–Venkatesh [DHRV22], under primality and ramification assumptions. Dihedral forms are classified by finite characters of $G_K := \mathrm{Gal}(\overline{K}/K)$ with $K/\mathbb{Q}$ quadratic: given a dihedral form f, there is a quadratic number field K and finite character χ of G_K such that $\rho = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\chi)$; conversely, given a character χ of G_K, there is a new form $f_\chi \in S_1(\Gamma_1(N), \mathbb{Z}[\chi])$ with q-expansion $f_\chi(z) = \sum_{n=1}^{\infty} a_n q^n$ such that $\sum_n a_n n^{-s} = L(s, \chi)$.

Theorem 3 (Darmon–Harris–Rotger–Venkatesh [DHRV22, Theorem 1.2])**.** *Let K be a quadratic number field of discriminant D_K and different $\mathcal{D}_K$, and let χ be a finite character of $\mathrm{Gal}(\overline{K}/K)$.*

- *If K is imaginary, assume that D_K is an odd prime and that χ is unramified;*
- *if K is real, assume that D_K is odd and that χ has conductor dividing $\mathcal{D}_K$.*

Then the Harris–Venkatesh conjecture is true for $f = f_\chi$.

Remark 4. Darmon–Harris–Rotger–Venkatesh [DHRV22, Section 1.3] also check that both sides of Conjecture 1 vanish when K is imaginary quadratic and p splits in K, so the Harris–Venkatesh conjecture holds in this trivial case. Consequently, we will assume that p is inert in K unless otherwise mentioned.

The methods of Darmon–Harris–Rotger–Venkatesh [DHRV22] cannot be directly generalized to ramified characters. A subsequent paper by Lecouturier [Lec22] uses the central L-value formulas of Ichino [Ich08] and Waldspurger [Wal85] to bypass part of the theta lifting arguments of [DHRV22] to prove new cases of a weaker unsigned Harris–Venkatesh conjecture: by requiring that $t = 1$ and ignoring the sign of the integer m, its argument assumes that the "antinorm" $\xi := \chi^{1-\mathrm{Frob}_\infty}$ is unramified instead of assuming that χ is unramified, and furthermore allows D_K to be composite when K is imaginary.

The purpose of this thesis is to translate the methods of Darmon–Harris–Rotger–Venkatesh [DHRV22] to the adelic language and then use the theory of theta lifts to treat composite discriminants and ramified characters in the imaginary case with full generality.

Main results

Our main result is the proof of Conjecture 1 for all dihedral weight-one forms in the imaginary case.

Theorem 5. *Let* K *be an imaginary quadratic number field and let* χ *be a finite character of* $\mathrm{Gal}(\overline{K}/K)$. *Then the Harris–Venkatesh conjecture is true for* $f = f_\chi$.

Remark 6. We expect the methods outlined here to be applicable to the real case of Conjecture 1 as well. This is part of an forthcoming work [Zha23c] with additional calculations that are necessary for indefinite theta series (cf. the additional constructions in [DHRV22, Section 3]).

In the imaginary dihedral case, $\mathrm{Ad}(\rho)$ decomposes as $\eta \oplus \mathrm{Ind}_{G_{\mathbb{Q}}}^{G_K}(\xi)$, where η is the quadratic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for the imaginary quadratic extension $K/\mathbb{Q}$ and the "antinorm" $\xi := \chi^{1-\mathrm{Frob}_\infty}$ is a ring class character of $\mathrm{Gal}(\overline{K}/K)$. In particular, ξ factors through $\mathrm{Gal}(H_c/K)$ where $c = c(\xi)$ is the conductor of ξ and H_c is the associated ring class field. Importantly in the dihedral case, the Stark conjecture is known for $\mathrm{Ad}(\rho)$; the unit u_f is an explicit elliptic unit in the imaginary case and an explicit fundamental unit in the real case. When χ is unramified and $\mathrm{disc}(K)$ is an odd prime, Darmon–Harris–Rotger–Venkatesh [DHRV22] proves Theorem 3 by computing the left-hand side and relating it to an elliptic unit $u_{\lambda,\xi} \in \mathcal{U}_\xi$ depending on an auxiliary prime λ with a splitting $\lambda = \mathfrak{l} \cdot \bar{\mathfrak{l}}$ in $\mathcal{O}_K$ such that $\xi(\mathfrak{l})$ generates $\mathrm{Im}(\xi)$. We extend this and define $u_\xi = \frac{m(\xi)}{1-\xi(\bar{\mathfrak{l}})} u_{\lambda,\xi}$, where,

$$m(\xi) = \begin{cases} \nu & \text{if } |\mathrm{Im}(\xi)| \text{ is a power of a prime } \nu, \\ 1 & \text{otherwise.} \end{cases}$$

We show that u_ξ is a unit independent of the choice of λ in Proposition 3.1. Furthermore, the Stark conjecture is known in this case and we show that u_{Stark} is the unique element of $\mathcal{U}(\mathrm{Ad}(\rho))$ such that,

$$u_{\mathrm{Stark}}(x_\infty) = \frac{h_K}{6m(\xi)w_K} u_\xi,$$

where $h_K = [H_1 : K]$ is the class number of K (with H_1 the Hilbert class field of K) and where w_K

is the number of roots of unity in K (see Proposition 3.2).

For the dihedral form $f = f_\chi$, we prove Theorem 5 by introducing and constructing a two-variable *optimal modular form* $f^{\mathrm{opt}}(z_1, z_2)$ on $X(N) \times X(N)$ generated by $f(z_1)f^*(z_2)$ under the action of Hecke operators, with its automorphic avatar given by the Equation 1.7. They are defined explicitly so that we are able to prove a refinement of the Harris–Venkatesh conjecture for the form f^{opt} with specified ratio and specified unit (u_f is proportional to u_ξ and u_{Stark}).

Theorem 7. *Let f be an imaginary dihedral modular form of weight 1 and level N, and let $f^{\mathrm{opt}}(z_1, z_2)$ be the optimal form associated to f. For all primes $p, \ell \geq 5$ coprime to N,*

$$\log_\ell \mathfrak{S}_p\big(f^{\mathrm{opt}}(z, pz)\big) = -\frac{[H_c : H_1] w_K}{2} \log_\ell \mathrm{Reg}_{\mathbb{F}_p^\times}(u_{\mathrm{Stark}}).$$

One of the main steps in the proof of Theorem 7 is to show that f^{opt} realizes the theta lifting (cf. Equation 2.21, Emerton [Eme02], Gross [Gro87, Proposition 5.6], Darmon–Harris–Rotger–Venkatesh [DHRV22, Sections 1.4 and 2.2]),

$$f^{\mathrm{opt}}(z, pz) = \Theta_p(\mathbb{1} \otimes \xi).$$

In fact, it is the unique solution that satisfies this equation for all primes p. In the unramified setting, all newforms in the unramified setting are actually optimal forms. The optimal form realizes the theta lifting $\Theta_p(\mathbb{1} \otimes \xi)$; this is their key property for the proof of Theorem 5. After this step, we can essentially follow the strategy of Darmon–Harris–Rotger–Venkatesh [DHRV22], reproducing and extending their chain of equalities with an explicit higher Eisenstein element $\Theta_p^*(\mathfrak{S}_p)$ (i.e. an element of a Hecke module annihilated by powers of the Eisenstein ideal, cf. Lecouturier [Lec21]),

$$\begin{aligned}
\log_\ell \mathfrak{S}_p\big(f^{\mathrm{opt}}(z,pz)\big) &= \log_\ell\big\langle f^{\mathrm{opt}}(z,pz),\mathfrak{S}_p\big\rangle \\
&= \log_\ell\langle \Theta_p(\mathbb{1}\otimes\xi),\mathfrak{S}_p\rangle \\
&= \log_\ell\big\langle \mathbb{1}\otimes\xi,\Theta_p^*(\mathfrak{S}_p)\big\rangle \\
&= -\frac{[H_c:K]}{12m(\xi)}\log_\ell \mathrm{Reg}_{\mathbb{F}_p^\times}(u_\xi) \\
&= -\frac{[H_c:H_1]w_K}{2}\log_\ell \mathrm{Reg}_{\mathbb{F}_p^\times}(u_{\mathrm{Stark}})
\end{aligned}$$

The construction of the optimal form f^{opt} allows us to furthermore obtain the following precise description of the constant of proportionality, which together with Theorem 7 implies Theorem 5.

Theorem 8. *Let* $\mathbb{Q}(\xi+\xi^{-1})$ *be the subring of* $\mathbb{C}$ *generated over* $\mathbb{Q}$ *by the values of* $\xi(\sigma)+\xi^{-1}(\sigma)$ *for all* $\sigma\in\mathrm{Gal}(H_c/K)$. *There exists an element* $\beta_\chi\in\mathbb{Q}(\xi+\xi^{-1})^\times$ *such that for almost all primes* $p,\ell\geq 5$ *coprime to* N*,*

$$\log_\ell \mathfrak{S}_p\big(\mathrm{Tr}_p^{Np}\big(f_\chi(z)f_{\chi^{-1}}(pz)\big)\big) = \beta_\chi \log_\ell \mathfrak{S}_p\big(f^{\mathrm{opt}}(z,pz)\big).$$

More precisely, there is a decomposition

$$\beta_\chi = \prod_{q\mid N}\beta_{\chi_q},$$

where β_{χ_q} *depends only on* χ_q*. Moreover, for odd primes* q *that are not ramified in both* K *and* χ*, we have the following explicit formula for* β_{χ_q}*:*

1. *If* q *is ramified in* K*, then*

$$\beta_{\chi_q} = 4.$$

2. If q *is inert in* K*, then*

$$\beta_{\chi_q} = \begin{cases} \frac{(q-1)^2}{q^2} & \text{if } \xi \text{ is unramified,} \\ \frac{\xi(-1)(q+1)^2}{q^2} & \text{if } \xi \text{ is ramified and } \xi^2 \text{ is unramified,} \\ \frac{\xi(-1)(q+1)}{q^2} & \text{if } \xi^2 \text{ is ramified.} \end{cases}$$

3. If q *is split in* K *(so* $K_q = \mathbb{Q}_q \oplus \mathbb{Q}_q$ *is split with uniformizers* ϖ_1, ϖ_2 *and* $\chi_q = (\chi_1, \chi_2)$*), then*

$$\beta_{\chi_q} = \begin{cases} \frac{(q-1)(q-\xi(\varpi_1))(q-\xi(\varpi_2))}{(q+1)q^2} & \text{if } \chi \text{ is ramified and } \xi \text{ is unramified,} \\ \frac{\chi(-1)(q-1)^2 q^{2o(\xi)-1}}{q^{2o(\xi)+1}-q^{2o(\xi)}+2} & \text{if } \xi \text{ is ramified, if } \xi^2 \text{ is unramified,} \\ & \text{and exactly one of the } \chi_i \text{ is ramified,} \\ \frac{\chi(-1)(q-1)^3 q^{2o(\xi)-2}}{q^{2o(\xi)+1}-q^{2o(\xi)}+2} & \text{if } \xi^2 \text{ is unramified} \\ & \text{and both } \chi_i \text{ are ramified,} \\ \frac{\chi(-1)(q-1)^2 q^{2o(\xi)+1}}{(q+1)(q^{2o(\xi)+1}-q^{2o(\xi)}+2)} & \text{if } \xi^2 \text{ is ramified} \\ & \text{and exactly one of the } \chi_i \text{ is ramified,} \\ \frac{\chi(-1)(q-1)^3 q^{2o(\xi)-2}}{(q+1)(q^{2o(\xi)+1}-q^{2o(\xi)}+2)} & \text{if } \xi^2, \chi_1, \text{ and } \chi_2 \text{ are all ramified.} \end{cases}$$

Remark 9. Almost all of the local β-factors from Theorem 8 are rational numbers, with the possible exception at a prime q that splits in K at which χ is ramified and $\xi = \chi^{1-\epsilon}$ is unramified. Even then, the β_{χ_q} factor is in a real subfield of the cyclotomic field generated by the values of ξ.

Remark 10. Combining Theorems 7 and 8, we obtain a rational version of the Harris–Venkatesh conjecture:

$$\log_\ell \mathfrak{S}_p\left(\mathrm{Tr}_p^{Np}\left(f_\chi(z) f_{\chi^{-1}}(pz)\right)\right) = \alpha_\chi \log_\ell \mathrm{Reg}_{\mathbb{F}_p^\times}(u_{\mathrm{Stark}}), \tag{1}$$

where the ratio is

$$\alpha_\chi := -\frac{[H_c : H_1] w_K}{2} \beta_\chi.$$

If we write $\alpha_\chi = \frac{a_\chi}{b_\chi}$ as a reduced fraction with $a_\chi \in \mathbb{Z}(\xi)$ and $b_\chi \in \mathbb{N}$, then b_χ divides 2(denominator of β_χ); therefore the integer m from Theorem 5 does as well. Furthermore, b_χ depends only on the set S of primes ramified in χ and the number of roots of unity in K. In a future work, we consider integral refinements of our results. Two questions of Harris motivate our results in this direction: in particular, we can realize α_χ as Hecke values, in terms of Hecke algebras; furthermore we can study the congruences of the units u_{ξ_1} and u_{ξ_1}, given congruences between χ_1 and χ_2.

To prove Theorem 8, we introduce two ingredients. The first is the *Harris–Venkatesh period*. Let Σ be the set of primes dividing N and consider the projective system $X_\Sigma = \varprojlim_m X(N^m)$ of modular curves unramified outside of Σ. Let R be a $\mathbb{Z}[1/N]$-algebra. Then the Harris–Venkatesh period is given on two copies of the space of weight-1 cusp forms unramified outside of Σ:

$$\mathcal{P}_{\mathrm{HV}} : H^0(X_{\Sigma,R}, \omega(-C_\Sigma)) \otimes H^0(X_{\Sigma,R}, \omega(-C_\Sigma)) \longrightarrow R.$$

Moreover, this pairing is invariant under the action of $\prod_{q|N} \mathrm{GL}_2(\mathbb{Q}_q)$. In this terminology,

$$\begin{aligned} \mathfrak{S}_p\big(\mathrm{Tr}_p^{Np}\big(f_\chi(z) f_{\chi^{-1}}(pz)\big)\big) &= [\Gamma(1) : \Gamma_0(N)] \cdot \mathcal{P}_{\mathrm{HV}}\big(f_\chi \otimes f_{\chi^{-1}}\big), \\ \mathfrak{S}_p\big(\mathrm{Tr}_p^{Np}(f^{\mathrm{opt}}(z, pz))\big) &= \mathcal{P}_{\mathrm{HV}}(f^{\mathrm{opt}}). \end{aligned}$$

The second ingredient is a *modulo-ℓ^t multiplicity-one argument*, which compares the Harris–Venkatesh period $\mathcal{P}_{\mathrm{HV}}$ of modular forms with a Rankin–Selberg period $\mathcal{P}_{\mathrm{RS}}$ for Whittaker functions at places at Σ. In particular, it gives an identity of two ratios,

$$\big[\mathcal{P}_{\mathrm{HV}}(f^{\mathrm{opt}}) : \mathcal{P}_{\mathrm{HV}}\big(f_\chi \otimes f_{\chi^{-1}}\big)\big] \equiv [\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}}) : \mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}})] \pmod{\ell^t},$$

with the left-hand side obtainable from the right-hand side by reduction modulo-ℓ^t. This then reduces the calculation of β_χ to a calculation of local Rankin–Selberg periods on $\mathrm{GL}_2 \times \mathrm{GL}_2$ (see Theorems 8.1 and 8.2).

Finally, we consider generalizations of the above results for non-dihedral forms. Let f be a newform of weight 1 and level N associated to a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C})$. We assume that f is locally dihedral, i.e. for every prime q dividing N, the restriction ρ_q on the decomposition group $\mathrm{Gal}(\overline{\mathbb{Q}_q}/\mathbb{Q}_q)$ is induced from a character χ_q of a quadratic extension $K_q/\mathbb{Q}_q$,

$$\rho_q = \mathrm{Ind}_{K_q}^{\mathbb{Q}_q}(\chi_q).$$

This local assumption is automatically satisfied for $q > 2$, and furthermore is satisfied for $q = 2$ when ρ_2 is reducible. (so if ρ_2 is reducible then f is locally dihedral). We can also define an optimal form $f^{\mathrm{opt}}(z_1, z_2)$ for locally dihedral forms in the same way that we did for dihedral forms, proving the same relation as the one in Theorem 8 (see Proposition 12.1). What is missing for locally dihedral forms is the analogue of Theorem 7, for which we make the following conjecture. For each q, let c_q be the conductor of the antinorm $\xi_q = \chi_q^{1-\mathrm{Frob}_\infty}$. For each q, define,

$$h_q := \left|(\mathcal{O}_{k,q}/c_q)^\times/(\mathbb{Z}_q/c_q)^\times\right|.$$

Conjecture 11. Let f be a locally dihedral modular form of weight 1 and level N, and let $f^{\mathrm{opt}}(z_1, z_2)$ be the optimal form associated to f. For all primes $p, \ell \geq 5$ coprime to N,

$$\log_\ell \mathfrak{S}_p(f^{\mathrm{opt}}(z, pz)) = -\prod_q h_q \log_\ell \mathrm{Reg}_{\mathbb{F}_p^\times}(u_{\mathrm{Stark}}),$$

Remark 12. Conjecture 11 is compatible with Theorem 7 and implies the locally dihedral case of the Harris–Venkatesh conjecture. In the dihedral case, the constant in Theorem 7 is just the product of these local factors,

$$\frac{[H_c : H_1]w_K}{2} = \prod_q h_q.$$

In the dihedral case, there is a global character ξ of conductor c and $[H_c : H_1]$ is the cardinality of the group,

$$\mathrm{Gal}(H_c/H_1) \xrightarrow{\sim} \widehat{\mathcal{O}}_K^\times/\mathcal{O}_K^\times \cdot \left(\widehat{\mathbb{Z}} + c\widehat{\mathcal{O}}_K\right)^\times.$$

Thus, $[H_c : H_1] = \frac{2}{w_K} \prod_q h_q$.

Outline

This thesis has been divided into two articles, [Zha23a] and [Zha23b]. The contents are thematically divided into three parts.

Part I covers the global theory, in particular developing the theory of the Harris–Venkatesh period. Section 1 includes various theta series calculations and a definition of optimal forms ϕ^{opt} in adelic automorphic language. Section 2 covers modular curves, including the definitions of optimal forms f^{opt} and the Harris–Venkatesh period $\mathcal{P}_{\mathrm{HV}}$. Section 3 defines the elliptic unit u_ξ while giving its precise relation to both Stark units and a higher Eisenstein element. Section 4 proves Theorem 7.

Part II covers the local theory, in particular applying the Rankin–Selberg method to this setting. Section 5 covers the theory of Whittaker and Kirillov models, in particular including the Rankin–Selberg method for $\mathrm{GL}_2 \times \mathrm{GL}_2$ following Jacquet [Jac72], and the definition of local optimal functions W^{opt}. Section 6 contains the calculations of the Rankin–Selberg inner product $\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}})$ for optimal forms, broken into cases based on whether the extension E/F is unramified, inert, or split. Section 7 contains the calculations of the Rankin–Selberg period $\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}})$ for newforms, broken into similar cases. Section 8 then combines these calculations to determine the ratio $[\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}}) : \mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}})]$ of Rankin–Selberg inner products of optimal forms and new forms.

Part III establishes a comparison between the Harris–Venkatesh periods and Rankin–Selberg periods. Section 9 gives a multiplicity-one argument comparing the Harris–Venkatesh period $\mathcal{P}_{\mathrm{HV}}$ and the Rankin–Selberg period $\mathcal{P}_{\mathrm{RS}}$. Section 10 proves Theorem 8 and Section 11 proves Theorem 5. Section 12 considers extensions of the main results to non-dihedral forms.

Part I. Global theory: Harris–Venkatesh periods

We start Part I with necessary background material: for automorphic forms, we largely follow [JL70, Jac72, Bum97, Zha01]; for modular forms and modular curves, we use [Shi94, DR73, KM85]. Three key ingredients that we introduce here are the definition of a unit u_ξ *independent of* λ, the definition of *optimal forms*, and the general definition of the *Harris–Venkatesh period*. At the end of Part I, we prove Theorem 7, applying the general method of Darmon–Harris–Rotger–Venkatesh [DHRV22] to u_ξ and optimal forms.

1 Automorphic forms

1.1 Background for GL_2

We start with some notation:

$$\widehat{\mathbb{Z}} := \varprojlim_{n} \mathbb{Z}/n\mathbb{Z} = \prod_{p} \mathbb{Z}_p, \qquad \widehat{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}.$$

Let $\mathbb{A} := \widehat{\mathbb{Q}} \times \mathbb{R}$ denote the ring of adèles of $\mathbb{Q}$. For any abelian group M, let $\widehat{M} := M \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. For any $\mathbb{Q}$-vector space V, let $V_{\mathbb{A}} := V \otimes_{\mathbb{Q}} \mathbb{A}$. We use superscripts to remove specified places and subscripts to restrict to specified places. For example, $\mathbb{A}^\infty = \widehat{\mathbb{Q}}$ and $\mathbb{A}_\infty = \mathbb{Q}_\infty = \mathbb{R}$.

Let $\psi : \mathbb{Q}\backslash\mathbb{A} \longrightarrow \mathbb{Q}$ be the standard additive measure:

$$\psi(x) = \prod_{p\leq\infty} \psi_p(x_p) = e^{2\pi i x_\infty} e^{-2\pi i (x^\infty \bmod \widehat{\mathbb{Z}})},$$

where we have used the identity $\mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} \widehat{\mathbb{Q}}/\widehat{\mathbb{Z}}$ by the Chinese remainder theorem.

For an algebraic group G over $\mathbb{Q}$, we denote its center by Z_G, denote the quotient $Z_G\backslash G$ by

$\mathrm{P}G$, and denote the set $G(\mathbb{Q})\backslash G(\mathbb{A})$ by $[G]$. For example, $[\mathrm{PGL}_2] = Z(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})$.

For a reductive group G over $\mathbb{Q}$, let $\mathcal{A}([G])$ denote its space of automorphic functions. These are smooth functions with some growth conditions on $G(\mathbb{Q})\backslash G(\mathbb{A})$. Let $\mathcal{A}_0([G])$ denote the subspace of cusp forms: the space of automorphic functions that vanish at cusps.

We will mainly focus on GL_2, so $\mathcal{A}_0([\mathrm{GL}_2])$ denotes the space of smooth functions $\varphi : \mathrm{GL}_2(\mathbb{A}) \longrightarrow \mathbb{C}$ invariant under left translation by $\mathrm{GL}_2(\mathbb{Q})$ that vanish at cusps:

$$\int_{[N]} \varphi(ng)dn = 0,$$

where dn is a Haar measure on $N(\mathbb{A})$ such that the volume of $N(\mathbb{Q})\backslash N(\mathbb{A})$ is 1. For such a function, we can define its Whittaker function:

$$W_\varphi(g) := \int_{[N]} \varphi(ng)\psi_N^{-1}(n)dn,$$

where ψ_N is the character on $N(\mathbb{A})$ via the canonical isomorphism

$$\begin{aligned} n : \mathbb{Q} &\xrightarrow{\sim} N \\ x &\longmapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Then we have the Fourier expansion

$$\varphi(g) = \sum_{a \in \mathbb{Q}^\times} W_\varphi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right).$$

Therefore the Fourier transform induces an embedding of representations of $\mathrm{GL}_2(\mathbb{A})$:

$$W : \mathcal{A}_0([\mathrm{GL}_2]) \longrightarrow \mathcal{W}(\psi) := \mathrm{Ind}_{N(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}(\psi_N).$$

By a cuspidal automorphic representation, we mean a subrepresentation $\pi \subset \mathcal{A}_0([\mathrm{GL}_2])$. We can embed π into $\mathcal{W}(\psi)$,

$$\pi \hookrightarrow \mathcal{W}(\psi).$$

If π is irreducible, then π is the restricted tensor product $\bigotimes_p \pi_p$. More precisely, there is a unique embedding of π into $\mathcal{W}(\psi)$ with image denoted its Whittaker model $\mathcal{W}(\pi, \psi)$, which has a decomposition

$$\mathcal{W}(\pi, \psi) = \bigotimes_p \mathcal{W}(\pi_p, \psi_p).$$

Each element in $\mathcal{W}(\pi, \psi)$ can be written as a finite linear combination of pure tensors $\bigotimes W_p$ such that for all but finitely many p, $W_p \in \mathcal{W}(\pi_p, \psi_p)$ is the normalized spherical element in the sense that W_p is invariant under $\mathrm{GL}_2(\mathbb{Z}_p)$ and $W_p(e) = 1$.

Let ω be the central character of π. Then define the Kirillov representation $\mathcal{K}(\omega, \psi)$ of $B(\mathbb{A})$ on $C^\infty(\mathbb{A}^\times)$ by the formula,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} f(x) = \omega(d)\psi\left(\frac{bx}{d}\right) f\left(\frac{ax}{d}\right).$$

Following Jacquet–Langlands [JL70], the restriction

$$\mathcal{W}(\pi, \psi) \longrightarrow \mathcal{K}(\omega, \psi),$$
$$W \longmapsto \kappa_W(x) := W(a(x)),$$

is injective. Let $\mathcal{K}(\pi, \psi)$ denote the image of this map.

New forms

Each irreducible cuspidal representation π has associated data (weight, level, central character, and new forms), defined as follows (cf. [Zha01, Section 2.3]).

1. The *weight* w of π is the minimal non-negative integer such that there exist a non-zero vector

$v \in \pi_\infty$ and a $\theta \in \mathbb{R}$ with,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} v = e^{iw\theta} v.$$

It is well-known that if w is the weight of the discrete series π_∞, then all other eigenvalues of $SO_2(\mathbb{R})$ are given by $\pm(w + 2k)$ for $k \in \mathbb{Z}_{\geq 0}$ (cf. [Bum97, Theorem 2.5.4(ii)] for the language of K-types).

2. The *level* N of π is the minimal positive integer such that $\pi^{U_1(N)}$ is non-zero, where

$$U_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathbb{Z}}) \,\middle|\, (c, d) \equiv (0, 1) \pmod{N\widehat{\mathbb{Z}}} \right\}.$$

   In that case, $\pi^{U_1(N)}$ is one-dimensional and is called the space of new forms.

3. the *central character* ω of π is the character of $[Z] \xrightarrow{\sim} \mathbb{Q}^\times \backslash \mathbb{A}^\times$ acting on π.

4. The *new vector* φ^{new} is a function in π whose Fourier transform $W^{\text{new}} = W_{\varphi^{\text{new}}}$ is a product of W_p^{new} (defined as before for $p < \infty$, and with W_∞^{new} required to take value 1 at the unit element e and have weight w under the action of SO_2).

Here we give two examples of new vectors in Whittaker models. The first one is the weight-k Whittaker function W_k for a positive integer k on $GL_2(\mathbb{R})$. By the Iwasawa decomposition, we need only specify its value on $a(y) = \left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $y \in \mathbb{R}^\times$,

$$W_k\big(a(y)\big) = \begin{cases} y^{k/2} & \text{if } y > 0 \\ 0 & \text{if } y < 0. \end{cases}$$

The second is the one for the unramified principal series $\pi(\chi_1, \chi_2)$. Again we need only consider its value at $a(p^n) = \left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)$:

$$W(a(p^n)) = p^{\frac{-n}{2}} \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2},$$

where $\alpha_1 = \chi_1(p)$ and $\alpha_2 = \chi_2(p)$.

1.2 Weil representations

Let (V, Q) be an orthogonal quadratic space over $\mathbb{Q}$ of even dimension m with bilinear form.

$$\langle x, y \rangle = Q(x+y) - Q(x) - Q(y).$$

Let $\mathrm{GO}(V)$ denote the group of similitudes on V with norm map $\nu : \mathrm{GO}(V) \longrightarrow \mathbb{G}_m$. Let $G = \mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GO}(V)$ be the fiber product of ν and $\det : \mathrm{GL}_2 \longrightarrow \mathbb{G}_m$. Then we may consider SL_2 and $\mathrm{O}(V)$ as subgroups of G. Let $\mathcal{S}(V_{\mathbb{A}})$ be the space of Schwartz functions on $V_{\mathbb{A}}$ and let $\psi : \mathbb{A}/\mathbb{Q} \longrightarrow \mathbb{C}$ be the standard character. Then we have a Weil representation r of $G(\mathbb{A})$ on $\mathcal{S}(V_{\mathbb{A}})$ by the following rules (cf. [Wal85, Section I], [HK92, Section 5], [HK04, Section 3], [YZZ13, Section 2.1]). To define this representation, we need the following special elements in GL_2:

$$d(a) := \begin{pmatrix} 1 & \\ & a \end{pmatrix},$$

$$m(a) := \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix},$$

$$n(b) := \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix},$$

$$w := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Then G is generated by elements $(d(\nu(h)), h)$ for $h \in \mathrm{GO}(V)$, $m(a)$, $n(b)$, and w.

1. For any $h \in \mathrm{GO}(V_{\mathbb{A}})$, $\Phi \in \mathcal{S}(V_{\mathbb{A}})$,

$$r\big(d(\nu(h)), h\big) \cdot \Phi(x) = |\nu(h)|^{\frac{-m}{4}} \Phi\big(h^{-1}x\big).$$

2. For any $a \in \mathbb{A}^{\times}$,

$$r\big(m(a)\big) \cdot \Phi(x) = \eta_V(a)|a|^{m/2}\Phi(ax),$$

where $\eta_V(a) = \big(a, (-1)^{m/2}\det(V)\big)$, or in other words,

$$\eta_V = \eta_{\mathbb{Q}\left(\sqrt{(-1)^{\frac{m}{2}}\det(V)}\right)}(a).$$

3. For any $b \in \mathbb{A}$,

$$r\big(n(b)\big) \cdot \Phi(x) = \psi\big(bQ(x)\big)\Phi(x).$$

4. For w as above,

$$r(w) \cdot \Phi(x) = \gamma \cdot \widehat{\Phi}(x),$$

where γ is an 8-th root of unity and $\widehat{\Phi}$ is the Fourier transform,

$$\widehat{\Phi}(x) = \int_{V_{\mathbb{A}}} \Phi(y)\psi(\langle x, y\rangle)dy.$$

From the definition, we see that $r(z, z)$ acts on $\mathcal{S}(V_{\mathbb{A}})$ by the character η_V. Indeed,

$$r(z,z)\Phi(x) = r\big(d(z^2)m(z), z\big)\Phi(x) = |z|^{-m/2}r(m(z))\Phi(z^{-1}x = \eta_V(z)\Phi(x).$$

1.3 Theta series

Let $\mathrm{GL}_2(\mathbb{A})^+$ denote the subgroup of $\mathrm{GL}_2(\mathbb{A})$ of elements with determinants in $\nu(\mathrm{GO}(V_{\mathbb{A}}))$. For any $\Phi \in \mathcal{S}(V_{\mathbb{A}})$, define the theta series (or theta kernel) automorphic form (cf. [YZZ13, Section 2.1]),

$$\theta(g,h,\Phi) := \sum_{x\in V} r(g,h)\Phi(x) \in \mathcal{A}\big(G(\mathbb{A})\big).$$

Let $\mathcal{A}(G(\mathbb{A}))^*$ be the dual space on the space of automorphic forms, which we call the space of automorphic distributions. Then for any distribution φ on $\mathrm{GO}(V)\backslash\mathrm{GO}(V_{\mathbb{A}})$, we can define a form on $\mathrm{GL}_2^+(\mathbb{Q})\backslash\mathrm{GL}_2^+(\mathbb{A})$ by integration,

$$\theta(g,\varphi,\Phi) := \int_{[O(V)]} \theta(g,hh_0,\Phi)\varphi(hh_0)dh, \tag{1.1}$$

where $h_0 \in \mathrm{GO}(V)$ is an element with norm $\det g$ to ensure that $hh_0 \in \mathrm{GO}(V)$. Now we extend $\theta(g,\varphi,\Phi)$ to a function on $\mathrm{GL}_2(\mathbb{A})$ by two rules:

1. $\theta(g,\varphi,\Phi)$ is invariant under the left action by $\mathrm{GL}_2(\mathbb{Q})$;
2. $\theta(g,\varphi,\Phi)$ is supported on $\mathrm{GL}_2(\mathbb{Q})\cdot\mathrm{GL}_2^+(\mathbb{A})$.

Now suppose that there is a character ω of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ such that for $z\in\mathbb{A}^\times, h\in\mathrm{GO}(V_{\mathbb{A}})$,

$$\varphi(zh) = \omega(z)\varphi(h).$$

Then we have,

$$\begin{aligned}\theta(zg,\varphi,\Phi) &= \int_{[O(V_{\mathbb{A}})]} \theta(zg,zhh_0)\varphi(zhh_0)dh_0\\ &= \eta_V(z)\omega(z)\int_{[O(V_{\mathbb{A}})]} \theta(g,hh_0)\varphi(hh_0)dh_0\\ &= \eta_V(z)\omega(z)\theta(g,\varphi,\Phi).\end{aligned}$$

Whittaker functions

In the following, we compute the Whittaker function of $\theta(g, \varphi, \Phi)$ when φ is an automorphic function on $[\mathrm{GSO}(V)]$:

$$W(g, \varphi, \Phi) := \int_{\mathbb{Q}\backslash\mathbb{A}} \theta(n(b)g, \varphi, \Phi)\psi(-b)db.$$

Proposition 1.1. *The function* $W(g, \varphi, \Phi)$ *is supported on*

$$\mathrm{GL}_2(\mathbb{A})_{Q(V_{\mathbb{A}})} := \{g \in \mathrm{GL}_2(\mathbb{A}) \,|\, \det g \in Q(V_{\mathbb{A}})\}.$$

Moreover for $g \in \mathrm{GL}_2(\mathbb{A})_{Q(V_{\mathbb{A}})}$ *with decomposition* $g = d(Q(v)^{-1})g_1$, *where* $v \in V_{\mathbb{A}}$ *and* $g_1 \in \mathrm{SL}_2(\mathbb{A})$, *we have the following expression:*

$$W(g, \varphi, \Phi) = |\det g|^{-\frac{m}{4}} \int_{O(V_{\mathbb{A}})/O(V_{v,\mathbb{A}})} r(g_1)\Phi(hv) \int_{[O(V_0)]} \varphi\left(uh_0^{-1}h^{-1}\right) du dh,$$

where

1. $V_{v,\mathbb{A}}$ *is the orthogonal complement of* v *in* $V_{\mathbb{A}}$*;*

2. $h_0 \in \mathrm{GO}(V_{\mathbb{A}})$ *such that* $v_0 := h_0^{-1}v \in V$, *which induces an isomorphism*

$$O(V_{\mathbb{A}})/O(V_{v,\mathbb{A}}) \xrightarrow{\sim} O(V_{\mathbb{A}})/O(V_{0,\mathbb{A}})$$
$$h \longmapsto h_0 h h_0^{-1},$$

   where V_0 *is the orthogonal complement of* v_0 *in* V*;*

3. dh *is a measure induced by the above isomorphism and the quotient measure of the measure on* $O(V_{\mathbb{A}})$ *by the measure on* $O(V_{0,\mathbb{A}})$ *so that the volume of* $[O(V_0)]$ *is* 1.

Proof. It is clear that the function $W(g, \varphi, \Phi)$ is also supported on $\mathrm{GL}_2(\mathbb{Q}) \cdot \mathrm{GL}_2^+(\mathbb{A})$. For $g \in \mathrm{GL}_2(\mathbb{Q}) \cdot \mathrm{GL}_2^+(\mathbb{A})$, we may write $g = d(a\nu(h_0))g_1$ for some $a \in \mathbb{Q}^\times$, $h_0 \in \mathrm{GO}(V_{\mathbb{A}})$, and

$g_1 \in \mathrm{SL}_2(\mathbb{A})$. Then we have

$$\begin{aligned} W(g, \varphi, \Phi) &= \int_{\mathbb{Q}\backslash\mathbb{A}} \theta\big(n(b)d\big(a\nu(h_0)\big)g_1, \varphi, \Phi\big)\psi(-b)db \\ &= \int_{\mathbb{Q}\backslash\mathbb{A}} \theta\big(n(ab)d\big(\nu(h_0)\big)g_1, \varphi, \Phi\big)\psi(-b)db \\ &= \int_{\mathbb{Q}\backslash\mathbb{A}} \theta\big(n(b)d\big(\nu(h_0)\big)g_1, \varphi, \Phi\big)\psi(-a^{-1}b)db \\ &= \int_{[O(V)]} \varphi(hh_0) \int_{\mathbb{Q}\backslash\mathbb{A}} \theta\big(n(b)d\big(\nu(h_0)\big)g_1, hh_0, \Phi\big)\psi(-a^{-1}b)dbdh. \end{aligned}$$

The second integral can be computed directly (for general g' and h'):

$$\begin{aligned} \int_{\mathbb{Q}\backslash\mathbb{A}} \theta(n(b)g', h', \Phi)\psi(-a^{-1}b)db &= \int_{\mathbb{Q}\backslash\mathbb{A}} \sum_{x\in V} \psi\big(\big(q(x) - a^{-1}\big)b\big)r(g', h')\Phi(x)db \\ &= \sum_{x\in V_a} r(g', h')\Phi(x), \end{aligned}$$

where V_a denote the subset of elements $x \in V$ with norm $q(x) = a^{-1}$. Define

$$\theta_a(g', h', \Phi) := \sum_{x\in V_a} r(g', h')\Phi(x).$$

Using $g' = d\big(\nu(h_0)\big)g_1$ and $h' = hh_0$, we have shown that

$$W(g, \varphi, \Phi) = \int_{[O(V)]} \theta_a(d(\nu(h_0))g_1, hh_0, \Phi)\varphi(hh_0)dh.$$

This shows that $W(g, \varphi, \Phi)$ is actually supported on $q(V^\times)\mathrm{GL}_2^+(\mathbb{A})$, where $V^\times$ is the subset of elements in V with non-zero norm. This proves the first part of Proposition 1.1.

For the second part of the Proposition 1.1, we use the fact that the V_a is an orbit of some $\nu_0 \in V_a$. Let V_0 be the orthogonal complement of ν_0 in V. Then we have

$$\theta_a(d(\nu(h_0)g_1, hh_0, \Phi) = \sum_{\gamma\in O(V_0)\backslash O(V)} |\det g|^{-\frac{m}{4}} r(g_1)\Phi\big(h_0^{-1}h^{-1}\gamma^{-1}\nu_0\big).$$

It follows that,

$$\begin{aligned} W(g,\varphi,\Phi) &= |\det g|^{-\frac{m}{4}} \int_{O(V_0)\backslash O(V_{\mathbb{A}})} r(g_1)\Phi\left(h_0^{-1}h^{-1}v_0\right)\varphi(hh_0)dh \\ &= |\det g|^{-\frac{m}{4}} \int_{O(V_{0,\mathbb{A}})\backslash O(V_{\mathbb{A}})} r(g_1)\Phi\left(h_0^{-1}h^{-1}v_0\right) \int_{[O(V_0)]} \varphi(uhh_0)dudh. \end{aligned}$$

A change of variables $h \mapsto h_0h^{-1}h_0^{-1}$ yields

$$W(g,\varphi,\Phi) = |\det g|^{-\frac{m}{4}} \int_{O(V_{\mathbb{A}})/h_0O(V_{0,\mathbb{A}})h_0^{-1}} r(g_1)\Phi\left(hh_0^{-1}v_0\right) \int_{[O(V_0)]} \varphi\left(uh_0h^{-1}\right)dudh.$$

Set $v = h_0^{-1}v_0$. Then $Q(v) = \nu(h_0)^{-1}a^{-1}$. Finally, change h_0 to h_0^{-1}. $\square$

It is quite useful to consider the Kirillov model, i.e the restriction of Whittaker functions at elements $g = a(x) = \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $x = Q(v)$. Assume that φ has a central character ω. Writing $g = Q(v)d(Q(v)^{-1})$ obtains the following.

Corollary 1.2. *Assume that φ has the central character ω. Then the Kirillov function $\kappa(x,\varphi,\Phi)$ for the theta series $\theta(g,\varphi,\Phi)$ is supported on $Q(V_{\mathbb{A}})$ with the following formula*

$$\kappa(x,\varphi,\Phi) = \eta_V\omega(x)|x|^{\frac{m}{4}} \int_{O(V_{\mathbb{A}})/O(V_{v,\mathbb{A}})} \Phi(hv) \int_{[O(V_0)]} \varphi\left(uh_0^{-1}h^{-1}\right)dudh,$$

where $v \in V_{\mathbb{A}}$ and $h_0 \in GO(V_{\mathbb{A}})$ such that $Q(v) = x$, $v_0 = h_0^{-1}v \in V$, and V_0 is the orthogonal complement ofv_0.

