

A homotopical description of Deligne–Mumford compactifications

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## **Abstract**

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In this thesis I will give a description of the Deligne–Mumford properad expressing it as the result of homotopically trivializing  $S^1$  families of annuli (with appropriate compatibility conditions) in the properad of smooth Riemann surfaces with parameterized boundaries. This gives an analog of the results of Drummond-Cole and Oancea–Vaintrob in the setting of properads. We also discuss a variation of this trivialization which gives rise to a new partial compactification of Riemann surfaces relevant to the study of operations on symplectic cohomology.

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## **Dedication**

To my grandparents,  
in Amdapur and in Nagpur.

# Chapter 1: Introduction

## 1.1 Context and Motivation

Two dimensional Topological Field Theories (TFTs), in the sense of [1, Section 2] (also commonly referred to as Topological Conformal Field Theories (TCFTs) [2]), are field theories with operations given by chains on the moduli spaces of Riemann surfaces. In this thesis we study homotopical aspects of the problem of extending operations in the closed sector of such a field theory from the uncompactified moduli spaces of Riemann surfaces of all genera to some of their (partial) compactifications. The compactifications we considered are those arising naturally in mirror symmetry and symplectic topology.

The motivation for considering such extensions comes from a proposal of M. Kontsevich for the construction of the so-called *categorical enumerative invariants* associated with a Calabi–Yau  $A_\infty$ -category, under suitable hypotheses (see [3, 2.23], [4, Section 11]). The Hochschild chain complex of a proper (respectively, smooth) Calabi–Yau  $A_\infty$ -category carries a chain-level two-dimensional right (respectively, left) positive-boundary TFT structure (see [5], [2] for the proper case and [6] for the smooth case). Recall that chain-level two-dimensional right (respectively, left) positive-boundary TFTs are field theories with operations given by chains on the moduli spaces of Riemann surfaces in which each connected component has at least one input (respectively, output). The TFT structure in particular induces a chain-level  $S^1$ -action on the Hochschild chains coming from the operations given by moduli spaces of annuli. Kontsevich proposed that under the assumption that this  $S^1$ -action is homotopically trivial, the TFT structure should extend to include operations coming from chains on the Deligne–Mumford compactification of the moduli spaces of Riemann surfaces. The desired categorical enumerative invariants can then be defined using the induced action of chains on the Deligne–Mumford compactifications.

If the Calabi–Yau  $A_\infty$ -category is Morita equivalent to the Fukaya category of a symplectic manifold or the category of coherent sheaves on a Calabi–Yau manifold then, under nice circumstances (for example when the open-closed map associated with the Fukaya category is an isomorphism), the categorical enumerative invariants are expected to recover, respectively, the Gromov–Witten (GW) invariants and the Bershadsky–Cecotti–Ooguri–Vafa (BCOV) invariants of the underlying manifolds. In such cases *homological mirror symmetry*, which asserts an equivalence between the Fukaya category and a *dg*-enhancement of the derived category of coherent sheaves of a Calabi–Yau mirror pair, can be used to deduce *enumerative mirror symmetry*, which relates the GW and BCOV invariants of the mirror pair.

*Remark 1.1.1.* 1. In what follows, we will mostly work with left positive-boundary TFTs. This is because of our interest in examples coming from symplectic topology which, as explained below in Section 1.5, admit only a left positive-boundary TFT structure in general. For this reason, below whenever we refer to a ‘positive-boundary TFT’ we will mean a *left* positive-boundary TFT, unless specified otherwise.

2. An alternate approach to defining the categorical enumerative invariants was carried out in [1] and further developed in [7]. This approach bypasses the use of Deligne–Mumford spaces by working with a Batalin–Vilkovisky algebra constructed using the uncompactified moduli spaces.

The genus 0,  $n$ -to-1 part of the extension described above was formulated by Kontsevich as the following conjecture: The structure of an algebra over the framed little disk operad along with a homotopy trivialization of the  $S^1$ -action is equivalent to the structure of an algebra over the Deligne–Mumford operad. Recall that an algebra over the framed little disk operad is equivalent to the data of the genus 0,  $n$ -to-1 part of a TFT. Various versions of this conjecture were discussed in [8] and [9], in the category of chain complexes over a characteristic 0 field. The statement in the category of topological spaces was proved in [10]. An extension to higher genus,  $n$ -to-1 operations was proved in [11] where the operad of framed little disks was replaced by the operad of Riemann surfaces with parametrized boundaries.

In this thesis we consider an extension of the above conjecture from the genus 0,  $n$ -to-1 part of a TFT to the entire connected, left positive-boundary part of the TFT. In particular, we shall consider operations coming from Riemann surfaces with possibly higher genus and with multiple inputs and multiple outputs. We do this using the language of ‘properads’.

## 1.2 Properads

Properads are a generalization of operads. Recall that operads encode algebraic structures involving operations with multiple inputs and a single output. Generalizing this, as we shall discuss in Chapter 2 below, properads encode algebraic structures involving operations that have multiple inputs and outputs. Moreover, these operations can be composed along multiple inputs and outputs. Roughly, a properad consists of (i) a spaces of operations with  $n_-$  inputs and  $n_+$  outputs for every choice of non-negative integers  $n_-$  and  $n_+$ , and (ii) the data of compositions of these operations by inserting outputs of one as the inputs of another.

In this thesis, we will consider a modification of the notion of properads which we call ‘input-output’ properads (*io-properads*). These are analogous to usual properads except that there are no operations having 0 inputs and 0 outputs. Consequently, the compositions in properads resulting in operations with no inputs and no outputs are also omitted from the structure. An io-properad is equivalent to the data of a properad whose space of operations in arity  $(0, 0)$  is the final object in the ambient category (a point in the case topological spaces and topological stacks). However, in this thesis we prefer to treat them as algebraic structure in their own right.

The restriction to io-properads has been done for the purpose of technical simplification of some parts of the proof. In the properads which concern us, this omission results in the loss of certain structures coming from operations parametrized by the moduli spaces of stable Riemann surfaces with no inputs and outputs. However, we do not expect the restriction to io-properads to be an essential one and expect our results to hold at the level of usual properads.

We now briefly describe io-properads of Riemann surfaces that are relevant to us. See Chapter 4 below for more details on these properads and the moduli spaces involved.

The data of operations in a TFT coming from connected Riemann surfaces naturally presents itself as representation of a certain properad. We denote the io-properad associated to it by  $\mathfrak{M}$ . The spaces of operations in  $\mathfrak{M}$  are given by the moduli spaces of Riemann surfaces having input and output boundary components, each of which is equipped with an analytic  $S^1$ -parameterization. We refer to  $\mathfrak{M}$  as the *TFT-properad*. There is an io-subproperad  $\partial_+\mathfrak{M}$  of the TFT-properad which encodes the structure given by the connected part of a positive-boundary TFT. It is obtained by considering only the operations indexed by Riemann surfaces which have at least one output. We refer to this io-properad as the  *$\partial_+$ TFT-properad*.

Moreover, to encode the operations coming from the compactified Riemann surfaces we use the io-properad  $\overline{\mathfrak{M}}$ , which we call the *Deligne–Mumford properad*. The spaces of operations in  $\overline{\mathfrak{M}}$  are given by the moduli spaces of stable nodal Riemann surfaces with input and output parametrized boundaries.

Theorem 1.4.2 will be concerned with a variant of  $\overline{\mathfrak{M}}$ , denoted  $\widehat{\mathfrak{M}}$ , based on a new partial compactification  $\widehat{\mathcal{M}}_{g,n_-,n_+}$  of the moduli spaces of Riemann surfaces with parametrized boundaries. This compactification is obtained by considering nodal Riemann surfaces with boundaries, which satisfy the condition that every irreducible component has at least one output boundary. We call it the *symplectic properad*. See Section 1.5 for more on this io-properad.

$\mathfrak{M}^{unst}$ ,  $\partial_+\mathfrak{M}^{unst}$ ,  $\widehat{\mathfrak{M}}^{unst}$ , and  $\overline{\mathfrak{M}}^{unst}$  will be the io-subproperads of  $\mathfrak{M}$ ,  $\partial_+\mathfrak{M}$ ,  $\widehat{\mathfrak{M}}$ , and  $\overline{\mathfrak{M}}$  consisting of (possibly nodal) Riemann surfaces in the unstable range

$$(g, n_-, n_+) \in \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (0, 0, 2), (0, 2, 0)\}.$$

These io-properads have no operations in arities other than those described by the above set. ( $\partial_+\mathfrak{M}^{unst}$  and  $\widehat{\mathfrak{M}}^{unst}$  have no operations in arity  $(1, 0)$  and  $(2, 0)$  either). Finally,  $\mathfrak{M}^{nop}$  will be the io-subproperad of  $\mathfrak{M}^{unst}$  obtained by excluding from  $\mathfrak{M}^{unst}$  the annuli with two input boundaries. In particular  $\mathfrak{M}^{nop}$  has operations in arity  $(0, 1)$ ,  $(1, 0)$ ,  $(1, 1)$  and  $(0, 2)$  and not operations

io-properad	(0,1)	(1,0)	(1,1)	(0,2)	(2,0)
$\mathfrak{M}^{unst}$					
$\partial_+ \mathfrak{M}^{unst}$	$\emptyset$				$\emptyset$
$\widehat{\mathfrak{M}}^{unst}$	$\emptyset$				$\emptyset$
$\overline{\mathfrak{M}}^{unst}$					
$\mathfrak{M}^{nop}$					$\emptyset$

Table 1.1: Summary of io-properads of surfaces in the unstable range. (Nodal entries in the table indicate that in the corresponding arities the respective properads also contain nodal curves in addition to the non-nodal curves. E.g.  $\widehat{\mathfrak{M}}^{unst}$  does not allow nodal annuli in arity (1, 1), but  $\overline{\mathfrak{M}}^{unst}$  does.)

in any other arities. (The superscript ‘nop’ here stands for ‘no pairing’. This is because we think of the operations corresponding to annuli with two inputs of as pairings.)

Table 1.1 summarizes the operations which are included in these io-properads of surfaces in the unstable range.

The main results of the thesis are statements about certain homotopy colimits of io-properads. In order to talk about the homotopy theory of these structures, we outline the construction of a model category structure on the category of topological io-properads. This is carried out in Chapter 3 below.

*Remark 1.2.1.* The structure of a TFT has the additional data of compositions coming from taking disjoint unions of Riemann surfaces. In order to capture this data, it is necessary to use a generalization of properads, known as *PROPs*. In addition to the compositions similar to those in properads, known as the *vertical compositions*, PROPs also have *horizontal compositions*. These latter compositions can be used to encode the operation of taking disjoint unions of Riemann surfaces. We however choose to work with properads instead of PROPs, since the homotopy theory of properads is easier to handle than that of PROPs.

### 1.3 Properads in Stacks

It is well known that the moduli spaces of stable curves admit natural lifts to topological stacks. One of the main results of this thesis concerns properad  $\overline{\mathfrak{M}}$  built out of these. As a consequence, we are required to deal with properads valued in topological stacks. This introduces an additional layer of complexity: since topological stacks form a 2-category, it is natural to consider properads in which compositions are associative only up to certain natural equivalences. To discuss the homotopy theory of such properads, instead of dealing with the homotopy theory of 2-properads, we find it convenient to use the language of  $\infty$ -properads. In, Chapter 8 we outline this formalism and describe how  $\overline{\mathfrak{M}}$  can be viewed as an  $\infty$ -properad.

However, for the most part we carry out our discussion without encountering this issue. Considering the properads in stacks is necessary only for Theorem 1.4.1 (2). Theorem 1.4.1 (1) as well as Theorem 1.4.2 deals only with moduli spaces which admit realizations in topological spaces, and hence can be stated and proved entirely in the category of properads in topological spaces. Specifically, throughout Chapters 2 to 6, we only work with properads in topological spaces.

On the other hand, the statement of Theorem 1.4.1 (2) is understood as a homotopy colimit computation in the category of properads in topological stacks. More precisely, as the ( $\infty$ -categorical) colimit in the  $\infty$ -category of  $\infty$ -properads in topological stacks. In the proof of Theorem 1.4.1 (2) in Chapter 7, which essentially is the only place where we deal with a properad in stacks, our computation of homotopy colimit yields a topological properad with the property that for any  $(n_-, n_+)$  the space of operations with  $n_-$  inputs and  $n_+$  outputs has the same weak homotopy type as the stack  $\overline{\mathcal{M}}_{n_-, n_+}$ , in the sense described in Chapter 8. The results outlined in Chapter 8 can then be used to argue that this homotopy colimit computation carried out in the category of topological properads can be upgraded to a  $\infty$ -categorical colimit of  $\infty$ -properads, with the resulting colimit homotopy equivalent (as an  $\infty$ -properad) to  $\overline{\mathfrak{M}}$ .



## 1.4 Statements of the Theorems

Our main results are as follows:

**Theorem 1.4.1.** 1.  $\mathfrak{M}$  is the homotopy colimit of the following diagram in the category of io-properads:

$$\begin{array}{ccc} \partial_+ \mathfrak{M}^{unst} & \longrightarrow & \mathfrak{M}^{nop} \\ \downarrow & & \\ \partial_+ \mathfrak{M} & & \end{array} \quad (1.4.1)$$

2.  $\overline{\mathfrak{M}}$  is the homotopy colimit of the following diagram in the category of io-properads:

$$\begin{array}{ccc} \mathfrak{M}^{unst} & \longrightarrow & \overline{\mathfrak{M}}^{unst} \\ \downarrow & & \\ \mathfrak{M} & & \end{array} \quad (1.4.2)$$

The first part describes the extension from a positive-boundary TFT to a TFT. This is achieved by the introduction of a *trace* operation (the point consisting the space  $\mathfrak{M}^{nop}(1, 0)$ ) dualizing the *cotrace* operation (the point consisting the space  $\mathfrak{M}^{nop}(0, 1)$ ).

The second part describes the extension of a TFT to an algebra over the Deligne–Mumford properad. An action of properad  $\mathfrak{M}$  in particular induces an action of the subproperad of disks and annuli. The second statement above can be interpreted as saying that, homotopically, an extension of this action to the properad  $\overline{\mathfrak{M}}^{unst}$  consisting of disks and possibly nodal annuli determines an extension of the TFT structure to an algebra over the Deligne–Mumford properad.

Note that spaces of operations in  $\mathfrak{M}^{unst}$  in arities  $(1, 1)$ ,  $(0, 2)$ , and  $(2, 0)$  are given by moduli spaces of annuli and are all homotopy equivalent to  $S^1$ . By fixing an annulus in any of these spaces and varying the parametrization of one of its boundaries by rotations, we obtain an  $S^1$ -parametrized family of operations whose inclusion inside the respective moduli space is a homotopy equivalence. On the other hand, the corresponding spaces of operations in  $\overline{\mathfrak{M}}^{unst}$ , given by moduli spaces of possibly nodal annuli, are all contractible. In particular, the inclusion  $\mathfrak{M}^{unst} \hookrightarrow \overline{\mathfrak{M}}^{unst}$  provides any  $S^1$ -family of annuli as described above with a null-homotopy inside  $\overline{\mathfrak{M}}^{unst}$ . Such a null-

homotopy can be obtained, for example, by simultaneously scaling the modulus of every annulus in the family, eventually making it infinite. Lastly, we note that operations in arity  $(0, 1)$  and  $(1, 0)$  in both  $\mathfrak{M}^{unst}$  and  $\overline{\mathfrak{M}}^{unst}$  are given by points and the inclusion  $\mathfrak{M}^{unst} \hookrightarrow \overline{\mathfrak{M}}^{unst}$  acts by identity on these.

Pushout of  $\mathfrak{M}$  along  $\mathfrak{M}^{unst} \hookrightarrow \overline{\mathfrak{M}}^{unst}$  thus has the effect of providing null-homotopies of the  $S^1$ -families of annuli in arities  $(1, 1)$ ,  $(0, 2)$ , and  $(2, 0)$ , described above. Since  $\mathfrak{M}^{unst} \hookrightarrow \overline{\mathfrak{M}}^{unst}$  is a map of properads, these homotopies are compatible, in an appropriate sense, with properad compositions with other  $S^1$ -families, as well as with the disks in arities  $(1, 0)$  and  $(0, 1)$ . In this sense, the second part of Theorem 1.4.1 can thus be interpreted as saying that, homotopically, an extension of  $\mathfrak{M}^{unst}$ -action to an  $\overline{\mathfrak{M}}^{unst}$ -action corresponds to providing homotopy trivializations of the  $S^1$ -families of operations in arities  $(1, 1)$ ,  $(0, 2)$ , and  $(2, 0)$ , such that these trivializations are compatible with properad compositions in the manner described above.

We also prove Theorem 1.4.2 which describes an extension of the positive-boundary TFT-properad to the symplectic properad. The motivation for studying such extensions comes from looking at the examples coming from symplectic cohomology (see 1.5 below).

**Theorem 1.4.2.**  *$\widehat{\mathfrak{M}}$  is the homotopy colimit of the following diagram in the category of io-properads*

$$\begin{array}{ccc} \partial_+ \mathfrak{M}^{unst} & \longrightarrow & \widehat{\mathfrak{M}}^{unst} \\ \downarrow & & \\ \partial_+ \mathfrak{M} & & \end{array} \quad (1.4.3)$$

Theorem 1.4.2 can be interpreted as saying that the data needed for the extension of an  $\partial_+ \mathfrak{M}$ -action to an  $\widehat{\mathfrak{M}}$ -action is precisely the data of extension of the induced action of the subproperad  $\partial_+ \mathfrak{M}^{unst}$  to  $\widehat{\mathfrak{M}}^{unst}$ . Note that in this case the space of arity  $(0, 2)$  operations of  $\widehat{\mathfrak{M}}^{unst}$  is again contractible, however, the space of operations in arity  $(1, 1)$  is homotopy equivalent to  $S^1$ . The extension from  $\partial_+ \mathfrak{M}^{unst}$  to  $\widehat{\mathfrak{M}}^{unst}$  thus corresponds to homotopy trivializing  $S^1$ -family of copairings coming from operations in arity  $(0, 2)$ , leaving the operations in arity  $(1, 1)$  unchanged.

In the work under preparation [12], it is shown that such trivializations are related to conjectures of Kontsevich regarding generalized versions of the categorical Hodge-de Rham degeneration for smooth and for proper dg-categories. These conjectures were proved in [12] for certain classes of Fukaya categories, but are known to be false in general [13].

## 1.5 Symplectic and Quantum cohomology

Symplectic cohomology and Quantum cohomology provide prototypical examples of the extensions described in the above theorems.

Symplectic cohomology  $SH^*(X)$  is an invariant associated with a symplectic manifold  $(X, \omega)$  with bounded geometry.  $SH^*(X)$  is known to admit a (positive boundary) topological quantum field theory (TQFT) structure. This was introduced in [14], with a detailed construction carried out in [15]. This structure is expected to lift to a chain-level action of the moduli spaces of Riemann surfaces with parametrized boundaries.

This structure is defined in terms of counts of maps  $u$  from a Riemann surface  $(\Sigma, j)$  to  $X$  which satisfy a perturbed Cauchy-Riemann equation:

$$(du - Z \otimes \beta)^{0,1} = 0 \tag{1.5.1}$$

Here  $Z$  and  $\beta$  are, respectively, auxiliary choices of a Hamiltonian vector field on  $X$  and a 1-form on  $\Sigma$ , both having a certain standard form near the boundaries of  $\Sigma$ . When  $X$  is not necessarily compact, we must in addition require that  $d\beta \leq 0$ . This is necessary to prevent a sequence of such maps from escaping to infinity, which in turn is needed to ensure compactness of the moduli space of these maps. Note that this condition imposes the restriction that  $n_+ \geq 1$ . Thus, in general, such operations can only be defined for surfaces which have at least one output and therefore we only expect a chain-level  $\partial_+ \mathfrak{M}$ -action on  $SH^*(X)$ . In particular, trace operations coming from disks with an input boundary are not defined on  $SH^*(X)$  in general.

The definition of these operations can be extended to include maps from nodal curves which

carry a 1-form  $\beta$  on its non-singular locus satisfying  $d\beta \leq 0$  and vanishing near the nodes. Curves appearing in the compactification  $\widehat{\mathcal{M}}_{g,n_-,n_+}$  satisfy this condition. The chain-level action of  $\partial_+\mathfrak{M}$  on the symplectic cohomology is thus expected to extend to an action of the properad  $\widehat{\mathfrak{M}}$ . Theorem 1.4.2 provides the extension statement at the level of underlying properads.

The action of the properad  $\partial_+\mathfrak{M}$  includes the data of copairings induced by the action of cylinders with both ends outputs. According to Theorem 1.4.2, the homotopy trivialization of this family of copairings, given by the construction of operations corresponding to nodal annuli with two outputs, is the homotopically essential data providing the extension from the  $\partial_+\mathfrak{M}$ -action to the  $\widehat{\mathfrak{M}}$ -action.

In the case when  $X$  is compact, the condition  $d\beta \leq 0$  may be dropped. As a consequence, operations can be defined using Riemann surfaces without any restriction on the number of outputs and thus the chain level  $\partial_+\mathfrak{M}$ -action is expected to extend to an  $\mathfrak{M}$ -action. The corresponding extension statement at the level of underlying properads is given by Theorem 1.4.1. The compactness of the ambient manifold  $X$  is reflected in the properad-level statement as the existence of the trace operations indexed by disks with an input boundary which dualize the cotrace operations mentioned in the previous paragraph.

Moreover, in this case, similar to the extension from  $\partial_+\mathfrak{M}$  to  $\widehat{\mathfrak{M}}$  described above, the action of  $\mathfrak{M}$  is expected to extend to an action of *all* stable curves in  $\overline{\mathfrak{M}}$ . As a result, one would obtain a chain-level action of the properad  $\overline{\mathfrak{M}}$  on  $SH^*(X)$ . This corresponds to the extension from  $\mathfrak{M}$  to  $\overline{\mathfrak{M}}$  described in second part of Theorem 1.4.1.

Recall that for a closed symplectic manifold  $X$ , the symplectic cohomology of  $X$  is isomorphic to its quantum cohomology  $QH^*(X)$ . The quantum cohomology itself is expected to admit a chain-level action of a properad  $\overline{\mathcal{M}}$ : the spaces of operations in  $\overline{\mathcal{M}}$  are given by the usual Deligne–Mumford moduli spaces of stable boundary-less nodal Riemann surfaces with input-output marked points. The compositions are given by simply concatenating stable curves along suitable marked points. The  $\overline{\mathcal{M}}$ -action is defined by counting pseudo-holomorphic maps from stable (boundary-less) Riemann surfaces with marked points into  $X$ , in a manner similar to the  $\partial_+\mathfrak{M}$ -action on  $SH^*(X)$ .

This action is equivalent to the data of a chain-level lift of the *Cohomological Field Theory* (CohFT) structure on  $H^*(X)$  ([16]). The properad  $\overline{\mathcal{M}}$  is weakly homotopy equivalent to  $\overline{\mathfrak{M}}$  (see Section 4.2) and the isomorphism between  $SH^*(X)$  and  $QH^*(X)$  should intertwine these actions. Using Theorem 1.4.1(2) it follows that the data of the CohFT structure on quantum cohomology of  $X$  is homotopically equivalent to the data of the  $\mathfrak{M}$ -action along with a (suitably compatible) homotopy trivializations of the operations coming from the  $S^1$ -families of annuli in  $\mathfrak{M}^{unst}$ .

## 1.6 Outline of the thesis

We start by giving a brief overview of properads in Chapter 2. Here we introduce io-properads as algebras over a monad  $\mathbb{G}$ . We also include a discussion of pushouts in the category of topological io-properads.

In Chapter 3 we describe a model structure on the category of topological io-properads using the description of properads as algebras over the monad  $\mathbb{G}$ . We then use the bar construction associated with monad  $\mathbb{G}$  to construct cofibrant resolutions of properads, and provide a description of homotopy pushouts of properads. Most of the technical details required to prove the statements appearing here are deferred to Appendix A.

In Chapter 4 we give the precise definitions of the properads which appear in the main theorems and of the moduli spaces on which these properads are based.

We then carry out the proof of Theorem 1.4.2 in Chapter 5. The technical heart of the proof is the constructions in Section 5.3, where local weak equivalences are proved by exhibiting explicit simplicial homotopies. These homotopies are the simplicial analogues of the geometric homotopies constructed in [11] for operads. Additional work is required in our case as we work with properads which have compositions indexed by directed graphs more general than trees and also because we are required to work with cofibrant resolutions which are of a simplicial nature as opposed to the geometric ones used in [11]

In Chapter 6 we prove the first part of Theorem 1.4.1. The pushout is computed with  $\mathfrak{M}^{unst}$  and  $\mathfrak{M}$  replaced by homotopy equivalent properads of spotted surfaces (see Section 6.1). The theorem is

then proved by the constructions similar to those in the proof of Theorem 1.4.2, with the weighted marked point playing a role analogous to nodes there.

In Chapter 7 the proof of the second part of Theorem 1.4.1 is provided. The proof in this case is obtained by combining the constructions used in the proofs of Theorem 1.4.2 and the first part of Theorem 1.4.1. After a few modifications (see Remarks 7.2.2 and 7.3.1), the proof follows a scheme very similar to that in the proof of Theorems 1.4.2 and 1.4.1(1).

Finally, in Chapter 8 we outline a formalism for  $\infty$ -io-properads and show that the homotopy colimits computed in earlier chapters in the category of topological properads remain valid in the context of  $\infty$ -properads.

## Chapter 2: An Overview of Properads

*Notation 2.0.1.* Let **Top** denote the category of compactly generated Hausdorff topological spaces. Let **TopSeq**<sup>io</sup> denote the category **Top** <sup>$\mathbb{N} \times \mathbb{N} \setminus (0,0)$</sup>  of bi-sequences of topological spaces indexed by  $\mathbb{N} \times \mathbb{N} \setminus (0,0)$ . Here the superscript ‘io’, short for ‘input-output’, refers to the fact that these sequences do not contain a  $(0,0)$  component. Below, we will often refer to these sequences as *topological io-sequences*.

In this chapter we present a brief overview of input-output properads in the category of topological spaces. For a detailed treatment, in the context of *usual* properads, see [17]. We shall define properads as algebras over a certain monad  $\mathbb{G}$ . This point of view will be convenient when we discuss the cofibrant resolutions of properads below.

### 2.1 Monads and algebras over them

**Definition 2.1.1.** A **monad** (also called a **triple**)  $T$  on a category  $C$ , is a functor  $T: C \rightarrow C$  along with natural transformations  $\mu: T \circ T \rightarrow T$  and  $\eta: \mathbb{1}_C \rightarrow T$  such that the following diagrams commute:

$$\begin{array}{ccc}
 T \circ T \circ T & \xrightarrow{\mu \circ 1} & T \circ T \\
 \downarrow 1 \circ \mu & & \downarrow \mu \\
 T \circ T & \xrightarrow{\mu} & T
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \mathbb{1}_C \circ T & \xrightarrow{\eta \circ 1} & T \circ T & \xleftarrow{1 \circ \eta} & T \circ \mathbb{1}_C \\
 & \searrow & \downarrow \mu & \swarrow & \\
 & & T & & 
 \end{array}$$

The natural transformations  $\mu$  and  $\eta$  are referred to as the composition and the unit of  $T$ , respectively. The first commutative diagram describes the associativity of  $\mu$ , whereas the second one describes the unitality of  $\eta$ .

An **algebra over**  $T$  is an object  $X \in C$  equipped with a map  $m: TX \rightarrow X \in C$  such that the

following diagrams commute:

$$\begin{array}{ccc}
TTX & \xrightarrow{\mu(X)} & TX \\
\downarrow Tm & & \downarrow m \\
TX & \xrightarrow{m} & T
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{\eta(X)} & TX \\
\parallel & & \downarrow m \\
& & X
\end{array}$$

We denote the category of  $T$ -algebras in  $C$  by **T-Alg**. For any  $X \in C$ ,  $TX$  is canonically a  $T$ -algebra. This defines the free  $T$ -algebra functor  $T: C \rightarrow \mathbf{T-Alg}$  which is left adjoint to the forgetful functor  $Forget: \mathbf{T-Alg} \rightarrow C$ .

## 2.2 Graphs

To define the monad  $\mathbb{G}$ , we need to fix some conventions regarding graphs.

By a *directed graph*  $G$  we mean the data of

- A finite set of vertices  $V(G)$
- A finite set of directed edges  $E(G)$ , equipped with source and target incidence maps  $s, t: E(G) \rightarrow V(G)$ .
- A set of input legs  $in(G)$  and output legs  $out(G)$  (also called external edges/tails), along with incidence maps  $t_l: in(G) \rightarrow V(G)$  and  $s_l: out(G) \rightarrow V(G)$ .
- An ordering on the sets  $in(G)$ ,  $out(G)$  and on  $in(v)$ ,  $out(v)$  for every  $v \in V(G)$ , where  $in(v)$  (respectively,  $out(v)$ ) denotes the set of edges and legs directed into (respectively, out of)  $v$ .

We say that  $G$  is connected if the geometric realization of  $G$  is connected. Here by the geometric realization of  $G$  we mean the topological space obtained by considering a union of points and intervals indexed by vertices and edges/legs of  $G$  respectively, with the endpoints of the intervals identified with the vertices according to the incidence maps.



By an *isomorphism of directed graphs*  $f: G \rightarrow H$ , we mean a collection of bijections

$$f_v: V(G) \rightarrow V(H), f_e: E(G) \rightarrow E(H), f_{l_{in}}: in(G) \rightarrow in(H), \text{ and } f_{l_{out}}: out(G) \rightarrow out(H)$$

which preserve the incidence maps and the orderings.

By a *directed cycle* in  $G$  we mean a sequence of edges  $e_1, \dots, e_n \in E(G)$  such that  $t(e_i) = s(e_{i+1})$  for  $1 \leq i \leq n-1$  and  $s(e_1) = t(e_n)$ .

We say that  $G$  is *input-output* if either  $in(G)$  or  $out(G)$  is non-empty i.e. if  $G$  has at least one input leg or output leg.

**Definition 2.2.1.** By an *input-output directed acyclic graph (ioda-graph)* we mean an input-output, connected directed graph which contains no directed cycles.

### 2.3 The monad $\mathbb{G}$

For an ioda-graph  $G$  and  $P \in \mathbf{TopSeq}^{\mathbf{i0}}$ , let

$$P(G) := \prod_{v \in V(G)} P(|in(v)|, |out(v)|)$$

denote the space of  $P$ -labelings of  $G$ . Denote by  $\mathcal{G}_{n_-, n_+}$  the collection of graphs one from each isomorphism class of ioda-graphs having  $n_-$  input and  $n_+$  output leaves. Set

$$\mathbb{G}_{n_-, n_+}(P) := \coprod_{G \in \mathcal{G}_{n_-, n_+}} P(G).$$

Denote by  $\mathbb{G}(P) \in \mathbf{TopSeq}^{\mathbf{i0}}$  the bi-sequence given by  $\mathbb{G}(P)(n_-, n_+) = \mathbb{G}_{n_-, n_+}(P)$ .

We now describe how to make  $\mathbb{G}$  into a monad. To do this we need to define operations  $\mu: \mathbb{G}\mathbb{G} \rightarrow \mathbb{G}$  and  $\eta: \mathbb{1} \rightarrow \mathbb{G}$ , satisfying the condition mentioned in Definition 2.1.1.

For  $P \in \mathbf{TopSeq}^{\text{io}}$ ,  $\mathbb{G}\mathbb{G}(P)$  is given by

$$\mathbb{G}\mathbb{G}(P)(n_-, n_+) = \coprod_{G \in \mathcal{G}_{n_-, n_+}} \mathbb{G}(P)(G).$$

Thus an element in  $\mathbb{G}\mathbb{G}(P)$  is represented by a  $\mathbb{G}(P)$ -labeling of an ioda-graph. Such a decorated graph can be identified with a ‘nested’ ioda-graph. Forgetting the nesting gives an ioda-graph with  $P$ -labelings. This defines the natural transformation  $\mu: \mathbb{G}\mathbb{G} \rightarrow \mathbb{G}$  (see [18] for details).

Let us now turn to the unit. Given  $P \in \mathbf{TopSeq}^{\text{io}}$ , consider the maps  $P(n_-, n_+) \rightarrow \mathbb{G}_{n_-, n_+}(P)$  which identify  $P(n_-, n_+)$  with  $P$ -labelings of the  $(n_-, n_+)$ -corolla, the unique graph in  $\mathcal{G}_{n_-, n_+}$  having a single vertex and no internal edges. This defines the natural transformation  $\eta: \mathbb{1} \rightarrow \mathbb{G}$ .

