The geometry of polymers and other results in the KPZ universality class

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Abstract

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This thesis investigates the geometry of polymers and other micellaneous results in the Kardar-Parisi-Zhang (KPZ) universality class. Directed polymers have enjoyed a rich history in both probability theory and mathematical physics and have connections to several families of statistical mechanical and random growth models that belong to the KPZ universality class [77]. In this thesis, we focus on 2 integrable polymer models, the (1+1)-dimensional continuum directed random polymer (CDRP) and the half-space log-gamma ($\mathcal{HSLG}$) polymer, and study their path properties. For the CDRP, we show both of its superdiffusivity and localization features. Namely, the annealed law of polymer of length t, upon $t^{2/3}$ superdiffusive scaling, is tight in the space of $C([0,1])$-valued random variables and the quenched law of any point distance pt from the origin on the path a point-to-point polymer (or the endpoint of a point-to-line polymer) concentrates in a $O(1)$ window around a random favorite point $\mathcal{M}_{p,t}$. The former marks the first pathwise tightness result for positive temperature models and the latter result confirms the "favorite region conjecture" for the CDRP. Moreover, we provide an explicit random density for the quenched distribution around the favorite point $\mathcal{M}_{p,t}$. The proofs of both results utilize connections with the KPZ equation and our techniques also allow us to prove properties of the KPZ equation itself, such as ergodicity and limiting Bessel behaviors around the maximum.

For the $\mathcal{HSLG}$ polymers, we combine our localization techniques from the CDRP and the recently developed $\mathcal{HSLG}$ line ensemble results [22, 27] with an innovative combinatorial

argument to obtain its limiting quenched endpoint distribution from the diagonal in the bound phase ($\alpha < 0$). This result proves Kardar's "pinning" conjecture in the case of $\mathcal{HSLG}$ polymers[158].

Finally, this thesis also contains two separate works on the tightness of the Bernoulli Gibbsian line ensemble under mild conditions and the upper-tail large deviation principle (LDP) of the asymmetric simple exclusion process (ASEP) with step initial data. In the first work, we prove that under a mild but uniform control of the one-point marginals of the top curve of the line ensemble, i.e. the shape of the top curve as approximately an inverse parabola and asymptotically covering the entire real line after scaling and recentering, the sequence of line ensembles is tight. With a characterization of [109], our tightness result implies the convergence of the Bernoulli Gibbsian line ensemble to the parabolic Airy line ensemble if the top curve converges to the parabolic Airy_2 process in the finite dimensional sense. Compared to a similar work of [93], our result applies to line ensembles with possibly random initial and terminal data, instead of a packed initial condition, and does not rely on exact formulas. In our work on the ASEP, we obtain the exact Lyapunov exponent for the height function of ASEP with step initial data and subsequently its upper-tail LDP, where the rate function matches with that of the TASEP given in a variational form in [156].

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Dedication

To Mom and Dad

Chapter 1: Introduction

The Kardar-Parisi-Zhang (KPZ) universality class describes a host of important probabilistic and physical models which are believed to display the same *universal* large-time fluctuation behaviors through the common scalings [160, 77, 139, 162, 169, 181, 186, 224]. This thesis focuses on only a few models known to belong in this universality class: the continuum directed random polymer, the half-space log-gamma polymer, the asymmetric simple exclusion process and in connection to these models, the KPZ equation and Gibbsian line ensembles. A theme in this thesis is the typical geometry of some of these models and a motif throughout is the KPZ equation. By way of our discussions, this thesis hopes to illustrate some connections and progress in the KPZ universality class.

The first part of the thesis focuses on directed polymer models in random media, which have a rich history in probability theory (see [129, 100, 65] and the references therein) and form an important group of the KPZ universality class. This type of models was first introduced in the statistical physics literature by [148] to study the domain walls of Ising models with impurities and was later mathematically reformulated as similar to random walks in random environments in [152, 44]. As a unifying framework, directed polymers have proved useful in studying a variety of different mathematical and physical problems [100, 50, 104, 156, 75]. In general, the model involves the following ingredients: a discrete or continuous space, a random environment specified by a discrete or continuous space-time noise, a reference path measure on the space and an inverse temperature parameter. Given the marginal measure of the random environment and the inverse temperature, the path measure becomes reweighted by its journey through the environment. This reweighted distrbution is called *polymer path measure* and a random path sampled through this distribution is refered to as a *polymer*.

Depending on the (inverse) temparature, the polymer path measure exhibits a *phase transi-*

tion phenomenon. In the high temperature regime, the polymer paths behave *diffusively* and fall into *weak disorder*. On the other hand, in the low temperature regime, diffusivity is no longer guaranteed. Instead, the polymers fall into *strong disorder* and two phenomenons are conjectured:

1. *Superdiffusivity:* the polymer measure in strong disorder belongs to the KPZ universality class and a path of length n has typical fluctuations of $O(n^{2/3})$ (compared to $O(\sqrt{n})$ diffusive behavior; see [148], [149], [160], [169], [179],[202], [155],[56], [188] for physics predictions and a few konwn examples).

2. *Localization:* upon fixing the environment, the quenched polymer measure coalesces around environments with large potential values and forms favorite corridors of width much smaller than $n^{2/3}$.

While the above conjectures have been demonstrated in a few disjoint cases, they are far from resolved. In this thesis, we first investigate the *continuum directed random polymer* (CDRP), which is a universal scaling of discrete directed polymers in the intermediate disorder regime and settle the above two conjectures for the CDRP. In a series of joint works with Sayan Das [89, 90], we have shown that the CDRP paths are superdiffusive and the quenched density of the path measure localizes within a region of stochastically bounded width around a random favorite point. Both of these results are derived from a connection between the CDRP and the KPZ equation. To show superdiffusivity, we prove a short-time local fluctuation result for the KPZ equation. To prove localization, we derive the Bessel convergence of two independent KPZ equations around their joint maximizer. Together with a previous one-point convergence result of the CDRP to the Tracy-Widom GUE distribution in [5], our works completely characterize the typical geometry of the CDRP. We refer the reader to Section 1.1.1 for more details.

From a technique perspective, our localization method in the proof of the above results does not rely on integrability other than the Gibbs property, which in the case of the CDRP is accessed through the KPZ line ensembles. Thus, our method can extend to other integrable models with similar setups. Indeed this proves to be true in our attempt to study the geometry of the *half-*

space log-gamma polymers ($\mathcal{HSLG}$ polymers) in the bound phase. The half-space polymers are a variant of the full-space polymers where the paths are restricted to stay on or below the diagonal. The weights on the diagonal differ from those in the bulk. Thus the structure of these polymer resembles the behavior of an interface in the presence of an attractive wall and is expected to exhibit a "depinning" phase transition contingent upon the diagonal weights[158]. In the bound phase, or when the diagonal weights are large, the endpoint of the polymer is expected to be pinned to the wall. This conjecture is consistent with the localization conjecture for full-space polymers.

When we set the weights to be log-gamma random variables, the polymer is endowed with an integrable strucutre. In a recent work by [22], a construction of the $\mathcal{HSLG}$ line ensemble has become available. With our localization technique, in a joint work with Sayan Das [91], we showed the endpoint of the $\mathcal{HSLG}$ polymer localizes around a window of width $O(1)$ around the diagonal in the bound phase and cofirmed the pinning conjecture of this model. Moreover, when combined with the diagonal free energy fluctuation result from [151], our resuls yields a similar Gaussian fluctuation around the diagonal. These results are explained in more detail in Section 1.1.5. Our localization techniques depend on the Gibbsian line ensembles as a principal tool and this line ensemble structure also appears in a large number of models beyond the Brownian bridge, the KPZ equation and the log-gamma polymers. In a joint work with a group of authors [108], we constructed a Bernoulli Gibbsian line ensemble, which is the law of n independent Bernoulli random walkers conditioned to not intersect. Under mild but uniform control of the one-point marginal of the top curve, i.e. the shape of the top curve as approximately an inverse parabola and assymptotically covering the entire real line after scaling and recentering, we proved the tightness of the line ensemble. Furthermore, with finite-dimensional convergence of the top curve to the parabolic Airy_2 process, the entire line ensemble converges to the parobolic Airy line ensemble. More discussions of this result are available in Section 1.2.

Finally, in the last part of the thesis, we discuss a separate result on the upper-tail large deviation principle (LDP) of the *asymmetric simple exclusion process* (ASEP) with step initial data. The ASEP is a pre-limit of the KPZ equation and a type of interacting particle system. The observable of interest in ASEP is the height function $H(0,t)$, which is the number of particles to the right of zero at time t. Its strong law of large number is well-known: $t^{-1}H(0,\frac{t}{q-p}) \to \frac{1}{4}$, where q and $p := 1-q$ are the right and left jump rates for the particles under an exponential clock. In addition, Tracy and Widom showed the convergence of the height function to the Tracy-Widom GUE fluctuation upon appropriate centering and scaling in a series of works [230, 229, 228]. A natural question that follows these fluctuation results is to inquire into its LDP, i.e. probability of when the event $-H_0(\frac{t}{q-p})+\frac{t}{4}$ has deviations of order t. The upper tail, $\mathbf{P}\left(-H_0(\frac{t}{q-p})+\frac{t}{4} > \frac{t}{4}y\right)$, corresponds to the ASEP being "too slow" and it is expected to be different from the lower tail $\mathbf{P}\left(-H_0(\frac{t}{q-p})+\frac{t}{4} < -\frac{t}{4}y\right)$ due to the nature of the speed process. Prior to our work, [85] obtained a one-sided bound for the upper-tail LDP of the ASEP via contour analysis. Through a different approach by Lyapunov exponents pioneered by [88, 128, 182], our result was able to produce the exact upper tail. Section 1.3 contains a summary of our technique and contributions.

1.1 Directed polymers in random environments

In the (1+1)-dimensional discrete case, directed polymers are modeled by up-right paths on the $\mathbb{Z}^2$ integer lattices and the environment is specified by a collection of zero-mean, i.i.d. random variables $\{\omega = \omega(i,j) | (i,j \in \mathbb{Z}_+ \times \mathbb{Z})\}$. Given the environment, the *energy* or *hamiltonian* of a n-step nearest neighbour random walk $(S_i)_{i=0}^n$ starting at the origin is given by $H_n^\omega(S) := \sum_{i=1}^n w(i,S_i)$. The **point-to-line** polymer on the set of all such paths is defined as

$$\mathbf{P}_{n,\beta}^\omega(S) = \frac{1}{Z_{n,\beta}^\omega} e^{\beta H_n^\omega(S)} \mathbf{P}(S),$$

where $\mathbf{P}(S)$ is the simple random walk measure, β is the inverse temperature and $Z_{n,\beta}^\omega$ is the partition function. For **point-to-point** polymers, $\mathbf{P}(S)$ is the random walk bridge measure.

A competition exists between the *entropy* of the paths and the *energy* of the random environment in the polymer path measure. Under this competition, two distinct regimes emerge depending on the inverse temperature β. When $\beta = 0$, the polymer measure is a simple random walk. Hence it's entropy-dominated and exhibits diffusive behaviors. We refer to this scenario as *weak disorder*. For $\beta > 0$, the polymer measure concentrates on paths with high energies and the diffusive behaviors cease to be guaranteed. This type of energy domination is known as *strong disorder* (see [64, 69, 171] and the references therein for more precise definitions).

In the strong disorder regime, instead of behaving diffusively, the polymer measure is believed to fall into the KPZ universality class (see [148, 149, 77, 169]) and their fluctuations are conjectured to be characterized by the fluctuation exponent $\chi = \frac{1}{3}$ and transversal exponent $\zeta = \frac{2}{3}$ [220, 4]:

$$\text{fluctuation of the endpoint of the path: } |S_n| \sim n^{\zeta},$$
$$\text{fluctuation of the log partition function: } [\log Z^{\omega}_{n,\beta} - \rho(\beta)n] \sim n^{\chi}.$$

This instance of the transversal exponent appearing larger than the diffusive scaling exponent $\frac{1}{2}$ is called *superdiffusivity*. Crucially, the conjectured values for χ and ζ satisfy the "KPZ relation":

$$\chi = 2\zeta - 1.$$

At the moment, rigorous results on either exponent or the KPZ relation have been scarce. For directed polymers, $\zeta = 2/3$ has only been obtained for log-gamma polymers in [220, 23]. Upper and lower bounds on ζ have been established in [202, 188] under additional weight assumptions. For zero-temperature models, $\zeta = \frac{2}{3}$ has been established in [155, 49, 140, 94, 28].

Another conjecture for polymers in strong disorder is the *localization* phenomenon (see [69, 33, 89] for partial surveys), which expects the quenched denseity of the midpoint or any specific point in the bulk of the path of length n to concentrate in a relavtively small region around a random point. The size of this region or corridor is believed to be much smaller than $n^{2/3}$ while the random point itself is of the order $n^{2/3}$. In particular, the *favorite region conjecture* speculates that the midpoint (or any other point) of the polymer is asymptotically localized in a region of stochastically bounded diameter (see [66, 33, 32, 17, 89] for related results). We emphasize that this localization phenonemon is not explained by the theory of KPZ universality class, which addresses the location of the corridor but not its width.

While much progress has taken place in advancing our understanding of these two conjectures, they are far from resolved. In a series of joint works with Sayan Das ([89, 90]), we settled the questions of superdiffusivity and localization for the continuous directed random polymers (CDRP). More specifically, we showed pathwise tightness of the continuous directed random polymers under superdiffusive scaling and pointwise localization of the CDRP in the sense of the favorite region conjecture. The pathwise result has never been previously proven for any discrete polymer model and the localization result has only been proven for the midpoint of stationary log-gamma polymer [67]. We will now introduce our first polymer model, the CDRP, in the next section and our results on localization and superdiffusivity are covered in Chapters 2 and 3 respectively.

1.1.1 The continuum directed random polymer

This section serves as a summary for Chapters 2 and 3. The *continuum directed random polymer* (CDRP) is a probability measure on the space of continuous functions $C([0,t]), t > 0$. Heuristically speaking, the random environment ξ is the space-time white noise, i.e. a random function with independent values at distinct space-time. One can realize ξ as a Gaussian process on $\mathbb{R}_+ \times \mathbb{R}$ with covariance structure

$$\mathbf{E}[\xi(s,y)\xi(t,x)] = \delta_{t=s}\delta_{x=y}.$$

The base measure on the paths is given by the law of Brownian bridges with endpoints $(0,0)$ and (t,x). The seminal works of [3, 4] showed that the CDRP can be obtained as a universal scaling limit of discrete directed polymers in the *intermediate disordered regime* and that the polymer path measure is *singular* with regard to the Brownian bridge. Instead, the CDRP is deteremined by its finite dimensional distributions given in terms of the *partition function*.

Definition 1.1.1 (Point-to-point CDRP). Conditioned on the white noise ξ, let $\mathbf{P}^{\xi}$ be a measure on $C([s,t])$ whose finite-dimensional distribution is given by

$$\mathbf{P}^{\xi}(X(t_1) \in dx_1, \ldots, X(t_k) \in dx_k) = \frac{1}{\mathcal{Z}(x,s;y,t)} \prod_{j=0}^{k} \mathcal{Z}(x_j, t_j, ; x_{j+1}, t_{j+1}) dx_1 \cdots dx_k. \quad (1.1.1)$$

for $s = t_0 \le t_1 < \cdots < t_k \le t_{k+1} = t$, with $x_0 = x$ and $x_{k+1} = y$.

Here for each $(x,s) \in \mathbb{R}\times\mathbb{R}_+$, the function $(y,t) \mapsto \mathcal{Z}(x,s;y,t)$ is the solution of the sotchastic heat equation (SHE) starting from location x at time s, i.e. the unique solution of

$$\partial_t \mathcal{Z} = \frac{1}{2}\partial_x^2 \mathcal{Z} + \mathcal{Z}\xi$$

with Dirac delta intial data $\lim_{t\downarrow s} \mathcal{Z}(x,s;y,t) = \delta(x-y)$ and the space-time white noise ξ. We denote $X \sim \mathsf{CDRP}_t$ if $X(\cdot)$ is a random continuous function on $[0,t]$ with $X(0) = X(t) = 0$ and its finite dimensional distributions satisty (1.1.1) condtioned on ξ. We will comment more on the connections between polymers, SHE and the KPZ equation in the next section but now we state our key results for point-to-point CDRP (the results for point-to-line CDRP are analogous.)

Theorem 1.1.2. *([89, 90]) For each $t > 0$, consider $X \sim \mathsf{CDRP}_t$.*

- *(Pathwise tightness) The annealed law of $(t^{-2/3}X(pt))_{p\in[0,1]}$ when viewed as a random variable in the space of $C([0,1])$ is tight as $t \to \infty$. As a process in p, $t^{-2/3}X(pt)$ converges weakly to a non-trivial distribution as $t \to \infty$.*

- *(Pointwise localization) For each $t > 0$ and $p \in [0,1]$, there exists a random variable $\mathcal{M}_{p,t}$ dependent only on the environment, such that $|X(pt) - \mathcal{M}_{p,t}| = O(1)$ as $t \to \infty$. Furthermore, the quenched density of $X(pt)$ when centered around $\mathcal{M}_{p,t}$ converges in distribution to an explicit random density proportional to $e^{-\sqrt{2}\mathcal{R}(x)}dx$ where $\mathcal{R}(x)$ is a standard two-sided Bessel process with diffusion coefficient 2.*

A similar localization result for the midpoint of point-to-point stationary log-gamma polymer was established in [67] by virtue of the Burke property of the model from [220]. This property allows one to express the quenched density of the midpoint as an exponent of a simple symmetric random walk (SSRW) and one can obtain the localization result through analyzing the behavior of SSRW around its maximizer. However, this technique in [67] is only restriced to the midpoint of the stationary log-gamma polymer alone.

1.1.2 Connections to the KPZ equation and the KPZ line ensemble

The principal tool we use to access the result in (1.1.2) is the Gibbs resampling property [74] enjoyed by the Kardar-Parisi-Zhang (KPZ) equation equation $\mathcal{H}(t,x) := \log \mathcal{Z}(0,0;x,t)$. It is a central object of the KPZ universality class as well as a motif throughout this thesis. The KPZ equation is a stochastic PDE first introduced in [160] and formally, it can be written as a PDE on the domain $\mathbb{R} \times \mathbb{R}_+$:

$$\partial_t \mathcal{H}(x,t) = \frac{1}{2}\partial_x^2 \mathcal{H}(x,t) + (\partial_x \mathcal{H}(x,t))^2 + \xi(t,x), \tag{1.1.2}$$

where ξ is a space-time white noise defined in Section 1.1.1.

In the mathematical physics literature, the KPZ equation arises universally as a scaling limit of a vast collection of models, including one-dimensional interface growth processes, interacting particle systems and random polymers in random environment [77]. In light of the physical relevance of the KPZ equation, we can interpret $\mathcal{H}$ as a time-evolving height profile. The equation formally implies that, starting from a given initial data $\mathcal{H}(x,0)$, the time evolution of the height profile is governed by three features: a smoothing mechanism associated with the heat operator, a slope-dependent growth governed by the nonlinear term and a random forcing given by the noise term.

Based on these features, it is predicted that the height profile will fluctuate around its mean like $t^{1/3}$ and the correlation of these fluctuations will be non-trivial on a spatial scale of $t^{2/3}$. However, inspite of the above interpretation, the KPZ equation in (1.1.2) is classically ill-posed due the presence of the nonlinear term (the function $\mathcal{H}$ is believed to be locally Brownian so it doesn't make sense to square its derivative). Instead, we consider the *Hopf-Cole solution* to the KPZ equation as $\mathcal{H}(x,t) := \log \mathcal{Z}(0,0;x,t)$ where $\mathcal{Z}$ solves the multiplicative-noise stochastic heat equation (SHE):

$$\partial_t \mathcal{Z}(x,t) = \partial_x^2 \mathcal{Z}(x,t) + \mathcal{Z}(x,t)\xi(t,x).$$

While scaling remain governed by the characteristic $1/3, 2/3$ exponents, the height function fluctuations detect differences in the inital data or geometry. The KPZ equation/SHE has six different types of initial data and the one that we will work with throughout this paper is the *narrow wedge* initial data, which corresponds to the SHE started from the Dirac delta function, i.e. $\mathcal{Z}(x,0) = \delta_{x=0}$. The narrow wedge initial condition endows the KPZ equation with an integrable or exactly solvable structure. An important result confirms that the KPZ equation with the narrow wedge initial data belongs to the KPZ universality class as $t \to \infty$ through computing the exact formulas for its one-point marginal distributions [5]:

$$t^{-1/3}(\mathcal{H}(0,t) + \frac{t}{24}) \xrightarrow{d} F_{GUE}.$$

Here F_{GUE} is the Tracy-Widom GUE distribution, first discovered to describe the largest eigenvalue of a random GUE matrix in [231], and later proved to be the limiting fluctuation statistics in the KPZ universality class in the works of [12, 155]. In addition, in this one-point convergence, we observe the usual KPZ $1/3$ scaling exponent.

Finally on the connections to polymers, via a version of the Feynman-Kac formula, we can interpret the solution to the SHE as the partition function of the CDRP (see [4]), which has appeared in (1.1.1) in the quenched density of $X(pt)$ in Section 1.1.1. Thus the Hopf-Cole solution to the KPZ equation corresponds to the free energy of the CDRP. Here, the initial data for the SHE corresponds to an initial potential which affects the starting position of the polymer and the narrow wedge initial data is equivalent to fixing the departure position at 0. It is easy to see that the joint maximizer $\mathcal{M}_{p,t}$ in Theorem1.1.2 should correspond to the random mode of the quenched density of $X(pt)$, which is the joint maximizer of two independent copies of the KPZ equations

$\operatorname{argmax}_{x\in\mathbb{R}} \mathcal{H}_1(x, pt)+\mathcal{H}_2(x, (1-p)t)$. The quenched density recentered around this random mode is proportional to

$$x \mapsto \mathcal{H}_1(\mathcal{M}_{p,t}, pt) - \mathcal{H}_1(\mathcal{M}_{p,t} + x, pt) + \mathcal{H}_2(\mathcal{M}_{p,t}, pt) - \mathcal{H}_2(\mathcal{M}_{p,t} + x, pt). \tag{1.1.3}$$

1.1.3 Gibbs property and the KPZ line ensemble

For each fixed $t > 0$, the process $\mathcal{H}(\cdot, t)$ can be viewed as the top curve of the *KPZ line ensemble* [74]. Conditioned on all the information outside, the law of the first k curves restricted to a fixed interval $[a, b]$ is absolutely continuous with regard to k Brownian bridges on $[a, b]$ with the same endpoints modulo an explicit Radon-Nikodym derivative. Note that here the interval is fixed. As we are working with a random interval around the random maximizer $\mathcal{M}_{p,t}$, traditional tools such that the usual Gibbs property on an interval or stochastic monotonicity cannot be applied to (1.1.3).

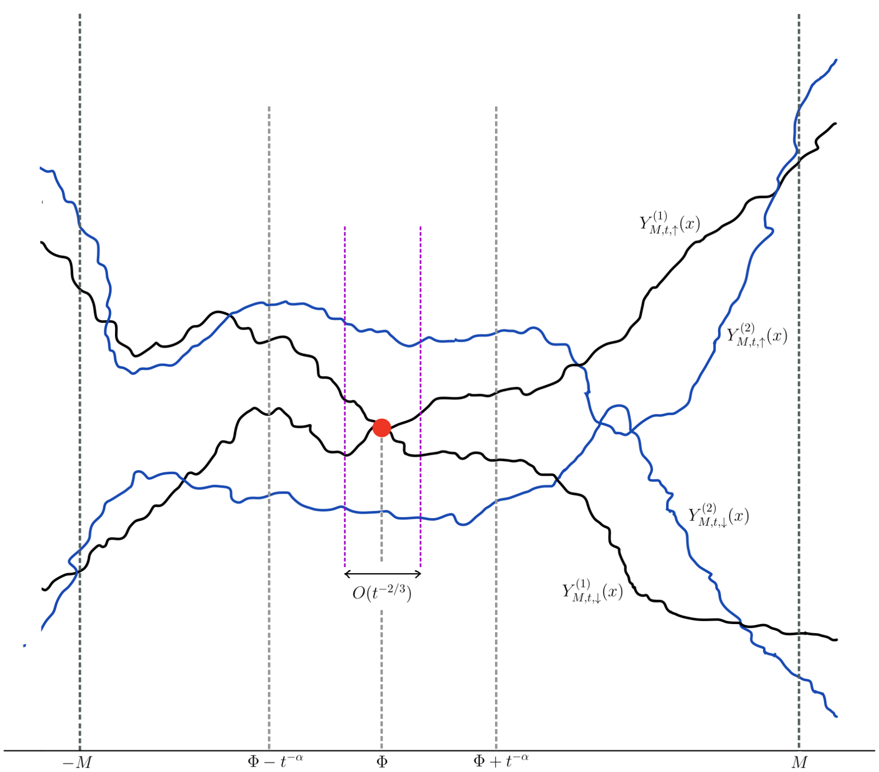

Figure 1.1: The Gibbs property around the random joint maximizer of the top curves (black) of two independent KPZ line ensembles. The blue curves are the associated second curves.

The solution we provide for this issue is a different version of the Gibbs resampling property with two copies of the KPZ equation around the joint maximizer (Fig. 1.1). It states that conditioned on the data at the random maximizer (location and the value of each KPZ equation at this location) as well as boundary data, the joint law around the joint maximizer of the 2 KPZ equations is the same as that of two independent pairs of nonintersecting Brownian bridges with appropriate endpoints. More specifically, we fix $K > 0$ and define a local joint maximizer $\mathcal{M}_{p,t}^{*} := \operatorname{argmax}_{|x|\le Kt^{2/3}}(\mathcal{H}_1(\cdot, pt) + \mathcal{H}_2(\cdot, (1-p)t))$. One can ensure that the local joint maximizer coincides with the global joint maximizer with high probability by choosing K large enough. Hence it suffices to work with the local maximizer. To embed this problem in a line ensemble framework, we consider two independent copies of KPZ line ensembles for $\mathcal{H}_1$ and $\mathcal{H}_2$ and first study the behavior of 2 independent Brownian motions around its joint maximizer. Then utilizing the existing Radon-Nikodym derivative of the KPZ line ensemble gives us an explicit way to resample the top curves of both line ensembles simultaneously over intervals of the form $[\mathcal{M}_{p,t}^{*} - a_t, \mathcal{M}_{p,t}^{*} + b_t] \subset [-Kt^{2/3}, Kt^{2/3}]$. This method could be adapted to solve similar problems for other integrable models with a line ensemble setup.