1.4 Theta series for one character

Let K be a quadratic field and $\chi : K^\times\backslash K_{\mathbb{A}}^\times \longrightarrow \mathbb{C}^\times$ be a finite character. Assume the following conditions.

1. χ is not of the form $\mu \circ N_{K/\mathbb{Q}}$, where $N_{K/\mathbb{Q}}$ is the norm of K over $\mathbb{Q}$.

2. If K is real, then the two components at the archimedean places have different signs.

Then we have an irreducible cuspidal representation $\pi(\chi)$ of GL_2 of weight 1. In the following, we want to construct new forms in $\pi(\chi)$ and optimal forms in $\pi(\chi) \otimes \pi(\chi^{-1})$ using theta liftings.

We start with the general quadratic space $V = (Ke, Q)$ under the action of K. Then $GO(V) = \langle K^\times, \iota \rangle$, where ι is an involution. In this case, ν is the usual norm $N = N_{K/\mathbb{Q}}$ of K over $\mathbb{Q}$. For each $\Phi \in \mathcal{S}(K_{\mathbb{A}}^\times)$, we obtain a theta series $\theta(g, \chi^c, \Phi) \in \mathcal{A}(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$. Its Whittaker function is supported by the subgroup $GL_2(\mathbb{A})^+$ of matrices with determinant in $N(K_{\mathbb{A}}^\times)$. By Proposition 1.1, we write $g = d(Q(h_0 e)^{-1})g_1$ with $h_0 \in K_{\mathbb{A}}^\times$ and $g_1 \in SL_2(\mathbb{A})$ to obtain

$$W(g, \chi, \Phi) = |\det g|^{-\frac{1}{2}} \int_{K_{\mathbb{A}}^1} r(g_1)\Phi(hh_0 e)\chi^c\left(h_0^{-1}h^{-1}\right)dh, \tag{1.2}$$

where K^1 is the subgroup of $K^\times$ of elements with norm 1. By Corollary 1.2, we have for $x = Q(h_0 e)$,

$$\kappa(x, \chi, \Phi) = |x|^{\frac{1}{2}} \int_{K_{\mathbb{A}}^1} \Phi(h_0 he)\chi(hh_0)dh. \tag{1.3}$$

Note that we used χ^c instead of χ for a neater zeta integral. More precisely,

$$\begin{aligned} Z(s, \theta(g, \chi^c, \Phi)) :&= \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \theta(a(x), \chi^c, \Phi)|x|^{s-\frac{1}{2}}dx \\ &= \int_{\mathbb{A}^\times} \kappa(a(x), \chi^c, \Phi)|s|^{s-\frac{1}{2}}dx \\ &= \int_{\mathbb{A}^\times} \Phi(xe)\chi(x)|x|^s dx \\ &=: Z(s, \chi, \Phi) \end{aligned}$$

The subrepresentation of $\mathcal{A}([GL_2])$ generated by $\theta(g, \chi, \Phi)$ is an irreducible representation denoted by $\pi(\chi)$. More precisely, this representation has a decomposition (cf. [Shi72, Equation 5.1]),

$$\pi(\chi) = \bigotimes_{p\leq\infty} \pi(\chi_p),$$

and $\pi(\chi_p)$ has Whittaker and Kirillov models generated respectively by the functions (cf. [Shi72, Equation 5.2])

$$W(g,\chi_p,\Phi_p) = |\det g|^{-\frac{1}{2}} \int_{K_p^1} r(g_1)\Phi(hh_0e)\chi_p^c\left(h_0^{-1}h^{-1}\right)dh, \tag{1.4}$$

$$\kappa(x,\chi_p,\Phi_p) = |x|^{\frac{1}{2}} \int_{K_p^1} \Phi(h_0he)\chi(hh_0)dh, \tag{1.5}$$

again with $x = Q(h_0e)$.

New forms

Now assume $V = (K, N)$, where $N = N_{K/\mathbb{Q}}$ is the norm of K over $\mathbb{Q}$. We construct a new form $\varphi^{\text{new}} \in \pi(\chi)$ by picking a standard

$$\Phi_\chi = \bigotimes_v \Phi_{\chi_v} \in \mathcal{S}(\mathbb{A}_K)$$

where the tensor product is over places of K. We pick Φ_{χ_v} as follows (cf. [Zha01, Section 2.1]):

1. If v is complex, $K_v \xrightarrow{\sim} \mathbb{C}$, and χ_v is trivial, take

$$\Phi_{\chi_v}(x+yi) = e^{-2\pi\left(x^2+y^2\right)}.$$

2. If v is real, $K_v = \mathbb{R}$, and $\chi_v(x) = \text{sgn}(x)^m$ with $m = 0, 1$, take

$$\Phi_{\chi_v}(x) = x^m e^{-\pi x^2}.$$

3. If v is finite and χ_v is unramified, take

$$\Phi_{\chi_v} = \mathbb{1}|_{\mathcal{O}_{K_v}}.$$

4. If ν is finite and χ_ν is ramified, take

$$\Phi_{\chi_\nu} = \chi_\nu^{-1}|_{\mathcal{O}_{K_\nu}^\times}.$$

For each place p of $\mathbb{Q}$, let

$$\Phi_{\chi_p} = \bigotimes_{\nu | p} \Phi_{\chi_\nu} \in \mathcal{S}(K_p).$$

In the real case, to avoid the need to remember signs, we use the following function instead:

$$\Phi_\infty(x, y) = \frac{1}{2}(x + y)e^{-\pi\left(x^2+y^2\right)}.$$

Then we have the following description of the Whittaker function $W(g, \chi_p^c, \Phi_{\chi_p})$ (cf. [Zha01, Section 2.3]).

1. If $p = \infty$, then $W(g, \chi_p^c, \Phi_{\chi_p})$ is the weight 1 form $W(g)$ in the following sense that,

$$W\left(z\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) = \operatorname{sgn}(z)^m \cdot y^{\frac{1}{2}}\Big|_{\mathbb{R}_+^\times} \cdot e^{i\theta},$$

   where $m = 0$ if K is imaginary, and $m = 1$ if K is real.

2. If p is not ramified in K, then $W(g, \chi_p^c, \Phi_{\chi_p})$ is the new form W_χ^{new} in $\pi(\chi_p)$ in the sense that it is invariant under $U_1(\pi^{c(\pi(\chi_p))})$ and takes value 1 at e.

3. If p is ramified in K, then $W(g, \chi_p^c, \Phi_{\chi_p})$ is the restriction of the new form W_χ^{new} on $\mathrm{GL}_2(\mathbb{Q}_p)^+$. One can also recover a new form by,

$$W_\chi^{\text{new}}(g) := W\big(g, \chi_p, \Phi_{\chi_p}\big) + W\big(ga(\epsilon_p), \chi_p, \Phi_{\chi_p}\big),$$

   where $\epsilon_p \in \mathbb{Z}_p^\times - \mathrm{N}(\mathcal{O}_{K_p}^\times)$.

Using Φ_χ, we get the theta series $\theta(g, \chi^c, \Phi_\chi)$. The new form is given by,

$$\varphi_\chi^{\text{new}}(g) = \sum_{\epsilon \in \widehat{\mathbb{Z}}^\times / N(\widehat{\mathcal{O}}_K)} \theta(g a(\epsilon), \chi^c, \Phi_\chi). \tag{1.6}$$

The sum in the right-hand side has 2^n many non-zero terms, where n is the number of primes ramified in K.

Optimal forms

Now we consider the tensor product representation, $\pi(\chi) \otimes_{\mathbb{Q}(\chi)} \pi(\chi^{-1})$. We already know that it has one distinguished element called the *new form*,

$$\varphi^{\text{new}} = \varphi_\chi^{\text{new}} \otimes \varphi_{\chi^{-1}}^{\text{new}}.$$

In the following, we construct another element called the *optimal form* φ^{opt} in this tensor product space that depends only on the "antinorm" $\xi := \chi^{1-\epsilon}$. Note that ξ is a ring class character. More precisely, let $c := c(\xi)$ be the conductor of ξ, i.e. the minimal integer such that ξ is trivial on $(1 + c(\xi)\widehat{\mathcal{O}}_K)^\times$. Then define the associated order of K as,

$$\mathcal{O}_{c(\xi)} = \mathbb{Z} + c(\xi)\mathcal{O}_K.$$

ξ is in fact trivial on $\widehat{\mathcal{O}}_{c(\xi)}^\times$. We can therefore view ξ as a character on

$$K^\times \backslash K_{\mathbb{A}}^\times / K_\infty^+ \widehat{\mathcal{O}}_{c(\xi)}^\times = K_+^\times \backslash \widehat{K}^\times / \widehat{\mathcal{O}}_{c(\xi)}^\times =: \text{Pic}^+\big(\mathcal{O}_{c(\xi)}\big),$$

where K_+ means K in the complex case and means a positive element in K in the real case.

Definition 1.3. Let $\mathcal{D}$ be the different ideal, i.e. the ideal generated by elements $x - \bar{x}$. Let δ be a generator of $\widehat{\mathcal{D}}$ in $\widehat{\mathcal{O}}_{c(\xi)}$. Now for each $\alpha \in \mathcal{O}_{c(\xi)}/\mathcal{D}$, define the function $\Phi_\alpha^{\text{opt}} = \Phi_{\alpha,\infty}^{\text{opt}} \otimes \Phi_\alpha^{\text{opt},\infty} \in \mathcal{S}(K_{\mathbb{A}})$ as follows.

1. $\Phi^{\mathrm{opt}}_{\alpha,\infty}$ is the new function for the character χ_∞, i.e. $e^{-2\pi|z|^2}$ in the complex case, and $\frac{1}{2}(x+y)e^{-\pi(x^2+y^2)}$ in the real case.

2. $\Phi^{\mathrm{opt},\infty}_{\alpha}$ is the characteristic function of

$$\widehat{\mathcal{O}}_c + \frac{\alpha}{\delta}.$$

Using the theta series $\theta(g, \chi, \Phi_\alpha)$, define the two-variable optimal form φ^{opt} as follows.

$$\varphi^{\mathrm{opt}}(g_1, g_2) := \sum_{\alpha \in \mathcal{O}_c/\mathcal{D}} \theta\Big(g_1, \chi, \Phi^{\mathrm{opt}}_{\alpha}\Big)\, \theta\big(g_2 \epsilon^\infty, \chi^{-1}, \Phi^{\mathrm{opt}}_{-\alpha}\big), \tag{1.7}$$

where ϵ^∞ is the element $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\widehat{\mathbb{Q}})$.

Remark 1.4. We chose the name "optimal form" here due to the relation to optimal embeddings. If B is the definite quaternion algebra with discriminant q and q is inert in K, then there is an embedding $K \hookrightarrow B$ and a maximal order $\mathcal{O}_B$ such that $\mathcal{O}_{c(\xi)} = \mathcal{O}_B \cap K$. In particular, $\mathcal{O}_{c(\xi)}$ is an optimal order in $\mathcal{O}_B$ and $\mathcal{O}_{c(\xi)} \hookrightarrow \mathcal{O}_B$ is an optimal embedding (cf. [Eic55, Section 3], [Gro87, Sections 1 & 3], and [Voi21, Section 30.3]). See [Voi21, Remark 30.3.17] for the history of the "optimal" terminology.

We can write down the optimal form's Kirillov functions. First, define

$$\kappa(x, \chi, \Phi_\alpha) = |x|^{\frac{1}{2}} \int_{K^1_{\mathbb{A}}} \Phi_\alpha(hh_0)\chi(hh_0)\,dh,$$

where $x = N(h_0)$. The Kirillov function for φ^{opt} is given by

$$\kappa^{\mathrm{opt}}(x_1, x_2) = \sum_{\alpha \in \mathcal{O}_c/\mathcal{D}} \kappa(x_1, \chi, \Phi^{\mathrm{opt}}_{\alpha})\kappa\big(x_2(-1)^\infty, \chi^{-1}, \Phi^{\mathrm{opt}}_{-\alpha}\big). \tag{1.8}$$

Comparison of models

Now we study the general quadratic space $V = (Ke, Q)$ under the action of K, so $GSO(V) = K^\times$. Then $\pi(\chi, \psi)$ can also be constructed by $\mathcal{S}(V(\mathbb{A}))$. More precisely by Equation 1.3, for each $\Phi \in \mathcal{S}(V(\mathbb{A}))$, the Kirillov function associated with the theta series $\theta(g, \chi^c, \Phi)$ is given by

$$\kappa(x, \chi^c, \Phi) = |x|^{\frac{1}{2}} \int_{K^1_{\mathbb{A}}} \Phi(tt_0 e)\chi\big(t_0^{-1}t^{-1}\big)\,dt,$$

where $x = Q(t_0 e)$.

Let $V' = (Ke', Q')$ be another quadratic space and $\iota : V'_{\mathbb{A}} \xrightarrow{\sim} V_{\mathbb{A}}$ be an isomorphism of $K_{\mathbb{A}}$-modules. Then we have a isomorphism,

$$\begin{aligned} \iota^* : \mathcal{S}(V_{\mathbb{A}}) &\xrightarrow{\sim} \mathcal{S}(V'_{\mathbb{A}}) \\ \Phi &\longmapsto \Phi \circ \iota. \end{aligned}$$

The Kirillov function's integral can be converted to an integral for $\iota^*\Phi$ as follows:

$$\begin{aligned} \kappa(x, \chi^c, \Phi) &= |x|^{\frac{1}{2}} \int_{K^1_{\mathbb{A}}} \iota^*\Phi\big(tt_0\iota^{-1}(e)\big)\chi\big(t_0^{-1}t^{-1}\big)\,dt, \\ &= |x|^{\frac{1}{2}} \big|tt_0\iota^{-1}(e)\big|^{-\frac{1}{2}} \kappa\big(Q\big(t_0\iota^{-1}(e)\big), \chi^c, \iota^*\Phi\big). \end{aligned}$$

Write $Q(\iota) = Q(e)/Q(\iota^{-1}e) \in K^\times_{\mathbb{A}}$. Then $Q(\iota)$ does not depend on the choice of e and is called the norm of the map ι. Then the above formula gives:

$$\kappa(x, \chi^c, \Phi) = |Q(\iota)|^{\frac{1}{2}} \kappa\big(xQ(\iota)^{-1}, \chi^c, \iota^*\Phi\big).$$

Since its Kirillov functions determine automorphic forms, we have proved the following.

Proposition 1.5. *Let V and V' be two quadratic spaces of dimension two with action by K. Let $\iota : V'_{\mathbb{A}} \longrightarrow V_{\mathbb{A}}$ be an isomorphism of $K_{\mathbb{A}}$ spaces with norm $Q(\iota)$. Then for any function $\Phi \in \mathcal{S}(V_{\mathbb{A}})$,* *we have*

$$\theta(g,\chi^c,\Phi) = |Q(\iota)|^{\frac{1}{2}}\theta\big(ga\big(Q(\iota)^{-1}\big),\chi^c,\iota^*\Phi\big).$$

For example, if we compare the theta functions defined by two opposite spaces $V_\pm := (V, \pm Q)$, then for each $\Phi \in \mathcal{S}(K)$, we get two theta series: $\theta_\pm(g,\chi^c,\Phi)$. We use the identity map $\iota : V_\mathbb{A} \longrightarrow V_\mathbb{A}$ for the quadratic space, so $Q(\iota) = -1$. Then we have:

$$\theta_-(g,\chi^c,\Phi) = \theta_+(g\epsilon,\chi^c,\Phi),$$

where $\epsilon = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

In the case that $V^\pm = (K, \pm N)$ and $\Phi = \Phi_\chi$, we see from the above identity that the Whittaker function $W_-(g,\chi_c,\Phi_\chi)$ of $\theta_-(g,\chi_c,\Phi_\chi)$ is still new at the finite part, but has weight -1 at ∞ with value

$$W_-(a(y)) = |y|^{\frac{1}{2}} \begin{cases} -y & \text{if } y < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus we also have,

$$\theta_-(g,\chi^c,\Phi) = \theta_+(g\epsilon_\infty,\chi^c,\Phi).$$

1.5 Theta series for two characters

Theta series for automorphic forms

Now we consider the theta lifting for $V = (B, N)$, with B a quaternion algebra over $\mathbb{Q}$ and with norm N given by the reduced norm on B. Then

$$GO(V) = \big\langle GSO(V) = B^\times \times B^\times/\Delta\mathbb{Q}^\times, \iota\big\rangle,$$

where $(b_1, b_2) \in B^\times \times B^\times$ brings $x \in V$ to $b_1 x b_2^{-1}$, and $\iota(x) = \bar{x}$. Let G denote the group over $\mathbb{Q}$ defined by

$$G := GL_2 \times_{\mathbb{G}_m} GSO(V).$$

Then we have a Weil representation of $G(\mathbb{A})$ on $\mathcal{S}(V(\mathbb{A}))$. For each $\Phi \in \mathcal{S}(V(\mathbb{A}))$, we have a theta series

$$\theta(g, h, \Phi) = \sum_{x \in V} r(g, h)\Phi(x).$$

Also for each automorphic form (or even each distribution) φ on $GO(V)$, we get a form on $GL(\mathbb{A})^+$ by

$$\theta(g, \varphi, \Phi) = \int_{[O(V)]} \theta(g, hh_0, \Phi)\varphi(hh_0)dh.$$

We want to interpret the theta liftings as Hecke operators. For any $g \in B^\times$, we define an operator $\rho(g)$ on $\mathcal{A}([B^\times])$ as usual:

$$\rho(g)\varphi(x) = \varphi(xg).$$

Now for any $x \in N(\mathbb{A}^\times)$ and $\Phi \in \mathcal{S}(B_{\mathbb{A}})$, we define the Hecke operator:

$$T_\Phi(x) = \int_{B^1_{\mathbb{A}}} \Phi(b_0 b)\rho(b_0 b)db \tag{1.9}$$

$$T^*_\Phi(x) = \int_{B^1_{\mathbb{A}}} \Phi\big(b^{-1}b_0^{-1}\big)\rho(b_0 b)db, \tag{1.10}$$

where $b_0 \in B^\times_{\mathbb{A}}$ such that $N(b_0) = x$.

Proposition 1.6. *Let* $\varphi = \varphi_1 \otimes \varphi_2$ *with* φ_i *automorphic forms on* $[B^\times]$ *with central characters* ω *and* ω^{-1}. *Then the Kirillov function* $\kappa(x, \varphi, \Phi)$ *is supported on* $N(B^\times_{\mathbb{A}})$ *with values given as follows:*

$$\kappa(x, \varphi, \Phi) = \omega(x)|x|\langle\varphi_1, T_\Phi(x)\varphi_2\rangle = \omega(x)|x|\langle T^*_\Phi(x)\varphi_1, \varphi_2\rangle,$$

where the pairing $[-, -]$ *is the bilinear form defined by*

$$\langle\varphi_1, \varphi_2\rangle = \int_{[B^\times/\mathbb{Q}^\times]} \varphi_1(u)\varphi_2(u)du.$$

Proof. By Corollary 1.2, if we take $x = Q(b_0)$ for some $b_0 \in B_{\mathbb{A}}^{\times}$, $h_0 = (1, b_0^{-1})$, and $\nu_0 = e$, then

$$\kappa(x, \varphi, \Phi) = \omega(x)|x| \int_{O(V_{\mathbb{A}})/O(V_{b_0,\mathbb{A}})} \Phi(hb_0) \int_{[O(V_0)]} \varphi\big(u \cdot (1, b_0) \cdot h^{-1}\big) du dh.$$

Here, $O(V) = B^{\times} \times_{\mathbb{Q}^{\times}} B^{\times}/\Delta\mathbb{Q}^{\times}$, and $O(V_{b_0,\mathbb{A}})$ consists of elements of the form $(b_0 b b_0^{-1}, b)$ for all $b \in B_{\mathbb{A}}^{\times}$. So we can use elements $(1, b^{-1})$ for $b \in B_{\mathbb{A}}^{1}$ to represent quotient elements. Then the above integral becomes:

$$\begin{aligned}
\kappa(x, \varphi, \Phi) &= \omega(x)|x| \int_{B_{\mathbb{A}}^{1}} \Phi(b_0 b) \int_{[B^{\times}/\mathbb{Q}^{\times}]} \varphi(u, ub_0 b) du db \\
&= \omega(x)|x| \int_{B_{\mathbb{A}}^{1}} \Phi(b_0 b) \int_{[B^{\times}/\mathbb{Q}^{\times}]} \varphi_1(u)\varphi_2(ub_0 b) du db \\
&= \omega(x)|x| \int_{B_{\mathbb{A}}^{1}} \Phi(b_0 b) \langle \varphi_1, \rho(b_0 b)\varphi_2 \rangle db \\
&= \omega(x)|x| \int_{B_{\mathbb{A}}^{1}} \Phi(b_0 b) \big\langle \rho\big(b^{-1} b_0^{-1}\big)\varphi_1, \varphi_2 \big\rangle db
\end{aligned}$$

The proposition follows from the last two identities. □

Theta series for two characters

Now let K be a quadratic field embedded into B. Then we have a decomposition $B = K + Kj$, which gives an orthogonal decomposition $V = V_1 + V_2$ for $V = (B, N)$. Then we have an embedding

$$GO(V_1) \times_{\mathbb{G}_m} GO(V_2) \subset GO(V).$$

The restriction to the connected component can be described as:

$$T := K^{\times} \times_{\mathbb{Q}^{\times}} K^{\times}/\Delta(\mathbb{Q}^{\times}) \xleftarrow{\sim} K^{\times} \times K^{\times}/\Delta(\mathbb{Q}^{\times}) \hookrightarrow B^{\times} \times B^{\times}/\Delta(\mathbb{Q}^{\times}).$$

The first map is given by,

$$\big(t_1/t_2, t_1\overline{t_2}\big) \longleftarrow (t_1, t_2).$$

Then we have two ways to describe an automorphic character for T in terms of two characters of $[K^\times]$: either as two automorphic characters ξ_1, ξ_2 of $\mathbb{A}_K^\times$ with the same restriction to $\mathbb{A}^\times$, or as the restriction to $[K^\times \times_{\mathbb{Q}^\times} K^\times]$ of a character $\chi_1 \otimes \chi_2$ on $[K^\times \times K^\times]$. Recalling that $\epsilon \in G_{\mathbb{Q}} - G_K$ and $\chi^\epsilon := \chi \circ \mathrm{ad}(\epsilon)$, the two descriptions are related in the following way,

$$\begin{aligned}\chi_1(t_1/t_2)\chi_2\big(t_1/\overline{t_2}\big) &= \xi_1(t_1)\xi_2(t_2),\\ \xi_1 &= \chi_1\chi_2,\\ \xi_2 &= \chi_1^{-1}\chi_2^{-\epsilon}.\end{aligned}$$

For an automorphic character $\xi = \xi_1 \otimes \xi_2$ and a function $\Phi \in \mathcal{S}(B_{\mathbb{A}})$, we define the theta lifting by,

$$\theta(g, \xi, \Phi) = \int_{[T]} \theta(g, t_0 t, \Phi)\xi(t_0 t)dt, \tag{1.11}$$

where $t_0 \in T(\mathbb{A})$ such that $N(t_0) = \det(g)$. This integration can be considered as the theta lifting for the distribution $\xi(t)dt$ on $[GSO(V)]$ defined by Equation 1.1.

Assume that $\Phi = \Phi_1 \otimes \Phi_2 \in \mathcal{S}(V_{\mathbb{A}}) = \mathcal{S}(V_{1,\mathbb{A}}) \otimes \mathcal{S}(V_{2,\mathbb{A}})$ is a decomposable function. Then for $h = (h_1, h_2) \in GO(V_1) \times_{\mathbb{G}_m} GO(V_2)$, we have,

$$\theta(g, h, \Phi) = \theta(g, h_1, \Phi_1) \cdot \theta(g, h_2, \Phi_2).$$

Thus if $\xi_1 \times \xi_2$ is the restriction of $\chi_1 \otimes \chi_2$, then we have,

$$\theta(g, \xi_1 \otimes \xi_2, \Phi) = \theta(g, \chi_1, \Phi_1) \cdot \theta(g, \chi_2, \Phi_2). \tag{1.12}$$

In the following, we assume that B is definite and K is imaginary.

Definition 1.7. Let $\mathcal{O}$ be an Eichler order of B, i.e. the intersection of two maximal orders in B.

Define the "standard" Schwartz function $\Phi_{\mathcal{O}} = \Phi_\infty \otimes \Phi^\infty$ as follows:

1. $\Phi_\infty(x) = e^{-2\pi|x|^2}$.

2. Φ^∞ is the characteristic function of $\widehat{\mathcal{O}}$.

We assume that ξ_1 and ξ_2 are finite characters with opposite restrictions on $\mathbb{A}^\times$. In this case, $\xi_1 \otimes \xi_2$ is the restriction of a finite character $\chi_1 \otimes \chi_2$. Under this assumption, the right-hand side Equation 1.12 shows that $\theta(g, \xi_1 \times \xi_2, \Phi)$ is a holomorphic form of weight 2. We can then apply Proposition 1.6 to obtain the following.

Proposition 1.8. *If $\Phi = \Phi_{\mathcal{O}}$ is standard as in Definition 1.7 with respect to an Eichler order $\mathcal{O}$ of B, then $\theta(g, \xi_1 \otimes \xi_2, \Phi)$ a holomorphic form of weight 2, level $U_1(\mathrm{disc}(\mathcal{O}))$, and central character ω.*

Let $M = \mathrm{disc}(\mathcal{O})$. Since $\theta(g, \xi, \Phi)$ is invariant under $U_1(M)$ and by the decomposition,

$$\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2(\mathbb{R})_+ U_1(M),$$

the value of $\theta(g, \xi, \Phi)$ is determined by its restriction on $\mathrm{GL}_2(\mathbb{R})_+$. Now we use the Whittaker decomposition:

$$\theta(g_\infty, \xi, \Phi) = \sum_{\lambda \in \mathbb{Q}^\times} W(a(\lambda) g_\infty, \xi, \Phi).$$

Since $W(g_\infty)$ has weight 2, $\theta(g, \xi, \Phi)$ is determined by the Kirillov function at the finite adèles. We would like to use Proposition 1.6 to write such a function, but there is a problem in defining the pairing and the Hecke action since the ξ_1, ξ_2 are distributions rather than automorphic functions.

1.6 Hecke operators

Following [DHRV22, Section 2.2] in the complex case, we define a projection map $[\cdot]$ from characters to automorphic forms. Let $\mathcal{A}(\omega^\pm)$ be the space of automorphic forms f on $[B]$ invariant under $U_1(M)$ and with the central character $\omega^\pm$ on $B_\infty^\times$. Then $\mathcal{A}(\omega^\pm)$ is a finite-dimensional space with the decomposition,

$$\mathcal{A}(\omega^{\pm}) = \mathcal{A}_{\infty} \otimes \mathcal{A}^{\infty}(\omega^{\pm}),$$

where $\mathcal{A}_{\infty}$ is the space of constant functions on $B_{\infty}^{\times}$. Analogously, define $\Xi(\omega^{\pm})$ to be the space of characters on $[K^{\times}]$ invariant under $U_K := U_0(M) \cap \widehat{K}^{\times}$ and with restriction $\omega^{\pm}$ on $\mathbb{A}^{\times}$. We only care about automorphic forms φ_1, φ_2 so that $\theta(g, \varphi_1 \otimes \varphi_2, \Phi)$ of $[B^{\times}]$ is a holomorphic form of weight 2 for $[GL_2]$. By Jacquet–Langlands [JL70], φ_1, φ_2 must be in the space of automorphic representations $\mathcal{A}(\omega^{\pm})$ of $[B]$. We want to define a projection map,

$$[\cdot] : \Xi(\omega^{\pm}) \longrightarrow \mathcal{A}(\omega^{\pm}),$$

such that,

$$\theta(g, \xi_1 \times \xi_2, \Phi_{\mathcal{O}}) = \theta(g, [\xi_1] \otimes [\xi_2], \Phi_{\mathcal{O}}). \tag{1.13}$$

Since we are dealing with the complex case, $\mathcal{A}(\omega^{+})$ and $\mathcal{A}(\omega^{-})$ are finite-dimensional and dual to each other. For any character $\xi^{\pm} \in \Xi(\omega^{\pm})$, we can define a linear functional on $\mathcal{A}(\omega^{\mp})$:

$$\begin{aligned} \mathcal{A}(\omega^{\mp}) &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \int_{[K^{\times}/\mathbb{Q}^{\times}]} \varphi(t)\xi^{\pm}(t)dt. \end{aligned} \tag{1.14}$$

Since $\mathcal{A}(\omega^{\pm})$ is finite-dimensional and dual to $\mathcal{A}(\omega^{\mp})$, the assignment of ξ to this functional gives the projection we want:

$$[-] : \Xi(\omega^{\pm}) \longrightarrow \mathcal{A}(\omega^{\pm}). \tag{1.15}$$

In particular, it satisfies Equation 1.13 since for all $\varphi \in \mathcal{A}(\omega^{\mp})$,

$$\int_{[B^{\times}/\mathbb{Q}^{\times}]} \varphi(x)[\xi](x)dx = \int_{[K^{\times}/\mathbb{Q}^{\times}]} \varphi(t)\xi(t).$$

Then by Propositions 1.6 and 1.8, we have the following.

Proposition 1.9. *Assume that* K *is imaginary. Let* $\Phi = \Phi_{\mathcal{O}}$ *be a standard function as in Defini-*

tion 1.7 with respect to an Eichler order and let ω *be a finite automorphic character of* $\mathbb{Q}^\times\backslash\mathbb{A}^\times$. *Then for* $\varphi_1 \in \mathcal{A}(\omega^+)$ *and* $\varphi_2 \in \mathcal{A}(\omega^{-1})$, $\theta(g, \varphi_1 \otimes \varphi_2, \Phi)$ *is holomorphic of weight 2, level* $U_1(\mathrm{disc}(\mathcal{O}))$*, and central character* ω. *Moreover, its Kirillov function is given by*

$$\kappa(x, \varphi_1 \otimes \varphi_2, \Phi_{\mathcal{O}}) = \omega(x)|x|\langle \varphi_1, T_{x^\infty,\Phi_{\mathcal{O}}^\infty}(\varphi_2)\rangle.$$

In particular for $\xi_1 \in \Xi^+(\omega)$ *and* $\xi_2 \in \Xi^-(\omega)$*, we have*

$$\kappa(x, \xi_1 \otimes \xi_2, \Phi_{\mathcal{O}}) = \omega(x)|x|\langle [\xi_1], T_{x^\infty,\Phi_{\mathcal{O}}^\infty}([\xi_2])\rangle.$$

1.7 A theta identity for the automorphic avatar of optimal forms

We again assume that K is imaginary and that B is definite. Then we get two characters $\xi_1 = \mathbb{1}$ and $\xi_2 = \xi := \chi^{1-c}$. In this case, ξ is a ring class character. Let $c(\xi) \in \mathbb{N}$ be the conductor of ξ, i.e. the minimal positive integer such that ξ is trivial over $(1 + c(\xi)\widehat{\mathcal{O}}_K)^\times$. Let $\mathcal{O}_{c(\xi)} = \mathbb{Z} + c(\xi)\mathcal{O}_K$ be the corresponding order. Then the discriminant $d(\xi)$ of $\mathcal{O}_{c(\xi)}$ is $c(\xi)^2\,\mathrm{disc}(K)$. Assume that $c(\xi)$ is coprime to $\mathrm{disc}(B)$.

Definition 1.10. An Eichler order $\mathcal{O}$ of B is ξ-optimal if the following conditions hold:

1. $\mathcal{O}_{c(\xi)} = K \cap \mathcal{O}$;

2. for each $q \nmid d(\xi)$, $\mathcal{O}_q = \mathcal{O}_{K,q} + \mathcal{O}_{K,q} j_q$ where $j_q \in B_q^\times$ such that $j_q x = \overline{x} j_q$ and $j_q^2 = \mathrm{disc}(\mathcal{O}_q)$.

We have the following description of optimal forms in terms of theta series on quaternion algebras.

Proposition 1.11. *Let* $\mathcal{O}$ *be a* ξ*-optimal Eichler order of* B *with discriminant* M *coprime to the discriminant of* K. *Then*

$$\theta(g, 1 \otimes \xi, \Phi_{\mathcal{O}}) = M^{-\frac{1}{2}}\varphi^{\mathrm{opt}}\big(g, ga(M^\infty)^{-1}\big).$$

Moreover, these are holomorphic with weight 2 and have their Kirillov function given by

$$\kappa(x, 1 \otimes \xi, \Phi_{\mathcal{O}}) = \omega(x)|x|\left\langle \mathbb{1}, T_{x^\infty, \Phi_{\hat{\mathcal{O}}}}(\xi)\right\rangle.$$

Proof. First, we decompose $\Phi_{\mathcal{O}}$ into a tensor product of functions in $\mathcal{S}(V_i)$. We need only do this locally.

If $q = \infty$, then there is a decomposition, $\Phi_\infty = \Phi_{1,\infty} \otimes \Phi_{2,\infty}$ with both

$$\Phi_{i,\infty}(x, y) = e^{-\pi(x^2+y^2)}.$$

If q is finite and does not divide $d(\xi)$, then we also have a decomposition $\Phi_q = \Phi_{1,q} \otimes \Phi_{2,q}$, with $\Phi_{1,q}$ the characteristic function of $\mathcal{O}_{K,q}$ and with $\Phi_{2,q}$ the characteristic function of $\mathcal{O}_{K,q} j_q$.

If q is finite and divides $d(\xi)$, then Φ_q is the characteristic function of the optimal lattice $\mathrm{End}(\mathcal{O}_{c(\xi)_q})$. Then $\bigotimes_{q|d(\xi)} \Phi_q$ is a sum ,

$$\sum_{\alpha \in \mathcal{O}_c/\delta\mathcal{O}_c} \Phi_{1,\alpha,d(\xi)} \otimes \Phi_{2,\alpha,d(\xi)},$$

where δ is a generator of the different ideal of $\mathcal{O}_c$ as before (e.g. if we write $\mathcal{O}_c = \mathbb{Z} + \mathbb{Z}t$ with $t \in \mathcal{O}_K$, then we can take $\delta = t - \bar{t}$).

Combining all of the above, we obtain that,

$$\Phi = \sum_{\alpha \in \mathcal{O}_c/\delta\mathcal{O}_c} \Phi_{1,\alpha} \otimes \Phi_{2,a},$$

such that $\Phi_{1,\alpha}$ is the same as $\Phi_\alpha^{\mathrm{opt}}$ (from Definition 1.3) and $\Phi_{2,\alpha}$ is the same as $\Phi_\alpha^{\mathrm{opt}}$ except at places not dividing $d(\xi)$. We then have

$$\theta(g, 1 \otimes \xi, \Phi) = \sum_{\alpha \in \mathcal{O}_c/\delta} \theta(g, \chi, \Phi_{1\alpha})\theta\left(g, \chi^{-1}, \Phi_{2\alpha}\right).$$

More precisely, we consider $V_1 := (K, N)$ and $V_2 = (Kj, -j^2 N)$. We define an isomorphism

$$\iota : V_{1,\mathbb{A}} \longrightarrow V_{2,\mathbb{A}}$$

as follows.

1. If $q = \infty$, then ι_∞ is the identity map. In particular, ι is an isometry and $\iota_\infty^*(\Phi_{2,\infty}) = \Phi_{1,\infty}$.
2. If $q \nmid d(\xi)$, then $\iota_q(x) = xj_q$ (with j_q as in Definition 1.10). Then $Q(\iota_q) = -j_q^2$ and $\iota_q^*(\Phi_{2,q}) = \Phi_{1,q}$.
3. If $q \mid d(\xi)$, then $\iota_q(x) = xj_q$ with $j_q^2 = 1$. Then $Q(\iota_q) = -j_q^2 = -1$ and $\iota_q^* \Phi_{2,\alpha,q} = \Phi_{1,-\alpha,q}$.

This shows that with the adéle $(-1)^\infty = (1, -1, -1, \dots)$,

$$Q(\iota) = (-1)^\infty \prod_{q \nmid d(\xi)} j_q^2,$$

and $\iota^* \Phi_{2,a} = \Phi_{-a}^{\mathrm{opt}}$. Then the isomorphism ι has norm $M = \mathrm{disc}(\mathcal{O})$ and so by Proposition 1.5, we have

$$\theta\big(g, \chi^{-1}, \Phi_{2,\alpha}\big) = |M|^{\frac{1}{2}} \theta\big(ga(M)^{-1} \epsilon^\infty, \chi^{-1}, \Phi_{-\alpha}^{\mathrm{opt}}\big).$$

In comparison with Equation 1.7, we get

$$\theta(g, 1 \otimes \xi, \Phi) = |M|^{\frac{1}{2}} \varphi^{\mathrm{opt}}\big(g, ga(M)^{-1}\big).$$

$\square$

2 Modular forms

2.1 Modular forms in finite level

We start with some background on the theory of modular forms in finite level, much of which can be found in the classical text of Shimura [Shi94, Chapter 6], Deligne–Rapoport [DR73, Chapters IV, VII], and Katz–Mazur [KM85, Chapters 3-4] (also cf. [DI95, Sections 7-9] and [Kat73, Section 1]).

For each positive integer n, we have a modular curve $Y(n)$ over $\mathbb{Z}[1/n]$ parametrizing isomorphism pairs $(E, \phi : (\mathbb{Z}/n\mathbb{Z})^2 \longrightarrow E[n])$. This curve is not geometrically connected. Over the complex numbers, we have a uniformization,

$$Y(n)(\mathbb{C}) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H} \times \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}),$$

so that a pair $(z, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))$ in the right-hand side gives a pair (E, ϕ) in the left-hand side,

$$\begin{aligned} E &= \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), \\ \phi(m_1, m_2) &= \left(\frac{am_1\tau + bm_2}{n}, \frac{cm_1\tau + dm_2}{n}\right). \end{aligned}$$

The set of connected components is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$, and the decomposition is given by

$$Y(n)(\mathbb{C}) = \Gamma(n)\backslash\mathcal{H} \times \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, a \in (\mathbb{Z}/n\mathbb{Z})^\times \right\}.$$

In fact, each of these connected components is defined over $\mathbb{Z}[1/n, \zeta_n]$, where ζ_n is a primitive n-th root of unity, and the corresponding component for each $a \in (\mathbb{Z}/n\mathbb{Z})^\times$ parametrizes pairs (E, ϕ) such that the Weil pairing $\langle \phi(1,0), \phi(0,1) \rangle$ is equal to ζ_n^a.

When $n \geq 3$, we have a universal elliptic curve $\mathcal{E}(n)$ on $Y(n)$ which can be constructed as follows,

$$\mathcal{E}(n) := \big(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})\big)\backslash\mathcal{H} \times \mathbb{C} \times \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}).$$

Here $(m_1, m_2, \gamma) \in \mathbb{Z}^2 \rtimes \mathrm{GL}_2$ acts on the right-hand side via,

$$(z, u, g) \longmapsto \big(\gamma z, j(\gamma, z)^{-1}(u + mz + n), \gamma g\big),$$

where for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})$,

$$j(\gamma, z) := |\det \gamma|^{-\frac{1}{2}}(cz + d).$$

Let ω denote the sheaf of relative invariant differentials on $\mathcal{E}(n)$. Then for any integer k, we have the space $H^0(Y(n), \omega^k)$ of *weakly holomorphic modular forms of weight* k.

Over the complex numbers, every such form can be written as a function $f(z, g)du^k$ on $\mathcal{H} \times \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ that is holomorphic in z and invariant under the action by every $\gamma \in \mathrm{GL}_2(\mathbb{Z})$:

$$f(\gamma z, \gamma g)d(\gamma u)^k = f(z, g)du^k.$$

Notice that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,

$$d(\gamma u) = j(\gamma, z)^{-1}du,$$

so then for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$f(\gamma z, \gamma g) = f(z, g)j(\gamma, z)^k.$$

The modular curve $Y(n)$ (resp. universal family $\mathcal{E}(n)$) can be extended into a projective curve $X(n)$ (resp. a generalized elliptic curve over $X(n)$) by adding cusps $C(n)$ (resp. $\mathbb{G}_m$). We can extend the sheaf ω to $X(n)$, and call $H^0(X(n), \omega^k)$ (resp. $H^0(X(n), \omega^k(-C(n)))$ the space of modular forms. (resp. cusp forms).