It is not difficult to verify that  $\mu$  and  $\eta$  satisfy the associativity and unitality conditions in Definition 2.1.1. We record here the conclusion for future reference:

**Lemma 2.3.1.**  *$(\mathbb{G}, \mu, \eta)$  is a monad.*

□

## 2.4 io-Properads as monad algebras

**Definition 2.4.1.** An *io-properad* in topological spaces is an algebra over the monad  $(\mathbb{G}, \mu, \eta)$  in the category  $\mathbf{TopSeq}^{\text{io}}$ .

We will denote the category of topological io-properads by  $\mathbb{G}\mathbf{Alg}$ .

*Remark 2.4.2.* In particular if  $P$  is an io-properad, for any ioda-graph  $G$  we have a map

$$P(G) \rightarrow P(|\text{in}(G)|, |\text{out}(G)|)$$

We will refer to it as the *properad composition indexed by  $G$*  and denote it by  $\mu_G$ . In particular, we have the following structure maps:

1. Using the graphs as in Figure 2.1(a), we obtain a left  $\Sigma_{n_-}$  and a right  $\Sigma_{n_+}$  action on  $P(n_-, n_+)$ .

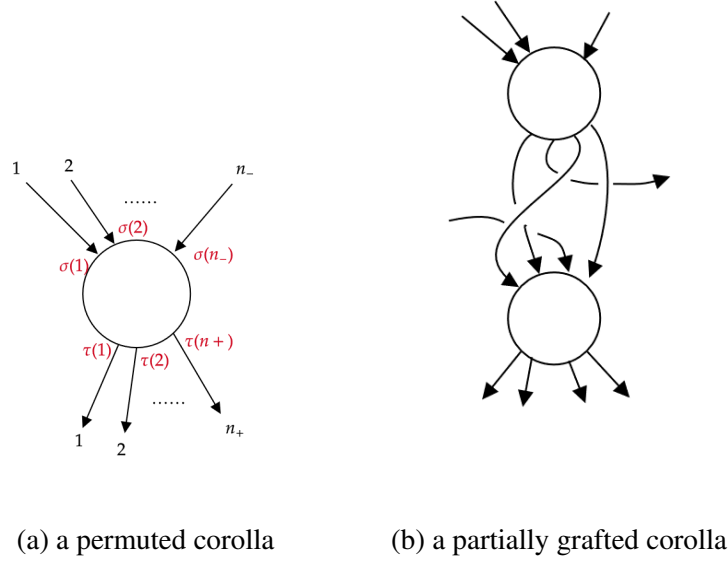


Figure 2.1

2. Using the graphs as in Figure 2.1(b), for any  $n_-, n_+, m_-, m_+ \geq 0$  and *non-empty* subsets  $s_-$  and  $s_+$  of  $[n_+]$  and  $[m_-]$  respectively, with an identification  $\phi: s_- \rightarrow s_+$ , we get a composition map

$$\circ_\phi: P(n_-, n_+) \times P(m_-, m_+) \rightarrow P(n_- + m_- - |s_+|, n_+ + m_+ - |s_-|)$$

provided  $n_-, m_- - |s_+|, n_+ - |s_-|$ , and  $m_+$  are not all simultaneously 0. This slightly unnatural constraint is imposed by the fact that we work with io-properads which have no  $(0, 0)$  component.

These compositions are associative and are equivariant with respect to the  $\Sigma \times \Sigma^{op}$ -actions, described in point (1), in a suitable sense.

Conversely, it can be shown that  $\Sigma \times \Sigma^{op}$ -actions as in point (1) and (associative,  $\Sigma \times \Sigma^{op}$ -equivariant) composition maps as in point (2) define the structure of a  $(\mathbb{G}, \mu, \eta)$ -algebra over  $P$ . We will not use this viewpoint and hence omit the details.

## 2.5 Pushouts of topological io-properads

The main results in this paper are statements about certain homotopy pushouts in the category of topological io-properads. An explicit description of pushouts in the category of topological io-properads is used in Chapters 5, 6, and 7 below to give a presentation of these homotopy pushouts. We now outline this description.

*Notation 2.5.1.* To distinguish colimits in the category of topological io-sequences from the colimits in topological io-properads, we will indicate the latter with a superscript  $\mathbb{G}\mathbf{Alg}$ . Below, whenever we use  $coeq$ ,  $\sqcup$ , and  $\coprod$  without a superscript  $\mathbb{G}\mathbf{Alg}$ , it is understood that they refer to the respective operations in  $\mathbf{TopSeq}^{\text{io}}$ .

**Lemma 2.5.2.** *Let  $P \leftarrow R \rightarrow Q$  be a diagram of topological io-properads. Then the pushout  $P \sqcup_R^{\mathbb{G}\mathbf{Alg}} Q$  can be obtained as the following coequalizer in  $\mathbf{TopSeq}^{\text{io}}$*

$$coeq \left[ \mathbb{G}(\mathbb{G}P \coprod_{\mathbb{G}R} \mathbb{G}Q) \rightrightarrows \mathbb{G}(P \coprod_R Q) \right]$$

with the io-properad structure induced from that of  $\mathbb{G}(P \sqcup_R Q)$ . Here,

1. the first arrow is given by applying functor  $\mathbb{G}$  to the map  $\mathbb{G}P \sqcup_{\mathbb{G}R} \mathbb{G}Q \rightarrow P \sqcup_R Q$  which itself is induced from the  $\mathbb{G}$ -algebra structure maps  $\mathbb{G}P \rightarrow P$ ,  $\mathbb{G}Q \rightarrow Q$ , and  $\mathbb{G}R \rightarrow R$
2. the second arrow is induced via the universal property of free  $\mathbb{G}$ -algebras using the map  $\mathbb{G}P \sqcup_{\mathbb{G}R} \mathbb{G}Q \rightarrow \mathbb{G}(P \sqcup_R Q)$  which in turn is the pushout of the maps given by applying functor  $\mathbb{G}$  to  $P \rightarrow P \sqcup_R Q$ ,  $Q \rightarrow P \sqcup_R Q$ , and  $R \rightarrow P \sqcup_R Q$  □

The above lemma can be proved by directly verifying that the coequalizer with the given  $\mathbb{G}$ -algebra structure satisfies the universal property of the pushout.

*Notation 2.5.3.* We say that a subgraph  $H$  of an ioda-graph  $G$  is *collapsible* if the graph  $G/H$ , obtained by replacing  $H$  with a single vertex, contains no directed cycles. More precisely: consider the topological io-sequence  $\mathcal{G} = \{\mathcal{G}_{n-, n+}\}$  (with the discrete topology), where, as in Section 2.3,  $\mathcal{G}$

is the collection containing one representative from each isomorphism class of ioda-graphs . It has a natural  $\mathbb{G}$ -algebra structure. We say that  $H \subset G$  is collapsible if  $\mu_K(\prod_{v \in V(K)} H_v) = G$  for some ioda-graphs  $K$  and  $H_v, v \in V(K)$ , such that  $H_v$ , considered as a subgraph of  $G$ , coincides with  $H$  for some  $v$ .

Unraveling the coequalizer in Lemma 2.5.2, we get that the topological io–sequence underlying  $P \sqcup_R^{\mathbb{G}\text{Alg}} Q$  can be described as the quotient

$$\frac{\mathbb{G}(P \sqcup Q)}{\sim}$$

where  $\sim$  is the equivalence relation generated by the following identifications:

1. Let  $G$  be a  $(P \sqcup Q)$ -labeled graph which has a collapsible subgraph  $H \subset G$  with all the vertices labeled by  $P$  (respectively,  $Q$ ). Then  $G$  is identified with the graph obtained by collapsing  $H$  to a single vertex labeled by the element of  $P$ (respectively,  $Q$ ) obtained by applying properad composition (indexed by  $H$ ) to the labels of  $H$ .
2. Suppose that the maps  $R \rightarrow P$  and  $R \rightarrow Q$  are denoted by  $i$  and  $j$  respectively. Let  $G$  be a  $(P \sqcup Q)$ -labeled graph with a vertex  $v \in G$  labeled by an element  $i(r) \in i(R) \subset P$  for some  $r \in R$ . Then  $G$  is identified with the graph obtained by switching the label of  $v$  to  $j(r) \in j(R) \subset Q$ .

This is analogous to the description of pushouts of operads presented in [11, Section 2.3].

## 2.6 Properads as algebras over a colored operad

We end this chapter by describing an alternate characterization of properads as algebras over a colored operad  $\mathbf{Prpd}_{\text{red}}$ . We will use this alternate description in Chapter 3 below to construct a model structure on topological io-properads and also in Chapter 8 to discuss the notion of an  $\infty$ -properad.

In this subsection we assume basic familiarity with colored operads and algebras over them. We refer the reader to [19, Section 1] for an exposition to these notions.

**Definition 2.6.1.** By a *vertex-ordered ioda-graph* we mean a graph as in Section 2.2, with the additional data of an ordering on the set of vertices. Isomorphisms of such graphs are required to preserve this ordering. We refer to isomorphisms of underlying ioda-graphs as *unordered-isomorphisms*.

### 2.6.1 Colored Operad $\mathbf{Prpd}_{\text{red}}$

Let us start by describing the set-valued colored operad  $\mathbf{Prpd}_{\text{red}}$  which encodes input-output properads. The colors of  $\mathbf{Prpd}_{\text{red}}$  are given by  $\mathbb{N} \times \mathbb{N} \setminus (0, 0)$ . Let  $\underline{n}_-^1, \underline{n}_-^2, \dots, \underline{n}_-^k$ , and  $\underline{n}_+$  be elements of  $\mathbb{N} \times \mathbb{N} \setminus (0, 0)$ . The operations corresponding to these colors,

$$\mathbf{Prpd}_{\text{red}}(\underline{n}_-^1, \underline{n}_-^2, \dots, \underline{n}_-^k; \underline{n}_+),$$

are given by the set of isomorphism classes of vertex-ordered ioda-graphs with input-output profile  $\underline{n}$  and where the input-output profiles of the vertices, in the vertex-ordering, are given by the sequence  $\underline{n}_-^1, \underline{n}_-^2, \dots, \underline{n}_-^k$ . The  $\Sigma$ -action on the operad is given by permuting the ordering of the vertices and the operad composition is given by graph substitution.

The following statement is not difficult to see:

**Lemma 2.6.2.** *A  $\mathbf{Prpd}_{\text{red}}$ -algebra in  $\mathbf{Top}$  is precisely a topological io-properad.* □

We note here a fact which will be important later in Chapter 8:

**Lemma 2.6.3.**  *$\mathbf{Prpd}_{\text{red}}$  is  $\Sigma$ -cofibrant, in other words the  $\Sigma$ -action on  $\mathbf{Prpd}_{\text{red}}$  is free.*

*Proof.* This follows from the fact that vertex-ordered ioda-graphs have no non-trivial unordered automorphism (since ioda-graphs have no non-trivial automorphisms). □

*Remark 2.6.4.* This is false for a *non-io* vertex-ordered graph. Figure 2.2 shows an example.

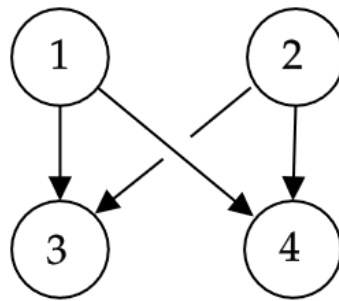


Figure 2.2: The *non-io* vertex-ordered graph shown above has a non-trivial unordered-automorphism which swaps vertices 1, 2 and 3,4 (at each vertex the order of edges is assumed to be the one induced from the planar embedding)

## Chapter 3: Model Category Structure and cofibrant resolutions

### 3.1 Model Category structure

Model categories provide a natural setting for discussing homotopy theory in an abstract categorical setting. A model structure on a category consists of three distinguished classes of morphisms: fibrations, cofibrations, and weak equivalences, satisfying certain conditions. A model structure in particular allows us to define homotopy limits/colimits in a category. For more details on model categories and homotopy limits/colimits see [20].

In order to make sense of the homotopy colimits in the category of topological io-properads which appear in Theorems 1.4.1 and 1.4.2, we equip it with a model category structure.

*Notation 3.1.1.* Unless specified otherwise, whenever we refer to the model structure on **Top** we mean the standard model structure with the weak equivalences given by weak homotopy equivalences and fibrations given by Serre fibrations [20, Section 17.2]. By the model structure on **TopSeq**<sup>io</sup> we will mean the induced model structure with weak equivalences and fibrations respectively given by component-wise weak equivalences and fibrations in **Top**.

Let  $\mathcal{O}$  be a colored operad in topological spaces with a set of colors  $C$ . Let  $\text{Alg}_{\mathcal{O}}(\mathbf{Top})$  denote the category of algebras over  $\mathcal{O}$ . Recall that any  $\mathcal{O}$ -algebra  $A$  has an underlying topological space  $A(c)$ , for every color  $c \in C$ .

Using [19, Theorem 2.1] for the monoidal model category **Top**, we have the following:

**Proposition 3.1.2.**  *$\text{Alg}_{\mathcal{O}}(\mathbf{Top})$  admits a cofibrantly generated model structure such that a map  $A \rightarrow B$  of  $\mathcal{O}$ -algebras is a weak equivalence (respectively, fibration) if and only if the underlying map of topological spaces  $A(c) \rightarrow B(c)$  is a weak homotopy equivalence (respectively, Serre fibration), for every  $c \in C$ .*



*Remark 3.1.3.* Using the general theory of model categories, it follows that the cofibrations in the above model structure are given by morphisms which have the left lifting property with respect to the class of acyclic fibrations (fibrations which are weak equivalences) of  $\mathcal{O}$ -algebras.

As a consequence, using Lemma 2.6.2, we have:

**Corollary 3.1.4.** *The category of topological io-properads has a cofibrantly generated model structure with weak equivalences (respectively, fibrations) given by the weak equivalences (respectively, fibrations) of the underlying topological io-sequences.*

*Remark 3.1.5.* As in Remark 3.1.3 above, cofibrations of io-properads are precisely the maps which have left lifting property with respect to acyclic fibrations of io-properads.

### 3.2 Cofibrant Resolutions via the Bar Construction

Homotopy colimits are defined in terms of the cofibrant resolutions of objects (and diagrams) in model categories. In the computations of homotopy colimits in Chapters 5, 7, and 6, we use an explicit description of the cofibrant resolutions to provide presentations of the homotopy colimits, which we now explain.

The cofibrant resolution we use is obtained by applying the bar construction to the monad  $\mathbb{G}$ :

#### 3.2.1 Monadic Bar Construction

Let  $(T, \mu, \eta)$  be a monad acting on a category  $\mathcal{C}$ . The monadic bar construction of  $T$  applied to a  $T$ -algebra is the analogue of the usual bar construction for algebras. Given a  $T$ -algebra  $X$ , its monadic bar construction  $B_\bullet(T, T, X)$  is a simplicial  $T$ -algebra with the  $n$ -simplices given by

$$B_n(T, T, X) := T^{n+1}(X)$$

The simplicial face maps are given by

$$d_i = T^i \mu T^{n-i-1} X, 0 \leq i \leq n$$

and the degeneracy maps are given by

$$s_i = T^{i+1} \eta T^{n-i} X, 0 \leq i \leq n.$$

The bar construction gives a simplicial resolution of the algebra  $X$  in the following sense: Let  $sC$  denote the category of simplicial objects in  $C$  and let the constant simplicial object associated to  $X$  be denoted by  $X_\bullet$ . Then,

**Proposition 3.2.1** ([21, Proposition 9.8]).  *$X_\bullet$  is a simplicial deformation retract of  $B_\bullet(T, T, X)$  in  $sC$ .* □

Applying this to the monad  $\mathbb{G}$  and an io-properad  $P$ , we obtain a simplicial resolution of  $P$ . When  $P$  is nice enough, it is possible to use this to obtain a cofibrant resolution of  $P$ . This is the cofibrant resolution we shall use below. The first step in this direction is to obtain an io-properad starting with the simplicial io-properad  $B_\bullet(T, T, X)$ . This is done by taking its *geometric realization*, which we now outline.

### 3.2.2 Geometric Realization

The category of topological io-properads is tensored over topological spaces. Roughly this means that

- for any two io-properads  $P_1$  and  $P_2$  we can endow the set of io-properad morphisms  $\mathbb{G}\mathbf{Alg}(P, Q)$  with a topology. Denote this topological space by  $[P, Q]$ , and
- for any io-properad  $P$  and a topological space  $Z$ , we can define their ‘tensor product’  $P \odot Z$ ,

such that a version of hom-tensor adjunction holds:

$$\mathbf{Top}(Z, [P, Q]) \simeq \mathbb{G}\mathbf{Alg}(P \odot Z, Q).$$

See Appendix A for more details on this.

Using this tensor product over topological spaces on the category of io-properads, we can define the geometric realization of a simplicial io-properad  $\{P_\bullet\}_{\bullet \geq 0}$  by the usual formula:

$$|P_\bullet|^{\mathbb{G}\mathbf{Alg}} = \int_{\Delta}^{\mathbb{G}\mathbf{Alg}} P_n \odot \Delta^n := \text{coeq}^{\mathbb{G}\mathbf{Alg}} \left[ \coprod_{\phi: [n] \rightarrow [m] \in \Delta}^{\mathbb{G}\mathbf{Alg}} P_m \odot \Delta^n \rightrightarrows \coprod_{[n] \in \Delta}^{\mathbb{G}\mathbf{Alg}} P_n \odot \Delta^n \right]. \quad (3.2.1)$$

Here  $\Delta$  denotes the simplex category, and  $\Delta^n \in \mathbf{Top}$  is the standard  $n$ -simplex. In the coequalizer, the first arrow is induced from the properad map given by the simplicial structure map  $P(\phi): P_m \rightarrow P_n$  corresponding to  $\phi: [n] \rightarrow [m]$ , and the second arrow is induced from the map  $\Delta(\phi): \Delta^n \rightarrow \Delta^m$  of topological spaces.

### 3.2.3 Cofibrancy of the bar construction

**Proposition 3.2.2.** *Let  $P$  be an io-properad which is cofibrant as a topological io-sequence. Then, the geometric realization  $|B_\bullet(\mathbb{G}, \mathbb{G}, P)|^{\mathbb{G}\mathbf{Alg}}$  is a cofibrant io-properad.*

The statement is proved by expressing the geometric realization as a sequence of pushouts in terms of the so-called *latching spaces* and reformulating the cofibrancy condition in terms of these successive pushouts. We defer the details to Appendix A.

We will also use a relative version of this statement:

**Proposition 3.2.3.** *Let  $P, Q$  be io-properads which are cofibrant as topological io-sequences, and let  $P \rightarrow Q$  be a map of io-properads such that the underlying map of topological io-sequences is a cofibration. Then, the induced map  $|B_\bullet(\mathbb{G}, \mathbb{G}, P)|^{\mathbb{G}\mathbf{Alg}} \rightarrow |B_\bullet(\mathbb{G}, \mathbb{G}, Q)|^{\mathbb{G}\mathbf{Alg}}$  is a cofibration of io-properads.*

The proof is provided in Appendix A.

### 3.2.4 Geometric realizations in topological io-sequences

It is possible to define the geometric realization of simplicial topological io-sequences using the same formula (3.2.1), with  $\odot$  replaced by component-wise product with topological spaces and

with the coproduct and the coequalizer taken in topological io-sequences instead of io-properads. This coincides with taking the usual geometric realization of simplicial topological spaces component-wise.

Let  $P_\bullet$  be a simplicial object in topological io-properads. Considering it as a simplicial object in topological io-sequences, we can take its geometric realization  $|P_\bullet|^{\mathbf{TopSeq}^{\text{io}}}$  in the category of topological io-sequences.

It is well known that  $| \cdot |^{\mathbf{TopSeq}^{\text{io}}}$  is a monoidal functor (i.e. satisfies  $|X_\bullet \times Y_\bullet| = |X_\bullet| \times |Y_\bullet|$ ), and hence it follows that  $|P_\bullet|^{\mathbf{TopSeq}^{\text{io}}}$  has a natural properad structure:

Given  $G \in \mathcal{G}$ , the composition along  $G$ ,  $\mu_G: G(|P_\bullet|) \rightarrow |P_\bullet|(in(G), out(G))$  is defined as the geometric realization of the corresponding map  $\mu_G: G(P_\bullet) \rightarrow P_\bullet(in(G), out(G))$ .

A priori it is not clear what the relation between the two geometric realizations is. However, we have the following (somewhat surprising) fact:

**Proposition 3.2.4** ([22, Theorem 7.5 (ii)]). *Let  $P_\bullet$  be a simplicial object in topological io-properads. Then, the topological io-properads obtained by taking the geometric realization of  $P_\bullet$  in the category of topological io-properads and the category of topological io-sequences are isomorphic.*

The proof, following [22], is given in Appendix A.

*Notation 3.2.5.* In the sequel, we will use  $| \cdot |$  to denote this common geometric realization. When it is necessary to emphasize the ambient category, we do so using superscripts  $\mathbf{TopSeq}^{\text{io}}$  and  $\mathbb{G}\mathbf{Alg}$ .

**Corollary 3.2.6.** *Let  $P$  be a topological io-properad which is cofibrant as a topological io-sequence, then  $|B_\bullet(\mathbb{G}, \mathbb{G}, P)|^{\mathbf{TopSeq}^{\text{io}}} \rightarrow P$  is a cofibrant resolution of  $P$ .*

*Proof.* Using Proposition 3.2.1 and taking geometric realizations, we get that  $|B_\bullet(\mathbb{G}, \mathbb{G}, P)|^{\mathbf{TopSeq}^{\text{io}}} \rightarrow |P_\bullet|^{\mathbf{TopSeq}^{\text{io}}} \simeq P$  is a homotopy equivalence. Also, using Proposition 3.2.2 and Proposition 3.2.4, it follows that  $|B_\bullet(\mathbb{G}, \mathbb{G}, P)|^{\mathbf{TopSeq}^{\text{io}}} \simeq |B_\bullet(\mathbb{G}, \mathbb{G}, P)|^{\mathbb{G}\mathbf{Alg}}$  is a cofibrant properad. This completes the proof that  $|B_\bullet(\mathbb{G}, \mathbb{G}, P)|^{\mathbf{TopSeq}^{\text{io}}} \rightarrow P$  is a cofibrant resolution of  $P$ .

□

Similarly using Proposition 3.2.3 we have

**Corollary 3.2.7.** *Let  $P \rightarrow Q$  be a map of io-properads as in Proposition 3.2.3, then  $|B_\bullet(\mathbb{G}, \mathbb{G}, P)|^{\text{TopSeq}^{\text{io}}} \rightarrow |B_\bullet(\mathbb{G}, \mathbb{G}, Q)|^{\text{TopSeq}^{\text{io}}}$  is a cofibration of io-properads.*

□

Using Corollaries 3.2.6 and 3.2.7, we get the following

**Lemma 3.2.8.** *Let  $P \leftarrow R \rightarrow Q$  be a diagram of io-properads, such that*

- *$P, Q, R$  are cofibrant as topological io-sequences, and*
- *$R \rightarrow P$  and  $R \rightarrow Q$  are cofibrations of topological io-sequences.*

*Then, the pushout of bar resolutions of  $P, Q$ , and  $R$*

$$|B_\bullet P| \coprod_{|B_\bullet R|} |B_\bullet Q|$$

*computes the homotopy pushout of  $P \leftarrow R \rightarrow Q$ .*

□

In Chapter 5, we will need to use this construction for a pushout diagram of properads where

- the map  $R \rightarrow Q$  is not a cofibration of topological io-sequences, and
- the topological io-sequence underlying  $P$  might not be cofibrant, but satisfies the property that  $P(n_-, n_+)$  has the homotopy type of a CW-complex for every  $(n_-, n_+)$ .

The comparison statements in Proposition 3.2.11 and 3.2.12 below, will be useful in this situation.

Before stating the propositions we start with some terminology:

*Notation 3.2.9.* We will say that a map of topological io-sequences is a *Hurewicz weak-equivalence* (respectively, *Hurewicz cofibration*) if it is component-wise a homotopy equivalence (respectively, Hurewicz cofibration) of topological spaces. We will say that a map of topological io-properads is a Hurewicz weak-equivalence if the underlying map of topological io-sequences is one.

*Remark 3.2.10.* Hurewicz weak-equivalences and cofibrations of topological io-sequences correspond to weak-equivalences and cofibrations in a certain model structure on topological io-sequences, induced from the Hurewicz (or Strøm) model structure on topological spaces. (See Section A.5 in Appendix A).

**Proposition 3.2.11.** *Let*

$$\begin{array}{ccccc} P & \longleftarrow & R & \longrightarrow & Q \\ \downarrow & & \downarrow & & \downarrow \\ P' & \longleftarrow & R' & \longrightarrow & Q' \end{array} \quad (3.2.2)$$

*be a map of pushout diagrams of io-properads. If*

1. *each vertical arrow is a Hurewicz weak-equivalence, and*
2.  *$R \rightarrow P$ ,  $R' \rightarrow P'$  are in fact Hurewicz cofibrations of io-topological io-sequences*

*Then*

$$|B_{\bullet}P| \coprod_{|B_{\bullet}R|}^{\mathbb{G}\mathbf{Alg}} |B_{\bullet}Q| \rightarrow |B_{\bullet}P'| \coprod_{|B_{\bullet}R'|}^{\mathbb{G}\mathbf{Alg}} |B_{\bullet}P'|$$

*is a Hurewicz weak-equivalence.*

Again we defer the proof of Proposition 3.2.11 to Appendix A.

**Proposition 3.2.12.** *Let  $P \leftarrow R \rightarrow Q$  be a pushout diagram of io-properads. If*

1.  *$R \rightarrow P$  and  $R \rightarrow Q$  are Hurewicz cofibrations of underlying topological io-sequences, and*
2. *for every  $(n_-, n_+)$ ,  $P(n_-, n_+)$ ,  $Q(n_-, n_+)$ , and  $R(n_-, n_+)$  have the homotopy type of a CW-complex,*

*then*

$$|B_{\bullet}P| \coprod_{|B_{\bullet}Q|} |B_{\bullet}R|$$

*computes the homotopy pushout of  $P \leftarrow R \rightarrow Q$ .*

*Proof.* Recall that **Top** admits a functorial cofibrant replacement given by

$$|\mathcal{S}ing(\_)|: \mathbf{Top} \rightarrow \mathbf{Top},$$

where  $\mathcal{S}ing(\_)$  is the singular simplicial set functor and  $|\_$  is the geometric realization functor. Applying it component-wise provides a functorial cofibrant replacement on **TopSeq**<sup>io</sup>. Since  $|\mathcal{S}ing(\_)|$  is symmetric monoidal on **Top**, for any topological io-properad  $X$ , the topological io-sequence  $|\mathcal{S}ing(X)|$  admits a canonical properad structure such that the map

$$|\mathcal{S}ing(X)| \rightarrow X$$

is a morphism of io-properads. Applying this cofibrant replacement functor to pushout diagram  $P \leftarrow R \rightarrow Q$ , we obtain a diagram of io-properads

$$\begin{array}{ccccc} |\mathcal{S}ing(P)| & \longleftarrow & |\mathcal{S}ing(R)| & \longrightarrow & |\mathcal{S}ing(Q)| \\ \downarrow & & \downarrow & & \downarrow \\ P & \longleftarrow & R & \longrightarrow & Q \end{array}, \quad (3.2.3)$$

such that the vertical arrows are weak-equivalences. Using the fact that  $P, Q$ , and  $R$  satisfy condition (2) mentioned in the statement of the proposition, it follows that the vertical arrows are Hurewicz weak-equivalences of properads. Moreover, note that maps  $|\mathcal{S}ing(R)| \rightarrow |\mathcal{S}ing(P)|$  and  $|\mathcal{S}ing(R)| \rightarrow |\mathcal{S}ing(Q)|$  are Hurewicz cofibrations. Thus the diagram (3.2.3) satisfies the hypothesis of Proposition 3.2.11 and hence we have a weak-equivalence of io-properads

$$|B_\bullet|\mathcal{S}ing(P)|| \coprod_{|B_\bullet|\mathcal{S}ing(R)||}^{\mathbb{G}Alg} |B_\bullet|\mathcal{S}ing(Q)|| \rightarrow |B_\bullet P| \coprod_{|B_\bullet R|}^{\mathbb{G}Alg} |B_\bullet P|$$

The proposition now follows by observing that

$$|B_\bullet|\mathcal{S}ing(P)|| \coprod_{|B_\bullet|\mathcal{S}ing(R)||}^{\mathbb{G}Alg} |B_\bullet|\mathcal{S}ing(Q)||$$

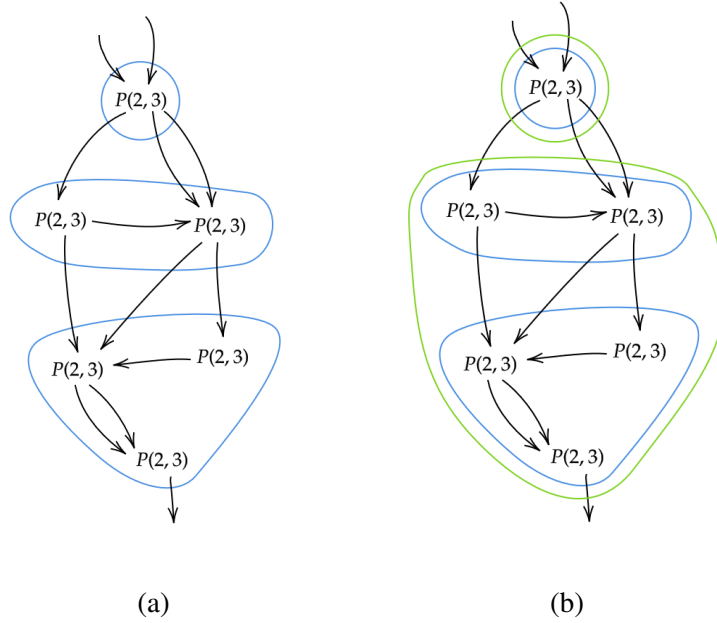


Figure 3.1: (b) shows a space of 2-simplices. (a) shows the result of applying  $d_0$  to simplices in (b)

computes the homotopy pushout of  $P \leftarrow R \rightarrow Q$ .  $\square$

### 3.2.5 Visualizing $B_\bullet(\mathbb{G}, \mathbb{G}, P)$

Here we provide some examples of how simplices in the bar construction and their face and degeneracy maps are visualized in the figures which will appear in the later chapters.

The  $n$ -simplices of  $B_\bullet(\mathbb{G}, \mathbb{G}, P)$  are given by  $\mathbb{G}^{n+1}P$ . We visualize points in this space as  $n$ -nested graphs, with each nesting indicated using a different color. For instance, elements in the space  $\mathbb{G}^2P$  of 1-simplices are denoted using 1-nested ioda-graphs. Figure 3.1(a) shows an example of such a simplex. The 1-nested graph there describes an element in  $\mathbb{G}^2P$  as follows: Considering each region in the outer nesting, indicated in blue, as a vertex, we obtain an ioda-graph. Moreover, each vertex of this ioda-graph has a labeling by a  $P$ -labeled graph given by the part of the nested graph lying inside the region.

Similarly, the 2-nested graph in Figure 3.1(b) represents a 2-simplex. Let us now describe the effect of applying the simplicial face maps to this simplex.



We start with the face map  $d_0$ . From its definition it follows that applying  $d_0$  to this simplex corresponds to applying the properad composition to the  $(\mathbb{G}P)$ -labels of the outermost nesting (along the graph described by the outermost nesting). This corresponds to simply forgetting the green nesting. The result is thus a 1-simplex having a shape as indicated in Figure 3.1(a). Similarly applying  $d_1$  to a simplex in Figure 3.1(b) corresponds to forgetting the blue nesting, thus giving a 1-simplex as in Figure 3.2(a).

On the other hand, applying  $d_2$  corresponds to performing properad composition at the innermost level of the nesting. It is thus given by replacing each blue region by the properad compositions of the  $P$ -labels lying within it. The result of applying  $d_2$  is thus a 1-simplex as in Figure 3.2(b). Note that unlike  $d_0$  and  $d_1$ ,  $d_2$  makes use of the io-properad structure of  $P$ .

Lastly, we look at an example of a degeneracy map. Let us consider the case of applying degeneracy map  $s_0$  to the simplices in Figure 3.1(b). This map is obtained by applying the unit  $\eta$  of monad  $\mathbb{G}$  to each region in the blue nesting. It thus corresponds to adding an additional nesting around each blue region, resulting in a 3-simplex having shape as shown in Figure 3.2(c). This additional nesting is shown in red in Figure 3.2(c).