1.1.4 Pathwise tightness for the CDRP

Note that the localization result in [89] explained above automatically implies pointwise tightness of $t^{-2/3}X(pt)$ for each $p \in [0, 1]$. To establish the pathwise tightness result in Theorem 1.1.2, we need a uniform control of the path fluctuations on the local level. With the long-time [5, 79, 80, 81] and short-time [87] tail estimates from existing literature, we are able to produce a quantitative modulus of continuity estimate for the CDRP, which leads to the final pathwise tightness result. This is the first pathwise tightness result for polymer models. The process-level convergence of $t^{-2/3}X(pt)$ to the geodesics of the directed landscape in [90] follows from a recent work of [241] that established the *convergence from the KPZ sheet to the Airy sheet*.

1.1.5 Localization of the half-space log-gamma polymer

Lastly in the vein of directed polymers, we build on the localization result in [89] to investigate the geometry of the half-space log-gamma polymer in the bound phase in an upcoming work with Sayan Das [91]. We now present a motivation and summary of this work, which is detailed in Chapter 4.

Half-space polymers are a variant of the full-space directed polymers where the paths are restricted to be on or below the diagonal. In terms of the environment, the weights placed on the diagonal differ from those in the bulk. From the physics perspective, the half-space polymer imitate the behavior of an interface in the presence of an attractive wall and this model has been linked to the studies of "wetting" phenomenon [1, 198, 51]. Depending on the strength of the diagonal weights, a phase transition known as "depinning" appears [158] where the model alternates between behaving like a full-space polymer or diffusively with Guassian fluctuations for the free energy and a coalescence of the endpoint to the diagonal.

For the *half-space log-gamma* ($\mathcal{HSLG}$) polymer, which is an integrable model, the diagonal weights are $\text{Gamma}^{-1}(\alpha+\theta)$ random variables and the bulk weights distribute as $\text{Gamma}^{-1}(2\theta)$. The "depinning" phase transition occurs at $\alpha = 0$ (see [158, 204, 33]). When $\alpha > 0$, [27, 22] showed that the polymer measure is unpinned and the endpoint lies in a $O(N^{2/3})$ window. For $\alpha \in (-\theta, 0)$, [27, 151] proved the free energy conjecture for the diagonal. However, the bounded endpoint conjecture for the $\mathcal{HSLG}$ polymer remains open and that's the goal of our project.

More specifically, let Π_N^{half} be the set of all upright lattice paths of length $2N-2$ starting from $(1,1)$ that are confined to the half-space $\mathcal{I}^-$ (see Figure 4.2). Given the weights in (4.1.1), the half-space log-gamma ($\mathcal{HSLG}$) polymer is a random measure on Π_N^{half} defined as

$$\mathbf{P}^W(\pi) = \frac{1}{Z(N)} \prod_{(i,j)\in\pi} W_{i,j} \cdot \mathbf{1}_{\pi\in\Pi_N^{\text{half}}}, \tag{1.1.4}$$

where $Z(N)$ is the normalizing constant. Our result states that

Theorem 1.1.3 (Bounded endpoint, [91])**.** *Fix $\theta > 0$ and $\alpha \in (-\theta, 0)$ and consider the random*

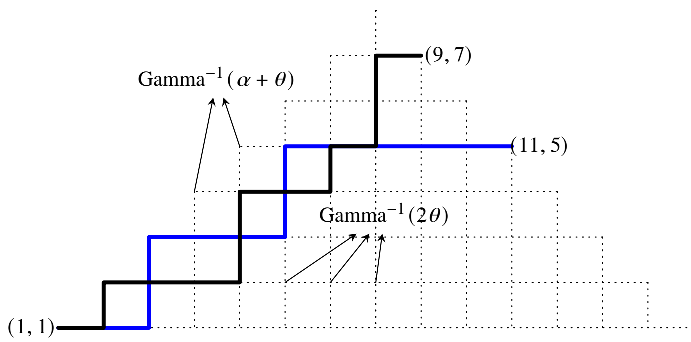

Figure 1.2: Two possible paths of length 14 in Π_8^{half} are shown in the figure.

measure $\mathbf{P}^W$ *from* (4.1.2). *For a path* $\pi \in \Pi_N^{\text{half}}$, *we denote* $\pi(2N-2)$ *as the height (i.e., y-coordinate) of the endpoint of the polymer. We have*

$$\limsup_{k\to\infty}\limsup_{N\to\infty}\mathbf{P}^W(\pi(2N-2)\le N-k)=0, \quad \textit{in probability}. \tag{1.1.5}$$

This is the first result to capture the "pinning" phenomenon of the half-space log-gamma polymer measure to the diagonal. Moreover, in [91], we deduce the following quenched distribution of the endpoint when viewed from around the diagonal.

Theorem 1.1.4 ([91])**.** *Fix* $\theta > 0$ *and* $\alpha \in (-\theta, 0)$ *and consider the random measure* $\mathbf{P}^W$ *from* (1.1.4). *Let* $(S_k)_{k\ge 0}$ *be a log-gamma random walk such that* $S_k := \sum_{i=1}^k X_i$ *and* $X_i := \log\Gamma(\theta+\alpha)/\Gamma(\theta-\alpha)$ *are i.i.d. random variables. Set* $Q := \sum_{p\ge 0} e^{-S_p}$. *For a path* $\pi \in \Pi_N^{\text{half}}$, *we denote* $\pi(2N-2)$ *as the height (i.e., y-coordinate) of the endpoint of the polymer. Then for each* $k \ge 1$, *as* $N \to \infty$, *we have the following multi-point convergence in distribution*

$$\left(\mathbf{P}^W(\pi(2N-2)=N-r)\right)_{r\in[\![0,k]\!]} \xrightarrow{d} \left(Q^{-1}\cdot e^{-S_r}\right)_{r\in[\![0,k]\!]}. \tag{1.1.6}$$

A stationary version of the half-space log-gamma polymer with $\alpha \in (-\theta, \theta)$ has been considered in the physics work of [24]. As a remark, we note that our above two theorems continue to hold for this stationary model.

Finally, our result also has implications on the one-point free-energy fluctuations near the diagonal. Given that the free-energy on the diagonal $\log Z(N,N)$ after appropriate recentering and scaling converges to $\mathcal{N}(0,1)$, we are able to derive that for $a_{N,i}/\sqrt{N} \to 0$, $\log Z(N+a_{N,i}, N-a_{N,i})$ also has one-point Gaussian convergences with the same recentering and scaling using a strong coupling result behind the proof of our localization theorem.

Similar to the our localization work on the CDRP in [89], our proof for the $\mathcal{HSLG}$ polymers relies on the Gibbsian line ensemble inputs for the $\mathcal{HSLG}$ polymers recently developed in [22] as well as point-to-(partial)line half-space log-partition functions in [27]. In addition to the localization techniques from [89], our argument crucially depends on an innovative combinatorial argument on the necessary separation between the top two curves in the $\mathcal{HSLG}$ line ensemble that bridges the micellaneous inputs and enables our proof.

1.2 The Bernoulli Gibbsian line ensemble

The Gibbsian line ensemble structure also appears in other continuous or discrete settings, one of which constructed is the *Bernoulli Gibbsian line ensemble* in [108], i.e., a collection of avoiding Bernoulli random walkers with a *Schur Gibbs property*. As a reminder, a Gibbs property refers to a type of resampling invaraince. One of the most well-known types of Gibbs property is the *Brownian Gibbs property*, i.e., the line ensemble is non-intersecting almost surely and the conditional law of consecutive line ensemble curves given the boundary data is that of non-intersecting Brownian bridges. The Schur Gibbs property is the discrete analog of the Brownian Gibbs property, which specifies the local conditional distribution to be that of Bernoulli random walk bridges. The Brownian Gibbsian line ensembles arise naturally in various models in statistical mechanics, integrable probability and mathematical physics. Examples of this type include the Dyson Brownian motion (which is the line of N independent one-dimensional Brownian motions all started at the origin and conditioned to never cross), Brownian last passage percolation [142, 143, 144, 141] and the parabolic Airy line ensemble [73, 204]. In particular, the Airy line ensemble has helped establish a lot of convergence results including the ones in this thesis and played a foundational role in the construction of the Airy sheet in [94].

The top curve of the parobolic Airy line ensemble is the parabolic Airy$_2$ process, which is a universal object in the KPZ unversality class, and the Airy line ensemble itself is believed to be a universal scaling limits of a variety of Gibbsian line ensembles. However, besides the case of a few examples with special integrable structures and initial conditions [93], the questions of convergence to the Airy line ensemble for general Gibbsian line ensembles with general boundary data had been wide open. The work in Chapter 5 sets out to investigate questions in this direction. Built upon a recent characterization result for the convegrence of Brownian Gibbsian line ensembles by [109], our work showed the convergence of the Bernoulli Gibbsian line ensemble to the parabolic Airy line ensemble under a mild but uniform control on the top curve.

Theorem 1.2.1. *[108] Consider a sequence of Bernoulli Gibbsian line ensembles L^N with N curves. Fix $\alpha > 0, p \in (0,1), \lambda > 0$. Let $\psi : \mathbb{N} \to (0,\infty)$ be a function such that $\lim_{N\to\infty} \psi(N) = \infty$, $a_N < -\psi(N)N^\alpha$ and $b_N > \psi(N)N^\alpha$. Suppose that there exists $\phi : (0,\infty) \to (0,\infty)$ and*

$$\sup_{n\in\mathbb{Z}} \limsup_{N\to\infty} \mathbf{P}\left(\left|N^{-\alpha/2}(L_1^N(nN^\alpha) - pnN^\alpha + \lambda n^2 N^{\alpha/2})\right| \geq \phi(\varepsilon)\right) \varepsilon$$

for any $\varepsilon > 0$. Then any subsequential limit L^∞ of L^N satisfies the Brownian Gibbs property. Moreover, if the top curve L_1^N conveges in finite dimensional distribution to the parabolic Airy$_2$ process, then L^N converges to the parobolic Airy line ensemble.

Unlike [93] which relied on exact formulas for the finite dimensional distributions for the random walkers for each fixed N and packed initial data to prove convergence for several discrete non-inersecting random walks, our result has much more relaxed boundary conditions and no dependence on exact formulas. Instead, we used a strong coupling result to compare the Bernoulli random walk bridge with Brownian bridges. The main technical result of this work is presented in Section 5.1.3 and we refer the reader to Chpater 5 for details.

1.3 The upper-tail large deviation principle of the ASEP

This section serves as a summary for Chapter 6.

The *asymmetric simple exclusion process* (ASEP) is a continuous-time Markov chain on particle configurations $\mathbf{x} = (x_1 > x_2 > \cdots)$ in $\mathbb{Z}$ that first appeared in [184, 222]. The process can be described as follows. Each site $i \in \mathbb{Z}$ can be occupied by at most one particle, which has an independent exponential clock with exponential waiting time of mean 1. When the clock rings, the particle jumps to the right with probability q or to the left with probability $p = 1 - q$. However, the jump is only permissible when the target site is unoccupied. For our purposes, it suffices to consider configurations with a rightmost particle. At any time $t \in \mathbb{R}_{>0}$, the process has the configuration $\mathbf{x}(t) = (x_1(t) > x_2(t) > \cdots)$ in $\mathbb{Z}$, where $x_j(t)$ denotes the location of the j-th rightmost particle at this time. When $q = 1$, the ASEP becomes the *total asymmetric simple exclusion process*. In this work, we focus on the ASEP with *step initial data*, where all the particles are lined up at zero or the left of zero.

The ASEP is one of the pre-limiting processes of the KPZ equation and a model in the KPZ unviersality class. A number of results are available about its statistics of interest, the height function $H_0(t)$, i.e. the number of particles to the right of zero at time t, including its law of large number and its central limit theorem [230, 229, 228]. The natural question to inquire next is its *large deviation principle* (LDP), which is concerned with the tails of the distribution $-H_0(\frac{t}{q-p})$ centered by its law of large numbers. The lower tail of the ASEP LDP $\mathbf{P}(-H_0(\frac{t}{q-p}) + \frac{t}{4} < \frac{yt}{4})$, which corresponds to the ASEP being "too fast", has been notoriously difficult. In general in the KPZ universality class, we only have the precise lower tail LDP of the KPZ equation with narrow-wedge initial data [232, 52] and the TASAP [155] although a number of results on the lower tail bounds have been developed in both math and physics works [219, 72, 168, 166, 176, 80]. For the upper tail $\mathbf{P}(-H_0(\frac{t}{q-p}) + \frac{t}{4} > \frac{yt}{4})$, prior to our work, the best available result is a one-sided bound from [85] obtained from contour analysis.

Our work on the precise upper-tail large deviation principle of the ASEP takes the route of Lyapunov exponent and intermittency in random media, which has been pioneered by [88, 128, 182] for the KPZ equation and its variants. Intermittency problem has been an active area of research in both mathematics and physics literature for the last few decades. Mathematically, it is characterized by rapid growth of moments of a process. Formally, we say a process Ψ_t is intermittent if $\lim_{t\to\infty} t^{-1}\log\mathbf{E}[\Psi_t^k]/k$ is strictly increasing where the limit $\lim_{t\to\infty} t^{-1}\log\mathbf{E}[\Psi_t^k]$ is known as its Lyapunov exponent. In terms of the KPZ equation, the physics work of [159] suggests that the SHE is intermittent. Thus, as the ASEP is a pre-limit of the KPZ equation, we expect that a similar intermittent phenomonon also exists. The main technical contribution of our work lies in computing the Lypunov exponent for $(p/q)^{H_0(t)}$ using known Fredholm determinant formulas [49]. This is the first instance of obtaining the Lyapunov exponent for the ASEP. After this step, a standard Legendre-Fenchel transform technique yields the precise upper-tail LDP for the ASEP in the fashion of [88, 128, 182]. Our result states

Theorem 1.3.1. *For $s \in (0,\infty)$ we have*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbf{E}[\tau^{sH_0(t)}] = -h_q(s) =: -(q-p)\frac{1-\tau^{\frac{s}{2}}}{1+\tau^{\frac{s}{2}}}. \tag{1.3.1}$$

For any $y \in (0,1)$ we have

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\left(-H_0(\tfrac{t}{\gamma}) + \tfrac{t}{4} > \tfrac{t}{4}y\right) = -[\sqrt{y} - (1-y)\tanh^{-1}(\sqrt{y})] =: -\Phi_+(y), \tag{1.3.2}$$

where $\gamma = 2q-1$. Furthermore, we have the following asymptotics near zero:

$$\lim_{y\to 0^+} y^{-3/2}\Phi_+(y) = \tfrac{2}{3}. \tag{1.3.3}$$

Note that our large deviation result is restricted to $y \in (0,1)$ as $\mathbb{P}(-H_0(\frac{t}{\gamma}) + \frac{t}{4} > \frac{t}{4}y) = 0$ for $y \geq 1$. Furthermore, although (1.3.2) makes sense when $q = 1$, one cannot recover it from (1.3.1), which only makes sense for $\tau = (1-q)/q \in (0,1)$. However, as mentioned before, [155] has already settled the $q = 1$ TASEP case and obtained the upper-tail rate function in a variational form, which matches with our rate function in (6.1.4). Finally, the large deviation bound in [85] coincides with the correct rate function Φ_+ defined in (1.3.2) only for $y \leq y_0 := \frac{1-2\sqrt{q(1-q)}}{1+2\sqrt{q(1-q)}}$. For any $y \in (y_0, 1)$, their result is suboptimal. Note that in the contour analysis of [85], one had to deform the contour to pass through critical points. Thus this threshold y_0 appeared due to the limitations of their choice of contour (see Figure 1.3). On the other hand, while our analysis

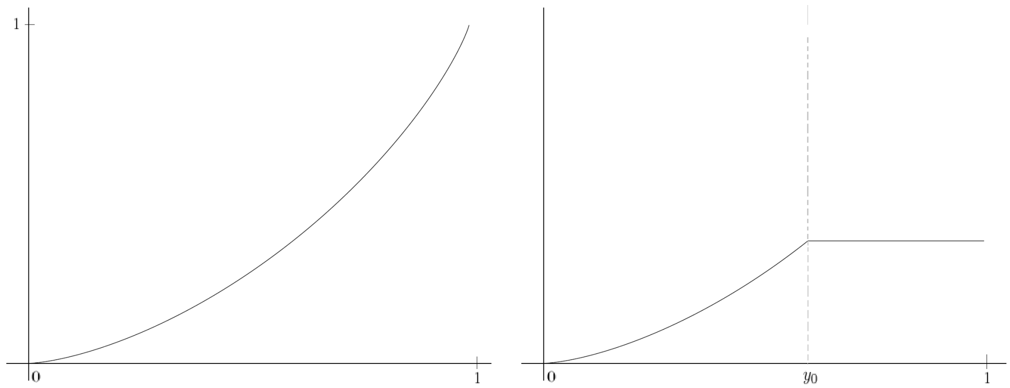

Figure 1.3: The figure on the left is the plot of $\Phi_+(y)$. The right one is the plot of $\widetilde{\Phi}_+(y)$.

followed a well-trodden approach, significant technical challenge remained in our steepest descent analysis of the Fredholm determinant as the underlying kernel is asymmetric and periodic and much more intricate than its KPZ counterpart. As we encountered infinitely many critical points from the periodic nature of the kernel, a major step in our proof was to prove that the contribution from only one of the critical points dominates those from the rest.

1.4 Other works

Lastly, in this section, we give the summary of a separate ongoing project not related to the KPZ universality class conducted during my PhD. This work is about the universal fluctuations of mean-field Ising models and is not included in the body of this thesis.

Consider the magnetization density $\sum_i^n v_i \sigma_i$ of an Ising model on an approximately d_n-regular graph G_n on n vertices. We show that in the high-temperature regime ($\beta < 1$), when the average degree $d_n \gg n^{1/3}$, if $\vec{v} = (v_i)_{i=1,\ldots,n}$ is a unit eigenvector of the scaled adjacency matrix A_n of G_n, the fluctuations converge to those of a normal random variable. As a corrollary of our theorem, we can deduce an error bound for the mean-field approximation of the partition funciton. Our techniques mainly use Stein's method of exchangeable pairs and can also extend to the case of multivariate convergence. Beyond the high temperature regime ($\beta < 1$), we also expect the aforementioned fluctuations to be universal throughout the ferromagnetic regime ($\beta > 0$) and the anti-ferromagnetic regime ($\beta < 0$). Both of these goals have been currently pursued in an ongoing collaboration with Sumit Mukhurjee.

The motivation of this project comes from the topic of variational inference (VI), which is a powerful techinique unibiqitous in probabilistic machine learning. It approximates an exact posterior $\mathbf{P}$ through minimizing the Kullback-Leibler (KL) divergence between a variational family $\mathcal{P}$ of distributions of the latent variables and the exact posterior, i.e.

$$\widehat{\mu} = \arg\min_{\mu \in \mathcal{P}} D(\mu || \mathbf{P}) = \arg\min_{\mu \in \mathcal{P}} \int \frac{\mathrm{d}\mu}{\mathrm{d}\mathbf{P}} \log \frac{\mathrm{d}\mu}{\mathrm{d}\mathbf{P}} \mathrm{d}\mu$$

(see [42, 41] and the references therein). One of the most popular variational families is the mean-field family, i.e. $\mathcal{P} = \mathcal{P}_{\text{prod}}(\mathbb{R}^p)$. It assumes independence between the latent parameters and has appeared in models including the Latent Dirichlet Allocation model (LDA), Bayesian mixture models and general linear models [41]. Given its simplicity and popularity, several questions naturally arise regarding the naive mean-field technique. For instance,

1. what's the constraint on the posterior $\mathbf{P}$ such that naive mean-field approximation is appropriate?

2. What is the size of the approximation error $D(\widehat{\mu} || \mathbf{P})$?

3. When the error $D(\widehat{\mu} || \mathbf{P})$ is small, can we understand $\mathbf{P}$ by analyzing $\widehat{\mu}$ instead, i.e. finding the law of large numbers and the central limit theorem under the posterior?

While there's rich literature in response to the first question (see [71], [120], [8], [243], [170]), the others are less explored (see [99]). Thus through this project, we hope to start addressing questions (2) and (3) for the Ising model on d_n-regular graphs, which has strong connections to machine learning.

Chapter 2: Localization of the continuum directed random polymer

2.1 Introduction

The continuum directed random polymer (CDRP) is a continuum version of the discrete directed polymer measures modeled by a path interacting with a space-time white noise that first appeared in [3]. It arises as a scaling limit of the 1+1 dimensional directed polymers in the "intermediate disorder regime" and can be defined through the Kardar-Parisi-Zhang (KPZ) equation with narrow wedge initial data (see Section 3.1.2). A folklore "favorite region" conjecture on directed polymers states that under strong disorder, the midpoint (or any other point) distribution of a point-to-point directed polymer is asymptotically localized in a region of stochastically bounded diameter (see [59],[33], Section 2.1.1).

In light of the "favorite region" conjecture, we initiate such study of the CDRP's long-time localization behaviors in this paper. Our main result, stated in Section 3.1.2, asserts that any point at a fixed proportional location on the point-to-point CDRP relative to it length converges to an explicit density function when centered around its almost surely unique random mode. The limiting density involves a two-sided 3D Bessel process with an appropriate diffusion coefficient (defined in Section 2.5.1).A similar result for the endpoint of point-to-line CDRP is also obtained, confirming the "favorite region" conjecture for the CDRP. In this process, through the connections between the CDRP and the KPZ equation with narrow wedge initial data, we have shown properties such as ergodicity and Bessel behaviors around the maximum for the latter. These and other results are summarized in Section 2.1.2 and explained in fuller detail in Section 3.1.2.

As an effort to understand the broader localization phenomena, our main theorems (Theorems 2.1.4, 2.1.5) confirm the "favorite region" conjecture for the first non-stationary integrable model, i.e. without stationary boundary conditions or the Burke property. The first rigorous localization type result for directed polymers in random environment appeared in [56], which proved the existence of “favorite sites" in the Gaussian environment, known as the *strong localization*. This notion of localization is weaker than the “favorite region" conjecture and the result of [56] has been later extended to general environments in [68]. As the techniques for both of these results relied on martingales, they are not applicable in proving the existence of “favorite sites” for the midpoint of the point-to-point polymers in our considerations. (For discussions on different notions of localizations, see Section 2.1.1.) The only other model where the “favorite region” conjecture has been proven so far is the one-dimensional stationary log-gamma polymer in [67]. A striking feature of this stationary model in [67] is its special integrable structure, which reduced the endpoint distribution to exponents of simple random walks [220]. As a consequence, the analysis of the limiting endpoint distribution around its mode in [67] was considerably simplified to that of a random walk seen from its infimum. In the case of the CDRP, the absence of a similar stationary boundary condition calls for an entirely new approach in showing the “favorite region" conjecture, which we discuss in this paper. Conversely, as we do not rely on integrability conditions other than the Gibbs property, our proof for the CDRP has the potential to generalize to other integrable models. Finally, accompanying our localization results, we also establish the convergence of the scaled favorite points to the almost sure unique maximizer of the Airy_2 process minus a parabola and the geodesics of the directed landscape respectively (see Theorem 2.1.8).

Beyond the polymer considerations, from the persepctive of the KPZ universality class, our paper is also an innovative application of several fundamental new techniques and results that have recently emerged in the community. These include the Brownian Gibbs resampling property [74], the weak convergence from the KPZ line ensemble to the Airy line ensemble [208], the tail estimates of the KPZ equation with narrow wedge initial data [79, 80, 81] as well as probabilistic properties of the Airy line ensemble from [96]. In particular, even though the Gibbs property has been utilized before in works such as [96, 55, 81, 82], in the CDRP case we overcome a unique challenge of quantifying the Gibbs property precisely on a *symmetric* random interval around the joint local maximizer of two independent copies of the KPZ equation with narrow wedge initial data. This issue is resolved after we prescribe the joint law of the KPZ equations around the desired interval. Additionally in our analysis, we treat the Radon-Nikodym derivative from the Gibbsian resampling directly and ensure that it converges exactly to 1. A more detailed description of our main technical innovations is available in Section 2.1.4.

Presently, we begin with an introduction on the CDRP's foundation - the discrete directed lattice polymers and related key concepts.

2.1.1 Introducing the CDRP through discrete directed lattice polymers

Directed polymers in random environments were first introduced in statistical physics literature by Huse and Henley [148] to study the phase boundary of the Ising model with random impurities. Later, it was mathematically reformulated as a random walk in a random environment by Imbrie and Spencer [152] and Bolthausen [44]. Since then immense progress has been made in understanding this model (see [65] for a general introduction and [129, 33] for partial surveys).

In the $(d+1)$- dimensional discrete polymer case, the random environment is specified by a collection of zero-mean i.i.d. random variables $\{\omega = \omega(i,j) \mid (i,j) \in \mathbb{Z}_+ \times \mathbb{Z}^d\}$. Given the environment, the energy of the n-step nearest neighbour random walk $(S_i)_{i=0}^n$ starting and ending at the origin (one can take the endpoint to be any suitable $\mathbf{x} \in \mathbb{R}^d$ as well) is given by $H_n^\omega(S) := \sum_{i=1}^n \omega(i, S_i)$. The **point-to-point** polymer measure on the set of all such paths is then defined as

$$\mathbf{P}_{n,\beta}^\omega(S) = \frac{1}{Z_{n,\beta}^\omega} e^{\beta H_n^\omega(S)} \mathbf{P}(S), \tag{2.1.1}$$

where $\mathbf{P}(S)$ is the uniform measure on set of all n-step nearest neighbour paths starting and ending at origin, β is the inverse temperature, and $Z_{n,\beta}^\omega$ is the partition function. Meanwhile, one can also consider the **point-to-line** polymer measures where the endpoint is 'free' and the reference measure $\mathbf{P}$ is given by n-step simple symmetric random walks. In the polymer measure, there is a competition between the *entropy* of paths and the *disorder strength* of the environment. Under this competition, depending on the inverse temperature β, two distinct regimes [69] have been shown to appear:

- *Weak Disorder*: When β is small or equivalently in high temperature regime, intuitively the disorder strength diminishes. The walk is dominated by the entropy and exhibits diffusive behaviors. This type of entropy domination is termed as *weak disorder*.

- *Strong Disorder*: If β is large and positive or equivalently the temperature is low but remains positive, the polymer measure concentrates on singular paths with high energies and the diffusive behavior is no longer guaranteed. This type of disorder strength domination is known as the *strong disorder*.

We refer the reader to [69] for the precise definitions of weak and strong disorder regimes. Moreover, [69] showed that there exists a critical inverse temperature $\beta_c(d)$, depending on the dimension d, such that weak disorder holds for $0 \leq \beta < \beta_c$ and strong disorder for $\beta > \beta_c$. In particular, when $d = 1$ or $d = 2$, $\beta_c = 0$, i.e. all positive β fall into the strong disorder regime for $d = 1, 2$.