Over the complex numbers, we have

$$X(n)(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Z})_+ \backslash \widehat{\mathcal{H}} \times \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}),$$

where $\widehat{\mathcal{H}}$ is the extended upper half-plane $\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. The cuspidal divisor is described as

$$C(n) = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{P}^1(\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}).$$

At each cusp c, there is a well-defined holomorphic coordinate q_γ on $X(N)$. If c is represented by

$(a, g) \in \mathbb{P}^1(\mathbb{Q}) \times GL_2(\mathbb{Z}/n\mathbb{Z})$ with stabilizer $N_{a,g}$ in $SL_2(\mathbb{Z})$, then q_c is represented by a generator of holomorphic functions invariant under $N_{a,g}$. For example if c is represented by $\infty \times e \in \mathbb{P}^1 \times GL_2(\mathbb{Z}/n\mathbb{Z})$, which has stabilizer $N(n\mathbb{Z})$, then we take $q_c = e^{\frac{2\pi i z}{n}}$. A modular form $f(z)du^k$, or rather $f(z)$, is then a weakly modular form with Taylor expansion at each cusp,

$$f(z) = \sum_{i \geq 0} a_{c,i} q_c^i.$$

Such a form is a cusp form if it vanishes at cusps, i.e. $a_{c,0} = 0$.

For a $\mathbb{Z}[1/n]$-algebra R, we respectively define the space of modular forms and the space of cusp forms,

$$M_k\big(U(n), R\big) = H^0\big(X(n)_R, \omega^k\big),$$
$$S_k\big(U(n), R\big) = H^0\big(X(n)_R, \omega^k\big(- C(n)\big)\big).$$

These modular forms have Taylor expansions at each cusp with $R[\zeta_n]$-coefficients. Now we have the following q-expansion principle (cf. [Kat73, Section 1.6]).

Proposition 2.1 (q-expansion principle)**.** *Let f be a modular form and c a cusp of $X(n)$. Assume that the q-series of f at c vanishes, then f vanishes on the connected component of $X(n)$ containing c.*

In practice, we only consider the cusp $c_u = (i\infty, a(u))$ for $u \in (\mathbb{Z}/n\mathbb{Z})^\times$ and write $a_f(\frac{i}{n}, u)$ for a_{i,c_u}. Then we have a q-expansion at c_u by

$$f(z) = \sum_{i \geq 0} a_f\left(\frac{i}{n}, u\right) q^{\frac{i}{n}}. \tag{2.1}$$

The advantage of this expression is the invariance under pull-back by $X(n') \longrightarrow X(n)$ for any multiple n' of n.

2.2 Modular forms in infinite level

Consider the projective limit of modular curves and cusp forms,

$$Y := \varprojlim_{n} Y(n)_{\mathbb{Q}},$$

$$X := \varprojlim_{n} X(n)_{\mathbb{Q}},$$

$$S_k := \varinjlim_{n} S_k\big(U(n), \mathbb{Q}\big).$$

Over the complex numbers, we have

$$Y(\mathbb{C}) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H} \times \mathrm{GL}_2(\widehat{\mathbb{Z}}),$$

$$X(\mathbb{C}) = \mathrm{SL}_2(\mathbb{Z})\backslash\widehat{\mathcal{H}} \times \mathrm{GL}_2(\widehat{\mathbb{Z}}).$$

The set of connected components is given by

$$\mathrm{GL}_2(\mathbb{Q})_+\backslash\mathrm{GL}_2(\widehat{\mathbb{Q}}) \xrightarrow{\ \sim\ } \mathbb{Q}_+^\times\backslash\widehat{\mathbb{Q}}^\times \simeq \widehat{\mathbb{Z}}^\times. \tag{2.2}$$

Using the identity,

$$\mathrm{GL}_2(\widehat{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{Q})_+\mathrm{GL}_2(\widehat{\mathbb{Z}}),$$

we can write the above as

$$Y(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{H}^\pm \times \mathrm{GL}_2(\widehat{\mathbb{Q}}),$$

$$X(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})\backslash\widehat{\mathcal{H}}^\pm \times \mathrm{GL}_2(\widehat{\mathbb{Q}}).$$

In this terminology, a cusp form $f \in S_k(\mathbb{C})$ is a function f on $\mathcal{H}^\pm \times \mathrm{GL}_2(\widehat{\mathbb{Q}})$ such that,

1. f is invariant under right translation of some open subgroup U of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$;

2. f is holomorphic in z;

3. for any $\gamma \in \mathrm{GL}_2(\mathbb{Q})$,

$$f(\gamma z, \gamma g) = j(\gamma, z)^k f(z, g);$$

4. f vanishes at each cusp.

q-expansions

Notice that the set of cusps in level U is given by

$$C_U(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathbb{P}^1(\mathbb{Q}) \times \mathrm{GL}_2(\widehat{\mathbb{Q}})/U \xrightarrow{\sim} \{i\infty\} \times B(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\widehat{\mathbb{Q}})/U.$$

Notice that $N(\mathbb{Q})$ is dense in $N(\widehat{\mathbb{Q}})$. So taking a limit gives,

$$C(\mathbb{C}) \xrightarrow{\sim} \{i\infty\} \times N(\widehat{\mathbb{Q}})M(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\widehat{\mathbb{Q}}),$$

where $M(\mathbb{Q})_+$ denotes the group of diagonal matrices in $\mathbb{Q}$ with positive determinant. Then for the last condition, we need only consider the cusp represented by $i\infty \times \mathrm{GL}_2(\widehat{\mathbb{Q}})/U$. Assume that c is represented by $(i\infty, gU)$, then we have the stabilizer,

$$N_c := N(\mathbb{Q}) \cap gUg^{-1} = N(m\mathbb{Z})$$

for some $m \in \mathbb{N}$. Then we have the q-expansion

$$f(z, g) = \sum_{n \geq 1} A_f\left(\frac{n}{m}, g\right) q^{\frac{n}{m}}.$$

So we defined a function $A_f : \mathbb{Q}_+^\times \times \mathrm{GL}_2(\widehat{\mathbb{Q}}) \longrightarrow \mathbb{C}$., which does not depend on the choice of U.

Relation to automorphic forms

Now we describe how to consider modular forms as automorphic forms (cf. [Cas73, Section 3], [Bum97, Section 3.6]). For a modular form f of weight k, we first define a function on $\mathrm{GL}_2(\mathbb{R})_+ \times$

$\mathrm{GL}_2(\widehat{\mathbb{Q}})$,

$$\varphi(g) := f(g_\infty(i), g^\infty) j(g_\infty, i)^{-k}, \tag{2.3}$$

where $g = (g_\infty, g^\infty)$. For $\gamma \in \mathrm{GL}_2(\mathbb{Q})_+$, we then have,

$$\begin{aligned}\varphi(\gamma g) &= f(\gamma g_\infty(i), \gamma g^\infty) j(\gamma g_\infty, i)^{-k} \\ &= f(\gamma g_\infty(i), \gamma g^\infty) j(\gamma, g_\infty(i))^{-k} j(g_\infty, i)^{-k} \\ &= f(g_\infty(i), g^\infty) j(g_\infty, i)^{-k} \\ &= \varphi(g).\end{aligned}$$

From the construction, it is clear that φ is invariant under $\mathbb{R}_+$ and has weight k under SO_2, as

$$j\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, i\right) = -(\sin\theta)i + \cos\theta = e^{-i\theta}.$$

So φ is invariant under $\mathrm{GL}_2(\mathbb{Q})_+$ on both factors. We then uniquely extend this function to a function on $\mathrm{GL}_2(\mathbb{A})$ that is invariant under left translation by defining $\varphi(g) := \varphi(hg)$ for $h \in \mathrm{GL}_2(\mathbb{Q})_-$ and $g \in \mathrm{GL}_2(\mathbb{A}) - (\mathrm{GL}_2(\mathbb{R})_+ \times \mathrm{GL}_2(\widehat{\mathbb{Q}}))$. With $z = x + yi$, we also can recover f from φ by,

$$f(z, g^\infty) = y^{-k/2} \varphi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, g^\infty\right). \tag{2.4}$$

Now we compare the Fourier expansion of both sides to get,

$$\sum_{r \in \mathbb{Q}^\times} A_f(r, g^\infty) e^{2\pi i r z} = y^{-k/2} \sum_{r \in \mathbb{Q}^\times} W_\varphi\left(\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, g^\infty\right),$$

where $W(g)$ is the Whittaker function for φ,

$$W_\varphi(g) := \int_{[N]} \varphi(ng)\psi^{-1}(n)dn.$$

Observe that

$$\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, g^\infty \right) = \left(\begin{pmatrix} ry & rx \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} r^\infty & 0 \\ 0 & 1 \end{pmatrix} g^\infty \right).$$

Therefore,

$$\begin{aligned} \sum_{r \in \mathbb{Q}_+^\times} A_f(r, g^\infty) e^{2\pi i r z} &= y^{-k/2} \sum_{r \in \mathbb{Q}^\times} W_\varphi \left(\begin{pmatrix} ry & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} r^\infty & rx \\ 0 & 1 \end{pmatrix} g^\infty \right). \\ &= y^{-k/2} \sum_{r \in \mathbb{Q}^\times} e^{2\pi i r x} W_\varphi \left(\begin{pmatrix} ry & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} r^\infty & 0 \\ 0 & 1 \end{pmatrix} g^\infty \right). \end{aligned}$$

Comparing the coefficients of $e^{2\pi i r x}$, we obtain:

$$W_\varphi \left(\begin{pmatrix} ry & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r^\infty & 0 \\ 0 & 1 \end{pmatrix} g^\infty \right) = \begin{cases} A_f(r, g^\infty) y^{k/2} e^{-2\pi r y} & \text{if } r > 0 \\ 0 & \text{if } r < 0 \end{cases}$$

For any positive integer k, we define a Whittaker function W_k on $GL_2(\mathbb{R})$ that is supported on $GL_2(\mathbb{R})_+$ with weight k and is invariant under $\mathbb{R}^\times$ such that

$$W_k \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} := y^{k/2} e^{2\pi i z}. \tag{2.5}$$

We have shown the following.

Proposition 2.2. *The Whittaker function W_φ of φ has the form:*

$$W_\varphi(g) = W_k(g_\infty) W_\varphi(g^\infty)$$

Moreover, the Equations 2.3 and 2.4 give a one-to-one correspondence between holomorphic forms of weight k and automorphic forms of weight k with the following compatibility of Fourier coeffi-

cients for any $r \in \mathbb{Q}_+^\times$*:*

$$A_f(r, g^\infty) = r^{k/2} W_\varphi\left(\begin{pmatrix} r^\infty & 0 \\ 0 & 1 \end{pmatrix} g^\infty\right),$$

In the correspondence of Proposition 2.2, we call f the *modular avatar* of φ, and φ the *automorphic avatar* of f.

2.3 Galois action and q-expansion principle

Let $S_k(\mathbb{C}) = S_k \otimes_{\mathbb{Q}} \mathbb{C}$ denote the space of weight-k cusp forms defined over $\mathbb{C}$ and let $\mathcal{A}_{0,k}([\mathrm{GL}_2])$ denote the space of cuspidal automorphic forms φ of weight k. Both have an action by $\mathrm{GL}_2(\widehat{\mathbb{Q}})$. Equations 2.3 and 2.4 define an isomorphism between these representations of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$:

$$S_k(\mathbb{C}) \xrightarrow{\sim} \mathcal{A}_{0,k}([\mathrm{GL}_2]).$$

Galois action

It is clear that $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ acts on $S_k(\mathbb{C})$; we can recover S_k as its invariants. This induces an action on the $\mathcal{A}_{0,k}([\mathrm{GL}_2])$. In the following, we want to write down the corresponding formula for the Galois action on $\mathcal{A}_{0,k}([\mathrm{GL}_2])$. First, we need to describe the Galois action on cusp forms in terms of q-expansions.

Notice that the set of cusps is defined over $\mathbb{Q}^{ab}$, the maximal abelian extension of $\mathbb{Q}$. The action of $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ on this set is given by the left action by an element $a_\sigma := \left(\begin{smallmatrix} \lambda_\sigma & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $\lambda : \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \xrightarrow{\sim} \widehat{\mathbb{Z}}^\times$ is the isomorphism induced from action on roots of unity and $\sigma \in \mathrm{Aut}(\mathbb{C})$. From the action of $\mathrm{Aut}(\mathbb{C})$ on the space of modular forms (via its action on cusps and coefficients), $f^\sigma(z, g)$ has Fourier coefficients given by

$$A_{f^\sigma}(r, g) = A_f\big(r, a_\sigma^{-1} g\big)^\sigma.$$

By Proposition 2.2,

$$W_\varphi(g^\infty) = A_f(1, g^\infty).$$

Thus an action of $\mathrm{Aut}(\mathbb{C})$ can be defined on the space $\mathcal{W}(\psi^\infty)$ of the Whittaker function on $\mathrm{GL}_2(\mathbb{A}_f)$ by

$$\begin{aligned} W &\longrightarrow W^\sigma \\ g^\infty &\longmapsto W\big(a_\sigma^{-1} g^\infty\big)^\sigma. \end{aligned} \tag{2.6}$$

q-expansion principle

By the q-expansion principle in finite level of Proposition 2.1, we have a q-expansion principle in infinite level as well: a modular form f vanishes on X if and only if its q-expansion vanishes on at least one cusp for each connected component of X.

By Equation 2.2, the set of connected components of X is given by X_u for $u \in \widehat{\mathbb{Z}}^\times$, the connected component of X containing the image of $(z, a(u)) \in \widehat{\mathcal{H}} \times \mathrm{GL}_2(\mathbb{A}_f)$. We can define the standard cusp on X_u by $c_u = (i\infty, a(u))$. Then by the q-expansion principle, f is determined by its q-expansion at the cusp c_u. More precisely (note the abuse of notation), we denote for each $u \in \widehat{\mathbb{Z}}^\times$,

$$\begin{aligned} f(z,u) &= f\big(z, a(u)\big), \\ a_f(r,u) &:= A_f\big(r, a(u)\big). \end{aligned}$$

Thus the q-expansion of f at $c(u)$ is given by

$$f(q,u) = \sum_{r \in \mathbb{Q}_+^\times} a_f(r,u) q^r.$$

The Galois action of $\mathrm{Aut}(\mathbb{C})$ on q-expansions can then be written for each $u \in \widehat{\mathbb{Z}}^\times$ as,

$$f^\sigma(q,u) = \sum_{r \in \mathbb{Q}_+^\times} a_f\big(r, \lambda_\sigma^{-1} u\big)^\sigma q^r. \tag{2.7}$$

Recall that on the automorphic side, we have the Kirillov model for a cuspidal automorphic form $\varphi \in \mathcal{A}_{0,k}([\mathrm{GL}_2])$:

$$\kappa_\varphi(x) = W_\varphi\big(a(x)\big).$$

By Proposition 2.2, we have the following relation:

$$a_f(r,u) = r^{\frac{1}{2}} \kappa_\varphi(ru). \tag{2.8}$$

Due to the decomposition $\mathbb{A}^{\infty,\times} = \mathbb{Q}_+^\times \times \widehat{\mathbb{Z}}^\times$, Equation 2.8 allows one to recover a_f and κ_φ from each other.

2.4 Newforms

Decompositions

We study the decomposition of S_k into the direct sum of irreducible representations of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$. We may do this by first working on $\mathbb{C}$ and then studying the action by $\mathrm{Aut}(\mathbb{C})$ later. Over the complex numbers, via Equations 2.4 and 2.3, there is an isomorphism:

$$S_k(\mathbb{C}) \xrightarrow{\sim} \mathcal{A}_{0,k}([\mathrm{GL}_2]). \tag{2.9}$$

We know that the right hand is a subspace $\mathcal{A}_0([\mathrm{GL}_2])$ of cusp forms which can be decomposed into irreducible representations:

$$\mathcal{A}_0([\mathrm{GL}_2]) = \bigoplus_\pi \pi,$$

where π range over all irreducible cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$. Notice that by their Whittaker functions, each π has a further decomposition into irreducible representation $\mathrm{GL}_2(\mathbb{Q}_p)$:

$$\pi = \bigotimes_{p \le \infty} \pi_p.$$

We define $W_k \subset \mathcal{W}(\psi_\infty)$ to be the representation of $\mathrm{GL}_2(\mathbb{R})$ generated by weight-k holomorphic Whittaker function W_k (defined in Equation 2.5). Then we have

$$\mathcal{A}_{0,k}([\mathrm{GL}_2]) = \bigoplus_{\pi_\infty = \pi_k} W_k \otimes \mathcal{W}(\pi^\infty, \psi^\infty) \xrightarrow{\sim} \bigoplus_{\pi_\infty = \pi_k} \mathcal{W}(\pi^\infty, \psi^\infty).$$

Combining this with the isomorphism from Equation 2.9, we obtain a decomposition of $S_k(\mathbb{C})$ into the direct sum of irreducible representations:

$$S_k(\mathbb{C}) \xrightarrow{\sim} \bigoplus_{\pi_\infty = \pi_k} \mathcal{W}(\pi^\infty, \psi^\infty). \tag{2.10}$$

Definition of newforms

For each irreducible representation $\pi = \mathcal{W}(\pi, \psi)$ on the right-hand side of Equation 2.10, there is a notion of level N and newform φ^{new}. For each positive integer N, define

$$U_1(N) := \left\{ u \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \,\middle|\, u \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Then there is a minimal N called the level of π such that $\pi^{U_1(N)} \neq 0$. For such N, $\dim \pi^{U_1(N)} = 1$. In the Whittaker model, we can normalize a form $W^{\text{new}} \in \pi^{U_1(N)}$ such that $W^{\text{new}}(e) = 1$. Thus we get a newform in $\pi \subset \mathcal{A}_{0,k}([\mathrm{GL}_2])$ by

$$\varphi^{\text{new}}(g) := \sum_{a \in \mathbb{Q}^\times} W^{\text{new}}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right).$$

Let ω be the central character of π. Then φ^{new} has character ω under the action by the larger group:

$$U_0(N) := \left\{ u \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \;\middle|\; u \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\} = \widehat{\mathbb{Z}}^\times \cdot U_1(N).$$

The isomorphism in Equation 2.10 gives a corresponding weight-k cusp form $f^{\text{new}} \in S_k(\mathbb{C})$. We show that f^{new} is the classical newform for $\Gamma_1(N)$ with nebentypus ω^∞, so we may equivalently consider the corresponding π and φ^{new} to be "newforms" (cf. [Cas73, Section 3]).

Proposition 2.3. *There are natural one-to-one correspondences between the following objects:*

1. *irreducible subrepresentations* π *of* $\mathcal{A}_{0,k}([\mathrm{GL}_2])$ *under* $\mathrm{GL}_2(\mathbb{A})$*;*

2. *newforms* φ^{new} *in* $\mathcal{A}_{0,k}([\mathrm{GL}_2])$*;*

3. *newforms* f^{new} *in* $S_k(\mathbb{C})$*;*

4. *irreducible subrepresentations* π^∞ *of* $S_k(\mathbb{C})$ *under* $\mathrm{GL}_2(\widehat{\mathbb{Q}})$.

Sketch of proof. First, since f^{new} is invariant under $U_1(N)$, we see that f^{new} is a modular form on the modular curve

$$X_{U_1(N),\mathbb{C}} := \mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathcal{H} \times \mathrm{GL}_2(\widehat{\mathbb{Q}})/U_1(N).$$

Use the decomposition $\mathrm{GL}_2(\widehat{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{Q})_+ \cdot U_1(N)$ to see that this modular curve is actually the classical modular curve

$$X_1(N) = \Gamma_1(N)\backslash\mathcal{H},$$

where

$$\Gamma_1(N) := \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod N \right\} = U_1(N) \cap \mathrm{GL}_2(\mathbb{Q})_+.$$

Therefore, f^{new} is a classical modular form for $\Gamma_1(N)$.

Second, since φ^{new} has character ω under $U_0(N)$, f^{new} has character ω under

$$\Gamma_0(N) := \left\{ \gamma \in SL_2(\mathbb{Z}) \,\middle|\, \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\} = U_0(N) \cap GL_2(\mathbb{Q})_+,$$

as follows:

$$\omega\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = \omega^\infty(d).$$

Therefore f^{new} has nebentypus ω^∞, i.e.

$$f^{\text{new}} \in S_k(\Gamma_1(N), \omega^\infty).$$

Third, φ^{new} is in the one-dimensional space $\pi^{U_1(N)}$, which is an eigenspace for the Hecke algebra

$$\mathbb{T}_1(N) := \mathbb{C}\left[U_1(N)\backslash GL_2(\widehat{\mathbb{Q}})/U_1(N)\right].$$

Thus f^{new} is an eigenform under the Hecke algebra

$$\mathbb{C}[\Gamma_1(N)\backslash GL_2(\mathbb{Q})/\Gamma_1(N)] \xrightarrow{\sim} \mathbb{C}\left[U_1(N)\backslash GL_2(\widehat{\mathbb{Q}})/U_1(N)\right] = \mathbb{T}_1(N).$$

This shows that f^{new} is an eigenform in $S_k(\Gamma_1(N), \omega^\infty)$.

Fourth, since $U_1(N)$ is the minimal level of φ^{new}, N is the minimal level f^{new}. This shows that f^{new} is a newform with level N.

Finally, since $\left(\begin{smallmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{smallmatrix}\right) \subset \Gamma_1(N)$, $f^{\text{new}}(z)$ has a q-expansion:

$$f^{\text{new}} = \sum_{n \geq 1} a_n q^n.$$

By Equation 2.8,

$$a_n = \sqrt{n} \cdot \kappa(n^\infty).$$

This shows that $a_1 = 1$ (recall that $W^{\text{new}}(e) = 1$). Combined with the previous steps, f^{new} is a normalized newform in $S_k(\Gamma_1(N), \omega^\infty)$. Furthermore, f^{new} generates the irreducible subrepresentation $\pi(f^{\text{new}})$ of $S_k(\mathbb{C})$ of $GL_2(\widehat{\mathbb{Q}})$ corresponding to π^∞ generated by φ^{new} (via the isomorphism in Equation 2.10).

Conversely, starting with a new form f^{new}, we can reverse the above procedure to construct a newform $\varphi^{\text{new}} \in \mathcal{A}_{0,k}([GL_2])$ in the sense that φ^{new} is an eigenform under $\mathbb{T}_1(N)$, with minimal level $U_1(N)$ and normalized so that $\kappa(1) = 1$. It is well-known that such an automorphic form generates an irreducible subrepresentation $\pi(\varphi^{\text{new}})$ of $\mathcal{A}_{0,k}[GL_2]$. $\square$

Rationality and integrality

The correspondences in Proposition 2.3 are $\mathrm{Aut}(\mathbb{C})$-equivariant, with the action on the automorphic side given by Equation 2.6. Then the objects in Proposition 2.3 also have the same field of definition. Such a field is largely easy to describe in terms of newforms (as modular forms): if f is a newform with q-expansion $\sum_n a_n q^n$, then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, the form f^σ will have q-expansion $\sum_n a_n^\sigma q^n$. In fact, if the q-expansion of f^σ is $\sum_n b_n q^n$, then

$$a_n = a_f(n, 1),$$
$$b_n = a_{f^\sigma}(n, 1).$$

By Equation 2.7,

$$b_n = a_{f^\sigma}(n, 1) = a_f\left(n, \lambda_\sigma^{-1}\right)^\sigma = a_f(n, 1)^\sigma = a_n^\sigma$$

where in the last step, we use the fact that f on $\mathcal{H} \times GL_2(\widehat{\mathbb{Q}})$ is invariant under $U_1(N)$. Therefore the field of definition of f is the subfield $\mathbb{Q}(f)$ of $\mathbb{C}/\mathbb{Q}$ generated by $\{a_n\}$.

Another advantage of using f for this description is that the coefficients a_n are always algebraic integers. They all come from Galois representations.

Theorem 2.4 (Eichler–Shimura [Eic57, Shi94] for $k = 2$, Deligne [Del71] for $k > 2$, Deligne–Serre [DS74] for $k = 1$)**.** *Let* f *be a newform with* q*-expansion* $\sum_n a_n q^n$*. Then there is a system of* ℓ*-adic Galois* *representations*

$$\rho_\ell : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \longrightarrow \mathrm{GL}_2(\mathbb{Q}_\ell),$$

such that for all $p \nmid \ell N$*,*

$$a_p = \mathrm{Tr}(\rho_\ell(\mathrm{Frob}_p)).$$

Then we can define the ring $\mathcal{O}(f)$ as the ring of integers of $\mathbb{Q}(f)$. Then all of the objects in Proposition 2.3 have an integral model defined over $\mathcal{O}(f)$.

2.5 The Harris–Venkatesh period

In the following, we extend the work of Harris–Venkatesh [HV19] and Darmon–Harris–Rotger–Venkatesh [DHRV22] about the Shimura class to the adelic setting.

Modular curves

Recall that for any positive integer N, there is a modular curve $X(N)$ defined over $\mathbb{Q}$. This curve is not geometrically defined over $\mathbb{Q}$. In fact, $\mathcal{O}(X(N))$ is the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta_N)$. This curve has a smooth model over $\mathbb{Z}[1/N]$, which we still denote by $X(N)$. If $N \geq 3$, then there is a bundle ω of weight-one forms on $X(N)$ and a bundle $\Omega = \Omega_{X(N)/\mathbb{Z}[1/N]}$ of relative differentials with the Kodaira–Spencer map,

$$\mathrm{KS} : \omega \otimes \omega \xrightarrow{\sim} \Omega(C(N)),$$

where $C(N)$ is the cuspidal divisor on $X(N)$. The Kodaira–Spencer map induces a pairing

$$H^0(X(N), \omega(-C(N))) \otimes_{\mathbb{Z}[1/N]} H^0(X(N), \omega(-C(N))) \longrightarrow H^0(X(N), \Omega(-C(N))). \tag{2.11}$$

This is the product map from the space of cuspidal one-forms to differential one-forms on $X(N)$. Serre duality defines another pairing,

$$H^0\big(X(N), \Omega_{X(N)}\big) \otimes_{\mathbb{Z}[1/N]} H^1\big(X(N), \mathcal{O}_{X(N)}\big) \longrightarrow H^1(X(N), \Omega) \xrightarrow{\sim} \mathbb{Z}[1/N, \zeta_N], \qquad (2.12)$$

where the last isomorphism is the trace map in Serre duality. Both of these pairings are compatible with pull-back maps and the action by $GL_2(\mathbb{Z}/N\mathbb{Z})$.

Now fix a finite set Σ of primes and consider the projective system of smooth curves over $\mathbb{Z}[1/\Sigma]$ indexed by positive integers N, with prime factors in Σ:

$$X_\Sigma(N) := X(N) \otimes_{\mathbb{Z}[1/N]} \mathbb{Z}[1/\Sigma].$$

Let X_Σ denote the limit of this projective system. Then X_Σ has an action by $GL_2(\mathbb{Q}_\Sigma)$, where $\mathbb{Q}_\Sigma$ is the product of $\mathbb{Q}_p$ with $p \in \Sigma$. Taking limits, we obtain the following two pairings.

$$H^0(X_\Sigma, \omega(-C_\Sigma)) \otimes_{\mathbb{Z}[1/\Sigma]} H^0(X_\Sigma, \omega(-C_\Sigma)) \longrightarrow H^0(X_\Sigma, \Omega(-C_\Sigma)). \qquad (2.13)$$

$$H^0(X_\Sigma, \Omega_{X_\Sigma}) \otimes_{\mathbb{Z}[1/\Sigma]} H^1(X_\Sigma, \mathcal{O}_{X_\Sigma}) \longrightarrow H^1(X_\Sigma, \Omega) \xrightarrow{\sim} \mathbb{Z}[1/\Sigma, \mu_\Sigma], \qquad (2.14)$$

where μ_Σ is the group of roots of unity whose order is divisible only by primes in Σ and where the last isomorphism is deduced from the inductive system of modified trace maps,

$$\begin{aligned} H^1(X(N, \Omega)) &\xrightarrow{\sim} \mathbb{Z}[1/N, \zeta_N] \\ x &\mapsto \frac{\left|SL_2\Big(\mathbb{Z}/\Big(\prod_{q\in\Sigma} q\Big)\mathbb{Z}\Big)\right|}{|SL_2(\mathbb{Z}/N\mathbb{Z})|} \cdot Tr(x). \end{aligned}$$

These pairings are compatible with the action by $GL_2(\mathbb{Q}_\Sigma)$. Notice that the action of $GL_2(\mathbb{Q}_\Sigma)$ on $\mathbb{Z}[1/\Sigma, \mu_\Sigma]$ is given by

$$GL_2(\mathbb{Q}_\Sigma) \xrightarrow{\det} \mathbb{Q}_\Sigma^\times \longrightarrow \mathbb{Z}_\Sigma^\times = Aut(\mu_\Sigma),$$

where the second step is taking the standard projection for each factor $\mathbb{Q}_p^\times$,

$$\mathbb{Q}_p^\times = p^{\mathbb{Z}} \times \mathbb{Z}_p^\times \longrightarrow \mathbb{Z}_p^\times.$$

Define a $\mathrm{GL}_2(\mathbb{Q}_\Sigma)$-invariant map

$$\tau : \mathbb{Z}[1/\Sigma, \mu_\Sigma] \longrightarrow \mathbb{Z}[1/\Sigma] \tag{2.15}$$

using the trace map $\mathrm{Tr}_{\mathbb{Q}(\mu_N)/\mathbb{Q}} : \mathbb{Q}(\mu_N) \to \mathbb{Q}$:

$$\tau(x) = \frac{\prod_{p\in\Sigma}(1-p)}{\phi(N)}\mathrm{Tr}_{\mathbb{Q}(\mu_N)/\mathbb{Q}}(x).$$

Concretely, we can compute the image of τ for any root of unity $\zeta \in \mu_N$ of order $N = \prod_{p\in\Sigma} p^{\alpha_p}$,

$$\tau(\zeta) = \begin{cases} \prod_{\alpha_p=0}(1-p) & \text{all } \alpha_p \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then we get a new pairing,

$$H^0(X_\Sigma, \Omega_{X_\Sigma}) \otimes_{\mathbb{Z}[1/\Sigma]} H^1(X_\Sigma, \mathcal{O}_{X_\Sigma}) \longrightarrow \mathbb{Z}[1/\Sigma], \tag{2.16}$$

compatible with the action by $\mathrm{GL}_2(\mathbb{Q}_\Sigma)$.

The Harris–Venkatesh period in finite level

Fix a pair of primes $p, \ell \geq 5$ and let ℓ^t be the highest power of ℓ dividing $p-1$. Fix a surjection

$$\log : \mathbb{F}_p^\times \longrightarrow R := \mathbb{Z}/\ell^t\mathbb{Z}.$$

For any positive integer N, we have a curve $X_0(p,N)$ defined by the open subset $U_0(p,N) := U_0(p) \cap U(N)$. In this setting, Harris–Venkatesh [HV19, Section 3.1] describes the Shimura class $\mathfrak{S}_p(N) \in H^1(X_0(p,N)_R, \mathcal{O})$ satisfying the properties,

1. $\mathfrak{S}_p(N)$ is invariant under $GL_2(\mathbb{Z}/N\mathbb{Z})$;
2. for any projection $\pi : X_0(p,N_2) \to X_0(p,N_1)$ with $N_1 \mid N_2$, we have that $\pi^*\mathfrak{S}_p(N_1) = \mathfrak{S}_p(N_2)$.

Notice that $U = U_0(p)$ has two embeddings into the maximal subgroup $U(1)$: the trivial embedding i_1 and the embedding i_2 obtained via conjugation by $\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$. This induces two projections,

$$\pi_1, \pi_2 : X_0(p,N) \longrightarrow X(N).$$

The pull-back on ω yields a pairing

$$\begin{aligned} H^0\Big(X(N)_R, \omega\big(-C(N)\big)\Big) \otimes H^0\Big(X(N)_R, \omega\big(-C(N)\big)\Big) &\longrightarrow H^0\Big(X(N)_R, \Omega\big(-C(N)\big)\Big) \\ \alpha \otimes \beta &\longmapsto \pi_1^*\alpha \cdot \pi_2^*\beta. \end{aligned}$$

Composition with the pairing $\langle -, \mathfrak{S}_p \rangle$ then gives a $GL_2(\mathbb{Z}/N\mathbb{Z})$-equivariant pairing on cuspidal one-forms with coefficients in R,

$$\begin{aligned} \mathcal{P}_{HV} : H^0\Big(X(N)_R, \omega\big(-C(N)\big)\Big) \otimes H^0\Big(X(N)_R, \omega\big(-C(N)\big)\Big) &\longrightarrow R[\zeta_N]. \\ \alpha \otimes \beta &\longmapsto \mathfrak{S}_p(\pi_1^*\alpha \cdot \pi_2^*\beta). \end{aligned}$$

We call $\mathcal{P}_{HV}$ the *Harris–Venkatesh period* (or *Harris–Venkatesh pairing*) *in level* N. It is compatible with pull-backs.

The Harris–Venkatesh period in infinite level

Fix a finite set Σ of primes not containing p and ℓ, and consider the projective system of curves $X_0(p, N)$ with all of the N contained in Σ. Let $X_{0,\Sigma}(p)$ denote its limit, which has a natural action by $\mathrm{GL}_2(\mathbb{Q}_\Sigma)$. Then the Harris–Venkatesh period in level N induces a pairing at infinite level:

$$H^0(X_{\Sigma,R}, \omega(-C_\Sigma)) \otimes H^0(X_{\Sigma,R}, \omega(-C_\Sigma)) \longrightarrow R[\mu_\Sigma]. \tag{2.17}$$

Composition of the pairing of Equation 2.17 with the map τ defined in Equation 2.15 yields the *Harris–Venkatesh period in infinite level*,

$$\mathcal{P}_{\mathrm{HV}} : H^0(X_{\Sigma,R}, \omega(-C_\Sigma)) \otimes H^0(X_{\Sigma,R}, \omega(-C_\Sigma)) \longrightarrow R. \tag{2.18}$$

Note that the left-hand side of Equation 2.18 consists of the spaces of cuspidal modular forms of weight 1 unramified outside of Σ.

The period in the setting of Harris–Venkatesh

The period $\mathcal{P}_{\mathrm{HV}}(\alpha \otimes \beta)$ is related to the setting of Harris–Venkatesh [HV19] in the following way. Let f be a cuspidal newform of weight 1 and level $\Gamma_1(N)$ with coefficients generating a subfield $\mathbb{Q}(f) \subset \mathbb{C}$ with ring of integers $\mathcal{O}(f)$. Assume that N is prime to $p\ell$. Then we may consider f and its dual f^* as elements of $H^0(X_1(N)_R, \omega(-C(N)))$. Then there is a form

$$G = \mathrm{Tr}_{\Gamma_0(N)/\Gamma_0(Np)}(f(z)f^*(pz)) \in H^0\big(X_0(p)_{\mathcal{O}(f)}, \Omega\big).$$

Pairing with the Shimura operator $\mathfrak{S}_p$ on $X_0(p)$ over R yields a value,

$$\mathfrak{S}_p(G) \in \mathcal{O}(f)/\ell^t.$$

This value is related to the Harris–Venkatesh period via

$$\begin{aligned}\mathfrak{S}_p(G) &= \sum_{\gamma \in U_0(p)/U_0(Np)} \mathcal{P}_{HV}(\gamma f \otimes \gamma f^*) \\ &= [U(1) : U_0(N)] \cdot \mathcal{P}_{HV}(f \otimes f^*).\end{aligned}$$

Since $[U(1) : U_0(N)]$ is not necessarily invertible in R (its order at ℓ is $\sum_{q|N} \mathrm{ord}_\ell(q+1)$), the value $\mathcal{P}_{HV}(f \otimes f^*)$ is more primitive than $\langle G, \mathfrak{S}_p \rangle$.

We finish this section with the following observation, which will not be used elsewhere in this article. See Vignéras [Vig89] for the definitions and details on the modular Steinberg representation St_p and the induced representation $i(\mu) = \mathrm{Ind}_B^G(\mu)$.

Lemma 2.5. *Let* $\mathrm{GL}_2(\mathbb{Q}^{p,\ell})$ *denote the group of finite adèles of* GL_2 *with trivial component at* p *and* ℓ*. For any integer* m *coprime to* ℓ*, let* $\overline{X}^\ell$ *denote the profinite modular curve* $\varprojlim_{(m,\ell)=1} \overline{X}(m)$ *over* $\mathrm{Spec}\,(k) = \mathrm{Spec}\,(\mathbb{Z}/\ell^t\mathbb{Z})$*. Let* $\mathfrak{S} \in H^1(\overline{X}^\ell, \mathcal{O})\langle 1 \rangle$ *denote the Shimura class and consider the representation* $\pi(\mathfrak{S})$ *of* $\mathrm{GL}_2(\mathbb{Q}^{p,\ell})$ *on the subspace of* $H^1(\overline{X}^\ell, \mathcal{O})\langle 1 \rangle$ *generated by* $\mathfrak{S}$*. Then there is an isomorphism of abstract representations*

$$\pi(\mathfrak{S}) \xrightarrow{\sim} \mathrm{St}_p \otimes \left(\bigotimes_{(q,p\ell)=1} \mathrm{triv}_q \right),$$

where triv_q *is the trivial representation of* $\mathrm{GL}_2(\mathbb{Q}_q)$ *and* St_p *is the modular Steinberg representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$.

Proof. The characterization for primes $q \neq p$ follows from the fact that the Shimura covering at level m' pulls back to the Shimura covering at level m whenever $m' \mid m$. To characterize the local component at p, it suffices to take $t = 1$. The representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathbb{Z}/\ell\mathbb{Z}$ generated by $\mathfrak{S}$ is a subquotient of the induced representation $i(\mu)$. Since $N \equiv 1 \pmod{\ell}$, it follows from a result of Vignéras [Vig89, Theorem 3(c)] that $i(\mu)$ is semisimple and $\dim i(\mu)^{K_0(p)} = 2$. On the other hand, $\mathfrak{S}_p$ is not invariant under $\mathrm{GL}_2(\mathbb{Z}_p)$ but is invariant under its Iwahori subgroup $K_0(p)$;

thus it must generate a representation isomorphic to St_p. $\square$

2.6 A theta identity for the modular avatar of optimal forms

Let $K/\mathbb{Q}$ be a quadratic extension and χ a character of $\mathrm{Gal}(\overline{K}/K)$. Then there is a newform f_χ corresponding to the induced Galois representation $\rho = \mathrm{Ind}_K^{\mathbb{Q}}(\chi)$. More precisely, the q-expansion $f_\chi = \sum_n a_\chi(n)q^n$ is determined by the equality of L-functions:

$$\sum a_\chi(n)n^{-s} = L(f,s) = L(\rho,s) = L(\chi,s) = \prod_{\wp \nmid c(\chi)} (1-\chi(\wp)N(\wp)^{-s})^{-1},$$

where $c(\chi) \subset \mathcal{O}_K$ is the conductor ideal of χ. Let φ_χ be the automorphic avatar of f_χ. Then φ_χ can also be defined as a theta lifting as in Equation 1.6.

More generally, for any function locally constant function $\Phi^\infty : \widehat{K} \longrightarrow \mathbb{C}$ with compact support we have a modular form $f_{\chi,\Phi^\infty}(z,u)$ whose automorphic avatar is $\theta(g,\chi^c,\Phi)$, where $\Phi = \Phi^\infty \otimes \Phi_\infty \in \mathcal{S}(K_{\mathbb{A}})$ with Φ_∞ the standard function (cf. Definition 1.7). By Equations 1.3 and 2.8, f_{χ,Φ^∞} has the q-expansion, for $u \in \widehat{\mathbb{Z}}^\times$,

$$f_{\chi,\Phi^\infty}(q,u) = \sum_{r\in\mathbb{Q}_+} a_{\chi,\Phi}(r,u)q^r,$$

with coefficient $a_{\chi,\Phi}(r,u)$ nonzero only if $ru = N(t_0)$ for some $t_0 \in \widehat{K}^\times$. In this case, it is given by,

$$a_{\chi,\Phi}(r,u) = \int_{\widehat{K}^1} \Phi^\infty(tt_0)\chi(tt_0)dt, \tag{2.19}$$

where $\widehat{K}^1$ is the subgroup of $\widehat{K}^\times$ of norm 1 and the measure is taken so that the volume of its maximal compact subgroup $\widehat{K}^1 \cap \widehat{\mathcal{O}}_K^\times$ is 1. For example, for f_χ, we take $\Phi^\infty = \bigotimes_{v\nmid\infty} \Phi_{\chi_v}$ defined in Section 1.4.