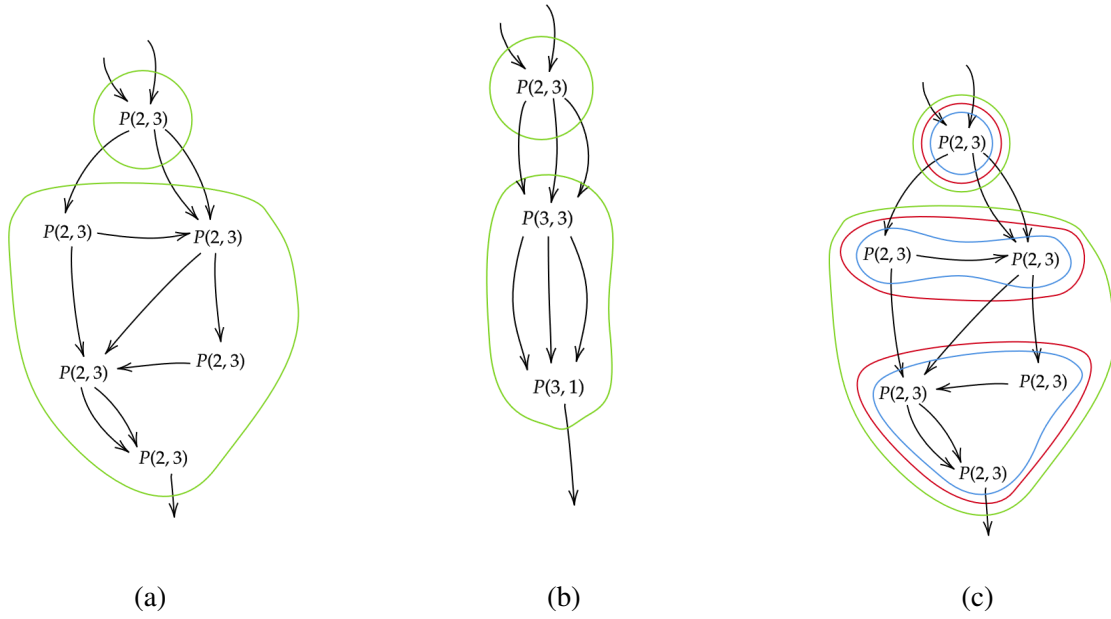


Figure 3.2: (a), (b), (c) show the result of applying, respectively,  $d_2$ ,  $d_0$ , and  $s_0$  to simplices in Figure 3.1(b)

## Chapter 4: Properads of Riemann surfaces

### 4.1 Some moduli spaces of Riemann surfaces

For  $g, n_-, n_+$  in the stable range  $2 - 2g - n_- - n_+ < 0$ , let  $\mathcal{M}_{g,n_-,n_+}$  denote the moduli space of genus  $g$  Riemann surfaces with  $n_-$  input and  $n_+$  output marked points. Let  $\overline{\mathcal{M}}_{g,n_-,n_+}$  be its Deligne–Mumford compactification, given by the moduli space of stable nodal curves with  $n_-$  input and  $n_+$  output marked points, and arithmetic genus  $g$ .

Let  $\mathcal{M}_{g,n_-,n_+}^{fr}$  denote the moduli space of genus  $g$  Riemann surfaces with  $n_-$  input and  $n_+$  output boundary components, each equipped with an analytic  $S^1$ -parametrization. The parametrization is assumed to be orientation preserving at the inputs and orientation reversing at the outputs. Let  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$  be the moduli space of stable nodal Riemann surfaces with  $n_-$  input and  $n_+$  output analytically  $S^1$ -parametrized boundaries. We assume that the nodes are away from the boundaries. Also, let  $\widehat{\mathcal{M}}_{g,n_-,n_+}^{fr}$  be the subspace of  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$  given by the moduli spaces of stable curves in which every irreducible component contains at least one output.

In addition, we also consider the moduli spaces  $\mathcal{M}_{g,n_-,n_+}^{fr}, \overline{\mathcal{M}}_{g,n_-,n_+}^{fr}, \widehat{\mathcal{M}}_{g,n_-,n_+}^{fr}$  for the unstable range

$$(g, n_-, n_+) \in \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (0, 0, 2), (0, 2, 0)\}.$$

For  $(g, n_-, n_+)$  given by  $\{(0, 1, 1), (0, 0, 2), \text{ and } (0, 1, 0)\}$ ,  $\mathcal{M}_{g,n_-,n_+}^{fr}$  are all identified with the space of annuli with  $S^1$ -parametrized boundary components. The spaces  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$  and  $\widehat{\mathcal{M}}_{0,0,2}^{fr}$  are obtained by compactifying these spaces by also including nodal annuli, thought of as annuli with modulus  $\infty$  (with boundaries suitably labeled as inputs and outputs). Consistent with the restriction that each irreducible component has an output, the space  $\widehat{\mathcal{M}}_{0,1,1}$  coincides with  $\mathcal{M}_{0,1,1}$  and  $\widehat{\mathcal{M}}_{0,2,0}^{fr}$  is empty.

For  $(g, n_-, n_+)$  equal to  $(0, 0, 1)$  or  $(0, 1, 0)$ , all the corresponding moduli spaces are identified with

moduli spaces of disks with a parametrized boundary and thus are given by a point. Moreover, in the case  $(g, n_-, n_+) = (0, 1, 1)$  for all the three types of moduli mentioned above, we also allow exceptional points corresponding to degenerate annuli with modulus 0. These points are added to ensure that the io-properads we consider below are unital.

Finally, denote by  $\mathcal{M}_{g,n_-,n_+}^{fr}(i)$  and  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}(i)$  the moduli space of, respectively, non-nodal and possibly nodal stable Riemann surfaces with arithmetic genus  $g$ ,  $n_-$  input boundaries,  $n_+$  output boundaries, and with  $i$  marked points disjoint from the nodes and the boundaries. For every  $1 \leq j \leq i$  we have forgetful maps

$$\pi_j: \mathcal{M}_{g,n_-,n_+}^{fr}(i) \rightarrow \mathcal{M}_{g,n_-,n_+}^{fr}(i-1) \text{ and } \pi_j: \overline{\mathcal{M}}_{g,n_-,n_+}^{fr}(i) \rightarrow \overline{\mathcal{M}}_{g,n_-,n_+}^{fr}(i-1),$$

given by forgetting the  $j$ th marked point and stabilizing (by collapsing any unstable components created) if necessary.

All the moduli spaces mentioned above are a priori topological stacks. However the following lemma shows that most of these are in fact topological spaces. This observation already occurs in [11, Section 3.1]:

**Lemma 4.1.1.** *The stacks  $\mathcal{M}_{g,n_-,n_+}^{fr}$  and  $\widehat{\mathcal{M}}_{g,n_-,n_+}^{fr}$  are represented by topological spaces provided  $(n_-, n_+) \neq (0, 0)$ . Moreover, the moduli spaces  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$  for  $(g, n_-, n_+)$  in the unstable range  $2g - 2 + n_+ + n_- < 0$ , are also represented by topological spaces.*

*Proof.* Notice that a Riemann surface with at least one parametrized boundary component has no non-trivial automorphisms: any automorphism of a Riemann surface with parametrized boundary preserves the boundary parametrization and hence, by analytic continuation, is forced to be the identity. Thus it follows that  $\mathcal{M}_{g,n_-,n_+}^{fr}$  for  $(n_-, n_+) \neq (0, 0)$  is in fact a topological space. Similar arguments can be used for nodal Riemann surfaces to show that  $\widehat{\mathcal{M}}_{g,n_-,n_+}^{fr}$  for  $(n_-, n_+) \neq (0, 0)$  and the moduli spaces  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$  in unstable range  $2g - 2 + n_- + n_+$  are also given by topological spaces.  $\square$

On the other hand  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$  are in general not spaces but only topological stacks. The stacky

points are given by surfaces containing irreducible components which have no boundaries and have a non-trivial conformal automorphism group.

## 4.2 Finite dimensional homotopy types

Even though we will not need this fact in what follows, let us note that although the spaces  $\mathcal{M}_{g,n_-,n_+}^{fr}$ ,  $\widehat{\mathcal{M}}_{g,n_-,n_+}^{fr}$ ,  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$  are infinite dimensional, they have finite dimensional homotopy types:

Let  $M_{g,n_-,n_+}^{fr}$  be the moduli space of Riemann surfaces with  $n_-$  input and  $n_+$  output boundary components with each boundary component carrying a marked point (and no analytic  $S^1$ -parametrization).

It can be shown that  $M_{g,n_-,n_+}^{fr}$  is represented by a topological space and is in fact a finite dimensional smooth manifold. Moreover,  $M_{g,n_-,n_+}^{fr}$  has the same homotopy type as  $\mathcal{M}_{g,n_-,n_+}^{fr}$ : there is a homotopy equivalence  $\mathcal{M}_{g,n_-,n_+}^{fr} \xrightarrow{\simeq} M_{g,n_-,n_+}^{fr}$  given by mapping a point in  $\mathcal{M}_{g,n_-,n_+}^{fr}$  to the underlying Riemann surface with boundary marked points given by base points of the analytic  $S^1$ -parametrizations of the boundaries.

Similarly, spaces  $\widehat{\mathcal{M}}_{g,n_-,n_+}^{fr}$  and  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$  have homotopy types of finite dimensional spaces  $\widehat{M}_{g,n_-,n_+}^{fr}$  and  $\overline{M}_{g,n_-,n_+}^{fr}$  given by moduli spaces of stable Riemann surfaces as in  $\widehat{\mathcal{M}}_{g,n_-,n_+}^{fr}$  and  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$ , but without the boundary parametrizations and with each boundary component carrying a marked point (see [23] for details on topology of the spaces  $\widehat{M}_{g,n_-,n_+}^{fr}$  and  $\overline{M}_{g,n_-,n_+}^{fr}$ ). The homotopy equivalences in these cases are realized by maps constructed similarly as before.

In the case of  $\mathcal{M}_{g,n_-,n_+}^{fr}$  and  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$  there are alternative descriptions of homotopy types that are more closely related to their appearance in the theory of closed string invariants of symplectic manifolds, namely symplectic cohomology and Gromov-Witten theory:

Let  $f\mathcal{M}_{g,n_-,n_+}^{\mathbb{R}}$  denote the moduli space of Riemann surfaces (without boundaries), with  $n_-$  input and  $n_+$  output marked points and with each marked point carrying a marker i.e. a ray in the tangent space of the marked point. This is a torus bundle over the usual moduli space of Riemann surfaces with marked points. It can be shown that the stack  $f\mathcal{M}_{g,n_-,n_+}^{\mathbb{R}}$  is represented by a finite dimensional smooth manifold. Further, there is a homotopy equivalence  $\mathcal{M}_{g,n_-,n_+}^{fr} \xrightarrow{\simeq} f\mathcal{M}_{g,n_-,n_+}^{\mathbb{R}}$  given by map-

ping a surface in  $\mathcal{M}_{g,n_-,n_+}^{fr}$  to the Riemann surface obtained by gluing-in unit disks at the boundary components using the boundary parametrizations (see Section 4.3 below), with the marked points at the centers of the disks and the markers determined by the positive real direction in each disc. Similarly, it can be shown that  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$  has the homotopy type of the stack  $\overline{\mathcal{M}}_{g,n_-,n_+}$  which is the finite dimensional moduli stack of stable Riemann surfaces with  $n_-$  input and  $n_+$  output marked points (with no markers), in other words the usual Deligne–Mumford compactification of the moduli space of Riemann surfaces (without boundaries). The map  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr} \rightarrow \overline{\mathcal{M}}_{g,n_-,n_+}$  is constructed as above by gluing unit disks along boundaries followed by collapsing any unstable spheres generated as a result of this gluing.

### 4.3 Gluing Riemann surfaces

Let  $\Sigma_1$  and  $\Sigma_2$  be two Riemann surfaces with parametrized boundaries, and let  $\gamma_1$  be an output boundary of  $\Sigma_1$  and  $\gamma_2$  an input boundary of  $\Sigma_2$ . Then there exists a unique complex structure on gluing  $\Sigma_1 \#_{\gamma_1, \gamma_2} \Sigma_2$ . For an identification of the boundaries along a diffeomorphism this is a classical fact. For an analytic identification, as in our case, it is possible to provide a simpler argument. We give an outline, the details are described in [11, Section 3.1]: the analytic  $S^1$ -parametrization of  $\gamma_1$  can be extended to an analytic identification of a neighborhood of the boundary with an annulus in  $\mathbb{C}$  of the form  $\{1 \leq |z| \leq 1 + \epsilon_1\}$ . Similarly a neighborhood of  $\gamma_2$  can be identified with  $\{1 - \epsilon_2 \leq |z| \leq 1\}$ . The germs of these identifications are uniquely determined. The gluing of these annuli is canonically identified with the annulus  $\{1 - \epsilon_2 \leq |z| \leq 1 + \epsilon_1\}$  and thus gets a unique complex structure. This can now be used to obtain a complex structure on the gluing  $\Sigma_1 \#_{\gamma_1, \gamma_2} \Sigma_2$ .

It is straightforward to generalize this to gluing along multiple boundaries and also to the gluing of nodal Riemann surfaces.

### 4.4 io-Properads of Riemann surfaces

We now describe the io-properads which will appear in our discussion:

- $\mathfrak{M}$  is the io-properad defined by  $\mathfrak{M}(n_-, n_+) = \coprod_{g \geq 0} \mathcal{M}_{g, n_-, n_+}^{fr}$ . The properad compositions are given by gluing the Riemann surfaces along suitable boundaries via the  $S^1$ -parametrization. We refer to  $\mathfrak{M}$  as the *TFT-properad*.
- $\overline{\mathfrak{M}}$  is the io-properad defined by  $\overline{\mathfrak{M}}(n_-, n_+) = \coprod_{g \geq 0} \overline{\mathcal{M}}_{g, n_-, n_+}^{fr}$ . The properad compositions are again given by gluing the nodal Riemann surfaces along suitable boundaries via the  $S^1$ -parametrization, followed by collapsing any unstable components created as a result of the gluing. We refer to this as the *Deligne–Mumford properad*.
- $\partial_+ \mathfrak{M}$  is the io-subproperad of  $\mathfrak{M}$  consisting of all operations which have at least one output i.e.  $\partial_+ \mathfrak{M}(n_-, n_+) = \coprod_{g \geq 0} \mathcal{M}_{g, n_-, n_+}^{\partial_+}$ , where

$$\mathcal{M}_{g, n_-, n_+}^{\partial_+} = \begin{cases} \emptyset & \text{if } n_+ = 0, \\ \mathcal{M}_{g, n_-, n_+}^{fr} & \text{otherwise.} \end{cases}$$

We refer to  $\partial_+ \mathfrak{M}$  as the  $\partial_+ \text{TFT-properad}$ .

- $\widehat{\mathfrak{M}}$  is the io-subproperad of  $\overline{\mathfrak{M}}$  defined by  $\widehat{\mathfrak{M}}(n_-, n_+) = \coprod_{g \geq 0} \widehat{\mathcal{M}}_{g, n_-, n_+}^{fr}$ . Note that similarly to  $\partial_+ \mathfrak{M}$ , the components with  $n_+ = 0$  are empty. We refer to  $\widehat{\mathfrak{M}}$  as the *symplectic properad*

Let  $\mathfrak{M}^{unst}$ ,  $\overline{\mathfrak{M}}^{unst}$ ,  $\widehat{\mathfrak{M}}^{unst}$ , and  $\partial_+ \mathfrak{M}^{unst}$  be the io-subproperads of  $\mathfrak{M}$ ,  $\overline{\mathfrak{M}}$ ,  $\widehat{\mathfrak{M}}$ , and  $\partial_+ \mathfrak{M}$  respectively, consisting of (possibly nodal) Riemann surfaces in the unstable range. More precisely, these properads are defined by:

- $\mathfrak{M}^{unst}$  is the io-subproperad of  $\mathfrak{M}$  which is empty in all components except the following:

$$\begin{aligned} \mathfrak{M}^{unst}(0, 1) &= \mathcal{M}_{0, 0, 1}^{fr}, & \mathfrak{M}^{unst}(1, 0) &= \mathcal{M}_{0, 1, 0}^{fr}, & \mathfrak{M}^{unst}(1, 1) &= \mathcal{M}_{0, 1, 1}^{fr}, \\ \mathfrak{M}^{unst}(0, 2) &= \mathcal{M}_{0, 0, 2}^{fr}, & \mathfrak{M}^{unst}(2, 0) &= \mathcal{M}_{0, 2, 0}^{fr}. \end{aligned}$$

- $\overline{\mathfrak{M}}^{unst}$  is the io-subproperad of  $\overline{\mathfrak{M}}$  which is empty in all components except the following:

$$\begin{aligned}\overline{\mathfrak{M}}^{unst}(0, 1) &= \mathcal{M}_{0,0,1}^{fr}, & \overline{\mathfrak{M}}^{unst}(1, 0) &= \mathcal{M}_{0,1,0}^{fr}, & \overline{\mathfrak{M}}^{unst}(1, 1) &= \mathcal{M}_{0,1,1}^{fr}, \\ \overline{\mathfrak{M}}^{unst}(0, 2) &= \mathcal{M}_{0,0,2}^{fr}, & \overline{\mathfrak{M}}^{unst}(2, 0) &= \mathcal{M}_{0,2,0}^{fr}.\end{aligned}$$

The properad operations are defined as in  $\overline{\mathfrak{M}}$ , i.e. by gluing along boundaries followed by stabilization.

- $\widehat{\mathfrak{M}}^{unst}$  is the io-subproperad of  $\widehat{\mathfrak{M}}$  which is empty in all components except the following:

$$\widehat{\mathfrak{M}}^{unst}(1, 1) = \widehat{\mathcal{M}}_{0,1,1}^{fr}, \quad \widehat{\mathfrak{M}}^{unst}(1, 1) = \widehat{\mathcal{M}}_{0,1,1}^{fr}, \quad \widehat{\mathfrak{M}}^{unst}(0, 2) = \widehat{\mathcal{M}}_{0,0,2}^{fr}.$$

- Finally  $\partial_+ \mathfrak{M}^{unst}$  is the io-subproperad of  $\partial_+ \mathfrak{M}$  which is empty in all components except the following

$$\partial_+ \mathfrak{M}^{unst}(0, 1) = \mathcal{M}_{0,0,1}^{fr}, \quad \partial_+ \mathfrak{M}^{unst}(1, 1) = \mathcal{M}_{0,1,1}^{fr}, \quad \partial_+ \mathfrak{M}^{unst}(0, 2) = \mathcal{M}_{0,0,2}^{fr}.$$

Clearly  $\partial_+ \mathfrak{M}^{unst}$  is a subproperad of  $\widehat{\mathfrak{M}}^{unst}$ ,  $\mathfrak{M}^{unst}$ , and  $\overline{\mathfrak{M}}^{unst}$ .

Lastly,  $\mathfrak{M}^{nop}$  is the io-subproperad of  $\mathfrak{M}$  which is empty in all components except the following:

$$\mathfrak{M}^{nop}(0, 1) = \mathcal{M}_{0,0,1}^{fr}, \quad \mathfrak{M}^{nop}(1, 0) = \mathcal{M}_{0,1,0}^{fr}, \quad \mathfrak{M}^{nop}(1, 1) = \mathcal{M}_{0,1,1}^{fr}, \quad \mathfrak{M}^{nop}(0, 2) = \mathcal{M}_{0,0,2}^{fr}.$$

In Section 5.1 below, we will also use the following modifications of io-properads  $\partial_+ \mathfrak{M}$ ,  $\partial_+ \mathfrak{M}^{unst}$ , and  $\widehat{\mathfrak{M}}^{unst}$  which we record here for convenience:

- $\partial_+ \mathbb{M}$  is the io-subproperad of  $\partial_+ \mathfrak{M}$  which coincides with  $\partial_+ \mathfrak{M}$  in all degrees except  $(1, 1)$ ,  $(0, 2)$ .

In these degrees the genus 0 components of  $\partial_+ \mathbb{M}(1, 1)$ ,  $\partial_+ \mathbb{M}(0, 2)$  are the subspaces of  $\mathcal{M}^{fr}(1, 1)$ ,  $\mathcal{M}^{fr}(0, 2)$ , respectively, containing only the exceptional annuli which have modulus 0.



- $\partial_+ \mathbb{M}^{unst}$  is the io-subproperad of  $\partial_+ \mathfrak{M}^{unst}$  defined in an analogous manner.
- $\widehat{\mathbb{M}}^{unst}$  is an io-subproperad of  $\widehat{\mathfrak{M}}^{unst}$ . In degree  $(1, 1)$ , similarly to the cases above, it only contains the exceptional annuli of modulus 0. However in this case, we define  $\widehat{\mathbb{M}}^{unst}(0, 2)$  to coincide with  $\widehat{\mathfrak{M}}^{unst}(0, 2)$ .

## Chapter 5: From the $\partial_+$ TFT-properad to the symplectic properad

In this chapter we present the proof of Theorem 1.4.2(1) . Let us start by recalling the statement:

**Theorem 5.0.1.**  $\widehat{\mathfrak{M}}$  is the homotopy colimit of the following diagram in the category of io-properads

$$\begin{array}{ccc} \partial_+ \mathfrak{M}^{unst} & \longrightarrow & \widehat{\mathfrak{M}}^{unst} \\ \downarrow & & \\ \partial_+ \mathfrak{M} & & \end{array} \quad (5.0.1)$$

### 5.1 The homotopy pushout

Let  $\partial_+ \mathbb{M}$ ,  $\partial_+ \mathbb{M}^{unst}$  and  $\widehat{\mathbb{M}}^{unst}$  be as described at the end of Chapter 4. Note that

- $\partial_+ \mathbb{M}(n_-, n_+)$ ,  $\partial_+ \mathbb{M}^{unst}(n_-, n_+)$  and  $\widehat{\mathbb{M}}^{unst}(n_-, n_+)$  have the homotopy type of CW-complexes ([24, Corollary 1]), and
- $\partial_+ \mathbb{M}^{unst} \rightarrow \partial_+ \mathbb{M}$  and  $\partial_+ \mathbb{M}^{unst} \rightarrow \widehat{\mathbb{M}}^{unst}$  are Hurewicz cofibrations of underlying topological io-sequences (see Notation 3.2.9).

Thus using Proposition 3.2.12, it follows that

$$\begin{array}{ccc} |B_\bullet \partial_+ \mathbb{M}| & \begin{array}{c} \text{GAlg} \\ \square \end{array} & |B_\bullet \widehat{\mathbb{M}}^{unst}| \\ & |B_\bullet \partial_+ \mathbb{M}^{unst}| & \end{array} \quad (5.1.1)$$

computes the homotopy pushout of  $\partial_+ \mathbb{M} \leftarrow \partial_+ \mathbb{M}^{unst} \rightarrow \widehat{\mathbb{M}}^{unst}$ . The analogous conclusion for the pushout

$$\begin{array}{ccc} |B_\bullet \partial_+ \mathfrak{M}| & \begin{array}{c} \text{GAlg} \\ \square \end{array} & |B_\bullet \widehat{\mathfrak{M}}^{unst}| \\ & |B_\bullet \partial_+ \mathfrak{M}^{unst}| & \end{array} \quad (5.1.2)$$

is not clear from the argument above, since the map  $\partial_+ \mathfrak{M}^{unst} \rightarrow \widehat{\mathfrak{M}}^{unst}$  is not a cofibration of topological io-sequences.

Now consider the inclusion of pushout diagrams:

$$\begin{array}{ccccc}
 \partial_+ \mathbb{M} & \longleftarrow & \partial_+ \mathbb{M}^{unst} & \longrightarrow & \widehat{\mathbb{M}}^{unst} \\
 \downarrow & & \downarrow & & \downarrow \\
 \partial_+ \mathfrak{M} & \longleftarrow & \partial_+ \mathfrak{M}^{unst} & \longrightarrow & \widehat{\mathfrak{M}}^{unst}
 \end{array} \tag{5.1.3}$$

Note that

- all the vertical maps are Hurewicz weak-equivalences of io-properads (see Notation 3.2.9), and
- the maps of topological io-sequences underlying  $\partial_+ \mathbb{M}^{unst} \rightarrow \partial_+ \mathbb{M}$  and  $\partial_+ \mathfrak{M}^{unst} \rightarrow \widehat{\mathfrak{M}}^{unst}$  are Hurewicz cofibrations.

**Lemma 5.1.1.** *The pushouts (5.1.1) and (5.1.2) are weakly homotopy equivalent. In particular, pushout (5.1.2) viz.*

$$\begin{array}{ccc}
 |B_\bullet \partial_+ \mathfrak{M}| & \begin{array}{c} \mathbb{G}\mathbf{Alg} \\ \square \\ |B_\bullet \partial_+ \mathfrak{M}^{unst}| \end{array} & |B_\bullet \widehat{\mathfrak{M}}^{unst}|
 \end{array}$$

*computes the homotopy colimit of (5.0.1)*

□

## 5.2 Pushout (5.1.2) is weak homotopy equivalent to $\widehat{\mathfrak{M}}$

From the computation (A.5.2), we have:

$$|B_\bullet \partial_+ \mathfrak{M}| \coprod_{|B_\bullet \partial_+ \mathfrak{M}^{unst}|}^{\mathbb{G}\mathbf{Alg}} |B_\bullet \widehat{\mathfrak{M}}^{unst}| \simeq \left| \{ \mathbb{G}(\mathbb{G}^\bullet \partial_+ \mathfrak{M} \coprod_{\mathbb{G}^\bullet \partial_+ \mathfrak{M}^{unst}} \mathbb{G}^\bullet \widehat{\mathfrak{M}}^{unst}) \} \right|$$

We prove Theorem 1.4.2(1) using the following strategy: We will show that the map

$$\pi: \left| \{ \mathbb{G}(\mathbb{G}^\bullet \partial_+ \mathfrak{M} \coprod_{\mathbb{G}^\bullet \partial_+ \mathfrak{M}^{unst}} \mathbb{G}^\bullet \widehat{\mathfrak{M}}^{unst}) \} \right| \rightarrow \widehat{\mathfrak{M}} \quad (5.2.1)$$

satisfies the property that every  $\Sigma \in \widehat{\mathfrak{M}}$  has a neighborhood  $U_\Sigma$  such that for any finite collection  $\Sigma_1, \dots, \Sigma_k$ ,

$$\pi^{-1}(U_{\Sigma_1, \dots, \Sigma_k}) \rightarrow U_{\Sigma_1, \dots, \Sigma_k} \text{ is a weak homotopy equivalence} \quad (5.2.2)$$

where  $U_{\Sigma_1, \dots, \Sigma_k} = U_{\Sigma_1} \cap \dots \cap U_{\Sigma_k}$ .

*Remark 5.2.1.* Note that here we are using a local criterion for weak equivalences (see for example, [25, Corollary 1.4]). We have to resort to this approach since  $\pi$  is not a fibration in general and hence just verifying the contractibility of fibers may not be sufficient to imply weak equivalence. However to understand the contracting homotopies constructed below, particularly those in Sections 5.3.6 and 5.3.7, it might be helpful to think of them as ways of extending the contracting homotopy of a given fiber to a neighborhood consisting of nearby fibers.

For every  $\Sigma \in \widehat{\mathfrak{M}}$  we shall in fact construct a neighborhood  $U_\Sigma$  satisfying the following properties:  $\pi^{-1}(U_\Sigma)$  has a filtration

$$W_{1,\Sigma} \subset \dots \subset W_{n,\Sigma} \dots \subset \pi^{-1}(U_\Sigma)$$

such that for each  $n$ ,

$$\pi|_{W_{n,\Sigma}}: W_{n,\Sigma} \rightarrow U_\Sigma$$

is a weak homotopy equivalence. We prove this by showing that for each  $n$ ,  $W_{n,\Sigma}$  has a further filtration

$$F_{n,\Sigma}^{(1)} \subset \dots \subset F_{n,\Sigma}^{(m)} \dots \subset W_{n,\Sigma}$$

such that

$$\begin{aligned} F_{n,\Sigma}^{(m)} \text{ admits a fiberwise contracting homotopy } & \text{inside } W_{n,\Sigma}, \\ \text{onto a section } \chi_{n,\Sigma}^{(m)}: U_\Sigma \rightarrow W_{n,\Sigma} \text{ of } \pi|_{W_{n,\Sigma}} \end{aligned} \quad (5.2.3)$$

Note that this implies (5.2.2) for any collection of elements of the cover.

### 5.2.1 Construction of $U_\Sigma$ , $W_{n,\Sigma}$ , and $F_{n,\Sigma}^{(m)}$

Let us start by describing the open set  $U_\Sigma$ . We use a local description of the moduli spaces as outlined in [26]. Let  $\widetilde{\Sigma} = \coprod \Theta_i$  be the normalization of  $\Sigma$ , where  $\Theta_i$  are smooth connected Riemann surfaces. For every node  $\nu$  of  $\Sigma$ , denote by  $\hat{\nu}, \check{\nu}$  its pre-images in  $\widetilde{\Sigma}$ . Let  $\mathcal{V}$  denote the collection of nodes of  $\Sigma$ . Let  $\widetilde{\mathcal{V}}$  be the collection of the points  $\hat{\nu}, \check{\nu}$  as  $\nu$  varies over  $\mathcal{V}$ . Let  $\widetilde{\mathcal{V}}_i$  denote the subcollection of such points on  $\Theta_i$ . Now, let  $U_{\Theta_i, \widetilde{\mathcal{V}}_i}$  be a small neighborhood of  $\Theta_i, \widetilde{\mathcal{V}}_i$  in the moduli space of Riemann surfaces with the same topological type as  $\Theta_i$  and with internal marked points indexed by  $\widetilde{\mathcal{V}}_i$ . Let  $C_{\Theta_i, \widetilde{\mathcal{V}}_i} \rightarrow U_{\Theta_i, \widetilde{\mathcal{V}}_i}$  be the universal curve over it, and let  $s_\zeta$  be the sections corresponding to the marked points  $\zeta \in \widetilde{\mathcal{V}}_i$ .

Consider the product family

$$C_{\widetilde{\Sigma}, \widetilde{\mathcal{V}}} = \prod C_{\Theta_i, \widetilde{\mathcal{V}}_i} \rightarrow U_{\widetilde{\Sigma}, \widetilde{\mathcal{V}}} = \prod U_{\Theta_i, \widetilde{\mathcal{V}}_i}$$

along with the sections  $s_\zeta, \zeta \in \widetilde{\mathcal{V}}$ . Now let  $z_\zeta$  be a holomorphic function in a neighborhood of  $\text{Im}(s_\zeta)$  which restricts to a holomorphic co-ordinate on each fiber. Shrinking  $U_{\Theta_i, \widetilde{\mathcal{V}}_i}$  and rescaling the  $z_\zeta$  if necessary, we may assume that the open sets  $R_\zeta = \{|z_\zeta| < 1\}$  contain no marked points other than  $\zeta$  and are isomorphic to  $U_{\Theta_i, \widetilde{\mathcal{V}}_i} \times \mathbb{D}$  over  $U_{\Theta_i, \widetilde{\mathcal{V}}_i}$ . Let  $t_\nu: \mathbb{D}^\mathcal{V} \rightarrow \mathbb{D}$  be the projection onto the  $\nu$ th component of  $\mathbb{D}^\mathcal{V}$ . Now, define  $U_\Sigma$  to be the neighborhood of  $\Sigma$  in  $\widehat{\mathfrak{M}}$  given by the product

$$U_\Sigma := U_{\widetilde{\Sigma}} \times \mathbb{D}^\mathcal{V}.$$

The restriction of the universal curve over  $U_\Sigma$

$$C_\Sigma \rightarrow U_\Sigma = U_{\tilde{\Sigma}} \times \mathbb{D}^{\mathcal{V}}$$

is obtained as follows: for every node  $v$  in  $\mathcal{V}$  remove the closed subsets

$$\{(z, \underline{t}) \in R_{\hat{v}} \times \mathbb{D}^{\mathcal{V}} \mid |z_{\hat{v}}(z)| \leq |t_v|\} \text{ and } \{(z, \underline{t}) \in R_{\check{v}} \times \mathbb{D}^{\mathcal{V}} \mid |z_{\check{v}}(z)| \leq |t_v|\}$$

from  $C_{\tilde{\Sigma}, \tilde{\mathcal{V}}} \times \mathbb{D}^{\mathcal{V}}$  and identify the rest of  $R_{\hat{v}}$  and  $R_{\check{v}}$  via

$$z_{\hat{v}} \cdot z_{\check{v}} = t_v.$$

Let us now turn to the construction of the filtrations  $W_{n, \Sigma}$  and  $F_{n, \Sigma}^{(m)}$  satisfying condition (5.2.3).