The rest of the article focuses on $d = 1$. While our previous discussion on weak disorder states that for $\beta = 0$, the paths fluctuations are of the order $\sqrt{n}$ via Brownian considerations, the situation is much more complex in the strong disorder regime. The following two phenomena are conjectured:

- *Superdiffusivity*: Under strong disorder, the polymer measure is believed to be in the KPZ universality class and paths have typical fluctuations of the order $n^{2/3}$. This phenomenon is known as superdiffusion and has been conjectured widely in physics literature (see [148], [149], [160], [169]). Although it has been rigorously proven in specific situations (see [179],[202], [155],[56], [188]), much remains unknown, especially for $d \geq 2$.

- *Localization and the "favorite region" conjecture*: The polymer exhibits certain localization phenomena under strong disorder. The quenched density of the midpoint of the path (or any specific point in the bulk of the path) is believed to be distributed in a relatively small region of order 1 around a random point which itself is of the order $n^{2/3}$ (see [59], [32] for partial survey.)

We remark that there exist many different notions of localizations. In addition to the one discussed above and the strong localization in [56] mentioned earlier, other notions of localization such as the atomic localization [235] and the geometric localization [33] were studied under a general abstract framework in [33] for simple random walks and were later extended to general reference walks in [32]. Both of [33] and [32] provide sufficient criteria for the existence of the 'favorite region' of order one for the endpoint in arbitrary dimension. However, despite the sufficiency of these criteria, it is yet unknown how to check them for standard directed polymers. We refer the readers to Bates' thesis [31] for a more detailed survey on this topic.

Even though the critical inverse temperature $\beta_c(1) = 0$ for $d = 1$, one might hope to scale the inverse temperature with the length of the polymer in a critical manner to capture the transition between weak and strong disorder. In this spirit, the seminal work of [4] considered *an intermediate disordered regime* where $\beta = \beta_n$ is taken to be $n^{-1/4}$, where n is the length of the polymer. [4] showed that the partition function Z^{ω}_{n,β_n} has a universal scaling limit given by the solution of the Stochastic Heat Equation (SHE) when ω has finite exponential moments. Furthermore, under the diffusive scaling, the polymer path itself converges to a universal object called the Continuous Directed Random Polymer (denoted as CDRP hereafter) which appeared first in [3] and depended on a continuum external environment given by the space-time white noise.

More precisely, given a white noise ξ on $[0, t] \times \mathbb{R}$, CDRP is a path measure on the space of $C([0, t])$ (continuous functions on $[0, t]$) for each realization of ξ. It was shown in [4] that conditioned on the environment, the CDRP is a continuous Markov process with the same quadratic variation as the Brownian motion but is singular w.r.t. the Brownian motion. Due to this singularity w.r.t. the Wiener measure, it is not clear how to express the CDRP path measure in a Gibbsian form similar to (2.1.1). Instead, using the partition functions as building blocks, one can construct a consistent family of finite dimensional distributions which uniquely specify the path measure (see [3] or Section 3.1.2 for more details).

As the CDRP sits between weak and strong disorder regimes, one expects it to exhibit weak disorder type behaviors in the short-time regime ($t \downarrow 0$) and strong ones in the long-time regime

($t \uparrow \infty$). Indeed, the log partition function of CDRP is Gaussian in the short time limit (see [5]), providing evidence for weak disorder. Upon varying the endpoint of the CDRP measure, the log partition function can be viewed as a random function of the endpoint and converges to the parabolic Airy$_2$ process under the $1:2:3$ KPZ scaling (see [208, 236]). Note that the KPZ scaling itself bears the characteristic $2/3$ spatial scaling. Thus, it provides evidence for the superdiffusivity in the strong disorder regime. However, the theory of universality class alone does not provide much insight into the possible localization phenomena of the CDRP measures.

2.1.2 Summary of Results

As we have explained in the previous section, the purpose of the present article is to study the localization phenomena for the long-time CDRP measure. The following summarizes our results, which we will elaborate on individually in Section 3.1.2. Our first two results affirm the 'favorite region' conjecture (discussed above) which has so far only been proven for the log-gamma polymer model in [67].

- For a point-to-point CDRP of length t, the quenched density of pt-point of the polymer with fixed $p \in (0,1)$ when centered around its almost sure unique mode (which is the maximizer of the probability density function) $\mathcal{M}_{p,t}$, converges weakly to a density proportional to $e^{-\mathcal{R}_2(x)}$. Here, $\mathcal{R}_2$ is a two-sided 3D-Bessel process with diffusion coefficient 2 defined in (2.5.2)(Theorem 2.1.4).

- Similarly, for a point-to-line CDRP of length t, the quenched density of the endpoint of the polymer when centered around its almost sure unique mode $\mathcal{M}_{*,t}$ converges weakly to a density proportional to $e^{-\mathcal{R}_1(x)}$, where $\mathcal{R}_1$ is a two-sided 3D-Bessel process with diffusion coefficient 1 (Theorem 2.1.5).

- The random mode $\mathcal{M}_{*,t}$ of the endpoint of point-to-line CDRP of length t upon $2^{-1/3}t^{2/3}$ scaling converges in law to the unique maximum of the Airy$_2$ process minus a parabola, whereas the random mode $\mathcal{M}_{p,t}$ of the pt point of point-to-point CDRP of length t upon

$t^{2/3}$ scaling converges to $\Gamma(p\sqrt{2})$, where $\Gamma(\cdot)$ is the geodesic from $(0,0)$ to $(0,\sqrt{2})$ of the Directed landscape (Theorem 2.1.8).

Next, the well-known KPZ equation with the narrow wedge initial data forms the log-partition function of the CDRP. Our main results below shed light on some local information about the KPZ equation:

- *Ergodicity*: The spatial increments of the KPZ equation with the narrow wedge initial data as time tends to infinity converges weakly to a standard two-sided Brownian motion (Theorem 2.1.11).
- The sum of two independent copies of the KPZ equation with the narrow wedge initial data when re-centered around its maximum converges to a two-sided 3D-Bessel process with diffusion coefficient 2 (Theorem 2.1.10).

These results provide a comprehensive characterization of the localization picture for the CDRP model. We present the formal statements of the results in the next subsection.

2.1.3 The model and the main results

In order to define the CDRP model we use the stochastic heat equation (SHE) with multiplicative noise as our building blocks. Towards this end, we consider a four-parameter random field $\mathcal{Z}(x,s;y,t)$ defined on

$$\mathbb{R}^4_{\uparrow} := \{(x,s;y,t) \in \mathbb{R}^4 : s < t\}.$$

For each $(x,s) \in \mathbb{R}\times\mathbb{R}$, the field $(y,t) \mapsto \mathcal{Z}(x,s;y,t)$ is the solution of the SHE starting from location x at time s, i.e., the unique solution of

$$\partial_t \mathcal{Z} = \tfrac{1}{2}\partial_{xx}\mathcal{Z} + \mathcal{Z}\cdot\xi, \qquad (y,t) \in \mathbb{R}\times(s,\infty),$$

with Dirac delta initial data

$$\lim_{t\downarrow s} \mathcal{Z}(x,s;y,t) = \delta(x-y).$$

Here $\xi = \xi(x,t)$ is the space-time white noise. The SHE itself enjoys a well-developed solution theory based on Itô integral and chaos expansion [34, 237] also [77, 206]. Moreover, the solution of the SHE is naturally connected to the partition functions of the directed polymers in continuum random environment via the Feynmann-Kac formula [149, 65]. In particular the four-parameter random field can be written in terms of chaos expansion as

$$\mathcal{Z}(x,s;y,t) = \sum_{k=0}^{\infty} \int_{\Delta_{k,s,t}} \int_{\mathbb{R}^k} \prod_{\ell=1}^{k+1} p(y_\ell - y_{\ell-1}, s_\ell - s_{\ell-1}) \xi(y_\ell, s_\ell) d\vec{y}\, d\vec{s}, \tag{2.1.2}$$

with $\Delta_{k,s,t} := \{(s_\ell)_{\ell=1}^k : s < s_1 < \cdots < s_k < t\}$, $s_0 = s, y_0 = x, s_{k+1} = t$, and $y_{k+1} = y$. Here $p(x,t) := (2\pi t)^{-1/2} \exp(-x^2/(2t))$ denotes the standard heat kernel. The field $\mathcal{Z}$ satisfies several other properties including the Chapman-Kolmogorov equations [3, Theorem 3.1]. Namely, for all $0 \le s < r < t$, and $x, y \in \mathbb{R}$ we have

$$\mathcal{Z}(x,s;y,t) = \int_{\mathbb{R}} \mathcal{Z}(x,s;z,r)\mathcal{Z}(z,r;y,t)dz. \tag{2.1.3}$$

For all $(x,s;y,t) \in \mathbb{R}^4_{\uparrow}$, we also set

$$\mathcal{Z}(x,s;*,t) := \int_{\mathbb{R}} \mathcal{Z}(x,s;y,t)dy. \tag{2.1.4}$$

Definition 2.1.1 (Point-to-point CDRP)**.** Conditioned on the white noise ξ, let $\mathbf{P}^\xi$ be a measure $C([s,t])$ whose finite dimensional distribution is given by

$$\mathbf{P}^\xi(X(t_1) \in dx_1, \ldots, X(t_k) \in dx_k) = \frac{1}{\mathcal{Z}(x,y;s,t)} \prod_{j=0}^{k} \mathcal{Z}(x_j, t_j, ; x_{j+1}, t_{j+1}) dx_1 \ldots dx_k. \tag{2.1.5}$$

for $s = t_0 \le t_1 < \cdots < t_k \le t_{k+1} = t$, with $x_0 = x$ and $x_{k+1} = y$.

The measure $\mathbf{P}^{\xi}$ also depends on x and y but we suppress it from our notations. We will also use the notation $\mathsf{CDRP}(x,s;y,t)$ and write $X \sim \mathsf{CDRP}(x,s;y,t)$ when $X(\cdot)$ is a random continuous function on $[s,t]$ with $X(s) = x$ and $X(t) = y$ and its finite dimensional distributions given by (3.1.5) conditioned on ξ.

Definition 2.1.2 (Point-to-line CDRP)**.** Conditioned on the white noise ξ, we also let $\mathbf{P}^{\xi}_{*}$ be a measure $C([s,t])$ whose finite dimensional distributions are given by

$$\mathbf{P}^{\xi}_{*}(X(t_1) \in dx_1, \ldots, X(t_k) \in dx_k) = \frac{1}{\mathcal{Z}(x,s;*,t)} \prod_{j=0}^{k} \mathcal{Z}(x_j, t_j, ; x_{j+1}, t_{j+1}) dx_1 \ldots dx_k. \quad (2.1.6)$$

for $s = t_0 \leq t_1 < \cdots < t_k \leq t_{k+1} = t$, with $x_0 = x$ and $x_{k+1} = *$.

Remark 2.1.3. Note that the Chapman-Kolmogorov equations (3.1.4) and (2.1.4) ensure that the finite dimensional distributions in (3.1.5) and (3.1.6) are consistent, and that $\mathbf{P}^{\xi}$ and $\mathbf{P}^{\xi}_{*}$ are probability measures. The measure $\mathbf{P}^{\xi}_{*}$ also depends on x but we again suppress it from our notations. We similarly use $\mathsf{CDRP}(x,y;*,t)$ to refer to $\mathbf{P}^{\xi}_{*}$.

Theorem 2.1.4 (Pointwise localization for point-to-point CDRP)**.** *Fix any $p \in (0,1)$. Let $X \sim \mathsf{CDRP}(0,0;0,t)$ and let $f_{p,t}(\cdot)$ denotes the density of $X(pt)$ which depends on the white noise ξ. Then, for all $t > 0$ the random density $f_{p,t}$ has almost surely a unique mode $\mathcal{M}_{p,t}$. Furthermore, as $t \to \infty$, we have the following convergence in law*

$$f_{p,t}(x + \mathcal{M}_{p,t}) \xrightarrow{d} r_2(x) := \frac{e^{-\mathcal{R}_2(x)}}{\int_{\mathbb{R}} e^{-\mathcal{R}_2(y)} dy}, \quad (2.1.7)$$

in the uniform-on-compact topology. Here $\mathcal{R}_2(\cdot)$ is a two-sided 3D-Bessel process with diffusion coefficient 2 defined in Definition 2.5.2.

Theorem 2.1.5 (Endpoint localization for point-to-line CDRP)**.** *Let $X \sim \mathsf{CDRP}(0,0;*,t)$ and let $f_t(\cdot)$ denotes the density of $X(t)$ which depends on the white noise ξ. Then for $t > 0$, the random density f_t has almost surely a unique mode $\mathcal{M}_{*,t}$. Furthermore, as $t \to \infty$, we have the following* *convergence in law*

$$f_{*,t}(x + \mathcal{M}_{*,t}) \xrightarrow{d} r_1(x) := \frac{e^{-\mathcal{R}_1(x)}}{\int_{\mathbb{R}} e^{-\mathcal{R}_1(y)} dy}, \tag{2.1.8}$$

in the uniform-on-compact topology. Here $\mathcal{R}_1(\cdot)$ is a two-sided 3D-Bessel process with diffusion coefficient 1 *defined in Definition 2.5.2.*

Remark 2.1.6. In Proposition 2.7.1 we show that for a two-sided 3D-Bessel process $\mathcal{R}_\sigma$ with diffusion coefficient $\sigma > 0$, $\int_{\mathbb{R}} e^{-\mathcal{R}_\sigma(y)} dy$ is finite almost surely. Thus $r_1(\cdot)$ and $r_2(\cdot)$ defined in (2.1.8) and (2.1.7) respectively are valid random densities. Theorems 2.1.4 and 2.1.5 derive explicit limiting probability densities for the quenched distributions of the endpoints of the point-to-line polymers and the pt-point of point-to-point polymers when centered around their respective modes, providing a complete description of the localization phenomena in the CDRP model. More concretely, it shows that the corresponding points are concentrated in a microscopic region of order one around their "favorite points" (see Corollary 2.7.3).

We next study the random modes $\mathcal{M}_{*,t}$ and $\mathcal{M}_{p,t}$. The "favorite point" $\mathcal{M}_{p,t}$ is of the order $t^{2/3}$ and converges in distribution upon scaling. The limit is given in terms of the directed landscape constructed in [94, 187] which arises as an universal full scaling limit of several zero temperature models [97]. Below we briefly introduce this limiting model in order to state our next result.

The directed landscape $\mathcal{L}$ is a random continuous function $\mathbb{R}^4_\uparrow \to \mathbb{R}$ that satisfies the metric composition law

$$\mathcal{L}(x, s; y, t) = \max_{z\in\mathbb{R}} \left[\mathcal{L}(x, s; z, r) + \mathcal{L}(z, r; y, t)\right], \tag{2.1.9}$$

with the property that $\mathcal{L}(\cdot, t_i; \cdot, t_i + s_i^3)$ are independent for any set of disjoint intervals $(t_i, t_i + s_i^3)$, and as a function in x, y, $\mathcal{L}(x, t; y, t + s^3) \stackrel{d}{=} s \cdot \mathcal{S}(x/s^2, y/s^2)$, where $\mathcal{S}(\cdot,\cdot)$ is a parabolic Airy Sheet. We will not define the parabolic Airy Sheet (see Definition 1.2 in [94]) here but we mention that $\mathcal{S}(0,\cdot) \stackrel{d}{=} \mathcal{A}(\cdot)$ where $\mathcal{A}$ is the parabolic Airy$_2$ process and $\mathcal{A}(x) + x^2$ is the (stationary) Airy$_2$

process constructed in [204]

Definition 2.1.7 (Geodesics of the directed landscape)**.** For $(x, s; y, t) \in \mathbb{R}^4_\uparrow$, a geodesic from (x, s) to (y, t) of the directed landscape is a random continuous function $\Gamma : [s, t] \to \mathbb{R}$ such that $\Gamma(s) = x$ and $\Gamma(t) = y$ and for any $s \le r_1 < r_2 < r_3 \le t$ we have

$$\mathcal{L}(\Gamma(r_1), r_1; \Gamma(r_3), r_3) = \mathcal{L}(\Gamma(r_1), r_1; \Gamma(r_2), r_2) + \mathcal{L}(\Gamma(r_2), r_2; \Gamma(r_3), r_3).$$

Thus geodesics precisely contain the points where the equality holds in (3.1.7). Given any $(x, s; y, t) \in \mathbb{R}^4_\uparrow$, by Theorem 12.1 in [94], it is known that almost surely there is a unique geodesic Γ from (x, s) to (y, t).

With the above definitions in place, we now state our favorite point scaling result.

Theorem 2.1.8 (Favorite Point Limit)**.** *Fix any $p \in (0, 1)$. Consider $\mathcal{M}_{p,t}$ and $\mathcal{M}_{*,t}$ defined almost surely in Theorems 2.1.4 and 2.1.5 respectively. As $t \to \infty$ we have*

$$2^{-1/3} t^{-2/3} \mathcal{M}_{*,t} \xrightarrow{d} \mathcal{M}, \qquad t^{-2/3} \mathcal{M}_{p,t} \xrightarrow{d} \Gamma(p\sqrt{2})$$

where $\mathcal{M}$ is the almost sure unique maximizer of the Airy$_2$ *process minus a parabola, and $\Gamma : [0, \sqrt{2}] \to \mathbb{R}$ is the almost sure unique geodesic of the directed landscape from $(0, 0)$ to $(0, \sqrt{2})$.*

Remark 2.1.9. Theorem 2.1.8 shows that the random mode fluctuates in the order of $t^{2/3}$. This corroborates the fact that CDRP undergoes superdiffusion as $t \to \infty$. We remark that the $\mathcal{M}_{*,t}$ convergence was anticipated in [190] modulo a conjecture about convergence of scaled KPZ equation to the parabolic Airy$_2$ process. This conjecture was later proved in [236, 208].

The proof of Theorem 2.1.4 relies on establishing fine properties of the partition function $\mathcal{Z}(x, t) := \mathcal{Z}(0, 0; x, t)$, or more precisely, properties of the log-partition function $\log \mathcal{Z}(x, t)$. For delta initial data, $\mathcal{Z}(x, t) > 0$ for all $(x, t) \in \mathbb{R} \times (0, \infty)$ almost surely [124]. Thus the logarithm of the partition function $\mathcal{H}(x, t) := \log \mathcal{Z}(x, t)$ is well-defined. It formally solves the KPZ equation:

$$\partial_t \mathcal{H} = \tfrac{1}{2}\partial_{xx}\mathcal{H} + \tfrac{1}{2}(\partial_x \mathcal{H})^2 + \xi, \qquad \mathcal{H} = \mathcal{H}(x,t), \quad (x,t) \in \mathbb{R} \times [0,\infty). \tag{2.1.10}$$

The KPZ equation was introduced in [160] to study the random growing interfaces and since then it has been extensively studied in both the mathematics and the physics communities. We refer to [123, 206, 77, 209, 58, 84] for surveys related to it.

It is worthwhile to note that as a stochastic PDE, (2.1.10) is ill-posed due to the presence of the nonlinear term $\frac{1}{2}(\partial_x \mathcal{H})^2$. The above notion of solutions coming from the logarithm of the solution of SHE is referred to as the Cole-Hopf solution. The corresponding initial data is called the narrow wedge initial data for the KPZ equation. Other notions of solutions, such as regularity structures [137, 136], paracontrolled distributions [131, 133], and energy solutions [130, 132], have been shown to coincide with the Cole-Hopf solution within the class of initial datas the theory applies.

Returning to Theorem 2.1.4, to prove this statement one needs to understand how multiple copies of the KPZ equation behave around its joint maximum. Towards this end, we present our first main result related to the KPZ equation that studies the limiting behavior of sum of two independent copies of KPZ equation re-centered around its joint maximum as $t \to \infty$.

Theorem 2.1.10 (Bessel behavior around joint maximum)**.** *Fix $k = 1$ or $k = 2$. Consider k independent copies of the KPZ equation $\{\mathcal{H}_i(x,t)\}_{i=1}^k$ started from the narrow wedge initial data. For each $t > 0$, almost surely, the process $x \mapsto \sum_{i=1}^k \mathcal{H}_i(x,t)$ has a unique maximizer, say $\mathcal{P}_{k,t}$. Furthermore, as $t \to \infty$, we have the following convergence in law*

$$R_k(x,t) := \sum_{i=1}^{k} \left[\mathcal{H}_i(\mathcal{P}_{k,t},t) - \mathcal{H}_i(x+\mathcal{P}_{k,t},t)\right] \xrightarrow{d} \mathcal{R}_k(x) \tag{2.1.11}$$

in the uniform-on-compact topology. Here $\mathcal{R}_k(x)$ is a two-sided Bessel process with diffusion coefficient k.

We also present the next theorem to complement the above result in Theorem 2.1.10. Theorem

2.1.11 is a by-product of our analysis and does not quite appear in the proof of Theorem 2.1.4. It captures the behaviors of the increments of $\mathcal{H}(\cdot, t)$.

Theorem 2.1.11 (Ergodicity of the KPZ equation)**.** *Consider the KPZ equation $\mathcal{H}(x,t)$ started from the narrow wedge initial data. As $t \to \infty$, we have the following convergence in law*

$$\mathcal{H}(x,t) - \mathcal{H}(0,t) \xrightarrow{d} B(x)$$

in the uniform-on-compact topology. Here $B(x)$ is a two-sided standard Brownian motion.

Remark 2.1.12. For a Brownian motion on a compact interval, it is well known that the law of the process when re-centered around its maximum is absolutely continuous w.r.t. Bessel process. In light of Theorem 2.1.11, it is natural to expect the Bessel process as a limit in Theorem 2.1.10. The diffusion coefficient is k because there are k independent copies of the KPZ equation.

Remark 2.1.13. We stress the fact that we prove (2.1.11) for $k = 1$ and $k = 2$ only. The $k = 1$ case is related to Theorem 2.1.5 whereas the $k = 2$ case is related to Theorem 2.1.4. Our proof strategy for Theorem 2.1.10 can also be adapted to prove the general case when $k \geq 3$. We explain later in Remark 2.4.12 what are the missing pieces for the proof of (2.1.11) for general k. Although Theorem 2.1.10 for general k is an interesting result in its own right, due to brevity and the lack of applications to our localization problem, we restrict ourselves only to when $k = 1, 2$.

A useful property in establishing the ergodicity of a given Markov process is the strong Feller property. For instance, the work of [138] introduced a framework to study the strong Feller property for singular SPDEs and established it for a multicomponent KPZ equation. One caveat of this framework is that [138] techniques and results are limited to only periodic boundary conditions, i.e. on torus domain, and are thus inaccessible for the KPZ equation with narrow-wedge initial data. For a more thorough discussion on the recent advances in singular stochastic PDEs, we refer the reader to [84]. It's also worth mentioning that as a consequence of the strong Feller property, [138] was able to conclude the Brownian bridge measure as the unique invariant measure for the

KPZ equation with periodic boundary conditions. The existence of the Brownian invariant measures for the KPZ equation, on the other hand, has been well-known since the work of [35] and proved via many different approaches such as renormalization [126] and paracontrolled distributions [133].For the spatial derivative of the KPZ equation, i.e. the Burgers' equation, long time properties such as ergodicity, synchronization and one-force-one-solution principle have also been studied extensively in the literature (see [212, 134, 115] and the references therein).

In addition to the strong Feller property, we can also probe the KPZ equation's ergodicity through its connection to the KPZ universality class. Often viewed as the fundamental positive temperature model at the heart of the KPZ universality class, the KPZ equation shares the same 1 : 2 : 3 scaling exponents and universal long-time behaviors expected or proven for other members of the class. A widely-held belief about the KPZ universality class is that under the 1 : 2 : 3 scaling and in the large scale limit, all models in the class converge to an universal scaling limit called the KPZ fixed point [94, 187]. In fact, this very conjecture has been recently proved for the KPZ equation in [208, 236]. As we utilize this result later, we recall a special case of the statement in [208] here. Consider the 1 : 2 : 3 scaling of the KPZ equation (the scaled KPZ equation)

$$\mathfrak{h}_t(x) := t^{-1/3}\left(\mathcal{H}(t^{2/3}x, t) + \tfrac{t}{24}\right).$$

Then we have that $2^{1/3}\mathfrak{h}_t(2^{1/3}x)$ converges to the parabolic Airy$_2$ process as $t \to \infty$. Note that the parabolic Airy$_2$ process is the marginal of the parabolic Airy Sheet, which is a canonical object in the construction of the KPZ fixed point and the related directed landscape (see [94, 208]).

On the KPZ fixed point level, questions about ergodicity and behavior around maximum are much better understood in the literature. Under the zero temperature setings, a plethora of results and techniques are available to address the ergodicity question for the KPZ fixed point. For instance, due to the 1 : 2 : 3 scaling invariance, ergodicity of the KPZ fixed point is equivalent to the local Brownian behavior ([187, Theorem 4.14 and 4.15]) or can be deduced in [201] using certain coupling that are applicable only in zero temperature settings.

On the other hand, [95] showed that local Brownianity and local Bessel behaviors around the maximizer hold for any process which is absolutely continuous w.r.t. Brownian motions on every compact set. The scaled KPZ equation is also known to have such property [74] and the question of ergodicity can be transformed into certain local Brownian behaviors of the scaled KPZ equation. Indeed we have

$$\mathcal{H}(x,t) - \mathcal{H}(0,t) = t^{-1/3}\left(\mathfrak{h}_t(t^{-2/3}x) - \mathfrak{h}_t(0)\right).$$

However a crucial difference for the KPZ equation is that the law of $\mathfrak{h}_t$ changes with respect to time and the diffusive scaling precisely depends on t. Therefore in such a scenario, it is unclear how to extend the soft techniques in [95, Lemma 4.3] to address the limiting local Brownian behaviors in above setting.

A recent line of inquiries regarding the behavior around the maxima is the investigation of the fractal nature of exceptional times for the KPZ fixed point with multiple maximizers [82, 92]. In [82], the authors computed the Hausdorff dimension of the set of times for the KPZ fixed point with at least two maximizers. [92] later extended the results in [82] to the case of exactly k maximizers. In [92], the author relied on a striking property of the KPZ fixed point. Namely, the evolution of the KPZ fixed point, when started with Bessel initial conditions, after recentering at the maximum becomes stationary in t. This property considerably simplified their analysis. Other initial data were then accessed through a transfer principle based on [216]. Unfortunately, such analogous properties for the KPZ equation are not yet known in the literature.

2.1.4 Proof Ideas

In this section we sketch the key ideas behind the proofs of our main results. For brevity, we present a heuristic argument for the proofs of Theorem 2.1.4 and the related Theorem 2.1.10 with the $k = 2$ case only. The proofs for the point-to-line case (Theorem 2.1.5) and the related $k = 1$ case of Theorem 2.1.10 and ergodicity (Theorem 2.1.11) follow from similar ideas. Meanwhile, the tools and methods related to the uniqueness and convergence of random modes (Theorem 2.1.8) are of a different flavor. As the corresponding arguments are relatively simple, we present them directly in Section 2.3.