Let $\pi^\infty(\chi)$ be the irreducible representation of $\mathrm{GL}_2(\widehat{\mathbb{Q}})$ of modular forms generated by f_χ. We will consider the tensor product of modular forms in two variables generated by f_ξ and $f_{\chi^{-1}}$:

$$\pi^\infty(\chi) \otimes \pi^\infty(\chi^{-1}).$$

Notice that since χ is unitary, $f_{\chi^{-1}}$ can be obtained from f_χ by complex conjugation on the coefficients in its q-expansion. In this space, we have an *optimal form* f^{opt} whose automorphic avatar φ^{opt} is given by Equation 1.7. For the q-expansion, we take $\Phi_\alpha^{\mathrm{opt},\infty}$'s as in Definition 1.3: let $c = c(\xi)$ denote the conductor of $\xi = \chi^{1-\epsilon}$ and write $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}t$ for some $t \in \mathcal{O}_c$; let $\delta = t - \bar{t}$, a generator of the different ideal δ of $\mathcal{O}_c$; then for each $\alpha \in \mathcal{O}_c/\delta\mathcal{O}_c$, we take Φ_α to be the characteristic function of

$$\widehat{\mathcal{O}}_c + \frac{\alpha}{\delta}.$$

Then by Equations 1.8, 2.8, and 5.5,

$$\begin{aligned} f^{\mathrm{opt}}(q_1, q_2, u_1, u_2) &= \sum_{r_1, r_2 \in \mathbb{Q}_+} a^{\mathrm{opt}}(r_1, r_2, u_1, u_2) q_1^{r_1} q_2^{r_2}, \\ a^{\mathrm{opt}}(r_1, r_2, u_1, u_2) &:= \sum_{\alpha \in \mathcal{O}_c/\delta} a_{\chi, \Phi_\alpha^{\mathrm{opt},\infty}}(r_1, u_1) a_{\chi^{-1}, \Phi_{-\alpha}^{\mathrm{opt},\infty}}(-r_2, u_2), \end{aligned} \tag{2.20}$$

where $u \in \widehat{\mathbb{Z}}^\times$ and the right-hand side is defined as in 2.19.

Notice that Equation 2.19 shows that $a_{\chi, \Phi_\alpha^{\mathrm{opt},\infty}}(r, u)$ is nonzero only if $ru = N(h_0)$ for some

$$h_0 \in \bigcup_\alpha \Big(\mathcal{O}_c + \frac{\alpha}{\delta}\Big) = \frac{1}{\delta}\mathcal{O}_c.$$

It follows that $a_{\chi, \Phi_\alpha^{\mathrm{opt},\infty}}(r, u)$ is nonzero only if $r \in M^{-1}\mathbb{Z}$, where $M = -\delta^2$ is the discriminant of $\mathcal{O}_c$. Thus, f^{opt} is a modular form on $X(M) \times X(M)$, with M the discriminant of $\mathcal{O}_c$, whose q-expansion in Equation 2.1 is given by Equation 2.20.

We embed imaginary K into a definite quaternion algebra B and fix an Eichler order $\mathcal{O}$ with discriminant M and a finite character of $[\mathbb{Q}^\times]$. As in Section 1.6, we define two spaces $\mathcal{A}^\pm = \mathcal{A}(\omega^\pm)$ of automorphic forms on $[B^\times]$ with central character $\omega^{\pm 1}$, invariant under $U_1(M)$, and invariant under the action by the maximal compact subgroup U_∞ of $B_\infty^\times$. As in Section 1.6, we define the analogous space $\Xi^\pm := \Xi(\omega^\pm)$ of Hecke characters of $[K^\times]$ and the projection from Equation 1.15:

$$[-] : \Xi^\pm \longrightarrow \mathcal{A}^\pm$$

We have a theta lift operator (cf. [DHRV22, Sections 1.4 and 2.2], [Eme02], [Gro87, Proposition 5.6]),

$$\begin{aligned} \Theta_M : \mathcal{A}^+ \otimes \mathcal{A}^- &\longrightarrow M_2(\Gamma_0(M)) \\ \varphi_1 \otimes \varphi_2 &\longmapsto \sum_{n\geq 0} \langle \varphi_1, T_n \varphi_n \rangle q^n, \end{aligned} \tag{2.21}$$

where the left hand side is the modular avatar of $\theta(g, \varphi_1 \otimes \varphi_2, \Phi_{\mathcal{O}})$ in Proposition 1.9. This applies in particular to $\xi^\pm \in \Xi^\pm$:

$$\Theta_M([\xi_1] \otimes [\xi_2]) = \sum_{n\geq 0} \langle [\xi_1], T_n[\xi_2] \rangle q^n,$$

If $\mathcal{O}$ is ξ-optimal as in Definition 1.10, then apply the above identity to pushforwards $\varphi_1 := [\mathbb{1}]$ and $\varphi_2 := [\xi]$. By Proposition 1.11 with $M = p$,

$$f^{\mathrm{opt}}(z, pz) = \Theta_p([\mathbb{1}] \times [\xi]). \tag{2.22}$$

In fact, a comparison of Fourier coefficients shows that the optimal form is uniquely determined as a two-variable modular form by its realization of the theta lifting $\Theta_p([\mathbb{1}] \times [\xi])$.

Proposition 2.6. *The optimal form* $f^{\mathrm{opt}}(z_1, z_2)$ *is uniquely determined by satisfying Equation 2.22 for all primes* $p \geq 5$.

Proof. Suppose that $g(z_1, z_2)$ is a two-variable modular form such that for all primes $p \geq 5$,

$$g(z, pz) = \Theta_p([\mathbb{1}] \times [\xi]).$$

Let $a_{m,n}, b_{m,n}$, and $c_{p,k}$ be the Fourier coefficients of $f^{\mathrm{opt}}(z_1, z_2), g(z_1, z_2)$, and $\Theta_p([\mathbb{1}] \times [\xi])$ respectively; write $q_1 = e^{2\pi i z_1}$, $q_2 = e^{2\pi i z_2}$, and $q = e^{2\pi i z}$ so that,

$$\begin{aligned}
f^{\mathrm{opt}}(z_1, z_2) &= \sum_{m,n\geq 0} a_{m,n} q_1^m q_2^n, \\
g(z_1, z_2) &= \sum_{m,n\geq 0} b_{m,n} q_1^m q_2^n, \\
\Theta_p([\mathbb{1}] \times [\xi]) &= \sum_{k\geq 0} c_{p,k} q^k.
\end{aligned}$$

Note that by Equation 2.22 and the assumption that $g(z, pz) = \Theta_p([\mathbb{1}] \times [\xi])$,

$$\begin{aligned}
f^{\mathrm{opt}}(z, pz) = &\Theta_p([\mathbb{1}] \times [\xi]) = g(z, pz), \\
\sum_{m,n\geq 0} a_{m,n} q^{m+pn} = &\sum_{k\geq 0} c_{p,k} q^k \ = \sum_{m,n\geq 0} b_{m,n} q^{m+pn}.
\end{aligned}$$

Comparing Fourier coefficients gives,

$$\sum_{m+pn=k} a_{m,n} = c_{p,k} = \sum_{m+pn=k} b_{m,n},$$

for all $k \geq 0$ and for all primes $p \geq 5$. In particular for all $k \geq 0$, taking any prime $p > k$ gives,

$$a_{k,0} = c_{p,k} = b_{k,0}.$$

We now show that $a_{m,n} = b_{m,n}$ for all $m, n \geq 0$ by induction on n. Suppose that $a_{m,n} = b_{m,n}$ for all $m \geq 0$ and $0 \leq n \leq n_0$. Fix any $m_0 \geq 0$ and any prime $p \geq \max(5, m_0)$. Taking $k = m_0 + p(n_0 + 1)$, the only solutions $(m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ to $m + pn = k$ have $n \leq n_0 + 1$, so

$$\sum_{n=0}^{n_0+1} a_{k-pn,n} = c_{p,k} = \sum_{n=0}^{n_0+1} b_{k-pn,n}.$$

But $a_{m,n} = b_{m,n}$ for each $m \geq 0$ and $n \in \{1, \ldots, n_0\}$, so $a_{m_0,n_0+1} = b_{m_0,n_0+1}$. $\square$

3 Elliptic units

In this setting with $K/\mathbb{Q}$ an imaginary quadratic number field, the space $\mathcal{U}_{\mathrm{Ad}(\rho)}$ of units and the reduction map have simple descriptions. First, $\mathrm{Ad}(\rho)$ depends only on the "antinorm" $\xi := \chi^{1-\epsilon}$ with,

$$\mathrm{Ad}(\rho) = \eta \oplus \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\xi),$$

where η is a quadratic character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ associated to $K/\mathbb{Q}$. We may realize $\mathrm{Ad}(\rho)$ on the following $\mathbb{Z}[\xi]$-module,

$$M := \mathbb{Z}[\xi]e_0 + \mathbb{Z}[\xi]e_1 + \mathbb{Z}[\xi]e_2,$$

where $g \in \mathrm{Gal}(\overline{K}/K)$ and ϵ respectively act as the matrices,

$$\begin{pmatrix} 1 & & \\ & \xi(g) & \\ & & \xi(g)^{-1} \end{pmatrix}, \quad \begin{pmatrix} -1 & & \\ & & 1 \\ & 1 & \end{pmatrix}.$$

Let $c = c(\xi)$ be the conductor of ξ and let H_c be the associated ring class field. Then ξ factors through $\mathrm{Gal}(H_c/K)$. We define the $\mathbb{Z}[\xi]$-module of units,

$$\mathcal{U}_{\mathrm{Ad}(\rho)} := \mathrm{Hom}_{G_{\mathbb{Q}}}\left(M, \mathcal{O}_{H_c}^{\times}\right) \xrightarrow{\sim} \mathcal{O}_K^1 \otimes \mathbb{Z}[\xi] \oplus \left(\mathcal{O}_{H_c}^{\times} \otimes \mathbb{Z}[\xi]\right)^{G_K},$$

where $\mathcal{O}_K^1$ is the kernel of the norm $\mathcal{O}_K^{\times} \longrightarrow \mathbb{Z}^{\times}$. We will mainly work on,

$$\mathcal{U}_{\xi} := \left(\mathcal{O}_{H_c}^{\times} \otimes \mathbb{Z}[\xi]\right)^{G_K},$$

which is the submodule of $\mathcal{O}_{H_c}^{\times} \otimes \mathbb{Z}[\xi]$ of elements u such that,

$$u^{\sigma} = \xi(\sigma)^{-1}u,$$

for all $\sigma \in \mathrm{Gal}(H_c/K)$.

Consider the distinguished element at the archimedean place corresponding to the unique complex conjugation of H_c, $x_\infty := 2\rho(\mathrm{Frob}_\infty) - \mathrm{Tr}(\rho(\mathrm{Frob}_\infty)) \in \mathrm{Ad}(\rho)$. Evaluation at x_∞ defines a map,

$$\begin{aligned} \mathcal{U}_{\mathrm{Ad}(\rho)} &\longrightarrow \mathcal{U}_\xi \\ u &\longmapsto u(x_\infty) \end{aligned}$$

Choose a prime $\mathfrak{p}$ of $\mathcal{O}_{H_c}$ over p. Then we have the reduction map,

$$\mathrm{red}_p : \mathcal{O}_{H_c}^\times \longrightarrow (\mathcal{O}_{H_c}/\mathfrak{p})^\times = \mathbb{F}_\mathfrak{p}^\times \xrightarrow{N} \mathbb{F}_p^\times.$$

This induces the regulator map $\mathrm{Reg}_{\mathbb{F}_p^\times}$ and the element $\log_\ell \mathrm{Reg}_{\mathbb{F}_p^\times} \in \mathrm{Hom}(\mathcal{U}_\xi, \mathbb{Z}/\ell^t\mathbb{Z} \otimes \mathbb{Z}[\xi])$. This map is equivalent to the reduction map in Darmon–Harris–Rotger–Venkatesh [DHRV22] by the same argument as the proof of [DHRV22, Lemma 5.6] (and its generalization by Lecouturier [Lec22, Theorem 2.5]).

Now we recall the elliptic units constructed by Darmon–Harris–Rotger–Venkatesh [DHRV22, Section 5.1] with an auxiliary prime $\lambda = \mathfrak{l}\bar{\mathfrak{l}}$ split in K and coprime to c. Consider the modular unit u_λ on $Y_0(\lambda)$ (denoted Δ_N on $Y_0(N)$ in [DHRV22, Section 4.4]),

$$u_\lambda(z) := \frac{\Delta(z)}{\Delta(\lambda z)} \in \mathcal{O}(Y_0(\lambda))^\times,$$

where $\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24}$ is the usual Ramanujan Δ function (or modular discrimant) with $q = e^{2\pi i z}$. Recall that $Y_0(\lambda)$ is the modular curve parametrizing isogenies $E_1 \longrightarrow E_2$ of elliptic curves of degree λ. For $i \in \{1, 2\}$, let π_i be the projection,

$$\begin{aligned} \pi_i : Y_0(\lambda) &\longrightarrow Y_0(1) \\ (E_1 \longrightarrow E_2) &\longmapsto [E_i]. \end{aligned}$$

Let $Z(1) \subset X(1)$ be the set of isomorphism classes of elliptic curves E with $\mathrm{End}(E) = \mathcal{O}_c$ and let $Z_0(\lambda) \subset X_0(\lambda)$ be the subset consisting of points $\phi : E_1 \longrightarrow E_2$ with both $\mathrm{End}(E_i) = \mathcal{O}_c$. Then all points of $Z(1)$ and $Z_0(\lambda)$ are defined over H_c. In particular, $u_p(x) \in H_c^\times$ for all $x \in Z_0(\lambda)$. We fix one point $x_c = \mathbb{C}/\mathcal{O}_c \in Z(1)$.

Notice that the projections π_i from $Y_0(\lambda) \longrightarrow Y_0(1)$ induce projections $\pi_i : Z_0(\lambda) \longrightarrow Z(1)$. Fixing a splitting $\lambda = \mathfrak{l} \cdot \bar{\mathfrak{l}}$ gives a lifting:

$$\begin{aligned} \eta_\lambda : Z(1) &\longrightarrow Z_0(\lambda) \\ E &\longmapsto (E \longrightarrow E/E[\mathfrak{l}]). \end{aligned}$$

We define elliptic units following Darmon–Harris–Rotger–Venkatesh [DHRV22, Definition 5.1],

$$u_{\xi,\lambda} := \sum_{\sigma \in \mathrm{Gal}(H_c/K)} u_\lambda(\eta_\lambda(x_c))^\sigma \otimes \xi(\sigma) \in H_c^\times \otimes \mathbb{Z}[\xi]. \tag{3.1}$$

Assume that $\xi(\mathfrak{l})$ generates the group $\mathrm{Im}(\xi)$. Let $m(\xi) = \mathrm{N}(1 - \xi(\bar{\mathfrak{l}}))$, which is equal to ν if $|\mathrm{Im}(\xi)|$ is a power of a prime ν, and is equal to 1 otherwise. Then define,

$$u_\xi := \frac{m(\xi)}{1 - \xi(\bar{\mathfrak{l}})} u_{\xi,\lambda}. \tag{3.2}$$

Proposition 3.1. u_ξ *is a unit independent of the choice of the auxiliary prime* λ.

Proof. u_ξ is clearly a unit since $u_{\xi,\lambda}$ is a unit, so we only need to show that u_ξ is independent of the choice of λ. By the definition of u_ξ, we need to show that for any two primes $\lambda_1 \neq \lambda_2$ split in K,

$$(1 - \xi(\bar{\mathfrak{l}}_2))u_{\xi,\lambda_1} = (1 - \xi(\bar{\mathfrak{l}}_1))u_{\xi,\lambda_2}$$

where $\mathfrak{l}_i$ is an invertible ideal in $\mathcal{O}_c$ and $\lambda_i \mathcal{O}_c = \mathfrak{l}_i \cdot \bar{\mathfrak{l}}_i$.

Consider the commutative diagram of isogenous elliptic curves,

$$\begin{array}{ccc} \mathbb{C}/\mathcal{O}_c & \xrightarrow{x_1} & \mathbb{C}/\mathfrak{l}_1^{-1} \\ \big\downarrow{\scriptstyle x_2} & & \big\downarrow{\scriptstyle x_4} \\ \mathbb{C}/\mathfrak{l}_2^{-1} & \xrightarrow{x_3} & \mathbb{C}/(\mathfrak{l}_1\mathfrak{l}_2)^{-1}. \end{array}$$

By construction, the diagonal isogeny has square-free degree $\lambda_1\lambda_2$ and thus has a cyclic kernel isomorphic to $(\mathbb{Z}/\lambda_1\mathbb{Z}) \times (\mathbb{Z}/\lambda_2\mathbb{Z})$. Then this diagonal isogeny defines a point x on $X_0(\lambda_1\lambda_2)$ and is represented by a point $z \in \mathcal{H}$ in the sense that x is represented by the $\lambda_1\lambda_2$-multiplication map,

$$x : \mathbb{C}/(\mathbb{Z}+\mathbb{Z}z) \longrightarrow \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\lambda_1\lambda_2 z).$$

The two isogenies x_1 and x_2 are given by modular subgroups of $\ker(x)$ of order λ_1 and λ_2 respectively. This implies the representations,

$$x_1 : \mathbb{C}/(\mathbb{Z}+\mathbb{Z}z) \longrightarrow \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\lambda_1 z),$$
$$x_2 : \mathbb{C}/(\mathbb{Z}+\mathbb{Z}z) \longrightarrow \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\lambda_2 z),$$
$$x_3 : \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\lambda_2 z) \longrightarrow \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\lambda_1\lambda_2 z),$$
$$x_4 : \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\lambda_1 z) \longrightarrow \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\lambda_1\lambda_2 z).$$

Then we have points $x_1, x_3 \in X_0(\lambda_1)$ and $x_2, x_4 \in X_0(\lambda_2)$ with representatives in $\mathcal{H}$: $z_1 = z_2 = z$, $z_3 = \lambda_2 z_1$, and $z_4 = \lambda_1 z_2$. By the definition of u_{λ_1} and u_{λ_2}, we have the relation,

$$u_{\lambda_1}(x_1)u_{\lambda_2}(x_4) = \frac{\Delta(z_1)}{\Delta(\lambda_1 z_1)}\frac{\Delta(z_4)}{\Delta(\lambda_2 z_4)} = \frac{\Delta(z)}{\Delta(\lambda_1\lambda_2 z)} = \frac{\Delta(z_2)}{\Delta(\lambda_2 z_2)}\frac{\Delta(z_3)}{\Delta(\lambda_1 z_3)} = u_{\lambda_2}(x_2)u_{\lambda_1}(x_3).$$

Moreover, by the theory of complex multiplication, all of these points are defined over H_c with the relations,

$$x_3 = x_1^{\mathrm{Frob}(\mathfrak{l}_2)},$$
$$x_4 = x_2^{\mathrm{Frob}(\mathfrak{l}_1)}.$$

Therefore we have

$$u_{\lambda_1}(x_1)^{1-\mathrm{Frob}(\mathfrak{l}_2)} = u_{\lambda_2}(x_2)^{1-\mathrm{Frob}(\mathfrak{l}_1)}.$$

Now we take the ξ-sum (as in Equation 3.2) to obtain,

$$\sum_{\sigma\in \mathrm{Gal}(H_c/K)} u_{\lambda_1}(x_1)^{(1-\mathrm{Frob}(\mathfrak{l}_2))\sigma} \otimes \xi(\sigma) = \sum_{\sigma\in \mathrm{Gal}(H_c/K)} u_{\lambda_2}(x_2)^{(1-\mathrm{Frob}(\mathfrak{l}_1))\sigma} \otimes \xi(\sigma).$$

Unfolding these sums, we obtain,

$$\left(1-\xi\left(\bar{\mathfrak{l}}_2\right)\right)u_{\xi,\lambda_1} = \left(1-\xi\left(\bar{\mathfrak{l}}_1\right)\right)u_{\xi,\lambda_2}.$$

□

3.1 Relation to the Stark unit

In this dihedral setting, the Stark conjecture is known due to the original work of Stark [Sta80]. $\mathrm{Ad}(\rho) = \eta \oplus \mathrm{Ind}_{G_K}^{G_\mathbb{Q}}(\xi)$ has rank 1 where the rank,

$$r(\mathrm{Ad}(\rho)) := \sum_{v|\infty} \dim\left(\mathrm{Ad}(\rho)^{\mathrm{Frob}_v}\right),$$

is also the order of vanishing of the Artin L-function $L(\mathrm{Ad}(\rho), s)$. We now give the explicit relation between the unit u_ξ and u_{Stark}.

Proposition 3.2. *Let* h_K *be the class number of* K *and and* W_K *be the number of roots of unity in* K. *The Stark element* u_{Stark} *is the unique element of* $\mathcal{U}(\mathrm{Ad}(\rho))$ *such that at the distinguished element of the unique archimedean place of* H_c,

$$u_{\mathrm{Stark}}(x_\infty) = \frac{h_K}{6m(\xi)w_K} u_\xi.$$

Remark 3.3. Proposition 3.2 answers a question of Gross about the relation between the Stark units from the Stark conjecture and the Stark units from the Harris–Venkatesh conjecture. It also implies [DHRV22, Lemma 5.6].

Remark 3.4. The conjugacy class of Frob_w does not change if $w = \infty$ is replaced by a finite place of H_c over a λ inert in K. If λ is inert in K with a unique prime $\mathfrak{l}$, then $\mathfrak{l}$ is completely split in H. Thus we have $x_w = x_\infty$ and thus,

$$u_{\text{Stark}}(x_w) = \frac{h_k}{6m(\xi)w_k} \otimes u_\xi.$$

If λ is split in K, then by an observation of Darmon–Harris–Rotger–Venkatesh [DHRV22, Section 1.3], both sides of the identity in Conjecture 1 vanish.

First, we recall an original result of Stark [Sta80] proving a weaker explicit version of his conjecture for CM characters using Kronecker's second limit formula. Let $\mathfrak{c}$ be the conductor of ξ, i.e. the maximal ideal of $\mathcal{O}_K$ such that ξ is trivial on $(1+\widehat{\mathfrak{c}})^\times \subset \widehat{\mathcal{O}_k}^\times$ as a character on $\widehat{K}^\times$. Let $E = K(\mathfrak{c})$ be the corresponding ray class field with Galois group $G(\mathfrak{c})$. Thus by class field theory, the Frobenius map induces an isomorphism

$$K^\times \backslash \widehat{K}^\times / (1+\widehat{\mathfrak{c}})^\times \xrightarrow{\sim} G(\mathfrak{c}).$$

Let c be the minimal positive integer divisible by $\mathfrak{c}$ and let $w(\mathfrak{c})$ be the number of roots of unity in k which are congruent to $1 \pmod{\mathfrak{c}}$.

Theorem 3.5 (Stark [Sta80, Lemma 7])**.** *Let K be an imaginary quadratic number field. If ξ is a non-trivial ramified character of* $\mathrm{Gal}(\overline{K}/K)$*, then there is an explicit element $\epsilon(\mathfrak{c}) \in K(\mathfrak{c})^\times$ such that,*

$$L'(\xi, 0) = -\frac{1}{6cw(\mathfrak{c})} \sum_{\sigma \in G(\mathfrak{c})} \xi(\sigma) \log|\epsilon(\mathfrak{c})^\sigma|.$$

Remark 3.6. Our notation here deviates from [Sta80]: our $\mathfrak{c}$, c, σ are respectively $\mathfrak{f}$, f, and $\mathfrak{c}$ in Stark's paper.

Stark proves Theorem 3.5 using the second Kronecker limit formula. For $u, v \in \mathbb{R}$, $z \in \mathcal{H}$, $q := e^{2\pi i z}$, and $\zeta := e^{2\pi i(uz+v)}$, we have the Siegel function,

$$g(u,v,z) := -iq^{\frac{1}{12}}\left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) \prod_{m=1}^{\infty}(1-q^m\zeta)(1-q^m\zeta^{-1}).$$

Again with H_c the corresponding ring class field to the conductor of ξ, fix a decomposition of $\mathcal{O}_{K(\mathfrak{c})} = \mathbb{Z} + z\mathbb{Z}$ and $\mathfrak{c} = c(\mathbb{Z} + z\mathbb{Z})$ with $z \in \mathcal{H} \cap K(\mathfrak{c})$. Then by [Sta80, Equations (9), (10), and (45)],

$$\epsilon(\mathfrak{c}) = g\left(0, \frac{1}{c}, z\right)^{12c}.$$

Stark described the Galois conjugates of $\epsilon(\mathfrak{c})$ as follows. For any $\sigma \in G$ represented by ideal $\mathfrak{a}$ prime to $\mathfrak{c}$, let $\mathfrak{b}$ be an ideal such that $\mathfrak{a}\mathfrak{b} = (\alpha(\sigma))$ and write $\mathfrak{c}\mathfrak{b} = c(\sigma)(\mathbb{Z} + z(\sigma)\mathbb{Z})$. Then $\alpha(\sigma)/c(\sigma) = u(\sigma)z(\sigma) + v(\sigma)$ for some rational numbers $u(\sigma)$ and $v(\sigma)$, and,

$$\epsilon(\mathfrak{c})^{\sigma} = g(u(\sigma), v(\sigma), z(\sigma))^{12c}.$$

We can give a precise version of Stark's theorem which is not stated by Stark nor Tate but which we deduce directly from Theorem 3.5. Define $\epsilon(c) := N_{K(\mathfrak{c})/H_c}(\epsilon(\mathfrak{c}))$.

Corollary 3.7. *Assume that χ is the ring class character of conductor $c > 1$. Then,*

$$L'(\chi, 0) = -\frac{1}{6c} \sum_{\sigma \in G(\mathfrak{c})} \chi(\sigma) \log|\epsilon(\mathfrak{c})^{\sigma}|,$$

with,

$$\epsilon(c) = \prod_{d|c} \Delta\left(q^{d}\right)^{c\mu\left(\frac{c}{d}\right)} \in H_c^{\times}.$$

Proof. By Theorem 3.5, we only need to compute the norm $\epsilon(c)$ of $\epsilon(\mathfrak{c})$. Let $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_{K(\mathfrak{c})}$. Then $\mathfrak{c} = c\mathcal{O}_{K(\mathfrak{c})}$ and we have,

$$\mathrm{Gal}(K(\mathfrak{c})/H_c) \xrightarrow{\sim} k^{\times}(\widehat{\mathbb{Z}} + c\widehat{\mathcal{O}}_K)^{\times}/k^{\times}(1+\widehat{\mathfrak{c}})^{\times} \xrightarrow{\sim} (\widehat{\mathbb{Z}} + c\widehat{\mathcal{O}}_K)^{\times}/(1+\widehat{\mathfrak{c}})^{\times} \xrightarrow{\sim} (\mathbb{Z}/c\mathbb{Z})^{\times}.$$

Thus every element in $\sigma \in \mathrm{Gal}(K(\mathfrak{c})/H_c)$ is represented by an ideal $\mathfrak{a}(\sigma) = n\mathcal{O}_K$ with n coprime to c. We take $\mathfrak{b}(\sigma) = m\mathcal{O}_K$ with $\alpha(\sigma) = mn$. Write $\mathcal{O}_K = \mathbb{Z} + z\mathbb{Z}$. Then $c(\sigma) = cm$,

$u(\sigma) = 0$, and $v(\sigma) = n/c$. It follows that,

$$N_{K(\mathfrak{c})/H_c}(\epsilon(\mathfrak{c})) = \prod_{n \in (\mathbb{Z}/c\mathbb{Z})^\times} g\Big(0, \frac{n}{c}, z\Big)^{12c}.$$

Let $\zeta_c = e^{2\pi i/c}$. Then $N_{K(\mathfrak{c})/H_c}(\epsilon(\mathfrak{c}))$ is a product of,

$$q^{c\phi(c)} \cdot N(1-\zeta_c)^{12c} \prod_m \prod_{n \in (\mathbb{Z}/c\mathbb{Z})^\times} (1 - q^m \zeta_c^n)^{24c},$$

where ϕ is the Euler totient function.

The term $m(c) := N(1-\zeta_c)$ is equal to 1 unless c is a prime, in which case it is equal to c. The other terms can be computed by Möbius inversion, $\phi(c) = \sum_{d|c} d\mu(c/d)$,

$$\prod_{n \in (\mathbb{Z}/c\mathbb{Z})^\times} (1 - \zeta_c^n T) = \prod_{d|c} \big(1 - T^d\big)^{\mu\left(\frac{c}{d}\right)}.$$

Thus we obtain

$$N_{K(\mathfrak{c})/H_c}\big(\epsilon(\mathfrak{c})\big) = m(c)^{12c} q^{c\phi(c)} \prod_{d|c} \prod_{m=1}^{\infty} \big(1 - q^{dm}\big)^{24c\mu\left(\frac{c}{d}\right)} = m(c)^{12c} \prod_{d|c} \Delta\big(q^d\big)^{c\mu\left(\frac{c}{d}\right)}.$$

□

We now prove a variation of Stark's theorem using our unit u_ξ from Equation 3.2.

Proposition 3.8. *Assume that ξ is a ring class character of conductor $c > 1$. Then,*

$$L'(\xi, 0) = -\frac{1}{6m(\xi)} \log(u_\xi).$$

Proof. Let $\tau = \mathrm{Frob}_{\mathfrak{l}} \in \mathrm{Gal}(H_c/K)$. For any $d \mid c$, let E_d denote the elliptic curve $\mathbb{C}/\mathcal{O}_d$ which is defined over H_c. Then we have the isogenies $E_d \longrightarrow E_d^\sigma$ with kernel $E[\mathfrak{l}]$. Then we have (an additive operation) for $\epsilon(c)$ (defined in Corollary 3.7),

$$\epsilon(c) - \epsilon(c)^{\tau} = c \sum_{d|c} \mu\Big(\frac{c}{d}\Big)\eta(d).$$

Take χ-sums to obtain,

$$\sum_{\sigma \in \mathrm{Gal}(H_c/K)} \big(\chi(\sigma) - \chi(\sigma\tau)\big)\epsilon(c) = c \sum_{d|c} \chi(\sigma)\eta(d)^{\sigma}.$$

It follows that,

$$\sum_{\sigma \in \mathrm{Gal}(H_c/K)} \chi(\sigma)\epsilon(c) = \frac{c}{1-\chi(\tau)} \sum_{d|c} \mu\Big(\frac{c}{d}\Big) \sum_{\sigma \in \mathrm{Gal}(H_c/K)} \chi(\sigma)\eta(d)^{\sigma}.$$

Since $\eta(d) \in H_d$, the last sum has a factor $\sum_{\sigma \in \mathrm{Gal}(H_c/H_d)} \chi(\sigma)$ which is zero if $d \neq c$ (as c is the conductor of χ). Thus we have shown that,

$$\sum_{\sigma \in \mathrm{Gal}(H_c/K)} \chi(\sigma)\epsilon(c) = \frac{c}{m(\chi)} u_{\chi}.$$

The desired identity follows from Corollary 3.7. $\square$

Proof of Proposition 3.2. We have,

$$L(\mathrm{Ad}(\rho), s) = L(\eta, s) \cdot L(\xi, s).$$

Applying Theorem 3.8, we see that ρ has rank 1 and,

$$L'(\mathrm{Ad}(\rho), 0) = L(\eta, 0) \cdot L'(\xi, 0) = \frac{h_K}{6m(\xi)w_K} \cdot \log(u_{\xi}),$$

where the log is with respect to a fixed embedding $H_c \hookrightarrow \mathbb{C}$. Also, since H_c is a CM field, there is a unique complex conjugation at ∞. Thus there is a single distinguished element $x_{\infty} \in M$.

Thus $u_{\text{Stark}} \in \mathcal{U}(\mathrm{Ad}\rho)$ is the unique element so that,

$$u_{\text{Stark}}(x_\infty) = \frac{h_K}{6m(\xi)w_K} \otimes u_\xi.$$

□

3.2 Relation to a higher Eisenstein element

We can slightly modify the unramified argument of Darmon–Harris–Rotger–Venkatesh [DHRV22, Proposition 5.2] to relate the unit u_ξ to the higher Eisenstein element Σ_1,

$$\left(1 - \xi(\bar{\mathfrak{l}})\right) \cdot (\Sigma_1, [\xi]) = -\frac{1}{6}\log(u_{\xi,\lambda}),$$

where $\Sigma_1 \in R[\mathrm{Pic}(\mathcal{O}_B)]$ is a higher Eisenstein element (cf. [Lec21], [DHRV22, Definition 4.6]) satisfying the equation,

$$(T_\nu - (\ell+1))\Sigma_1 = (\nu - 1)\log(\nu)\Sigma_0,$$

for any prime ν. Since $\xi(\mathfrak{l}) \neq 1$, we have

$$\langle \Sigma_1, [\xi] \rangle = -\frac{1}{6m(\xi)}\log(u_\xi). \tag{3.3}$$

4 Proof of Theorem 7

In this section, we give a proof of Theorem 7 in the CM case (i.e. imaginary K). The method here largely follows the method of Darmon–Harris–Rotger–Venkatesh [DHRV22]. Throughout this proof, we let $c = c(\xi)$ be the conductor of ξ and consider the order,

$$\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K.$$

Let H_c denote the ring class field corresponding to $\mathcal{O}_c$ via class field theory,

$$\mathrm{Gal}(H_c/K) \xleftarrow{\sim} K^\times\backslash\widehat{K}^\times/\widehat{\mathcal{O}}_c^\times \xrightarrow{\sim} \mathrm{Pic}(\mathcal{O}_c).$$

4.1 Theta liftings for a definite quaternion algebra

Let B be the definite quaternion algebra with discriminant p. Since p inert in K, we have an embedding $K \hookrightarrow B$. We can then pick a maximal order $\mathcal{O}_B$, which is optimal for the order $\mathcal{O}_c$ because $\mathcal{O}_c = K \cap \mathcal{O}_B$. Then the embedding $K \hookrightarrow B$ induces a map,

$$\mathrm{Pic}(\mathcal{O}_c) := K^\times\backslash\widehat{K}^\times/\widehat{\mathcal{O}}_c^\times \xrightarrow{\iota} \mathrm{Pic}(B) := B^\times\backslash\widehat{B}^\times/\widehat{\mathcal{O}}_B^\times.$$

Darmon–Harris–Rotger–Venkatesh [DHRV22, Section 2.1] defines $\mathrm{Pic}(B)$ to be the set of equivalence classes of oriented maximal orders, where an oriented maximal order $(\mathcal{M}, \sigma)$ of B is a maximal order $\mathcal{M} \subset B$ with a homomorphism $\sigma : \mathcal{M} \longrightarrow \mathbb{F}_{p^2}$. This is in bijection with the set of isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}_p}$ by a result of Deuring [Deu41] (cf. [Voi21, Corollary 42.3.7]). We briefly show that this agrees with our definition of $\mathrm{Pic}(B) := B^\times\backslash\widehat{B}^\times/\widehat{\mathcal{O}}_B^\times$.

Lemma 4.1. *There is a bijection*

$$\begin{aligned} f : B^\times\backslash\hat{B}^\times/(\hat{\mathbb{Q}}^\times \cdot \hat{\mathcal{O}}_B^\times) &\longleftrightarrow \{\textit{equivalence classes of oriented maximal orders of } B\} \\ [g] &\longmapsto (\mathrm{Ad}(g)\mathcal{M}, \sigma) \end{aligned}$$

where $\mathrm{Ad}(g)\mathcal{M} := g\mathcal{M}g^{-1}$.

To see that the map f is a bijection, we look at the local picture due to the following correspondence.

$$\begin{tikzcd}
B/\mathbb{Q} & \hat{B}/\hat{\mathbb{Q}} \\
\mathbb{Z}\text{-lattices} \arrow[u, hook] \arrow[r, leftrightarrow] & \hat{\mathbb{Z}}\text{-lattices} \arrow[u, hook] \\
\text{oriented maximal orders of } B \arrow[u, hook] \arrow[r, leftrightarrow] & \text{oriented maximal orders of } \hat{B} \arrow[u, hook]
\end{tikzcd}$$

Locally, the map f is an isomorphism due to the following fact.

Lemma 4.2. *Let q be any prime. For a quaternion algebra $B_q/\mathbb{Q}_q$, all oriented maximal orders are $B_q^\times$-conjugates and the stabilizer of each oriented maximal order is $\mathbb{Q}_q^\times \cdot \mathcal{O}_{B_q}^\times$.*

Proof. Suppose B is split. Let $\mathcal{M}$ be an oriented maximal order in B. Then $\mathcal{M} = \text{End}(\Lambda)$ where Λ is the lattice

$$\Lambda := \left\{ \sum a_i v_i \mid a_i \in \mathcal{M}, v_i \in \mathbb{Z}_q^2 \right\},$$

since $\mathcal{M}\Lambda \subset \Lambda$ implies that $\mathcal{M} \subset \text{End}(\Lambda)$ but $\mathcal{M}$ is maximal. In this situation, we have the following diagram.

$$\begin{tikzcd}
M_2(\mathbb{Z}_q) \arrow[r, hook] \arrow[d, "\wr"] & M_2(\mathbb{Q}_q) \arrow[d, equal] & \mathcal{M} \arrow[l, hook'] \arrow[d, equal] \\
\text{End}(\mathbb{Z}_q^2) \arrow[r, hook] & \text{End}(\mathbb{Q}_q^2) & \text{End}(\Lambda) \arrow[l, hook']
\end{tikzcd}$$

Since $\Lambda \subset \mathbb{Q}_q^2$ is a lattice, there is a $g \in \text{GL}_2(\mathbb{Q}_q)$ such that $\Lambda = g\mathbb{Z}_q^2$.

For $\gamma \in \mathcal{M}$, $\gamma\Lambda \subset \Lambda$. Consequently, one can deduce the chain of equivalent statements:

$$\begin{aligned}
\gamma g\mathbb{Z}_q^2 &\subset g\mathbb{Z}_q^2, \\
g^{-1}\gamma g\mathbb{Z}_q^2 &\subset \mathbb{Z}_q^2, \\
g^{-1}\gamma g &\in M_2(\mathbb{Z}_q), \\
\gamma &\in gM_2(\mathbb{Z}_q)g^{-1}.
\end{aligned}$$

Therefore, $\mathcal{M} \subset gM_2(\mathbb{Z}_q)g^{-1}$. Since $\mathcal{M}$ is maximal, $\mathcal{M} = \text{Ad}(g)M_2(\mathbb{Z}_q)$.

Nonsplit: this case is trivial because there is only one order with two possible orientations. Just check the index-2 subgroup, which are the elements of even valuation. $\square$

Proof of Lemma 4.1. Surjectivity of f follows immediately from the definition of f and the fact that all local oriented maximal orders are conjugates by Lemma 4.2.

Injectivity of f follows from the fact that stabilizer of each local oriented maximal order is $\mathbb{Q}_q^\times \cdot \mathcal{O}_{B_q}^\times$ by Lemma 4.2. If $f([g_1]) = f([g_2])$, then $[g_1]$ and $[g_2]$ give the same equivalence class of oriented maximal orders and there is a $\gamma \in B^\times$ such that

$$\begin{tikzcd}
g_1\widehat{\mathcal{M}}g_1^{-1} \arrow[r, equal] \arrow[d, "\mathrm{Ad}(g_1^{-1})"] & \gamma g_2\widehat{\mathcal{M}}g_2^{-1}\gamma^{-1} \arrow[d, "\mathrm{Ad}(g_2^{-1}\gamma^{-1})"] \\
\widehat{\mathcal{M}} \arrow[d, "\sigma"] & \widehat{\mathcal{M}} \arrow[d, "\sigma"] \\
\mathbb{F}_{q^2} & \mathbb{F}_{q^2}
\end{tikzcd}$$

Then $\mathrm{Ad}(h)$ fixes $\widehat{\mathcal{M}}$, where $h := g_1^{-1}\gamma g_2$. By the second part of the lemma, h is in $\mathbb{Q}_q^\times \cdot \mathcal{O}_{B_q}^\times$. Since $g_1 = \gamma g_2 h^{-1}$, we have equality of $[g_1] = [g_2]$ in $B^\times \backslash B_q^\times / (\mathbb{Q}_q^\times \cdot \mathcal{O}_{B_q}^\times)$. $\square$

Remark 4.3. The $\hat{\mathbb{Q}}^\times$ in $B^\times \backslash \hat{B}^\times / (\hat{\mathbb{Q}}^\times \cdot \hat{\mathcal{O}}_B^\times)$ is unnecessary, i.e. $B^\times \backslash \hat{B}^\times / (\hat{\mathcal{O}}_B^\times)$. $\hat{\mathbb{Q}}^\times = \mathbb{Q}^\times * \hat{\mathbb{Z}}^\times$ since the class number is 1; $\mathbb{Q}^\times$ is in $B^\times$; and $\hat{\mathbb{Z}}^\times$ is in $\hat{\mathcal{O}}_B{}^\times$.