Let  $\underline{x}$  be an element in  $\mathbb{G}\partial_+\mathfrak{M} \coprod_{\mathbb{G}\partial_+\mathfrak{M}^{unst}} \widehat{\mathbb{G}\mathfrak{M}^{unst}}$ . It is an  $(\partial_+\mathfrak{M} \coprod_{\partial_+\mathfrak{M}^{unst}} \widehat{\mathfrak{M}^{unst}})$ -labeled graph, say with underlying ioda-graph  $G$  and labeling  $\prod_{v \in V(G)} \Sigma_v$ . Consider the map

$$\mathbb{G}^1\partial_+\mathfrak{M} \coprod_{\mathbb{G}^1\partial_+\mathfrak{M}^{unst}} \mathbb{G}^1\widehat{\mathfrak{M}^{unst}} \rightarrow \widehat{\mathfrak{M}}$$

induced by properad compositions. It maps  $\underline{x}$  to the Riemann surface  $\Sigma_{\underline{x}} = \#_{v \in V(G)} \Sigma_v$  obtained by gluing the surfaces  $\Sigma_v$  as prescribed by the edges of  $G$ . In particular, we get an analytic embedding

$$\gamma_{\underline{x}}: \coprod_{E(G) \cup \text{in}(G) \cup \text{out}(G)} S^1 \rightarrow \Sigma_{\underline{x}}$$

whose image lies away from the nodes of  $\Sigma_{\underline{x}}$ .

Now consider the analogous maps

$$\mathbb{G}^{n+1}\partial_+\mathfrak{M} \coprod_{\mathbb{G}^{n+1}\partial_+\mathfrak{M}^{unst}} \mathbb{G}^{n+1}\widehat{\mathfrak{M}^{unst}} \rightarrow \widehat{\mathfrak{M}}$$

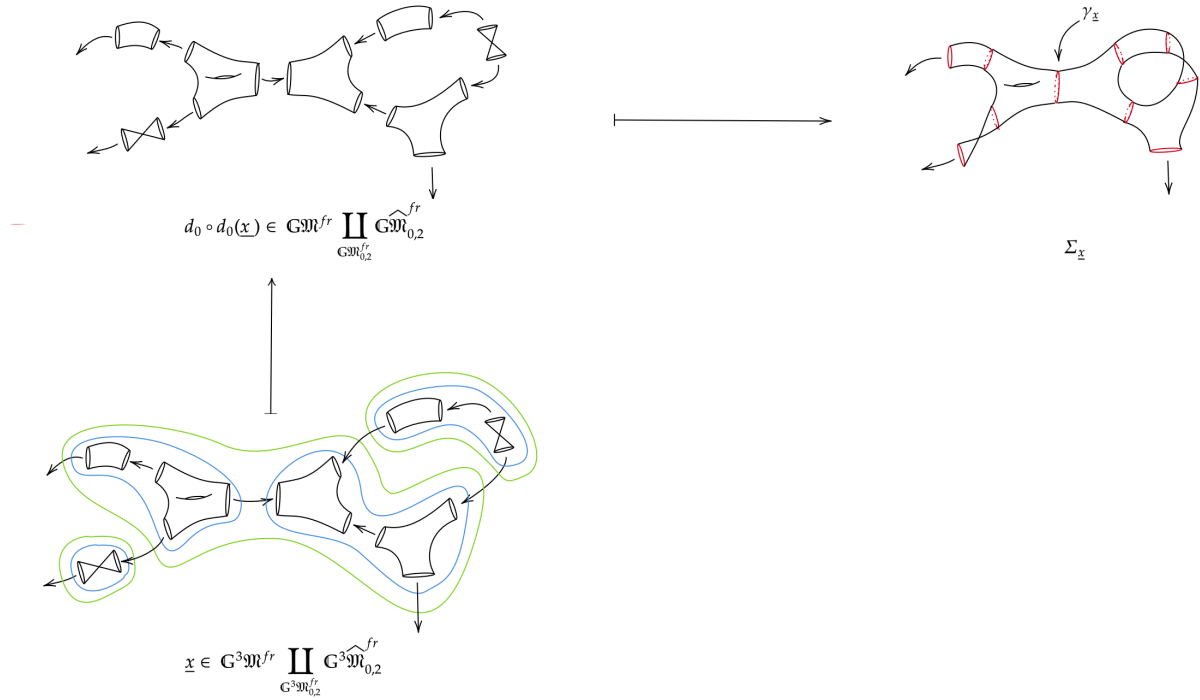


Figure 5.1: Construction of the embedding  $\gamma_{\underline{x}}$  for an  $\underline{x}$  as shown in the bottom left corner. The surface  $\Sigma_{\underline{x}}$  obtained by gluing the inner-most labels of  $\underline{x}$  is shown in the top right corner. The image of  $\gamma_{\underline{x}}$  inside  $\Sigma_{\underline{x}}$  is indicated in red.

induced by (iterated) properad compositions. Note that these maps factor as

$$\mathbb{G}^{n+1} \partial_+ \mathfrak{M} \coprod_{\mathbb{G}^{n+1} \partial_+ \mathfrak{M}^{unst}} \mathbb{G}^{n+1} \widehat{\mathfrak{M}}^{unst} \rightarrow \mathbb{G}^1 \partial_+ \mathfrak{M} \coprod_{\mathbb{G}^1 \partial_+ \mathfrak{M}^{unst}} \mathbb{G}^1 \widehat{\mathfrak{M}}^{unst} \rightarrow \widehat{\mathfrak{M}}$$

where the first map is the composition of simplicial degeneracy maps  $\underbrace{d_0 \circ \dots \circ d_0}_{n\text{-times}}$  and corresponds to forgetting all the nestings in the underlying graph of an element, and the second map is as described above. In particular, to any

$$\underline{x} \in \mathbb{G}^{n+1} \partial_+ \mathfrak{M} \coprod_{\mathbb{G}^{n+1} \partial_+ \mathfrak{M}^{unst}} \mathbb{G}^{n+1} \widehat{\mathfrak{M}}^{unst},$$

we can assign an embedding  $\gamma_{\underline{x}}$  as in 5.2.1. For an example see Figure 5.1.

Recall the description of the family of curves  $C_{\Sigma}$  over  $U_{\Sigma}$  given above. For each  $n, m$  and  $\nu$ , let

$A_{n,\nu}^{(m)} \subset C_\Sigma$  be the subset defined, in terms of this descriptions, by

$$A_{n,\nu}^{(m)} = \left\{ (z, \underline{t}) \in R_{\hat{\nu}} \times \mathbb{D}^{\tilde{\mathcal{V}}} \left| |t_\nu| \in \left[ 0, \frac{1}{n^2} \right] \text{ and } |z_{\hat{\nu}}(z)| \in \left[ \frac{|t_\nu|}{\frac{1}{n} \left( 1 + \frac{1}{m} \right)}, \frac{1}{n} \left( 1 + \frac{1}{m} \right) \right] \right. \right\}.$$

Set

$$A_n^{(m)} = \bigcup_{\nu \in \mathcal{V}} A_{n,\nu}^{(m)}.$$

Define  $\mathbb{F}_{n,\Sigma}^{(m)}$  to be the simplicial subspace of  $\mathbb{G}^{\bullet+1} \partial_+ \mathfrak{M} \amalg_{\mathbb{G}^{\bullet+1} \partial_+ \mathfrak{M}^{unst}} \mathbb{G}^{\bullet+1} \widehat{\mathfrak{M}}^{unst}$  consisting of simplices  $\underline{x}$  such that

1. the associated embedding  $\gamma_{\underline{x}}$  is disjoint from  $A_n^{(m)}$ , and
2. if  $\Sigma_u$  is one of the inner-most labels of  $\underline{x}$  (i.e. an element of  $\underbrace{d_0 \circ \dots \circ d_0}_{n\text{-times}}$ , the labeled graph obtained by forgetting all the nestings of the graph underlying  $\underline{x}$ ) which, considered as a subset of  $\Sigma_{\underline{x}}$ , intersects  $A_n^{(m)}$ , then each component of  $\Sigma_u \setminus A_n^{(m)}$  has at least one output boundary.

(5.2.4)

We then define  $F_{n,\Sigma}^{(m)}$  to be an open subset of  $\left| \mathbb{G}^{\bullet+1} \partial_+ \mathfrak{M} \amalg_{\mathbb{G}^{\bullet+1} \partial_+ \mathfrak{M}^{unst}} \mathbb{G}^{\bullet+1} \widehat{\mathfrak{M}}^{unst} \right|$  which fiber-wise deformation retracts onto this subcomplex.

Finally, we set

$$\mathbb{W}_{n,\Sigma} = \bigcup_m \mathbb{F}_{n,\Sigma}^{(m)} \quad \text{and} \quad W_{n,\Sigma} = \bigcup_m F_{n,\Sigma}^{(m)}.$$

Note that  $W_{n,\Sigma}$  deformation retracts onto the geometric realization of  $\mathbb{W}_{n,\Sigma}$ .

Let  $U_\Sigma^{sing}$  denote the subset

$$\{(z, \underline{t}) \mid |t_\nu| = 0 \text{ for some } \nu\}.$$

This is the locus of nodal Riemann surfaces in  $U_\Sigma$ . Denote by  $U_\Sigma^n$  the neighborhood of  $U_\Sigma^{sing}$  given



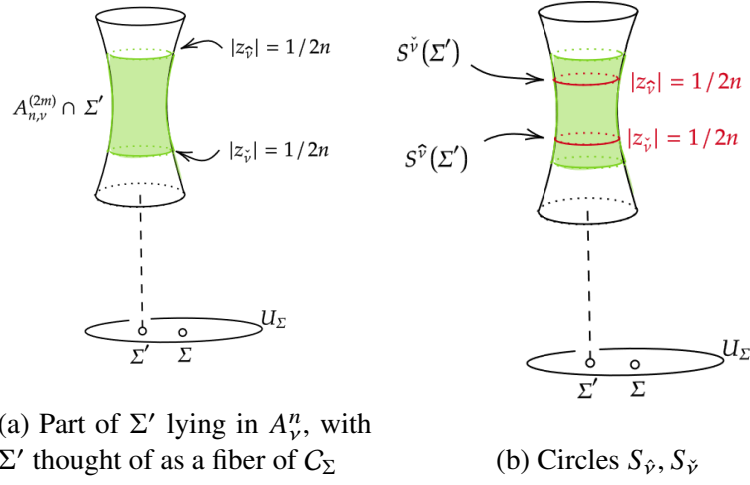


Figure 5.2

by

$$\{(z, \underline{t}) \mid |t_\nu| \leq \frac{1}{n^2} \text{ for some } \nu\}.$$

In the constructions below we will also use the subcomplex  $\mathbb{F}_{n,\Sigma}^{(m)} \cap \pi^{-1}(U_\Sigma^n)$ . Denote this by  $\mathbb{F}_{n,\Sigma}^{(m)}|_{U_\Sigma^n}$ .  $F^n(U_\Sigma) \cap \pi^{-1}(U_\Sigma^n)$  give an open subset of  $\pi^{-1}(U_\Sigma)$  fiberwise deformation retracting onto this subcomplex.

### 5.3 Proof of condition (5.2.3)

Let us now turn to the fiberwise contractions. We begin by describing a number of preliminary notions and maps:

#### 5.3.1 Annuli $A_v$ and curves $S^{\check{v}}, S^{\hat{v}}$

For any  $\Sigma' \in U_\Sigma^n$ , denote by  $A_v(\Sigma')$  the annulus

$$A_{n,v}^{(2m)} \cap \Sigma'$$

where we think of  $\Sigma'$  as sitting inside the universal curve  $C_\Sigma$  as a fiber. Denote by  $S^{\check{\nu}}, S^{\hat{\nu}}$  the boundaries of this annulus given by

$$S^{\hat{\nu}}(\Sigma') = \left\{ (z, t_{\hat{\nu}}(\Sigma')) \mid |z_{\hat{\nu}}(z)| = \frac{1}{n} \left( 1 + \frac{1}{2m} \right) \right\}, S^{\check{\nu}}(\Sigma') = \left\{ (z, t_{\check{\nu}}(\Sigma')) \mid |z_{\check{\nu}}(z)| = \frac{|t_{\hat{\nu}}|(\Sigma')}{\frac{1}{n} \left( 1 + \frac{1}{2m} \right)} \right\} \subset \Sigma'$$

Now consider the graph  $G_{\Sigma'}$ , with

- vertices given by the connected components of  $\Sigma' \setminus \bigcup_{\mathcal{V}} \partial A_{\nu}(\Sigma')$ , and
- edges correspond to the connected components of  $\bigcup_{\mathcal{V}} \partial A_{\nu}(\Sigma')$ , with all edges directed away from vertices labeled by the annuli  $A_{\nu}(\Sigma')$ .

Denote by  $\underline{x}_{\Sigma'}$  the 0-simplex with the underlying graph  $G_{\Sigma'}$  and with the labeling of each vertex given by the corresponding connected component of  $\Sigma' \setminus \bigcup_{\mathcal{V}} \partial A_{\nu}(\Sigma')$ .

### 5.3.2 Graphs $H_{\nu}$

Let  $\underline{x}$  be an element in the space of  $k$ -simplices  $(\mathbb{F}_{n,\Sigma}^{(m)}|_{U_{\Sigma}^n})^k$ . Denote the underlying ioda-graph by  $G$ . Let  $\underline{x}_{\nu}$  denote the label of  $\nu \in V(G)$  in  $\underline{x}$  and let  $\Sigma'_{\nu}$  be the surface obtained by gluing the inner-most labels of  $\in x_{\nu}$ . Since  $\underline{x}$  lies in  $\mathbb{F}_{n,\Sigma}^{(m)}$ , it follows that for any  $\nu \in \mathcal{V}$ , the intersection  $A_{\nu}(\Sigma') \cap \Sigma'_{\nu}$  is either empty or  $A_{\nu}(\Sigma')$ . We can then construct a graph  $H_{\nu}$  as follows:

- the vertices of  $H_{\nu}$  are given by the connected components of  $\Sigma'_{\nu} \setminus \bigcup_{\mathcal{V}} \partial A_{\nu}$ , and
- the edges of  $H_{\nu}$  are given by the connected components of  $\bigcup_{\mathcal{V}} \partial A_{\nu}$  lying inside  $\Sigma'_{\nu}$ , with each edge directed away from the vertex corresponding to a component of the form  $A_{\nu}$  for some  $\nu$ .

The following observation will turn out to be important in the construction of the cut maps and cut homotopies below in Sections 5.3.3 and 5.3.4:

**Lemma 5.3.1.** *Every vertex in  $H_{\nu}$  has at least one output.*

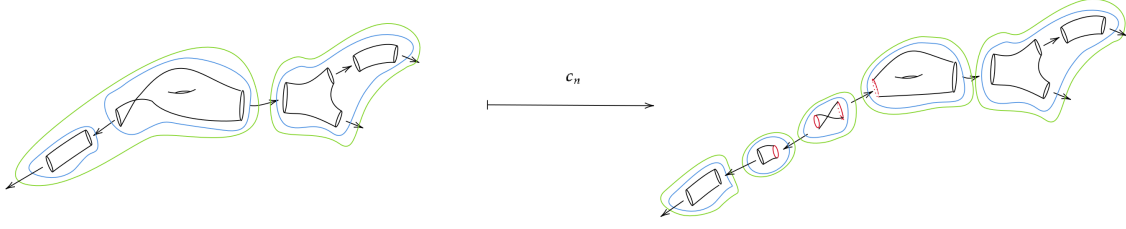


Figure 5.3: An example of cut map  $c_n$  for  $n = 2$

*Proof.* The proof follows by observing that  $\Sigma'_v$  is obtained by composing labels of a subgraph of  $\underbrace{d_0 \circ \dots \circ d_0}_{n\text{-times}}(\underline{x})$ , each of which has at least one output and moreover satisfies the condition mentioned in 5.2.4, (2).  $\square$

### 5.3.3 The cut maps

We define a simplicial map

$$c: \mathbb{F}_{n,\Sigma}^{(m)}|_{U_\Sigma^n} \rightarrow \mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}$$

which will be used in the definitions of homotopies below. Intuitively, it corresponds to making ‘cuts’ to the underlying Riemann surface along the circles  $S^\hat{v}$  and  $S^\check{v}$  (see Figure 5.3). Maps

$$c_r: (\mathbb{F}_{n,\Sigma}^{(m)}|_{U_\Sigma^n})^r \rightarrow (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n})^r$$

are defined inductively as follows:

- Base case: Let  $\underline{x} \in (\mathbb{F}_{n,\Sigma}^{(m)}|_{U_\Sigma^n})^0$  and let  $G$  be the graph underlying  $\underline{x}$ . For a vertex  $v \in G$  let  $\Sigma_v$  and  $H_v$  be as described above in Section 5.3.2. Consider an element  $\underline{y}_v \in \mathbb{W}_{n,\Sigma_v}^0$  whose underlying ioda-graph is  $H_v$  and whose vertex labels are given by the corresponding connected components of  $\Sigma_v \setminus \bigcup_{\gamma} \partial A_\gamma$ . The fact that such a labeled ioda-graph indeed lies in  $\mathbb{W}_{n,\Sigma_v}^0$  follows from Lemma 5.3.1.

We then define  $c_0(\underline{x})$  as the graph composition

$$c_0(\underline{x}) := \mu_G(\prod \underline{y}_v). \quad (5.3.1)$$

- Assume inductively that  $c_i$  for  $0 \leq i < r$ , have been defined and satisfy the following property: For any  $\underline{x} \in (\mathbb{W}_{n,\Sigma}|U_\Sigma^n)^i$  with underlying graph  $G$ ,

$$c_i(\underline{x}) = \mu_G(\prod \underline{y}_v)$$

for some  $\underline{y}_v \in (\mathbb{W}_{n,\Sigma}|U_\Sigma^n)^i$ . Notice that the condition is satisfied for  $i = 0$ .

Now suppose  $\underline{x} \in (\mathbb{W}_{n,\Sigma}|U_\Sigma^n)^r$ , having underlying ioda-graph  $G$ . Consider the element  $c_{r-1}(d_0\underline{x}) \in (\mathbb{W}_{n,\Sigma}|U_\Sigma^n)^{r-1}$ . From the induction hypothesis we know that  $c_{r-1}(d_0\underline{x}) = \mu_{G_{d_0\underline{x}}}(\prod \underline{y}_w'')$  for some  $\underline{y}_w''$ ,  $w \in V(G_{d_0\underline{x}})$ , where  $G_{d_0\underline{x}}$  is the ioda-graph underlying  $d_0\underline{x}$ . But since  $G_{d_0\underline{x}} = \mu_G(\prod G_v)$  for some graphs  $G_v$ , we can express  $c_{r-1}(d_0\underline{x})$  as a graph composition  $\mu_G(\prod \underline{y}_v')$  for some  $\underline{y}_v' \in (\mathbb{W}_{n,\Sigma}|U_\Sigma^n)^{r-1}$ . For  $v \in G$ , consider an element  $\underline{y}_v \in \Sigma^r$  defined as follows: the ioda-graph underlying  $\underline{y}_v$  is  $H_v$ . The label of a vertex  $u \in H_v$  in  $\underline{y}_v$  is the labeled subgraph of  $\underline{y}_v'$  consisting of vertices and edges mapping under  $\pi$  into the connected component  $\Sigma_u$  of  $\Sigma \setminus \bigcup_{\mathcal{V}} \partial A_v$  corresponding to  $u$ , as in Section 5.3.2. Then, we set

$$c_r(\underline{x}) = \mu_G(\prod \underline{y}_v).$$

#### 5.3.4 The cut homotopies

We define a fiber preserving homotopy

$$\Phi_t: F_{n,\Sigma}^{(m)}|U_\Sigma^n \rightarrow W_{n,\Sigma}|U_\Sigma^n$$

from the inclusion  $F_{n,\Sigma}^{(m)}|_{U_\Sigma^n} \hookrightarrow W_{n,\Sigma}|_{U_\Sigma^n}$  to the geometric realization of the cut map, defined above in 5.3.3. It turns out to be easier to write down the expression for the inverse homotopy

$$\widetilde{\Phi}_t: F_{n,\Sigma}^{(m)}|_{U_\Sigma^n} \rightarrow W_{n,\Sigma}|_{U_\Sigma^n}$$

going from the geometric realization of the cut map to the inclusion  $F_{n,\Sigma}^{(m)}|_{U_\Sigma^n} \hookrightarrow W_{n,\Sigma}|_{U_\Sigma^n}$ .

$\widetilde{\Phi}$  is defined as the geometric realization of a simplicial homotopy

$$\widetilde{\phi}: \mathbb{F}_{n,\Sigma}^{(m)}|_{U_\Sigma^n} \rightarrow \mathbb{W}_{n,\Sigma}|_{U_\Sigma^n},$$

which in turn consists of maps  $\widetilde{\phi}_j: (\mathbb{F}_{n,\Sigma}^{(m)}|_{U_\Sigma^n})^r \rightarrow (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n})^{r+1}$  for  $0 \leq j \leq r$ , satisfying the usual conditions of a simplicial homotopy.

- **Base Case:** Let  $\underline{x} \in (\mathbb{F}_{n,\Sigma}^{(m)}|_{U_\Sigma^n})^0$  and let  $G$  be the graph underlying  $\underline{x}$ . Suppose that  $c_0(\underline{x}) = \mu_G(\prod \underline{y}_v)$ . Then, we define

$$\widetilde{\phi}_0(\underline{x}) := \mu_G(\prod \eta(\underline{y}_v)),$$

where we recall from Section 2.3 that  $\eta: \mathbb{1}_{\mathbf{TopSeq}^{\text{io}}} \rightarrow \mathbb{G}$  is the unit of the monad  $\mathbb{G}$ .

- Assume inductively that the maps  $\widetilde{\phi}_j$  have been defined for all degrees  $i < r$  and satisfy the following property similar to the one used above in 5.3.3: For any  $\underline{x} \in (\mathbb{F}_{n,\Sigma}^{(m)}|_{U_\Sigma^n})^i$ , with the underlying graph  $G$ ,

$$\widetilde{\phi}_j(\underline{x}) = \mu_G(\prod \underline{y}_v)$$

for some  $\underline{y}_v \in (\mathbb{F}_{n,\Sigma}^{(m)}|_{U_\Sigma^n})^i$ . The maps  $\widetilde{\phi}_j: (\mathbb{F}_{n,\Sigma}^{(m)}|_{U_\Sigma^n})^r \rightarrow (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n})^{r+1}$  are then defined as follows for  $\underline{x} \in (\mathbb{F}_{n,\Sigma}^{(m)}|_{U_\Sigma^n})^r$  with underlying ioda-graph  $G$ :

- $j = 0$ : For  $\underline{x} \in (\mathbb{F}_{n,\Sigma}^{(m)})^r$ , suppose that  $c_r(\underline{x}) = \mu_G(\prod \underline{y}_v)$ . Then define

$$\widetilde{\phi}_j(\underline{x}) := \mu_G(\prod \eta(\underline{y}_v)).$$

- $j > 0$ : From the condition in the inductive hypothesis we have  $\tilde{\phi}_{j-1}(d_0 \underline{x}) = \mu_G(\prod \underline{y}_{\underline{v}})$ , for some  $\underline{y}_{\underline{v}}$ . Then define

$$\tilde{\phi}_j(\underline{x}) := \mu_G(\prod \eta(\underline{y}_{\underline{v}})).$$

### 5.3.5 The homotopy $\Psi$

Let  $\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M$  be the subcomplex of  $\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}$  consisting of simplices  $\underline{x}$  in which the annuli  $A_\nu$  are isolated in the following sense: if  $\underline{x}_\nu$  is the label of a vertex in  $\underline{x}$ ,  $\Sigma_\nu$  the surface obtained by gluing the inner-most labels of  $\underline{x}_\nu$ , then

$$\Sigma_\nu \cap A_\nu \neq \emptyset \Rightarrow \underline{x}_\nu = \eta^r(A_\nu), \text{ for any } \nu \in \mathcal{V}.$$

Let  $W_{n,\Sigma}|_{U_\Sigma^n}^M$  be the geometric realization of  $\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M$ .

Let  $\chi_{n,\Sigma}^M: U_\Sigma^n \rightarrow W_{n,\Sigma}|_{U_\Sigma^n}^M$  be a section of  $\pi$  over  $U_\Sigma^n$  defined as follows: the image of  $\chi_{n,\Sigma}^M$  is contained inside the geometric realization of the 0-skeleton  $(\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n})^0$  with the associated map given by

$$\chi_{n,\Sigma}^M: \Sigma' \mapsto \underline{x}_{\Sigma'}, \text{ for every } \Sigma' \in U_\Sigma^n,$$

where  $\underline{x}_{\Sigma'}$  is a 0-simplex as defined in 5.3.1. Denote the constant simplicial space corresponding to the image of  $\chi_{n,\Sigma}^M$  by  $\mathbb{X}_{n,\Sigma}^M$  (recall a constant simplicial space is a simplicial space with all face and degeneracy maps given by identities).

We will construct a fiber-preserving homotopy  $\Psi$  contracting  $W_{n,\Sigma}|_{U_\Sigma^n}^M$  onto the image of  $\chi_{n,\Sigma}^M$ . This will again be a geometric realization of a simplicial homotopy  $\psi$ , which will be constructed by specifying extra degeneracies and augmentation on  $\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M$  (cf. [27, Section 4.5]). For this we need to define:

1. Extra degeneracies:  $s_{-1}: (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^r \rightarrow (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^{r+1}$  for all  $r \geq -1$ , where  $(\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^{-1} := \mathbb{X}_{n,\Sigma}^M$ , and
2. Augmentation:  $d_0: (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^0 \rightarrow (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^{-1}$ ,

where  $(\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^{-1} = \mathbb{X}_{n,\Sigma}^M$ , satisfying:

1.  $d_0 s_{-1}$  is identity on  $(\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^r$

2. for  $r \geq 1, 0 \leq i, j \leq r$ ,

$$(a) \quad d_{i+1} s_{-1} = s_{-1} d_i$$

$$(b) \quad s_{j+1} s_{-1} = s_{-1} s_j$$

on  $(\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^r$ .

The contracting homotopy  $\psi$  underlying  $\Psi$  is then obtained by setting  $\psi_i : (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^r \rightarrow (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^{r+1}$

to be  $\underbrace{s_0 \dots s_0}_{i\text{-times}} \underbrace{s_{-1} d_0 \dots d_0}_{i\text{-times}}$  for  $0 \leq i \leq r$ .

- Extra degeneracies: We define  $s_{-1} : (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^r \rightarrow (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^{r+1}$ . Let  $\underline{x} \in (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^r$  and  $\Sigma_{\underline{x}}$  the Riemann surface obtained by gluing the inner-most labels of  $\underline{x}$ . Let  $G_{\Sigma_{\underline{x}}}$  be the graph for the Riemann surface  $\Sigma_{\underline{x}}$  as defined in 5.3.1. From the definition of  $\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M$ , it follows that the vertices of  $\underline{x}$  can be regrouped into subgraphs  $\underline{y}_v \in (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^r$  labeled by  $v \in G_{\Sigma_{\underline{x}}}$  so that  $\underline{x} = \mu_{G_{\Sigma_{\underline{x}}}}(\prod \underline{y}_v)$ . Then set

$$s_{-1}(\underline{x}) := \mu_{G_{\Sigma_{\underline{x}}}}\left(\prod \eta(\underline{y}_v)\right).$$

- Augmentation: Note that for  $\underline{x} \in (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^0$ ,  $d_1(s_{-1}\underline{x}) = \underline{x}_{\Sigma_{\underline{x}}} \in \mathbb{X}_{n,\Sigma}^M$ . We define  $d_0 : (\mathbb{W}_{n,\Sigma}|_{U_\Sigma^n}^M)^0 \rightarrow \mathbb{X}_{n,\Sigma}^M$  as

$$d_0 = d_1 \circ s_{-1}.$$

### 5.3.6 The homotopy $\Psi^{ns}$

We construct another fiber-preserving homotopy, this time contracting  $W_{n,\Sigma}^{ns} = W_{n,\Sigma} \cap \pi^{-1}(U_\Sigma^{ns})$  onto a section  $\chi^{ns}$  of  $\pi$  over  $U_\Sigma^{ns}$ . Here

$$U_\Sigma^{ns} = \{(z, \underline{t}) | t_v \neq 0 \text{ for all } v\} \subset U_\Sigma \times \mathbb{D}^{\mathcal{V}} = U_\Sigma$$

is the subspace of non-singular curves in  $U_\Sigma$ . Let  $\mathbb{W}_{n,\Sigma}^{ns}$  be the corresponding subcomplex of  $\mathbb{W}_{n,\Sigma}$ .

The section  $\chi_\Sigma^{ns}$  is contained in the 0-skeleton of  $\mathbb{W}_{n,\Sigma}^{ns}$  and the corresponding map is given by

$$\Sigma' \mapsto \tilde{x}_{\Sigma'},$$

where  $\tilde{x}_{\Sigma'}$  is the 0-simplex such that the underlying graph has a single vertex, no internal edges, and the label of the vertex is given by  $\Sigma'$ . Note that  $\Sigma'$  is non-singular and hence  $\tilde{x}_{\Sigma'} \in \mathbb{G}\partial_+\mathfrak{M} \subset (\mathbb{W}_{n,\Sigma})^0$ .

Denote by  $\mathbb{X}_\Sigma^{ns}$  the simplicial subspace of  $\mathbb{W}_{n,\Sigma}^{ns}$  corresponding to the image  $\chi_\Sigma^{ns}$ . It is a constant simplicial space and is contained inside the 0-skeleton of  $\mathbb{W}_{n,\Sigma}^{ns}$ . Again we shall construct the contraction by specifying extra degeneracies and augmentation as in 5.3.5. We continue using the same notation for these as in 5.3.5.

- Extra degeneracies: In this case the map  $s_{r+1}: (\mathbb{W}_{n,\Sigma}^{ns})^r \rightarrow (\mathbb{W}_{n,\Sigma}^{ns})^{r+1}$  is defined by

$$s_{-1}(\underline{x}) = \eta(\underline{x}).$$

- The augmentation  $d_0: (\mathbb{W}_{n,\Sigma}^{ns})^0 \rightarrow \mathbb{X}_\Sigma^{ns}$  is defined by

$$d_0 := d_1 \circ s_{-1}.$$

The extra degeneracies  $s_{-1}$ 's and augmentation  $d_0$  satisfy conditions analogous to those mentioned in 5.3.5. The simplicial homotopy  $\psi^{ns}$  underlying  $\Psi^{ns}$  is obtained from  $s_{r+1}$ 's and  $d_0$  by formulas similar to those in 5.3.5.

### 5.3.7 The contraction for proving condition (5.2.3)

We are finally ready to define the homotopy needed to show that condition (5.2.3) is satisfied:

We construct a homotopy  $\Omega$  from the inclusion  $F_{n,\Sigma}^{(m)} \hookrightarrow W_{n,\Sigma}$  to a map which takes  $F_{n,\Sigma}^{(m)}$  onto a



section of  $\pi$  over  $U_\Sigma$ . The section coincides with  $\chi_{n,\Sigma}^M$  near the singular locus  $U_\Sigma^{sing}$ , and with  $\chi_\Sigma^{ns}$  away from it.

Fix a continuous function  $\zeta : U_\Sigma \rightarrow [0, 1]$  such that

$$\begin{aligned}\zeta &\equiv 1 \text{ on } \bigcup_{\mathcal{V}} \{|t_\nu| \leq \frac{1}{2n^2}\} \text{ and} \\ \zeta &\equiv 0 \text{ outside of } U_\Sigma^n = \bigcup_{\mathcal{V}} \{|t_\nu| \leq \frac{1}{n^2}\}\end{aligned}$$

Set  $\kappa = \zeta \circ \pi$ . The homotopy  $\Omega$  is defined as follows:

$$\begin{aligned}\text{On } \pi^{-1}\left(\bigcup_{\mathcal{V}} \{|t_\nu| \leq \frac{1}{2n^2}\}\right) : \Omega(\underline{x}, t) &= \begin{cases} \Phi(\underline{x}, 2t), & t < 1/2 \\ \Psi(\Phi(\underline{x}, 1), 2t - 1) & t \geq 1/2 \end{cases} \\ \text{On } \pi^{-1}(U_\Sigma^{ns}) : \Omega(\underline{x}, t) &= \begin{cases} \Phi(\underline{x}, 2t), & t < \kappa(\underline{x})/2 \\ \Phi(\Psi^{ns}(\underline{x}, \frac{t-\kappa/2}{1-\kappa/2}), \kappa(\underline{x})) & t \geq \kappa(\underline{x})/2 \end{cases}\end{aligned}$$

To ensure that  $\Omega$  is well-defined it suffices to check that the two definitions agree on points lying over  $U_\Sigma^{ns} \cap \bigcup_{\mathcal{V}} \{|t_\nu| \leq \frac{1}{2n^2}\}$

$$\Psi(\Phi(\underline{x}, 1), t) = \Phi(\Psi^{ns}(\underline{x}, t), 1) \text{ for all } t$$

Since  $\Phi(\underline{x}, 1) = c(\underline{x})$ , this amounts to checking for every  $r$  and  $0 \leq i \leq r$ ,

$$\psi_i c_r = c_{r+1} \psi_i^{ns}$$

on  $r$ -simplices lying over  $U_{n,\Sigma}^{ns} \cap \bigcup_{\mathcal{V}} \{|t_\nu| \leq \frac{1}{2n^2}\}$ . Since  $c$  is simplicial, from the definitions of  $\psi_i, \psi_i^{ns}$  we are reduced to checking

$$\psi_0 c_r = c_{r+1} \psi_0^{ns}$$

This can be verified directly from the definition of  $\phi_0$  and  $\psi_0^{ns}$ .