Recall from the statement of Theorem 2.1.4 that $f_{p,t}$ denotes the quenched density of $X(pt)$ where $X \sim \mathsf{CDRP}(0,0;0,t)$. To simplify our discussion below, we let $p = \frac{1}{2}$ and replace t by $2t$. From (3.1.5) we have

$$f_{\frac{1}{2},2t}(x) = \frac{\mathcal{Z}(0,0;x,t)\mathcal{Z}(x,t;0,2t)}{\mathcal{Z}(0,0;0,2t)}.$$

Recall the chaos expansion for $\mathcal{Z}(x,s;y,t)$ from (3.1.3). Note that $\mathcal{Z}(0,0;x,t)$ and $\mathcal{Z}(x,t;0,2t)$ are independent as they uses different sections of the noise ξ. By a change of variable and symmetry, we obtain that $\mathcal{Z}(x,t;0,2t)$ is same in distribution as $\mathcal{Z}(0,0;x,t)$ as a process in x. Thus as a process in x, $\mathcal{Z}(0,0;x,t)\mathcal{Z}(x,t;0,2t) \stackrel{d}{=} e^{\mathcal{H}_1(x,t)+\mathcal{H}_2(x,t)}$ were $\mathcal{H}_1(x,t)$ and $\mathcal{H}_2(x,t)$ are independent copies of the KPZ equation with narrow wedge initial data. This puts Theorem 2.1.4 in the framework of Theorem 2.1.10. Viewing the density around its unique random mode $\mathcal{M}_{\frac{1}{2},2t}$ (that is the maximizer), we may thus write $f_{\frac{1}{2},2t}(x+\mathcal{M}_{\frac{1}{2},2t})$ as

$$\frac{e^{-R_2(x,t)}}{\int_{\mathbb{R}} e^{-R_2(y,t)}dy},$$

where $R_2(x,t)$ is defined in (2.1.11). For simplicity, let us use the notation $\mathcal{P} = \mathcal{M}_{\frac{1}{2},2t}$.

The rest of the argument hinges on the following two main facts:

(i) *Bessel convergence*: $R_2(x,t)$ converges weakly to 3D-Bessel process with diffusion coefficient 2 in the uniform-on-compact topology (Theorem 2.1.10).

(ii) *Controlling the tails*: By taking K large, $\int_{[-K,K]^c} e^{-R_2(y,t)}dy$ can be made arbitrarily small for all large t (see Proposition 2.7.2 for precise statement).

Theorem 2.1.4 can then be deduced from the above two items by standard analysis. We now explain the ideas behind items (i) and (ii). The principal tool of our analysis is the Gibbsian line ensemble – an object bearing an integrable origin but are largely used in probabilistic setting. More precisely, we use the *KPZ line ensemble* (recalled in Proposition 3.5.1), a set of random continuous functions whose lowest indexed curve is same in distribution as the narrow wedge solution of the KPZ equation. The law of the lowest indexed curve enjoys a certain Gibbs property which is known as **H**-Brownian Gibbs property. Roughly, this property states that the law of the lowest indexed curve conditioned on an interval depends only on the curve indexed one below and the starting and ending points. Furthermore, this conditional law is absolutely continuous w.r.t. a Brownian bridge of the same starting and ending points with an explicit expression of the Radon-Nikodym derivative.

We recast the problem in item (i) in the language of Gibbsian line ensemble. Note that $R_2(x,t)$ is a sum of two independent KPZ equations viewed from joint maximum (see (2.1.11)). To access the distribution of $R_2(x,t)$ one then needs a precise description of the conditional joint law of the top curves of two independent copies of the KPZ line ensemble on random intervals determined by the location of the joint maximizer. In view of this, to establish item (i) we prove the following two items:

(a) Two Brownian bridges when viewed around joint maximum can be appropriately given by two pairs of non-intersecting Brownian bridges to either side of the maximum (Proposition 2.4.9).

(b) For a suitable $K(t) \uparrow \infty$, the Radon-Nikodym derivatives associated with the KPZ line ensembles (see (2.2.3) for the precise expression of Radon-Nikodym derivative) on the random interval $[\mathcal{P} - K(t), \mathcal{P} + K(t)]$ containing the maximizer goes to 1.

Combining the above two ideas, we can conclude the joint law of

$$(D_1(x,t), D_2(x,t)) := (\mathcal{H}_1(\mathcal{P},t) - \mathcal{H}_1(\mathcal{P}+x,t), \mathcal{H}_2(\mathcal{P}+x,t) - \mathcal{H}_2(\mathcal{P},t)) \tag{2.1.12}$$

on $x \in [-K(t), K(t)]$ is close to two-sided pair of non-intersecting Brownian bridges with the same starting point and appropriate endpoints. Upon taking $t \to \infty$, one obtains a two-sided Dyson Brownian motion $(\mathcal{D}_1, \mathcal{D}_2)$ defined in Definition 2.5.1 as a distributional limit. Proposition

2.6.1 is the precise rendering of this fact. Finally a 3D-Bessel process emerges as the difference of two parts of the Dyson Brownian motion: $\mathcal{D}_1(\cdot) - \mathcal{D}_2(\cdot)$ (see Lemma 2.5.3).

Before expanding upon items (a) and (b), let us explain why we adopted an approach substantially different from the existing methods. Since the random interval in question includes joint maximizers, it is not a stopping domain and one cannot utilize classical properties such as the strong Gibbs property for KPZ line ensemble. In the context of the KPZ fixed point [82] used Gibbs property on random intervals defined to the right of the maximizer as one of the ingredients of their proof. There they relied on a path decomposition of Markov processes at certain spatial times from [189]. The result of [189] argues that conditioned on the maximizer, the process to the *right* of the maximizer is Markovian. However in our context, the intervals around the maximum is *symmetric*. Thus the abstract setup of [189] is not suited for our case. Thus, the precise description of the law given for the Brownian bridges in item (a) is indispensable to our argument.

To go from Brownian laws to KPZ laws, one needs an exact comparison between the two. Traditional tools such as stochastic monotonicity for the KPZ line ensembles are known to help obtain one-sided bounds for monotone events. Especially in the context of tail estimates of the KPZ equation, such tools reduce the problem to the setting of Brownian bridges, which can be treated classically. However, this approach only produces a one-sided bound and is thus insufficient for the precise convergence we wish to obtain. Hence we establish the exact comparison between the two laws by treating the Radon-Nikodym derivative directly.

To describe the result in item (a), consider two independent Brownian bridges $\overline{B}_1$ and $\overline{B}_2$ on $[0,1]$ both starting and ending at zero. See Figure 2.1. Let $M =: \operatorname{argmax}(\overline{B}_1(x)+\overline{B}_2(x))$. We wish to study the conditional law of $(\overline{B}_1, \overline{B}_2)$ given the max data: $(M, \overline{B}_1(M), \overline{B}_2(M))$. The key fact from Proposition 2.4.9 is that conditioned on the max data

$$(\overline{B}_1(M) - \overline{B}_1(M-x), \overline{B}_2(M-x) - \overline{B}_2(M))_{x\in[0,M]}, \quad (\overline{B}_1(M) - \overline{B}_1(x), \overline{B}_2(x) - \overline{B}_2(M))_{x\in[M,1]}$$

are independent and each is a non-intersecting Brownian bridge with appropriate end points (see

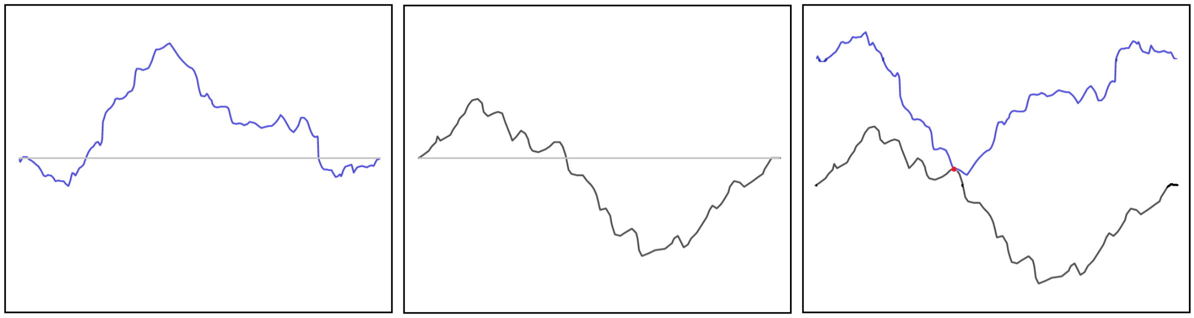

Figure 2.1: First idea for the proof: The first two figures depicts two independent Brownian bridges 'blue' and 'black' on $[0, 1]$ both starting and ending at zero. We flip the blue one and shift it appropriately so that when it is superimposed with the black one, the blue curve always stays above the black one and touches the black curve at exactly one point. The superimposed figure is shown in third figure. The red point denotes the 'touching' point or equivalent the joint maximizer. Conditioned on the max data, the trajectories on the left and right of the red points are given by two pairs of non-intersecting Brownian bridges with appropriate end points.

Definition 2.4.4). The key proof idea is to show such a decomposition at the level of discrete random walks, then take diffusive limits to get the same for Brownian motions and finally for Brownian bridges after conditioning. The details are all presented in Section 2.4.

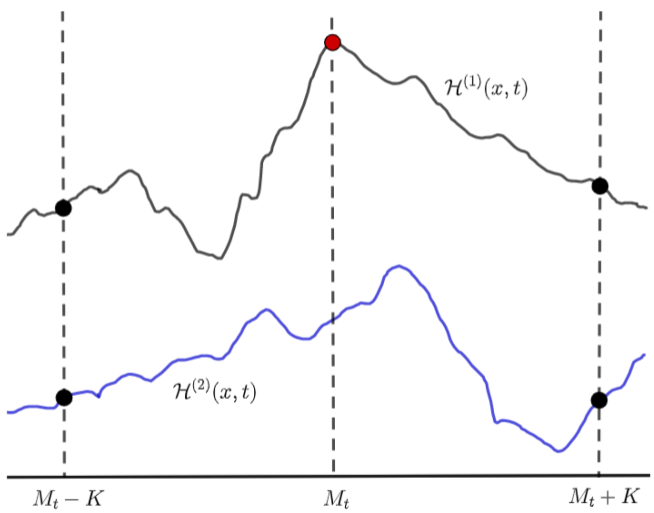

Figure 2.2: Second idea for the proof: For all "good" boundary data and max data, with high probability, there is an uniform separation of order $t^{1/3}$ between the first two curves on the random interval $[M_t - K, M_t + K]$.

To illustrate the idea behind item (b), let us consider an easier yet pertinent scenario. Let

$\mathcal{H}^{(1)}(x,t)$ and $\mathcal{H}^{(2)}(x,t)$) be the first two curves of the KPZ line ensemble. Let $M_t = \operatorname{argmax} \mathcal{H}^{(1)}(x,t)$. We consider the interval $I_t := [M_t - K, M_t + K]$. See Figure 2.2. We show that

1. Maximum is not too high: $\mathcal{H}^{(1)}(M_t, t) - \mathcal{H}^{(1)}(M_t \pm K, t) = O(1)$,
2. The gap at the end points is sufficiently large: $\mathcal{H}^{(1)}(M_t \pm K, t) - \mathcal{H}^{(2)}(M_t \pm K, t) = O(t^{1/3})$.
3. The fluctuations of the second curve on I_t are $O(1)$.

Under the above favorable class of boundary data: $\mathcal{H}^{(1)}(M_t \pm K, t), \mathcal{H}^{(2)}(\cdot, t)$ and the max data: $(M_t, \mathcal{H}^{(1)}(M_t, t))$, we show that the conditional fluctuations of the first curve are $O(1)$. This forces a uniform separation between the first two curves throughout the random interval I_t. Consequently the Radon-Nikodym derivative converges to 1 as $t \to \infty$ (see (2.2.3) for the precise expression of Radon-Nikodym derivative).

In order to conclude such a statement rigorously, we rely on tail estimates for the KPZ equation as well as some properties of the Airy line ensemble which are the distributional limits of the scaled KPZ line ensemble defined in (2.2.6). We review in depth of the necessary tools in Section 3.2. We remark that the rigorous argument for the Radon-Nikodym derivative present in the proof of Theorem 2.1.4 (Proposition 2.6.1 to be precise) is slightly different and more involved. Indeed, one needs to consider another copy of line ensemble and argue that similar uniform separation holds for both when viewed around the joint maximum $\mathcal{P}$. We also take $K = K(t) \uparrow \infty$ and the separation length is consequently different.

We have argued so far that $(D_1(x,t), D_2(x,t))$ (defined in (2.1.12)) jointly converges to a two-sided Dyson Brownian motion. This convergence holds in the uniform-on-compact topology. However, this does not address the question about behavior of the tail integral

$$\int_{[-K,K]^c} e^{D_2(y,t) - D_1(y,t)} \mathrm{d}y$$

that appears in item (ii).

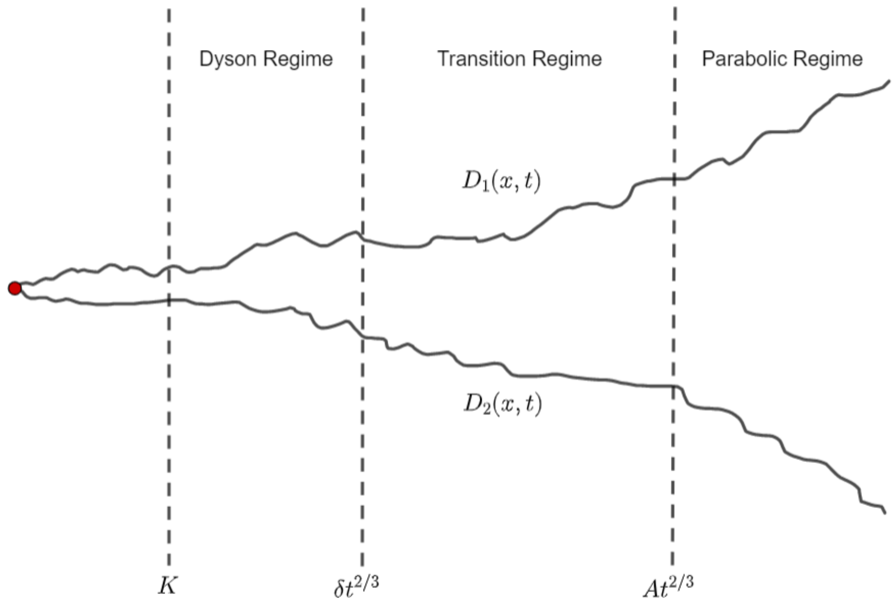

Figure 2.3: Third idea for the proof: The three regimes

To control the tail, we divide the tail integral into three parts based on the range of integration (See Figure 2.3):

- *Dyson regime:* The law of $(\mathcal{D}_1(x,t), \mathcal{D}_2(x,t))$ on the interval $[0, \delta t^{2/3}]$ is comparable to that of the Dyson Brownian motions for small enough δ and for all large t. Under the Dyson Brownian motion law, it is known that with high probability $\mathcal{D}_1(x) - \mathcal{D}_2(x) \geq \varepsilon |x|^{1/4}$ for all large enough $|x|$. This property translates to $(D_1(x,t), D_2(x,t))$ and provides a decay estimate for our integral over this interval.

- *Parabolic Regime:* The maximizer $\mathcal{P}$ can be shown to lie in a window of order $t^{2/3}$ region with high probability. On the other hand, the KPZ equation upon centering is known to have a parabolic decay: $\mathcal{H}(x,t) + \frac{t}{24} \approx -\frac{x^2}{2t} + O(t^{1/3})$. Thus taking A large enough ensures with high probability $D_1(x,t) \approx \frac{x^2}{4t}$ and $D_2(x,t) \approx -\frac{x^2}{4t}$ on the interval $[At^{2/3}, \infty)$. These estimates give the rapid decay of our integral in this regime.

- *Transition Regime:* In between the two regimes, we use soft arguments related to non-intersecting brownian bridges to ensure that with high probability $D_1(x,t) - D_2(x,t)$ is at least $\rho t^{1/3}$ uniformly on $[\delta t^{2/3}, At^{2/3}]$.

Proposition 2.5.6 and Proposition 2.7.2 together are the rigorous manifestations of the above idea. In the proof of Proposition 2.5.6 we provide decay estimates in the Dyson and transition regimes together for Brownian objects. Later in the proof of Proposition 2.7.2 we show how estimates in Proposition 2.5.6 translates to D_1, D_2 for those two regimes which we collectively refer to as “shallow tail regime" (see Figure 2.10). The parabolic regime (which we refer to as the “deep tail" later in Section 2.7) is also handled in Proposition 2.7.2.

Outline

The remainder of the paper is organized as follows. Section 3.2 reviews some of the existing results related to the KPZ line ensemble and its zero temperature counterpart, the Airy line ensemble. With the necessary background, we then prove the existence and uniqueness of random modes in Theorem 2.1.8 in Section 2.3. Section 2.4 is dedicated to the behaviors of the Brownian bridges around their joint maximum. Two key objects emerge in this Section: the Bessel bridges and the non-intersecting Brownian bridges. Several properties of these two objects are subsequently proved in Section 2.5. The proofs of Theorems 2.1.10 and 2.1.11 comprise section 2.6. Finally in Section 2.7, we complete the proofs of Theorems 2.1.4 and 2.1.5. Appendix 2.8 contains a convergence result about non-intersecting random walks used in Section 2.4. We defer the proof to the appendix as the arguments are standard.

Acknowledgements

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2.2 Basic framework and tools

Remark on Notations

Throughout this paper we use $\mathrm{C} = \mathrm{C}(\alpha, \beta, \gamma, \ldots) > 0$ to denote a generic deterministic positive finite constant that may change from line to line, but dependent on the designated variables $\alpha, \beta, \gamma, \ldots$. We will often write C_α in case we want to stress the dependence of the constant to the variable α. We will use serif fonts such as $\mathsf{A}, \mathsf{B}, \ldots$ to denote events as well as $\mathsf{CDRP}, \mathsf{DBM} \ldots$ to denote laws. The distinction will be clear from the context. The complement of an event A will be denoted as $\neg\mathsf{A}$.

In this section, we present the necessary background on the directed landscape and Gibbsian line ensembles including the Airy line ensemble and the KPZ line ensemble as well as known results on these objects that are crucial in our proofs.

2.2.1 The directed landscape and the Airy line ensemble

We recall the definition of the directed landscape and several related objects from [94, 96]. The directed landscape is the central object in the KPZ universality class constructed as a scaling limit of the Brownian Last Passage percolation (BLPP). We recall the setup of the BLPP below to define the directed landscape.

Definition 2.2.1 (Directed landscape)**.** Consider an infinite collection $B := (B_k(\cdot))_{k\in\mathbb{Z}}$ of independent two-sided Brownian motions with diffusion coefficient 2. For $x \le y$ and $n \le m$, the last passage value from (x, m) to (y, n) is defined by

$$B[(x, m) \to (y, n)] = \sup_{\pi} \sum_{k=n}^{m} [B_k(\pi_k) - B_k(\pi_{k-1})],$$

where the supremum is over all $\pi \in \Pi_{m,n}(x,y) := \{\pi_m \le \cdots \le \pi_n \le \pi_{n-1} \mid \pi_m = x, \pi_{n-1} = y\}$. Now for any $(x,s;y,t) \in \mathbb{R}^4_\uparrow$, we denote $(x,s)_n := (s + 2xn^{-1/3}, -\lfloor sn \rfloor)$ and $(y,t)_n := (t + 2yn^{-1/3}, -\lfloor tn \rfloor)$ and define

$$\mathcal{L}_n(x,s;y,t) := n^{1/6} B_n[(x,s)_n \to (y,t)_n] - 2(t-s)n^{2/3} - 2(y-x)n^{1/3}.$$

The directed landscape $\mathcal{L}$ is the distributional limit of $\mathcal{L}_n$ as $n \to \infty$ with respect to the uniform convergence on compact subsets of $\mathbb{R}^4_\uparrow$. By [94], the limit exists and is unique.

The marginal $\mathcal{A}_1(x) := \mathcal{L}(0,0;x,1)$ is known as the parabolic Airy$_2$ process. In [204] the Airy$_2$ process $\mathcal{A}_1(x) + x^2$ was constructed as the scaling limit of the polynuclear growth model. At the same time, $\mathcal{A}_1(x)$ can also be viewed as the top curve of the Airy line ensemble, which we define formally below in the approach of [73].

Definition 2.2.2 (Brownian Gibbs Property)**.** Recall the general notion of line ensembles from Section 2 in [73]. Fix $k_1 \le k_2$ with $k_1, k_2 \in \mathbb{N}$ and an interval $(a,b) \in \mathbb{R}$ and two vectors $\vec{x}, \vec{y} \in \mathbb{R}^{k_2-k_1+1}$. Given two measurable functions $f, g : (a,b) \to \mathbb{R} \cup \{\pm\infty\}$, let $\mathbb{P}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}_{\text{nonint}}$ be the law of $k_2 - k_1 + 1$ many independent Brownian bridges (with diffusion coefficient 2) $\{B_i : [a,b] \to \mathbb{R}\}_{i=k_1}^{k_2}$ with $B_i(a) = x_i$ and $B_i(b) = y_i$ conditioned on the event that

$$f(x) > B_{k_1}(x) > B_{k_1+1}(x) > \cdots > B_{k_2}(x) > g(x), \quad \text{for all } x \in [a,b].$$

Then the $\mathbb{N} \times \mathbb{R}$ indexed line ensemble $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \ldots)$ is said to enjoy the *Brownian Gibbs property* if, for all $K = \{k_1, \ldots, k_2\} \subset \mathbb{N}$ and $(a,b) \subset \mathbb{R}$, the following distributional equality holds:

$$\mathrm{Law}\Big(\mathcal{L}_{K\times(a,b)} \text{ conditioned on } \mathcal{L}_{\mathbb{N}\times\mathbb{R}\setminus K\times(a,b)}\Big) = \mathbb{P}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}_{\text{nonint}},$$

where $\vec{x} = (\mathcal{L}_{k_1}(a), \ldots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \ldots, \mathcal{L}_{k_2}(b))$, $\mathcal{L}_{k_1-1} = f$ (or ∞ if $k_1 = 1$) and $\mathcal{L}_{k_2+1} = g$.

Definition 2.2.3 (Airy line ensemble)**.** The Airy line ensemble $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots)$ is the unique $\mathbb{N} \times \mathbb{R}$-indexed line ensemble satisfying Brownian Gibbs property whose top curve $\mathcal{A}_1(\cdot)$ is the parabolic Airy$_2$ process. The existence and uniqueness of $\mathcal{A}$ follow from [73] and [109] respectively.

The Airy line ensemble is in fact a strictly ordered line ensemble in the sense that almost surely,

$$\mathcal{A}_k(x) > \mathcal{A}_{k+1}(x) \text{ for all } k \in \mathbb{N}, x \in \mathbb{R}. \tag{2.2.1}$$

(2.2.1) follows from the Brownian Gibbs property and the fact that for each $x \in \mathbb{R}$, $(\mathcal{A}_k(x) + x^2)_{k\geq 1}$ is equal in distribution to the Airy point process. The latter is strictly ordered. In [96], the authors studied several probabilistic properties of the Airy line ensembles such as tail estimates and modulus of continuity. Below we state an extension of one of such results used later in our proof.

Proposition 2.2.4. *Fix $k \geq 1$. There exists a universal constant $\mathrm{C}_k > 0$ such that for all $m > 0$ and $R \geq 1$ we have*

$$\mathbf{P}\left(\sup_{\substack{x\neq y\in[-R,R]\\ |x-y|\leq 1}} \frac{|\mathcal{A}_k(x)+x^2-\mathcal{A}_k(y)-y^2|}{\sqrt{|x-y|}\log^{\frac{1}{2}}\frac{2}{|x-y|}} \geq m\right) \leq \mathrm{C}_k \cdot R\exp\left(-\tfrac{1}{\mathrm{C}_k}m^2\right). \tag{2.2.2}$$

Proof. Fix $k \geq 1$. By [96, Lemma 6.1] there exists a constant C_k such that for all $x, y \in \mathbb{R}$ with $|x-y| \leq 1$, we have

$$\mathbf{P}\left(|\mathcal{A}_k(x)+x^2-\mathcal{A}_k(y)-y^2| \geq m\sqrt{x-y}\right) \leq \mathrm{C}_k\exp\left(-\tfrac{1}{\mathrm{C}_k}m^2\right).$$

Thus applying Lemma 3.3 in [96] (with $d = 1, T = [-R, R], r_1 = 1, \alpha_1 = \frac{1}{2}, \beta_1 = 2$) and adjusting the value of C_k yields (2.2.2). □

2.2.2 KPZ line ensemble

Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \ldots)$ be an $\mathbb{N} \times \mathbb{R}$-indexed line ensemble. Fix $k_1 \leq k_2$ with $k_1, k_2 \in \mathbb{N}$ and an interval $(a, b) \in \mathbb{R}$ and two vectors $\vec{x}, \vec{y} \in \mathbb{R}^{k_2-k_1+1}$. Given a continuous function $\mathbf{H} : \mathbb{R} \to [0, \infty)$ (Hamiltonian) and two measurable functions $f, g : (a, b) \to \mathbb{R} \cup \{\pm\infty\}$, the law $\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ on $\mathcal{L}_{k_1}, \ldots, \mathcal{L}_{k_2} : (a, b) \to \mathbb{R}$ has the following Radon-Nikodym derivative with respect to $\mathbb{P}_{\text{free}}^{k_1,k_2,(a,b),\vec{x},\vec{y}}$, the law of $k_2 - k_1 + 1$ many independent Brownian bridges (with diffusion coefficient 1) taking values $\vec{x}$ at time a and $\vec{y}$ at time b:

$$\frac{d\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}}{d\mathbb{P}_{\text{free}}^{k_1,k_2,(a,b),\vec{x},\vec{y}}}(\mathcal{L}_{k_1}, \ldots, \mathcal{L}_{k_2}) = \frac{\exp\left\{ - \sum_{i=k_1}^{k_2+1} \int \mathbf{H}\big(\mathcal{L}_i(x) - \mathcal{L}_{i-1}(x)\big) dx \right\}}{Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}}, \tag{2.2.3}$$

where $\mathcal{L}_{k_1-1} = f$, or ∞ if $k_1 = 1$; and $\mathcal{L}_{k_2+1} = g$. Here, $Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ is the normalizing constant which produces a probability measure. We say $\mathcal{L}$ enjoys the **H**-*Brownian Gibbs property* if, for all $K = \{k_1, \ldots, k_2\} \subset \mathbb{N}$ and $(a, b) \subset \mathbb{R}$, the following distributional equality holds:

$$\text{Law}\Big(\mathcal{L}_{K\times(a,b)} \text{ conditioned on } \mathcal{L}_{\mathbb{N}\times\mathbb{R}\setminus K\times(a,b)}\Big) = \mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g},$$

where $\vec{x} = (\mathcal{L}_{k_1}(a), \ldots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \ldots, \mathcal{L}_{k_2}(b))$, and where again $\mathcal{L}_{k_1-1} = f$, or ∞ if $k_1 = 1$; and $\mathcal{L}_{k_2+1} = g$.