Let $\mathbb{Z}[\mathrm{Pic}(B)]$ denote the space of $\mathbb{Z}$-valued functions on $\mathrm{Pic}(B)$ (denoted as $\mathrm{Div}(\mathcal{E})$ in Darmon–Harris–Rotger–Venkatesh [DHRV22, Section 2.2]). It can also be viewed as a subspace of $\mathcal{A}^+ = \mathcal{A}^-$ with trivial central character ω. It is equipped with an action by Hecke algebra $\mathbb{T}$ and a pairing (the correspondence and height pairing respectively of [Gro87, Section 4]),

$$\langle -, - \rangle : \mathbb{Z}[\mathrm{Pic}(B)] \otimes \mathbb{Z}[\mathrm{Pic}(B)] \longrightarrow \mathbb{Z}.$$

Let Σ_0 be the function corresponding to the measure. Then Σ_0 generates the Eisenstein subspace (cf. [DHRV22, Equation 88]),

$$T_\ell \Sigma_0 = (\ell + 1)\Sigma_0.$$

We have a theta lifting from 2.21,

$$\begin{aligned}\Theta_p : \mathbb{Z}[\mathrm{Pic}(B)] \otimes_{\mathbb{T}} \mathbb{Z}[\mathrm{Pic}(B)] &\longrightarrow M_2(\Gamma_0(p)) \\ \varphi_1 \otimes \varphi_2 &\longmapsto \frac{1}{2}\langle \varphi_1, \Sigma_0\rangle\langle \varphi_2, \Sigma_0\rangle + \sum_{n\geq 1}\langle T_n\varphi_1, \varphi_2\rangle q^n,\end{aligned}$$

where the constant term calculation is from Emerton [Eme02] and Gross [Gro87, Proposition 5.6] (cf. [DHRV22, Equation 16]).

By 2.22, we have

$$f^{\mathrm{opt}}(z, pz) = \Theta_p(\mathbb{1} \otimes \xi). \tag{4.1}$$

4.2 Proof of Theorem 7 for definite theta series

By Equation 4.1,

$$\log_\ell \mathfrak{S}_p(f^{\mathrm{opt}}(z, pz)) = \log_\ell(\langle f^{\mathrm{opt}}(z, pz), \mathfrak{S}_p\rangle) = \log_\ell(\langle \Theta_p(\mathbb{1} \otimes \xi), \mathfrak{S}_p\rangle) = \log_\ell\big(\big\langle \mathbb{1} \otimes \xi, \Theta_p^*(\mathfrak{S}_p)\big\rangle\big),$$

where Θ_p^* is the adjoint operator of Θ_p,

$$\Theta_p^* : M_0(p)_R^* \longrightarrow (R[\mathrm{Pic}(B)] \otimes_{\mathbb{T}} R[\mathrm{Pic}(B)])^*.$$

Darmon–Harris–Rotger–Venkatesh [DHRV22, Theorem 5.4] showed that,

$$\Theta_p^*(\mathfrak{S}_p) = \frac{1}{2}(\Sigma_1 \otimes \Sigma_0 + \Sigma_0 \otimes \Sigma_1) \pmod{\Sigma_0 \otimes \Sigma_0}.$$

Now we pair both sides with $\mathbb{1} \otimes \xi$. Notice that $\langle \Sigma_0, \xi\rangle = 0$ and $\langle \Sigma_0, \mathbb{1}\rangle = h(\mathcal{O}_c)$, the class number of $\mathcal{O}_c$. Therefore, we have,

$$\log_\ell \mathfrak{S}_p(f^{\mathrm{opt}}(z, pz)) = \frac{1}{2}h(\mathcal{O}_c)\log_\ell(\langle \Sigma_1, \xi\rangle).$$

Apply Equation 3.3 to obtain the equality,

$$\log_\ell \mathfrak{S}_p(f^{\mathrm{opt}}(z, pz)) = -\frac{h(\mathcal{O}_c)}{12m(\xi)} \log_\ell \mathrm{Reg}_{\mathbb{F}_p^\times}(u_\xi).$$

Finally, we can use Proposition 3.2 to obtain (using $h_k = [H_1 : K]$ for the Hilbert class field H_1),

$$\log_\ell \mathfrak{S}_p(f^{\mathrm{opt}}(z, pz)) = -\frac{[H_c : H_1]w_K}{2} \log_\ell \mathrm{Reg}_{\mathbb{F}_p^\times}(u_{\mathrm{Stark}}).$$

Part II. Local theory: Rankin–Selberg periods

Let p be any prime, F be a p-adic field, E/F be a semisimple F-algebra of dimension 2 with trace map $\mathrm{Tr} : \mathrm{E} \longrightarrow \mathrm{F}$ and norm map $\mathrm{N} : \mathrm{E}^\times \longrightarrow \mathrm{F}^\times$. We have three cases:

1. E is split: $\mathrm{E} = \mathrm{F} \oplus \mathrm{F}$;

2. E/F is an inert quadratic extension;

3. E/F is a ramified quadratic extension.

Let χ be a character of $\mathrm{E}^\times$. Using the Weil representation, we obtain an irreducible representation $\pi(\chi)$ of $\mathrm{GL}_2(\mathrm{F})$ with central character $\omega = \eta \cdot \chi|_{\mathrm{F}^\times}$, where η is a quadratic character on $\mathrm{F}^\times$ with kernel $\mathrm{N}(\mathrm{E}^\times)$. All irreducible representations π of $\mathrm{GL}_2(\mathrm{F})$ can be obtained in this way, except for some supercuspidal representations π when $\mathrm{p} = 2$ ([Kut84, Corollary 4.3]). The dual representation of $\pi(\chi)$ is equal to $\pi(\chi^{-1})$. So we have a one-dimensional invariant quotient:

$$\mathcal{P} : \pi(\chi) \otimes \pi(\chi^{-1}) \longrightarrow \big(\pi(\chi) \otimes \pi(\chi^{-1})\big)_{\mathrm{GL}_2(\mathrm{F})}.$$

In Section 5, we construct two canonical elements: the *new vector* W^{new} and the *optimal vector* W^{opt} in $\pi(\chi) \otimes \pi(\chi^{-1})$. To do so, we first recall some of the local GL_2 and $\mathrm{GL}_2 \times \mathrm{GL}_2$ theory of Whittaker models, Kirillov models, and Rankin–Selberg convolutions following Jacquet–Langlands [JL70] and Jacquet [Jac72]. In our quadratic setting, we use local theta liftings and explicitly realize the pairing $\mathcal{P}$ as the Rankin–Selberg period $\mathcal{P}_{\mathrm{RS}}$ in terms of Rankin–Selberg zeta integrals $\mathrm{Z}(s, W)$ at $s = 1$ to study the ratio

$$[\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}}) : \mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}})] \in \mathbb{C} \cup \{\infty\}.$$

In Section 6, we calculate Rankin–Selberg zeta integrals evaluated at the optimal form W^{opt}. In Section 7, we calculate Rankin–Selberg zeta integrals evaluated at the newform W^{new}, define some models of $\pi(\chi)$ over a natural ring $\mathbb{Z}[\pi(\phi)]$, and show that

$$[\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}}) : \mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}})] = [A : B],$$

for some $A, B \in \mathbb{Z}[\pi(\chi)]$. In Section 8, the main results of Part II are collected in Theorem 8.1, demonstrating the rationality of this ratio, and Theorem 8.2, which explicitly calculates this ratio in several cases.

5 Whittaker functions

5.1 Whittaker and Kirillov models for GL_2

Let ψ be a non-trivial additive character of F and let $\mathcal{W}(\psi)$ denote the induced representation $\mathcal{W}(\psi) := \mathrm{Ind}_{N(F)}^{\mathrm{GL}_2(F)}(\psi)$. $\mathcal{W}(\psi)$ is the space of functions W on $\mathrm{GL}_2(F)$ such that,

$$W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \psi(x)W(g),$$

with action by $\mathrm{GL}_2(F)$ via translation.

Let π be an irreducible infinite-dimensional representation of $\mathrm{GL}_2(F)$. We can embed π into $\mathcal{W}(\psi)$:

$$\pi \hookrightarrow \mathcal{W}(\psi).$$

This embedding is unique up to scaling (due to [GK75, JL70], cf. [Bum97, Theorem 4.1.2], [Zha21, Section 5]). Therefore, we have a well-defined subspace $\mathcal{W}(\pi, \psi) \subset \mathcal{W}(\psi)$ called the Whittaker model of π. If we change ψ to another character $\psi_a(x) = \psi(ax)$, then we have an isomorphism,

$$\begin{aligned} \mathcal{W}(\psi) &\longrightarrow \mathcal{W}(\psi_a) \\ W(g) &\longmapsto W\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right). \end{aligned}$$

Thus, without loss of generality, we can assume that ψ has order 0 in the sense that $\mathcal{O}_F$ is the maximal fractional ideal of F over which ψ is trivial.

Define the map,

$$\begin{aligned} a : F^\times &\longrightarrow \mathrm{GL}_2(F) \\ x &\longmapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

According to Jacquet–Langlands [JL70, Section 2] (cf. [Bum97, Section 4.4]), the following restriction map is injective,

$$W(g) \longmapsto \kappa(x) := W\big(a(x)\big).$$

Let $\mathcal{K}(\pi, \psi)$ be the image of this map with the induced action by $\mathrm{GL}_2(F)$. This is called the Kirillov model. The action of the Borel subgroup of $\mathrm{GL}_2(F)$ on $\mathcal{K}(\pi, \psi)$ is as follows (cf. [Jac72, Section 14], [Bum97, Equation 4.25]),

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \kappa(x) = \omega(d)\psi\left(\frac{bx}{d}\right)\kappa\left(\frac{ax}{d}\right).$$

The action of $w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is not easy to write down (cf. [Bum97, Section 4.7]).

If π is changed to $\pi \otimes \mu$ for a character μ of $F^\times$ via $\det : \mathrm{GL}_2(F) \longrightarrow F^\times$, then $\mathcal{W}(\pi \otimes \mu, \psi) = \mathcal{W}(\pi, \psi) \otimes \mu$.

Let ϖ be the uniformizer of F. For any non-negative integer i, define $U_0(\varpi^i)$ and $U_1(\varpi^i)$ as the following subgroups of $\mathrm{GL}_2(\mathcal{O}_F)$ (cf. [Zha01, Section 2.3]),

$$U_0(\varpi^i) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c \equiv 0 \pmod{\varpi^i} \right\},$$
$$U_1(\varpi^i) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| (c, d) \equiv (0, 1) \pmod{\varpi^i} \right\}.$$

Then for any irreducible representation π of $\mathrm{GL}_2(F)$, we define the level $o = o(\pi)$ of π to be the minimal non-negative integer i such that $\pi(\chi)^{U_1(\pi^i)} \neq 0$. At the level of π, there is a unique element $W^{\text{new}} \in \mathcal{W}(\pi)^{U_1(\pi^o)}$ such that $W^{\text{new}}(e) = 1$.

If ψ is changed to ψ_a for some $a \in \mathcal{O}_F^\times$, then $W^{\text{new}}(g)$ is changed to $W\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right)$. However, if π is changed to $\pi \otimes \mu$, there is no simple formula to write down the change to W^{new}.

Following Jacquet–Langlands [JL70, Theorem 2.18] (cf. [Jac72, Section 14], [Bum97, Proposition 4.7.5]), we have the zeta integral,

$$Z(s, W) := \int_{F^\times} W\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} da,$$

where da is a Haar measure on $F^\times$ normalized such that $\mathrm{vol}(\mathcal{O}_F^\times) = 1$. These integrals are absolutely convergent when $\Re(s) \gg 0$. The values of these integrals define a fractional ideal of $\mathbb{C}[q^\pm s]$ with a generator $L(s, \pi)$,

$$L(s, \pi) = Z(s, W^{\text{new}}).$$

Then we can define the normalized zeta integral,

$$\Psi(s, W) = \frac{Z(s, W)}{L(s, \pi)} \in \mathbb{C}[q^{\pm s}].$$

For example, if $\pi = \pi(\chi_1, \chi_2)$ is a principal series with χ_1, χ_2 unramified and $\alpha_i = \chi_i(\varpi)$ (cf.

[JL70, Section 3]), the above identity is equivalent to,

$$\frac{1}{(1-\alpha_1 q^{-s})(1-\alpha_2 q^{-s})} = \sum_{n=0}^{\infty} W^{\text{new}}\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right) q^{-n\left(s-\frac{1}{2}\right)}.$$

It follows that W^{new} is the unique element in $\mathcal{W}(\psi)$ which is invariant under $\mathrm{GL}_2(\mathcal{O}_F)$ with central character $\chi_1\chi_2$ and takes values,

$$W^{\text{new}}\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right) = q^{\frac{-n}{2}} \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2}.$$

5.2 Rankin–Selberg periods for $\mathrm{GL}_2 \times \mathrm{GL}_2$

Let π be an irreducible infinite-dimensional representation of $\mathrm{GL}_2(F)$ with contragredient representation $\widetilde{\pi}$. Then we have a canonical $\mathrm{GL}_2(F)$-invariant pairing:

$$\mathcal{P} : \pi \otimes \widetilde{\pi} \longrightarrow \mathbb{C}.$$

We realize the representations π and $\widetilde{\pi}$ with respective Whittaker models $\mathcal{W}(\pi,\psi)$ and $\mathcal{W}(\widetilde{\pi},\psi)$. We want to explicitly construct a $\mathrm{GL}_2(F)$-invariant pairing,

$$\mathcal{P}_{\mathrm{RS},\psi} : \mathcal{W}(\pi,\psi) \otimes \mathcal{W}(\widetilde{\pi},\psi) \longrightarrow \mathbb{C},$$

using the Rankin–Selberg method. This pairing will induce a unique isomorphism with a compatible linear form:

$$\pi \otimes \widetilde{\pi} \xrightarrow{\sim} \mathcal{W}(\pi,\psi) \otimes \mathcal{W}(\widetilde{\pi},\psi).$$

First, consider the right-hand side as a space of functions on $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. Then we restrict this space to the diagonal:

$$\begin{aligned}\Delta^* : \mathcal{W}(\pi, \psi) \otimes \mathcal{W}(\widetilde{\pi}, \psi) &\longrightarrow \mathrm{Ind}_{N(F)Z(F)}^{\mathrm{GL}_2(F)}(\mathbb{1}),\\ W_1(g_1) \otimes W_2(g_2) &\longmapsto W_1(g)W_2(\epsilon' g),\end{aligned}$$

where $\epsilon' := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We use the Iwasawa decomposition

$$\mathrm{GL}_2(F) = B(F)\mathrm{GL}_2(\mathcal{O}_F)$$

to define a function $f(g, s)$ on $N(F)Z(F)\backslash\mathrm{GL}_2(F) \times \mathbb{C}$ by

$$f\left(\begin{pmatrix} a & x \\ & b \end{pmatrix} k, s\right) = \left|\frac{a}{b}\right|^s,$$

for $k \in \mathrm{GL}_2(\mathcal{O}_F)$.

Following Jacquet [Jac72, Section 14], we consider the zeta integral

$$Z(s, W_1, W_2) := \int_{N(F)Z(F)\backslash\mathrm{GL}_2(F)} W_1(g)W_2(\epsilon' g)f(g, s)dg. \tag{5.1}$$

Here dg is the quotient measure on $N(F)Z(F)\backslash\mathrm{GL}_2(F)$ constructed from the Haar measures on $\mathrm{GL}_2(F)$ and its various subgroups. Write

$$g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} z & \\ & z \end{pmatrix}\begin{pmatrix} a & \\ & 1 \end{pmatrix} k,$$

with $a, x, z \in F$ and $k \in \mathrm{GL}_2(\mathcal{O}_F)$. Then

$$dg = dxdz\frac{da}{|a|}dk,$$

where dx is a measure on F so that $\mathrm{vol}(\mathcal{O}_F) = 1$, dz and da are measures on $F^\times$ so that $\mathrm{vol}(\mathcal{O}_F^\times) =$

1, and dk is a measure on $\mathrm{GL}_2(\mathcal{O}_F)$ with $\mathrm{vol}(\mathrm{GL}_2(\mathcal{O}_F)) = 1$.

Remark 5.1. Note that we use $\epsilon' := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ instead of $\epsilon := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (ϵ is called η by Jacquet [Jac72, p.11]) in order to notationally avoid the repeated appearance of $\chi(-1)$ in our calculations.

The above zeta integral is absolutely convergent when $\Re(s) \geq 0$, and has values forming a fractional ideal of $\mathbb{C}[q^{\pm s}]$ with generator,

$$(1 + q^{-s})L\big(s, \mathrm{Ad}(\pi)\big).$$

We can then define the normalized zeta integral (cf. [Jac72, Theorem 14.7]):

$$\Psi(s, W_1, W_2) = \frac{Z(s, W_1, W_2)}{(1 + q^{-s})L\big(s, \mathrm{Ad}(\pi)\big)} \in \mathbb{C}\big[q^{\pm s}\big]. \tag{5.2}$$

We define the invariant form $\mathcal{P}_{\mathrm{RS},\psi}$ by

$$\mathcal{P}_{\mathrm{RS},\psi}(W_1 \otimes W_2) := Z(1, W_1, W_2).$$

Note that up to an ϵ-factor of $\mathrm{Ad}(\pi)$, we may replace $\Psi(1, W_1, W_2)$ by $\Psi(0, W_1, W_2)$, which is a regularization for the usual divergent integral,

$$\int_{N(F)Z(F)\backslash \mathrm{GL}_2(F)} W_1(g)W_2(\epsilon' g)dg.$$

Note that if we change ψ to ψ_a, then the above pairings are compatible with the isomorphism

$$\mathcal{W}(\pi, \psi) \otimes \mathcal{W}(\widetilde{\pi}, \psi) \xrightarrow{\sim} \mathcal{W}(\pi, \psi_a) \otimes \mathcal{W}(\widetilde{\pi}, \psi_a).$$

So we may drop the ψ subscript from $\mathcal{P}_{\mathrm{RS},\psi}$. Also, we note that the form $W^{\mathrm{new}} \otimes \widetilde{W}^{\mathrm{new}}$ is invariant under the canonical isomorphisms with respect to different ψ's.

If π is unitary, then there is a positive definite $\mathrm{GL}_2(F)$-invariant Hermitian pairing,

$$\langle -,- \rangle_0 : \mathcal{W}(\pi,\psi) \otimes \mathcal{W}(\pi,\psi) \longrightarrow \mathbb{C},$$

such that $\langle W^{\mathrm{new}}, W^{\mathrm{new}} \rangle_0 = 1$. We can write such a pairing in terms of $\mathcal{P}_{\mathrm{RS}}$. For any $W \in \mathcal{W}(\pi,\psi)$, define $W^{\epsilon} \in \mathcal{W}(\overline{\pi},\psi)$,

$$W^{\epsilon}(g) := \overline{W(\epsilon g)}. \tag{5.3}$$

Then we have a non-degenerate $\mathrm{GL}_2(F)$-invariant Hermitian pairing:

$$\begin{aligned} \langle -,- \rangle_1 : \mathcal{W}(\pi,\psi) \otimes \mathcal{W}(\pi,\psi) &\longrightarrow \mathbb{C} \\ (W_1, W_2) &\longmapsto \mathcal{P}_{\mathrm{RS}}\big(W_1, \overline{W}_2\big). \end{aligned}$$

Thus it is a real multiple of $\langle -,- \rangle_0$. This gives the following fact.

Proposition 5.2. *If π is unitary and $W \neq 0$, then $\mathcal{P}_{\mathrm{RS}}(W \otimes W^{\epsilon}) \neq 0$ and is real.*

For example, we can compute the pairing for new forms in the case $\pi = \pi(\chi_1,\chi_2)$ with χ_1,χ_2 unramified. Write $\alpha_i = \chi_i(\varpi)$ and $\beta_i = \widetilde{\chi_i}(\varpi)$. Then the newforms W^{new} and $\widetilde{W}^{\mathrm{new}}$ for π and $\widetilde{\pi}$ take the values,

$$\begin{aligned} W^{\mathrm{new}}\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right) &= \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} q^{\frac{-n}{2}}, \\ \widetilde{W}^{\mathrm{new}}\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix}\right) &= \frac{\beta_1^{n+1} - \beta_2^{n+1}}{\beta_1 - \beta_2} q^{\frac{-n}{2}}. \end{aligned}$$

Bringing this into the formula for $Z(s, W^{\text{new}}, \widetilde{W}^{\text{new}})$ (Equation 5.1) yields,

$$
\begin{aligned}
Z(s, W^{\text{new}}, \widetilde{W}^{\text{new}}) &= \int_{ZN\backslash GL_2(F)} W^{\text{new}}(g)\widetilde{W}^{\text{new}}(\epsilon' g) f(g,s)\,dg \\
&= \sum_n q^n W^{\text{new}} \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} \widetilde{W}^{\text{new}} \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} q^{-ns} \\
&= \sum_{n=0}^{\infty} \frac{\alpha_1^{n+1} - \alpha_2^{n+1}}{\alpha_1 - \alpha_2} \frac{\beta_1^{n+1} - \beta_2^{n+1}}{\beta_1 - \beta_2} q^{-ns} \\
&= \frac{1}{(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)} \cdot \\
&\quad \left(\frac{\alpha_1\beta_1 q^{-s}}{1 - \alpha_1\beta_1 q^{-s}} - \frac{\alpha_1\beta_2 q^{-s}}{1 - \alpha_1\beta_2 q^{-s}} - \frac{\alpha_2\beta_1 q^{-s}}{1 - \alpha_2\beta_1 q^{-s}} + \frac{\alpha_2\beta_2 q^{-s}}{1 - \alpha_2\beta_2 q^{-s}} \right) \\
&= \frac{1 - \alpha_1\alpha_2\beta_1\beta_2 q^{-2s}}{(1 - \alpha_1\beta_1 q^{-s})(1 - \alpha_1\beta_2 q^{-s})(1 - \alpha_2\beta_1 q^{-s})(1 - \alpha_2\beta_2 q^{-s})} \\
&= \frac{1 + q^{-s}}{(1 - q^{-s})\left(1 - \frac{\alpha_1}{\alpha_2} q^{-s}\right)\left(1 - \frac{\alpha_2}{\alpha_1} q^{-s}\right)} \\
&= (1 + q^{-s}) L(s, \mathrm{Ad}(\pi)).
\end{aligned}
$$

Therefore, $\Psi(s, W^{\text{new}}, \widetilde{W}^{\text{new}}) = 1$ by Equation 5.2.

5.3 Theta liftings

Let (V, Q) be an orthogonal quadratic space over F of even dimension m with the bilinear form,

$$\langle x, y \rangle = Q(x + y) - Q(x) - Q(y).$$

Let $GO(V)$ be the group of similitudes on V with norm map $\nu : GO(V) \longrightarrow \mathbb{G}_m$. Let $G = GL_2 \times_{\mathbb{G}_m} GO(V)$ be the fiber product of ν and $\det : GL_2 \longrightarrow \mathbb{G}_m$. Then we may consider SL_2 and $O(V)$ to be normal subgroups of G with respective quotients isomorphic to $GO(V)$ and $GL_2(F)^+$, the subgroup of elements $g \in GL_2(F)$ such that $\det g \in \nu(GO(V))$.

Let $\mathcal{S}(V)$ be the space of Schwartz functions on V. Then we have a Weil representation r of $G(F)$ on $\mathcal{S}(V)$ with respect to the character $\psi : F \longrightarrow \mathbb{C}^\times$ (cf. [Wal85, Section 1], [YZZ13, Section 2.1]). To describe this representation, we need the following special elements in GL_2:

$$d(\lambda) := \begin{pmatrix} 1 & \\ & \lambda \end{pmatrix}, \qquad a(\lambda) := \begin{pmatrix} \lambda & \\ & 1 \end{pmatrix},$$
$$m(\lambda) := \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}, \qquad n(b) := \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix},$$
$$w := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Then G is generated by the elements $m(\lambda)$, $n(b)$, w, and $(d(\nu(h)), h)$ for $h \in \mathrm{GO}(V)$. We describe r by the following.

1. For any $h \in \mathrm{GO}(V)$, $\Phi \in \mathcal{S}(V)$,

$$r\big(d(\nu(h), h\big) \cdot \Phi(x) = |\nu(h)|^{\frac{-m}{4}} \Phi\big(h^{-1}x\big).$$

2. For any $\lambda \in F^\times$,

$$r\big(m(\lambda)\big) \cdot \Phi(x) = \eta_V(\lambda)|\lambda|^{\frac{m}{2}} \Phi(\lambda x),$$

where $\eta_V(\lambda) = (\lambda, (-1)^{m/2} \det(V))$.

3. For any $b \in F$,

$$r\big(n(b)\big) \cdot \Phi(x) = \psi\big(bQ(x)\big)\Phi(x).$$

4. For w as above,

$$r(w) \cdot \Phi(x) = \gamma \cdot \widehat{\Phi}(x),$$

where γ is an 8-th root of unity and $\widehat{\Phi}$ is the Fourier transform,

$$\widehat{\Phi}(x) = \int_V \Phi(y)\psi(\langle x, y\rangle)dy.$$

Remark 5.3. We follow the convention of using $d(\nu(h))$ (e.g. Harris–Kudla [HK91, Section 3.2], [HK04, Equation 1.1], Yuan–Zhang–Zhang [YZZ13, Section 2.1.3]) instead of $a(\nu(h))$ (e.g. Jacquet–Langlands [JL70, Chapter 1]). Similar to Remark 5.1, this is a notational decision that affects the appearance of $\chi(-1)$ in later calculations. We will also omit the r where the context is clear, for example simply writing $w\Phi$ for $r(w)\cdot\Phi$ and $n(b)\Phi$ for $r(n(b))\cdot\Phi$.

Remark 5.4. The 8-th root of unity γ in the action of $r(w)$ on a Schwartz function Φ is called the Weil index. It is dependent on (V, Q), but is equal to -1 for nonsplit quaternion algebras and 1 for split quaternion algebras (cf. [Wei64, Chapter II]) so it can be omitted for most of our purposes.

From the definition, we see that $r(z,z)$ acts on $\mathcal{S}(V)$ by character η_V. In particular,

$$\begin{aligned} r(z,z)\cdot\Phi(x) &= r\left(d\left(z^2\right)m(z), z\right)\cdot\Phi(x) \\ &= |z|^{-m/2} r(m(z))\cdot\Phi\left(z^{-1}x\right) \\ &= \eta_V(z)\Phi(x). \end{aligned}$$

5.4 Quadratic cases

We start with the general quadratic space $V = (Ee, Q)$ with an action of E, where E is a semisimple algebra over F and Q is a multiple of the norm $N = N_{E/F}$ of E over F. Then $GO(V) = \langle E^\times, \iota\rangle$ where ι is an involution. In this case, ν is the usual norm N of E over F.

Let $\chi : E^\times \longrightarrow \mathbb{C}^\times$ be a character. For each $\Phi \in \mathcal{S}(E)$, we obtain a Whittaker function supported by the subgroup $GL_2(F)^+$ of matrices with determinant in $N(E^\times)$. Thus $GL_2(F)^+ = GL_2(F)$ if $E = F \oplus F$; otherwise, $GL_2(F)^+$ is an index-2 subgroup of $GL_2(F)$. More precisely, for $g \in GL_2(F)^+$, we write $g = d(Q(t_0e)^{-1})g_1$ with $h_0 \in E^\times$ and $g_1 \in SL_2(F)$, and we have

$$W(g,\chi,\Phi) = |\det g|^{-\frac{1}{2}} \int_{E^1} r(g_1)\Phi(tt_0e)\chi^c\left(t_0^{-1}t^{-1}\right)dt, \tag{5.4}$$

where E^1 is the subgroup of $E^\times$ with norm 1, $\chi^c = \chi \circ c$ with $c \in \mathrm{Aut}(E/F)$ the non-trivial involution, and dt is a Haar measure on E^1 such that $\mathrm{vol}(\mathcal{O}_E^1) = 1$. The corresponding Kirillov functions are given by

$$\kappa(x,\chi,\Phi)=|x|^{\frac{1}{2}}\int_{E^1}\Phi(t_0te)\chi(tt_0)dt, \tag{5.5}$$

where $x=Q(t_0e)$.

Remark 5.5. Note that the above construction is compatible with the construction in the global situation (Equations 1.2 and 1.3) by Equations 1.4 and 1.5.

The subrepresentation of $\mathrm{GL}_2(F)$ generated by $W(g,\chi,\Phi)$ is an irreducible representation denoted by $\pi(\chi)$. The set of such functions is an explicit local theta lifting $\theta(\chi,\psi_c)$. We may consider the functional

$$\Phi\longmapsto W(g,\chi,\Phi)$$

as an element of,

$$\mathrm{Hom}_{E^\times\times_{\mathbb{G}_m}\mathrm{GL}_2(F)}(\mathcal{S}(V)\otimes\chi,\theta(\chi,\psi_c)),$$

where χ and $\theta(\chi,\psi_c)$ are considered as representations via projections to $E^\times$ and $\mathrm{GL}_2(F)^+\subset\mathrm{GL}_2(F)$ respectively.

The subspace $\theta(\chi,\psi_c)$ is stable under the right translation by $\mathrm{GL}_2(F)^+$. We consider $\theta(\chi,\psi_c)$ as a subspace of functions on $\mathrm{GL}_2(F)$ supported on $\mathrm{GL}_2(F)^+$ and define $\mathcal{W}(\chi,\psi_c)$ to be the space of Whittaker functions on $\mathrm{GL}_2(F)$ induced by such functions. More precisely, if $E=F\oplus F$, then $\mathcal{W}(g,\chi)=\theta(g,\chi,\psi_c)$; otherwise, let $h\in\mathrm{GL}_2(F)-\mathrm{GL}_2(F)^+$, then $\mathcal{W}(g,\chi)$ consists of functions

$$W(g)=W_1(g)+W_2(gh),$$

for $W_1,W_2\in\theta(g,\chi,\psi_c)$.

The space $\mathcal{W}(\chi)$ forms an irreducible representation of $\mathrm{GL}_2(F)$ denoted by $\pi(\chi)$. This space has the following properties.

1. The central character of $\pi(\chi)$ is $\omega:=\eta\cdot\chi|_{F^\times}$;

2. If $E=F\oplus F$ and $\chi=(\chi_1,\chi_2)$ then $\pi(\chi)=\pi(\chi_1,\chi_2)$ is a principal series;

3. If $\chi = \omega \circ N$, then $\chi = (\omega, \omega \cdot \eta)$ is a principal series;

4. If E is not split and χ does not factor through N, then $\pi(\chi)$ is supercuspidal.

New forms

We assume that $V = (E, N)$ with E a semisimple algebra over F. For the induced representation $\pi(\chi)$, the level c and the newform W_χ^{new} can be constructed explicitly from the Whittaker function $W(g, \chi, \Phi_\chi)$ by the following two steps.

1. Define Φ_χ according to Tate's thesis [Tat67].

   (a) If E is a field extension, and χ is unramified, then define Φ_χ to be the characteristic function of $\mathcal{O}_E$;

   (b) If E is a field extension and χ is ramified, then define Φ_χ to be the restriction of χ^{-1} on $\mathcal{O}_E^\times$;

   (c) If $E = F \oplus F$, $\chi = (\chi_1, \chi_2)$, then define $\Phi_\chi = \Phi_{\chi_1} \otimes \Phi_{\chi_2}$, where for each i,

$$\Phi_{\chi_i} = \begin{cases} \mathbb{1}|_{\mathcal{O}_F} & \text{if } \chi_i \text{ is unramified,} \\ \chi_i^{-1}|_{\mathcal{O}_F^\times} & \text{otherwise.} \end{cases}$$

   Then $W(g, \chi, \Phi_\chi)$ is already a newform $W_\chi^{\text{new}}(g)$ when E/F is not ramified.

2. If E/F is ramified, then the new form in $\pi(\chi)$ has the form,

$$W_\chi^{\text{new}}(g) = W(g, \chi, \Phi_\chi) + W(ga(\epsilon), \chi, \Phi_\chi),$$

   where $\epsilon \in \mathcal{O}_F^\times - N(\mathcal{O}_E^\times)$.

This shows, in particular, the following formula:

Proposition 5.6. *The inert product of newforms can be computed from theta forms as follows,*

$$\mathcal{P}_{\mathrm{RS}}\left(W^{\mathrm{new}}_{\chi} \otimes W^{\mathrm{new}}_{\chi^{-1}}\right) = e_{E/F}\mathcal{P}_{\mathrm{RS}}\big(W(g, \chi^{-1}, \Phi_{\chi^{-1}}) \otimes W(g, \chi, \Phi_{\chi})\big),$$

where

$$e_{E/F} := \begin{cases} 1 & \textit{if } E/F \textit{ is unramified,} \\ 2 & \textit{if } E/F \textit{ is ramified.} \end{cases}$$

Optimal forms

Now we want to construct an optimal element in $\mathcal{W}(\chi, \psi) \otimes \mathcal{W}(\chi^{-1}, \psi)$. Let $\xi = \chi^{1-c}$ be the "antinorm" character on $E^{\times}$ that sends $x \mapsto \chi(x/\overline{x})$. We may also consider ξ as the restriction of χ^{-1} on E^1. Then ξ is a ring class character; it is trivial on $(\mathcal{O}_F + \varpi^o\mathcal{O}_E)^{\times}$ for some non-negative integer o, where ϖ is the uniformizer of E. The minimal such number is called the order $o(\xi)$ of ξ, and $c = c(\xi) := \varpi^{o(\xi)}$ is called the conductor of ξ (note the abuse of notation with the Galois conjugation c in the definition of ξ). We write

$$\mathcal{O}_c = \mathcal{O}_F + c\mathcal{O}_E,$$

for the associated order of E.

Let $\delta \in \mathcal{O}_c$ be a generator of the different ideal $\mathcal{D}$ of $\mathcal{O}_c$, namely the ideal generated by $x - \overline{x}$ for all $x \in \mathcal{O}_c$. Then for each $a \in \mathcal{O}_c/\delta$, we define the function Φ^{opt}_a to be the characteristic function of

$$\mathcal{O}_c + \frac{a}{\delta} \subset E.$$

Letting $\epsilon := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the optimal form is defined as follows.

Definition 5.7. We define the one-variable optimal function W^{opt}_a on $\mathrm{GL}_2(F)$, for $a \in \mathcal{O}_c/\delta$, and the two-variable optimal function W^{opt} on $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$ as follows,

$$\begin{aligned} W_a^{\text{opt}}(g) &:= W(g, \chi, \Phi_a^{\text{opt}}), \\ W^{\text{opt}}(g_1, g_2) &:= \sum_{a \in \mathcal{O}_c/\delta} W_a^{\text{opt}}(g_1) \otimes W_{-a}^{\text{opt}}(g_2 \epsilon). \end{aligned}$$

One of our main objectives is to study the period $\mathcal{P}_{\text{RS}}(W^{\text{opt}})$. First, we prove a non-vanishing result.

Proposition 5.8. *If χ is unitary, then*

$$W^{\text{opt}}(g_1, g_2) = \chi(-1) \sum_{a \in \mathcal{O}_c/\delta} W_a^{\text{opt}}(g_1) \otimes W_a^{\text{opt},\epsilon}(g_2).$$

Furthermore, $W_1^{\text{opt}} \neq 0$ and $\mathcal{P}_{\text{RS}}(W^{\text{opt}}) \in \mathbb{R}^{\times}$.

Proof. First, let us write the Kirillov functions for W^{opt}:

$$\kappa^{\text{opt}}(x_1, x_2) := \sum_{a \in \mathcal{O}_c/\delta} \kappa(x_1, \chi, \Phi_a^{\text{opt}}) \cdot \kappa\big(-x_2, \chi^{-1}, \Phi_{-a}^{\text{opt}}\big).$$

Since $\Phi_{-a}(x) = \Phi_a(-x)$, we can use Equation 5.5 with a change of variables $t \mapsto -t$ to get,

$$\kappa^{\text{opt}}(x_1, x_2) := \chi(-1) \sum_{a \in \mathcal{O}_c/\delta} \kappa(x_1, \chi, \Phi_a^{\text{opt}}) \cdot \kappa\big(-x_2, \chi^{-1}, \Phi_a^{\text{opt}}\big).$$

Since χ is unitary,

$$\kappa\big(-x_2, \chi^{-1}, \Phi_a^{\text{opt}}\big) = \overline{\kappa\big(-x_2, \chi, \Phi_a^{\text{opt}}\big)} = \kappa^{\epsilon}(x_2, \chi, \Phi_a),$$

where $\kappa^{\epsilon}(x, \chi_2, \Phi_a)$ is the Kirillov function associated to $W^{\epsilon}(g, \chi, \Phi_a)$ defined in Equation 5.3. Thus,

$$\kappa^{\text{opt}}(x_1, x_2) := \chi(-1) \sum_{a \in \mathcal{O}_c/\delta} \kappa(x_1, \chi, \Phi_a^{\text{opt}}) \cdot \kappa^{\epsilon}(x_2, \chi, \Phi_a^{\text{opt}}).$$

It follows that,

$$\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}}) = \chi(-1)\sum_{a\in\mathcal{O}_c/\delta}\mathcal{P}_{\mathrm{RS}}(W_a^{\mathrm{opt}}\otimes W_a^{\mathrm{opt},\epsilon}).$$

For the non-vanishing of W_1^{opt}, take $x = \mathrm{N}(\delta^{-1})$ in Equation 5.5 to obtain

$$\kappa_1^{\mathrm{opt}}(x) = \chi(\delta)^{-1}|\delta|^{-\frac{1}{2}}\int_{(1+\delta\mathcal{O}_c)^1}\xi(t) = \chi(\delta)^{-1}|\delta|^{-\frac{1}{2}}\mathrm{vol}\big((1+\delta\mathcal{O}_c)^1\big)\neq 0.$$

The last part of the proposition follows from the previous two parts and Proposition 5.2. $\square$

Comparison of models

We study the general quadratic space $V = (Ee, Q)$ as a linear space over E so that $\mathrm{GSO}(V) = E^\times$. Then $\mathcal{W}(\chi,\psi)$ can also be constructed by $\mathcal{S}(V)$. More precisely, by Equation 5.5, the Kirillov function associated with the theta series $\theta(g,\chi^c,\Phi)$ for each $\Phi\in\mathcal{S}(V(\mathbb{A}))$ is given by

$$\kappa(x,\chi^c,\Phi) = \eta_V(x)|x|^{\frac{1}{2}}\int_{E^1}\Phi(tt_0e)\chi(t_0t)dt,$$

where $x = Q(t_0e)$.

Let $V' = (Ee', Q')$ be another quadratic space and let $\iota : V' \xrightarrow{\sim} V$ be the isomorphism such that $\iota(e') = e$. Define $\iota^*\Phi := \Phi\circ\iota\in\mathcal{S}(V')$. Then we have

$$\begin{aligned}\kappa(x,\chi^c,\Phi) &= |x|^{\frac{1}{2}}\int_{E^1}\iota^*\Phi\big(tt_0\iota^{-1}(e)\big)\chi\big(t_0^{-1}t^{-1}\big)dt,\\ &= |x|^{\frac{1}{2}}\big|tt_0\iota^{-1}(e)\big|^{-\frac{1}{2}}\kappa\big(Q'\big(t_0\iota^{-1}(e)\big),\chi^c,\iota^*\Phi\big).\end{aligned}$$

Write $Q(\iota) := Q(e)/Q'(\iota^{-1}e)\in F^\times$. Then the above formula gives:

$$\kappa(x,\chi^c,\Phi) = |Q(\iota)|^{\frac{1}{2}}\kappa\big(xQ(\iota)^{-1},\chi^c,\iota^*\Phi\big). \tag{5.6}$$

For example, if we compare the Whittaker functions defined by two opposite spaces $V_\pm := (E,\pm\mathrm{N})$, we get two Whittaker functions $W_\pm(g,\chi^c,\Phi)$ for $\Phi\in\mathcal{S}(E)$. We use the identity map

$\iota : V_+ \longrightarrow V_-$ for the quadratic space, so $Q(\iota) = -1$. Then

$$W_-(g, \chi^c, \Phi) = W_+(g\epsilon, \chi^c, \Phi),$$

where $\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Then from Definition 5.7, we can instead write

$$W^{\mathrm{opt}}(g_1, g_2) = \sum_{a \in \mathcal{O}_c/\delta} W_+(g_1, \chi, \Phi_a^{\mathrm{opt}}) \cdot W_-\big(g_2, \chi^{-1}, \Phi_{-a}^{\mathrm{opt}}\big). \tag{5.7}$$

As in Section 1.4, we call W^{opt} an optimal form due to the connection with optimal orders and optimal embeddings (cf. Remark 1.4). We identify $B = \mathrm{End}_F(E)$ as usual and define an optimal order $\mathcal{O}_B^{\mathrm{opt}} = \mathrm{End}_{\mathcal{O}_F}(\mathcal{O}_c)$. Let Φ^{opt} be the characteristic function of $\mathcal{O}_B^{\mathrm{opt}}$.