This shows that the filtration  $\{F_{n,\Sigma}^{(m)}\}$  constructed in Section 5.2.1 satisfies the condition (5.2.3) and thus completes the proof of the first part of Theorem 1.4.2.

## Chapter 6: From the $\partial_+$ TFT-properad to the TFT-properad

In this chapter we prove the first part of Theorem 1.4.1. Let us start by recalling the statement:

**Theorem 6.0.1.**  *$\mathfrak{M}$  is the homotopy colimit of the following diagram in the category of io-properads*

$$\begin{array}{ccc} \partial_+ \mathfrak{M}^{unst} & \longrightarrow & \mathfrak{M}^{nop} \\ \downarrow & & \\ \partial_+ \mathfrak{M} & & \end{array} \quad (6.0.1)$$

To compute the homotopy pushout of the diagram (6.0.1), we first replace  $\mathfrak{M}$  and  $\mathfrak{M}^{unst}$  with homotopy equivalent properads  $\mathcal{M}$  and  $\mathcal{M}^{nop}$  which are more suitable for the computation. A suitable modification of the strategy used for proving Theorem 1.4.2(1) is then used to complete the proof.

### 6.1 Properads $\mathcal{M}^{nop}$ and $\mathcal{M}$

Define  $\widetilde{\mathcal{M}}_{g,n_-,n_+}^{fr}(k)$  to be the moduli space of tuples  $(\Sigma, \underline{s}, \underline{t})$ , where

- $\Sigma \in \mathcal{M}_{g,n_-,n_+}^{fr}$ ,
- $\underline{s} = (s_1, \dots, s_k)$  a tuple of  $k$  labeled marked points in the interior of  $\Sigma$ , and
- $\underline{t} = (t_1, \dots, t_k) \in [0, 1]^k$ , such that  $t_i = 1$  for at least one  $i$  whenever  $n_+ = 0$ .

Define  $\mathcal{M}_{g,n_-,n_+}^{fr}(k)$  to be the quotient

$$\mathcal{M}_{g,n_-,n_+}^{fr}(k) := \widetilde{\mathcal{M}}_{g,n_-,n_+}^{fr}(k) / \Sigma_k$$

where  $\Sigma_k$  acts on  $\mathcal{M}_{g,n_-,n_+}^{fr}(k)$  by permuting the labels of the marked points and the co-ordinates of  $\underline{t}$ . We can think of  $\mathcal{M}_{g,n_-,n_+}^{fr}(k)$  as the moduli space of Riemann surfaces with  $k$  unlabeled marked

points such that each point carries a weight in  $[0, 1]$  and at least one of the weights is positive when the surface has no outputs. We refer to these marked points as *weighted marked points* and to a point in  $\mathcal{M}_{g,n_-,n_+}^{fr}(k)$  as a ‘*spotted surface*’.

Now define

$$\mathcal{M}_{g,n_-,n_+}^{fr} := \coprod_{k \geq 0} \mathcal{M}_{g,n_-,n_+}^{fr}(k) / \sim$$

where  $\sim$  identifies a spotted surface with the one obtained by forgetting the marked points which have weight 0. More precisely,  $\mathcal{M}_{g,n_-,n_+}^{fr}$  is obtained as the colimit

$$\mathcal{M}_{g,n_-,n_+}^{fr} = \varinjlim [F_0 \mathcal{M}_{g,n_-,n_+}^{fr} \rightarrow F_1 \mathcal{M}_{g,n_-,n_+}^{fr} \rightarrow \dots]$$

where the spaces  $F_k \mathcal{M}_{g,n_-,n_+}^{fr}$  are defined inductively, satisfying the following properties:

1.  $F_0 \mathcal{M}_{g,n_-,n_+}^{fr} = \mathcal{M}_{g,n_-,n_+}^{fr}(0)$
2.  $F_k \mathcal{M}_{g,n_-,n_+}^{fr}$  admits a map from  $[\mathcal{M}_{g,n_-,n_+}^{fr}(k+1)]^-$ , where  $[\mathcal{M}_{g,n_-,n_+}^{fr}(k+1)]^-$  is the subspace of  $\mathcal{M}_{g,n_-,n_+}^{fr}(k+1)$  of spotted surfaces where at least one marked point has weight 0.
3.  $F_{k+1} \mathcal{M}_{g,n_-,n_+}^{fr}$  is obtained inductively from  $F_k \mathcal{M}_{g,n_-,n_+}^{fr}$  as the pushout

$$\begin{array}{ccc} [\widetilde{\mathcal{M}}_{g,n_-,n_+}^{fr}(k+1)]^- & \longrightarrow & \widetilde{\mathcal{M}}_{g,n_-,n_+}^{fr}(k+1) \\ \downarrow & & \downarrow \\ F_k \mathcal{M}_{g,n_-,n_+}^{fr} & \longrightarrow & F_{k+1} \mathcal{M}_{g,n_-,n_+}^{fr} \end{array}$$

*Notation 6.1.1.* 1. The image of a tuple  $(\Sigma, \underline{x}, \underline{t})$  in  $\mathcal{M}_{g,n_-,n_+}^{fr}$  will be denoted by  $[\Sigma, \underline{x}, \underline{t}]$

2. We fix notation for some surfaces which appear frequently in the later sections: A spotted surface given by a disk with an output (respectively, input) boundary along with a single positive weight marked point will be called a positive cup (respectively, cap). Note that the marked point in a positive weight cup necessarily has weight 1.

**Proposition 6.1.2.** *The forgetful map  $\mathcal{M}_{g,n_-,n_+}^{fr} \rightarrow \mathcal{M}_{g,n_-,n_+}^{fr}$  is a weak homotopy equivalence.*

*Proof.* *Case  $n_+ > 0$ :* In this case can we consider the section of the forgetful map given by mapping a Riemann surface  $\Sigma$  to a spotted surface with underlying Riemann surface  $\Sigma$  and with no marked points. Then, there is a homotopy contracting  $\mathcal{M}_{g,n_-,n_+}^{fr}$  onto the image of this section given by shrinking the weights of all the marked points to 0.

*General Case:* The map  $\mathcal{M}_{g,n_-,n_+}^{fr} \rightarrow \mathcal{M}_{g,n_-,n_+}^{fr}$  is a fibration and hence it suffices to show that the fiber of this map is contractible.

Fix a surface  $\Sigma \in \mathcal{M}_{g,n_-,n_+}^{fr}$ . We will show that the fiber over  $\Sigma$  has an open cover  $\{W_x\}$ , indexed by points  $x \in \Sigma$ , such that every finite intersection  $W_{x_1, \dots, x_n} := W_{x_1} \cap \dots \cap W_{x_n}$  is non-empty and contractible.

The open sets  $W_x$  are defined to be subsets of the fiber consisting of spotted surfaces which have no marked point with weight  $\geq 1/3$  at  $x$ . It is clear that  $\{W_x\}$  forms an open cover of the fibre, with every finite intersection non-empty.

Let  $J = \{x_1, \dots, x_n\}$  be a finite subspace of  $\Sigma$ . We now turn to showing contractibility of the finite intersection  $W_J = W_{x_1} \cap \dots \cap W_{x_n}$ . To show this we will exhibit a filtration

$$F_1 \subset F_2 \subset \dots \subset F_n \subset \dots \text{ of } W_{x_1, \dots, x_n}$$

such that the inclusion of each  $F_n$  is null-homotopic inside  $W_J$ .

The filtration is defined as follows: Choose a conformal embedding  $\phi_{x_1} : \mathbb{D} \rightarrow \Sigma$  with  $\phi_{x_1}(0) = x_1$ . For  $n \in \mathbb{N}$ , define  $F_n$  to be the subspace of the fiber consisting of spotted surfaces for which all the marked points with weight  $\geq 1/3$  are contained inside  $\Sigma \setminus \phi_{x_1}(|z| \leq 1/n)$ . The subspaces  $F_n$  give an increasing and exhausting sequence of open subsets of  $W_J$ .

We now prove that each  $F_n$  is null-homotopic *inside*  $W_J$ . Consider the point  $x_0 \in \Sigma$  given by  $x_0 = \phi_{x_1}(1/2n)$ , and the spotted surface in  $W_J$  which has a single weight 1 marked point at  $x_0$  viz.  $[\Sigma, (x_0), (1)]$ . We show that  $F_n$  can be contracted to this spotted surface *inside*  $W_J$ .

The contraction homotopy is given by concatenation of 3 homotopies  $H_1, H_2, H_3$ :

1.  $H_1$ : Consider a function  $\tau : [0, 1] \rightarrow [0, 1]$  which is identically 0 on  $[0, 1/3]$  and identically

1 on  $[2/3, 1]$ . Let  $f_1$  be the map  $[(\Sigma, \underline{x}, (t_1, \dots, t_k))] \mapsto [(\Sigma, \underline{x}, (\tau(t_1), \dots, \tau(t_k)))]$ . Then  $H_1$  is a homotopy from the identity to  $f_1$ . The homotopy is simply given by linearly interpolating weights of the marked points .

2.  $H_2$ : Now consider the map  $f_2$  which takes any surface  $[(\Sigma, \underline{x}, \underline{t})]$  in  $F_n$  to the surface obtained by adding the marked point  $x_0$  with weight 1 to  $f_1([(\Sigma, \underline{x}, \underline{t})])$ . (In particular, note that  $f_2$  takes values outside  $F_n$ .) Define  $H_2$  as the homotopy between  $f_1$  and  $f_2$ , given by interpolating the weight of  $\gamma_0$  from 0 to 1. The well-definedness (and continuity) of  $f_2$  and  $H_2$  follows from the restriction in the definition of the subset  $F_n$  that any marked point contained inside  $\phi_{x_1}(\{|z| \leq \frac{1}{n}\})$  has weight  $< 1/3$ .
3.  $H_3$ : Finally,  $H_3$  is a homotopy from  $f_2$  to the constant map  $f_3$  with value  $[\Sigma, (x_0), (1)]$ . The homotopy  $H_3$  is given by decreasing the weights of marked point outside  $\phi_{x_1}(\{|z| \leq \frac{1}{n}\})$  to 0.

□

The properads  $\mathcal{M}$  and  $\mathcal{M}^{nop}$  are now defined as follows:

$\mathcal{M}$  is defined in a manner similar to  $\mathfrak{M}$  with the space  $\mathcal{M}_{g, n_-, n_+}^{fr}$  replaced by the corresponding spaces  $\mathcal{M}_{g, n_-, n_+}^{fr}$ .

$\mathcal{M}^{nop}$  is the io-subproperad of  $\mathcal{M}$  defined as follows:

- $\mathcal{M}^{nop}(0, 1)$  is the subspace of  $\mathcal{M}_{0,0,1}^{nop}$  having a positive weight cap i.e. a disk with output having a single weight 1 spot.
- $\mathcal{M}^{nop}(1, 0) = \mathcal{M}_{0,1,1}$  is the subspace of  $\mathcal{M}_{0,1,0}^{nop}$  consisting of disks with inputs having a single marked point . Note that the weight of the spot is allowed to vary in  $[0, 1]$ .
- $\mathcal{M}^{nop}(1, 1) = \mathcal{M}_{0,1,1}$  i.e. the subspace of  $\mathcal{M}_{0,1,1}$  spotted surfaces having no positive weight marked points . Similarly,

- $\mathcal{M}^{nop}(0, 2) = \mathcal{M}_{0,0,2}$  i.e. the subspace of  $\mathcal{M}_{0,0,2}$  spotted surfaces having no positive weight marked points .

The properad compositions are given by the maps induced by the gluing of Riemann surfaces underlying the spotted surfaces.

The following is a consequence of Proposition 6.1.2:

**Corollary 6.1.3.** *Maps*

$$\mathcal{M}^{nop} \rightarrow \mathfrak{M}^{nop} \quad \text{and} \quad \mathcal{M} \rightarrow \mathfrak{M}$$

*are weak equivalences of io-properads.*

□

## 6.2 Homotopy pushout of (6.0.1)

As mentioned before, to find a model for the homotopy pushout we shall consider the diagram

$$\partial_+ \mathfrak{M} \leftarrow \partial_+ \mathfrak{M}^{nop} \rightarrow \mathcal{M}^{nop} \tag{6.2.1}$$

which is weakly equivalent to (6.0.1). It follows that

$$|B_\bullet \partial_+ \mathfrak{M}| \coprod_{|B_\bullet \partial_+ \mathfrak{M}^{nop}|}^{\mathbb{G}\mathbf{Alg}} |B_\bullet \mathcal{M}^{nop}|$$

computes the homotopy pushout of (6.2.1) and thus of (6.0.1). From the computation (A.5.2), we have:

$$|B_\bullet \partial_+ \mathfrak{M}| \coprod_{|B_\bullet \partial_+ \mathfrak{M}^{nop}|}^{\mathbb{G}\mathbf{Alg}} |B_\bullet \mathcal{M}^{nop}| = |\mathbb{G}(\mathbb{G}^\bullet \partial_+ \mathfrak{M} \sqcup_{\mathbb{G}^\bullet \partial_+ \mathfrak{M}^{nop}} \mathbb{G}^\bullet \mathcal{M}^{nop})|$$

To complete the proof of Theorem 1.4.1 it now suffices to show that:

$$\pi : |\mathbb{G}(\mathbb{G}^\bullet \partial_+ \mathfrak{M} \sqcup_{\mathbb{G}^\bullet \partial_+ \mathfrak{M}^{nop}} \mathbb{G}^\bullet \mathcal{M}^{nop})| \rightarrow \mathcal{M} \tag{6.2.2}$$

### 6.3 Map (6.2.2) is a weak equivalence of properads

The strategy of proof is similar to that of Section 5.2. In this case instead of cuts made around nodal points, we will construct cuts around positive weight marked points. This gives a canonical way of decomposing an element of  $\mathcal{M}$  into a surface in  $\partial_+ \mathfrak{M}$  and a collection positive weight caps and cups. As in Section 5.2, some care will be needed to extend this decomposition continuously to a neighborhood of the spotted surface. Cut homotopies, analogous to those in Section 5.2, will then be constructed to homotope the points in the pushout to the zero simplices corresponding to such decompositions.

To highlight the parallels between the formal structure of the argument here and in Section 5.2, we will use the same notation as in Section 5.2 for the analogous notions here. As we shall see, having suitably adopted various definitions to the current context, many of the maps and homotopies will be given by the same expressions as in Section 5.2.

As before, we will find a cover  $\{U_\lambda\}_{\lambda \in \Lambda}$  of  $\mathcal{M}_{g,n_-,n_+}^{fr}$  such that for any finite subset  $J = \{\lambda_1, \dots, \lambda_k\} \subset \Lambda$ ,

$$\pi^{-1}|_{U_J} : \pi^{-1}(U_J) \rightarrow U_J$$

is a weak-equivalence, where  $U_J := U_{\lambda_1} \cap \dots \cap U_{\lambda_k}$ .

We will in fact show that each  $J$ ,  $U_J$  has a filtration

$$W_{1,J} \subset \dots \subset W_{n,J} \subset \dots \pi^{-1}(U_J)$$

such that

$$\pi|_{W_{n,J}} : W_{n,J} \rightarrow U_J \text{ is a weak homotopy equivalence.} \quad (6.3.1)$$

Similarly to Section 5.2, we prove this by showing that each  $W_{n,J}$  has a further filtration

$$F_{n,J}^{(1)} \subset \dots \subset F_{n,J}^{(m)} \subset W_{n,J},$$



satisfying a condition analogous to (5.2.3), with  $W_{n,\Sigma}$  there replaced by  $W_{n,J}$ .

*Remark 6.3.1.* Notice that unlike Section 5.2,  $\chi_n$  is not a section of  $\pi$  in this case. In particular the homotopies as in (5.2) which we construct, will not be fiber-preserving. Thus, in contrast to Section 5.2, it is not sufficient to work with a single  $U_\lambda$  and instead it becomes necessary to construct the homotopies over all finite intersections  $U_J$ .

### 6.3.1 Construction of $U_\lambda$ , $W_{n,J}$ , and $F_{n,J}^{(m)}$

We start by describing the spaces  $U_\lambda$ . Let  $U_{\alpha,i}$  be the open subset of  $\mathcal{M}_{g,n_-,n_+}^{fr}$  consisting of spotted surfaces containing exactly  $i$  marked points with weight  $> \alpha$  and no marked point with weight  $= \alpha$ . Also, let  $Z_{\alpha,i} \subset U_{\alpha,i}$  be the subspace of spotted surfaces containing exactly  $i$  positive weight marked points all of which have weight 1. There exists a deformation retraction  $r_{\alpha,i}: U_{\alpha,i} \rightarrow Z_{\alpha,i}$  obtained by homotoping the weight  $w_\gamma$  for an marked point  $\gamma$  to

$$\begin{cases} 0 & \text{if } w_\gamma < \alpha \\ 1 & \text{if } w_\gamma > \alpha. \end{cases}$$

Notice that we have a covering map  $\mathcal{M}_{n_-,n_+}^{fr}(i) \rightarrow Z_{\alpha,i}$ , where are the spaces  $\mathcal{M}_{n_-,n_+}^{fr}(i)$  are as defined at the end of Section 4.1. The fiber over a spotted surface in  $Z_{\alpha,i}$  is given by different ways of labeling the marked points. Consider a covering  $\{W_{\alpha,i}^\delta\}_{\delta \in \Delta_{\alpha,i}}$  of  $\mathcal{M}_{g,n_-,n_+}^{fr}(i)$ , where  $\Delta_{\alpha,i}$  is some indexing set, satisfying the following conditions:

- $W_{\alpha,i}^\delta$  maps homeomorphically onto an open subset of  $Z_{\alpha,i}$ .
- Over  $W_{\alpha,i}^\delta$ , there exists a continuous choice of analytically embedded disks around the marked points in the following sense: Let  $C_{g,n_-,n_+}(i) \rightarrow \mathcal{M}_{g,n_-,n_+}^{fr}(i)$  denote the universal curve. Then, there exists maps

$$\sigma_{\alpha,i}^\delta: W_{\alpha,i}^\delta \times \sqcup_{j=1}^i \mathbb{D} \rightarrow C_{g,n_-,n_+}(i)|_{W_{\alpha,i}^\delta} \quad (6.3.2)$$

over  $W_{\alpha,i}^\delta$ , such that  $\sigma_{\alpha,i}^\delta$  is a fiberwise embedding which maps  $0 \in \mathbb{D}$  in the  $j$ th disk in the second factor to the  $j$ th marked point of the fiber. Moreover, we assume that the embeddings given by  $\sigma_{\alpha,i}^\delta$  are disjoint from the boundaries.

Let  $Z_{\alpha,i}^\delta$  be the image of  $W_{\alpha,i}^\delta$  in  $Z_{\alpha,i}$ . The collection  $\{Z_{\alpha,i}^\delta\}$  forms a cover of  $Z_{\alpha,i}$ . Define  $U_{\alpha,i}^\delta = r_{\alpha,i}^{-1}(Z_{\alpha,i}^\delta)$ , where  $r_{\alpha,i}: U_{\alpha,i} \rightarrow Z_{\alpha,i}$  is the retraction mentioned above. The collection  $U_{\alpha,i}^\delta, \delta \in \Lambda_{\alpha,i}$  gives a covering of  $U_{\alpha,i}$ . Putting all the  $\{U_{\alpha,i}^\delta\}$ 's together as  $\alpha, n$ , and  $\delta$  vary in  $(0, 1), \mathbb{Z}_{\geq 0}$ , and  $\Delta_{\alpha,i}$  respectively, we obtained the desired cover  $\{U_\lambda\}_{\lambda \in \Lambda}$  of  $\mathcal{M}_{g,n-,n+}^{fr}$ .

For a finite collection  $J \subset \Lambda$ , we now turn to constructing a filtrations  $W_{n,J}$  and  $F_{n,\Sigma}^{(m)}$  satisfying a condition analogous to (6.3.1). Let  $\alpha_1, \dots, \alpha_k$  be the constants for  $J$  as discussed above. Without loss of generality, assume that  $0 = \alpha_0 < \alpha_1 \leq \dots \leq \alpha_k < \alpha_{k+1} = 1$ . From the definitions of the sets  $U_{\lambda_i}$  it is clear that for any surface in the intersection  $U_J$  the number of marked points with weight in  $> \alpha_r$  stays constant, viz.  $i_r$ , for every  $r$ .

We now construct a filtration  $F_{n,J}^{(m)}$  satisfying (6.3.1). Analogous to Section 5.2.1, the filtered pieces are defined to be a neighborhood deformation retracting onto of a simplicial subcomplex  $\mathbb{F}_{n,J}^{(M)}$  constructed as follows: Consider the fiberwise embeddings

$$(\sigma_{\alpha_1,i_1}^{\lambda_1})^{(m)}: W_{\alpha_1,i_1}^{\lambda_1} \times \left\{ |z| \leq \frac{1}{n} \left( 1 + \frac{1}{m} \right) \right\} \rightarrow C_{g,n-,n+}(i)|_{W_{\alpha,i}^\lambda}$$

obtained by restricting the maps  $\sigma_{\alpha_1,i_1}^{\lambda_1}$  from (6.3.2). Let  $\phi_{\alpha_1,i_1}^{\lambda_1}: U_{\alpha_1,i_1}^{\lambda_1} \rightarrow W_{\alpha_1,i_1}^{\lambda_1}$  denote the composition of the retraction  $r_{\alpha_1,i_1}: U_{\alpha_1,i_1}^\lambda \rightarrow Z_{\alpha_1,i_1}^\lambda$  and homeomorphism  $Z_{\alpha_1,i_1}^\lambda \rightarrow W_{\alpha_1,i_1}^\lambda$ . Pulling back the maps  $(\sigma_{\alpha_1,i_1}^{\lambda_1})^{(m)}$  by  $\phi_{\alpha_1,i_1}^{\lambda_1}$  we obtain

$$(\phi_{\alpha_1,i_1}^{\lambda_1})^*(\sigma_{\alpha_1,i_1}^{\lambda_1})^{(m)}: U_{\alpha_1,i_1}^{\lambda_1} \times \left\{ |z| \leq \frac{1}{n} \left( 1 + \frac{1}{m} \right) \right\} \rightarrow (\phi_{\alpha_1,i_1}^{\lambda_1})^*C_{g,n-,n+}(n)|_{W_{\alpha,i}^\lambda}$$

Denote the image of this map along  $A_{n,j}^{(m)}$  and set  $A_n^{(m)} = \bigcup_j A_{n,j}^{(m)}$ .

Similar to Section 5.2.1, given an element  $\underline{x} \in \mathbb{G}^{n+1} \partial_+ \mathfrak{M} \amalg_{\mathbb{G}^{n+1} \partial_+ \mathfrak{M}^{nop}} \mathbb{G}^{n+1} \mathcal{M}^{nop}$ , with underlying

ioda-graph  $G$ , we can define an analytic embedding

$$\gamma_{\underline{x}}: \coprod_{E(G) \cup \text{in}(G) \cup \text{out}(G)} S^1 \rightarrow \Sigma_{\underline{x}}$$

with  $\Sigma_{\underline{x}}$  the Riemann surface obtained by gluing the inner-most labels of  $\underline{x}$  (ignoring any marked point present on the labels). Let  $\mathbb{F}_{n,J}^{(m)}$  denote the subcomplex of  $\mathbb{G}^\bullet \partial_+ \mathfrak{M} \amalg_{\mathbb{G}^\bullet \partial_+ \mathfrak{M}^{nop}} \mathbb{G}^\bullet \mathcal{M}^{nop}$  consisting of simplices  $\underline{x}$  such that the corresponding embedding  $\gamma_{\underline{x}}$  is disjoint from  $A_n^{(m)}$ . We then set  $F_{n,J}^{(m)}$  to be the open subset of  $|\mathbb{G}^{n+1} \partial_+ \mathfrak{M} \amalg_{\mathbb{G}^{n+1} \partial_+ \mathfrak{M}^{nop}} \mathbb{G}^{n+1} \mathcal{M}^{nop}|$  corresponding to this subcomplex. Set

$$\mathbb{W}_{n,J} = \bigcup_m \mathbb{F}_{n,j}^{(m)} \text{ and } W_{n,J} = \bigcup_m F_{n,j}^{(m)}.$$

Note that  $W_{n,J}$  deformation retracts onto the subcomplex

*Notation 6.3.2.* Denote the simplicial complex underlying  $\pi^{-1}(U_{\lambda_1, \dots, \lambda_k})$  by  $\mathbb{W}_{n,J}$ .

### 6.3.2 Construction of maps $\chi_n: U_J \rightarrow F^n \pi^{-1}(U_J)$

We now construct maps  $\chi_n$  as in (6.3.1). Note that unlike Section 5.2,  $\chi_n$  will not be a section in this case of  $\pi$ . The image of the map lies in the subspace of 0-skeleton of  $\pi^{-1}U_J$ .

Recall that from the construction of  $U_{\lambda_1}$  it follows that any surface with weighted marked point in  $U_{\lambda_1}$  has  $i_1$  marked point with weight  $> \alpha_1$  and there is a way of labeling these marked point as  $1, \dots, i_1$  with the labels varying continuously over  $U_{\lambda_1}$ .

Let  $\underline{\Sigma}$  be an element of  $U_J$  and let  $\Sigma$  be the Riemann surface underlying  $\underline{\Sigma}$ . We identify  $\Sigma$  with the fiber over  $\underline{\Sigma}$  of the pullback of  $C_{g,n_-,n_+}(i_1)$  to  $U_J$ . Let  $x_1, \dots, x_{i_1}$  denote the corresponding marked points on  $\Sigma$ . Denote by  $A_j^n(\underline{\Sigma}) \subset \Sigma$  the image of the (pullback of) map  $(\sigma_{\alpha, i_1}^{\lambda_1})^n$  restricted to the fiber above  $\underline{\Sigma}$  and by  $w_j(\Sigma)$  the weight of the  $j$ th marked point,  $0 \leq j \leq i_1$ . Set

$$\underline{A}_j^n(\underline{\Sigma}) := \begin{cases} [A_{n,j}^{(m)}(\underline{\Sigma}), (x_j), \rho(w_j)] \in \mathcal{M}_{0,0,1}^{fr} & \text{for } w_j < \alpha_n \\ [A_{n,j}^{(m)}(\underline{\Sigma}), (x_j), \rho(w_j) = 1] \in \mathcal{M}_{0,1,0}^{fr} & \text{for } w_j > \alpha_n \end{cases}$$

where  $\rho$  is the step function defined on  $[0, 1] \setminus \{\alpha_1, \dots, \alpha_k\}$ , given by:

$$\rho(x) := \begin{cases} 0 & \text{for } x \in (0, \alpha_1) \\ \frac{\alpha_r + \alpha_{r+1}}{2} & \text{for } x \in (\alpha_r, \alpha_{r+1}), 0 < r < n \\ 1 & \text{for } x \in (\alpha_k, 1) \end{cases} \quad (6.3.3)$$

(the parametrization of the boundary  $\partial A_{n,j}^{(m)}$  is given by the embeddings  $\sigma_{\alpha_1, i_1}^{\lambda_1}$ ).

Consider the complement  $\Sigma \setminus \bigcup_j \partial A_{n,j}^{(m)}(\Sigma)$ . Denote by  $G_J(\Sigma)$  the graph with vertices given by the connected component of  $\Sigma \setminus \bigcup_j \partial A_{n,j}^{(m)}(\Sigma)$  and edges given by the components of  $\partial A_{n,j}^{(m)}(\Sigma)$ , directed away from the vertices corresponding to disks  $A_{n,j}^{(m)}(\Sigma)$  if  $w_j < \alpha_n$  and towards them if  $w_j > \alpha_n$ .

Consider a labeling of  $G_J(\Sigma)$  with labels of vertices given by the corresponding connected components of  $\Sigma \setminus \bigcup \partial A_{n,j}^{(m)}$ . The boundaries are considered as inputs or outputs according to the orientations of the edges mentioned above.

This gives an element in the 0-skeleton  $(\mathbb{W}_{n,J})^0$ . Denote this element by  $x_{\underline{\Sigma}}$ .

We then define the map  $\chi_n: U_J \rightarrow W_{n,J}$  by

$$\underline{\Sigma} \mapsto x_{\underline{\Sigma}}$$

## 6.4 Verifying condition 6.3.1

We construct on  $W_{n,J}$  a homotopy between the identity and  $\chi_n \circ \pi$ . We start by describing notions analogous to those in Section 5.3.

### 6.4.1 The graphs $K_v$

These are analogous to graphs  $H_v$  defined in Section 5.3.2.

Let  $\underline{x}$  be an element in  $\mathbb{W}_{n,J}$  with underlying graph  $G$ , and let  $\underline{x}_v$  be the label of  $v \in V(G)$  in  $\underline{x}$ .

Let  $\underline{\Sigma}_{\underline{x}}$  and  $\underline{\Sigma}_{\underline{x}_v}$  denote the spotted surfaces obtained by gluing the inner-most labels of  $\underline{x}$  and  $\underline{x}_v$

respectively.

Then, denote by  $K_v$  the graph with:

- vertices given by the connected components of  $\Sigma_{\underline{x}_v} \setminus \bigcup_n \partial A_{n,j}^{(m)}(\Sigma_{\underline{x}})$ , and
- edges given by connected components of  $\bigcup_n \partial A_{n,j}^{(m)}(\Sigma_{\underline{x}})$ . The edges corresponding to  $\partial A_{n,j}^{(m)}$  are directed away from the disk  $A_{n,j}^{(m)}(\Sigma_{\underline{x}})$  if  $w_j < \alpha_n$  and towards it if  $w_j > \alpha_n$ .

#### 6.4.2 Cut map $c: F_{n,J}^{(m)} \rightarrow W_{n,J}$

As before  $c$  is defined as the geometric realization of a map of simplicial spaces given by  $c_r: \mathbb{F}_{n,J}^{(m)} \rightarrow (\mathbb{W}_{n,J})^r$ . Maps  $c_r$  are defined inductively as follows:

- $r = 0$ : Let  $\underline{x} \in (\mathbb{W}_{n,J})^0$  be an element with underlying graph  $G$ . Then  $c_0(\underline{x})$  is defined by the same expression as in (5.3.1), where now  $\underline{y}_v, v \in V(G)$  are defined as follows: let  $\underline{x}_v$  denote the label of  $v$  in  $\underline{x}$ . Then  $\underline{y}_v$  is defined to be the element of  $(\mathbb{W}_{n,J})^0$  with underlying ioda-graph  $K_v$  as above and with the labels of vertices in  $K_v$  given as follows:
  - For vertices corresponding to connected components  $\Sigma_{\underline{x}} \setminus \bigcup_i \partial A_{n,j}^{(m)}$  other than the disks  $A_{n,j}^{(m)}$ , the label is given by the underlying Riemann surface, treating the boundaries as inputs or outputs according to the edges of  $K_v$  and forgetting any marked point present on this surface (equivalently by the spotted surfaces containing no positive weight marked point corresponding to the connected component).
  - For vertices corresponding to the annuli  $A_{n,j}^{(m)}$ , the labels are given by the Riemann surface with weighted marked point given by the annuli  $A_{n,j}^{(m)}$  and with both boundaries treated as inputs.
- The inductive step for constructing the map  $c_r$  assuming the existence of  $c_i$  for  $i < r$  is performed exactly as in section 5.3.3, with  $(\mathbb{W}_{\Sigma})^i, H_v$ , and  $A_v$  respectively replaced by  $(\mathbb{W}_{n,J})^i, k_v$ , and  $A_{n,j}^{(m)}(\Sigma)$ .