In the following text, we consider a specific class of **H** such that

$$\mathbf{H}_t(x) = t^{2/3} e^{t^{1/3} x}. \tag{2.2.4}$$

The next proposition then recalls the unscaled and scaled KPZ line ensemble constructed in [74] with $\mathbf{H}_t$-Brownian Gibbs property.

Proposition 2.2.5 (Theorem 2.15 in [74]). *Let* $t \geq 1$. *There exists an* $\mathbb{N} \times \mathbb{R}$*-indexed line ensemble* $\mathcal{H}_t = \{\mathcal{H}_t^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$ *such that:*

(a) the lowest indexed curve $\mathcal{H}_t^{(1)}(x)$ is equal in distribution (as a process in x) to the Cole-Hopf solution $\mathcal{H}(x,t)$ of the KPZ equation started from the narrow wedge initial data and the line ensemble $\mathcal{H}_t$ satisfies the $\mathbf{H}_1$-Brownian Gibbs property;

(b) the scaled KPZ line ensemble $\{\mathfrak{h}_t^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$, defined by

$$\mathfrak{h}_t^{(n)}(x) := t^{-1/3}\Big(\mathcal{H}_t^{(n)}\big(t^{2/3}x\big) + t/24\Big) \tag{2.2.5}$$

satisfies the $\mathbf{H}_t$-Brownian Gibbs property. Furthermore, for any interval $(a,b)\subset\mathbb{R}$ and $\varepsilon>0$, there exists $\delta>0$ such that, for all $t\geq 1$,

$$\mathbf{P}\left(Z_{\mathbf{H}_t}^{1,1,(a,b),\mathfrak{h}_t^{(1)}(a),\mathfrak{h}_t^{(1)}(b),\infty,\mathfrak{h}_t^{(2)}} < \delta\right) \leq \varepsilon,$$

where $Z_{\mathbf{H}_t}^{1,1,(a,b),\mathfrak{h}_t^{(1)}(a),\mathfrak{h}_t^{(1)}(b),\infty,\mathfrak{h}_t^{(2)}}$ is the normalizing constant defined in (2.2.3).

Remark 2.2.6. In part (3) of Theorem 2.15 [74] it is erroneously mentioned that the scaled KPZ line ensemble satisfies $\mathbf{H}_t$-Brownian Gibbs property with $\mathbf{H}_t(x) = e^{t^{1/3}x}$ (instead of $\mathbf{H}_t(x) = t^{2/3}e^{t^{1/3}x}$ from (2.2.4)). This error was reported by Milind Hegde and has been acknowledged by the authors of [74], who are currently preparing an errata for the same.

More recently, it has also been shown in [107] that the KPZ line ensemble as defined in Proposition 3.5.1 is unique as well. We will make extensive use of this scaled KPZ line ensemble $\mathfrak{h}_t^{(n)}(x)$ in our proofs in later sections. For $n=1$, we also adopt the shorthand notation:

$$\mathfrak{h}_t(x) := \mathfrak{h}_t^{(1)}(x) = t^{-1/3}\left(\mathcal{H}(t^{2/3}x,t) + \tfrac{t}{24}\right). \tag{2.2.6}$$

Note that for t large, the Radon-Nikodym derivative in (2.2.3) attaches heavy penalty if the curves are not ordered. Thus, intuitively at $t\to\infty$, one expects to get completely ordered curves, where the $\mathbf{H}_t$-Brownian Gibbs property will be replaced by the usual Brownian Gibbs property (see Definition 2.2.2) for non-intersecting Brownian bridges. Thus it's natural to expect the scaled KPZ line ensemble to converge to the Airy line ensemble. Along with the recent progress on the tightness of KPZ line ensemble [242] and characterization of Airy line ensemble [109], this remarkable result has been recently proved in [208].

Proposition 2.2.7 (Theorem 2.2 (4) in [208])**.** *Consider the KPZ line ensemble and the Airy line ensemble defined in Proposition 3.5.1 and Definition 2.2.3 respectively. For any $k \geq 1$, we have*

$$(2^{1/3}\mathfrak{h}_t^{(i)}(2^{1/3}x))_{i=1}^k \xrightarrow{d} (\mathcal{A}_i(x))_{i=1}^k,$$

in the uniform-on-compact topology.

The $2^{1/3}$ factor in Proposition 2.2.7 corrects the different diffusion coefficient used when we define the Brownian Gibbs property and $\mathbf{H}_t$ Brownian Gibbs property. We end this section by recalling several known results and tail estimates for the scaled KPZ equation with narrow wedge initial data.

Proposition 2.2.8. *Recall $\mathfrak{h}_t(x)$ from* (2.2.6)*. The following results hold:*

(a) For each $t > 0$, $\mathfrak{h}_t(x) + x^2/2$ is stationary in x.

(b) Fix $t_0 > 0$. There exists a constant $\mathrm{C} = \mathrm{C}(t_0) > 0$ such that for all $t \geq t_0$ and $m > 0$ we have

$$\mathbf{P}(|\mathfrak{h}_t(0)| \geq m) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^{3/2}\right).$$

(c) Fix $t_0 > 0$ and $\beta > 0$. There exists a constant $\mathrm{C} = \mathrm{C}(\beta, t_0) > 0$ such that for all $t \geq t_0$ and $m > 0$ we have

$$\mathbf{P}\left(\sup_{x\in\mathbb{R}} \left(\mathfrak{h}_t(x) + \tfrac{x^2}{2}(1-\beta)\right) \geq m\right) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^{3/2}\right).$$

The results in Proposition 2.2.8 is a culmination of results from several papers. Part (a) follows from [5, Corollary 1.3 and Proposition 1.4]. The one-point tail estimates for KPZ equation are obtained in [79, 80]. One can derive part (b) from those results or can combine the statements of Proposition 2.11 and 2.12 in [81] to get the same. Part (c) is Proposition 4.2 from [81].

2.3 Uniqueness and convergence of random modes

In this section we prove the uniqueness of random modes that appears in Theorems 2.1.4 and 2.1.5 and prove Theorem 2.1.8 which claims the convergences of random modes to appropriate limits. The following lemma settles the uniqueness question.

Lemma 2.3.1. *Fix $p \in (0,1)$ and $t > 0$. Recall $f_{p,t}$ and $f_{*,t}$ from Theorem 2.1.4 and 2.1.5. Then $f_{*,t}$ has almost surely a unique mode $\mathcal{M}_{*,t}$ and $f_{p,t}$ has almost surely a unique mode $\mathcal{M}_{p,t}$. Furthermore for any $t_0 > 0$, there exist a constant $\mathrm{C}(p, t_0) > 0$ such that for all $t > t_0$ we have*

$$\mathbf{P}(t^{-2/3}|\mathcal{M}_{p,t}| > m) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right), \quad \textit{and} \quad \mathbf{P}(t^{-2/3}|\mathcal{M}_{*,t}| > m) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right). \tag{2.3.1}$$

Proof. We first prove the point-to-point case. Fix $p \in (0,1)$ and set $q = 1-p$. Take $t > 0$. Throughout the proof $\mathrm{C} > 0$ will depend on p, we won't mention this further.

Note that (3.1.5) implies that the density $f_{p,t}(x)$ is proportional to $\mathcal{Z}(0,0;x,pt)\mathcal{Z}(x,pt;0,t)$ and that $\mathcal{Z}(0,0;x,pt)$ and $\mathcal{Z}(x,pt;0,t)$ are independent. By time reversal property of SHE we have $\mathcal{Z}(x,pt;0,t) \stackrel{d}{=} \mathcal{H}(x,qt)$ as functions in x. Using the $1:2:3$ scaling from (2.2.6) we may write

$$f_{p,t}(x) \stackrel{d}{=} \frac{1}{\widetilde{Z}_{p,t}} \exp\left(t^{1/3}p^{1/3}\mathfrak{h}_{pt,\uparrow}(p^{-2/3}t^{-2/3}x) + t^{1/3}q^{1/3}\mathfrak{h}_{qt,\downarrow}(q^{-2/3}t^{-2/3}x)\right) \tag{2.3.2}$$

where $\mathfrak{h}_{t,\uparrow}(x)$ and $\mathfrak{h}_{t,\downarrow}(x)$ are independent copies of the scaled KPZ line ensemble $\mathfrak{h}_t(x)$ defined in (2.2.6) and $\widetilde{Z}_{p,t}$ is the normalizing constant. Thus it suffices to study the maximizer of

$$\mathcal{S}_{p,t}(x) := p^{1/3}\mathfrak{h}_{pt,\uparrow}(p^{-2/3}x) + q^{1/3}\mathfrak{h}_{qt,\downarrow}(q^{-2/3}x). \tag{2.3.3}$$

Note that maximizer of $f_{p,t}$ can be retrieved from that of $\mathcal{S}_{p,t}$ by a $t^{-2/3}$ scaling.

We first claim that for all $m > 0$ we have

$$\mathbf{P}(\mathsf{A}_1) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right), \quad \text{where } \mathsf{A}_1 := \left\{\mathfrak{h}_{pt,\uparrow}(p^{-2/3}x) > \mathfrak{h}_{pt,\uparrow}(0) \text{ for some } |x| > m\right\} \tag{2.3.4}$$

$$\mathbf{P}(\mathsf{A}_2) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right), \quad \text{where } \mathsf{A}_2 := \left\{\mathfrak{h}_{qt,\downarrow}(q^{-2/3}x) > \mathfrak{h}_{qt,\downarrow}(0) \text{ for some } |x| > m\right\}. \tag{2.3.5}$$

Let us prove (2.3.4). Define

$$\mathsf{D}_1 := \left\{\sup_{x\in\mathbb{R}}\left(\mathfrak{h}_{pt,\uparrow}(p^{-2/3}x) + \frac{x^2}{4p^{4/3}}\right) \le \frac{m^2}{8p^{4/3}}\right\}, \quad \mathsf{D}_2 := \left\{|\mathfrak{h}_{pt,\uparrow}(0)| \le \frac{m^2}{16p^{4/3}}\right\}.$$

Note that on D_2, $\mathfrak{h}_{pt,\uparrow}(0) \in [-\frac{m^2}{16p^{4/3}}, \frac{m^2}{16p^{4/3}}]$, whereas on D_1, for all $|x| > m$ we have

$$\mathfrak{h}_{pt,\uparrow}(p^{-2/3}x) < \tfrac{m^2}{8p^{4/3}} - \tfrac{m^2}{4p^{4/3}} = -\tfrac{m^2}{8p^{4/3}}.$$

Thus $\mathsf{A}_1 \subset \neg\mathsf{D}_1 \cup \neg\mathsf{D}_2$ where A_1 is defined in (2.3.4). On the other hand, by Proposition 2.2.8(c) with $\beta = \frac{1}{2}$ and Proposition 2.2.8 (b) we have

$$\mathbf{P}(\mathsf{D}_1) > 1 - \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right), \quad \mathbf{P}(\mathsf{D}_2) > 1 - \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right).$$

Hence by union bound we get $\mathbf{P}(\mathsf{A}_1) \le \mathbf{P}(\neg\mathsf{D}_1) + \mathbf{P}(\neg\mathsf{D}_2) \le \mathrm{C}\exp(-\frac{1}{\mathrm{C}}m^3)$. This proves (2.3.4). Proof of (2.3.5) is analogous.

Now via the Brownian Gibbs property $\mathfrak{h}_t$ is absolute continuous w.r.t. Brownian motion on every compact interval. Hence for each $t > 0$, $\mathcal{S}_{p,t}(x)$ defined in (2.3.3) has a unique maximum on any compact interval almost surely. But due to the bounds in (2.3.4) and (2.3.5), we see that

$$\mathbf{P}\left(\mathcal{S}_{p,t}(x) > \mathcal{S}_{p,t}(0) \text{ for some } |x| > m\right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right). \tag{2.3.6}$$

Thus $\mathcal{S}_{p,t}(\cdot)$ has a unique maximizer almost surely. By the definitions of $f_{p,t}(x)$ and $\mathcal{S}_{p,t}(x)$ from

(2.3.2) and (2.3.3), this implies $f_{p,t}(x)$ also has a unique maximizer $\mathcal{M}_{p,t}$ and we have that

$$\mathcal{M}_{p,t} \stackrel{d}{=} t^{2/3} \operatorname*{argmax}_{x\in\mathbb{R}} \mathcal{S}_{p,t}(x). \tag{2.3.7}$$

In view of (2.3.6), the above relation (2.3.7) leads to the first inequality in (2.3.1).

For the point-to-line case, note that via (3.1.6) and (2.2.6), $f_{*,t}(x)$ is proportional to $\exp(t^{1/3}\mathfrak{h}_t(t^{-2/3}x))$. The proofs of uniqueness of the maximizer and the second bound in (2.3.1) then follow by analogous arguments. This completes the proof. □

In the course of proving the above lemma, we have also proved an important result that connects the random modes to the maximizers of the KPZ equations. We isolate this result as a separate lemma.

Lemma 2.3.2. *Consider three independent copies $\mathcal{H}, \mathcal{H}_\uparrow, \mathcal{H}_\downarrow$ of the KPZ equation started from the narrow wedge initial data. The random mode $\mathcal{M}_{p,t}$ of $f_{p,t}$ (defined in statement of Theorem 2.1.4) is same in distribution as the maximizer of*

$$\mathcal{H}_\uparrow(x, pt) + \mathcal{H}_\downarrow(x, qt).$$

Similarly one has that the random mode $\mathcal{M}_{*,t}$ of $f_{*,t}$ (defined in statement of Theorem 2.1.5) is same in distribution as the maximizer of $\mathcal{H}(x,t)$.

Proof of Theorem 2.1.8. Due to the identity in (2.3.7) we see that $t^{-2/3}\mathcal{M}_{p,t}$ is same in distribution as

$$\operatorname*{argmax}_{x\in\mathbb{R}} S_{p,t}(x)$$

where $\mathcal{S}_{p,t}(x)$ is defined in (2.3.3). By Proposition 2.2.7 we see that as $t\to\infty$

$$\mathcal{S}_{p,t}(x) \xrightarrow{d} 2^{-1/3}\left(p^{1/3}\mathcal{A}_{1,\uparrow}(2^{-1/3}p^{-2/3}x) + q^{1/3}\mathcal{A}_{1,\downarrow}(2^{-1/3}q^{-2/3}x)\right)$$

in the uniform-on-compact topology where $\mathcal{A}_{1,\uparrow}, \mathcal{A}_{1,\downarrow}$ are independent parabolic Airy$_2$ processes. Note that the expression in the r.h.s. of the above equation is the same as

$$\mathcal{A}(x) := 2^{-1/2} \left(\mathcal{A}_{\uparrow}^{(p\sqrt{2})}(x) + \mathcal{A}_{\downarrow}^{(q\sqrt{2})}(x) \right) \tag{2.3.8}$$

where $\mathcal{A}_{\uparrow}^{(p\sqrt{2})}(x), \mathcal{A}_{\downarrow}^{(q\sqrt{2})}(x)$ are independent Airy sheets of index $p\sqrt{2}$ and $q\sqrt{2}$ respectively. By Lemma 9.5 in [94] we know that $\mathcal{A}(x)$ has almost surely a unique maximizer on every compact set. Thus,

$$\operatorname*{argmax}_{x\in[-K,K]} \mathcal{S}_{p,t}(x) \xrightarrow{d} \operatorname*{argmax}_{x\in[-K,K]} \mathcal{A}(x). \tag{2.3.9}$$

Finally the decay bounds for the maximizer of $\mathcal{S}_{p,t}(x)$ from Lemma 2.3.1 and the decay bounds for the maximizer of $\mathcal{A}$ from [207] allow us to extend the weak convergence to the case of maximizers on the full line. However, by the definition of the geodesic of the directed landscape from Definition 3.1.6, we see that $\Gamma(p\sqrt{2}) \stackrel{d}{=} \operatorname{argmax}_{x\in\mathbb{R}} \mathcal{A}(x)$. This concludes the proof for the point-to-point case. For the point-to-line case, following Lemma 2.3.2 and recalling again the scaled KPZ line ensemble from (2.2.6), we have

$$2^{-1/3}t^{-2/3}\mathcal{M}_{*,t} = \operatorname*{argmax}_{x\in\mathbb{R}} \mathcal{H}(2^{1/3}t^{2/3}x, t) = \operatorname*{argmax}_{x\in\mathbb{R}} \left(t^{1/3}\mathfrak{h}_t(2^{1/3}x) - \tfrac{t}{24} \right) = \operatorname*{argmax}_{x\in\mathbb{R}} 2^{1/3}\mathfrak{h}_t(2^{1/3}x).$$

From Proposition 2.2.7 we know $2^{1/3}\mathfrak{h}_t(2^{1/3}x)$ converges in distribution to $\mathcal{A}_1(x)$ in the uniform-on-compact topology. Given the decay estimates for $\mathcal{M}_{*,t}$ from (2.3.1) and decay bounds for the maximizer of $\mathcal{A}_1$ from [94], we thus get that $\operatorname{argmax}_{x\in\mathbb{R}} 2^{1/3}\mathfrak{h}_t(2^{1/3}x)$ converges in distribution to $\mathcal{M}$, the unique maximizer of the parabolic Airy$_2$ process. This completes the proof. □

2.4 Decomposition of Brownian bridges around joint maximum

The goal of this section is to prove certain decomposition properties of Brownian bridges around the joint maximum outlined in Proposition 2.4.8 and Proposition 2.4.9. To achieve this goal, we first discuss several Brownian objects and their related properties in Section 2.4.1 which will be foundational for the rest of the paper. Then we prove Proposition 2.4.8 and 2.4.9 in the ensuing subsection. We refer to Figure 2.4 for the structure and various Brownian laws convergences in this and the next sections. The notation $p_t(y) := (2\pi t)^{-1/2} e^{-y^2/(2t)}$ for the standard heat kernel will appear throughout the rest of the paper.

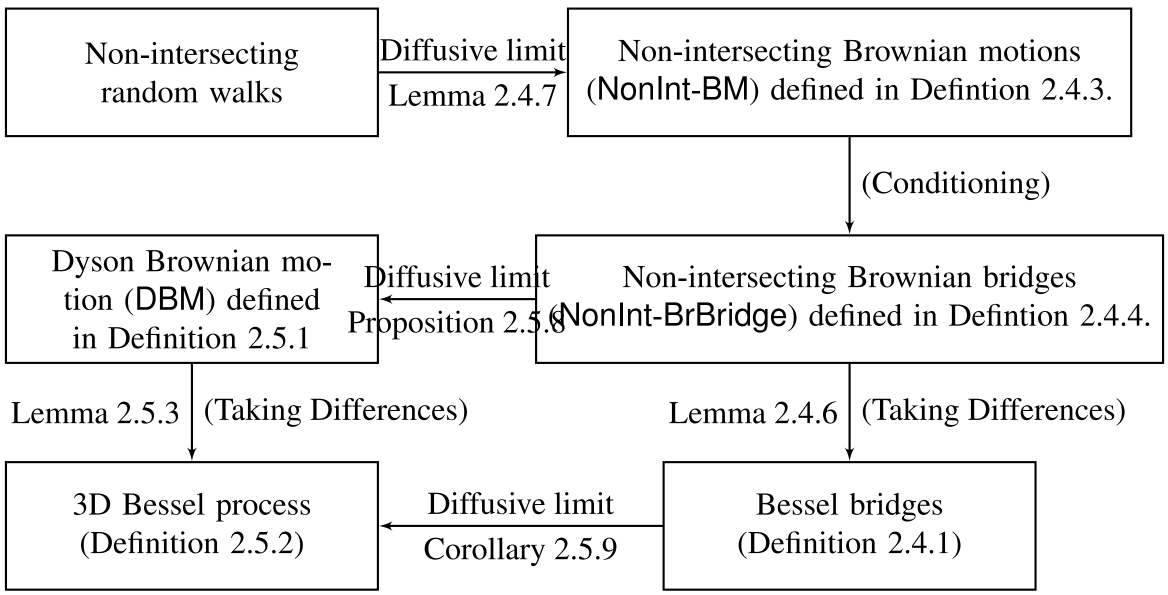


Figure 2.4: Relationship between different laws used in Sections 2.4 and 2.5.

2.4.1 Brownian objects

In this section we recall several objects related to Brownian motion, including the Brownian meanders, Bessel bridges, non-intersecting Brownian motions and non-intersecting Brownian bridges.

Definition 2.4.1 (Brownian meanders and Bessel bridges)**.** Given a standard Brownian motion $B(\cdot)$ on $[0,1]$, a standard Brownian meander $\mathfrak{B}_{\text{me}} : [0,1] \to \mathbb{R}$ is a process defined by

$$\mathfrak{B}_{\text{me}}(x) = (1-\theta)^{-\frac{1}{2}} |B(\theta + (1-\theta)x)|, \quad x \in [0,1],$$

where $\theta = \sup\{x \in [0,1] \mid B(x) = 0\}$. In general, we say a process $\mathfrak{B}_{\text{me}} : [a,b] \to \mathbb{R}$ is a Brownian meander on $[a,b]$ if

$$\mathfrak{B}'_{\text{me}}(x) := (b-a)^{-\frac{1}{2}}\mathfrak{B}_{\text{me}}(a + x(b-a)), \quad x \in [0,1]$$

is a standard Brownian meander. A Bessel bridge $\mathcal{R}_{\text{bb}}$ on $[a,b]$ ending at $y > 0$ is a Brownian meander $\mathfrak{B}_{\text{me}}$ on $[a,b]$ subject to the condition (in the sense of Doob) $\mathfrak{B}_{\text{me}}(b) = y$.

A Bessel bridge can also be realized as conditioning a 3D Bessel process to end at some point and hence the name. As we will not make use of this fact, we do not prove this in the paper.

Lemma 2.4.2 (Transition densities for Bessel Bridge)**.** *Let V be a Bessel bridge on $[0,1]$ ending at a. Then for $0 < t < 1$,*

$$\mathbf{P}(V(t) \in dx) = \frac{x}{at}\frac{p_t(x)}{p_1(a)}[p_{1-t}(x-a) - p_{1-t}(x+a)]dx, \quad x \in [0,\infty).$$

For $0 < s < t < 1$ and $x > 0$,

$$\mathbf{P}(V(t) \in dy \mid V(s) = x) = \frac{[p_{t-s}(x-y) - p_{t-s}(x+y)][p_{1-t}(y-a) - p_{1-t}(y+a)]}{[p_{1-s}(x-a) - p_{1-s}(x+a)]}dy, \quad y \in [0,\infty).$$

Proof. We recall the joint density formula for Brownian meander W on $[0,1]$ from [150]. For $0 = t_0 < t_1 < t_2 < \cdots < t_k \le 1$:

$$\mathbf{P}(W(t_1) \in dx_1, \ldots, W(t_k) \in dx_k) = \frac{x_1}{t}p_{t_1}(x_1)\Psi(\tfrac{x_k}{\sqrt{1-t_k}})\prod_{j=1}^{k-1} g(x_j, x_{j+1}; t_{j+1} - t_j)\prod_{j=1}^{k} dx_j$$

where

$$g(x_j, x_{j+1}; t_{j+1} - t_j) := [p_{t_{j+1}-t_j}(x_j - x_{j+1}) - p_{t_{j+1}-t_j}(x_j + x_{j+1})],$$
$$\Psi(x) := (2/\pi)^{\frac{1}{2}}\int_0^x e^{-u^2/2}du.$$

The joint density is supported on $[0,\infty)^k$. We now use Doob-h transform to get the joint density for Bessel bridge on $[0,1]$ ending at a. For $0 = t_0 < t_1 < t_2 < \cdots < t_k < 1$:

$$\mathbf{P}(V(t_1) \in dx_1, \ldots, V(t_k) \in dx_k) = \frac{x_1}{at_1} \frac{p_{t_1}(x_1)}{p_1(a)} \prod_{j=1}^{k} g(x_j, x_{j+1}; t_{j+1} - t_j) \prod_{j=1}^{k} dx_j$$

where $x_{k+1} = a$ and $t_{k+1} = 1$. Formulas in Lemma 2.4.2 is obtained easily from the above joint density formula. □

Definition 2.4.3 (Non-intersecting Brownian motions)**.** We say a random continuous function $W(t) = (W_1(t), W_2(t)) : [0,1] \to \mathbb{R}^2$ is a pair of non-intersecting Brownian motion (NonInt-BM in short) if its distribution is given by the following formulas:

(a) We have for any $y_1, y_2 \in \mathbb{R}$

$$\mathbf{P}(W_1(1) \in \mathrm{d}y_1, W_2(1) \in \mathrm{d}y_2) = \frac{\mathbf{1}\{y_1 > y_2\}(y_1 - y_2)p_1(y_1)p_1(y_2)}{\int_{r_1>r_2}(r_1 - r_2)p_1(r_1)p_1(r_2)\mathrm{d}r_1\mathrm{d}r_2}\mathrm{d}y_1\mathrm{d}y_2. \tag{2.4.1}$$

(b) For $0 < t < 1$, we have

$$\begin{aligned}&\mathbf{P}(W_1(t) \in \mathrm{d}y_1, W_2(t) \in \mathrm{d}y_2)\\ &= \frac{\mathbf{1}\{y_1 > y_2\}(y_1 - y_2)p_t(y_1)p_t(y_2) \int_{r_1>r_2} \det(p_{1-t}(y_i - r_j))_{i,j=1}^2 \mathrm{d}r_1\mathrm{d}r_2}{t \int_{r_1>r_2}(r_1 - r_2)p_1(r_1)p_1(r_2)\mathrm{d}r_1\mathrm{d}r_2}\mathrm{d}y_1\mathrm{d}y_2.\end{aligned} \tag{2.4.2}$$

(c) For $0 < s < t \le 1$ and $x_1 > x_2$, we have

$$\begin{aligned}&\mathbf{P}(W_1(t) \in \mathrm{d}y_1, W_2(t) \in \mathrm{d}y_2 | W_1(s) = x_1, W_2(s) = x_2)\\ &= \mathbf{1}\{y_1 > y_2\}\frac{\det(p_{t-s}(y_i - x_j))_{i,j=1}^2 \int_{r_1>r_2} \det(p_{1-t}(y_i - r_j))_{i,j=1}^2 \mathrm{d}r_1\mathrm{d}r_2}{\int_{r_1>r_2} \det(p_{1-s}(x_i - r_j))_{i,j=1}^2 \mathrm{d}r_1\mathrm{d}r_2}\mathrm{d}y_1\mathrm{d}y_2.\end{aligned} \tag{2.4.3}$$

We call $W^{[0,M]}$ a NonInt-BM on $[0,M]$ if $(M^{-1/2}W_1^{[0,M]}(Mx), M^{-1/2}W_2^{[0,M]}(Mx))$ is a NonInt-BM on $[0,1]$.