Lemma 5.9. *We have the following identity in $\mathcal{S}(B) = \mathcal{S}(V_+) \otimes \mathcal{S}(V_-)$,*

$$\Phi^{\mathrm{opt}} = \sum_a \Phi_a^{\mathrm{opt}} \otimes \Phi_{-a}^{\mathrm{opt}}.$$

Proof. First, let us describe $\mathcal{O}_B^{\mathrm{opt}}$ precisely. An element $x + yj \in B$ is in $\mathcal{O}_B$ if and only if

$$(x + yj)(1) = x + y \in \mathcal{O}_c,$$
$$(x + yj)(\varpi) = x\varpi + y\overline{\varpi} \in \mathcal{O}_c.$$

These conditions mean that $x + y \in \mathcal{O}_c$ and $x, y \in \mathcal{O}_c/(\varpi - \overline{\varpi})$. So we have that

$$\mathcal{O}_B^{\mathrm{opt}} = \mathcal{O}_c + \mathcal{O}_c \frac{1 - j}{\varpi - \overline{\varpi}}.$$

Let $\delta = \varpi - \overline{\varpi}$ be a generator of the different ideal of $\mathcal{O}_c$. Concretely, if E/F is a field extension, then δ is a generator of the different ideal multiplied by $\varpi^{o(\xi)}$. If $E = F \oplus F$, then δ equals $(1, -1)$ multiplied by $\varpi^{o(\xi)}$.

From the above description, it is clear that $\mathcal{O}_B^{\mathrm{opt}}$, as a subset of B, is the disjoint union of the product,

$$(\mathcal{O}_c + a/\delta) \times (\mathcal{O}_c - a/\delta)j.$$

The lemma then follows. $\square$

6 Rankin–Selberg periods of optimal forms

6.1 A formula for the Rankin–Selberg zeta integral

Let E/F be a semisimple algebra of degree 2 and χ be a character of $E^\times$. We have constructed Whittaker models $\mathcal{W}(\chi^\pm)$ for $\mathrm{GL}_2(F)^+$ via theta liftings on quadratic spaces $V_1 = (E, N)$ and $V_2 = (E, -N)$. More precisely for functions $\Phi_i \in \mathcal{S}(V_i)$, we obtained functions (cf. Equation 5.4),

$$\begin{aligned}
W_1(g) &= W(g, \chi_1, \nu_1, \Phi_1) \\
&= |\det g|^{-\frac{1}{2}} \int_{E^1} r(g_1)\Phi_1\big(t_0^{-1}t_1^{-1}\big)\chi^c(t_0t_1)dt_1, \\
W_2(\epsilon' g) &= W(\epsilon' g, \chi_2, \nu_2, \Phi_2) \\
&= |\det g|^{-\frac{1}{2}} \int_{E^1} r(g_1)\Phi_2\big(t_0^{-1}t_2^{-1}\big)\chi^{-c}(t_0t_2)dt_2,
\end{aligned}$$

where $t_0 \in E^\times$, $g_1 \in \mathrm{SL}_2(F)$ such that $g = d(N(t_0))g_1$, and $\epsilon' g = d(-N(t_0))g_1$.

With the above, we compute the zeta integral from Equation 5.1 to obtain,

$$\begin{aligned}
Z(s, W_1, W_2) &= \int_{N(F)Z(F)\backslash \mathrm{GL}_2(F)^+} W_1(g)W_2(\epsilon' g)f(g,s)dg \\
&= \int_{N(F)Z(F)\backslash \mathrm{GL}_2(F)^+} |\det g|^{-1} f(g,s)dg \\
&\quad \cdot \int_{E^1\times E^1} r(g_1)\Phi\big(t_0^{-1}\big(t_1^{-1} + t_2^{-1}j\big)\big)\chi^c\left(\frac{t_1}{t_2}\right)dt_1dt_2.
\end{aligned}$$

To simplify this integral, we set

$$\widetilde{\Phi}(x) = \int_{SL_2(\mathcal{O}_F)} r(k)\Phi(x)dk,$$

where the volume form is taken to be one. Then the integral expression of $Z(s, W_1, W_2)$ becomes,

$$\int_{N(F)Z(F)\backslash GL_2(F)^+/SL_2(\mathcal{O}_F)} |\det g|^{-1} f(g,s)dg \int_{E^1\times E^1} r(g_1)\widetilde{\Phi}\big(t_0^{-1}\big(t_1^{-1}+t_2^{-1}j\big)\big)\chi^c\left(\frac{t_1}{t_2}\right)dt_1dt_2.$$

Using the Iwasawa decomposition,

$$GL_2(F)^+ = N(F)Z(F)d\big(N\big(E^\times\big)\big)SL_2(\mathcal{O}_F),$$

the first integral becomes, $\int_{N(E^\times)}|h|^{-1}|h|^{-s}|h|dh$. Noting that $T = E^\times \times_{F^\times} E^\times \subset GO(V)$ and setting $\sigma(t) = \sigma(t_1,t_2) = \chi^c(t_2/t_1)$, we may rewrite the expression for $Z(s, W_1, W_2)$ as

$$\int_T |\nu(t)|^s \widetilde{\Phi}(t(1+j))\sigma(t)dt.$$

For the purposes of calculation, we can take a model of T,

$$\begin{aligned} E^\times \times E^1 &\xrightarrow{\sim} T, \\ (t_1,t_2) &\longmapsto (t_1, t_1t_2). \end{aligned}$$

In summary, we have demonstrated the following expression for our Rankin–Selberg zeta integral. Recall that $\xi = \chi^{1-c}$ can be viewed as the restriction $\chi^{-1}\big|_{E^1}$.

Proposition 6.1. *Let $W \in \mathcal{W}(\chi,\psi)\otimes\mathcal{W}(\chi^{-1},\psi)$ and let $\Phi \in \mathcal{S}(B)$. Then*

$$Z(s,W) = Z(s,\Phi) := \int_{E^\times} |t_1|^s \int_{E^1} \widetilde{\Phi}(t_1(1+t_2j))\xi(t_2)dt_2dt_1.$$

Corollary 6.2. *Let* $\mathbb{Q}(\xi, \widetilde{\Phi})$ *be the subfield of* $\mathbb{C}$ *generated by the values of* ξ *and* $\widetilde{\Phi}$*. Then,*

$$\mathcal{P}_{\mathrm{RS}}(W) \in \mathbb{Q}\left(\xi, \widetilde{\Phi}\right).$$

The remainder of Section 6 is a calculation of the zeta integral $Z(s, W^{\mathrm{opt}})$ for the optimal form W^{opt} defined by Equation 5.7. This is eventually used to prove Theorem 8. By Lemma 5.9, W^{opt} is defined by an *optimal function* Φ^{opt} with respect to ξ, which is the characteristic function of,

$$\mathcal{O}_B^{\mathrm{opt}} = \mathcal{O}_c + \mathcal{O}_c \frac{1-j}{\varpi - \overline{\varpi}},$$

where ϖ is the uniformizer of E. Note that $\widetilde{\Phi}^{\mathrm{opt}} = \Phi^{\mathrm{opt}}$, and that $t_1 + t_2 j \in \mathcal{O}_B$ if and only if

$$t_1 \in \delta^{-1}\mathcal{O}_c,$$
$$t_2 \in -1 + t_1^{-1}\mathcal{O}_c.$$

Thus we have the following expression.

Proposition 6.3.

$$Z(s, W^{\mathrm{opt}}) = Z(s, \Phi^{\mathrm{opt}}) = \chi(-1) \int_{\delta^{-1}\mathcal{O}_c} |t_1|^s dt_1 \int_{\left(1+t_1^{-1}\mathcal{O}_c\right)^1} \xi(t_2) dt_2.$$

Combining Propositions 6.3 and 5.8, we obtain the following rationality statement.

Corollary 6.4. *Let* $\mathbb{Q}(\xi + \xi^{-1})$ *denote the ring generated by values of* $\xi + \xi^{-1}$*. Then,*

$$\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}}) \in \mathbb{Q}\left(\xi + \xi^{-1}\right).$$

Furthermore, if ξ *is unitary, then,*

$$\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}}) \in \mathbb{Q}\left(\xi + \xi^{-1}\right)^{\times}.$$

We divide our remaining calculations of $Z(s, W^{\text{opt}})$ and $\mathcal{P}_{\text{RS}}(W^{\text{opt}})$ into three cases, when ξ is: unramified; ramified and E/F is inert; ramified and E/F is split.

6.2 Unramified calculation

In this subsection, we calculate $\mathcal{P}_{\text{RS}}(W^{\text{opt}})$ when ξ is unramified (i.e. as a character from $E^\times$ to $\mathbb{C}^\times$, ξ can be factored as $\omega \circ \mathrm{N}$). Recall that $\pi = \pi(\chi)$. Denote the uniformizers of $\mathcal{O}_F$ and $\mathcal{O}_E$ by ϖ_F and ϖ_E respectively.

Proposition 6.5. *Assume that $F/\mathbb{Q}_p$ is unramified if $p = 2$. If ξ is unramified, then,*

$$Z(s, W^{\text{opt}}) = \begin{cases} (1+q^{-s})L\big(s, \mathrm{Ad}(\pi)\big) & \text{if } E/F \text{ is not ramified,} \\ \frac{q^s(1+q^{-s})}{2(1-q^{-s})} & \text{if } E/F \text{ is ramified and } p \neq 2, \\ \frac{q^{2s}(1+q^{-s})}{2(1-q^{-s})} & \text{if } E/F \text{ is ramified and } p = 2. \end{cases}$$

In particular,

$$\mathcal{P}_{\text{RS}}(W^{\text{opt}}) = Z(1, W^{\text{opt}}) = \begin{cases} \big(1+q^{-1}\big)L\big(1, \mathrm{Ad}(\pi)\big) & \text{if } E/F \text{ is not ramified,} \\ \frac{q\left(1+q^{-1}\right)}{2(1-q^{-1})} & \text{if } E/F \text{ is ramified and } p \neq 2, \\ \frac{q^2\left(1+q^{-1}\right)}{2(1-q^{-1})} & \text{if } E/F \text{ is ramified and } p = 2. \end{cases}$$

Proof. First, consider the case that E/F is inert. Then $\xi = 1$ and δ is invertible. Let η be the quadratic character associated to E/F. The integral reduces to

$$Z(s, \Phi^{\text{opt}}) = \int_{|t_1| \leq 1} |t_1|^s \, dt_1 = \zeta_E(s) = \zeta(s) L(s, \eta).$$

Then $L(s, \mathrm{Ad}(\pi)) = L(s, \eta)^2 \zeta(s)$, and

$$\Psi(s, \Phi^{\text{opt}}) = 1,$$

(recall the definition of the normalized zeta integral in Equation 5.2).

Second, we assume that E/F is split: $E = F \oplus F$. Use coordinates (u_1, u_2) for t_1 and (v, v^{-1}) for t_2. After composition with a character $\omega \circ N$, we may assume that $\chi = (\mu, 1)$ with μ unramified. In this case, δ is still invertible. Thus we have,

$$\begin{aligned}
Z(s, \Phi^{\text{opt}}) &= \int_{\substack{|u_1|\le 1\\ |u_2|\le 1}} |u_1 u_2|^s \int_{|u_2|\le|v|\le|u_1|^{-1}} \mu(v)\,dv\,du_1\,du_2 \\
&= \sum_{m,n\ge 0} q^{-s(m+n)} \sum_{-m\le \ell\le n} \mu(\varpi_F^{\ell}) \\
&= \frac{1}{1-\mu(\varpi_F)} \sum_{m,n\ge 0} q^{-s(m+n)}\big(\mu(\varpi_F)^{-m} - \mu(\varpi_F)^{n+1}\big) \\
&= \frac{1}{\big(1-\mu(\varpi_F)\big)(1-q^{-s})}\left(\frac{1}{1-\mu(\varpi_F)^{-1}q^{-s}} - \frac{\mu(\varpi_F)}{1-\mu(\varpi_F)q^{-s}}\right) \\
&= \frac{1+q^{-s}}{(1-q^{-s})(1-\mu(\varpi_F)q^{-s})(1-\mu(\varpi_F)^{-1}q^{-s})} \\
&= (1+q^{-s})\zeta(s)L(s,\mu)L(s,\mu^{-1})
\end{aligned}$$

With our definition, $L(s, \mathrm{Ad}(\pi)) = \zeta(s)L(s,\mu)L(s,\mu^{-1})$, so,

$$\Psi(s, \Phi^{\text{opt}}) = 1.$$

Finally, consider the case when E/F is ramified. Again, $\xi = \chi|_{E^1} = 1$. Then we have that

$$Z(s, \Phi^{\text{opt}}) = \sum_{n\ge -\text{ord}(\delta)} q^{-ns}\,\text{vol}\big((1+\varpi_E^{-n}\mathcal{O}_E)^1\big).$$

We can compute the volume case-by-case. First, if $n \ge 0$, then $\text{vol}((1+\varpi_E^{-n}\mathcal{O}_E)^1) = 1$. Second, if $n = -1$, then E^1 is the union of $\pm(1+\varpi_E\mathcal{O}_E)^1$. In this case,

$$\text{vol}\big((1+\varpi_E^{1}\mathcal{O}_E)^1\big) = \begin{cases} \frac{1}{2} & \text{if } p \ne 2, \\ 1 & \text{if } p = 2. \end{cases}$$

Finally, we treat the case $n < -1$. If $p = 2$ (and F is unramified over $\mathbb{Q}_2$ by assumption), then $\mathrm{ord}(\delta) = 2$. In this case, $E^1 \pmod{\varpi_E}$ is generated by 1 and $1 + \varpi_E$. Thus $\mathrm{vol}((1 + \varpi_E^2 \mathcal{O}_E)^1) = 1/2$.

Collecting the terms for all n, we have the following formula for $Z(s, \Phi^{\mathrm{opt}})$ when E/F is ramified. If $p \neq 2$, then,

$$\begin{aligned} Z(s, \Phi^{\mathrm{opt}}) &= \frac{1}{2} q^s + \sum_{n \geq 0} q^{-ns} \\ &= \frac{q^s(1 + q^{-s})}{2(1 - q^{-s})}. \end{aligned}$$

If $p = 2$, then,

$$\begin{aligned} Z(s, \Phi^{\mathrm{opt}}) &= \frac{1}{2} q^{2s} + q^s + \sum_{n \geq 0} q^{-ns} \\ &= \frac{q^{2s}(1 + q^{-s})}{2(1 - q^{-s})}. \end{aligned}$$

$\square$

6.3 Ramified calculation: E/F inert

In this subsection, we calculate $\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}})$ when ξ is ramified and E/F is inert.

Proposition 6.6. *If ξ is ramified and E/F is inert with $p \neq 2$, then,*

$$Z(s, \Phi^{\mathrm{opt}}) = \chi(-1)\left(1 + q^{-1}\right)^{-2} q^{2o(\xi)(s-1)}.$$

In particular,

$$\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}}) = Z(1, \Phi^{\mathrm{opt}}) = \chi(-1)\left(1 + q^{-1}\right)^{-2}.$$

Proof. Recall that we have,

$$Z(s,\Phi^{\mathrm{opt}}) = \chi(-1)\int_{\delta^{-1}\mathcal{O}_c} |t_1|^s \int_{\left(1+t_1^{-1}\mathcal{O}_c\right)^1} \xi(t_2)dt_2dt_1.$$

Here, $c = c(\xi) = \varpi^{o(\xi)}$ is the conductor of ξ with associated order $\mathcal{O}_c = \mathcal{O}_{o(\xi)}$, where ϖ is the uniformizer of E and,

$$\mathcal{O}_k := \mathcal{O}_F + \varpi^k\mathcal{O}_F\epsilon.$$

We can write $\mathcal{O}_E = \mathcal{O}_F + \mathcal{O}_F\epsilon$ with $\epsilon^2 \in \mathcal{O}_F^\times$. Then $\delta = 2\varpi^{o(\xi)}\epsilon$, so,

$$\delta^{-1}\mathcal{O}_c = \mathcal{O}_{-c} := \mathcal{O}_{-o(\xi)}.$$

Consider the $t_1 \in \mathcal{O}_E$. Then the double integral in the expression for $Z(s,\Phi^{\mathrm{opt}})$ becomes,

$$\begin{aligned}\sum_{n\geq 0} q^{-2ns}\int_{\mathcal{O}_E^\times}\int_{(1+\varpi^{-n}u\mathcal{O}_c)^1}\xi(t_2)dt_2du &= \sum_{n\geq 0} q^{-2ns}\mathrm{vol}\left(\mathcal{O}_c^\times\right)\sum_{u\in\mathcal{O}_E^\times/\mathcal{O}_c^\times}\int_{(1+\varpi^{-n}u\mathcal{O}_c)^1}\xi(t_2)dt_2\\ &= \mathrm{vol}\left(\mathcal{O}_c^\times\right)\sum_{n\geq 0} q^{-2ns}\int_{\left(1+\varpi^{-n}\mathcal{O}_E^\times\mathcal{O}_c\right)^1}\xi(t_2)dt_2\end{aligned}$$

Notice that $\mathcal{O}_E^\times\mathcal{O}_c = \mathcal{O}_E$. Thus $1+\varpi_E^{-n}\mathcal{O}_E^\times\mathcal{O}_c = 1+\varpi_E^{-n}\mathcal{O}_E \supset E^1$, and the last integral vanishes. So there is no contribution to $Z(s,\Phi^{\mathrm{opt}})$ from $t \in \mathcal{O}_E$.

Now consider the remaining contribution to $Z(s,W^{\mathrm{opt}})$ from $t_1 \notin \mathcal{O}_E$. Then we can write $t_1 = \varpi^{-i}\epsilon u$ with $u \in \mathcal{O}_k^\times$ for $1 \leq k \leq o(\xi)$. The remaining double integrals become

$$\begin{aligned}Z(s,\Phi^{\mathrm{opt}}) &= \chi(-1)\sum_{k=1}^{o(\xi)} q^{2ks}\int_{u\in\mathcal{O}_k^\times} du\int_{(1+\varpi^k\epsilon u\mathcal{O}_c)^1}\xi(t_2)dt_2 \qquad (6.1)\\ &= \chi(-1)\mathrm{vol}\left(\mathcal{O}_c^\times\right)\sum_{k=1}^{o(\xi)} q^{2ks}\int_{\left(1+\varpi^k\epsilon\mathcal{O}_k^\times\mathcal{O}_c\right)^1}\xi(t_2)dt_2.\end{aligned}$$

To further compute the remaining integral, we use the decomposition

$$\mathcal{O}_c = \mathcal{O}_F + \varpi^{o(\xi)}\mathcal{O}_E = \varpi^{o(\xi)}\mathcal{O}_E \cup \bigcup_{i=1}^{o(\xi)} \varpi^{o(\xi)-i}\mathcal{O}_i^\times.$$

Then,

$$\begin{aligned}
\mathcal{O}_k^\times \mathcal{O}_c &= \varpi^{o(\xi)}\mathcal{O}_E \cup \bigcup_{i=1}^{k} \varpi^{o(\xi)-i}\mathcal{O}_i^\times \cup \bigcup_{j=k+1}^{o(\xi)} \varpi^{o(\xi)-j}\mathcal{O}_k^\times \\
&= \varpi^{o(\xi)-k}\mathcal{O}_k \cup \bigcup_{i=0}^{o(\xi)-k-1} \left(\varpi^i\mathcal{O}_k - \varpi^{i+1}\mathcal{O}_{k-1}\right).
\end{aligned}$$

For a set X, let $\mathbb{1}_X$ denote its characteristic function. We have,

$$\begin{aligned}
\mathbb{1}_{\left(1+\varpi^k\epsilon\mathcal{O}_k^\times\mathcal{O}_c\right)^1} &= \mathbb{1}_{\left(1+\varpi^{o(\xi)}\epsilon\mathcal{O}_k\right)^1} + \sum_{i=0}^{o(\xi)-k-1} \mathbb{1}_{(1+\varpi^{k+i}\epsilon\mathcal{O}_k)^1} + \sum_{j=0}^{o(\xi)-k-1} \mathbb{1}_{(1+\varpi^{k+j+1}\epsilon\mathcal{O}_{k-1})^1} \\
&= \sum_{i=k}^{o(\xi)} \mathbb{1}_{(1+\varpi^i\epsilon\mathcal{O}_k)^1} + \sum_{j=k+1}^{o(\xi)} \mathbb{1}_{(1+\varpi^j\epsilon\mathcal{O}_{k-1})^1}
\end{aligned}$$

In particular, the integral becomes,

$$\int_{\left(1+\varpi^k\epsilon\mathcal{O}_k^\times\mathcal{O}_c\right)^1} \xi(t_2)dt_2 = \sum_{i=k}^{o(\xi)} \int_{(1+\varpi^i\epsilon\mathcal{O}_k)^1} \xi(t_2)dt_2 - \sum_{j=k+1}^{o(\xi)} \int_{(1+\varpi^j\epsilon\mathcal{O}_{k-1})^1} \xi(t_2)dt_2.$$

Next, we apply the following lemma.

Lemma 6.7. *For any integer* $i \geq 0$*,*

$$\left(1+\varpi^i\mathcal{O}_E\right)^1 = \left(1+\varpi^i\epsilon\mathcal{O}_i\right)^1.$$

Proof of Lemma 6.7. This is trivial when $i = 0$. Assume that $i > 0$ and let $x \in (1+\varpi^i\mathcal{O}_E)^1$. Then we can write for $\alpha, \beta \in \mathcal{O}_F$,

$$x = 1 + \varpi^i\alpha + \varpi^i\beta\epsilon.$$

Factor out $1+\varpi^i\alpha$ to obtain for $\gamma \in \mathcal{O}_F$,

$$x = \left(1+\varpi^i\alpha\right)\frac{1+\varpi^i\alpha+\varpi^i\beta\epsilon}{1+\varpi^i\alpha} = \left(1+\varpi^i\alpha\right)\left(1+\varpi^i\gamma\epsilon\right),$$

for some $\gamma \in \mathcal{O}_F$. Now we take the norm $N_{E/F}$ of both sides, noting that $1+\varpi^i\alpha \in F$ and the conjugate of $1+\varpi^i\gamma\epsilon$ is $1-\varpi^i\gamma\epsilon$,

$$1 = \left(1+2\varpi^i\alpha+\varpi^{2i}\alpha^2\right)\left(1-\varpi^{2i}\gamma^2\epsilon^2\right).$$

Since 2 is invertible in $\mathcal{O}_F$, $\alpha \in \varpi^i\mathcal{O}_F$. This shows that $x \in 1+\varpi^i\epsilon\mathcal{O}_i$. $\square$

By Lemma 6.7, we may replace $(1+\varpi^i\epsilon\mathcal{O}_k)^1$ and $(1+\varpi^j\epsilon\mathcal{O}_{k-1})^1$ in the integrals by $(1+\varpi^i\mathcal{O}_E)^1$ and $(1+\varpi^j\mathcal{O}_E)^1$. Since $o(\xi)$ is the order of ξ, we need only consider $k \geq o(\xi)$ (the $k < c$ integral terms vanish) in the sum,

$$\sum_{k=1}^{o(\xi)} q^{2ks}\int_{\left(1+\varpi^k\epsilon\mathcal{O}_k^\times\mathcal{O}_c\right)^1}\xi(t_2)\,dt_2.$$

For $k = o(\xi)$, the integral is given by vol$(1+\varpi^{o(\xi)}\mathcal{O}_E)^1$. Consequently,

$$Z(s,\Phi) = \chi(-1)\mathrm{vol}\left(\mathcal{O}_c^\times\right)q^{2o(\xi)s}\mathrm{vol}\left(\left(1+\varpi^{o(\xi)}\mathcal{O}_E\right)^1\right).$$

To compute the volume of $\mathcal{O}_E^\times/\mathcal{O}_c^\times$, we observe that $\mathcal{O}_E^\times$ and $\mathcal{O}_c^\times$ both contain $1+\varpi^{o(\xi)}\mathcal{O}_E$, factor both the top and bottom, and then find the cardinality,

$$\begin{aligned}\mathrm{vol}\left(\mathcal{O}_E^\times/\mathcal{O}_c^\times\right) &= \mathrm{vol}\left(\left(\mathcal{O}_E/\varpi^{o(\xi)}\right)^\times/\left(\mathcal{O}_F/\varpi^{o(\xi)}\right)^\times\right)\\ &= \frac{\left(q^2-1\right)q^{2o(\xi)-2}}{(q-1)q^{o(\xi)-1}}\\ &= \left(1+q^{-1}\right)q^{o(\xi)}.\end{aligned}$$

We find the volume of $\mathcal{O}_E^1/\big(1+\varpi^{o(\xi)}\mathcal{O}_E\big)^1$ in a similar manner.

$$\begin{aligned}\mathrm{vol}\big(\mathcal{O}_E^1/\big(1+\varpi^{o(\xi)}\mathcal{O}_E\big)^1\big) &= \mathrm{vol}\big(\big(\mathcal{O}_E/\varpi^{o(\xi)}\big)^1\big)\\ &= \mathrm{vol}\big(\big(\mathcal{O}_E/\varpi^{o(\xi)}\big)^\times/\big(\mathcal{O}_F/\varpi^{o(\xi)}\big)^\times\big)\\ &= \big(1+q^{-1}\big)q^{o(\xi)}.\end{aligned}$$

With our normalization such that $\mathrm{vol}(\mathcal{O}_E^\times)=1$ and $\mathrm{vol}(\mathcal{O}_E^1)=1$, we have that,

$$Z(s,\Phi)=\chi(-1)\big(1+q^{-1}\big)^{-2}q^{2o(\xi)(s-1)}.$$

$\square$

6.4 Ramified calculation: E/F split

In this subsection, we calculate $\mathcal{P}_{RS}(W^{opt})$ when ξ is ramified and E/F is split.

Proposition 6.8. *If ξ is ramified and E/F is split with $p\neq 2$, then,*

$$Z(s,\Phi^{opt})=\chi(-1)\big(1-q^{-1}\big)^{-2}\left(\frac{2q^{-2o(\xi)-s}}{1-q^{-s}}+q^{2o(\xi)(s-1)}\right).$$

In particular,

$$\mathcal{P}_{RS}(W^{opt})=Z(1,W^{opt})=\xi(-1)\big(1-q^{-1}\big)^{-3}\big(1-q^{-1}+2q^{-2o(\xi)-1}\big).$$

Proof. Recall that we have,

$$Z(s,\Phi^{opt})=\chi(-1)\int_{\delta^{-1}\mathcal{O}_c}|t_1|^s\int_{\big(1+t_1^{-1}\mathcal{O}_c\big)^1}\xi(t_2)dt_2dt_1.$$

We have $\mathcal{O}_c=\mathcal{O}_F+\varpi^{o(\xi)}\mathcal{O}_F\epsilon$, $\delta=\varpi^{o(\xi)}\epsilon$, and,

$$\delta^{-1}\mathcal{O}_c=\mathcal{O}_{-c}=\mathcal{O}_F+\varpi^{-o(\xi)}\epsilon\mathcal{O}_F.$$

Let us first consider the contribution to $Z(s, \Phi^{\mathrm{opt}})$ from the integral over $t_1 \in \mathcal{O}_E$,

$$\chi(-1)\mathrm{vol}\big(\mathcal{O}_c^\times\big) \sum_{m,n\geq 0} q^{-(m+n)s} \int_{\left(1+(\varpi^{-m},\varpi^{-n})\mathcal{O}_E^\times\mathcal{O}_c\right)^1} \xi(t_2)\,dt_2.$$

We can decompose $\mathcal{O}_c$ as

$$\mathcal{O}_c = \mathcal{O}_F + \varpi^{o(\xi)}\mathcal{O}_E = \varpi^{o(\xi)}\mathcal{O}_E \cup \bigcup_{k=0}^{o(\xi)-1} \varpi^k \mathcal{O}_{o(\xi)-k}^\times.$$

For a set X, let $\mathbb{1}_X$ denote its characteristic function. Then

$$\begin{aligned}
\mathbb{1}_{\mathcal{O}_E^\times\mathcal{O}_c} &= \mathbb{1}_{\varpi^{o(\xi)}\mathcal{O}_E} + \sum_{k=0}^{o(\xi)-1} \mathbb{1}_{\varpi^k\mathcal{O}_E^\times} \\
&= \mathbb{1}_{\varpi^{o(\xi)}\mathcal{O}_E} + \sum_{k=0}^{o(\xi)-1} \left(\mathbb{1}_{\varpi^k\mathcal{O}_E} - \mathbb{1}_{(\varpi^{i+1},\varpi^k)\mathcal{O}_E} - \mathbb{1}_{(\varpi^k,\varpi^{k+1})\mathcal{O}_E} + \mathbb{1}_{(\varpi^{k+1},\varpi^{k+1})\mathcal{O}_E}\right) \\
&= \mathbb{1}_{\mathcal{O}_E} + 2\sum_{i=1}^{o(\xi)} \mathbb{1}_{\varpi^i\mathcal{O}_E} - \sum_{j=0}^{o(\xi)-1} \left(\mathbb{1}_{(\varpi^{j+1},\varpi^j)\mathcal{O}_E} + \mathbb{1}_{(\varpi^j,\varpi^{j+1})\mathcal{O}_E}\right).
\end{aligned}$$

In particular, the integral becomes,

$$\begin{aligned}
\int_{\left(1+(\varpi^{-m},\varpi^{-n})\mathcal{O}_E^\times\mathcal{O}_c\right)^1} \xi(t_2)\,dt_2 = \int_{(\varpi^{-m},\varpi^{-n})\mathcal{O}_E} \xi(t_2)\,dt_2 + 2\sum_{i=1}^{o(\xi)} \int_{(\varpi^{i-m},\varpi^{i-n})\mathcal{O}_E} \xi(t_2)\,dt_2 \qquad (6.2)\\
- \sum_{j=0}^{o(\xi)-1} \left(\int_{(\varpi^{j+1-m},\varpi^{j-n})\mathcal{O}_E} \xi(t_2)\,dt_2 + \int_{(\varpi^{j-m},\varpi^{j+1-n})\mathcal{O}_E} \xi(t_2)\,dt_2\right).
\end{aligned}$$

So we need to compute, for integers a and b,

$$\int_{(1+(\varpi^a,\varpi^b)\mathcal{O}_E)^1} \xi(t)\,dt.$$

Lemma 6.9.

$$\int_{(1+(\varpi^a,\varpi^b)\mathcal{O}_E)^1} \xi(t)dt = \begin{cases} q^{-\max(a,b)}(1-q^{-1})^{-1} & \textit{if } \max(a,b) \geq o(\xi), \\ 0 & \textit{otherwise}. \end{cases}$$

Proof of Lemma 6.9. Using the coordinates (t, t^{-1}), the integral is then over t such that,

$$t = 1 + \varpi^a x,$$
$$t^{-1} = 1 + \varpi^b y,$$

for $x, y \in \mathcal{O}_F$.

If $\max(a,b) \leq 0$, then these conditions become,

$$|\varpi|^{-b} \leq |t| \leq |\varpi|^a.$$

Hence the set of such t is stable under multiplication by $\mathcal{O}_F^\times$. The integral vanishes in this case since χ is ramified.

If $\max(a,b) > 0$, then the integral is given by,

$$\int_{\left(1+\varpi^{\max(a,b)}\mathcal{O}_E\right)^1} \xi(t)dt = \begin{cases} \mathrm{vol}\left(1+\varpi^{\max(a,b)}\mathcal{O}_F\right) & \text{if } \max(a,b) \geq o(\xi), \\ 0 & \text{if } 0 < \max(a,b) \leq 0. \end{cases}$$

Evaluating the volume finishes the lemma. $\square$

By Lemma 6.9, the terms of Equation 6.2 either individually vanish or cancel completely with each other, unless either $m = 0$ and $n > 0$ or $m > 0$ and $n = 0$. In particular,

$$\int_{\left(1+(\varpi^{-m},\varpi^{-n})\mathcal{O}_E^\times\mathcal{O}_c\right)^1} \xi(t_2)dt_2 = \begin{cases} q^{-o(\xi)}(1-q^{-1})^{-1} & \text{if } mn = 0 \text{ and } m+n > 0, \\ 0 & \text{otherwise}. \end{cases}$$

Then the sum over all such m and n is twice the geometric series in q^{-s} from fixing m or n to be zero,

$$\sum_{m,n\geq 0} q^{-(m+n)s} \int_{\left(1+(\varpi^{-m},\varpi^{-n})\mathcal{O}_E^\times \mathcal{O}_c\right)^1} \xi(t_2)dt_2 = 2\frac{q^{-s}}{1-q^{-s}} q^{-o(\xi)}\left(1-q^{-1}\right)^{-1}.$$

The remaining calculation in the $t_1 \in \mathcal{O}_E$ case is $\mathrm{vol}(\mathcal{O}_c^\times)$, which was done in the inert case (cf. proof of Proposition 6.6),

$$\begin{aligned}\mathrm{vol}\left(\mathcal{O}_E^\times/\mathcal{O}_c^\times\right) &= \mathrm{vol}\left(\left(\mathcal{O}_E/\varpi^{o(\xi)}\right)^\times/\left(\mathcal{O}_F/\varpi^{o(\xi)}\right)^\times\right)\\ &= \frac{\left(q^2-1\right)q^{2o(\xi)-2}}{(q-1)q^{o(\xi)-1}}\\ &= \left(1+q^{-1}\right)q^{o(\xi)}.\end{aligned}$$

Therefore the contribution from $t_1 \in \mathcal{O}_E$ to $Z(s, W^{\mathrm{opt}})$ is,

$$\begin{aligned}\chi(-1)\mathrm{vol}\left(\mathcal{O}_{o(\xi)}^\times\right) \sum_{m,n\geq 0} q^{-(m+n)s} \int_{\left(1+(\varpi^{-m},\varpi^{-n})\mathcal{O}_E^\times \mathcal{O}_c\right)^1} \xi(t_2)dt_2&\\ = 2\chi(-1)\left(1-q^{-1}\right)^{-2}\frac{q^{-2o(\xi)-s}}{1-q^{-s}}&.\end{aligned}$$

Now we consider the remaining contribution to $Z(s, W^{\mathrm{opt}})$ from $t_1 \notin \mathcal{O}_E$. We use,

$$\mathcal{O}_c = \varpi^{o(\xi)}\mathcal{O}_E \cup \bigcup_{k=1}^{o(\xi)} \varpi^{o(\xi)-k}\mathcal{O}_k^\times,$$

to get,

$$\delta^{-1}\mathcal{O}_c - \mathcal{O}_E = \bigcup_{k=1}^{o(\xi)} \varpi^{-k}\epsilon\mathcal{O}_k^\times.$$

Then we have that the $t_1 \notin \mathcal{O}_E$ contribution to $Z(s, W^{\mathrm{opt}})$ is,

$$\chi(-1)\int_{\delta^{-1}\mathcal{O}_c-\mathcal{O}_E} |t_1|^s \int_{\left(1+t_1^{-1}\mathcal{O}_c\right)^1} \xi(t_2)dt_2dt_1 = \chi(-1)\sum_{k=1}^{o(\xi)} q^{2ks}\mathrm{vol}\left(\mathcal{O}_c^\times\right)\int_{\left(1+\varpi^k\epsilon\mathcal{O}_k^\times\mathcal{O}_c\right)^1} \xi(t_2)dt_2.$$

This is the same expression from Equation 6.1 that we calculated in the proof of Proposition 6.6,

$$\chi(-1)\sum_{k=1}^{o(\xi)} q^{2ks}\mathrm{vol}(\mathcal{O}_c^\times)\int_{(1+\varpi^k\epsilon\mathcal{O}_k^\times\mathcal{O}_c)^1}\xi(t_2)dt_2 = \chi(-1)(1-q^{-1})^{-2}q^{2o(\xi)(s-1)}$$

Combining the contributions from $t_1 \in \mathcal{O}_E$ and $t_1 \in \delta^{-1}\mathcal{O}_c - \mathcal{O}_E$, we have shown that,

$$Z(s,\Phi^{\mathrm{opt}}) = \chi(-1)(1-q^{-1})^{-2}\left(\frac{2q^{-2o(\xi)-s}}{1-q^{-s}} + q^{2o(\xi)(s-1)}\right).$$

□

7 Rankin–Selberg periods of newforms

In the following, we calculate the zeta integral and Rankin–Selberg period for newforms, and in particular compare them with the zeta integrals and Rankin–Selberg period for optimal forms from Section 6.

Note that in this section, q denotes the cardinality of the residue field $\mathbb{F}_q$ of the p-adic field F, rather than the prime in the previous sections. Let $\Phi = \Phi_1 \otimes \Phi_2$ with Φ_1 and Φ_2 standard functions for the characters χ and χ^{-1} respectively (cf. Section 5.4). We again view the "antinorm" $\xi = \chi^{1-c}$ as the restriction of χ^{-1} on E^1. Recall from Proposition 6.1 that,

$$Z(s,\Phi) = \int_{E^\times} |t_1|^s \int_{E^1} \widetilde{\Phi}(t_1(1+t_2 j))\xi(t_2)dt_2dt_1,$$

where,

$$\widetilde{\Phi} = \int_{SL_2(\mathcal{O}_F)} r(k)\Phi dk.$$

7.1 Unramified calculation

First, we assume that χ is unramified. For the precise calculation, we do not treat the E/F ramified case when $p = 2$.

Proposition 7.1. *If χ is unramified, then $\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}}) \in \mathbb{Q}(\xi + \xi^{-1})^{\times}$. Furthermore,*

$$Z(s, W^{\mathrm{new}}) = \begin{cases} Z(s, W^{\mathrm{opt}}) & \text{if } E/F \text{ is not ramified}, \\ 4(q+1)^{-1} Z(s, W^{\mathrm{opt}}) & \text{if } E/F \text{ is ramified and } p \neq 2. \end{cases}$$

In particular,

$$\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}}) = \begin{cases} \mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}}) & \text{if } E/F \text{ is not ramified}, \\ 4(q+1)^{-1} \mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}}) & \text{if } E/F \text{ is ramified and } p \neq 2. \end{cases}$$

Proof. With Φ_i as the characteristic function of $\mathcal{O}_E$ (cf. Section 5.4), Φ is the characteristic function of $\mathcal{O}_B^{\mathrm{new}} := \mathcal{O}_E + \mathcal{O}_E j$ in the definite quaternion algebra $B = E + Ej$.

If E/F is unramified, then $\Phi = \Phi^{\mathrm{opt}}$ and the result follows from Proposition 6.5.

We assume that E/F is ramified. We start by computing $\widetilde{\Phi}$. Let $d_E \mathcal{O}_F$ be the discriminant ideal of E/F. We claim that Φ is invariant under the subgroup

$$U_0(d_E) = \left\{ \begin{pmatrix} a & b \\ d_E c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) \right\}.$$

Section 5, U_0 is in SL_2 instead of Notice that $U_0(d_E)$ is generated by $B(\mathcal{O}_F)$ and $wB(d_E)w$ (recall that $w := \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix} \right)$). Thus it suffices to show that Φ is invariant under $B(\mathcal{O}_F)$ and $w\Phi$ is invariant under $N(d_E \mathcal{O}_F)$ (where N is the subgroup of upper triangular matrices with 1's on the diagonal). The $B(\mathcal{O}_F)$-invariance is clear, since Φ is the characteristic function of the $\mathcal{O}_E$-module $\mathcal{O}_B^{\mathrm{new}} = \mathcal{O}_E + \mathcal{O}_E j$. For the second invariance, notice that $w\Phi = \widehat{\Phi} = \widehat{\Phi}_1 \otimes \widehat{\Phi}_2$. Then we need to calculate the Fourier transform for the characteristic function Φ_i of $\mathcal{O}_E$ with the character $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$,

$$\widehat{\Phi}_i(x) = \int_{\mathcal{O}_E} \psi_E(xy)\, dy.$$

For a fixed x, this is an integration of the character on E over a lattice. So $\widehat{\Phi}$ is the characteristic function of the dual lattice $(\mathcal{O}_B^{\text{new}})^\vee$ of $\mathcal{O}_B^{\text{new}}$, multiplied by $\text{vol}(\mathcal{O}_B^{\text{new}})$. Let $\delta_{E/F}$ be the different ideal of $\mathcal{O}_E$, then we have that $(\mathcal{O}_B^{\text{new}})^\vee = \delta_{E/F}^{-1}\mathcal{O}_B^{\text{new}})$. To compute the volume, we use the general formula for a lattice Λ in B,

$$1 = \text{vol}(\Lambda)\text{vol}\big(\Lambda^\vee\big) = \text{vol}(\Lambda)\text{vol}(\Lambda)\big[\Lambda^\vee : \Lambda\big].$$

Therefore,

$$\text{vol}(\mathcal{O}_B^{\text{new}}) = \big[(\mathcal{O}_B^{\text{new}})^\vee : \mathcal{O}_B^{\text{new}}\big]^{-\frac{1}{2}} = \big|\delta_{E/F}\big|_E = |d_E|_F.$$

Thus we have shown that,

$$\widehat{\Phi} = |d_E|_{\delta_{E/F}^{-1}\mathcal{O}_B}.$$

It follows that $\widehat{\Phi}$ is invariant under $N(d_E\mathcal{O}_F)$. Hence Φ is invariant under $U_0(d_E)$.