### 6.4.3 Map $f_\rho$ and homotopy $H_\rho$

Recall the step function  $\rho$  from Section 6.3.3. We define  $f_\rho: \mathbb{W}_{n,J} \rightarrow \mathbb{W}_{n,J}$  to be the geometric realization of the map of simplicial spaces which changes the label of a simplex by applying  $\rho$  to the weights of the marked point in the label. Function  $\rho$  is continuous on its domain and thus the induced map  $f_\rho: \mathbb{W}_{n,J} \rightarrow \mathbb{W}_{n,J}$  is continuous as well.

Let  $H_\rho$  be the homotopy from the identity to  $f_\rho$  obtained by linearly interpolating the weights of the marked point in the labels.

### 6.4.4 Cut homotopy $\Phi_t: F_{n,J}^{(m)} \rightarrow W_{n,J}$

We now construct a homotopy from  $f_\rho$  to the cut map  $c$ .

It defined exactly as the cut homotopy in Section 5.3.4, using the same formulas as there, provided

1. we replace the subcomplex  $\mathbb{W}_{n,\Sigma}$  there with subcomplex  $\mathbb{W}_{n,J}$
2. the simplicial components  $c_i$  of the cut maps being used are as in Section 6.4.2 here, instead of Section 5.3.3.

### 6.4.5 The Homotopy $\Psi$

Finally we have a homotopy  $\Psi$  from  $c$  to the map  $\chi_n \circ \pi$ . The homotopy is again defined exactly as in Section 5.3.5, with the following straightforward adjustments:

1. Subcomplex  $\mathbb{W}_\Sigma^M$  is replaced by the subcomplex  $\mathbb{W}_{n,J}^n$  given by simplices with labels  $\underline{x}$ , such that for a vertex  $v$  with label  $\underline{x}_v$  in  $\underline{x}$ ,

$$\pi(\underline{x}_v) \cap A_{n,j}^{(m)}(\underline{\Sigma}) \neq \emptyset \Rightarrow \underline{x}_v = \eta^r(A_{n,j}^{(m)}(\underline{\Sigma})) \text{ for some } r$$

2. Section  $\chi_{n,\Sigma}^M$  is replaced by  $\chi_n$  as in 6.3.2.
3. graph  $G_{\pi(\underline{x})}$  is replaced by  $G_J(\underline{x})$  as in 6.3.2.

The homotopy  $\Psi$  is then defined using extra degeneracies and augmentations as in 5.3.5.

#### 6.4.6 Contraction homotopy proving condition (6.3.1)

This is just the concatenation of the cut homotopy  $\Phi$ , from identity to the cut map, and homotopy  $\Psi$ , from  $c$  to  $\chi_n \circ \pi$ .

This completes the proof of condition 6.3.1 and hence of the proof of Theorem 1.4.1(1).

## Chapter 7: From the TFT-properad to the Deligne–Mumford properad

We now prove that homotopy trivializing the io-subproperad of annuli  $\mathfrak{M}^{unst}$  in  $\mathfrak{M}$  gives the Deligne–Mumford properad  $\overline{\mathfrak{M}}$ .

Let us recall the statement of the Theorem:

**Theorem 7.0.1.**  *$\overline{\mathfrak{M}}$  is the homotopy colimit of the following diagram in the category of io-properads*

$$\begin{array}{ccc} \mathfrak{M}^{unst} & \longrightarrow & \overline{\mathfrak{M}}^{unst} \\ \downarrow & & \\ \mathfrak{M} & & \end{array} \quad (7.0.1)$$

As explained below in Section 8.4, properad  $\overline{\mathfrak{M}}$  takes values in stacks. In this chapter we shall show that the homotopy colimit of the diagram (7.0.1) in the category of topological properads has the property that for every  $(n_-, n_+)$  the corresponding space of operations has the same homotopy type as the stack  $\overline{\mathfrak{M}}(n_-, n_+) = \coprod_{g \geq 0} \mathcal{M}_{g, n_-, n_+}^{fr}$ , in the sense of Section 8.2.2. The statement in the category of io-properads then follows using Proposition 8.3.1.

A combination of the strategies used for proving Theorem 1.4.2 and Theorem 1.4.1(1) will be used for proving Theorem 7.0.1. To highlight the similarity between the argument here and in Section 5.2 as well as Section 6.3, we will use the same notation as in these sections for the analogous notions here. As outlined in Remark 7.2.2 below, to account for the appearance of composition of a sequence of nodal cylinders as well as nodal cylinders composed with disks with positive weight marked points, some changes are needed when constructing the open sets and filtration as in Section 5.2.1. In the rest of the section we carry out this proof.



## 7.1 Homotopy pushout of (7.0.1)

Similar to Section 6.3, it turns out that for computing the pushout 7.0.1, it is convenient to replace  $\mathfrak{M}, \mathfrak{M}^{unst}, \overline{\mathfrak{M}}$  and  $\overline{\mathfrak{M}}^{unst}$  with homotopy equivalent properads  $\mathcal{M}, \mathcal{M}^{unst}, \overline{\mathcal{M}}$  and  $\overline{\mathcal{M}}^{unst}$ . The definitions of these moduli spaces are of similar flavor as those in Section 6.1 using spotted surfaces, but are slightly different. We now describe these:

Moduli spaces  $\mathcal{M}_{g,n_-,n_+}$ , underlying properad  $\mathcal{M}$ , are defined analogously as the moduli spaces  $\mathcal{M}_{g,n_-,n_+}$  except the following: We do not impose the restriction that an irreducible component without an output necessarily has a marked point of weight 1, instead we require that if  $(g, n_-, n_+) = (0, 0, 1)$  or  $(0, 1, 0)$ , i.e. if with the Riemann surface underlying the spotted surface is given by a disk, then it contains at least one weight 1 marked point.

Properad  $\overline{\mathcal{M}}$  is built using moduli spaces  $\overline{\mathcal{M}}_{g,n_-,n_+}$  which are defined similarly as  $\mathcal{M}_{g,n_-,n_+}$ , with the only difference being that we define it using stable nodal surfaces with weighted marked points instead of smooth surfaces. We continue to impose the restriction that the spotted surface contains at least one weight 1 marked point when  $(g, n_-, n_+) = (0, 0, 1)$  or  $(0, 1, 0)$ . Note that the identification used for defining  $\overline{\mathcal{M}}$  by forgetting weight 0 marked points may involve collapsing unstable components.

Finally,  $\mathcal{M}^{unst}$  and  $\overline{\mathcal{M}}^{unst}$  are defined as the subproperad consisting of (possibly nodal) spotted surfaces such that the underlying surface, obtained by forgetting all the marked points, is a (possibly nodal) annulus, or a disk.

The following proposition is proved using an argument similar to the one used for Proposition 6.1.2:

**Proposition 7.1.1.** *The maps*

$$\mathcal{M} \rightarrow \mathfrak{M}, \quad \mathcal{M}^{unst} \rightarrow \mathfrak{M}^{unst}, \quad \overline{\mathcal{M}} \rightarrow \overline{\mathfrak{M}} \quad \text{and} \quad \overline{\mathcal{M}}^{unst} \rightarrow \overline{\mathfrak{M}}^{unst},$$

*given by mapping a spotted surface to the underlying nodal Riemann surface, are a homotopy equivalence of properads.*

□

Thus to compute the homotopy pushout of diagram (7.0.1), we use the homotopy equivalent pushout diagram

$$m \leftarrow m^{unst} \rightarrow \overline{m}^{unst}$$

Using Proposition 3.2.8 and 3.2.11, and an argument similar to the one used in Section 5.1, we know that the homotopy pushout of the diagram (7.0.1) coincides with the topological io-properad

$$\left| \mathbb{G} \left( \mathbb{G}^\bullet m \coprod_{\mathbb{G}^\bullet \overline{m}^{unst}} \mathbb{G}^\bullet \overline{m}^{unst} \right) \right|.$$

Consider the induced map

$$\left| \mathbb{G} \left( \mathbb{G}^\bullet m \coprod_{\mathbb{G}^\bullet \overline{m}^{unst}} \mathbb{G}^\bullet \overline{m}^{unst} \right) \right| \rightarrow \overline{m}. \quad (7.1.1)$$

We now show that this is a weak equivalence of io-properads in stacks, in the sense of Section 8.2.2 below.

## 7.2 (7.1.1) is a weak-equivalence

As in Section 5.2, we will prove that for every stable nodal spotted surface  $\underline{\Sigma}$ , there is a chart  $U_{\underline{\Sigma}}$  of  $\overline{m}$  such that for the atlas

$$U = \coprod_{\underline{\Sigma}_1, \dots, \underline{\Sigma}_k \in \overline{m}} U_{\underline{\Sigma}_1, \dots, \underline{\Sigma}_k} \rightarrow \overline{m}$$

where  $U_{\underline{\Sigma}_1, \dots, \underline{\Sigma}_k} = U_{\underline{\Sigma}_1} \times_{\overline{m}} \dots \times_{\overline{m}} U_{\underline{\Sigma}_k}$ , and for the pullback

$$\pi: \left| \mathbb{G} \left( \mathbb{G}^\bullet m \coprod_{\mathbb{G}^\bullet \overline{m}^{unst}} \mathbb{G}^\bullet \overline{m}^{unst} \right) \right| \times_{\overline{m}} U \rightarrow U$$

of (7.1.1) to this atlas, the following property holds: for every finite collection  $\underline{\Sigma}_1, \dots, \underline{\Sigma}_k$ ,

$$\pi^{-1}(U_{\underline{\Sigma}_1, \dots, \underline{\Sigma}_k}) \rightarrow U_{\underline{\Sigma}_1, \dots, \underline{\Sigma}_k} \text{ is a weak homotopy equivalence.} \quad (7.2.1)$$

We will in fact show that open sets  $\pi^{-1}(U_{\underline{\Sigma}})$  have a covering  $\{W_{n,\underline{\Sigma}}\}_{n \in \mathbb{N}}$  such that for every  $I \subset \mathbb{N}$  non-empty,

$$\pi|_{W_{I,\underline{\Sigma}}} : W_{I,\underline{\Sigma}} \rightarrow U_{\underline{\Sigma}} \text{ is a weak homotopy equivalence,}$$

where  $W_{I,\underline{\Sigma}} = \bigcap_{n \in I} W_{n,\underline{\Sigma}}$ .

As in Section 5.2 we shall prove this by exhibiting a filtration

$$F_{I,\underline{\Sigma}}^{(1)} \subset \dots \subset F_{I,\underline{\Sigma}}^{(m)} \dots \subset W_{I,\underline{\Sigma}}$$

which satisfies a condition analogous to Condition (5.2.3), with  $W_{n,\underline{\Sigma}}$  there replaced by  $W_{I,\underline{\Sigma}}$ , and section  $\chi_{n,\underline{\Sigma}}^{(m)}$  there replaced by a section  $\chi_{n,\underline{\Sigma}}^{(m)} : U_{\underline{\Sigma}} \rightarrow W_{I,\underline{\Sigma}}$ .

### 7.2.1 Construction of $U_{\underline{\Sigma}}$ , $W_{n,\underline{\Sigma}}$ , and $F_{I,\underline{\Sigma}}^{(m)}$

Suppose that  $\underline{\Sigma}$  is a nodal spotted surface. Let  $i$  be the number of positive weight marked points on  $\underline{\Sigma}$  and  $0 < \alpha < 1$  be such that there exists no marked point with weight  $\leq \alpha$  in  $\underline{\Sigma}$ .  $\Sigma$  be the nodal Riemann surface underlying  $\underline{\Sigma}$ .

The open set  $U_{\underline{\Sigma}}$  is defined to be the set of nodal spotted surfaces satisfying the following conditions

1. The underlying nodal Riemann surface lies in an open set  $U_{\Sigma}$  constructed exactly as in Section 5.2.1, the only difference being this time we work with  $\underline{\Sigma}$  in  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}(i)$ , the moduli space of stable nodal Riemann surfaces with  $i$  marked points, instead of  $\widehat{\mathcal{M}}_{g,n_-,n_+}^{fr}$ .
2. Shrinking  $U_{\Sigma}$  if necessary, we may assume that there exists a choice of charts  $\sigma_{\Sigma',j}$ ,  $1 \leq j \leq i$  centered at the  $j$ th marked point in each surface  $\Sigma' \in U_{\Sigma}$ , varying continuously with  $\Sigma'$ . Then, we assume that there are no marked points with weight  $\geq \alpha$  lying outside the images of these charts.

The open sets  $W_{n,\underline{\Sigma}}$  are constructed as follows:

For a simplex

$$\underline{x} \in \mathbb{G}^{n+1} m \coprod_{\mathbb{G}^{n+1} \overline{m}^{unst}} \mathbb{G}^{n+1} \overline{m}^{unst}$$

define  $\gamma_{\underline{x}}$  as in Section 5.2.1.

*Remark 7.2.1.* In the current situation, the map  $\gamma_{\underline{x}}$  may not be an embedding and moreover its image may not be disjoint from the nodes or the marked points . More precisely, the image of a component  $S^1$  under  $\gamma_{\underline{x}}$  is either disjoint from the nodes and marked points , or is contained inside one. Components mapping into nodes arise when we consider simplices  $\underline{x}$  for which the associated labeled graph contains a string of nodal annuli. Such strings do not appear in graphs considered in Section 5.2.1 since any vertex labeled by a nodal annulus there is of type  $(0, 2)$ . Components mapping into marked points arise from simplices for which the associated labeled graph contains a path with one of the vertices labeled by a nodal annulus and which terminates in a vertex labeled by a disk with a positive weight marked point .

For each  $n$  and  $\nu \in \mathcal{V}$ , let  $K_{n,\nu}^{(m)}$  be the open subset of the universal curve  $C_\Sigma$  over  $U_\Sigma$  defined by

$$K_{n,\nu}^{(m)} = \left\{ (z, \underline{t}) \in R_{\hat{\nu}} \times \mathbb{D}^{\mathcal{V}} \left| |t_\nu| \in \left[ 0, \frac{1}{4n^2} \right], \text{ and } |z_{\hat{\nu}}(z)| \in \left[ \frac{|t_\nu|}{\frac{1}{n-1} \left( 1 + \frac{1}{m} \right)}, \frac{|t_\nu|}{\frac{1}{n} \left( 1 - \frac{1}{m} \right)} \right] \cup \left[ \frac{1}{n} \left( 1 - \frac{1}{m} \right), \frac{1}{n-1} \left( 1 + \frac{1}{m} \right) \right] \right\}.$$

Set

$$K_n^{(m)} := \bigcup_{\nu \in \mathcal{V}} K_{n,\nu}^{(m)}.$$

Also for every  $n$  and  $1 \leq j \leq i$ , let  $L_{n,j}^{(m)} \subset C_\Sigma$  be the open set defined by the union of images

$$L_{n,j}^{(m)} := \bigcup_{\Sigma' \in U_\Sigma} \sigma_{\Sigma',j} \left[ \frac{1}{n} \left( 1 - \frac{1}{m} \right) < |z| < \frac{1}{n-1} \left( 1 + \frac{1}{m} \right) \right]$$

Define  $\mathbb{F}_{n,\underline{\Sigma}}^{(m)}$  to be the simplicial subspace of  $\mathbb{G}^{n+1}\mathcal{M} \amalg_{\mathbb{G}^{n+1}\mathcal{M}^{unst}} \mathbb{G}^{n+1}\overline{\mathcal{M}}^{unst}$  consisting of simplices  $\underline{x}$  such that the associated embedding  $\gamma_{\underline{x}}$  is disjoint from  $K_n^{(m)} \cup L_n^{(m)}$ . Set  $\mathbb{F}_{I,\underline{\Sigma}}^{(m)}$  to be the

intersection

$$\mathbb{F}_{I,\underline{\Sigma}} := \bigcap_{n \in I} \mathbb{F}_{n,\underline{\Sigma}}.$$

Set  $F_{n,\underline{\Sigma}}^{(m)}$  and  $F_{I,\underline{\Sigma}}^{(m)}$  to be the open sets of  $\left| \mathbb{G}^{n+1} \mathcal{M} \amalg_{\mathbb{G}^{n+1} \mathcal{M}^{unst}} \mathbb{G}^{n+1} \overline{\mathcal{M}}^{unst} \right|$  corresponding to these subcomplexes. We set

$$\mathbb{W}_{n,\underline{\Sigma}} := \bigcup_{m \in \mathbb{N}} F_{n,\underline{\Sigma}}^{(m)} \quad \text{and} \quad \mathbb{W}_{I,\underline{\Sigma}} = \bigcap_{n \in I} \mathbb{W}_{n,\underline{\Sigma}}$$

Finally, we define

$$W_{n,\underline{\Sigma}} = \bigcup_{m \in \mathbb{N}} F_{n,\underline{\Sigma}}^{(m)}.$$

$W_{n,\underline{\Sigma}}$  deformation retracts onto the geometric realization of the subcomplex  $\mathbb{W}_{n,\underline{\Sigma}}$ .

Note that the open subsets

$$F_{I,\underline{\Sigma}}^{(m)} := \bigcap_{n \in I} F_{n,\underline{\Sigma}}^{(m)}$$

deformation retract onto the subcomplexes  $\mathbb{F}_{I,\underline{\Sigma}}$  and provide a filtration of the subspace

$$W_{I,\underline{\Sigma}} = \bigcap_{n \in I} W_{n,\underline{\Sigma}}$$

*Remark 7.2.2.* Note that this is analogous to the construction of open sets  $A_n^\gamma$  and  $\mathbb{W}_{n,\underline{\Sigma}}$  in 5.2.1, except that in this case instead of an annulus containing the node we have a union of two annuli, one on each side of the node, and both disjoint from the node. This is necessary since the open sets constructed using annuli  $A_n^\gamma$  as in 5.2.1 do not cover the entire geometric realization in the current context: for all  $\underline{x}$  which contains a sequence of nodal annuli as mentioned in Remark 7.2.1, the image of  $\gamma_{\underline{x}}$  intersects annuli  $A_n^\gamma$  for any  $n$ . In order to construct a filtration which covers such points and moreover gives an open subset in the geometric realization, it is natural to work with the sets  $K_n^\gamma$  considered above.

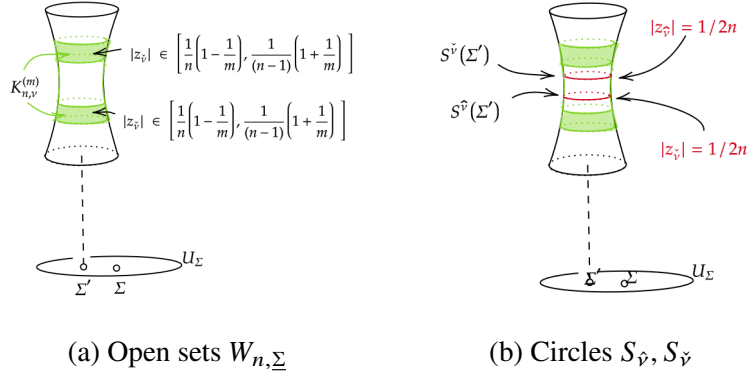


Figure 7.1

### 7.3 Proof of Condition (5.2.3)

Let us now turn to the fiberwise contraction.

Fix any  $n_0 \in I$ . Let  $\underline{\Sigma}' \in U_{\underline{\Sigma}}$  and let  $S^{\hat{\nu}}(\underline{\Sigma}')$ , and  $S^{\check{\nu}}(\underline{\Sigma}')$  be circles defined by

$$S^{\hat{\nu}}(\underline{\Sigma}') = \left\{ (z, t_{\hat{\nu}}(\underline{\Sigma}')) \mid |z_{\hat{\nu}}(z)| = \frac{1}{n_0} \left( 1 + \frac{1}{2m} \right) \right\}, S^{\check{\nu}}(\underline{\Sigma}') = \left\{ (z, t_{\check{\nu}}(\underline{\Sigma}')) \mid |z_{\check{\nu}}(z)| = \frac{|t_{\hat{\nu}}(\underline{\Sigma}')|}{\frac{1}{n_0} \left( 1 + \frac{1}{2m} \right)} \right\} \subset \underline{\Sigma}'$$

Denote by  $A_{\nu}(\underline{\Sigma}')$  the annulus bounded by  $S^{\hat{\nu}}$  and  $S^{\check{\nu}}$  in  $\underline{\Sigma}'$ . Similarly, let  $S^{\hat{j}}(\Sigma')$  and  $S^{\check{j}}(\Sigma')$  denote the circles

$$S^{\hat{j}}(\Sigma') = \sigma_{\Sigma', j} \left( \left\{ |z| = \frac{1}{n_0} \left( 1 - \frac{1}{m} \right) \right\} \right) \text{ and } S^{\check{j}}(\Sigma') = \sigma_{\Sigma', j} \left( \left\{ |z| = \frac{1}{n_0 - 1} \left( 1 + \frac{1}{m} \right) \right\} \right)$$

and  $A_j(\Sigma')$  be the annulus bounded by these.

Having thus defined  $A_{\nu}(\Sigma')$  and  $A_j(\Sigma')$ , rest of the argument for verifying Condition (5.2.3), including the construction of cut maps, cut homotopies, and the contraction proving Condition (5.2.3), proceeds by combining the constructions in Sections 5.3 and 6.3 in a straightforward manner, with cuts made around both the nodes and weight 1 marked points. The corresponding maps and homotopies can be defined using exactly the same expressions as in Section 5.3, provided we adopt the definition of graphs  $H_{\nu}$  constructed in 5.3.2 as follows:

### 7.3.1 Modification of graphs from Section 5.3.2

Let  $\underline{x}$  be an element in  $\mathbb{W}_{I,\underline{\Sigma}}$  with underlying graph  $G$ , and let  $\underline{x}_v$  be the label of  $v \in V(G)$  in  $\underline{x}$ . Let  $\underline{\Sigma}_{\underline{x}}$  and  $\underline{\Sigma}_{\underline{x}_v}$  denote the spotted surfaces obtained by gluing the inner-most labels of  $\underline{x}$  and  $\underline{x}_v$  respectively.

Then, in this case we define  $H_v$  to be the graph with:

- vertices given by the connected components of  $\underline{\Sigma}_{\underline{x}_v} \setminus \left( \bigcup_v \partial A_v(\underline{\Sigma}_{\underline{x}}) \cup \bigcup_j \partial A_j(\underline{\Sigma}_{\underline{x}}) \right)$ , and
- edges given by connected components of  $\bigcup_v \partial A_v(\underline{\Sigma}_{\underline{x}}) \cup \bigcup_j \partial A_j(\underline{\Sigma}_{\underline{x}})$ . The edges corresponding to  $\partial A_v(\underline{\Sigma}_{\underline{x}})$  are directed away from the annulus  $A_{n,j}^v(\underline{\Sigma}_{\underline{x}})$  and those corresponding to  $\partial A_j(\underline{\Sigma}_{\underline{x}})$  are directed away from the disk  $A_j(\underline{\Sigma}_{\underline{x}})$ .

*Remark 7.3.1.* Unlike the graphs  $H_v$  constructed in 5.3.2, analogously defined graphs here will not satisfy the property that every vertex has an output (see Remark 5.3.1). This was needed to ensure that the cut maps and homotopies constructed there land in appropriate targets. This itself was necessary since properad  $\partial_+ \mathfrak{M}$  has no operations with  $n_+ = 0$ . However in the current case, properad  $\mathfrak{M}$  has operations in all components and thus the constraint on graphs  $H_v$  mentioned above is not relevant for the construction of cut maps and homotopies in this case.

Having thus adopted the definition of graphs  $H_v$ , the rest of the argument for proving condition 5.2.3 carries over from Section 5.3, with the same formulae defining the maps and homotopies in this case. This concludes the proof of condition Condition (5.2.3), and hence of Theorem 7.0.1

## Chapter 8: Homotopy colimits of properads in stacks

In this chapter we outline a formalism for  $\infty$ -io-properads and explain how to view  $\overline{\mathfrak{M}}$  as an example of such a properad. Moreover, we explain how to interpret the earlier statements about homotopy pushouts of io-properads in the context of  $\infty$ -io-properads.

In this chapter we use the language of  $\infty$ -operads and  $\infty$ -categories. The main references for this part are [28, Section 2.1], and [29, Section 5.2]. We shall also use the theory of homotopy type of stacks and singular simplicial complexes of stacks developed in [30] and [31].

### 8.1 $\infty$ -properads

As in the case of operads, there are now a number of models available for up-to-homotopy properads. In [32], a model for  $\infty$ -properads in terms of graphical sets (presheaves on a certain category of graphs) was constructed, generalizing the dendroidal sets approach to  $\infty$ -operads. A model category structure was constructed on the category of simplicial properads in [33], generalizing an analogous structure on the category of simplicial operads. Another approach is to use the fact that ordinary properads can be expressed as algebras over a colored operad. A model for  $\infty$ -properads can then be obtained by considering algebras over the  $\infty$ -operad associated with this operad.

We will follow this last approach to set up a category of  $\infty$ -io-properads. The reason behind this choice is that in the restricted setting of io-properads, the results of [34] can be used to prove Proposition 8.1.2. It gives a convenient way for comparing the homotopy categories, and in particular the homotopy colimits, of ordinary io-properads with those of  $\infty$ -io-properads. Proposition 8.1.2 uses the fact that we are working with io-properads: the corresponding theorem for non-io-properads is known to be false (see Remark 8.1.3 below).



### 8.1.1 $\infty$ -properads

As described in [28], we can associate to any colored operad  $O$  a category  $O^\otimes$  with a map  $O^\otimes \rightarrow \mathcal{F}in_*$  such that the associated map of nerves  $NO^\otimes \rightarrow N\mathcal{F}in_*$  defines an  $\infty$ -operad. We denote this  $\infty$ -operad simply as  $O^\otimes$  and refer to it as the  $\infty$ -operad associated to  $O$ . For later reference, we note here that

- objects of  $O^\otimes$  are sequences  $c_1, \dots, c_k$  of colors of  $O$ , and
- morphisms in  $O^\otimes$  from  $\{c_i\}_{i \in [k]}$  to  $\{d_j\}_{j \in [l]}$  are given by a map  $\alpha: [k] \rightarrow [l]$  along with an element

$$\phi_j \in O(\{c_i\}_{i \in \alpha^{-1}(j)}; d_j) \text{ for every } j \quad (8.1.1)$$

Given a symmetric monoidal category  $(C, \otimes)$ , we have an associated colored operad  $O_C$ . We will denote the corresponding category  $O_C^\otimes$  simply as  $C^\otimes$ . Here,

- objects of  $C^\otimes$  are given by sequences of  $c_1, \dots, c_n$  of objects in  $C$  and
- morphisms in  $C^\otimes$  between  $\{c_i\}_{i \in [k]}$  and  $\{d_j\}_{j \in [l]}$  are given by a maps  $\alpha: \phi: [k] \rightarrow [l]$  along with an element

$$\phi_j \in C\left(\bigotimes_{i \in \alpha^{-1}(j)} c_i; d_j\right) \text{ for every } j \quad (8.1.2)$$

Notice that the data of an  $O$ -algebra in  $C$  is equivalent to an  $\infty$ -operad map  $O^\otimes \rightarrow C^\otimes$ .

More generally when  $C$  is a symmetric monoidal  $\infty$ -category, there is an associated  $\infty$ -operad  $C^\otimes \rightarrow N\mathcal{F}in_*$ . Now, recall the set-valued, colored operad  $\mathbf{Prpd}_{\text{red}}$  defined in Section 2.6 above. Note that an algebra over  $\mathbf{Prpd}_{\text{red}}$  in  $\mathbf{Top}$  is precisely a topological properad. Let  $\mathbf{Prpd}_{\text{red}}^\otimes$  denote the  $\infty$ -operad associated to  $\mathbf{Prpd}_{\text{red}}$ .

**Definition 8.1.1.** An  $\infty$ -io-properad in a symmetric monoidal  $\infty$ -category  $C$  is defined to be a morphism of  $\infty$ -operads  $\mathbf{Prpd}_{\text{red}}^\otimes \rightarrow C^\otimes$ , in other words a  $\mathbf{Prpd}_{\text{red}}^\otimes$ -algebra in  $C$ .

These algebras can be assembled into an  $\infty$ -category which we denote by  $\text{AlgPrpd}_{\text{red}}^{\otimes}(C)$ .

### 8.1.2 The comparison functor

Recall that to any model category  $A$  we can associate an  $\infty$ -category  $N(A^c)[W^{-1}]$ , where

1.  $A^c$  denotes the full subcategory of cofibrant objects in  $A$ ,
2.  $N(A^c)$  denotes the  $\infty$ -category obtained by taking the nerve of the ordinary category  $A^c$ ,
3.  $W$  is the collection of weak equivalences in  $A^c$ , and
4.  $N(A^c)[W^{-1}]$  denotes the ( $\infty$ -categorical) localization of the  $\infty$ -category  $N(A^c)$  at  $W$ .

(see [28, Definition 1.3.4.15 and Remark 1.3.4.16]). This is an  $\infty$ -categorical enhancement of the homotopy category of  $A$ . In particular, homotopy colimits in  $A$  correspond to the  $\infty$ -categorical colimits in  $N(A^c)[W^{-1}]$ .

We denote the  $\infty$ -categories associated with the model categories **Top** and  $\text{AlgPrpd}_{\text{red}}(\mathbf{Top})$  by  $\mathbf{Top}_{\infty}$  and  $\text{AlgPrpd}_{\text{red}}(\mathbf{Top})_{\infty}$ .

We have a comparison map

$$\text{AlgPrpd}_{\text{red}}(\mathbf{Top})_{\infty} \rightarrow \text{AlgPrpd}_{\text{red}}^{\otimes}(\mathbf{Top}_{\infty}) \quad (8.1.3)$$

between  $\infty$ -categories.

**Proposition 8.1.2.** *The comparison functor (8.1.3) is an equivalence of  $\infty$ -categories.*

*Proof.* This is an analogue for  $\infty$ -io-properads of [35, Theorem 7.2.4], which itself is a corollary of [34, Theorem 7.11]. As in [35], the main step is the verification of the symmetric flatness hypothesis in [34, Theorem 7.11], which in our case follows from Lemma 2.6.3.  $\square$

*Remark 8.1.3.* In Proposition 8.1.2 we use the fact that we are working with input-output properads. The corresponding statement is not true for ordinary properads (see [35, Theorem 7.2.5]). However, even without the full strength of the proposition, it may still be possible to show that the

homotopy colimits appearing in our theorems continue to be preserved in the context of ordinary  $\infty$ -properads. We do not pursue this here.

## 8.2 Topological stacks and their homotopy theory

In this section we recall the definition of topological stacks and construct an  $\infty$ -category modeling their homotopy theory.

### 8.2.1 Topological stacks

Recall that a *groupoid* is a category in which every morphism is an isomorphism. Groupoids form a (strict) 2-category. An equivalence of groupoids is an equivalence of the underlying categories.

Denote by  $\mathcal{PShv}(\mathbf{Grpd})$  the (strict) 2-category of presheaves of groupoids over topological spaces. By the Yoneda Lemma, we have a fully faithful embedding

$$\mathbf{Top} \rightarrow \mathcal{PShv}(\mathbf{Grpd}).$$

More precisely, for any topological space  $T$  and a presheaf of groupoid  $\mathcal{X}$ , we have an isomorphism of groupoids

$$\mathcal{X}(T) \simeq \mathcal{PShv}(\mathbf{Grpd})(\mathcal{X}, T).$$

Say that a map of presheaves  $\mathcal{X} \rightarrow \mathcal{Y}$  is an equivalence if for every topological space  $T$ , the corresponding maps  $\mathcal{X}(T) \rightarrow \mathcal{Y}(T)$  are equivalences of groupoids. Two presheaves are called equivalent if they are related by a zig-zag of equivalences.

**Definition 8.2.1.** A *stack* over topological spaces is a presheaf of groupoids  $\mathcal{X}$  such that for any topological space  $T$  and an open cover  $\{U_\alpha\}$  of  $T$ ,

$$\mathcal{X}(T) \rightarrow \varinjlim \left[ \prod_{\alpha} \mathcal{X}(U_{\alpha}) \rightrightarrows \prod_{\alpha, \beta} \mathcal{X}(U_{\alpha\beta}) \Rrightarrow \prod_{\alpha, \beta, \gamma} \mathcal{X}(U_{\alpha\beta\gamma}) \right]$$

is an equivalence of groupoids. Here  $\varinjlim$  is the 2-categorical limit in the category of groupoids.

Let  $\mathcal{S}tk$  denote the full subcategory spanned by stacks in  $\mathcal{PShv}(\mathbf{Grpd})$ . Equivalences of stacks are precisely those of the underlying presheaves of groupoids.