Definition 2.4.4 (Non intersection Brownian bridges)**.** A 2-level non-intersecting Brownian bridge (NonInt-BrBridge in short) $V = (V_1, V_2)$ on $[0, 1]$ ending at (z_1, z_2) with $(z_1 \neq z_2)$ is a NonInt-BM on $[0, 1]$ defined in Definition 2.4.3 subject to the condition (in the sense of Doob) $V(1) = z_1, V(1) = z_2$. It is straight forward to check we have the following formulas for the distribution of V:

(a) For $0 < t < 1$, we have

$$\mathbf{P}(V_1(t) \in \mathrm{d}y_1, V_2(t) \in \mathrm{d}y_2) = \frac{\mathbf{1}\{y_1 > y_2\}(y_1 - y_2)p_t(y_1)p_t(y_2)}{t(z_1 - z_2)p_1(z_1)p_1(z_2)} \det(p_{1-t}(y_i - z_j))_{i,j=1}^2 \mathrm{d}y_1\mathrm{d}y_2.$$

(b) For $0 < s < t \leq 1$ and $x_1 > x_2$, we have

$$\begin{aligned}\mathbf{P}(V_1(t) \in \mathrm{d}y_1, V_2(t) \in \mathrm{d}y_2 | V_1(s) = x_1, V_2(s) = x_2)& \\ = \frac{\det(p_{t-s}(y_i - x_j))_{i,j=1}^2 \det(p_{1-t}(y_i - z_j))_{i,j=1}^2}{\det(p_{1-s}(x_i - z_j))_{i,j=1}^2}\mathrm{d}y_1\mathrm{d}y_2.&\end{aligned}$$

Just like NonInt-BM, we call $V^{[0,M]}$ a NonInt-BrBridge on $[0, M]$ if $(\frac{1}{\sqrt{M}}V_1^{[0,M]}(Mx), \frac{1}{\sqrt{M}}V_2^{[0,M]}(Mx))$ is a NonInt-BrBridge on $[0, 1]$.

Remark 2.4.5. It is possible to specify the distributions for a n-level non-intersecting Brownian bridge. However, the notations tend to get cumbersome due to the possibility of some paths sharing the same end points. We refer to Definition 8.1 in [108] for a flavor of such formulas. We remark that in this paper we will focus exclusively on the $n = 2$ case with distinct endpoints.

The following Lemma connects NonInt-BrBridge with Bessel bridges.

Lemma 2.4.6 (NonInt-BrBridge to Bessel bridges)**.** *Let $V = (V_1, V_2)$ be a* NonInt-BrBridge *on $[0, 1]$ ending at (z_1, z_2) with $z_1 > z_2$. Then, as functions in x, we have $V_1(x) - V_2(x) \stackrel{d}{=} \sqrt{2}\mathcal{R}_{\mathrm{bb}}(x)$ where $\mathcal{R}_{\mathrm{bb}} : [0, 1] \to \mathbb{R}$ is a Bessel bridge (see Definition 2.4.1) ending at $(z_1 - z_2)/\sqrt{2}$.*

The proof of Lemma 2.4.6 is based on the following technical lemma that discusses how NonInt-BM comes up as a limit of non-intersecting random walks.

Lemma 2.4.7. *Let X_j^i be i.i.d.* $\mathrm{N}(0,1)$ *random variables. Let* $S_0^{(i)} = 0$ *and* $S_k^{(i)} = \sum_{j=1}^k X_j^i$. *Consider* $Y_n(t) = (Y_{n,1}(t), Y_{n,2}(t)) := (\frac{S_{nt}^{(1)}}{\sqrt{n}}, \frac{S_{nt}^{(2)}}{\sqrt{n}})$ *an* $\mathbb{R}^2$ *valued process on* $[0,1]$ *where the in-between points are defined by linear interpolation. Then conditioned on the non-intersecting event* $\Lambda_n := \cap_{j=1}^n \{S_j^{(1)} > S_j^{(2)}\}$, $Y_n \xrightarrow{d} W$, *where* $W(t) = (W_1(t), W_2(t))$ *is distributed as* NonInt-BM *defined in Definition 2.4.3.*

We defer the proof of this lemma to the Appendix as it roughly follows standard calculations based on the Karlin-McGregor formula [161].

Proof of Lemma 2.4.6. Let X_j^i to be i.i.d. $\mathrm{N}(0,1)$ random variables. Let $S_0^{(i)} = 0$ and $S_k^{(i)} = \sum_{j=1}^k X_j^i$. Set $Y_n(t) = (\frac{S_{nt}^{(1)}}{\sqrt{n}}, \frac{S_{nt}^{(2)}}{\sqrt{n}})$ an $\mathbb{R}^2$ valued process on $[0,1]$ where the in-between points are defined by linear interpolation. By Lemma 2.4.7, conditioned on the non-intersecting event $\Lambda_n := \cap_{j=1}^n \{S_j^{(1)} > S_j^{(2)}\}$, Y_n converges to $W = (W_1, W_2)$, a NonInt-BM on $[0,1]$ defined in Definition 2.4.3. On the other hand, classical results from [150] tell us, $(S_{nt}^{(1)} - S_{nt}^{(2)})/\sqrt{n}$ conditioned on Λ_n converges weakly to $\sqrt{2}\mathfrak{B}_{\mathrm{me}}(t)$, where $\mathfrak{B}_{\mathrm{me}}(\cdot)$ is a Brownian meander defined in Definition 2.4.1. The $\sqrt{2}$ factor comes because $S_k^{(1)} - S_k^{(2)}$ is random walk with variance 2. Thus

$$W_1(\cdot) - W_2(\cdot) \stackrel{d}{=} \sqrt{2}\mathfrak{B}_{\mathrm{me}}(\cdot).$$

From [150], $\mathfrak{B}_{\mathrm{me}}$ is known to be Markov process. Hence the law of $W_1 - W_2$ depends on $(W_1(1), W_2(1))$ only through $W_1(1) - W_2(1)$. In particular conditioning on $(W_1(1) = z_1, W_2(1) = z_2)$, for any $z_1 > z_2$, makes W to be a NonInt-BrBridge on $[0,1]$ ending at (z_1, z_2) and the conditional law of $\frac{1}{\sqrt{2}}(W_1 - W_2)$ is then a Bessel bridge ending at $\frac{1}{\sqrt{2}}(z_1 - z_2)$. This completes the proof. □

2.4.2 Decomposition Results

In this section we prove two path decomposition results around the maximum for a single Brownian bridge and for a sum of two Brownian bridges. The first one is for a single Brownian bridge.

Proposition 2.4.8 (Bessel bridge decomposition)**.** *Let $\overline{B} : [a,b] \to \mathbb{R}$ be a Brownian bridge with $\overline{B}(a) = x$ and $\overline{B}(b) = y$. Let M be the almost sure unique maximizer of $\overline{B}$. Consider $W_\ell : [a,M] \to \mathbb{R}$ defined as $W_\ell(x) = \overline{B}(M) - \overline{B}(M+a-x)$, and $W_{\mathrm{r}} : [M,b] \to \mathbb{R}$ defined as $W_{\mathrm{r}}(x) = \overline{B}(M) - \overline{B}(x)$. Then, conditioned on $(M, \overline{B}(M))$,*

(a) $W_\ell(\cdot)$ and $W_{\mathrm{r}}(\cdot)$ are independent.

(b) $W_\ell(\cdot)$ is a Bessel bridge on $[a,M]$ starting at zero and ending at $\overline{B}(M) - x$.

(c) $W_{\mathrm{r}}(\cdot)$ is a Bessel bridge on $[M,b]$ starting at zero and ending at $\overline{B}(M) - y$.

Recall that the Bessel bridges are defined in Definition 2.4.1.

Proof. We will prove the result for $a = 0$, $b = 1$ and $x = 0$; the general case then follows from Brownian scaling and translation property of bridges. We recall the classical result of Brownian motion decomposition around maximum from [101]. Consider a map $\Upsilon : C([0,1]) \to C([0,1]) \times C([0,1])$ given by

$$(\Upsilon f)_-(t) := M^{-\frac{1}{2}}[f(M) - f(M(1-t))], \quad t \in [0,1],$$
$$(\Upsilon f)_+(t) := (1-M)^{-\frac{1}{2}}[f(M) - f(M+(1-M)t)], \quad t \in [0,1],$$

where $M = M(f) := \inf\{t \in [0,1] \mid f(s) \le f(t), 0 \le s \le 1\}$ is the left-most maximizer of f. We set $(\Upsilon f)_- \equiv (\Upsilon f)_+ \equiv 0$ if $M = 0$ or $M = 1$. We also define

$$(\Upsilon^M f)_-(t) := M^{1/2}(\Upsilon f)_-(\tfrac{t}{M}), \quad t \in [0,M],$$
$$(\Upsilon^M f)_+(t) := (1-M)^{\frac{1}{2}}(\Upsilon f)_+(\tfrac{t-M}{1-M}), \quad t \in [M,1].$$

Given a standard Brownian motion B on $[0,1]$, by Theorem 1 in [101], $\Upsilon(B)$ is independent of $M = M(B)$ and $\Upsilon(B)_-$ and $\Upsilon(B)_+$ are independent Brownian meanders on $[0,1]$. By the Brownian scaling and the fact that $\Upsilon(B)$ is independent of $M(B)$, conditioned on $M(B)$, we see that $(\Upsilon^M B)_-$ and $(\Upsilon^M B)_+$ are independent Brownian meanders on $[0,M]$ and $[M,1]$ respectively. Observe that

$(\Upsilon^M f)_-(M) = f(M)$ and $(\Upsilon^M f)_+(1) = f(M) - f(1)$ for any $f \in C([0,1])$. Thus conditioning on $(B(M) = v, B(1) = y)$ is equivalent to conditioning on $((\Upsilon^M B)_-(M) = v, (\Upsilon^M B)_+(1) = v - y)$. By definition the conditional law of Brownian meanders upon conditioning their end points are Bessel bridges. Thus conditioning on $(M = m, B(M) = v, B(1) = y)$, we see that $(\Upsilon^M B)_-$ and $(\Upsilon^M B)_+$ are independent Bessel bridges on $[0, M]$ and $[M, 1]$ ending at v and $v - y$ respectively. But the law of a Brownian motion conditioned on $(M = m, B(M) = v, B(1) = y)$ is same as the law of a Brownian bridge $\overline{B}$ on $[0,1]$ starting at 0 and ending at y, conditioned on $(M(\overline{B}) = m, \overline{B}(M) = v)$. Identifying $(\Upsilon^M \overline{B})_-$ and $(\Upsilon^M \overline{B})_+$ with W_ℓ and W_{r} gives us the desired result. □

The next proposition show that for two Brownian bridges the decomposition around the joint maximum is given by non-intersecting Brownian bridges.

Proposition 2.4.9 (Non-intersecting Brownian bridges decomposition)**.** *Let $\overline{B}_1, \overline{B}_2 : [a,b] \to \mathbb{R}$ be independent Brownian bridges such that $\overline{B}_i(a) = x_i, \overline{B}_i(b) = y_i$. Let M be the almost sure unique maximizer of $(\overline{B}_1(x) + \overline{B}_2(x))$ on $[0,1]$. Define $\overline{V}_\ell(x) : [0, M-a] \to \mathbb{R}^2$ and $\overline{V}_r : [0, b-M] \to \mathbb{R}^2$ as follows:*

$$\overline{V}_\ell(x) := (\overline{B}_1(M) - \overline{B}_1(M-x), -\overline{B}_2(M) + \overline{B}_2(M-x))$$
$$\overline{V}_r(x) := (\overline{B}_1(M) - \overline{B}_1(M+x), -\overline{B}_2(M) + \overline{B}_2(M+x))$$

Then, conditioned on $(M, \overline{B}_1(M), \overline{B}_2(M))$,

(a) $\overline{V}_\ell(\cdot)$ and $\overline{V}_r(\cdot)$ are independent.

(b) $\overline{V}_\ell(\cdot)$ is a NonInt-BrBridge *on $[0, M-a]$ ending at $(\overline{B}_1(M) - x_1, x_2 - \overline{B}_2(M))$.*

(c) $\overline{V}_r(\cdot)$ is a NonInt-BrBridge *on $[0, b-M]$ ending at $(\overline{B}_1(M) - y_1, y_2 - \overline{B}_2(M))$.*

Recall that NonInt-BrBridge*s are defined in Definition 2.4.4.*

As in the proof of Proposition 2.4.8, to prove Proposition 2.4.9 we rely on a decomposition result for Brownian motions instead. To state the result we introduce the Ω map which encodes the trajectories of around the joint maximum of the sum of two functions.

Definition 2.4.10. For any $f = (f_1, f_2) \in C([0,1] \to \mathbb{R}^2)$, we define $\Omega f \in C([-1,1] \to \mathbb{R}^2)$ as follows:

$$(\Omega f)_1(t) = \begin{cases} [f_1(M) - f_1(M(1+t))]M^{-1/2} & -1 \le t \le 0 \\ [f_1(M) - f_1(M + (1-M)t)](1-M)^{-1/2} & 0 \le t \le 1 \end{cases}$$

$$(\Omega f)_2(t) = \begin{cases} -[f_2(M) - f_2(M(1+t))]M^{-1/2} & -1 \le t \le 0 \\ -[f_2(M) - f_2(M + (1-M)t)](1-M)^{-1/2} & 0 \le t \le 1 \end{cases}$$

where $M = \inf\{t \in [0,1] : f_1(s) + f_2(s) \le f_1(t) + f_2(t), \forall s \in [0,1]\}$ is the left most maximizer. We set $(\Omega f) \equiv (0,0)$ if $M = 0$ or 1 on $[0,1]$. With this we define two functions in $C([0,1] \to \mathbb{R}^2)$ as follows

$$(\Omega f)_+(x) := ((\Omega f)_1(x), (\Omega f)_2(x)), \quad x \in [0,1]$$
$$(\Omega f)_-(x) := ((\Omega f)_1(-x), (\Omega f)_2(-x)), \quad x \in [0,1].$$

We are now ready to state the corresponding result for Brownian motions.

Lemma 2.4.11. *Suppose $B = (B_1, B_2)$ are independent Brownian motions on $[0,1]$ with $B_i(0) = x_i$. Let*

$$M = \underset{t \in [0,1]}{\operatorname{argmax}} (B_1(t) + B_2(t)).$$

Then $(\Omega B)_+, (\Omega B)_-$ are independent and distributed as non-intersecting Brownian motions on $[0,1]$ (see Definition 2.4.3). Furthermore, $(\Omega B)_+, (\Omega B)_-$ are independent of M.

We first complete the proof of Proposition 2.4.9 assuming the above Lemma.

Proof of Proposition 2.4.9. Without loss of generality, we set $a = 0$ and $b = 1$. Let $B_1, B_2 : [0,1] \to \mathbb{R}$ be two independent Brownian bridges with $B_i(0) = x_i$ and denote $M = \operatorname{argmax}_{x \in [0,1]} B_1(x) + B_2(x)$. Consider

$$V_\ell(x) := (B_1(M) - B_1(M - x), -B_2(M) + B_2(M - x))_{x \in [0,M]}$$

$$V_r(x) := (B_1(M) - B_1(M+x), -B_2(M) + B_2(M+x))_{x\in[0,1-M]}$$

By Lemma 2.4.11, conditioned on M and after Brownian re-scaling, we have where V_r, V_ℓ are conditionally independent and $V_r \sim W^{[0,1-M]}$ and $V_\ell \sim W^{[0,M]}$ where $W^{[0,\rho]}$ denote a NonInt-BM on $[0,\rho]$ defined in Definition 2.4.4. To convert the above construction to Brownian bridges, we observe that the map

$$(B_1(M), B_2(M), B_1(1), B_2(1)) \leftrightarrow (V_r(1-M), V_\ell(M))$$

is bijective. Indeed, we have that

$$\begin{pmatrix} B_1(M) = b_1, \ B_2(M) = b_2 \\ B_1(1) = y_1, \ B_2(1) = y_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} V_r(1-M) = (b_1 - y_1, -b_2 + y_2) \\ V_\ell(M) = (b_1 - x_1, -b_2 + x_2) \end{pmatrix}.$$

Thus conditioned on $(M = m, B_i(M) = b_i, B_i(1) = y_i)$, $V_r(\cdot)$ is now a NonInt-BrBridge Brownian bridge on $[0, 1-m]$ ending at $(b_1 - y_1, -b_2 + y_2)$ and $V_\ell(\cdot)$ is a NonInt-BrBridge on $[0, m]$ ending at $(b_1 - x_1, -b_2 + x_2)$ where both are conditionally independent of each other. But the law of a Brownian motions conditioned on $(M = m, B_i(M) = b_i, B_i(1) = y_i)$ is same as the law of a Brownian bridges $\overline{B}$ on $[0,1]$ starting at (x_1, x_2) and ending at (y_1, y_2), conditioned on $(M = m, \overline{B}_i(M) = b_i)$. Thus this leads to the desired decomposition for Brownian bridges. □

Let us now prove Lemma 2.4.11. The proof of Lemma 2.4.11 follows similar ideas from [101] and [150]. To prove such a decomposition holds, we first show it at the level of random walks. Then we take diffusive limit to get the same decomposition for Brownian motions.

Proof of Lemma 2.4.11. Let $X_j^{(i)} \overset{i.i.d.}{\sim} \mathrm{N}(0,1)$, $i = 1, 2$, $j \geq 1$ and set $S_k^{(i)} = \sum_{j=1}^n X_j^{(i)}$. Define

$$M_n := \operatorname*{argmax}_{k=1}^{n} \{S_k^{(1)} + S_k^{(2)}\},$$

and let $A_j^{(i)}$ be subsets of $\mathbb{R}$. Define

$$\mathcal{I} := \mathbf{P}\Bigg(\bigcap_{\substack{j=0\\ i=1,2}}^{k-1}\{S_k^{(i)} - S_j^{(i)} \in (-1)^{i+1}A_{k-j}^{(i)}\} \cap \bigcap_{\substack{j=k+1\\ i=1,2}}^{n}\{S_k^{(i)} - S_j^{(i)} \in (-1)^{i+1}A_j^{(i)}\} \cap \{M_n = k\}\Bigg). \quad (2.4.4)$$

Noting that the event $\{M_n = k\}$ is the same as

$$\bigcap_{\substack{j=0\\ i=1,2}}^{k-1}\{S_k^{(1)} + S_k^{(2)} > S_j^{(1)} + S_j^{(2)}\} \bigcap_{\substack{j=k+1\\ i=1,2}}^{n}\{S_k^{(1)} + S_k^{(2)} > S_j^{(1)} + S_j^{(2)}\},$$

we have

$$\begin{aligned}\text{r.h.s of (2.4.4)} = \mathbf{P}\Bigg(&\bigcap_{\substack{j=0\\ i=1,2}}^{k-1}\{S_k^{(i)} - S_j^{(i)} \in (-1)^{i+1}A_{k-j}^{(i)}, S_k^{(1)} + S_k^{(2)} > S_j^{(1)} + S_j^{(2)}\}\\ &\cap \bigcap_{\substack{j=k+1\\ i=1,2}}^{n}\{S_k^{(i)} - S_j^{(i)} \in (-1)^{i+1}A_j^{(i)}, S_k^{(1)} + S_k^{(2)} > S_j^{(1)} + S_j^{(2)}\}\Bigg).\end{aligned}$$

We also observe that the pairs $(S_k^{(1)} - S_j^{(1)}, S_k^{(2)} - S_j^{(2)})_{j=0}^{k-1}$ and $(S_k^{(1)} - S_j^{(1)}, S_k^{(2)} - S_j^{(2)})_{j=k+1}^{n}$ are independent of each other and as X_j^i is symmetric

$$\begin{aligned}(S_k^{(1)} - S_j^{(1)}, S_k^{(2)} - S_j^{(2)})_{j=0}^{k-1} &\overset{(d)}{=} (S_{k-j}^{(1)}, -S_{k-j}^{(2)})_{j=0}^{k-1}\\ (S_k^{(1)} - S_j^{(1)}, S_k^{(2)} - S_j^{(2)})_{j=k+1}^{n} &\overset{(d)}{=} (S_{j-k}^{(1)}, -S_{j-k}^{(2)})_{j=k+1}^{n}.\end{aligned}$$

Thus,

$$\mathcal{I} = \mathbf{P}\Bigg(\bigcap_{\substack{j=0\\ i=1,2}}^{k-1}\{S_j^{(i)} \in A_j^{(i)}, S_j^{(1)} > S_j^{(2)}\}\Bigg) \cdot \mathbf{P}\Bigg(\bigcap_{\substack{j=1\\ i=1,2}}^{n-k}\{S_j^{(i)} \in A_j^{(i)}, S_j^{(1)} > S_j^{(2)}\}\Bigg). \quad (2.4.5)$$

Based on (2.4.5), we obtain that

$$\frac{\mathcal{I}}{\mathbf{P}(M_n = k)} = \mathbf{P}\Big(\bigcap_{\substack{j=0\\ i=1,2}}^{k-1}\{S_j^{(i)} \in A_j^{(i)}\} | \bigcap_{j=1}^{k}\{S_j^{(1)} > S_j^{(2)}\}\Big) \cdot \mathbf{P}\Big(\bigcap_{\substack{j=1\\ i=1,2}}^{n-k}\{S_j^{(i)} \in A_j^{(i)}\} | \bigcap_{j=1}^{n-k}\{S_j^{(1)} > S_j^{(2)}\}\Big) \tag{2.4.6}$$

where we utilize the fact $\mathbf{P}(M_n = k) = \mathbf{P}(\cap_{j=1}^{k} S_j^{(1)} > S_j^{(2)})\mathbf{P}(\cap_{j=1}^{n-k} S_j^{(1)} > S_j^{(2)})$. The above splitting essentially shows that conditioned on the maximizer, the left and right portion of the maximizer are independent non-intersecting random walks.

We now consider $Z_n(t) = (\frac{S_{nt}^{(1)}}{\sqrt{n}}, \frac{S_{nt}^{(2)}}{\sqrt{n}})$ on $[0, 1]$ where it is linearly interpolated in between. By Donsker's invariance principle, $Z_n \Rightarrow B = (B_1, B_2)$ independent Brownian motions on $[0, 1]$. Recall Ω from Definition 2.4.10. Clearly $\mathbf{P}(B \in \text{Discontinuity of } \Omega) = 0$, so

$$(\Omega Z_n)_+ \Rightarrow (\Omega B)_+ \text{ and } (\Omega Z_n)_- \Rightarrow (\Omega B)_-.$$

On the other hand, following (2.4.6) we see that conditioned on $M_n = k$, $(\Omega Z_n)_+ \overset{(d)}{=} Y_{n-k}$ and $(\Omega Z_n)_- \overset{(d)}{=} Y_k$ are independent where $Y_n(\cdot)$ is the linearly interpolated non-intersecting random walk defined in Proposition 2.4.7. As $k, n \to \infty$, $Y_k(\cdot), Y_{n-k}(\cdot) \xrightarrow{d} W$ where W is the non-intersecting Brownian motion on $[0, 1]$ defined in Definition 2.4.3. At the same time, $\frac{M_n}{n} \Rightarrow M$, which has density $\propto \frac{1}{\sqrt{t(1-t)}}$ on $[0, 1]$. Thus, $(\Omega B)_+, (\Omega B)_-, M$ are independent and $(\Omega B)_+ \overset{(d)}{=} (\Omega B)_- \overset{(d)}{=} W$. □

Remark 2.4.12. We expect similar decomposition results to hold for 3 or more Brownian motions or bridges around the maximizer of their sums. More precisely, if M is the maximizer of $B_1(x) + B_2(x) + B_3(x)$, where B_i are independent Brownian motion on $[0, 1]$, we expect the law of

$$(B_1(M) - B_1(M + x), B_2(M) - B_2(M + x), B_3(M) - B_3(M + x))$$

to be again Brownian motions but their sum conditioned to be positive (its singular conditioning; so requires some care to define properly). Indeed, such a statement can be proven rigorously at the level of random walks. Then a possible approach is to take diffusive limit of random walks under conditioning and prove existence of weak limits. Due to lack of results for such conditioning event, proving such a statement require quite some technical work. Since it is extraneous for our purpose, we do not pursue this direction here.

2.5 Bessel bridges and non-intersecting Brownian bridges

In this section, we study diffusive limits and separation properties of Bessel bridges and non-intersecting Brownian bridges. The central object that appears in this section is the Dyson Brownian motion [118] which are intuitively several Brownian bridges conditioned on non-intersection. In Section 2.5.1, we recall Dyson Brownian motion and study different properties of it. In Section 2.5.2 we prove a technical estimate that indicates the two parts of non-intersecting Brownian bridges have uniform separation and derive the diffusive limits of non-intersecting Brownian bridges. The precise renderings of these results are given in Proposition 2.5.6 and Proposition 2.5.8.

2.5.1 Diffusive limits of Bessel bridges and NonInt-BrBridge

We first recall the definition of Dyson Brownian motion. Although they are Brownian motions conditioned on non-intersection, since the conditioning event is singular, such an interpretation needs to be justified properly. There are several ways to rigorously define the Dyson Brownian motion, either through the eigenvalues of Hermitian matrices with Brownian motions as entries or as a solution of system of stochastic PDEs. In this paper, we recall the definition via specifying the entrance law and transition densities (see [196] and [238, Section 3] for example).

Definition 2.5.1 (Dyson Brownian motion)**.** A 2-level Dyson Brownian motion $\mathcal{D}(\cdot) = (\mathcal{D}_1(\cdot), \mathcal{D}_2(\cdot))$

is an $\mathbb{R}^2$ valued process on $[0,\infty)$ with $\mathcal{D}_1(0) = \mathcal{D}_2(0) = 0$ and with the entrance law

$$\mathbf{P}\left(\mathcal{D}_1(t) \in \mathrm{d}y_1, \mathcal{D}_2(t) \in \mathrm{d}y_2\right) = \mathbf{1}\{y_1 > y_2\}\frac{(y_1-y_2)^2}{t}p_t(y_1)p_t(y_2)\mathrm{d}y_1\mathrm{d}y_2, \quad t > 0. \tag{2.5.1}$$

For $0 < s < t < \infty$ and $x_1 > x_2$, its transition densities are given by

$$\begin{aligned}\mathbf{P}\left(\mathcal{D}_1(t) \in \mathrm{d}y_1, \mathcal{D}_2(t) \in \mathrm{d}y_2 \mid \mathcal{D}_1(s) = x_1, \mathcal{D}_2(s) = x_2\right)\\ = \mathbf{1}\{y_1 > y_2\}\frac{y_1-y_2}{x_1-x_2}\det(p_{t-s}(x_i-y_j))_{i,j=1}^2\mathrm{d}y_1\mathrm{d}y_2.\end{aligned} \tag{2.5.2}$$

The above formulas can be extended to n-level Dyson Brownian motions with (see [238, Section 3]) but for the rest of the paper we only require the $n = 2$ case. So, we will refer to the 2-level object defined above loosely as Dyson Brownian motion or DBM in short.