Using the Bruhat decomposition with ϖ the uniformizer of E,

$$SL_2(\mathcal{O}_F) = wN(\varpi\mathcal{O}_F/d_E)wU_0(d_E) \,\cup\, N(\mathcal{O}_F/d_E)wU_0(d_E),$$

we compute,

$$\begin{aligned}\widetilde{\Phi} &= \int_{SL_2(\mathcal{O}_F)} r(k)\widehat{\Phi}\,dk \\ &= \big(|d_E/\varpi|^{-1} + |d_E|^{-1}\big)^{-1}\left(w\sum_{b\in\varpi\mathcal{O}_F/d_E} n(b)\widehat{\Phi} + \sum_{b\in\mathcal{O}_F/d_E} n(b)\widehat{\Phi}\right).\end{aligned}$$

From this description of $\widehat{\Phi}$, we see that $\sum_{b\in\mathcal{O}_F/d_E} n(b)\widehat{\Phi}$ is supported on $\delta_E^{-1}(\mathcal{O}_E + \mathcal{O}_E j)$ with value

$$|d_E|\sum_{b\in\mathcal{O}_F/d_E} \psi\big(b\big(N_{E/F}(x) - N_{E/F}(y)\big)\big),$$

for $x+yj \in \delta_E^{-1}(\mathcal{O}_E+\mathcal{O}_E j)$. This integral defines the characteristic function of $N_{E/F}(x)-N_{E/F}(y) \in$

$\mathcal{O}_F$. Thus $\widetilde{\Phi}$ is the characteristic function of a subset of B of elements with the form

$$\delta_E^{-1}(x + yj),$$

for $x, y \in \mathcal{O}_E$ with $N_{E/F}(x) - N_{E/F}(y) \in d_E\mathcal{O}_F$. By Corollary 6.2, we have that $\mathcal{P}_{RS}(W^{\text{new}}) \in \mathbb{Q}(\xi)$. To see that it does not vanish, use projection to the space of newforms by integration over $U_1(\varpi_E^o)$, where o is the order of $\pi(\chi)$.

For the precise calculation of $\widetilde{\Phi}$ with E/F ramified, we have $p \neq 2$ by assumption. Then d_E is a prime in $\mathcal{O}_F$. For $x, y \in \mathcal{O}_E$ with $N_{E/F}(x) - N_{E/F}(y) \in d_E\mathcal{O}_F$, we have $x = \pm y \pmod{\varpi_E}$. Then we have that $\sum_{b \in \mathcal{O}_F/\varpi_E} n(b)\widehat{\Phi}$ is the characteristic function of the union of the two sets,

$$\mathcal{O}_B^{\pm} := \mathcal{O}_E + \mathcal{O}_E j + \frac{(1 \pm j)}{\varpi_E}\mathcal{O}_E.$$

These are two maximal orders of B with intersection $\mathcal{O}_B^{\text{new}} = \mathcal{O}_E + \mathcal{O}_E j$. Let $\Phi^{\pm}$ denote the characteristic function of $\mathcal{O}_B^{\pm}$. Recall that Φ is the characteristic function of $\mathcal{O}_B^{\text{new}}$. Then,

$$\sum_{b \in \mathcal{O}_F/\varpi_E} n(b)\widehat{\Phi} = \Phi^+ + \Phi^- - \Phi.$$

This shows that,

$$\widetilde{\Phi} = (q+1)^{-1}(\Phi^+ + \Phi^-).$$

Now the result follows from Proposition 6.5 for the two optimal functions $\Phi^{\pm}$, with the factors of 2 arising from Proposition 5.6 □

7.2 Ramified calculation: E/F inert

Next, we assume that χ is ramified and E/F is inert.

Proposition 7.2. *If χ is ramified and E/F is inert, then,*

$$\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}}) = Z(1,\Phi) = \begin{cases} 1 & \textit{if } \xi^2 \textit{ is unramified} \\ \frac{1}{q+1} & \textit{if } \xi^2 \textit{ is ramified.} \end{cases}$$

Proof. Let $o(\chi)$ be the order of χ, namely, the minimal integer such that χ is non-trivial on $1+\varpi^{o(x)}\mathcal{O}_E$. Let $c(\chi) := \varpi^{o(x)}$ be the conductor of χ. Again, the strategy to evaluate $Z(1,\Phi)$ is to use Proposition 6.1 and a description of $\widetilde{\Phi}$ (also using a description of $\widehat{\Phi}$).

Since E/F is inert, the Φ_i are the restrictions of χ^{-1} and χ on $\mathcal{O}_E^\times$ respectively (cf. Section 5.4). Then Φ is invariant under $B(\mathcal{O}_F)$. Thus it is invariant under some $U_0(\varpi^k)$ for some k, which we call the level of Φ. To determine such k, let us compute $\widehat{\Phi} = \widehat{\Phi}_1 \otimes \widehat{\Phi}_2$.

$$\widehat{\Phi}_1(x) = \int_{\mathcal{O}_E^\times} \chi^{-1}(u)\psi_E(xu)du,$$
$$\widehat{\Phi}_2(y) = \int_{\mathcal{O}_E^\times} \chi(v)\psi_E(-yv)dv,$$

where the measure is additive so that $\mathrm{vol}(\mathcal{O}_E) = 1$ and $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$. These are Gaussian integrals, and their values are essentially given by ϵ-factors defined as follows,

$$\epsilon(\chi,\psi) := \int_{\mathcal{O}_E^\times} \chi(\varpi^{-o(x)}u)\psi_E(\varpi^{-o(x)}u)\,du. \tag{7.1}$$

Lemma 7.3. *Let $\chi : E^\times \longrightarrow \mathbb{C}^\times$ be a multiplicative character of order $o(\chi) > 0$, and let $\psi_E : E \longrightarrow \mathbb{C}^\times$ be an additive character of order 0. Then we have the following two identities,*

1.
$$\int_{\mathcal{O}_E^\times} \chi(u)\psi_E(xu)du = \chi^{-1}(x)\epsilon(\chi,\psi)\big|_{\varpi^{-o(x)}\mathcal{O}_E^\times}$$

2.
$$\epsilon(\chi,\psi)\epsilon(\chi^{-1},\psi^{-1}) = \left|\varpi_E^{o(x)}\right|_E.$$

Proof of Lemma 7.3. It is easy to see that $\chi(u)$ does not change if we replace u by $u+v$ with $v \in c(\chi)\mathcal{O}_E$. Thus it has a factor,

$$\int_{c(\chi)\mathcal{O}_E} \psi(vx)\,dv.$$

It follows that the integral $\int_{\mathcal{O}_E^\times} \chi(u)\psi_E(xu)\,du \neq 0$ only if $x \in \varpi_E^{-o(\chi)}\mathcal{O}_E$. Furthermore if $x \in \varpi^{1-o(\chi)}\mathcal{O}_E$, then $\psi(xu)$ does not change if we replace u by $u(1+\varpi_E^{o(\chi)-1}\mathcal{O}_F)$ Thus, the integral has a factor

$$\int_{1+\varpi_E^{o(\chi)-1}\mathcal{O}_F} \chi(v)\,dv = 0.$$

It follows that the the function $\int_{\mathcal{O}_E^\times} \chi(u)\psi_E(xu)\,du$ is supported on $\varpi_E^{-o(\chi)}\mathcal{O}_E^\times$, with value $\chi^{-1}(x)\epsilon(\chi,\psi)$. This proves the first identity.

For the second identity, we need to calculate the product of $\epsilon(\chi,\psi)$ and $\epsilon(\chi^{-1},\psi^{-1})$,

$$\begin{aligned}\epsilon(\chi,\psi)\epsilon(\chi^{-1},\psi^{-1}) &= \int_{(\mathcal{O}_E^\times)^2} \chi\left(\frac{u}{v}\right)\psi\left(\varpi_E^{-o(\chi)}(u-v)\right)du\,dv\\ &= \int_{(\mathcal{O}_E^\times)^2} \chi(w)\psi\left(\varpi_E^{-o(\chi)}(w-1)v\right)dv\,dw,\end{aligned}$$

where we wrote $u = vw$ with $w \in \mathcal{O}_E^\times$ in the last step. The integration over v is given by,

$$\int_{\mathcal{O}_E^\times} \psi\left(\varpi_E^{-o(\chi)}(w-1)v\right)dv = \int_{\mathcal{O}_E} \psi\left(\varpi_E^{-o(\chi)}(w-1)v\right)dv - \int_{\varpi_E\mathcal{O}_E} \psi\left(\varpi_E^{-o(\chi)}(w-1)v\right)dv,$$

with,

$$\int_{\mathcal{O}_E} \psi\left(\varpi_E^{-o(\chi)}(w-1)v\right)dv = \begin{cases} 1 & \text{if } w-1 \in \varpi_E^{o(\chi)}\mathcal{O}_E,\\ 0 & \text{otherwise,}\end{cases}$$

$$\int_{\varpi_E\mathcal{O}_E} \psi\left(\varpi_E^{-o(\chi)}(w-1)v\right)dv = \begin{cases} -|\varpi_E| & \text{if } w-1 \in \varpi_E^{o(\chi)-1}\mathcal{O}_E,\\ 0 & \text{otherwise,}\end{cases}$$

Then the full double integral is,

$$\begin{aligned}\epsilon(\chi,\psi)\epsilon(\chi^{-1},\psi^{-1}) &= \int_{1+\varpi_E^{o(\chi)}\mathcal{O}_E}\chi(w)\,dw + \int_{1+\varpi_E^{o(\chi)-1}\mathcal{O}_E}\chi(w)\,dw\\ &= \mathrm{vol}\Big(1+\varpi_E^{o(\chi)}\mathcal{O}_E\Big)+0\\ &= \Big|\varpi_E^{o(\chi)}\Big|_E \mathrm{vol}(\mathcal{O}_E)\\ &= \Big|\varpi_E^{o(\chi)}\Big|_E,\end{aligned}$$

where we used that dw is the additive measure on E. $\square$

By Lemma 7.3, we have the following,

$$\begin{aligned}w\Phi &= \widehat{\Phi}\\ &= \epsilon(\chi,\psi)\epsilon(\chi^{-1},\psi^{-1})\big(\chi\otimes\chi^{-1}\big)\big|_{\varpi^{-o(\chi)}(\mathcal{O}_E^\times+\mathcal{O}_E^\times j)}\\ &= q^{-2o(\chi)}\big(\chi\otimes\chi^{-1}\big)\big|_{\varpi^{-o(\chi)}(\mathcal{O}_E^\times+\mathcal{O}_E^\times j)}.\end{aligned}$$

It follows that Φ is invariant under $U_0(\varpi^{2o(\chi)})$. So we can take $k=2o(\chi)$.

Now we calculate $\widetilde{\Phi}$ using the Bruhat decomposition,

$$\mathrm{SL}_2(\mathcal{O}_F) = wN\big(\mathcal{O}_F/\varpi^k\big)wU_0\big(\varpi^k\big)\ \cup\ N\big(\varpi\mathcal{O}_F/\varpi^k\big)wU_0\big(\varpi^k\big),$$

so,

$$\widetilde{\Phi}(x) = \big(q^k+q^{k-1}\big)^{-1}\left(w\sum_{b\in\varpi\mathcal{O}_F/\varpi^k} r(b)w\Phi(x) + \sum_{b\in\mathcal{O}_F/\varpi^k} r(b)w\Phi(x)\right).$$

The two sums are respectively equal to,

$$\begin{aligned}\sum_{b\in\mathcal{O}_F/\varpi^k} r(b)\widehat{\Phi}(x+yj) &= \sum_{b\in\mathcal{O}_F/\varpi^k}\psi\big(b(N(x)-N(y))\big)\widehat{\Phi}(x+yj).\\ \sum_{b\in\varpi\mathcal{O}_F/\varpi^k} r(b)\widehat{\Phi}(x+yj) &= \sum_{b\in\varpi\mathcal{O}_F/\varpi^k}\psi\big(b(N(x)-N(y))\big)\widehat{\Phi}(x+yj).\end{aligned}$$

These sums are non-zero only if $N(x)-N(y) \in \mathcal{O}_F$ and $N(x)-N(y) \in \varpi^{-1}\mathcal{O}_F$, respectively. Write $x = \varpi^{-k/2}u$ and $y = \varpi^{-k/2}v$ with $u, v \in \mathcal{O}_E^\times$. Then these conditions are that $N(uv^{-1}) \in 1+\varpi^k\mathcal{O}_F$ and $N(uv^{-1}) \in 1+\varpi^{k-1}\mathcal{O}_F$, respectively. So,

$$\widetilde{\Phi} = q^{-k}(q+1)^{-1}\Big(w\big(\chi\otimes\chi^{-1}\big)\big|_{\Omega_1} + q\Big(\chi\otimes\chi^{-1}\big|_{\Omega_0}\Big)\Big),$$

where for integers $i \in \{0, 1\}$,

$$\Omega_i := \Big\{(u, v) \in \varpi^{-\frac{k}{2}}\big(\mathcal{O}_E^\times \times \mathcal{O}_E^\times\big)\;\Big|\; N\Big(\frac{u}{v}\Big) \in 1+\varpi^{k-i}\mathcal{O}_F\Big\}.$$

To further describe $\widetilde{\Phi}$, we need to calculate $w(\chi\otimes\chi^{-1})\big|_{\Omega_1}$.

Lemma 7.4. *Define the function* Φ_n *supported on* $\varpi^{-n}(\mathcal{O}_E^\times \times \mathcal{O}_E^\times)$,

$$\Phi_n(x, y) := q^{k-2n}\chi\Big(\frac{y}{x}\Big)f_n\Big(N\Big(\frac{y}{x}\Big)\Big),$$

where f_n *is a function on* $\mathcal{O}_F^\times$ *defined by,*

$$f_n(x) := \Big|\big(\mathcal{O}_F/\big(\varpi^{k-1}\big)\big)^\times\Big|^{-1} \sum_{\substack{\omega:\,(\mathcal{O}_F/(\varpi^{k-1}))^\times\to\mathbb{C}^\times \\ o(\chi\omega)=\frac{k}{2}+n}} \omega(x).$$

Then,

$$w\big(\chi\otimes\chi^{-1}\big)\big|_{\Omega_1} = \sum_n \Phi_n.$$

Proof of Lemma 7.4. If we change variables $(u, v) \mapsto (\varpi^{-k/2}u, \varpi^{-k/2}v)$, then Ω_1 is replaced by the following subgroup of $(\mathcal{O}_E^\times)^2$,

$$G := \mathrm{Ker}\begin{pmatrix} (\mathcal{O}_E^\times)^2 \longrightarrow (\mathcal{O}_F/\varpi^{k-1})^\times \\ (u, v) \longmapsto N\Big(\dfrac{v}{u}\Big) \end{pmatrix}.$$

With the aforementioned change of variables for $(u,v) \in \Omega_1$ to G,

$$\begin{aligned} w\big(\chi \otimes \chi^{-1}\big)\big|_{\Omega_1}(x,y) &= \int_{\Omega_1} \chi\Big(\frac{u}{v}\Big)\psi_E(xu - yv)\,du\,dv \\ &= q^{2k}\int_G \chi\Big(\frac{u}{v}\Big)\psi\Big(\varpi^{-\frac{k}{2}}xu - \varpi^{-\frac{k}{2}}yv\Big)\,du\,dv. \end{aligned}$$

Notice that the characteristic function of G in $(\mathcal{O}_E^\times)^2$ is given by

$$\big|(\mathcal{O}_F/(\varpi^{k-1}))^\times\big|^{-1} \sum_{\omega\colon (\mathcal{O}_F/(\varpi^{k-1}))^\times \to \mathbb{C}^\times} \omega\Big(N\Big(\frac{v}{u}\Big)\Big).$$

Then we obtain,

$$\begin{aligned} w\big(\chi \otimes \chi^{-1}\big)\big|_{\Omega_1}(x,y) = q^{2c}\big|\big(\mathcal{O}_F/\big(\varpi^{k-1}\big)\big)^\times\big|^{-1} \\ \cdot \sum_{\omega\colon (\mathcal{O}_F/(\varpi^{k-1}))^\times \to \mathbb{C}^\times} \int_{(\mathcal{O}_E^\times)^2} \chi\omega\Big(\frac{u}{v}\Big)\psi\Big(\varpi^{-\frac{k}{2}}xu - \varpi^{-\frac{k}{2}}yv\Big)\,du\,dv. \end{aligned}$$

The last integral is the product of two integrals over $\chi\omega$ and $(\chi\omega)^{-1}$. It is non-vanishing only if $\mathrm{ord}(x) = \mathrm{ord}(y) = k/2 - o(\chi\omega)$, in which case, it is given by,

$$(\chi\omega)^{-1}\Big(\varpi^{-\frac{k}{2}}x\Big) \cdot \epsilon(\chi\omega,\psi) \cdot \chi\omega\Big(\varpi^{-\frac{k}{2}}y\Big) \cdot \epsilon\big((\chi\omega)^{-1},\psi^{-1}\big) = q^{-2o(\chi\omega)}\chi\omega\Big(\frac{y}{x}\Big).$$

So $w\big(\chi \otimes \chi^{-1}\big)\big|_{\Omega_1}(x,y) \neq 0$ only if $\mathrm{ord}(x) = \mathrm{ord}(y) = -n'$ for some n'. In this case, it is given by,

$$\begin{aligned} w\big(\chi \otimes \chi^{-1}\big)\big|_{\Omega_1}(x,y) &= q^{2k}\big|\big(\mathcal{O}_F/\big(\varpi^{k-1}\big)\big)^\times\big|^{-1} \sum_{\substack{\omega\colon (\mathcal{O}_F/(\varpi^{k-1}))^\times \to \mathbb{C}^\times \\ o(\chi\omega) = \frac{k}{2} + n'}} q^{-2o(\chi\omega)}\chi\omega\Big(\frac{y}{x}\Big) \\ &= q^{k-2n}\chi\Big(\frac{y}{x}\Big)f_{n'}\Big(N\Big(\frac{y}{x}\Big)\Big) \\ &= \Phi_{n'}(x,y). \end{aligned}$$

We can conclude the lemma claim for general (x,y) by the vanishing of Φ_n in the other cases. $\square$

By Lemma 7.4

$$\widetilde{\Phi} = q^{-k}(q+1)^{-1}\left(\sum_n \Phi_n + q\left(\chi \otimes \chi^{-1}|_{\Omega_0}\right)\right).$$

We apply this to the equality from Proposition 6.1,

$$Z(s,\Phi) = \int_{E^\times} |t_1|^s dt_1 \int_{E^1} \widetilde{\Phi}(t_1(1+t_2 j))\xi(t_2)dt_2.$$

By Lemma 7.4, we only need to look at Φ_n and $(\chi \otimes \chi^{-1})\big|_{\Omega_0}$,

$$\begin{aligned} Z(s,\Phi_n) &= q^{k-2n}\int_{\varpi^{-n}\mathcal{O}_E^\times} |t_1|^s dt_1 \int_{E^1} \chi\left(\frac{t_1 t_2}{t_1}\right)\chi^{-1}(t_2) f_n(1) dt_2 \\ &= f_n(1) q^{2n(s-1)+c} \\ Z\left(s, \chi \otimes \chi^{-1}|_{\Omega_0}\right) &= \int_{\varpi^{-\frac{k}{2}}\mathcal{O}_E^\times} |t_1|^s dt_1 \int_{E^1} \chi\left(\frac{t_1}{t_1 t_2}\right)\chi^{-1}(t_2)dt_2 \\ &= \begin{cases} q^{ks} & \text{if } \xi^2 = 1, \\ 0 & \text{otherwise.} \end{cases}. \end{aligned}$$

In summary, we have shown that,

$$(q+1)Z(s,\Phi) = \begin{cases} \sum_n f_n(1) q^{2n(s-1)} + q^{1+k(s-1)} & \text{if } \xi^2 = 1, \\ \sum_n f_n(1) q^{2n(s-1)} & \text{otherwise.} \end{cases}$$

Since $\sum_n f_n(1) = 1$, set $s = 1$ to obtain,

$$Z(1,\Phi) = \begin{cases} 1 & \text{if } \xi^2 = 1, \\ \frac{1}{q+1} & \text{if } \xi^2 \neq 1. \end{cases}$$

□

7.3 Fully ramified calculation: E/F split

Next, we assume that χ is ramified and $E/F = F \oplus F$ is split. In particular, we can write $\chi = (\chi_1, \chi_2)$. There are two cases: either both χ_1 and χ_2 are ramified, or exactly one of them is ramified. Here, we consider the "fully ramified" case that both χ_i are ramified.

Proposition 7.5. *If $\chi = (\chi_1, \chi_2)$ with χ_1 and χ_2 ramified and $E = F \oplus F$, then,*

$$\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}}) = Z(1, \Phi) = \begin{cases} 1 & \text{if } \xi^2 \text{ is unramified,} \\ \frac{1}{q+1} & \text{if } \xi^2 \text{ is ramified.} \end{cases}$$

Proof. Let o_1, o_2 be the orders of χ_1 and χ_2 respectively. Take Φ_χ to be χ^{-1} restricted to $\mathcal{O}_E^\times$, and $\Phi = \Phi_\chi \otimes \Phi_{\chi^{-1}}$. Then Φ is supported on

$$\mathcal{O}_F^\times \times \mathcal{O}_F^\times + (\mathcal{O}_F^\times \times \mathcal{O}_F^\times)j,$$

with value

$$\Phi((x_1, x_2) + (y_1, y_2)j) = \chi_1\left(\frac{y_1}{x_1}\right)\chi_2\left(\frac{y_2}{x_2}\right).$$

Again, this is invariant under $B(\mathcal{O}_F)$. To find its level, we compute the Fourier transform of Φ:

$$\widehat{\Phi}(x_1, x_2, y_1, y_2) = \int_{(\mathcal{O}_F^\times)^4} \chi_1\left(\frac{v_1}{u_1}\right)\chi_2\left(\frac{v_2}{u_2}\right)\psi(u_1x_1 + u_2x_2 - v_1y_1 - v_2y_2)du_1du_2dv_1dv_2.$$

This is the product of four Gaussian integrals. So we apply Lemma 7.3. $\widehat{\Phi}$ is supported on

$$\Omega := \varpi^{-o_1}\mathcal{O}_F^\times \times \varpi^{-o_2}\mathcal{O}_F^\times + (\varpi^{-o_1}\mathcal{O}_F^\times \times \varpi^{-o_2}\mathcal{O}_F^\times)j,$$

with value

$$\widehat{\Phi}((x_1, x_2) + (y_1, y_2)j) = \chi_1\left(\frac{x_1}{y_1}\right)\chi_2\left(\frac{x_2}{y_2}\right)q^{-o_1-o_2}.$$

This description shows that $\widehat{\Phi}$ is invariant under $N(\varpi^k\mathcal{O}_F)$ for $k = o_1 + o_2$. Thus we have shown that Φ is invariant under $U_0(\varpi^k)$.

Next, we calculate $\widetilde{\Phi} = \int_{SL_2(\mathcal{O}_F)} r(k)\Phi$. We again use the Bruhat decomposition,

$$SL_2(\mathcal{O}_F) = wN\big(\varpi\mathcal{O}_F/\varpi^k\big)wU_0\big(\varpi^k\big) \,\cup\, N\big(\mathcal{O}_F/\varpi^k\big)wU_0\big(\varpi^k\big).$$

It follows that,

$$\widetilde{\Phi} = \big(q^{k-1}+q^k\big)^{-1}\left(w\sum_{b\in\varpi\mathcal{O}_F/\varpi^k} r\big(n(b)\big)\widehat{\Phi} + \sum_{b\in\mathcal{O}_F/\varpi^k} r\big(n(b)\big)\widehat{\Phi}\right).$$

As before, the two sums can be rewritten so that

$$\widetilde{\Phi} = q^{-k}(q+1)^{-1}\Big(w\big(\chi\otimes\chi^{-1}\big)\big|_{\Omega_1} + q\big(\chi\otimes\chi^{-1}\big)\big|_{\Omega_0}\Big),$$

where for integers $i\in\{0,1\}$,

$$\Omega_i := \left\{(u_1,v_1)+(u_2,v_2)j\in\Omega \;\middle|\; \frac{u_1v_1}{u_2v_2}\in 1+\varpi^{k-i}\mathcal{O}_F\right\}.$$

To further describe $\widetilde{\Phi}$, we need to calculate $w(\chi_1\otimes\chi_1^{-1})_{\Omega_1}$.

Lemma 7.6. *Define the function $\Phi_{m,n}$ supported on $(\varpi^{-m}\mathcal{O}_F^\times\times\varpi^{-n}\mathcal{O}_F^\times)^2$ by,*

$$\Phi_{m,n}(x,y) := q^{k-m-n}\chi_1\left(\frac{y_1}{x_1}\right)\chi_2\left(\frac{y_2}{x_2}\right)f_{m,n}\left(\frac{y_1y_2}{x_1x_2}\right),$$

where $f_{m,n}$ is a function on $\mathcal{O}_F^\times$ defined by

$$f_{m,n}(x) := \big|\big(\mathcal{O}_F/\varpi^{k-1}\big)^\times\big|^{-1}\sum_{\substack{\omega:\,(\mathcal{O}_F/\varpi^{k-1})^\times\to\mathbb{C}^\times\\ \mathrm{ord}(\chi_1\omega)=m+o_1\\ \mathrm{ord}(\chi_2\omega)=n+o_2}}\omega(x).$$

Then,

$$w(\chi \otimes \chi^{-1})\big|_{\Omega_1} = \sum_{m,n} \Phi_{m,n}.$$

Proof of Lemma 7.6. From the definition, $w(\chi \otimes \chi^{-1})\big|_{\Omega_1}$ is given by,

$$w\big(\chi \otimes \chi^{-1}\big)\big|_{\Omega_1}(x, y) = \int_{\Omega_1} \chi_1\left(\frac{u_1}{v_1}\right)\chi_2\left(\frac{u_2}{v_2}\right)\psi(u_1x_1 + u_2x_2 - v_1y_1 - v_2y_2)du_1du_2dv_1dv_2.$$

If we change variables,

$$(u_1, u_2, v_1, v_2) \longmapsto (\varpi^{-o_1}u_1, \varpi^{-o_2}u_2, \varpi^{-o_1}v_1, \varpi^{-o_2}v_2),$$

then Ω_1 is replaced by the following subgroup of $\mathcal{O}_F^\times$,

$$G := \operatorname{Ker}\left(\begin{array}{rcl} (\mathcal{O}_E^\times)^4 & \longrightarrow & \left(\mathcal{O}_F/\varpi^{k-1}\right)^\times \\ (u_1, u_2, v_1, v_2) & \longmapsto & \dfrac{u_1u_2}{v_1v_2} \end{array}\right).$$

With the aforementioned change of variables for $(u_1, u_2, v_1, v_2) \in \Omega_1$ to G,

$$w\big(\chi \otimes \chi^{-1}\big)\big|_{\Omega_1}(x, y) = q^{2k} \int_G \chi_1\left(\frac{u_1}{v_1}\right)\chi_2\left(\frac{u_2}{v_2}\right)\psi(\alpha)du_1du_2dv_1dv_2,$$

where,

$$\alpha := \varpi^{-o_1}u_1x_1 + \varpi^{-o_2}u_2x_2 - \varpi^{-o_1}v_1y_1 - \varpi^{-o_2}v_2y_2.$$

Notice that the characteristic function of G for $(u_1, u_2, v_1, v_2 \in (\mathcal{O}_E^\times)^4$ is given by,

$$\left|\left(\mathcal{O}_F/\varpi^{k-1}\right)^\times\right|^{-1} \sum_{\omega\colon (\mathcal{O}_F/\varpi^{k-1})^\times \to \mathbb{C}^\times} \omega\left(\frac{u_1u_2}{v_1v_2}\right).$$

Thus,

$$w(\chi\otimes\chi^{-1})\big|_{\Omega_1}(x,y)=q^{2k}\left|(\mathcal{O}_F/\varpi^{k-1})^\times\right|^{-1}$$
$$\cdot\sum_{\omega\colon(\mathcal{O}_F/\varpi^{k-1})^\times\to\mathbb{C}^\times}\int_{(\mathcal{O}_F^\times)^4}\omega\chi_1\left(\frac{u_1}{v_1}\right)\omega\chi_2\left(\frac{u_2}{v_2}\right)\psi(\alpha)du_1du_2dv_1dv_2.$$

Now we apply Lemma 7.3 to obtain that the integral is non-vanishing only if

$$\mathrm{ord}(x_1)=\mathrm{ord}(y_1)=o_1-o(\chi_1\omega),$$
$$\mathrm{ord}(x_2)=\mathrm{ord}(y_2)=o_2-o(\chi_2\omega),$$

in which case, it is given by (cf. Equation 7.1 for the ϵ-factor),

$$\omega\chi_1\left(\frac{y_1}{x_1}\right)\omega\chi_2\left(\frac{y_2}{x_2}\right)\epsilon(\omega\chi_1,\psi)\epsilon(\omega\chi_2,\psi)\epsilon((\omega\chi_1)^{-1},\psi^{-1})\epsilon((\omega\chi_2)^{-1},\psi^{-1})$$
$$=\chi_1\left(\frac{y_1}{x_1}\right)\chi_2\left(\frac{y_2}{x_2}\right)\omega\left(\frac{y_1y_2}{x_1x_2}\right)q^{-o(\chi_1\omega)-o(\chi_2\omega)}.$$

Thus $w(\chi\otimes\chi^{-1})\big|_{\Omega_1}(x,y)\neq 0$ only if there is some (m',n') such that

$$\mathrm{ord}(x_1)=\mathrm{ord}(y_1)=-m',$$
$$\mathrm{ord}(y_1)=\mathrm{ord}(y_2)=-n',$$

in which case, it is given by,

$$\begin{aligned}
w\left(\chi\otimes\chi^{-1}\right)\big|_{\Omega_1}(x,y) &= q^{2k}\left|\left(\mathcal{O}_F/\varpi^{k-1}\right)^\times\right|^{-1} \\
&\quad\cdot \sum_{\substack{\omega:\left(\mathcal{O}_F/\varpi^{k-1}\right)^\times\to\mathbb{C}^\times \\ \mathrm{ord}(\chi_1\omega)=m'+o_1 \\ \mathrm{ord}(\chi_2\omega)=n'+o_2}} \chi_2^{-1}\left(\frac{x_1}{y_1}\right)\chi_1^{-1}\left(\frac{x_2}{y_2}\right)\omega\left(\frac{x_1x_2}{y_1y_2}\right)q^{-k-m'-n'} \\
&= q^{k-m'-n'}\chi_1\left(\frac{y_1}{x_1}\right)\chi_2\left(\frac{y_2}{x_2}\right)f_{m',n'}\left(\frac{y_1y_2}{x_1x_2}\right) \\
&= \Phi_{m',n'}(x,y).
\end{aligned}$$

We can conclude the lemma claim for general (x, y) by the vanishing of $\Phi_{m,n}$ in the other cases.

□

By Lemma 7.6,

$$\widetilde{\Phi} = q^{-k}(q+1)^{-1}\left(\sum_{m,n}\Phi_{m,n} + q\left(\chi\otimes\chi^{-1}\right)\big|_{\Omega_0}\right)$$

We apply this to the equality from Proposition 6.1,

$$Z(s,\Phi) = \int_{E^\times}|t_1|^s\int_{E^1}\widetilde{\Phi}(t_1(1+t_2j))\xi(t_2)dt_2dt_1.$$

By Lemma 7.4, we only need to look at $\Phi_{m,n}$ and $(\chi\otimes\chi^{-1})\big|_{\Omega_0}$,

$$\begin{aligned}
Z(s,\Phi_{m,n}) &= q^{k-m-n}\int_{\varpi^{-m}\mathcal{O}_F^\times\times\varpi^{-n}\mathcal{O}_F^\times}|t_1|^s\int_{\mathcal{O}_E^1}\xi\left(\frac{t_1}{t_1t_2}\right)\xi(t_2)f_n(1)dt_2dt_1 \\
&= f_{m,n}(1)q^{(m+n)(s-1)+k}, \\
Z\left(s,(\chi\otimes\chi^{-1})\big|_{\Omega_0}\right) &= \int_{\varpi^{-o_1}\mathcal{O}_F^\times\times\varpi^{-o_2}\mathcal{O}_F^\times}|t_1|^s\int_{\mathcal{O}_E^1}\xi^{-1}\left(\frac{t_1}{t_1t_2}\right)\xi(t_2)dt_2dt_1 \\
&= \begin{cases} q^{ks} & \text{if } \xi|_{\mathcal{O}_E^1}{}^2 = 1, \\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

In summary, we have shown that,

$$(q+1)Z(s,\Phi) = \begin{cases} \sum_{m,n} f_{m,n}(1)q^{(m+n)(s-1)} + q^{1+k(s-1)} & \text{if } \xi^2\big|_{\mathcal{O}_E^\times} = 1, \\ \sum_{m,n} f_{m,n}(1)q^{(m+n)(s-1)} & \text{if } \xi^2\big|_{\mathcal{O}_E^\times} \neq 1. \end{cases}$$

Since $\sum_{m,n} f_{m,n}(1) = 1$, set $s = 1$ to obtain,

$$Z(1,\Phi) = \begin{cases} 1 & \text{if } \xi^2 \text{ is ramified,} \\ \frac{1}{q+1} & \text{if } \xi^2 \text{ is unramified.} \end{cases}$$

□

7.4 Semi-ramified calculation: E/F split

We finish Section 7 with the last remaining case. Assume again that $\chi = (\chi_1, \chi_2)$ is ramified, $E/F = F \oplus F$ is split. In particular, consider the "semi-ramified" case wherein exactly one of χ_1 and χ_2 is ramified. Without loss of generality, we assume that χ_1 is ramified and χ_2 is unramified.

Proposition 7.7. *If $\chi = (\chi_1, \chi_2)$ with χ_1 ramified and χ_2 unramified and $E = F \oplus F$, then,*

$$\mathcal{P}_{RS}(W^{\text{new}}) = Z(1,\Phi) = \begin{cases} \frac{q}{q-1} & \textit{if } \xi^2 \textit{ is unramified,} \\ \frac{q^3}{q^2-1} & \textit{if } \xi^2 \textit{ is ramified.} \end{cases}$$

Proof. We take Φ_χ to be the restriction of $\chi_1^{-1} \otimes \mathbb{1}$ on $\mathcal{O}_F^\times \times \mathcal{O}_F$ and take,

$$\Phi = \Phi_\chi \otimes \Phi_{\chi^{-1}}.$$

It is clear that Φ is invariant under $B(\mathcal{O}_F)$. To get the level k of Φ, we need to calculate the Fourier transform of Φ, which is given by $\widehat{\Phi}_\chi \otimes \widehat{\Phi}_{\chi^{-1}}$. A standard calculation using Lemma 7.3 shows that,

$$\widehat{\Phi} = q^{-o(\chi)}\left(\chi_1|_{\varpi^{-o(\chi_1)}\mathcal{O}_F^\times} \times \mathbb{1}_{\mathcal{O}_F} \times \chi_1^{-1}|_{\varpi^{-o(\chi_1)}\mathcal{O}_F^\times} \times \mathbb{1}_{\mathcal{O}_F}\right).$$

This shows that Φ has level $k = o(\chi_1)$.

We again use the Bruhat decomposition,

$$\mathrm{SL}(\mathcal{O}_F) = wN(\varpi\mathcal{O}_F/\varpi^k)wU_0(\varpi^k) \cup N(\mathcal{O}_F/\varpi^k)wU_0(\varpi^k).$$

It follows that,

$$\widetilde{\Phi} = \left(q^k + q^{k-1}\right)^{-1}\left(w\sum_{b\in\varpi\mathcal{O}_F/\varpi^k} n(b)w\Phi + \sum_{b\in\mathcal{O}_F/\varpi^k} n(b)w\Phi\right).$$

As before,

$$\widetilde{\Phi} = q^{-k}(1+q)^{-1}\left(w\chi_1 \otimes \chi_1^{-1}|_{\Omega_1} + q\chi_1 \otimes \chi_1^{-1}|_{\Omega_0}\right),$$

where for integers $i \in \{0, 1\}$,

$$\Omega_i := \left\{(u_1, v_1, u_2, v_2) \in (\varpi^{-k}\mathcal{O}_F^\times \times \mathcal{O}_F)^2 \,\middle|\, u_1u_2 - v_1v_2 \in \varpi^{-i}\mathcal{O}_F\right\},$$

Note that,

$$\chi_1 \otimes \chi_1^{-1}(u_1, u_2, v_1, v_2) := \chi_1\left(\frac{x_1}{y_1}\right).$$

To further describe $\widetilde{\Phi}$, we need to calculate $w(\chi_1 \otimes \chi_1^{-1})_{\Omega_1}$.

Lemma 7.8. *Define the function* Φ_i *supported on* $(\mathcal{O}_F^\times \times \varpi^{-n}\mathcal{O}_F^\times)^2$,

$$\Phi_{m,n}(x, y) := \chi_1\left(\frac{y_1}{x_1}\right)q^{k-m-n}f_{m,n}\left(\frac{y_1y_2}{x_1x_2}\right),$$

where $f_{m,n}$ *is a function on* $\mathcal{O}_F^\times$ *defined by,*

$$f_{m,n}(x) := \left|\left(\mathcal{O}_F/\varpi^{k-1-m}\right)^\times\right|^{-1} \sum_{\substack{\omega:\left(\mathcal{O}_F/\varpi^{k-1-m}\right)^\times \to \mathbb{C}^\times \\ o(\omega)=-m+n}} \omega(x).$$

Then,

$$w\left(\chi_1 \otimes \chi_1^{-1}\right)\big|_{\Omega_1} = \sum_{m=0}^{k-1} \sum_{n=m}^{k-1} \Phi_{m,n}.$$

Proof of Lemma 7.8. From the definition, $w(\chi_1 \otimes \chi_1^{-1})_{\Omega_1}$ is given by,

$$w\left(\chi_1 \otimes \chi_1^{-1}\right)\big|_{\Omega_1}(x, y) = \int_{\Omega_1} \chi_1\left(\frac{u_1}{v_1}\right)\psi(u_1x_1 + u_2x_2 - v_1y_1 - v_2y_2)du_1du_2dv_1dv_2.$$

We substitute variables:

$$(u_1, u_2, v_1, v_2) \longmapsto \left(\frac{u_1}{\varpi^k}, u_2, \frac{v_1}{\varpi^k}, v_2\right).$$

Then Ω_1 changes to a subset D of $(\mathcal{O}_F^\times \times \mathcal{O}_F)^2$,

$$D := \left\{(u_1, v_1, u_2, v_2) \in (\mathcal{O}_F^\times \times \mathcal{O}_F)^2 \,\middle|\, u_1u_2 - v_1v_2 \in \varpi^{k-1}\mathcal{O}_F\right\}.$$

Then,

$$w\left(\chi_1 \otimes \chi_1^{-1}\right)\big|_{\Omega_1}(x_1, x_2, y_1, y_2) = q^{2k} \int_D \chi_1\left(\frac{u_1}{v_1}\right)\psi(\alpha_0)du_1du_2dv_1dv_2,$$

where

$$\alpha_j := \varpi^{-k}u_1x_1 + \varpi^j u_2x_2 - \varpi^{-k}v_1y_1 - \varpi^j v_2y_2.$$

We further decompose D into a disjoint union of D_m for $m \in \{0, \dots, k-1\}$, with D_{k-1} defined by the condition $(u_2, v_2) \in \varpi^{k-1}\mathcal{O}_F \times \varpi^{k-1}\mathcal{O}_F$, and D_m with $m \in [0, k-2]$ defined by the condition $(u_2, v_2) \in \varpi^m(\mathcal{O}_F^\times \times \mathcal{O}_F^\times)$.