*Remark 8.2.2.* Conventionally a stack is defined as a category fibered in groupoids which satisfies a descent condition. Let  $\mathcal{G}rpd/\mathbf{Top}$  denote the 2-category of categories fibered in groupoids over  $\mathbf{Top}$ . It was proved in [36], that there are model category structures on  $\mathcal{G}rpd/\mathbf{Top}$  and  $\mathcal{PShv}(\mathbf{Grpd})$  such that the weak equivalences in both categories are given by the corresponding notions of equivalences. Moreover, there is an adjunction

$$\Gamma: \mathcal{G}rpd/\mathbf{Top} \rightleftarrows \mathcal{PShv}(\mathbf{Grpd}): p$$

which is a Quillen equivalence with respect to these model structures. Furthermore, this adjunction restricts to one between the category of categories fibered in groupoids over topological spaces which satisfy a descent condition and the category of stacks in the sense of Definition 8.2.1. Thus both these approaches for defining stacks turn out to be equivalent.

For later reference, we note that the functor  $\Gamma$  is given by

$$\Gamma: \mathcal{E} \mapsto (T \mapsto \mathcal{G}rpd/\mathbf{Top}_{\mathcal{G}rpd}(\mathbf{Top}/T, \mathcal{E})).$$

Here ,

1.  $\mathbf{Top}/T$  given by the category of topological spaces with a map to  $T$ . Note that this is a category fibered in groupoids over  $\mathbf{Top}$ .
2.  $\mathcal{G}rpd/\mathbf{Top}_{\mathcal{G}rpd}(\_, \_)$  denotes the groupoid of 1-morphisms in  $\mathcal{G}rpd/\mathbf{Top}$ .

Recall that a topological groupoid  $\mathbb{X}$  is a groupoid such that the set of objects and the set of morphisms are topological spaces, and moreover all the structure maps (source, target, identities, compositions, and inverses) are continuous. Let  $\mathbf{TopGrpd}$  be the category of topological groupoids.

Given a topological groupoid  $\mathbb{X}$ , we can define a presheaf given by

$$T \mapsto [\mathbb{X}](T) := \mathbf{TopGrpd}_{Grpd}(T, \mathbb{X})$$

where  $\mathbf{TopGrpd}_{Grpd}$  denotes the groupoid of  $\mathbf{TopGrpd}$ -morphisms from  $T$  to  $\mathbb{X}$ . This presheaf can be *sheafified* to obtain a stack  $[\mathbb{X}]$ , known as the quotient stack of  $\mathbb{X}$ , by

$$[\mathbb{X}]: T \rightarrow \lim_{\substack{U \subset T \\ \text{open}}} \mathbf{TopGrpd}_{Grpd}(U, \mathbb{X})$$

**Definition 8.2.3.** By a *topological stack* we mean a stack which is equivalent to the quotient stack of some topological groupoid.

### 8.2.2 Homotopy type of a topological stack

Following [30, Section 5] we say that a map of topological spaces  $f: X \rightarrow Y$  is *shrinkable* if it has a section  $s: Y \rightarrow X$  such that  $X$  admits a fiberwise relative deformation retract onto  $s(Y)$ . We say that a map  $f: X \rightarrow Y$  is *locally shrinkable* if  $Y$  has an open cover  $\{U_\alpha\}$  such that  $f|_{f^{-1}(U_\alpha)}$  is shrinkable for each  $\alpha$ .

Say that a representable map of topological stacks  $f: \mathcal{X} \rightarrow \mathcal{Y}$  is locally shrinkable if for any topological space  $T$  and a map  $g: T \rightarrow \mathcal{Y}$ , the pullback  $g \times_{\mathcal{Y}} f: T \times_{\mathcal{Y}} \mathcal{X} \rightarrow T$  is locally shrinkable. We say that a representable map  $\mathcal{X} \rightarrow \mathcal{Y}$  of topological stacks is a *universal weak equivalence* if for any topological space  $T$  with a map  $T \rightarrow \mathcal{Y}$ , the induced map  $T \times_{\mathcal{Y}} \mathcal{X} \rightarrow T$  is a weak equivalence. In particular, any locally shrinkable map is a universal weak equivalence.

Finally, we say that a stack  $\mathcal{X}$  admits an *atlas* if there exists a representable map  $U \rightarrow \mathcal{X}$  from a topological space  $U$ , such that for any topological space  $T$  and a map  $T \rightarrow \mathcal{X}$  the pullback  $T \times_{\mathcal{X}} U \rightarrow T$  admits local sections. It follows from the definition of topological stacks that every topological stack admits an atlas.

Note that if  $V \rightarrow \mathcal{Y}$  is an atlas for  $\mathcal{Y}$ , a representable map of stacks  $\mathcal{X} \rightarrow \mathcal{Y}$  is locally shrinkable

(respectively, a universal weak equivalence) if and only if the induced map of topological spaces  $T \times_{\mathcal{Y}} \mathcal{X} \rightarrow \mathcal{X}$  is locally shrinkable (respectively, a weak equivalence).

**Definition 8.2.4.** We say that a topological space  $X$  *represents the weak homotopy type* of a stack  $\mathcal{X}$  if there exists a universal weak homotopy equivalence  $X \rightarrow \mathcal{X}$ .

Note that such a representative, if it exists, is unique up to weak homotopy equivalence. We now state a theorem, due to Noohi, guaranteeing the existence of such a space:

**Theorem 8.2.5** ([31, Theorem 3.4]). *Let  $\mathcal{X}$  be a topological stack. Then there exists a topological space  $X$  along with a map  $X \rightarrow \mathcal{X}$  which is locally shrinkable and in particular a universal weak equivalence.*

From Theorem 8.2.5 it follows that the weak homotopy type of every topological stack can be represented by a topological space.

Given a map of stacks  $f: \mathcal{X} \rightarrow \mathcal{Y}$  and a topological space  $Y$  representing the weak homotopy type of  $\mathcal{Y}$  via a map  $Y \rightarrow \mathcal{Y}$ , the pullback  $X := Y \times_{\mathcal{Y}} \mathcal{X} \rightarrow \mathcal{X}$  represents the weak homotopy type of  $\mathcal{X}$  and moreover, we have an induced map  $\tilde{f}: X \rightarrow Y$ . In Section 8.2.4 below, we shall construct a model for the homotopy theory of topological stacks obtained by inverting maps  $f$  such that the induced maps  $\tilde{f}$  on representatives of the homotopy types are weak homotopy equivalences.

### 8.2.3 Singular simplicial complexes of presheaves of groupoid

Let  $sSet$  denote the category of simplicial sets. We now define a functor

$$Sing: \mathcal{PShv}(\mathbf{Grpd}) \rightarrow sSet.$$

For an  $\mathcal{X} \in \mathcal{PShv}(\mathbf{Grpd})$ , define  $\mathcal{X}_{\Delta}$  to be the simplicial groupoid

$$\mathcal{X}_{\Delta}: n \mapsto \mathcal{X}(\Delta^n).$$

Let  $N(\mathcal{X}_\Delta)$  denote the bisimplicial set obtained by taking level-wise nerve of  $\mathcal{X}_\Delta$ . Finally, define  $\mathcal{S}ing(\mathcal{X})$  by

$$\mathcal{S}ing(\mathcal{X}) := \text{Diag}(N(\mathcal{X}_\Delta)),$$

where  $\text{Diag}: bsSet \rightarrow sSet$  is the diagonal functor defined by  $\{\mathcal{Y}_{m,n}\}_{m \geq 0, n \geq 0} \mapsto \{\mathcal{Y}_{n,n}\}_{n \geq 0}$ .

We note here a useful property of  $\mathcal{S}ing$ :

**Lemma 8.2.6.**  *$\mathcal{S}ing$  preserves limits. In particular,  $\mathcal{S}ing$  preserves products.* □

#### 8.2.4 $\infty$ -category of stacks

Given two stacks  $\mathcal{X}$  and  $\mathcal{Y}$ , we can consider the mapping stack  $\text{Maps}(\mathcal{X}, \mathcal{Y})$ . The underlying presheaf of groupoids is given by

$$T \mapsto \mathcal{S}tk_{Grpd}(\mathcal{X} \times T, \mathcal{Y}),$$

where  $\mathcal{S}tk_{Grpd}$  denotes the groupoid of stack morphisms. Using the functor  $\mathcal{S}ing$  above we can associate with this stack a simplicial set  $\mathcal{S}ing(\text{Maps}(\mathcal{X}, \mathcal{Y}))$ . Using Lemma 8.2.6, it follows that this provides a simplicial enrichment of the category of stacks. Denote this simplicial category by  $\mathcal{S}tk^\Delta$ .

We can associate with this a fibrant simplicial category by taking the Kan replacements of the simplicial morphism spaces: Recall that the category  $sSet$  has a symmetric monoidal fibrant replacement functor given by

$$X \mapsto X^\wedge = \text{Maps}(\Delta_\bullet, |X|)$$

Using this functor, we replace the  $sSet$ -morphism spaces  $\mathcal{S}ing(\text{Maps}(\mathcal{X}, \mathcal{Y}))$  by their fibrant replacement  $\mathcal{S}ing(\text{Maps}(\mathcal{X}, \mathcal{Y}))^\wedge$ . This gives us a fibrant simplicial enrichment of the category of stacks. Taking the homotopy coherent nerve of this category gives an  $\infty$ -category  $\widetilde{\mathcal{S}tk}_\infty$ . As a model for the homotopy theory of topological stacks, we shall use a category  $\mathcal{S}tk_\infty$  which is obtained by further localization of  $\widetilde{\mathcal{S}tk}_\infty$  at the class of morphisms mentioned at the end of Section

8.2.2 above.

Let  $\mathbf{Top}^\Delta$  denote the simplicially enriched category of spaces and  $\widetilde{\mathbf{Top}}_\infty$  the corresponding  $\infty$ -category of spaces. Note that

$$\mathbf{Top}_\infty = \widetilde{\mathbf{Top}}_\infty[S^{-1}],$$

where  $S$  denotes the class of weak homotopy equivalences in  $\widetilde{\mathbf{Top}}_\infty$  (see for example, [28, Theorem 1.3.4.20]). For topological spaces  $X, Y$ , the mapping stack  $\mathbf{Maps}(X, Y)$  is the stack associated with the space of continuous maps  $\mathbf{Top}(X, Y)$  with the usual compact-open topology. Thus, the functor

$$\widetilde{\mathbf{Top}}_\infty \rightarrow \widetilde{\mathcal{S}tk}_\infty$$

obtained by mapping a topological space to the stack associated to it, is fully faithful.

We now state Lemma 8.2.7 and Proposition 8.2.8 which are general category theoretic results. These are essentially  $\infty$ -categorical refinements of some results from [30, Section 7]. We shall use  $\infty$ -categorical generalizations of the 1-categorical arguments in loc. cit. to prove their  $\infty$ -categorical enhancements.

Recall that for an  $\infty$ -category  $C$  and a collection of morphisms  $\mathcal{W}$  in  $C$ ,  $C[W^{-1}]$  denotes the  $\infty$ -categorical localization of  $C$  at  $W$ .

**Lemma 8.2.7** ([30, Lemma 7.5]). *Let  $F: \mathcal{D} \rightleftarrows \mathcal{E}: G$  be an adjunction between  $\infty$ -categories. Let  $S \subset \mathcal{D}$  and  $T \subset \mathcal{E}$  be collection of morphisms such that  $F(S) \subset T$  and  $G(T) \subset S$ . Then, there is an induced adjunction*

$$\widetilde{F}: \mathcal{D}[S^{-1}] \rightleftarrows \mathcal{E}[T^{-1}]: \widetilde{G}$$

*Moreover, if  $F$  is fully-faithful then so is  $\widetilde{F}$ .*

*Proof.* The existence of the adjoint follows using [37, Proposition 7.1.14]. If  $F$  is fully-faithful it follows that the unit of the adjunction  $\mathbb{1}_{\mathcal{E}} \Rightarrow G \circ F$  is an isomorphism of  $\infty$ -functors. In this case, the unit of the adjunction induced on localized categories is also an isomorphism and it follows that the functor  $\widetilde{F}$  is fully-faithful as well.  $\square$



Let us now turn to Proposition 8.2.8. The set up for the proposition is as follows: Let  $\mathcal{B} \rightarrow \mathcal{C}$  be a fully-faithful functor between  $\infty$ -categories and let  $R$  be a class of morphisms in  $\mathcal{B}$  which contains all identities, is closed under compositions, and is closed under pullbacks (see [37, Definition 7.2.14] for the precise meanings of these terms). Denote by  $\widetilde{R}$  the class of morphisms  $f: y \rightarrow x$  in  $\mathcal{C}$  satisfying the property that for any  $p: t \rightarrow x$  with  $t \in \mathcal{B}$ , the pullback  $t \times_x y \rightarrow t$  of  $f$  along  $p$  is in  $R$ .

**Proposition 8.2.8** ([30, Lemma 7.1, 7.2]). *In the setting described in the previous paragraph, suppose that for every  $x \in \mathcal{C}$ , there exists an object  $\theta(x)$  in  $\mathcal{B}$  along with a map  $\theta(x) \rightarrow x$  lying in  $\widetilde{R}$ . Then, the functor  $\mathcal{B} \rightarrow \mathcal{C}$  induces a functor  $i: \mathcal{C}[R^{-1}] \rightarrow \mathcal{B}[R^{-1}]$  which is fully faithful and has a right adjoint  $\mathcal{B}[R^{-1}] \rightarrow \mathcal{C}[R^{-1}]$  which naturally extends  $\theta$ . Further, the functors  $i$  and  $\theta$  induce an equivalence between  $\mathcal{C}[\widetilde{R}^{-1}]$  and  $\mathcal{B}[R^{-1}]$ .*

*Proof.* Let  $\mathcal{H}$  denote the homotopy category of spaces and let  $h\mathcal{B}[R^{-1}]$  and  $h\mathcal{C}[R^{-1}]$  denote the  $\mathcal{H}$ -enriched homotopy categories of  $\mathcal{B}[R^{-1}]$  and  $\mathcal{C}[R^{-1}]$  respectively, as described in [29, Section 1.1.5]. Let  $hi: h\mathcal{B}[R^{-1}] \rightarrow h\mathcal{C}[R^{-1}]$  be the functor induced by inclusion  $i: \mathcal{B} \rightarrow \mathcal{C}$ . We shall prove that  $hi$  is fully faithful and has a right adjoint

$$h\theta: h\mathcal{C}[R^{-1}] \rightarrow h\mathcal{B}[R^{-1}]$$

given by  $\theta(\_)$  on objects. It will follow that  $i$  is fully-faithful and using [29, Proposition 5.2.2.12] we get that  $i$  has a right adjoint

$$\theta: \mathcal{C}[R^{-1}] \rightarrow \mathcal{B}[R^{-1}],$$

Using the proof of that proposition it follows that we may assume that  $\theta$  is given by  $\theta(\_)$  on objects. Finally, the fact that  $i$  and  $\theta$  induce an equivalence of categories  $\mathcal{C}[\widetilde{R}^{-1}]$  and  $\mathcal{B}[R^{-1}]$  follows using Lemma 8.2.7.

Let us now turn to the construction of the functor  $h\theta$ . In order to carry out the construction we use the description of morphism spaces in localizations given by the theory of *Calculus of Fractions* discussed in [37, Section 7.2].

Using Theorem 7.2.16 and Remark 7.2.10 from [37] it follows that the class  $R$  of morphisms has a right calculus of fractions given by the maximal putative right calculus of fraction at each object and the homotopy types of the morphism spaces in the localization are given by the following homotopy equivalences of simplicial sets

$$C[R^{-1}](x, y) \simeq \text{Span}_C^R(x, y) \text{ and}$$

$$\mathcal{B}[R^{-1}](x, y) \simeq \text{Span}_{\mathcal{B}}^R(x, y),$$

Here  $\text{Span}_C^R(x, y)$  (respectively,  $\text{Span}_{\mathcal{B}}^R(x, y)$ ) denotes the  $\infty$ -category of diagrams of the form  $x \xleftarrow{r} \tilde{x} \xrightarrow{f} y$  in  $C$  (respectively,  $\mathcal{B}$ ) with  $r$  in  $R$  (see [37, Remark 7.2.10] for more on the Span-categories). Applying [37, Lemma 7.2.15] to the map  $\theta(y) \rightarrow y$  and using the fact that the pull-back of  $\theta(y) \rightarrow y$  along any  $x \rightarrow y$  with  $x \in \mathcal{B}$  lies in  $R$ , it follows that the map

$$i_{x,y}: \text{Span}_{\mathcal{B}}^R(x, \theta(y)) \rightarrow \text{Span}_C^R(x, y)$$

given by inclusion  $\mathcal{B} \rightarrow C$  followed by post-composition with the map  $\theta(y) \rightarrow y$ , has a right adjoint  $g_{x,y}$ . Moreover, on the level of objects  $g_{x,y}$  is given by mapping a span  $x \xleftarrow{r} z \xrightarrow{f} y$  in  $C$  to the span  $x \xleftarrow{r'} z' \xrightarrow{f'} \theta(y)$  in  $\mathcal{B}$  where  $f'$  is obtained by considering the cartesian square

$$\begin{array}{ccc} z' & \xrightarrow{f'} & \theta(y) \\ \downarrow s & & \downarrow \\ z & \xrightarrow{f} & y \end{array}$$

and  $r' = r \circ s$ . It follows that for any  $x \in \mathcal{B}$  and  $y \in C$ , the map

$$i_{x,y}: \mathcal{B}[R^{-1}](x, \theta(y)) \rightarrow C[R^{-1}](x, y)$$

induces a homotopy equivalence of the simplicial sets underlying the  $\infty$ -categories, with the homotopy inverse given by the  $g_{x,y}$ .

We now describe the  $\mathcal{H}$ -enriched functor  $h\theta: hC[R^{-1}] \rightarrow h\mathcal{B}[R^{-1}]$  : On objects it is given by  $\theta(\_)$ , as mentioned above. On morphisms we define  $h\theta_{x,y}$  by the composition

$$hC[R^{-1}](x, y) \rightarrow hC[R^{-1}](\theta(x), y) \xrightarrow{g_{\theta(x), y}} h\mathcal{B}[R^{-1}](\theta(x), \theta(y))$$

where the first arrow is induced by pre-composing with  $\theta(x) \rightarrow x$ . To verify compatibility with composition note that we have the following commutative diagram in the homotopy category of spaces  $\mathcal{H}$ :

$$\begin{array}{ccc}
hC[R^{-1}](x, y) \times hC[R^{-1}](y, z) & \longrightarrow & hC[R^{-1}](x, z) \\
\downarrow & & \downarrow \\
hC[R^{-1}](\theta(x), y) \times hC[R^{-1}](y, z) & \longrightarrow & hC[R^{-1}](\theta(x), z) \\
\uparrow i_{\theta(x), y} \times \mathbb{1} & & \parallel \\
h\mathcal{B}[R^{-1}](\theta(x), \theta(y)) \times hC[R^{-1}](y, z) & & \\
\downarrow & & \parallel \\
h\mathcal{B}[R^{-1}](\theta(x), \theta(y)) \times hC[R^{-1}](\theta(y), z) & \longrightarrow & hC[R^{-1}](\theta(x), z) \\
\uparrow \mathbb{1} \times i_{\theta(y), z} & & \uparrow i_{\theta(x), z} \\
h\mathcal{B}[R^{-1}](\theta(x), \theta(y)) \times h\mathcal{B}[R^{-1}](\theta(y), \theta(z)) & \longrightarrow & h\mathcal{B}[R^{-1}](\theta(x), \theta(z))
\end{array}$$

Here all the unlabeled vertical arrows are induced by suitable pre/post-compositions and the horizontal arrows are given by compositions in suitable categories. Notice that the maps  $i_{\theta(x), y} \times \mathbb{1}$ ,  $\mathbb{1} \times i_{\theta(y), z}$ , and  $i_{\theta(x), z}$  are homotopy equivalences with inverses  $g_{\theta(x), y} \times \mathbb{1}$ ,  $\mathbb{1} \times g_{\theta(y), z}$ , and  $g_{\theta(x), z}$  respectively. Thus inverting these maps and composing the vertical arrows we get the following commutative diagram :

$$\begin{array}{ccc}
hC[R^{-1}](x, y) \times hC[R^{-1}](y, z) & \longrightarrow & hC[R^{-1}](x, z) \\
\downarrow h\theta_{x, y} \times h\theta_{y, z} & & \downarrow h\theta_{x, z} \\
h\mathcal{B}[R^{-1}](\theta(x), \theta(y)) \times h\mathcal{B}[R^{-1}](\theta(y), \theta(z)) & \longrightarrow & h\mathcal{B}[R^{-1}](\theta(x), \theta(z))
\end{array}$$

This completes the proof of compatibility of  $h\theta$  with composition and thus of the fact that  $h\theta$

defines a functor. It is clear from the construction of  $h\theta$  that it is right adjoint to  $hi$ . That  $hi$ , and thus  $i$ , is fully-faithful follows from the adjunction using the fact that  $\theta(y) \rightarrow y$  is an isomorphism in  $\mathcal{B}[R^{-1}]$  for any  $y \in B$ .  $\square$

Now, let  $R$  denote the class of locally shrinkable morphisms in  $\widetilde{\mathbf{Top}}_\infty$ . We have the following

**Proposition 8.2.9** ([30, Proposition 8.1]).  *$i: \widetilde{\mathbf{Top}}_\infty \rightarrow \widetilde{\mathcal{S}tk}_\infty$  induces a fully faithful functor  $\widetilde{\mathbf{Top}}_\infty[R^{-1}] \rightarrow \widetilde{\mathcal{S}tk}_\infty[R^{-1}]$  which has a right adjoint  $\theta: \widetilde{\mathbf{Top}}_\infty \rightarrow \widetilde{\mathcal{S}tk}_\infty$ .*

*Proof.* This is an  $\infty$ -categorical restatement of [30, Proposition 8.1]. The proof follows by applying Lemma 8.2.8 to the inclusion  $\widetilde{\mathbf{Top}}_\infty \rightarrow \widetilde{\mathcal{S}tk}_\infty$ .  $\square$

Finally, recall that  $S$  denotes the class of weak homotopy equivalences in  $\widetilde{\mathbf{Top}}_\infty$  and  $\widetilde{\mathbf{Top}}_\infty[S^{-1}] \simeq \mathbf{Top}_\infty$ . Denote by  $S_{\mathcal{S}tk}$  the class of maps in  $\widetilde{\mathcal{S}tk}_\infty$  whose image under  $\theta$  lies in  $\mathcal{S}tk$  and by  $\mathcal{S}tk_\infty$  the localization  $\mathcal{S}tk[S_{\mathcal{S}tk}^{-1}]$ . Then, using Lemma 8.2.7 and Proposition 8.2.9 we have:

**Corollary 8.2.10** ([30, Corollary 8.3]).  *$i$  and  $\theta$  as in Proposition 8.2.9 induce a pair of adjoint functors*

$$i: \mathbf{Top}_\infty \rightleftarrows \mathcal{S}tk_\infty: \theta$$

*which are in fact equivalences of categories. In particular,  $i: \mathbf{Top}_\infty \rightarrow \mathcal{S}tk_\infty$  is fully faithful and preserves colimits.*

$\square$

### 8.3 $\infty$ -io-Propserads in $\mathcal{S}tk_\infty$

Both  $\mathbf{Top}_\infty$  and  $\mathcal{S}tk_\infty$  admit finite products and the inclusion  $\mathbf{Top}_\infty \hookrightarrow \mathcal{S}tk_\infty$  preserves finite products. Let  $\mathbf{Top}_\infty^\otimes$  and  $\mathcal{S}tk_\infty^\otimes$  be the symmetric monoidal categories associated with the  $\infty$ -categories  $\mathbf{Top}_\infty$  and  $\mathcal{S}tk_\infty$  with the monoidal structure give by finite products. Using Corollary 8.2.10 it follows that we have an equivalence of symmetric monoidal  $\infty$ -categories

$$\mathbf{Top}_\infty^\otimes \hookrightarrow \mathcal{S}tk_\infty^\otimes$$

Thus we have,

**Proposition 8.3.1.** *The inclusion  $\mathbf{Top}_\infty \hookrightarrow \mathcal{Stk}_\infty$  induces a functor of operad algebras*

$$Alg_{\mathbf{Prpd}_{\text{red}}^\otimes}(\mathbf{Top}_\infty^\otimes) \hookrightarrow Alg_{\mathbf{Prpd}_{\text{red}}^\otimes}(\mathcal{Stk}_\infty^\otimes)$$

which is an equivalence of categories and in particular preserves colimits.  $\square$

#### 8.4 $\overline{\mathfrak{M}}$ as an $\infty$ -io-properad in $\mathcal{Stk}_\infty$

Recall that  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$  is the moduli stack of genus  $g$  stable nodal Riemann surfaces with  $n_-$  input and  $n_+$  output parametrized boundaries. As a category fibered in groupoids this stack is the category with the

- objects given by the families  $C \rightarrow T$  of nodal stable Riemann surfaces of genus  $g$  with  $n_-$  input and  $n_+$  output boundary components, over topological spaces, and
- morphisms given by pullback diagrams of such families.

The functor  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr} \rightarrow \mathbf{Top}$  is given by mapping a family  $(C \rightarrow T)$  to its base  $T$ .

The presheaf of groupoids underlying  $\overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$ , obtained by applying functor  $\Gamma$  described in Remark 8.2.2, is thus given by

$$T \mapsto \mathcal{Grpd}/\mathbf{Top}_{Grpd}(\mathbf{Top}/T, \overline{\mathcal{M}}_{g,n_-,n_+}^{fr})$$

*Notation 8.4.1.* Let  $\overline{\mathcal{M}}_{n_-,n_+}^{fr}$  denote the stack  $\coprod_{g \geq 0} \overline{\mathcal{M}}_{g,n_-,n_+}^{fr}$ .

We now outline how to view  $\overline{\mathfrak{M}}$  as a  $\mathbf{Prpd}_{\text{red}}^\otimes$ -algebra using these moduli spaces. We shall in fact show that  $\overline{\mathfrak{M}}$  can be realized as an algebra over  $\mathbf{Prpd}_{\text{red}}$ , interpreted as a 2-operad, in the (strict) 2-category of stacks  $\mathcal{Stk}$ . More concretely, we will construct a lax 2-functor of categories over  $\mathcal{Fin}_*$

$$\overline{\mathfrak{M}}_2: \mathbf{Prpd}^\otimes \rightarrow \mathcal{Stk}^\otimes.$$

Here

1.  $\mathcal{Stk}^\otimes$  is the category over  $\mathcal{Fin}_*$  associated to the strict 2-category  $\mathcal{Stk}$  of stacks as in (8.1.2), where the monoidal product is given by product of stacks. (The construction in (8.1.2) is described for 1-categories, but generalizes to the case of strict 2-categories in a straightforward manner).
2.  $\mathbf{Prpd}_{\text{red}}^\otimes$  is the category over  $\mathcal{Fin}_*$  associated with the colored operad  $\mathbf{Prpd}_{\text{red}}$  as in (8.1.1).

For an object  $\underline{n}^1, \underline{n}^2, \dots, \underline{n}^k$  in  $\mathbf{Prpd}^\otimes$  define

$$\overline{\mathfrak{M}}_2(\underline{n}^1, \underline{n}^2, \dots, \underline{n}^k) := \{\overline{\mathfrak{M}}_{\underline{n}^i}^{fr}\}_{i=1}^k$$

Action of the functor  $\overline{\mathfrak{M}}_2$  on morphisms is defined as follows: as described in 8.1.1, the morphisms in the category  $\mathbf{Prpd}^\otimes$  are given by a sequence of vertex-ordered ioda-graphs. We describe how  $\overline{\mathfrak{M}}_2$  acts on the morphisms of type  $\mathbf{Prpd}^\otimes(\underline{n}^1, \underline{n}^2, \dots, \underline{n}^k; \underline{n}_+)$ , in other words on morphisms given by a single vertex-ordered ioda-graphs. The definition extends to morphisms described by disconnected graphs in a straightforward manner.

Let  $G$  be a (connected) vertex-ordered ioda-graph describing a morphism form  $\underline{n}^1, \underline{n}^2, \dots, \underline{n}^k$  to  $\underline{n}_+$ .

We define a map

$$\mu_G(\overline{\mathfrak{M}}_2) : \overline{\mathfrak{M}}_2(\underline{n}^1, \underline{n}^2, \dots, \underline{n}^k) = \prod_i \overline{\mathfrak{M}}_{\underline{n}^i}^{fr} \rightarrow \overline{\mathfrak{M}}_2(\underline{n}_+) = \overline{\mathfrak{M}}_{\underline{n}^1}^{fr}$$

Let  $T$  be any test space, we will describe the functor

$$\mu_G(\overline{\mathfrak{M}}_2)(T) : \prod_i \overline{\mathfrak{M}}_{\underline{n}^i}^{fr}(T) \rightarrow \overline{\mathfrak{M}}_2(\underline{n}_+)(T)$$

Here,

1.  $\prod_i \overline{\mathfrak{M}}_{\underline{n}^i}^{fr}(T)$  is the groupoid of tuples  $\{C^i \rightarrow T\}_{i=1}^n$  such that the curves in  $C^i$  have input-output profile  $\underline{n}^i$

2.  $\overline{\mathfrak{M}}_2(\underline{n}_+)(T)$  is the groupoid of families  $C_{\underline{n}_+} \rightarrow T$  with input-output profile  $\underline{n}_+$

The groupoid map  $\mu_G(\overline{\mathfrak{M}}_2)(T)$  is induced by mapping a tuple of families over  $T$  as in (1) to the family obtained by gluing them as prescribed by the edges of  $G$  (followed by stabilization).

Now let  $G$  and  $H$  be two graphs describing a pair of composable morphisms in  $\mathbf{Prpd}^\otimes$ . Given a test space  $T$  and a tuple of families of curves  $\{C_v \rightarrow T\}_{v \in V(G)}$  indexed by the vertices of  $G$ , we note that the two families over  $T$  obtained by

- first gluing the families along edges of  $G$  and then gluing the resulting families along  $H$ , and
- gluing the families along the composed graph  $H \circ G$

are canonically isomorphic. Thus there exist canonical isomorphisms

$$\mu_H(\overline{\mathfrak{M}}_2)(T) \circ \mu_G(\overline{\mathfrak{M}}_2)(T) \Rightarrow \mu_{H \circ G}(T).$$

These isomorphisms are natural in  $T$  and hence it follows that there exists a *unique* 2-isomorphism  $\alpha: \overline{\mathfrak{M}}_2(H) \circ \overline{\mathfrak{M}}_2(G) \Rightarrow \overline{\mathfrak{M}}_2(H \circ G)$ .

Thus we have a lax 2-functor

$$\overline{\mathfrak{M}}_2: \mathbf{Prpd}^\otimes \rightarrow \mathcal{S}tk^\otimes$$

as desired.

From the construction of  $\overline{\mathfrak{M}}_2$  it is clear that the induced map of  $\infty$ -categories over  $\mathcal{F}in_*$  preserves inert morphisms and thus defines a map of  $\infty$ -operads.

Note that  $\mathcal{S}tk$  is a subcategory of the  $\infty$ -category  $\mathcal{S}tk_\infty$  and the inclusion  $\mathcal{S}tk \hookrightarrow \mathcal{S}tk_\infty$  induces a map of  $\infty$ -operads  $\mathcal{S}tk^\otimes \rightarrow \mathcal{S}tk_\infty^\otimes$ . Composing  $\overline{\mathfrak{M}}_2$  with this map then gives a map the desired map of  $\infty$ -operads:

$$\overline{\mathfrak{M}}: \mathbf{Prpd}^\otimes \rightarrow \mathcal{S}tk^\otimes$$

exhibiting  $\overline{\mathfrak{M}}$  as a  $\mathbf{Prpd}_{\text{red}}^\otimes$ -algebra.

## Appendix A: Proofs from Chapter 3

The aim of this appendix is to fill in technical details which were left out in Chapter 3.2.

In Section A.1 we explain in detail the fact that the category of topological io-properads is tensored and cotensored over the category of topological spaces. We also prove Lemma A.1.1 which describes the compatibility of the tensor and cotensor operations with the model structure on io-properads. In Section A.2, we recall the definition of geometric realization from Section 3.2, and provide its alternate description in terms of *latching spaces* which was alluded to there. In Section A.3, we use this description to prove Proposition 3.2.2 and Corollary 3.2.6 from Section 3.2. In Section A.4 we provide the proof of Proposition 3.2.4 asserting that the geometric realizations of simplicial io-properads in the categories of topological io-sequences and io-properads coincide. Finally, in Section A.5, we provide the details of the proof of Proposition 3.2.11.