We next define the Bessel processes via specifying the entrance law and transition densities which are also well known in literature (see [211, Chapter VI.3]).

Definition 2.5.2 (Bessel Process)**.** A 3D Bessel process $\mathcal{R}_1$ with diffusion coefficient 1 is an $\mathbb{R}$-valued process on $[0,\infty)$ with $\mathcal{R}_1(0) = 0$ and with the entrance law

$$\mathbf{P}(\mathcal{R}_1(t) \in dy) = \tfrac{2y^2}{t}p_t(y)\mathrm{d}y, \quad x \in [0,\infty), \quad t > 0.$$

For $0 < s < t < \infty$ and $x > 0$, its transition densities are given by

$$\mathbf{P}(\mathcal{R}_1(t) \in dy \mid \mathcal{R}_1(s) = x) = \frac{y}{x}[p_{t-s}(x-y) - p_{t-s}(x+y)]dy, \quad y \in [0,\infty).$$

More generally, $\mathcal{R}_\sigma(\cdot)$ is a 3D Bessel process with diffusion coefficient $\sigma > 0$ if $\sigma^{-1/2}\mathcal{R}_\sigma(\cdot)$ is a 3D Bessel process with diffusion coefficient 1.

In this paper we will only deal with 3-dimensional Bessel processes. Thus we will just loosely refer to the above processes as Bessel processes.

DBM is directly linked with Bessel processes. Indeed the difference of the two paths of DBM is known (see [112] for example) to be a 3D Bessel process with diffusion coefficient 2. This fact can be proven easily via SPDE or the Hermitian matrices interpretation of DBM. Since we use this result repeatedly in later sections we record it as a lemma below.

Lemma 2.5.3 (Dyson to Bessel)**.** *Let $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2)$ be a* DBM*. Then, as a function in x, we have $\mathcal{D}_1(x) + \mathcal{D}_2(x) \stackrel{d}{=} \sqrt{2}B(x)$ and $\mathcal{D}_1(x) - \mathcal{D}_2(x) \stackrel{d}{=} \mathcal{R}_2(x)$ where $B(x)$ is a Brownian motion and $\mathcal{R}_2 : [0,\infty) \to \mathbb{R}$ is a Bessel process (see Definition 2.5.2) with diffusion coefficient* 2*.*

We end this subsection by providing two lemmas that compare the densities of NonInt-BrBridge and DBM.

Lemma 2.5.4. *Suppose the pair of random variables (U_1, U_2) has joint probability density function:*

$$\mathbf{P}(U_1 \in \mathrm{d}y_1, U_2 \in \mathrm{d}y_2) = \frac{(y_1 - y_2)^2}{t} p_t(y_1) p_t(y_2), \ y_1 > y_2. \tag{2.5.3}$$

Conditioning on (U_1, U_2), we consider a NonInt-BrBridge *(V_1, V_2) on $[0,t]$ ending at (U_1, U_2), see Definition 2.4.4. Then unconditionally, (V_1, V_2) is equal in distribution as* DBM *$(\mathcal{D}_1, \mathcal{D}_2)$ on $[0,t]$. (see Definition 2.5.1).*

Lemma 2.5.5. *Fix $\delta, M > 0$. Consider a* NonInt-BrBridge *(V_1, V_2) on $[0,1]$ ending at (a_1, a_2) (see Definition 2.4.4), where $a_1 > a_2$. Then, there exists a constant $\mathrm{C}_{M,\delta} > 0$ such that for all $t \in (0, \delta)$, $y_1 > y_2$ and $-M \le a_2 < a_1 \le M$,*

$$\frac{\mathbf{P}(V_1(t) \in \mathrm{d}y_1, V_2(t) \in \mathrm{d}y_2)}{\mathbf{P}(\mathcal{D}_1(t) \in \mathrm{d}y_1, \mathcal{D}_2(t) \in \mathrm{d}y_2)} \le \mathrm{C}_{M,\delta}, \tag{2.5.4}$$

where $(\mathcal{D}_1, \mathcal{D}_2)$ is a DBM *defined in Definition 2.5.1.*

Proof of Lemma 2.5.4. To show that (V_1, V_2) is equal in distribution to $(\mathcal{D}_1, \mathcal{D}_2)$ on $[0,t]$, it suffices to show that (V_1, V_2) has the same finite dimensional distribution as $(\mathcal{D}_1, \mathcal{D}_2)$ on $[0,t]$. Fix any $k \in \mathbb{N}$, and $0 < s_1 < \ldots < s_k < t$ and $y_1 > y_2$. Using Brownian scaling and the formulas from

Definition 2.4.4 we have

$$\mathbf{P}\Big(\bigcap_{i=1}^{k}\{V_1(s_i)\in \mathrm{d}x_{i,1}, V_2(s_i)\in \mathrm{d}x_{i,2}\}|U_1=y_1,U_2=y_2\Big)$$
$$=\frac{(x_{1,1}-x_{1,2})}{s_1}p_{s_1}(x_{1,1})p_{s_1}(x_{1,2})\prod_{m=1}^{k-1}\det(p_{s_{m+1}-s_m}(x_{m+1,i}-x_{m,j}))_{i,j=1}^{2}$$
$$\cdot\frac{\det(p_{t-s_k}(x_{k,i}-y_j))_{i,j=1}^{2}}{\frac{1}{t}(y_1-y_2)p_t(y_1)p_t(y_2)}\prod_{i=1}^{k}\mathrm{d}x_{i,1}\mathrm{d}x_{i,2},$$

where the above density is supported on $\{x_{i,1}>x_{i,2}\mid i=1,2,\dots,k\}$. For convenience, in the rest of the calculations, we drop $\prod_{i=1}^{k}\mathrm{d}x_{i,1}\mathrm{d}x_{i,2}$ from the above formula. In view of the marginal density of (U_1,U_2) given by (2.5.3), we thus have that

$$\mathbf{P}\Big(\bigcap_{i=1}^{k}\{V_1(s_i)\in \mathrm{d}x_{i,1}, V_2(s_i)\in \mathrm{d}x_{i,2}\}\Big)$$
$$=\int_{y_1>y_2}\mathbf{P}\Big(\bigcap_{i=1}^{k}\{V_1(s_i)\in \mathrm{d}x_{i,1}, V_2(s_i)\in \mathrm{d}x_{i,2}\}|U_1=y_1,U_2=y_2\Big)\frac{(y_1-y_2)^2}{t}p_t(y_1)p_t(y_2)\mathrm{d}y_1\mathrm{d}y_2$$
$$=\frac{(x_{1,1}-x_{1,2})}{s_1}p_{s_1}(x_{1,1})p_{s_1}(x_{1,2})\prod_{m=1}^{k-1}\det(p_{s_{m+1}-s_m}(x_{m+1,i}-x_{m,j}))_{i,j=1}^{2}$$
$$\cdot\int_{y_1>y_2}(y_1-y_2)\det(p_{t-s_k}(x_{k,i}-y_j))_{i,j=1}^{2}\mathrm{d}y_1\mathrm{d}y_2.$$

But given the transition densities for DBM from (2.5.2). we know that

$$\int_{y_1>y_2}(y_1-y_2)\det(p_{t-s_k}(x_{k,i}-y_j))_{i,j=1}^{2}\mathrm{d}y_1\mathrm{d}y_2=x_{k,1}-x_{k,2}.$$

Plugging this back we get

$$\mathbf{P}\Big(\bigcap_{i=1}^{k}\{V_1(s_i)\in \mathrm{d}x_{i,1}, V_2(s_i)\in \mathrm{d}x_{i,2}\}\Big)$$
$$=\frac{(x_{1,1}-x_{1,2})^2}{s_1}p_{s_1}(x_{1,1})p_{s_1}(x_{1,2})\prod_{m=1}^{k-1}\frac{x_{m+1,1}-x_{m+1,2}}{x_{m,1}-x_{m,2}}\det(p_{s_{m+1}-s_m}(x_{m+1,i}-x_{m,j}))_{i,j=1}^{2}.$$

Using the entrance law and transition densities formulas for DBM from Definition 2.5.1, we see that the above formula matches with the finite dimensional density formulas for DBM. This completes the proof. □

Proof of Lemma 2.5.5. Fix any arbitrary $y_1 > y_2$ and $t \in (0, \delta)$ Recall the density formulas for NonInt-BrBridge and DBM from Definitions 2.4.4 and 2.5.1. We have

$$\text{l.h.s of (2.5.4)} = \frac{\det(p_{1-t}(y_i - a_j))_{i,j=1}^2}{(y_1 - y_2)(a_1 - a_2)p_1(a_1)p_1(a_2)} \tag{2.5.5}$$

$$= \frac{p_{1-t}(y_1 - a_2)p_{1-t}(y_2 - a_1)}{(y_1 - y_2)(a_1 - a_2)p_1(a_1)p_1(a_2)} \left[e^{\frac{(y_1-y_2)(a_1-a_2)}{1-t}} - 1\right]. \tag{2.5.6}$$

If $(y_1 - y_2)(a_1 - a_2) \geq 1 - t$, then

$$\text{r.h.s. of (2.5.5)} \leq \frac{\det(p_{1-t}(y_i - a_j))_{i,j=1}^2}{(1-t)p_1(a_1)p_1(a_2)} \leq \tfrac{1}{(1-t)^2}e^{\frac{a_1^2+a_2^2}{2}} \leq \tfrac{1}{(1-\delta)^2}e^{M^2}.$$

If $(y_1 - y_2)(a_1 - a_2) \leq 1 - t$, we utilize the elementary inequality that $\gamma(e^{\frac{1}{\gamma}} - 1) \leq e - 1$, for all $\gamma \geq 1$. Indeed, taking $\gamma = \frac{1-t}{(y_1-y_2)(a_1-a_2)} \geq 1$ in this case we have

$$\text{r.h.s. of (2.5.6)} \leq \frac{p_{1-t}(y_1 - a_2)p_{1-t}(y_2 - a_1)}{(1-t)p_1(a_1)p_1(a_2)}(e - 1) \leq \tfrac{2}{(1-t)^2}e^{\frac{a_1^2+a_2^2}{2}} \leq \tfrac{2}{(1-\delta)^2}e^{M^2}.$$

Combining both cases yields the desired result. □

2.5.2 Uniform separation and diffusive limits

The main goal of this subsection is to prove Proposition 2.5.6 and Proposition 2.5.8. Proposition 2.5.6 highlights a uniform separation between the two parts of the NonInt-BrBridge defined in Definition 2.4.4, while Proposition 2.5.8 shows DBMs are obtained as diffusive limits of NonInt-BrBridges.

Proposition 2.5.6. *Fix $M > 0$. Let $(V_1^{(n)}, V_2^{(n)})$ be a sequence of* NonInt-BrBridge*s (see Definition 2.4.4) on $[0, 1]$ beginning at 0 and ending at $(a_1^{(n)}, a_2^{(n)})$. Suppose that $a_1^{(n)} - a_2^{(n)} > \frac{1}{M}$ and*

$|a_i^{(n)}| \le M$ *for all* n *and* $i = 1, 2$. *Then for all* $\rho > 0$, *we have*

$$\limsup_{\theta\to\infty}\limsup_{n\to\infty} \mathbf{P}\left(\int_{\theta}^{n} \exp\left(-\sqrt{n}\big[V_1^{(n)}(\tfrac{y}{n}) - V_2^{(n)}(\tfrac{y}{n})\big]\right) \mathrm{d}y \ge \rho\right) = 0. \tag{2.5.7}$$

Recall that by Lemma 2.4.6, the difference of the two parts of NonInt-BrBridge is given by a Bessel bridge (upto a constant). Hence we can recast the above result in terms of separations between Bessel bridges from the x-axis as well.

Corollary 2.5.7. *Fix* $M > 0$. *Let* $\mathcal{R}_{\text{bb}}^{(n)}$ *be a sequence of Bessel bridges (see Definition 2.4.1) on* $[0, 1]$ *beginning at 0 and ending at* $a^{(n)}$. *Suppose that* $M > a_1^{(n)} > \frac{1}{M}$ *for all n. Then for all* $\rho > 0$, *we have*

$$\limsup_{\theta\to\infty}\limsup_{n\to\infty} \mathbf{P}\left(\int_{\theta}^{n} \exp\left(-\sqrt{n}\mathcal{R}_{\text{bb}}^{(n)}(\tfrac{y}{n})\right) \mathrm{d}y \ge \rho\right) = 0.$$

Proof of Proposition 2.5.6. We fix $\delta \in (0, \frac{1}{4})$. To prove the inequality in (2.5.7), we divide the integral from θ to n into two parts: $(\theta, n\delta)$ and $[n\delta, n)$ for some $\delta \in (0, 1)$ and n large and prove each one separately. For the interval $(n\delta, n)$ interval, we use the fact that the non-intersecting Brownian bridges $V_1^{(n)}(y), V_2^{(n)}(y)$ are separated by a uniform distance when away from 0. For the smaller interval $(\theta, n\delta)$ close to 0, we define a $\mathsf{Gap}_{n,\theta,\delta}$ event that occurs with high probability and utilize Lemmas 2.5.4 and 2.5.5 to transform the computations of NonInt-BrBridge into those of the DBM to simplify the proof.

We now fill out the details of the above road-map. First, as $(V_1^{(n)}, V_2^{(n)})$ are non-intersecting Brownian bridges on $[0, 1]$ starting from 0 and ending at two points which are within $[-M, M]$ and are separated by at least $\frac{1}{M}$, for every $\lambda, \delta > 0$, there exists $\alpha(M, \delta, \lambda) > 0$ small enough such that

$$\mathbf{P}\left(V_1^{(n)}(y) - V_2^{(n)}(y) \ge \alpha, \forall y \in [\delta, 1]\right) \ge 1 - \lambda. \tag{2.5.8}$$

(2.5.8) implies that with probability at least $1-\lambda$,

$$\int_{n\delta}^{n} \exp\left(-\sqrt{n}\big[V_1^{(n)}(\tfrac{y}{n}) - V_2^{(n)}(\tfrac{y}{n})\big]\right) \mathrm{d}y \le (n-n\delta)e^{-\sqrt{n}\alpha} \tag{2.5.9}$$

which converges to 0 as $n \to \infty$. Next we define the event

$$\mathsf{Gap}_{n,\theta,\delta} := \left\{\sqrt{n}\big[V_1^{(n)}(\tfrac{y}{n}) - V_2^{(n)}(\tfrac{y}{n})\big] \ge y^{\frac{1}{4}}, \forall y \in [\theta, n\delta]\right\}.$$

We claim that $\neg\mathsf{Gap}_{n,\theta,\delta}$ event is negligible in the sense that

$$\limsup_{\theta\to\infty}\limsup_{n\to\infty} \mathbf{P}(\neg\mathsf{Gap}_{n,\theta,\delta}) = 0. \tag{2.5.10}$$

Note that on $\mathsf{Gap}_{n,\theta,\delta}$ event, we have

$$\int_{\theta}^{n\delta} \exp\left(-\sqrt{n}\big[V_1^{(n)}(\tfrac{y}{n}) - V_2^{(n)}(\tfrac{y}{n})\big]\right) \mathrm{d}y \le \int_{\theta}^{n\delta} \exp(-y^{1/4})\mathrm{d}y \tag{2.5.11}$$

which goes to zero as $n \to \infty$, followed by $\theta \to \infty$. In view of the probability estimates from (2.5.8) and (2.5.9), combining (2.5.10) and (2.5.11) yields

$$\limsup_{\theta\to\infty}\limsup_{n\to\infty} \mathbf{P}\left(\int_{\theta}^{n} \exp\left(-\sqrt{n}\big[V_1^{(n)}(\tfrac{y}{n}) - V_2^{(n)}(\tfrac{y}{n})\big]\right) \mathrm{d}y \ge \rho\right) \le \lambda. \tag{2.5.12}$$

Since λ is arbitrary, (2.5.12) completes the proof. Hence it suffices to show (2.5.10). Towards this end, by the properties of the conditional expectation, if we condition on the values of $V_1^{(n)}(2\delta), V_2^{(n)}(2\delta)$, we have that

$$\begin{aligned}\mathbf{P}(\neg\mathsf{Gap}_{n,\theta,\delta}) &= \mathbf{E}\left[\mathbf{P}\left(\neg\mathsf{Gap}_{n,\theta,\delta} | V_1^{(n)}(2\delta), V_2^{(n)}(2\delta)\right)\right] \\ &= \int_{y_1>y_2} \mathbf{P}_{y_1,y_2}(\neg\mathsf{Gap}_{n,\theta,\delta})\mathbf{P}(V_1^{(n)}(2\delta)\in \mathrm{d}y_1, V_2^{(n)}(2\delta)\in\mathrm{d}y_2)\end{aligned} \tag{2.5.13}$$

where $\mathbf{P}_{y_1,y_2}$ is the conditional law of $\mathsf{NonInt\text{-}BrBridge}$ conditioned on $(V_1^{(n)}(2\delta) = y_1, V_2^{(n)}(2\delta) =$

y_2). Note that $\mathsf{Gap}_{n,\theta,\delta}$ event depends only on the $[0,\delta]$ path of the NonInt-BrBridge. Thus by Markovian property of the NonInt-BrBridge, $\mathbf{P}_{y_1,y_2}(\mathsf{Gap}_{n,\theta,\delta})$ can be computed by assuming the NonInt-BrBridge is on $[0,2\delta]$ and ends at (y_1,y_2).

On the other hand, $\mathbf{P}(V_1^n(2\delta)\in \mathrm{d}y_1, V_2^n(2\delta)\in \mathrm{d}y_2)$ is the probability density function of the marginal density of NonInt-BrBridge on $[0,1]$. Via Lemma 2.5.5, this is comparable to the density of $(\mathcal{D}_1(2\delta),\mathcal{D}_2(2\delta))$, where $\mathcal{D}$ follows DBM law defined in Definition 2.5.1. Thus by (2.5.4) the r.h.s of (2.5.13) is bounded from above by

$$\begin{aligned}\text{r.h.s of (2.5.13)} &\le \mathrm{C}_{M,2\delta}\int \mathbf{P}_{y_1,y_2}(\neg\mathsf{Gap}_{n,\theta,\delta})\mathbf{P}(\mathcal{D}_1(2\delta)\in \mathrm{d}y_1,\mathcal{D}_2(2\delta)\in \mathrm{d}y_2)\mathrm{d}y_1\mathrm{d}y_2\\ &= \mathrm{C}_{M,2\delta}\cdot \mathbf{P}_{\mathrm{Dyson}}(\neg\mathsf{Gap}_{n,\theta,\delta}).\end{aligned} \tag{2.5.14}$$

Here the notation $\mathbf{P}_{\mathrm{Dyson}}$ means the law of the paths (V_1,V_2) is assumed to follow DBM law. With this notation, the last equality of (2.5.14) follows from Lemma 2.5.4. From the density formulas of DBM from Definition 2.5.1, it is clear that DBM is invariant under diffusive scaling, i.e.

$$\sqrt{n}(\mathcal{D}_1(\tfrac{\cdot}{n}),\mathcal{D}_2(\tfrac{\cdot}{n}))\stackrel{d}{=}(\mathcal{D}_1(\cdot),\mathcal{D}_2(\cdot)) \tag{2.5.15}$$

and by Lemma 2.5.3, $\mathcal{D}_1(\cdot)-\mathcal{D}_2(\cdot)=\mathcal{R}_2(\cdot)$, a 3D Bessel process with diffusion coefficient 2. Thus, we obtain that for any $n\in\mathbb{N}$,

$$\mathbf{P}_{\mathrm{Dyson}}(\neg\mathsf{Gap}_{n,\theta,\delta})\le \mathbf{P}(\mathcal{R}_2(y)\le y^{1/4},\ \text{ for some } y\in[\theta,\infty)). \tag{2.5.16}$$

Meanwhile, Motoo's theorem [191] tells us that

$$\limsup_{\theta\to\infty}\mathbf{P}(\mathcal{R}_2(y)\le y^{1/4},\ \text{ for some } y\in[\theta,\infty))=0. \tag{2.5.17}$$

Hence (2.5.14), (2.5.16) and (2.5.17) imply (2.5.10). This completes the proof. □

We now state our results related to the diffusive limits of NonInt-BrBridge (defined in Definition 2.4.4) and Bessel bridges (defined in Definition 2.4.1) with varying endpoints.

Proposition 2.5.8. *Fix $M > 0$. Let $V^{(n)} = (V_1^{(n)}, V_2^{(n)}) : [0, a_n] \to \mathbb{R}$ be a sequence of* NonInt-BrBridge*s (defined in Definition 2.4.4) with $V_i^{(n)}(0) = 0$ and $V_i^{(n)}(a_n) = z_i^{(n)}$. Suppose that for all $n \geq 1$ and $i = 1, 2$, $M > a_n > \frac{1}{M}$ and $|z_i^{(n)}| < \frac{1}{M}$. Then as $n \to \infty$ we have:*

$$\sqrt{n}\big(V_1^{(n)}(\tfrac{t}{n}), V_2^{(n)}(\tfrac{t}{n})\big) \xrightarrow{d} (\mathcal{D}_1(t), \mathcal{D}_2(t))$$

in the uniform-on-compact topology. Here $\mathcal{D}$ is a DBM *defined in Definition 2.5.1.*

In view of Lemma 2.4.6 and Lemma 2.5.3, Proposition 2.5.8 also leads to the following corollary.

Corollary 2.5.9. *Fix $M > 0$. Let $\mathcal{R}_{\mathrm{bb}}^{(n)} : [0, a_n] \to \mathbb{R}$ be a sequence of Bessel bridges (defined in Definition 2.4.1) with $\mathcal{R}_{\mathrm{bb}}^{(n)}(0) = 0$ and $\mathcal{R}_{\mathrm{bb}}^{(n)}(a_n) = z^{(n)}$. Suppose for all $n \geq 1$, $M > a_n > \frac{1}{M}$ and $|z^{(n)}| < \frac{1}{M}$. Then as $n \to \infty$ we have:*

$$\sqrt{n}\mathcal{R}_{\mathrm{bb}}^{(n)}(\tfrac{t}{n}) \xrightarrow{d} \mathcal{R}_1(t)$$

in the uniform-on-compact topology. Here $\mathcal{R}_1$ is a Bessel process with diffusion coefficient 1, defined in Definition 2.5.2.

Proof of Proposition 2.5.8. For convenience, we drop the superscript (n) from V_1, V_2 and z_i's. We proceed by showing convergence of one-point densities and transition densities of $\sqrt{n}(V_1(\frac{t}{n}), V_2(\frac{t}{n}))$ to that of DBM and then verifying the tightness of the sequence. Fix any $t > 0$. For each fixed $y_1 > y_2$, it is not hard to check that we have as $n \to \infty$

$$\frac{a_n\sqrt{n}\det(p_{a_n-\frac{t}{n}}(\frac{y_i}{\sqrt{n}} - z_j))_{i,j=1}^2}{(z_1 - z_2)p_{a_n}(z_1)p_{a_n}(z_2)} \to y_1 - y_2. \tag{2.5.18}$$

uniformly over $a_n \in [\frac{1}{M}, M]$ and $z_1, z_2 \in [-M, M]$.

Utilizing the one-point densities and transition densities formulas for NonInt-BrBridge of length 1 in Definition 2.4.4, we may perform a Brownian rescaling to get analogous formulas for V_1, V_2 which are NonInt-BrBridge of length a_n. Then by a change of variable, the density of $(\sqrt{n}V_1(\frac{t}{n}), \sqrt{n}V_2(\frac{t}{n}))$ is given by

$$\frac{a_n(y_1-y_2)p_t(y_1)p_t(y_2)}{t(z_1-z_2)p_{a_n}(z_1)p_{a_n}(z_2)}\sqrt{n}\det(p_{a_n-\frac{t}{n}}(\tfrac{y_i}{\sqrt{n}}-z_j))_{i,j=1}^2.$$

Using (2.5.18) we see that for each fixed $y_1 > y_2$, the above expression goes to $\frac{(y_1-y_2)^2}{t}p_t(y_1)p_t(y_2)$ which matches with (2.5.1).