For the region D_{k-1}, the variables u_1, v_1 are completely free in $\mathcal{O}_F^\times$:

$$D_{k-1} = \mathcal{O}_F^\times \times \varpi^{k-1}\mathcal{O}_F \times \mathcal{O}_F^\times \times \varpi^{k-1}\mathcal{O}_F.$$

The integral in the D_{k-1}-component $\Phi_{k-1}(x,y)$ of $w(\chi_1 \otimes \chi_1^{-1})\big|_{\Omega_1}$ is therefore the product of four integrals, two of them Gaussian and two of them simple integrals of ψ's,

$$\int_{D_{k-1}} \chi_1(u_1)\psi(\varpi^{-k}u_1x_1)\,du_1,$$
$$\int_{D_{k-1}} \psi(u_2x_2)\,du_2,$$
$$\int_{D_{k-1}} \chi_1\left(\frac{1}{v_1}\right)\psi\left(\frac{1}{\varpi^{-k}v_1y_1}\right)dv_1,$$
$$\int_{D_{k-1}} \psi\left(\frac{1}{v_2y_2}\right)dv_2.$$

Apply Lemma 7.3 to obtain that the integral over D_{k-1} is non-vanishing only if

$$(x_1,x_2,y_1,y_2) \in \mathcal{O}_F^\times \times \varpi^{1-k}\mathcal{O}_F \times \mathcal{O}_F^\times \times \varpi^{1-k}\mathcal{O}_F,$$

in which case the four integral values are respectively given by (cf. Equation 7.1 for the ϵ-factor),

$$\epsilon(\chi_1,\psi)\chi_1^{-1}(\varpi^{-k}x_1),$$
$$q^{1-k},$$
$$\epsilon(\chi_1^{-1},\psi^{-1})\chi_1(\varpi^{-k}y_1),$$
$$q^{1-k}.$$

Thus the full integral in the $m = k-1$ case is given by,

$$\Phi_{k-1}(x,y) := q^{2k}\int_{D_{k-1}} \chi_1\left(\frac{u_1}{v_1}\right)\psi(\alpha_0)\,du_1\,du_2\,dv_1\,dv_2 = q^{2-k}\chi_1\left(\frac{y_1}{x_1}\right).$$

Consequently,

$$\Phi_{k-1}(x,y) = \begin{cases} q^{2-k}\chi_1\left(\frac{y_1}{x_1}\right) & \text{if } (x_1,x_2,y_1,y_2) \in \left(\mathcal{O}_F^\times \times \varpi^{1-k}\mathcal{O}_F\right)^2, \\ 0 & \text{otherwise.} \end{cases}$$

Now we consider the D_m-component $\Phi_m(x,y)$ with $m \in \{0,\dots,k-2\}$. With the change of variables,

$$(u_1,u_2,v_1,v_2) \longmapsto (u_1,u_2\varpi^m,v_1,v_2\varpi^m),$$

D_m is changed to the following subgroup of $(\mathcal{O}_F^\times)^4$,

$$G_m := \left\{(u_1,u_2,v_1,v_2) \in (\mathcal{O}_F^\times)^4 \,\middle|\, \frac{u_1u_2}{v_1v_2} \in 1+\varpi^{k-1-m}\mathcal{O}_F\right\}.$$

Then,

$$\begin{aligned}\Phi_m(x,y) :=& q^{2k}\int_{D_m}\chi_1\left(\frac{u_1}{v_1}\right)\psi(\alpha_0)\,du_1\,du_2\,dv_1\,dv_2 \\ =& q^{2k-2m}\int_{G_m}\chi_1\left(\frac{u_1}{v_1}\right)\psi(\alpha_m)\,du_1\,du_2\,dv_1\,dv_2.\end{aligned}$$

Again, notice that the characteristic function of G_i is given by,

$$\left|\left(\mathcal{O}_F/\varpi^{k-1-m}\right)^\times\right| \sum_{\omega\colon(\mathcal{O}_F/\varpi^{k-1-m})^\times\to\mathbb{C}^\times} \omega\left(\frac{u_1u_2}{v_1v_2}\right).$$

Then we obtain,

$$\begin{aligned}\Phi_m(x,y) = & q^{2k-2m}\left|\left(\mathcal{O}_F/\varpi^{k-1-m}\right)^\times\right| \\ &\cdot \sum_{\omega\colon(\mathcal{O}_F/\varpi^{k-1-m})^\times\to\mathbb{C}^\times}\int_{(\mathcal{O}_F^\times)^4}\omega\chi_1\left(\frac{u_1}{v_1}\right)\omega\left(\frac{u_2}{v_2}\right)\psi(\alpha_m)\,du_1\,du_2\,dv_1\,dv_2.\end{aligned}$$

The above integral is the product of four Gaussian integrals. Applying Lemma 7.3, we have that it is non-zero only when,

$$\begin{aligned}
\operatorname{ord}(x_1) &= \operatorname{ord}(y_1) = 0 \\
\operatorname{ord}(x_2) &= \operatorname{ord}(y_2) = -m - o(\omega),
\end{aligned}$$

in which case, their respective values are,

$$\begin{aligned}
&\epsilon(\omega\chi_1, \psi) \cdot (\omega\chi_1)^{-1}\big(\varpi^{-k}x_1\big), \\
&\epsilon(\omega, \psi) \cdot \omega^{-1}(\varpi^{-m}x_2), \\
&\epsilon((\omega\chi_1)^{-1}, \psi^{-1}) \cdot (\omega\chi_1)\big(\varpi^{-k}y_1\big), \\
&\epsilon(\omega^{-1}, \psi^{-1}) \cdot \omega(\varpi^{-m}y_2).
\end{aligned}$$

Their product is given by,

$$q^{-k-o(\omega)}\chi_1\left(\frac{y_1}{x_1}\right)\omega\left(\frac{y_1 y_2}{x_1 x_2}\right).$$

Thus, $q^{2k}\int_{D_m} \neq 0$ only if $(x, y) \in (\mathcal{O}_F^\times \times \varpi^{-n}\mathcal{O}_F^\times)^2$ for some $n' \in \{m, \ldots, k-1\}$, in which case, it is given by,

$$\begin{aligned}
\chi_2^{-1}\left(\frac{x_1}{y_1}\right) q^{2k-2n'} \left|\left(\mathcal{O}_F/\varpi^{k-1-m}\right)^\times\right|^{-1} \sum_{\substack{\omega\colon \left(\mathcal{O}_F/\varpi^{k-1-m}\right)^\times \to \mathbb{C}^\times \\ o(\omega) = n'-m}} & q^{-k+m-n'}\omega\left(\frac{x_1 x_2}{y_1 y_2}\right) \\
&= \chi_1\left(\frac{x_1}{y_1}\right) q^{k-m-n'} f_{m,n}\left(\frac{y_1 y_2}{x_1 x_2}\right) \\
&= \Phi_{m,n'}.
\end{aligned}$$

By the non-vanishing of $\Phi_{m,n}$ for $n \neq n'$, we have that,

$$\Phi_m = \sum_{n=m}^{k-1} \Phi_{m,n}.$$

In particular,

$$w(\chi_1 \otimes \chi_1^{-1})\big|_{\Omega_1} = \sum_{m=0}^{k-1} \Phi_m = \sum_{m=0}^{k-1}\sum_{n=m}^{k-1} \Phi_{m,n}.$$

$\square$

By Lemma 7.8,

$$\widetilde{\Phi} = q^{-k}(q+1)^{-1}\left(\sum_{m=0}^{k-1}\sum_{n=m}^{k-1} \Phi_{m,n} + q(\chi_1^{-1} \otimes \chi_1)\big|_{\Omega_0}\right).$$

We apply this to the equality from Proposition 6.1,

$$Z(s, \Phi) = \int_{E^\times} |t_1|^s \int_{E^1} \widetilde{\Phi}(t_1, t_1 t_2)\xi(t_2)dt_2 dt_1.$$

By Lemma 7.8, we only need to look at $\Phi_{m,n}$ and $q(\chi_1^{-1} \otimes \chi_1)\big|_{\Omega_0}$ For $m \in \{1, \ldots, k-1\}$ (using the special calculation for $m = k-1$), the $\Phi_{m,n}$ terms are as follows,

$$\begin{aligned} Z(s, \Phi_{m,n}) &= q^{ns} q^{k-m-n} f_{m,n}(1) \\ &= q^{k-m+n(s-1)} f_{m,n}(1), \\ Z(s, \Phi_{k-1,k-1}) &= q^{2-k} \sum_{n \geq 1-k} q^{-ns} \\ &= \frac{q^{2-k+(k-1)s}}{1-q^{-s}}. \end{aligned}$$

To calculate $Z(s, q(\chi_1^{-1} \otimes \chi_1)\big|_{\Omega_0})$, notice that $(t_1, t_1 t_2) \in \Omega_0$ for $t_1 \in E^\times$ and $t_2 \in E^1$ if and only if $t_1 \in \varpi^{-k}\mathcal{O}_F^\times \times \mathcal{O}_F$ and $t_2 \in \mathcal{O}_E^1$. Then we can write,

$$Z\Big(s, q(\chi_1^{-1} \otimes \chi_1)\big|_{\Omega_0}\Big) = q \int_{\varpi^{-k}\mathcal{O}_F^\times \times \mathcal{O}_F} |t_1|^s \int_{\mathcal{O}_E^1} \xi(t_2)^2 dt_2 dt_1.$$

The first integral equals $q^{ks}(1-q^{-s})^{-1}$. The second integral equals $\int_{\mathcal{O}_F^\times} \chi_1^{-2}(t)dt$, which is non-vanishing only if χ_1^2 is unramified, in which case its value is 1. Thus we have,

$$Z\Big(s, q\big(\chi_1^{-1} \otimes \chi_1\big)\big|_{\Omega_0}\Big) = \frac{q^{1+ks}}{1-q^{-s}} \begin{cases} 1 & \text{if } \chi_1^2\big|_{\mathcal{O}_F^\times} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In summary, we have shown that,

$$(q+1)Z(s,\Phi) = \begin{cases} \frac{q^{(k-1)(s-2)}}{1-q^{-s}} + \sum_{m=0}^{k-2} q^{-m} \sum_{n=m}^{k-1} q^{n(s-1)} f_{m,n}(1) + \frac{q^{1+k(s-1)}}{1-q^{-s}} & \text{if } \chi_1^2\big|_{\mathcal{O}_F^\times} = 1, \\ \frac{q^{(k-1)(s-2)}}{1-q^{-s}} + \sum_{m=0}^{k-2} q^{-m} \sum_{n=m}^{k-1} q^{n(s-1)} f_{m,n}(1) & \text{if } \chi_1^2\big|_{\mathcal{O}_F^\times} \neq 1. \end{cases}$$

Since $\sum_n f_{m,n}(1) = 1$, set $s = 1$ to obtain,

$$(q+1)Z(1,\Phi) = \begin{cases} \frac{q^{1-k}}{1-q^{-1}} + \sum_{m=0}^{k-2} q^{-m} + \frac{q}{1-q^{-1}} & \text{if } \chi_1^2\big|_{\mathcal{O}_F^\times} = 1, \\ \frac{q^{1-k}}{1-q^{-1}} + \sum_{m=0}^{k-2} q^{-m} & \text{if } \chi_1^2\big|_{\mathcal{O}_F^\times} \neq 1. \end{cases}$$

$$Z(1,\Phi) = \begin{cases} (1-q^{-1})^{-1} & \text{if } \chi_1^2 \text{ is unramified,} \\ q^{-1}\big(1-q^{-2}\big)^{-1} & \text{if } \chi_1^2 \text{ is ramified,} \end{cases}$$

$$= \begin{cases} \frac{q}{q-1} & \text{if } \xi^2 \text{ is unramified} \\ \frac{q^3}{q^2-1} & \text{if } \xi^2 \text{ is ramified.} \end{cases}$$

□

Remark 7.9. The proof of Proposition 7.7, uses the fact that the sum of $f_{m,n}(1)$ is 1. A more exact formula for each $f_{m,n}(1)$ can be computed as follows,

$$f_{m,n}(1) = \big|(\mathcal{O}_F/\varpi^{k-1-m})^\times\big|^{-1} \left(\left|\left(\mathcal{O}_F/\varpi^{n-m}\right)^\times\right| - \big|(\mathcal{O}_F/\varpi^{n-m-1})^\times\big|\right)$$

By direct evaluation,

$$|(\mathcal{O}_F/\varpi^j)^\times| = \begin{cases} q^{j-1}(q-1) & \text{if } j > 0, \\ 1 & \text{if } j = 0, \\ 0 & \text{if } j < 0, \end{cases}$$

so it follows that for $m \in \{1, \dots, k-2\}$ and $n \geq m$,

$$f_{m,n}(1) = \begin{cases} q^{n-k}(q-1) & \text{if } n > m+1, \\ \frac{q-2}{q-1} q^{-k+n+1} & \text{if } n = m+1, \\ \frac{1}{(q-1)} q^{-k+n+2} & \text{if } n = m. \end{cases}$$

8 Comparison of Rankin–Selberg periods of optimal forms and newforms

We obtain the main results of Part II by combining Corollary 6.4 and Propositions 6.5, 6.6, 6.8, 7.1, 7.2, 7.5, 7.7.

Note that in this section, q denotes the cardinality of the residue field $\mathbb{F}_q$ of the p-adic field F, rather than the prime in the previous sections. Recall that $\xi := \chi^{1-c}$ can be viewed as the restriction of χ^{-1} on E^1. In general, ϖ refers to the uniformizer ϖ_E of $\mathcal{O}_E$. In the split case $E = F \oplus F$, ϖ_1 and ϖ_2 refers to the uniformizers of each component. Our first result is the following rationality statement.

Theorem 8.1. *Let F be a p-adic field, E/F be a quadratic semisimple algebra, χ be a character of $E^\times$, and ξ be the "antinorm" χ^{1-c}. Let $\mathbb{Q}(\xi + \xi^{-1})$ be the subfield of $\mathbb{C}$ generated by values of $\xi + \xi^{-1}$. Then,*

1. *$\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}}) \in \mathbb{Q}(\xi + \xi^{-1})^\times$ and $\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}}) \in \mathbb{Q}(\xi + \xi^{-1})$;*
2. *if ξ is unitary, then $\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}}) \in \mathbb{Q}(\xi + \xi^{-1})^\times$.*

For the following theorem , we also need to expand the factor from Proposition 6.5 when ξ is unramified,

$$\begin{aligned}\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}}) &= \left(1+q^{-1}\right)L(1,\mathrm{Ad}(\pi)) \\ &= \frac{q+1}{q-1}L(1,\xi).\end{aligned}$$

In particular, if E/F is inert, then

$$\begin{aligned}\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}}) &= \frac{q^2(q+1)}{(q-1)(q-\xi(\varpi))(q-\xi(\overline{\varpi}))} \\ &= \frac{q^2}{(q-1)^2}.\end{aligned}$$

Theorem 8.2. *Let F be a $\mathfrak{p}$-adic field with residue field $\mathbb{F}_q$, E/F be a quadratic semisimple algebra, χ be a character of $E^\times$, and ξ be the "antinorm" χ^{1-c}. Assume the following conditions,*

(a) if $p = 2$, then both E/F and χ are unramified,

(b) if E/F is ramified, then χ is unramified.

Then the ratio $[\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}}) : \mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}})]$ is given as follows.

1. *If E/F is ramified, then,*

$$[\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}}) : \mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}})] = \frac{4}{q+1}.$$

2. *If E/F is inert, then,*

$$[\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}}) : \mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}})] = \begin{cases} 1 & \text{if } \chi \text{ is unramified,} \\ \frac{(q-1)^2}{q^2} & \text{if } \chi \text{ is ramified and } \xi \text{ is unramified,} \\ \frac{\xi(-1)(q+1)^2}{q^2} & \text{if } \xi \text{ is ramified and } \xi^2 \text{ is unramified,} \\ \frac{\xi(-1)(q+1)}{q^2} & \text{if } \xi^2 \text{ is ramified.} \end{cases}$$

3. If $\mathsf{E} = \mathsf{F} \oplus \mathsf{F}$ *is split, then for* $\chi = (\chi_1, \chi_2)$,

$$[\mathcal{P}_{\mathrm{RS}}(W^{\mathrm{new}}) : \mathcal{P}_{\mathrm{RS}}(W^{\mathrm{opt}})] = \begin{cases} 1 & \text{if } \chi \text{ is unramified,} \\ \frac{(q-1)(q-\xi(\varpi_1))(q-\xi(\varpi_2))}{(q+1)q^2} & \text{if } \chi \text{ is ramified and } \xi \text{ is unramified,} \\ \frac{\chi(-1)(q-1)^2 q^{2o(\xi)-1}}{q^{2o(\xi)+1}-q^{2o(\xi)}+2} & \text{if } \xi \text{ is ramified, if } \xi^2 \text{ is unramified} \\ & \text{and exactly one of the } \chi_i \text{ is ramified,} \\ \frac{\chi(-1)(q-1)^3 q^{2o(\xi)-2}}{q^{2o(\xi)+1}-q^{2o(\xi)}+2} & \text{if } \xi^2 \text{ is unramified,} \\ & \text{and both } \chi_i \text{ are ramified,} \\ \frac{\chi(-1)(q-1)^2 q^{2o(\xi)+1}}{(q+1)(q^{2o(\xi)+1}-q^{2o(\xi)}+2)} & \text{if } \xi^2 \text{ is ramified} \\ & \text{and exactly one of the } \chi_i \text{ is ramified,} \\ \frac{\chi(-1)(q-1)^3 q^{2o(\xi)-2}}{(q+1)(q^{2o(\xi)+1}-q^{2o(\xi)}+2)} & \text{if } \xi^2, \chi_1, \text{ and } \chi_2 \text{ are all ramified.} \end{cases}$$

Part III. Arithmetic theory: Harris–Venkatesh vs. Rankin–Selberg

In Part III, we compare the Harris–Venkatesh period $\mathcal{P}_{\mathrm{HV}}$ and the Rankin–Selberg period $\mathcal{P}_{\mathrm{RS}}$. In particular, we prove a multiplicity-one theorem after reduction modulo ℓ^t in order to compare the ratios,

$$[\mathcal{P}_{\mathrm{HV}}(cf^{\mathrm{opt}}) : \mathcal{P}_{\mathrm{HV}}(f^{\mathrm{new}})], \qquad [\mathcal{P}_{\mathrm{RS}}(cf^{\mathrm{opt}}) : \mathcal{P}_{\mathrm{RS}}(f^{\mathrm{new}})],$$

of periods from the Harris–Venkatesh and Rankin–Selberg periods.

We then deduce Theorems 5 and 8. At the end of Part III, we look at how generalizations of these results apply to locally dihedral forms.

9 Liftings of pairings

Let F be a p-adic field, q be the cardinality of the residue field of F, $A \subset \mathbb{C}$ be a principal ideal domain such that p is invertible, $G = \mathrm{GL}_2(F)$, and π be an infinite dimensional irreducible representation of G over $\mathbb{C}$.

We say that π has an A-*model* $\pi_A \subset \pi$ if π_A is an A-module such that $\pi_A \otimes_A \mathbb{C} \xrightarrow{\sim} \pi$, π_A is G-stable, and π_A^H is free of finite type for every compact open subgroup $H \leq G$ (cf. [Vig89]).

Recall that π has a subspace π^{new} of new forms of dimension 1 over $\mathbb{C}$, defined as the subset of vectors fixed by the subgroup

$$U_1(\varpi^k) = \left\{ \gamma \in \mathrm{GL}_2(\mathcal{O}_F) \;\middle|\; \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{\varpi^k} \right\}.$$

If π_A is an A-model of π, then $\pi_A^{\mathrm{new}} := \pi_A \cap \pi^{\mathrm{new}}$ is an A-module of finite type such that $\pi_A^{\mathrm{new}} \otimes_A \mathbb{C} = \pi^{\mathrm{new}}$. Thus it is free of rank 1. In this section, we study the pairings of A-models and their reductions when these models are generated by new vectors. First, we construct some pairings.

Proposition 9.1. *Let π_1, π_2 be two infinite-dimensional irreducible representations of* $\mathrm{GL}_2(F)$ *that are dual to each other in the sense that*

$$\mathrm{Hom}_{\mathbb{C}[\mathrm{GL}_2(F)]}(\pi_1 \otimes \pi_2, \mathbb{C}) \neq 0.$$

Let $\pi_{1,A}, \pi_{2,A}$ be A-models of π_1, π_2 respectively such that both $\pi_{i,A}$ are generated by newforms $v_{1,A}^{\mathrm{new}}, v_{2,A}^{\mathrm{new}}$. There is a unique element $\mathcal{P}_0 \in \mathrm{Hom}_{A[\mathrm{GL}_2(F)]}(\pi_{1,A} \otimes \pi_{2,A}, A)$ such that $\mathcal{P}_0(v_{1,A}^{\mathrm{new}}, v_{2,A}^{\mathrm{new}}) = q - 1$.

The main tool that we use to prove this proposition is the Haar measure on $U_0(\varpi^o)$, where o is the order of χ_i (recall that they are dual to each other)

Lemma 9.2. *There is a Haar measure dh on $U_0(\varpi^o)$ with values in $\mathbb{Z}[1/q]$ and total volume $q-1$.*

Proof. Let H be the maximal pro-p subgroup of $U_0(\varpi^o)$. Then H has the form

$$H = \left\{ \gamma \in U_0(\varpi^o) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{\varpi} \right\}.$$

Thus there is Haar measure valued in $\mathbb{Z}[1/q]$ such that for any open sugroup I of H, $\mathrm{vol}(I) = |H/I|^{-1}$. Then the total mass of G is $|G/H| = q - 1$. □

Proof of Proposition 9.1. As π_1, π_2 are dual to each other, there is a non-trivial pairing $\mathcal{P} \in \mathrm{Hom}_{\mathbb{C}[\mathrm{GL}_2]}(\pi_1 \otimes \pi_2, \mathbb{C})$. We want to study the value of this pairing on π_A. It suffices to consider the value $\mathcal{P}(g_1 v_1^{\mathrm{new}}, g_2 v_2^{\mathrm{new}}) \neq 0$ for each pair $g_1, g_2 \in \mathrm{GL}_2(F)$. By invariance under $\mathrm{GL}_2(F)$, we have that

$$\mathcal{P}(g_1 v_1^{\mathrm{new}}, g_2 v_2^{\mathrm{new}}) = \mathcal{P}\big(v_1^{\mathrm{new}}, g_1^{-1} g_2 v_2^{\mathrm{new}}\big).$$

Integrating over $U_0(\varpi^k)$ and using Lemma 9.2,

$$(q-1)\mathcal{P}\big(v_{1,A}^{\mathrm{new}}, g_1^{-1}g_2v_{2,A}^{\mathrm{new}}\big) = \int_{U_0(\varpi^o)} \mathcal{P}\big(hv_1^{\mathrm{new}}, hg_1^{-1}g_2v_2^{\mathrm{new}}\big)\,dh$$
$$= \mathcal{P}\left(v_1^{\mathrm{new}}, \int_{U_0(\varpi^o)} hg_1^{-1}g_2v_2^{\mathrm{new}}\,dh\right).$$

The last integral defines an element in π_A^{new}, so it can be written as $\lambda v_2^{\mathrm{new}}$ for some $\lambda \in A$. Thus,

$$(q-1)\mathcal{P}(g_1v_{1,A}^{\mathrm{new}}, g_2v_{2,A}^{\mathrm{new}}) = \lambda\mathcal{P}(v_1^{\mathrm{new}}, v_2^{\mathrm{new}}).$$

It follows that $\mathcal{P}(v_1^{\mathrm{new}}, v_2^{\mathrm{new}}) \neq 0$. Then define $\mathcal{P}_0$ by

$$\mathcal{P}_0 := \frac{q-1}{\mathcal{P}(v_1^{\mathrm{new}}, v_2^{\mathrm{new}})}\mathcal{P},$$

so

$$\mathcal{P}_0(g_1v_{1,A}^{\mathrm{new}}, g_2v_{2,A}^{\mathrm{new}}) = \lambda \in A.$$

Uniqueness comes from the definition of $\mathcal{P}_0$. $\square$

The main result of this section is the following multiplicity-one type statement.

Proposition 9.3. *Let π_1, π_2 be two infinite-dimensional irreducible representations of* $\mathrm{GL}_2(F)$ *that are dual to each other in the sense that*

$$\mathrm{Hom}_{\mathbb{C}[\mathrm{GL}_2(F)]}(\pi_1 \otimes \pi_2, \mathbb{C}) \neq 0.$$

Let $\pi_{1,A}, \pi_{2,A}$ be A-models of π_1, π_2 respectively such that $\pi_{i,A}$ are respectively generated by newforms $v_1^{\mathrm{new}}, v_2^{\mathrm{new}}$, let $A \twoheadrightarrow B$ be a surjective homomorphism of rings, and denote $\pi_{i,B} := \pi_i \otimes_A B$,

$v_{i,B}^{\mathrm{new}} := v_i^{\mathrm{new}} \otimes 1$. *Then the cokernel of the homomorphism,*

$$\mathrm{Hom}_{A[\mathrm{GL}_2(F)]}(\pi_{1,A} \otimes \pi_{2,A}, A) \longrightarrow \mathrm{Hom}_{B[\mathrm{GL}_2(F)]}(\pi_{1,B} \otimes \pi_{2,B}, B),$$
$$\mathcal{P} \longmapsto \mathcal{P} \otimes B$$

is annihilated by $(q-1)^2$. *More precisely, for any* $\mathcal{P}_B \in \mathrm{Hom}_{B[\mathrm{GL}_2(F)]}(\pi_{1,B} \otimes \pi_{2,B}, B)$, *we have*

$$(q-1)^2 \cdot \mathcal{P}_B = (q-1) \cdot \mathcal{P}_B(v_{1,B}^{\mathrm{new}}, v_{2,B}^{\mathrm{new}}) \cdot \mathcal{P}_0 \otimes B,$$

where $\mathcal{P}_0$ *is defined in Proposition 9.1.*

We first need the following vanishing lemma.

Lemma 9.4. *Let* $\mathcal{P} \in \mathrm{Hom}_{B[\mathrm{GL}_2(F)]}(\pi_{1,B} \otimes \pi_{2,B}, B)$ *such that*

$$\mathcal{P}(v_{1,B}^{\mathrm{new}}, v_{2,B}^{\mathrm{new}}) = 0.$$

Then $(q-1)\mathcal{P} = 0$.

Proof of Lemma 9.4. By the same argument as in the proof of Lemma 9.2, we have for any $g_1, g_2 \in \mathrm{GL}_2(F)$,

$$(q-1)\mathcal{P}\big(v_{1,B}^{\mathrm{new}}, g_1^{-1}g_2 v_{2,B}^{\mathrm{new}}\big) = \mathcal{P}\left(v_{1,B}^{\mathrm{new}}, \int_{U_1(\varpi^k)} h g_1^{-1} g_2 v_{2,B}^{\mathrm{new}}\, dh\right).$$

The last integral is the image of $\int_{U_1(\varpi^k)} h g_1^{-1} g_2 v_2^{\mathrm{new}}\, dh = \lambda v_2^{\mathrm{new}}$. Thus,

$$(q-1)\mathcal{P}\big(v_{1,B}^{\mathrm{new}}, g_1^{-1}g_2 v_{2,B}^{\mathrm{new}}\big) = \lambda \mathcal{P}(v_{1,B}^{\mathrm{new}}, v_{2,B}^{\mathrm{new}}) = 0.$$

$\square$

Proof of Proposition 9.3. Let $a \in A$ be a lift of $b := \mathcal{P}_B(v_{1,B}^{\mathrm{new}}, v_{2,B}^{\mathrm{new}}) \in B$. Then

$$\mathcal{Q} := (q-1)\mathcal{P}_B - a\mathcal{P}_0 \otimes B \in \mathrm{Hom}_{B[\mathrm{GL}_2(F)]}(\pi_{1,B} \otimes \pi_{2,B}, B),$$

vanishes at $(v_{1,B}^{\text{new}} \otimes v_{2,B}^{\text{new}})$. By Lemma 9.4, $(q-1)\mathcal{Q} = 0$. $\square$

10 Proof of Theorem 8

We need to compare the values of the two pairs:

$$\mathfrak{S}_p(f^{\text{opt}}(z, pz)), \qquad \mathfrak{S}_p\big(\mathrm{Tr}_N\big(f_\chi(z) f_{\chi^{-1}}(pz)\big)\big).$$

We may write both sides in terms of the Harris–Venkatesh period from Equation 2.18 on the space of cusp forms of weight 1:

$$\mathcal{P}_{\mathrm{HV}} : H^0(X_{\Sigma,R}, \omega(-C_\Sigma)) \otimes H^0(X_{\Sigma,R}, \omega(-C_\Sigma)) \longrightarrow R,$$

where Σ is the set of primes dividing N, the level of the form f_χ. Thus we need to compare the following periods:

$$\mathcal{P}_{\mathrm{HV}}(cf^{\text{opt}}), \qquad \mathcal{P}_{\mathrm{HV}}(f^{\text{new}})$$

where c is a constant such that cf^{opt} has an integral q-expansion, and $f^{\text{new}} = f_\chi \otimes f_{\chi^{-1}}$.

Let $\pi(\chi)_\Sigma$ denote the subspace of $H^0(X_\Sigma, \omega(-C_\Sigma))$ generated by f_χ over $\mathbb{Z}[1/N, \chi]$. Then have a decomposition of $\pi(\chi)_\Sigma$ into representations of $\mathrm{GL}_2(\mathbb{Q}_\Sigma) = \prod_{q|N} \mathrm{GL}_2(\mathbb{Q}_q)$:

$$\pi(\chi)_\Sigma = \bigotimes_{q|N} \pi(\chi_q)_{\mathbb{Z}[1/N,\chi]}.$$

Then over $\mathbb{C}$, taking Whittaker functions, we have

$$\pi(\chi)_{\Sigma,\mathbb{C}} \xrightarrow{\sim} \bigotimes_{q|N} \mathcal{W}(\chi_q, \psi_q).$$

By Proposition 9.1, there is a pairing

$$\mathcal{P}_0 : \pi(\chi)_\Sigma \otimes \pi(\chi^{-1})_\Sigma \longrightarrow \mathbb{Z}\left[\frac{1}{N}, \chi\right]$$

such that

$$\mathcal{P}_0\big(f_\chi \otimes f_{\chi^{-1}}\big) = \prod_{q|N}(q-1).$$

By the multiplicity of the pairings, we have

$$\mathcal{P}_0 = \frac{\prod_{q|N}(q-1)}{\mathcal{P}_{\mathrm{RS}}\big(f_\chi \otimes f_{\chi^{-1}}\big)}\mathcal{P}_{\mathrm{RS}}$$

By Proposition 9.3, $\mathcal{P}_{\mathrm{RS}}$ and $\mathcal{P}_{\mathrm{HV}}$ are related as follows. We have the following relation between $\mathcal{P}_{\mathrm{RS}}$ and $\mathcal{P}_{\mathrm{HV}}$:

$$\begin{aligned}\prod_{q|N}(q-1)^2 \cdot \mathcal{P}_{\mathrm{HV}} &= \prod_{q|N}(q-1)\mathcal{P}_{\mathrm{HV}}\big(f_\chi \otimes f_{\chi^{-1}}\big) \cdot \mathcal{P}_0 \\ &= \prod_{q|N}(q-1)\mathcal{P}_{\mathrm{HV}}\big(f_\chi \otimes f_{\chi^{-1}}\big)\left(\frac{\prod_{q|N}(q-1)}{\mathcal{P}_{\mathrm{RS}}\big(f_\chi \otimes f_{\chi^{-1}}\big)} \cdot \mathcal{P}_{\mathrm{RS}}\right) \pmod{\ell^t}.\end{aligned}$$

In particular, this implies the equality of ratios,

$$[\mathcal{P}_{\mathrm{HV}}(cf^{\mathrm{opt}}) : \mathcal{P}_{\mathrm{HV}}(f^{\mathrm{new}})] = [\mathcal{P}_{\mathrm{RS}}(cf^{\mathrm{opt}}) : \mathcal{P}_{\mathrm{RS}}(f^{\mathrm{new}})].$$

By Theorem 8.1, $[\mathcal{P}_{\mathrm{RS}}(f^{\mathrm{opt}}) : \mathcal{P}_{\mathrm{RS}}(f^{\mathrm{new}})]$ is actually in $\mathbb{Q}(\xi + \xi^{-1})^\times$, the field generated by the values of $\xi + \xi^{-1}$. This together with Theorem 8.2 gives Theorem 8.

11 Proof of Theorem 5

Take the denominator in 8 after applying the multiplicity-one argument of Section 9. $[\mathcal{P}_{\mathrm{HV}}(cf^{\mathrm{opt}}) : \mathcal{P}_{\mathrm{HV}}(f^{\mathrm{new}})]$ is a rational number $\frac{a}{b}$ and the conjecture is about new forms; to get m, take $\frac{b}{a}$ and multiply both sides by a to get an algebraic integer while a on the right-hand side is absorbed into the unit u.

12 Generalizations to locally dihedral forms

In this section, we consider the extension of some of our results to non-dihedral forms. Let f be a newform of weight 1 and level N associated to a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C})$. Our basic assumption is that f is locally dihedral: for every prime q dividing N, the restriction ρ_q on the decomposition group $\mathrm{Gal}(\overline{\mathbb{Q}_q}/\mathbb{Q}_q)$ is induced from a character χ_q of a quadratic extension $K_q/\mathbb{Q}_q$:

$$\rho_q = \mathrm{Ind}_{K_q}^{\mathbb{Q}_q}(\chi_q).$$

This assumption is automatically satisfied when $q \neq 2$ or ρ_2 is reducible. For f, $K_N := \prod_{q|N} K_q$ is a quadratic extension of $\mathbb{Q}_N := \prod_{q|N} \mathbb{Q}_q$ and $\chi_N := \prod_{q|N} \chi_q$ is a quadratic character.

12.1 Optimal modular forms

Under the above assumption, we can define a two-variable modular form $f^{\mathrm{opt}}(z_1, z_2)$ in the space of $f(z_1)f^*(z_2)$ generated by $\mathrm{GL}_2(\mathbb{Q}_N)$ analogously to Equation 1.7 but using Whittaker functions where f^* is the dual form to f. More precisely, let φ and φ^* be the automorphic avatars of f and f^*, and let $W(g)$ and $W^*(g)$ be their Whittaker coefficients:

$$\varphi(g) = \sum_{a \in \mathbb{Q}^\times} W\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right),$$
$$\varphi^*(g) = \sum_{a \in \mathbb{Q}^\times} W^*\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right).$$

Then we have decompositions into products of local newforms:

$$W(g) = \prod_{q \leq \infty} W_q(g_q),$$
$$W^*(g) = \prod_{q \leq \infty} W_q^*(g_q).$$

To construct $f^{\mathrm{opt}}(z_1, z_2)$, it suffices to construct the Whittaker coefficients $W^{\mathrm{opt}}(g_1, g_2)$ of its automorphic avatar $\varphi^{\mathrm{opt}}(g_1, g_2)$:

$$\varphi^{\mathrm{opt}}(g_1, g_2) = \sum_{a,b\in\mathbb{Q}^\times} W^{\mathrm{opt}}\left(\begin{pmatrix} a & \\ & 1\end{pmatrix} g_1, \begin{pmatrix} b & \\ & 1\end{pmatrix} g_2\right).$$

We will construct the local Whittaker functions W_p^{opt} and then put them together:

$$W^{\mathrm{opt}}(g_1, g_2) = \prod_{q\leq\infty} W_q^{\mathrm{opt}}(g_{1,q}, g_{2,q}).$$

For $q \nmid N$, we take

$$W_q^{\mathrm{opt}}(g_1, g_2) = W_q(g_1)W_q^*(g_2).$$

For $q \mid N$, we want to construct an optimal element in $\mathcal{W}(\chi_q, \psi_q) \otimes \mathcal{W}(\chi_q^{-1}, \psi_q)$. Let $\xi_q = \chi_q^c \cdot \chi_q^{-1}$ be the character on $K_q^\times$ which brings $x \mapsto \chi(\overline{x}/x)$. We may also consider ξ_q as the restriction of χ_q on K_q^1. Then ξ_q is a ring class character: it is trivial on $(\mathbb{Z}_q + \varpi^{o(\xi_q)}\mathcal{O}_{K,q})^\times$ for some non- minimal number $o(\xi)$ called the order of ξ. We write

$$\mathcal{O}_{o(\xi)} = \mathbb{Z}_q + \varpi^{o(\xi)}\mathcal{O}_{K,q},$$

for the associated order.

Let $\delta_q \in \mathcal{O}_{c,q}$ be a generator of the different ideal $\mathcal{D}_q$ of $\mathcal{O}_{c,q}$, namely the ideal generated by $x - \overline{x}$ for all $x \in \mathcal{O}_{c,q}$. Then for each $a \in \mathcal{O}_{c,q}/\delta_q$, we define the function $\Phi_{a,q}^{\mathrm{opt}}$ to be the characteristic function of,

$$\mathcal{O}_{c,q} + \frac{a}{\delta_q} \subset K_q.$$

We define the one-variable optimal function $W_{a,q}^{\mathrm{opt}}$ for $a \in \mathcal{O}_{c,q}/\delta_q$ and the two-variable optimal function $W_{\mathfrak{q}}^{\mathrm{opt}}$ (cf. Definition 5.7),

$$W_{\mathfrak{a},\mathfrak{q}}^{\mathrm{opt}}(g) := W(g, \chi, \Phi_{\mathfrak{a},\mathfrak{q}})$$
$$W_{\mathfrak{p}}^{\mathrm{opt}}(g_1, g_2) := \sum_{\mathfrak{a} \in \mathcal{O}_{c,q}/\delta_q} W_{\mathfrak{a},\mathfrak{q}}^{\mathrm{opt}}(g_1) \otimes W_{-\mathfrak{a},\mathfrak{q}}^{\mathrm{opt}}(g_2 \epsilon).$$

12.2 Comparison of Harris–Venkatesh periods

For any primes $p, \ell \geq 5$ coprime to N, we want to compare Harris–Venkatesh periods:

$$[\Gamma(1) : \Gamma_0(N)] \cdot [\mathcal{P}_{\mathrm{HV}}(f^{\mathrm{new}}) : \mathcal{P}_{\mathrm{HV}}(f^{\mathrm{opt}})].$$

By the multiplicity one argument of Section 9, it is equal to the ratio of Rankin–Selberg periods

$$[\mathcal{P}_{\mathrm{RS}}(f^{\mathrm{opt}}) : \mathcal{P}_{\mathrm{RS}}(f^{\mathrm{new}})].$$

By Theorem 8.1 and 8.2, this ratio is in $\mathbb{Q}(\xi_N + \xi_N^{-1})^\times$ and has a precise formula.

Proposition 12.1. *If f is a locally dihedral newform of weight 1 and level N with associated quadratic character χ_N, then there exists an element $\beta_{\chi_N} \in \mathbb{Q}(\xi + \xi^{-1})^\times$ such that for almost all primes $p, \ell \geq 5$ coprime to N,*

$$\log_\ell \mathfrak{S}_p\big(\mathrm{Tr}_q^{Np}(f(z) f^*(pz))\big) = \beta_{\chi_N} \log_\ell \mathfrak{S}_p(f^{\mathrm{opt}}(z, pz)).$$

Furthermore, there is a decomposition,

$$\beta_{\chi_N} = \prod_{q \mid N} \beta_{\chi_q},$$

where β_{χ_q} depends only χ_q. Moreover, for odd primes q that are not simultaneously ramified in both K and χ, we have the following explicit formula for β_{χ_q}:

1. *If* q *is ramified in* K*, then*

$$\beta_{\chi_q} = 4.$$

2. *If* q *is inert in* K*, then*

$$\beta_{\chi_q} = \begin{cases} \frac{(q-1)^2}{q^2} & \text{if } \xi \text{ is unramified,} \\ \frac{\xi(-1)(q+1)^2}{q^2} & \text{if } \xi \text{ is ramified and } \xi^2 \text{ is unramified,} \\ \frac{\xi(-1)(q+1)}{q^2} & \text{if } \xi^2 \text{ is ramified.} \end{cases}$$

3. *If* q *is split in* K *(so* $K_q = \mathbb{Q}_q \oplus \mathbb{Q}_q$ *is split with uniformizers* ϖ_1, ϖ_2 *and* $\chi_q = (\chi_1, \chi_2)$*), then*

$$\beta_{\chi_q} = \begin{cases} \frac{(q-1)(q-\xi(\varpi_1))(q-\xi(\varpi_2))}{(q+1)q^2} & \text{if } \chi \text{ is ramified and } \xi \text{ is unramified,} \\ \frac{\chi(-1)(q-1)^2 q^{2o(\xi)-1}}{q^{2o(\xi)+1}-q^{2o(\xi)}+2} & \text{if } \xi \text{ is ramified, if } \xi^2 \text{ is unramified} \\ & \text{and exactly one of the } \chi_i \text{ is ramified,} \\ \frac{\chi(-1)(q-1)^3 q^{2o(\xi)-2}}{q^{2o(\xi)+1}-q^{2o(\xi)}+2} & \text{if } \xi^2 \text{ is unramified,} \\ & \text{and both } \chi_i \text{ are ramified,} \\ \frac{\chi(-1)(q-1)^2 q^{2o(\xi)+1}}{(q+1)(q^{2o(\xi)+1}-q^{2o(\xi)}+2)} & \text{if } \xi^2 \text{ is ramified} \\ & \text{and exactly one of the } \chi_i \text{ is ramified,} \\ \frac{\chi(-1)(q-1)^3 q^{2o(\xi)-2}}{(q+1)(q^{2o(\xi)+1}-q^{2o(\xi)}+2)} & \text{if } \xi^2, \chi_1, \text{ and } \chi_2 \text{ are all ramified.} \end{cases}$$

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