### A.1 Tensor and Cotensor over topological spaces

The category of io-properads is tensored and cotensored over topological spaces. This means that, along with the bifunctor

$$[\_, \_]: \mathbb{G}\mathbf{Alg} \times \mathbb{G}\mathbf{Alg}^{op} \rightarrow \mathbf{Top}$$

providing an enrichment of the category of io-properads over topological spaces, there are two additional bifunctors

$$\odot: \mathbb{G}\mathbf{Alg} \times \mathbf{Top} \rightarrow \mathbb{G}\mathbf{Alg}, \text{ and}$$

$$(\_)-: \mathbb{G}\mathbf{Alg} \times \mathbf{Top}^{op} \rightarrow \mathbb{G}\mathbf{Alg}$$



such that there are isomorphisms

$$\mathbb{G}\mathbf{Alg}(P \odot Z, Q) \simeq \mathbf{Top}(Z, [P, Q]) \simeq \mathbb{G}\mathbf{Alg}(P, Q^Z),$$

natural in all the variables involved. These bifunctors are constructed as follows:

The enrichment of topological io-properads over topological spaces is given by endowing the properad hom-spaces with the natural topology obtained by identifying them as subspaces of the hom-spaces of topological io-sequences. In other words,  $[P, Q]$  is the space of io-properad maps from  $P$  to  $Q$  with the product compact-open topology. Equivalently, if  $[ , ]^{\mathbf{TopSeq}^{\text{io}}}$  denotes the enrichment of topological io-sequences over topological spaces,  $[P, Q]$  is defined as the equalizer

$$[P, Q] \rightarrow [P, Q]^{\mathbf{TopSeq}^{\text{io}}} \rightrightarrows [\mathbb{G}P, Q]^{\mathbf{TopSeq}^{\text{io}}},$$

where the first arrow is induced by pullback along the properad structure map  $\mathbb{G}P \rightarrow P$  and the second map is the composition  $[P, Q]^{\mathbf{TopSeq}^{\text{io}}} \rightarrow [\mathbb{G}P, \mathbb{G}Q]^{\mathbf{TopSeq}^{\text{io}}} \rightarrow [\mathbb{G}P, Q]^{\mathbf{TopSeq}^{\text{io}}}$

The operation  $\odot$  can be constructed as

$$P \odot Z := \text{coeq} \left[ \mathbb{G}(\mathbb{G}P \times Z) \rightrightarrows \mathbb{G}(P \times Z) \right].$$

Here, for  $X$  a topological io-sequence and  $Z$  a topological space,  $X \times Z$  denotes the topological io-sequence given by component-wise product  $\{X(n_-, n_+) \times Z\}_{n_-, n_+}$ . The coequalizer can be taken in either topological io-properads or topological io-sequences, since they both coincide in this case. The first arrow is induced from the properad structure map  $\mathbb{G}P \rightarrow P$ , whereas the second arrow is induced from  $\mathbb{G}P \times Z \rightarrow \mathbb{G}(P \times Z)$  using the universal property of free properads.

The operation  $(\_)-$  can be constructed as follows: The underlying sequence of  $P^Z$  is given by  $P^Z = \{\mathbf{Top}(Z, P(n_-, n_+))\}_{n_-, n_+}$  and the properad structure is induced from that of  $P$ .

These operations are compatible with the model structure in the following sense:

**Lemma A.1.1** ([38]). *The following equivalent conditions are satisfied:*

1. *If  $i: Y \rightarrow Z$  is a cofibration of topological spaces and  $j: P \rightarrow Q$  is a cofibration of io-properads, then*

$$P \odot Z \begin{array}{c} \text{\textcircled{G}Alg} \\ \square \\ P \odot Y \end{array} Q \odot Y \rightarrow Q \odot Z$$

*is a cofibration of io-properads. Moreover this map is a weak equivalence whenever  $i$  or  $j$  is.*

2. *If  $i: Y \rightarrow Z$  is a cofibration of topological spaces and  $j: P \rightarrow Q$  is a fibration of io-properads, then*

$$P^Z \rightarrow P^Y \times_{Q^Y} Q^Z$$

*is a fibration of properads. Moreover this map is a weak equivalence whenever  $i$  or  $j$  is.*

3. *If  $j: P \rightarrow Q$  is a cofibration and  $k: R \rightarrow S$  a fibration of io-properads, then*

$$[Q, R] \rightarrow [P, R] \times_{[P, S]} [Q, S]$$

*is a fibration. Again, this map is a weak equivalence whenever  $j$  or  $k$  is.*

*Proof.* We explain the proof of (2) and the implication (2)  $\Rightarrow$  (1), since this is the only part we shall use.

(2): Note that we need to show  $P^Z \rightarrow P^Y \times_{Q^Y} Q^Z$  is a fibration of underlying topological io-sequences. The fibered product  $P^Y \times_{Q^Y} Q^Z$  coincides with the fibered product of the underlying topological io-sequences. Let  $V \rightarrow W$  be an acyclic cofibration in topological io-sequences. We need to show that any square

$$\begin{array}{ccc} V & \longrightarrow & P^Z \\ \downarrow & & \downarrow \\ W & \longrightarrow & P^Y \times_{Q^Y} Q^Z \end{array}$$

admits a lift  $W \rightarrow P^Z$  (in topological io-sequences). This is equivalent to the statement that any

square

$$\begin{array}{ccc} W \times Y \sqcup_{V \times Y} V \times Z & \longrightarrow & P \\ \downarrow & & \downarrow \\ W \times Z & \longrightarrow & Q \end{array}$$

admits a lift  $W \times Z \rightarrow P$  (in topological io-sequences). But this follows from the fact that

- $P \rightarrow Q$  is a fibration of io-properads, and hence of topological io-sequences as well
- $Y \rightarrow Z$  is a cofibration of topological spaces and  $V \rightarrow W$  is an acyclic cofibration of topological io-sequences, and hence  $W \times Y \sqcup_{V \times Y} V \times Z \rightarrow W \times Z$  is an acyclic cofibration in topological io-sequences.

The conclusion when  $i$  or  $j$  is a weak equivalence follows by a similar argument, starting with a cofibration  $V \rightarrow W$  which is not necessarily a weak equivalence. This completes the proof of (2).

(2)  $\Rightarrow$  (1): To show that  $P \odot Z \sqcup_{P \odot Y}^{\mathbf{GAlg}} Q \odot Y \rightarrow Q \odot Z$  is a cofibration of topological io-properads, we show that for any acyclic fibration  $E \rightarrow F$  in io-properads the square

$$\begin{array}{ccc} P \odot Z \sqcup_{P \odot Y}^{\mathbf{GAlg}} Q \odot Y & \longrightarrow & E \\ \downarrow & & \downarrow \\ Q \odot Z & \longrightarrow & F \end{array}$$

admits a lift  $Q \odot Z \rightarrow E$  (in io-properads). This is equivalent to

$$\begin{array}{ccc} P & \longrightarrow & E^Z \\ \downarrow & & \downarrow \\ Q & \longrightarrow & F^Y \times_{F^Y} F^Z \end{array}$$

admitting a lift  $Q \rightarrow E^Z$  (in io-properads). But this is a consequence of (2) combined with the fact that  $P \rightarrow Q$  is a properad cofibration.

Again, the conclusion when  $i$  or  $j$  is a weak equivalence follows by a similar argument, starting with a fibration  $E \rightarrow F$  which is not necessarily a weak equivalence.  $\square$

## A.2 Geometric realization

Now, using this tensor product on the category of io-properads over topological spaces, we can define the geometric realization of a simplicial io-properad  $\{P_\bullet\}_{\bullet \geq 0}$  by the usual formula:

$$|P_\bullet|^{\mathbb{G}\mathbf{Alg}} = \int_{\Delta}^{\mathbb{G}\mathbf{Alg}} P_n \odot \Delta^n := \text{coeq}^{\mathbb{G}\mathbf{Alg}} \left[ \coprod_{\phi: [n] \rightarrow [m] \in \Delta}^{\mathbb{G}\mathbf{Alg}} P_m \odot \Delta^n \rightrightarrows \coprod_{[n] \in \Delta}^{\mathbb{G}\mathbf{Alg}} P_n \odot \Delta^n \right] \quad (\text{A.2.1})$$

Here  $\Delta$  denotes the simplex category, and  $\Delta^n$  is the standard  $n$ -simplex. In the coequalizer, the first arrow is induced from the simplicial structure map  $P^*(\phi): P_m \rightarrow P_n \in \mathbb{G}\mathbf{Alg}$  corresponding to  $\phi: [n] \rightarrow [m]$  and the second arrow is induced from the map  $\Delta(\phi): \Delta^n \rightarrow \Delta^m \in \mathbf{Top}$ .

There is an alternate description of  $| \cdot |^{\mathbb{G}\mathbf{Alg}}$  as an iterated pushout in terms of the so called ‘latching spaces’:

Define the  $n$ th *latching space* of  $P_\bullet$  to be the coequalizer

$$L_n P_\bullet := \text{coeq}^{\mathbb{G}\mathbf{Alg}} \left[ \coprod_{0 \leq i < j \leq n}^{\mathbb{G}\mathbf{Alg}} P_{n-1} \rightrightarrows \coprod_{0 \leq i \leq n}^{\mathbb{G}\mathbf{Alg}} P_n \right].$$

The two arrows are defined as follows: on the  $(i, j)$ th summand, the first arrow is induced by the degeneracy map  $s_i: P_{n-1} \rightarrow P_n$  into the  $j$ th summand and the second arrow is induced from  $s_{j-1}: P_{n-1} \rightarrow P_n$  into the  $i$ th summand. Note that there is a canonical map

$$L_n P_\bullet \rightarrow P_{n+1}.$$

$L_n P_\bullet$  can be thought of intuitively as “the space of degenerate simplices in  $P_{n+1}$ ”.

Define  $| \cdot |_{(n)}^{\mathbb{G}\mathbf{Alg}}$  inductively as:

- $|P_\bullet|_{(0)}^{\mathbb{G}\mathbf{Alg}} = P_0$ , and

- $|P_\bullet|^{\mathbb{G}\mathbf{Alg}}_{(n+1)}$  is defined to be the pushout

$$\begin{array}{ccc}
L_n P_\bullet \odot \Delta^{n+1} \sqcup_{L_n P_\bullet \odot \partial \Delta^{n+1}}^{\mathbb{G}\mathbf{Alg}} P_{n+1} \odot \partial \Delta^{n+1} & \longrightarrow & P_{n+1} \odot \Delta^{n+1} \\
\downarrow & & \downarrow \\
|P_\bullet|^{\mathbb{G}\mathbf{Alg}}_{(n)} & \longrightarrow & |P_\bullet|^{\mathbb{G}\mathbf{Alg}}_{(n+1)}
\end{array} \tag{A.2.2}$$

Then we have  $|P_\bullet|^{\mathbb{G}\mathbf{Alg}} = \varinjlim_n |P_\bullet|^{\mathbb{G}\mathbf{Alg}}_{(n)}$ .

Using this description we can now reformulate the cofibrancy condition on  $|P_\bullet|^{\mathbb{G}\mathbf{Alg}}$  in terms of the pushouts (A.2.2):

**Lemma A.2.1.** *If  $L_n P_\bullet \rightarrow P_{n+1}$  is a cofibration of io-properads for every  $n$ , then  $|P_\bullet|^{\mathbb{G}\mathbf{Alg}}$  is a cofibrant io-properad.*

*Proof.* It suffice to prove that the upper horizontal arrow in the diagram A.2.2 is a cofibration for all  $n$ . This follows from the compatibility statements from Lemma A.1.1 since  $\partial \Delta^{n+1} \hookrightarrow \Delta^{n+1}$  is a cofibration of topological spaces and  $P \rightarrow Q$  is a cofibration of io-properads.  $\square$

We have a relative version of Lemma A.2.1:

**Lemma A.2.2.** *If  $P_\bullet \rightarrow Q_\bullet$  is a map of simplicial io-properads such that  $L_n Q_\bullet \sqcup_{L_n P_\bullet}^{\mathbb{G}\mathbf{Alg}} P_{n+1} \rightarrow Q_{n+1}$  is a cofibration of properads for every  $n$ , then  $|P_\bullet|^{\mathbb{G}\mathbf{Alg}} \rightarrow |Q_\bullet|^{\mathbb{G}\mathbf{Alg}}$  is a cofibration of io-properads.*

*Proof.* Using the filtrations on  $|P_\bullet|$  and  $|Q_\bullet|$  described by the diagram (A.2.2), it suffices to prove that

$$|P_\bullet|^{\mathbb{G}\mathbf{Alg}}_{(n+1)} \sqcup_{|P_\bullet|^{\mathbb{G}\mathbf{Alg}}_{(n)}}^{\mathbb{G}\mathbf{Alg}} |Q_\bullet|^{\mathbb{G}\mathbf{Alg}}_{(n)} \rightarrow |Q_\bullet|^{\mathbb{G}\mathbf{Alg}}_{(n+1)} \tag{A.2.3}$$

is a cofibration of io-properads. For convenience, set

$$L_n^\Delta P_\bullet := L_n P_\bullet \odot \Delta^{n+1} \bigsqcup_{L_n P_\bullet \odot \partial \Delta^{n+1}}^{\mathbb{G}\mathbf{Alg}} P_{n+1} \odot \partial \Delta^{n+1}, \text{ and}$$

$$L_n^\Delta Q_\bullet := L_n Q_\bullet \odot \Delta^{n+1} \bigsqcup_{L_n Q_\bullet \odot \partial \Delta^{n+1}}^{\mathbb{G}\mathbf{Alg}} Q_{n+1} \odot \partial \Delta^{n+1}.$$

Then, considering the map of diagrams (A.2.2) induced by  $P_\bullet \rightarrow Q_\bullet$ , it can be proved that (A.2.3) is equivalent to showing that

$$L_n^\Delta Q_\bullet \bigsqcup_{L_n^\Delta P_\bullet}^{\mathbb{G}\mathbf{Alg}} P_{n+1} \odot \Delta^{n+1} \rightarrow Q_{n+1} \odot \Delta^{n+1} \quad (\text{A.2.4})$$

is a cofibration of io-properads. To prove (A.2.4) observe that

$$L_n^\Delta Q_\bullet \bigsqcup_{L_n^\Delta P_\bullet}^{\mathbb{G}\mathbf{Alg}} P_{n+1} \odot \Delta^{n+1} = \left( L_n Q_\bullet \bigsqcup_{L_n P_\bullet}^{\mathbb{G}\mathbf{Alg}} P_{n+1} \right) \odot \Delta^{n+1} \bigsqcup_{(L_n Q_\bullet \bigsqcup_{L_n P_\bullet}^{\mathbb{G}\mathbf{Alg}} P_{n+1}) \odot \partial \Delta^{n+1}}^{\mathbb{G}\mathbf{Alg}} Q_n \odot \partial \Delta^{n+1}$$

(A.2.4) now follows by Lemma A.1.1 from the hypothesis that  $L_n Q_\bullet \sqcup_{L_n P_\bullet}^{\mathbb{G}\mathbf{Alg}} P_{n+1} \rightarrow Q_{n+1}$  is a cofibration.  $\square$

### A.3 Cofibrancy of the bar construction

We now provide the proof of Proposition 3.2.2. Recall that for an io-properad  $P$  its bar construction is a simplicial properad which we denote by  $B_\bullet(\mathbb{G}, \mathbb{G}, P)$ .

Let us start by recalling the statement of Proposition 3.2.2:

**Proposition A.3.1.** *If the topological io-sequence underlying a io-properad  $P$  is cofibrant, then  $|B(\mathbb{G}, \mathbb{G}, P)|^{\mathbb{G}\mathbf{Alg}}$  is cofibrant in the category of io-properads.*

*Proof.* The latching objects for the bar construction are given by

$$L_n B(\mathbb{G}, \mathbb{G}, P) = \text{coeq}^{\mathbb{G}\mathbf{Alg}} \left[ \coprod_{0 \leq i < j \leq n}^{\mathbb{G}\mathbf{Alg}} \mathbb{G}^n P \rightrightarrows \coprod_{0 \leq i \leq n}^{\mathbb{G}\mathbf{Alg}} \mathbb{G}^{n+1} P \right].$$

The free  $\mathbb{G}$ -algebra functor, being the left adjoint in the pair  $\mathbb{G}: \mathbf{TopSeq}^{\text{io}} \rightleftarrows \mathbb{G}\mathbf{Alg}$ : Forget, commutes with coprouducts and coequalizer. Thus,

$$L_n B(\mathbb{G}, \mathbb{G}, P) = \mathbb{G} \left( \text{coeq} \left[ \coprod_{0 \leq i < j \leq n} \mathbb{G}^{n-1} P \rightrightarrows \coprod_{0 \leq i \leq n} \mathbb{G}^n P \right] \right).$$

Set

$$K_n P := \text{coeq} \left[ \coprod_{0 \leq i < j \leq n} \mathbb{G}^{n-1} P \rightrightarrows \coprod_{0 \leq i \leq n} \mathbb{G}^n P \right].$$

Then, to show that  $L_n B(\mathbb{G}, \mathbb{G}, X) \rightarrow B_{n+1}(\mathbb{G}, \mathbb{G}, P) = \mathbb{G}^{n+2} P$  is a cofibration of io-properads, it suffices to show that  $K_n P \rightarrow \mathbb{G}^{n+1} P$  is a cofibration of topological io-sequences. But, it is not difficult to see that this map identifies  $K_n P$  with the subspace of  $\mathbb{G}^{n+1} P$  given by  $\bigcup_{0 \leq i \leq n} s_i(\mathbb{G}^n P)$ , where  $s_i$  are the simplicial degeneracy maps.

Moreover, note that  $\mathbb{G}^{n+1} P$  is a disjoint union of spaces indexed by  $(n+1)$ -nested graphs and  $s_i(\mathbb{G}^n P)$  is the union corresponding to a subset of this indexing set. Since the topological io-sequence underlying  $P$  is cofibrant, it follows that the inclusion  $\bigcup_{0 \leq i \leq n} s_i(\mathbb{G}^n P) \hookrightarrow \mathbb{G}^{n+1} P$  is a cofibration of topological io-sequences.  $\square$

A similar argument using Lemma A.2.2 gives the relative version of Proposition A.3.1. We omit the details:

**Proposition A.3.2.** *Let  $P, Q$  be properads which are cofibrant as topological io-sequences and let  $P \rightarrow Q$  be a map of io-properads such that the underlying map of topological io-sequences is a cofibration. Then,  $|B(\mathbb{G}, \mathbb{G}, P)|^{\mathbb{G}\mathbf{Alg}} \rightarrow |B(\mathbb{G}, \mathbb{G}, Q)|^{\mathbb{G}\mathbf{Alg}}$  is a cofibration of properads.*

$\square$

#### A.4 Comparison of geometric realizations in topological io-sequences and in topological io-properads

The aim of this section is to provide the proof of Proposition 3.2.4. In this section we follow the discussion in [22, Section 7].

Let us start by recalling some notation: Let  $P_\bullet$  be a simplicial topological io-properad. Denote by  $|P_\bullet|^{\mathbb{G}\mathbf{Alg}}$  and  $|P_\bullet|^{\mathbf{TopSeq}^{\text{io}}}$  the geometric realizations of  $P$  in the category of properads and the category of topological io-sequences, respectively. As a consequence of the fact that the geometric realization of simplicial spaces is a monoidal functor, it follows that  $|P_\bullet|^{\mathbf{TopSeq}^{\text{io}}}$  carries a natural properad structure induced from the structure maps of the simplicial properad  $P_\bullet$ .

Proposition 3.2.4 asserts that the io-properads given by geometric realizations of  $P_\bullet$  in the category of io-properads and in the category of topological io-sequences are isomorphic.

*Proof of Proposition 3.2.4.* Recall that

$$P_n \odot \Delta^n = \text{coeq} \left[ \mathbb{G}(\mathbb{G}P_n \times \Delta^n) \rightrightarrows \mathbb{G}(P_n \times \Delta^n) \right]$$

(where the coequalizers taken in io-properads and io-sequences coincide).

The maps  $P_n \times \Delta^n \rightarrow |P_\bullet|^{\mathbf{TopSeq}^{\text{io}}}$  induce properad maps  $P_n \odot \Delta^n \rightarrow |P_\bullet|^{\mathbf{TopSeq}^{\text{io}}}$ . Moreover, these maps are compatible with the simplicial face and degeneracy maps, and hence we get a  $\mathbb{G}$ -algebra map

$$\text{coeq} \left[ \coprod_{m,n}^{\mathbb{G}\mathbf{Alg}} P_m \odot \Delta^n \rightrightarrows \coprod_n^{\mathbb{G}\mathbf{Alg}} P_n \odot \Delta^n \right] = |P_\bullet|^{\mathbb{G}\mathbf{Alg}} \rightarrow |P_\bullet|^{\mathbf{TopSeq}^{\text{io}}}.$$

Now, notice that for properad  $P$ ,

$$|\mathbb{G}P_\bullet|^{\mathbf{TopSeq}^{\text{io}}} \simeq \mathbb{G}|P_\bullet|^{\mathbf{TopSeq}^{\text{io}}}. \quad (\text{A.4.1})$$



This holds since  $| \cdot |^{\mathbf{TopSeq}^{\mathbf{io}}}$ , being a left adjoint ( $| \cdot |^{\mathbf{TopSeq}^{\mathbf{io}}}: \mathbf{TopSeq}^{\mathbf{io}} \rightleftarrows \mathbf{TopSeq}^{\mathbf{io}}: (\_ )^{\Delta^\bullet}$ ), commutes with colimits and as noted above it also commutes with finite products.

Furthermore, for any  $X \in \mathbf{TopSeq}^{\mathbf{io}}$ , we have

$$|\mathbb{G}X_\bullet|^{\mathbb{G}\mathbf{Alg}} \simeq \mathbb{G}|X_\bullet|^{\mathbf{TopSeq}^{\mathbf{io}}}. \quad (\text{A.4.2})$$

To see this, first note that for a topological io-sequence  $X$

$$\mathbb{G}X_n \odot \Delta^n = \text{coeq} \left[ \mathbb{G}(\mathbb{G}(\mathbb{G}X_n) \times \Delta^n) \rightrightarrows \mathbb{G}(\mathbb{G}X_n \times \Delta^n) \right] \simeq \mathbb{G}(X_n \times \Delta^n).$$

This implies that

$$\begin{aligned} \coprod_{m,n}^{\mathbb{G}\mathbf{Alg}} \mathbb{G}X_n \odot \Delta^m &\simeq \mathbb{G} \left( \coprod_{m,n} X_n \times \Delta^m \right), \text{ and} \\ \coprod_n^{\mathbb{G}\mathbf{Alg}} \mathbb{G}X_n \odot \Delta^n &\simeq \mathbb{G} \left( \coprod_n X_n \times \Delta^n \right). \end{aligned}$$

Since  $\mathbb{G}$  commutes with colimits, this gives us

$$\begin{aligned} \text{coeq}^{\mathbb{G}\mathbf{Alg}} \left[ \coprod \mathbb{G}X_n \odot \Delta^m \rightrightarrows \coprod \mathbb{G}X_n \odot \Delta^n \right] &\simeq \\ \mathbb{G} \left( \text{coeq} \left[ \coprod X_n \times \Delta^m \rightrightarrows \coprod X_n \times \Delta^n \right] \right). \end{aligned}$$

Thus, it follows that  $|\mathbb{G}X_\bullet|^{\mathbb{G}\mathbf{Alg}} \simeq \mathbb{G}|X_\bullet|^{\mathbf{TopSeq}^{\mathbf{io}}}$  for any topological io-sequence  $X$ . Finally, for any simplicial properad  $P_\bullet$

$$\mathbb{G}(\mathbb{G}P) \rightrightarrows \mathbb{G}P \rightarrow P \quad (\text{A.4.3})$$

is a coequalizer diagram (in io-properads as well as in topological io-sequences). Since  $| \cdot |^{\mathbb{G}\mathbf{Alg}}$  is a left adjoint ( $| \cdot |^{\mathbb{G}\mathbf{Alg}}: s\mathbb{G}\mathbf{Alg} \rightleftarrows \mathbb{G}\mathbf{Alg}: (\_ )^{\Delta^\bullet}$ ), it commutes with the coequalizer. Thus, using

(A.4.3), we have the following commutative diagram in the category of io-properads:

$$\begin{array}{ccccc}
|\mathbb{G}(\mathbb{G}P)|^{\mathbb{GAlg}} & \rightrightarrows & |\mathbb{G}P|^{\mathbb{GAlg}} & \longrightarrow & |P_{\bullet}|^{\mathbb{GAlg}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{G}\mathbb{G}|P_{\bullet}|^{\text{TopSeq}^{\text{io}}} & \rightrightarrows & \mathbb{G}|P_{\bullet}|^{\text{TopSeq}^{\text{io}}} & \longrightarrow & |P_{\bullet}|^{\text{TopSeq}^{\text{io}}}
\end{array}$$

where both the rows are coequalizer diagrams mentioned above. Further, using (A.4.1) and (A.4.2) it follows that the first two vertical maps are isomorphisms. It thus follows that the rightmost vertical arrow is also an isomorphism, proving the proposition.  $\square$

### A.5 Proof of Proposition 3.2.11

Let us first recall the statement of Proposition 3.2.11:

**Proposition A.5.1.** *Let*

$$\begin{array}{ccccc}
P & \longleftarrow & R & \longrightarrow & Q \\
\downarrow & & \downarrow & & \downarrow \\
P' & \longleftarrow & R' & \longrightarrow & Q'
\end{array} \tag{A.5.1}$$

*be a map of pushout diagrams of io-properads. If*

1. *each vertical arrow is a Hurewicz weak-equivalence, and*
2.  *$R \rightarrow P$ ,  $R' \rightarrow P'$  are Hurewicz cofibrations of topological io-sequences*

*Then*

$$|B_{\bullet}P| \coprod_{|B_{\bullet}R|}^{\mathbb{GAlg}} |B_{\bullet}Q| \rightarrow |B_{\bullet}P'| \coprod_{|B_{\bullet}R'|}^{\mathbb{GAlg}} |B_{\bullet}P'|$$

*is a Hurewicz weak-equivalence.*

Note that here we continue to follow the convention mentioned in Notation 3.2.5, namely  $|\_|$  denotes the common geometric realization in topological io-properads and topological io-sequences.

### A.5.1 Hurewicz model structure on topological io-sequences

Before beginning the proof of Proposition 3.2.11, we start with a short discussion of the Hurewicz model structure on topological io-sequences, alluded to in Remark 3.2.10:

There exists a model structure on the category **Top**, called the *Hurewicz or Strøm model structure*, in which the weak-equivalences, fibrations, and cofibrations are given, respectively, by homotopy equivalences, Hurewicz fibrations, and Hurewicz cofibrations (see [20, Section 17.1]). There is an induced model structure on **TopSeq**<sup>io</sup> with weak-equivalences, fibrations, and cofibrations defined component-wise. We will refer to this as the Hurewicz model structure on **TopSeq**<sup>io</sup>.

With these definitions, the Hurewicz weak equivalences and cofibrations defined in Notation 3.2.9 correspond precisely to the weak-equivalences and cofibrations in the Hurewicz model structure on topological io-sequence.

### A.5.2 Proof of Proposition 3.2.11

Note that,

$$\begin{aligned} |B_{\bullet}P| \coprod_{|B_{\bullet}R|}^{\mathbb{G}\mathbf{Alg}} |B_{\bullet}Q| &\simeq \left| B_{\bullet}P \coprod_{B_{\bullet}R}^{\mathbb{G}\mathbf{Alg}} B_{\bullet}Q \right| \simeq \left| \mathbb{G}^{\bullet+1}P \coprod_{\mathbb{G}^{\bullet+1}R}^{\mathbb{G}\mathbf{Alg}} \mathbb{G}^{\bullet+1}Q \right| \\ &\simeq \left| \mathbb{G}(\mathbb{G}^{\bullet}P \coprod_{\mathbb{G}^{\bullet}R} \mathbb{G}^{\bullet}Q) \right| \end{aligned} \quad (\text{A.5.2})$$

Here the first equality holds since  $| \cdot |$  is a left adjoint in the pair  $| \cdot | : \rightleftarrows \mathbb{G}\mathbf{Alg} : (\_)^{\Delta^{\bullet}}$  and hence commutes with colimits. Similarly the last equality holds since  $\mathbb{G}$  is a left adjoint in the pair  $\mathbb{G} : \mathbf{TopSeq}^{\text{io}} \rightleftarrows \mathbb{G}\mathbf{Alg} : \text{Forget}$  and hence commutes with colimits. Similarly,

$$|B_{\bullet}P'| \coprod_{|B_{\bullet}P'|}^{\mathbb{G}\mathbf{Alg}} |B_{\bullet}Q'| \simeq \left| \mathbb{G}(\mathbb{G}^{\bullet}P' \coprod_{\mathbb{G}^{\bullet}R'} \mathbb{G}^{\bullet}Q') \right|$$

For the rest of the proof, we shall consider the category of topological io-sequences equipped with the Hurewicz model structure (see Section A.5.1 above). On the category of simplicial topological

io-sequences we consider the Reedy model structure induced from the Hurewicz model structure on topological io-sequences (see [39] for details on Reedy model structures).

Consider the simplicial topological io-sequence

$$\mathbb{G}(\mathbb{G}^\bullet P \coprod_{\mathbb{G}^\bullet R} \mathbb{G}^\bullet Q).$$

Its  $(n + 1)$ -simplices are given by a disjoint union of spaces indexed by certain  $(n + 1)$ -nested ioda-graphs. As in the proof of Proposition A.3.1, it can be seen that the latching map

$$L_n \left( \mathbb{G}(\mathbb{G}^\bullet P \coprod_{\mathbb{G}^\bullet R} \mathbb{G}^\bullet Q) \right) \rightarrow \mathbb{G}(\mathbb{G}^{n+1} P \coprod_{\mathbb{G}^{n+1} R} \mathbb{G}^{n+1} Q) \quad (\text{A.5.3})$$

identifies the latching space with the disjoint union of spaces corresponding to a subset of this indexing set. Using the fact that all topological io-sequences are cofibrant in the Hurewicz model structure, it then follows that map (A.5.3) is a Hurewicz cofibration as well. We conclude that

$$\mathbb{G}(\mathbb{G}^\bullet P \coprod_{\mathbb{G}^\bullet R} \mathbb{G}^\bullet Q)$$

is a Reedy cofibrant simplicial topological io-sequence. Similarly, it follows that

$$\mathbb{G}(\mathbb{G}^\bullet P' \coprod_{\mathbb{G}^\bullet R'} \mathbb{G}^\bullet Q')$$

is also Reedy cofibrant. Applying [39, Corollary 10.6], it suffice to show that the map

$$\mathbb{G}(\mathbb{G}^\bullet P \coprod_{\mathbb{G}^\bullet R} \mathbb{G}^\bullet Q) \rightarrow \mathbb{G}(\mathbb{G}^\bullet P' \coprod_{\mathbb{G}^\bullet R'} \mathbb{G}^\bullet Q')$$

is a Reedy weak-equivalence i.e. a level-wise Hurewicz weak-equivalence. It follows from the

definition of  $\mathbb{G}$  that it preserves Hurewicz weak-equivalences and thus it suffices to verify that

$$\mathbb{G}^\bullet P \coprod_{\mathbb{G}^\bullet R} \mathbb{G}^\bullet Q \rightarrow \mathbb{G}^\bullet P' \coprod_{\mathbb{G}^\bullet R'} \mathbb{G}^\bullet Q'$$

is a Hurewicz weak-equivalence.

Applying  $\mathbb{G}^\bullet$  to the diagram (A.5.1), we get a diagram of topological io-sequences

$$\begin{array}{ccccc} \mathbb{G}^\bullet P & \longleftarrow & \mathbb{G}^\bullet R & \longrightarrow & \mathbb{G}^\bullet Q \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{G}^\bullet P' & \longleftarrow & \mathbb{G}^\bullet R' & \longrightarrow & \mathbb{G}^\bullet Q' \end{array} \quad (\text{A.5.4})$$

satisfying the following conditions:

- the vertical maps are Hurewicz weak equivalences, and
- $\mathbb{G}^\bullet R \rightarrow \mathbb{G}^\bullet P$ ,  $\mathbb{G}^\bullet R' \rightarrow \mathbb{G}^\bullet P'$  are Hurewicz cofibrations of topological io-sequences.

The Hurewicz model structure on **TopSeq**<sup>10</sup> is left proper (see for example [20, Proposition 15.4.2]) and hence the pushout of any map along a cofibration coincides with its homotopy pushout . In particular, the pushouts of both the rows in (A.5.4) coincide with their homotopy pushouts. Since the vertical arrows in (A.5.4) are Hurewicz weak-equivalences, it then follows that the two pushouts are Hurewicz weak-equivalent. This completes the proof of the proposition.

□

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