Similarly for the transition probability, letting $0 < s < t < a_n$, $y_1 > y_2$ and $x_1 > x_2$, we have

$$\begin{aligned}
&\mathbf{P}\left(\sqrt{n}V_1(\tfrac{t}{n}) \in \mathrm{d}y_1, \sqrt{n}V_2(\tfrac{t}{n}) \in \mathrm{d}y_2 \mid \sqrt{n}V_1(\tfrac{s}{n}) \in \mathrm{d}x_1, \sqrt{n}V_2(\tfrac{s}{n}) \in \mathrm{d}x_2\right) \\
&= \det(p_{t-s}(y_i-x_j))_{i,j=1}^2 \frac{\det(p_{a_n-\frac{t}{n}}(\frac{y_i}{\sqrt{n}}-z_j))_{i,j=1}^2}{\det(p_{a_n-\frac{s}{n}}(\frac{x_i}{\sqrt{n}}-z_j))_{i,j=1}^2}\mathrm{d}y_1\mathrm{d}y_2.
\end{aligned} \tag{2.5.19}$$

Applying (2.5.18) we see that as $n \to \infty$

$$\text{r.h.s of (2.5.19)} \to \det(p_{t-s}(x_i-y_j))_{i,j=1}^2 \cdot \frac{y_1-y_2}{x_1-x_2}.$$

which matches with (2.5.2). This verifies the finite dimensional convergence by Scheffe's theorem. For tightness we will show that there exists a constant $\mathrm{C}_{K,M} > 0$ such that for all $0 < s < t < K$,

$$\sum_{i=1}^{2}\mathbf{E}\left[\left(\sqrt{n}V_i(\tfrac{t}{n}) - \sqrt{n}V_i(\tfrac{s}{n})\right)^4\right] \le \mathrm{C}_{K,M}(t-s)^2. \tag{2.5.20}$$

We compute the above expectation by comparing with DBM as was done in the proof of Proposition 2.5.6. Using definition of the conditional expectation we have

$$\begin{aligned}
&\mathbf{E}\left[\left(\sqrt{n}V_i(\tfrac{t}{n}) - \sqrt{n}V_i(\tfrac{s}{n})\right)^4\right] \\
&= \int_{y_1>y_2}\mathbf{E}\left[\left(\sqrt{n}V_i(\tfrac{t}{n}) - \sqrt{n}V_i(\tfrac{s}{n})\right)^4 \mid V_1(\tfrac{K}{n}) = y_1, V_1(\tfrac{K}{n}) = y_2\right]\mathbf{P}(V_1(\tfrac{K}{n}) \in \mathrm{d}y_1, V_2(\tfrac{K}{n}) \in \mathrm{d}y_2)
\end{aligned}$$

$$\leq \mathrm{C}_{K,M} \int_{y_1>y_2} \mathbf{E}\left[\left(\sqrt{n}V_i(\tfrac{t}{n}) - \sqrt{n}V_i(\tfrac{s}{n})\right)^4 \mid V_1(\tfrac{K}{n}) = y_1, V_1(\tfrac{K}{n}) = y_2\right] \mathbf{P}(\mathcal{D}_1(\tfrac{K}{n}) \in \mathrm{d}y_1, \mathcal{D}_2(\tfrac{K}{n}) \in \mathrm{d}y_2)$$

where the last inequality follows from Lemma 2.5.5 by taking n large enough. Here $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2)$ follows DBM law. Due to Lemma 2.5.4 and (2.5.15) the last integral above is precisely $\mathbf{E}[(\mathcal{D}_i(t) - \mathcal{D}_i(s))^4]$. Hence it suffices to show

$$\mathbf{E}[(\mathcal{D}_i(t) - \mathcal{D}_i(s))^4] \leq \mathrm{C}(t-s)^2. \tag{2.5.21}$$

By Lemma 2.5.3, we see $\sqrt{2}B(x) := \mathcal{D}_1(x) + \mathcal{D}_2(x)$ and $\sqrt{2}\mathcal{R}(x) := \mathcal{D}_1(x) - \mathcal{D}_2(x)$ are a standard Brownian motion and a 3D Bessel process with diffusion coefficient 1 respectively. We have

$$\mathbf{E}[(\mathcal{D}_i(t) - \mathcal{D}_i(s))^4] \leq \mathrm{C}\left[\mathbf{E}[(\mathcal{R}(t) - \mathcal{R}(s))^4] + \mathbf{E}[(B(t) - B(s))^4]\right].$$

We have $\mathbf{E}[(B(t) - B(s))^4] = 3(t-s)^2$, whereas for $\mathcal{R}(\cdot)$, we use Pitman's theorem [211, Theorem VI.3.5], to get that $\mathcal{R}(t) \stackrel{d}{=} 2M(t) - B(t)$, where B is a Brownian motion and $M(t) = \sup_{u \leq t} B(u)$. Thus,

$$\begin{aligned}
\mathbf{E}[(\mathcal{R}(t) - \mathcal{R}(s))^4] &\leq \mathrm{C}\left[\mathbf{E}[(M(t) - M(s))^4] + \mathbf{E}[(B(t) - B(s))^4]\right] \\
&\leq \mathrm{C}\left[\mathbf{E}\Big[\big(\sup_{s \leq u \leq t} B(u) - B(s)\big)^4\Big] + \mathbf{E}[(B(t) - B(s))^4]\right].
\end{aligned}$$

Clearly both the expressions above are at most $\mathrm{C}(t-s)^2$. This implies (2.5.21) completing the proof. □

2.6 Ergodicity and Bessel behavior of the KPZ equation

The goal of this section is to prove Theorems 2.1.10 and 2.1.11. As the proof of the latter is shorter and illustrates some of the ideas behind the proof of the former, we first tackle Theorem 2.1.11 in Section 2.6.1. After that in Section 2.6.2, we state a general version of the $k = 2$ case of Theorem 2.1.10, namely Proposition 2.6.1. This proposition will then be utilized in the proof of

Theorem 2.1.4. Finally in Section 2.6.3, we show how to obtain Theorem 2.1.10 from Proposition 2.6.1.

2.6.1 Proof of Theorem 2.1.11

For clarity we divide the proof into several steps.

Step 1. In this step, we introduce necessary notations used in the proof and explain the heuristic idea behind the proof.

Fix any $a > 0$. Consider any Borel set A of $C([-a,a])$ which is also a continuity set of a two-sided Brownian motion $B(x)$ restricted to $[-a,a]$. By Portmanteau theorem, it suffices to show

$$\mathbf{P}((\mathcal{H}(\cdot,t) - \mathcal{H}(0,t) \in A) \to \mathbf{P}(B(\cdot) \in A). \tag{2.6.1}$$

For simplicity let us write $\mathbf{P}_t(A) := \mathbf{P}((\mathcal{H}(\cdot,t) - \mathcal{H}(0,t) \in A)$. Using (2.2.6) we have $\mathcal{H}(x,t) - \mathcal{H}(0,t) = t^{1/3}(\mathfrak{h}_t(t^{-2/3}x) - \mathfrak{h}_t(0))$. Recall that $\mathfrak{h}_t(\cdot) = \mathfrak{h}_t^{(1)}(\cdot)$ can be viewed as the top curve of the KPZ line ensemble $\{\mathfrak{h}_t^{(n)}(\cdot)\}_{n\in\mathbb{N}}$ which satisfies the $\mathbf{H}_t$-Brownian Gibbs property with $\mathbf{H}_t$ given by (2.2.4).

Note that at the level of the scaled KPZ line ensembles we are interested in understanding the law of $\mathfrak{h}_t^{(1)}(\cdot)$ restricted to a very small interval: $x \in [-t^{-2/3}a, t^{-2/3}a]$. At such a small scale, we expect the Radon-Nikodym derivative appearing in (2.2.3) to be very close to 1. Hence the law of top curve should be close to a Brownian bridge with appropriate end points. To get rid of the endpoints we employ the following strategy, which is also illustrated in Figure 2.5 and its caption.

- We start with a slightly larger but still vanishing interval $I_t := (-t^{-\alpha}, t^{-\alpha})$ with $\alpha = \frac{1}{6}$ say. We show that conditioned on the end points $\mathfrak{h}_t^{(1)}(-t^{-\alpha}), \mathfrak{h}_t^{(1)}(t^{-\alpha})$ of the first curve and the second curve $\mathfrak{h}_t^{(2)}$, the law of $\mathfrak{h}_t^{(1)}$ is close to that of a Brownian bridge on I_t starting and ending at $\mathfrak{h}_t^{(1)}(-t^{-\alpha})$ and $\mathfrak{h}_t^{(1)}(t^{-\alpha})$ respectively.

- Once we probe further into an even narrower window of $[-t^{2/3}a, t^{2/3}a]$, the Brownian bridge no longer feels the effect of the endpoints and one gets a Brownian motion in the limit.

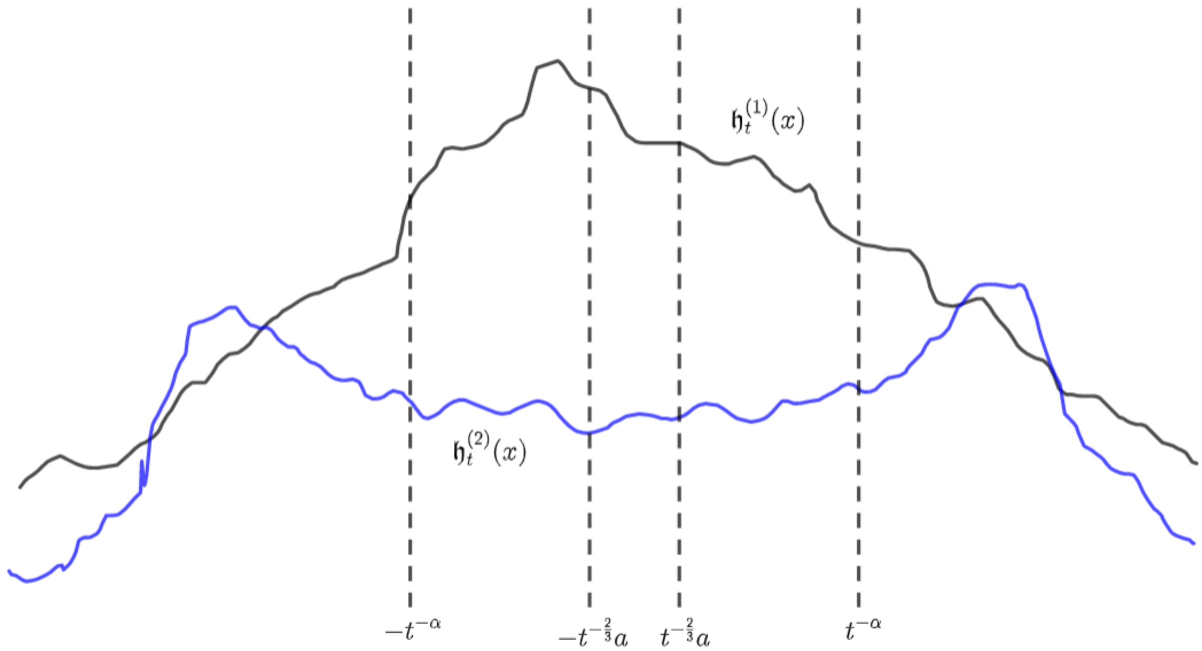


Figure 2.5: Illustration of the proof of Theorem 2.1.11. In a window of $[t^{-\alpha}, t^{\alpha}]$, the curves $\mathfrak{h}_t^{(1)}(x), \mathfrak{h}_t^{(2)}(x)$ attains an uniform gap with high probability. This allows us to show law of $\mathfrak{h}_t^{(1)}$ on that small patch is close to a Brownian bridge. Upon zooming in a the tiny interval $[-t^{2/3}a, t^{2/3}a]$ we get a two-sided Brownian bridge as explained in **Step 1** of the proof.

Step 2. In this step and next step, we give a technical roadmap of the heuristics presented in **Step 1**. Set $\alpha = \frac{1}{6}$ and consider the small interval $I_t = (t^{-\alpha}, t^{-\alpha})$. Let $\mathcal{F}$ be the σ-field generated by

$$\mathcal{F} := \sigma\left(\{\mathfrak{h}_t^{(1)}(x)\}_{x \in I_t^c}, \{\mathfrak{h}_t^{(n)}(\cdot)\}_{n \geq 2}\right). \tag{2.6.2}$$

Fix any arbitrary $\delta > 0$ and consider the following three events:

$$\mathsf{Gap}_t(\delta) := \left\{\mathfrak{h}_t^{(2)}(-t^{-\alpha}) \leq \min\{\mathfrak{h}_t^{(1)}(t^{-\alpha}), \mathfrak{h}_t^{(1)}(-t^{-\alpha})\} - \delta\right\}, \tag{2.6.3}$$

$$\mathsf{Rise}_t(\delta) := \left\{\sup_{x \in I_t} \mathfrak{h}_t^{(2)}(x) \leq \tfrac{1}{4}\delta + \mathfrak{h}_t^{(2)}(-t^{-\alpha})\right\}, \tag{2.6.4}$$

$$\mathsf{Tight}_t(\delta) := \left\{-\delta^{-1} \leq \mathfrak{h}_t^{(1)}(t^{-\alpha}), \mathfrak{h}_t^{(1)}(-t^{-\alpha}) \leq \delta^{-1}\right\}. \tag{2.6.5}$$

Note that all the above events are measurable w.r.t. $\mathcal{F}$. A visual interpretation of the above events are given later in Figure 2.6. Since the underlying curves are continuous almost surely, while specifying events over I_t, such as the $\mathsf{Rise}_t(\delta)$ event defined in (2.6.4), one may replace I_t by its closure $\overline{I}_t = [-t^{-\alpha}, t^{-\alpha}]$ as well; the events will remain equal almost surely. We will often make use of this fact, and will not make a clear distinction between I_t and $\overline{I}_t$.

We set

$$\mathsf{Fav}_t(\delta) := \mathsf{Gap}_t(\delta) \cap \mathsf{Rise}_t(\delta) \cap \mathsf{Tight}_t(\delta). \tag{2.6.6}$$

The $\mathsf{Fav}_t(\delta)$ event is a favorable event in the sense that given any $\varepsilon > 0$, there exists $\delta_0 \in (0,1)$ such that for all $\delta \in (0, \delta_0)$

$$\liminf_{t\to\infty} \mathsf{Fav}_t(\delta) \geq 1 - \varepsilon. \tag{2.6.7}$$

We will prove (2.6.7) in **Step 4**. For the moment, we assume this and continue with our calculations. We now proceed to find tight upper and lower bounds for $\mathbf{P}_t(A) = \mathbf{P}((\mathcal{H}(\cdot,t) - \mathcal{H}(0,t) \in A)$. Recall the σ-field $\mathcal{F}$ from (2.6.2). Note that using the tower property of the conditional expectation we have

$$\mathbf{P}_t(A) = \mathbf{E}\left[\mathbf{P}_t(A \mid \mathcal{F})\right] \geq \mathbf{E}\left[\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_t(A \mid \mathcal{F})\right]. \tag{2.6.8}$$

$$\mathbf{P}_t(A) = \mathbf{E}\left[\mathbf{P}_t(A \mid \mathcal{F})\right] \leq \mathbf{E}\left[\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_t(A \mid \mathcal{F})\right] + \mathbf{P}\left(\neg\mathsf{Fav}_t(\delta)\right). \tag{2.6.9}$$

Applying the $\mathbf{H}_t$-Brownian Gibbs property for the interval I_t we have

$$\mathbf{P}_t(A \mid \mathcal{F}) = \mathbf{P}_{\mathbf{H}_t}^{1,1,I_t,\mathfrak{h}_t(-t^{-\alpha}),\mathfrak{h}_t(t^{-\alpha}),\mathfrak{h}_t^{(2)}}(A) = \frac{\mathbf{E}_{\text{free},t}\left[W\mathbf{1}_A\right]}{\mathbf{E}_{\text{free},t}\left[W\right]}, \tag{2.6.10}$$

where

$$W := \exp\left(-t^{2/3}\int_{t^{-\alpha}}^{t^{-\alpha}} e^{t^{1/3}(\mathfrak{h}_t^{(2)}(x)-\mathfrak{h}_t^{(1)}(x))}\mathrm{d}x\right) \tag{2.6.11}$$

and $\mathbf{P}_{\mathrm{free},t} := \mathbf{P}_{\mathrm{free}}^{1,1,I_t,\mathfrak{h}_t(-t^{-\alpha}),\mathfrak{h}_t(t^{-\alpha})}$ and $\mathbf{E}_{\mathrm{free},t} := \mathbf{E}_{\mathrm{free}}^{1,1,I_t,\mathfrak{h}_t(-t^{-\alpha}),\mathfrak{h}_t(t^{-\alpha})}$ are the probability and the expectation operator respectively for a Brownian bridge $B_1(\cdot)$ on I_t starting at $\mathfrak{h}_t(-t^{-\alpha})$ and ending at $\mathfrak{h}_t(t^{-\alpha})$. Note that the second equality in (2.6.10) follows from (2.2.3). We now seek to find upper and lower bounds for r.h.s. of (2.6.10). For W, we have the trivial upper bound: $W \le 1$. For the lower bound, we claim that there exists $t_0(\delta) > 0$, such that for all $t \ge t_0$, we have

$$\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_{\mathrm{free},t}(W \ge 1-\delta) \ge \mathbf{1}\{\mathsf{Fav}_t(\delta)\}(1-\delta). \tag{2.6.12}$$

Note that (2.6.12) suggests that the W is close to 1 with high probability. This is the technical expression of the first conceptual step that we highlighted in **Step 1**. In the similar spirit for the second conceptual step, we claim that there exists $t_0(\delta) > 0$, such that for all $t \ge t_0$, we have

$$\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\left|\mathbf{P}_{\mathrm{free},t}(A) - \gamma(A)\right| \le \mathbf{1}\{\mathsf{Fav}_t(\delta)\}\cdot\delta, \tag{2.6.13}$$

where $\gamma(A) := \mathbf{P}(B(\cdot) \in A) \in [0,1]$. We remark that the l.h.s. of (2.6.12) and (2.6.13) are random variables measurable w.r.t. $\mathcal{F}$. The inequalities above hold pointwise. We will prove (2.6.12) and (2.6.13) in **Step 5** and **Step 6** respectively. We next complete the proof of the Theorem 2.1.11 assuming the above claims.

Step 3. In this step we assume (2.6.7), (2.6.12), and (2.6.13) and complete the proof of (2.6.1). Fix any $\varepsilon \in (0,1)$. Get a $\delta_0 \in (0,1)$, so that (2.6.7) is true for all $\delta \in (0,\delta_0)$. Fix any such $\delta \in (0,\delta_0)$. Get $t_0(\delta)$ large enough so that both (2.6.12) and (2.6.13) hold for all $t \ge t_0$. Fix any such $t \ge t_0$.

As $W \le 1$, we note that on the event $\mathsf{Fav}_t(\delta)$,

$$\frac{\mathbf{E}_{\mathrm{free},t}\,[W\mathbf{1}_A]}{\mathbf{E}_{\mathrm{free},t}\,[W]} \ge \mathbf{E}_{\mathrm{free},t}\,[W\mathbf{1}_A]$$

$$\geq (1-\delta)\mathbf{P}_{\text{free},t}\left(A \cap \{W \geq 1-\delta\}\right)$$
$$\geq (1-\delta)\mathbf{P}_{\text{free},t}(A) - (1-\delta)\mathbf{P}_{\text{free},t}(W < 1-\delta)$$
$$\geq (1-\delta)\mathbf{P}_{\text{free},t}(A) - (1-\delta)\delta,$$

where in the last line we used (2.6.12). Plugging this bound back in (2.6.8) we get

$$\begin{aligned}
\mathbf{P}_t(A) &\geq (1-\delta)\mathbf{E}\left[\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_{\text{free},t}(A)\right] - (1-\delta)\delta \\
&\geq (1-\delta)\mathbf{E}\left[\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\gamma(A) - \delta\right] - (1-\delta)\delta \\
&= \gamma(A)(1-\delta)\mathbf{P}(\mathsf{Fav}_t(\delta)) - 2\delta(1-\delta).
\end{aligned}$$

where the inequality in the penultimate line follows from (2.6.13). Taking lim inf both sides as $t \to \infty$, in view of (2.6.7) we see that

$$\liminf_{t\to\infty} \mathbf{P}_t(A) \geq (1-\delta)(1-\varepsilon)\gamma(A) - 2\delta(1-\delta).$$

Taking $\liminf_{\delta\downarrow 0}$ and using the fact that ε is arbitrary, we get that $\liminf_{t\to\infty} \mathbf{P}_t(A) \geq \gamma(A)$. Similarly for the upper bound, on the event $\mathsf{Fav}_t(\delta)$ we have

$$\frac{\mathbf{E}_{\text{free},t}\left[W\mathbf{1}_A\right]}{\mathbf{E}_{\text{free},t}\left[W\right]} \leq \frac{\mathbf{P}_{\text{free},t}(A)}{(1-\delta)\mathbf{P}_{\text{free},t}(W \geq 1-\delta)} \leq \frac{1}{(1-\delta)^2}\mathbf{P}_{\text{free},t}(A),$$

where we again use (2.6.12) for the last inequality. Inserting the above bound in (2.6.9) we get

$$\begin{aligned}
\mathbf{P}_t(A) &\leq \frac{1}{(1-\delta)^2}\mathbf{E}\left[\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_{\text{free},t}(A)\right] + \mathbf{P}\left(\neg\mathsf{Fav}_t(\delta)\right) \\
&\leq \frac{\delta}{(1-\delta)^2} + \frac{1}{(1-\delta)^2}\mathbf{E}\left[\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\gamma(A)\right] + \mathbf{P}\left(\neg\mathsf{Fav}_t(\delta)\right) \\
&\leq \frac{\delta}{(1-\delta)^2} + \frac{1}{(1-\delta)^2}\gamma(A) + \mathbf{P}\left(\neg\mathsf{Fav}_t(\delta)\right).
\end{aligned}$$

The inequality in the penultimate line above follows from (2.6.13). Taking lim sup both sides as

$t \to \infty$, in view of (2.6.7) we see that

$$\limsup_{t\to\infty} \mathbf{P}_t(A) \le \frac{\delta}{(1-\delta)^2} + \frac{1}{(1-\delta)^2}\gamma(A) + \varepsilon.$$

As before taking $\limsup_{\delta\downarrow 0}$ and using the fact that ε is arbitrary, we get that $\limsup_{t\to\infty} \mathbf{P}_t(A) \le \gamma(A)$. With the matching upper bound for lim inf derived above, we thus arrive at (2.6.1), completing the proof of Theorem 2.1.11.

Step 4. In this step we prove (2.6.7). Fix any $\delta > 0$. Recall the distributional convergence of KPZ line ensemble to Airy line ensemble from Proposition 2.2.7. By the Skorokhod representation theorem, we may assume that our probability space are equipped with $\mathcal{A}_1(x)$ $\mathcal{A}_2(x)$, such that as $t \to \infty$, almost surely we have

$$\max_{i=1,2} \sup_{x\in[-1,1]} |2^{1/3}\mathfrak{h}_t^{(i)}(2^{1/3}x) - \mathcal{A}_i(x)| \to 0. \tag{2.6.14}$$

For $i = 1, 2$, consider the event

$$\mathsf{Fluc}_t^{(i)}(\delta) := \left\{\sup_{x\in I_t} |\mathfrak{h}_t^{(i)}(x) - \mathfrak{h}_t^{(i)}(-t^{-\alpha})| \le \tfrac{1}{4}\delta\right\}. \tag{2.6.15}$$

See Figure 2.6 and its caption for an interpretation of this event. We claim that for every $\delta > 0$,

$$\liminf_{t\to\infty} \mathbf{P}\left(\mathsf{Fluc}_t^{(i)}(\delta)\right) = 1. \tag{2.6.16}$$

Let us complete the proof of (2.6.7) assuming (2.6.16). Fix any $\varepsilon > 0$. Note that $\{|\mathfrak{h}_t^{(1)}(-t^{-\alpha}) - \mathfrak{h}_t^{(1)}(t^{-\alpha})| \le \frac{1}{4}\delta\} \supset \mathsf{Fluc}_t^{(1)}(\delta)$. Recall $\mathsf{Gap}_t(\delta)$ from (2.6.3). Observe that

$$\begin{aligned}\neg\mathsf{Gap}_t(\delta) \cap \left\{|\mathfrak{h}_t^{(1)}(-t^{-\alpha}) - \mathfrak{h}_t^{(1)}(t^{-\alpha})| \le \tfrac{1}{4}\delta\right\} &\subset \left\{\mathfrak{h}_t^{(2)}(-t^{\alpha}) - \mathfrak{h}_t^{(1)}(-t^{-\alpha}) \ge -\tfrac{5}{4}\delta\right\} \\ &\subset \left\{\inf_{x\in[-1,0]} [\mathfrak{h}_t^{(2)}(x) - \mathfrak{h}_t^{(1)}(x)] \ge -\tfrac{5}{4}\delta\right\}.\end{aligned}$$

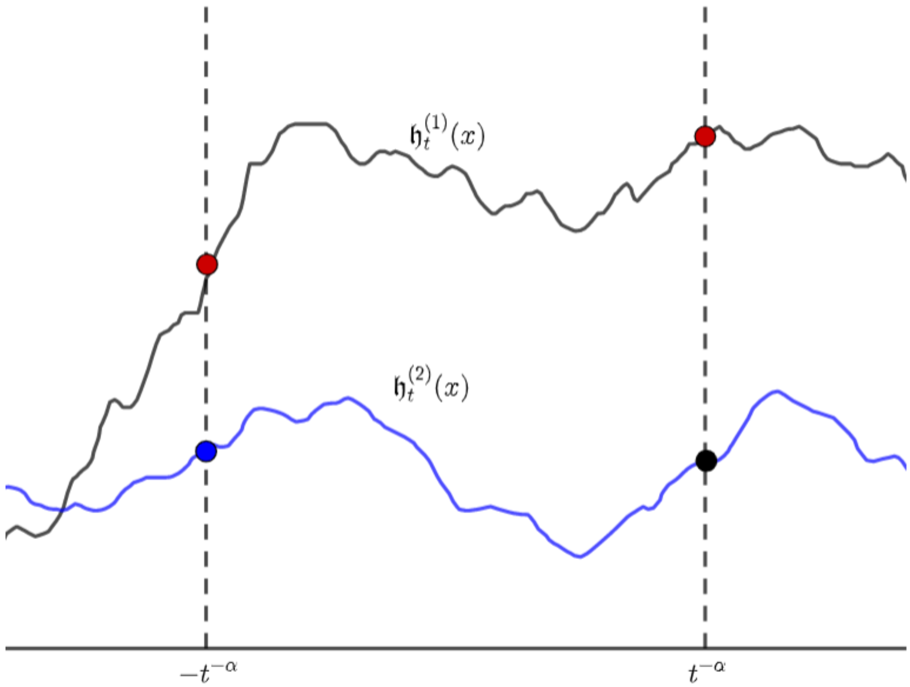


Figure 2.6: In the above figure $\mathsf{Gap}_t(\delta)$ defined in (2.6.3) denotes the event that the value of the blue point is smaller than the value of each of the red points at least by δ, The $\mathsf{Rise}_t(\delta)$ event defined in (2.6.4) requires *no* point on the whole blue curve (restricted to $I_t = (-t^{-\alpha}, t^{-\alpha})$) exceed the value of the blue point by a factor $\frac{1}{4}\delta$ (i.e., there is no significant rise). The $\mathsf{Tight}_t(\delta)$ defined in (2.6.5) event ensures the value of the red points are within $[-\delta^{-1}, \delta^{-1}]$. The $\mathsf{Fluc}_t^{(i)}(\delta)$ event defined in (2.6.15) signifies every value of every point on the i-th curve (restricted to I_t) is within $\frac{1}{4}\delta$ distance away from its value on the left boundary: $\mathfrak{h}_t^{(1)}(-t^{-\alpha})$. Finally, $\mathsf{Sink}_t(\delta)$ event defined in (2.6.20) denotes the event that no point on the black curve (restricted to I_t) drops below the value of the red points by a factor larger than $\frac{1}{4}\delta$, (i.e., there is no significant sink).

Using these two preceding set relations, by union bound we have

$$\begin{aligned}\mathbf{P}\left(\neg\mathsf{Gap}_t(\delta)\right) &\le \mathbf{P}\left(|\mathfrak{h}_t^{(1)}(-t^{-\alpha}) - \mathfrak{h}_t^{(1)}(t^{-\alpha})| \ge \tfrac{1}{4}\delta\right) + \mathbf{P}\left(\neg\mathsf{Gap}_t(\delta) \cap |\mathfrak{h}_t^{(1)}(-t^{-\alpha}) - \mathfrak{h}_t^{(1)}(t^{-\alpha})| \le \tfrac{1}{4}\delta\right) \\ &\le \mathbf{P}\left(\neg\mathsf{Rise}_t^{(1)}(\delta)\right) + \mathbf{P}\left(\inf_{x\in[-1,0]}[\mathfrak{h}_t^{(2)}(x) - \mathfrak{h}_t^{(1)}(x)] \ge -\tfrac{5}{4}\delta\right).\end{aligned}$$

As $t \to \infty$, the first term goes to zero due (2.6.16) and by Proposition 2.2.7, the second term goes to

$$\mathbf{P}\left(\inf_{x\in[-1,0]}[\mathcal{A}_2(2^{-1/3}x) - \mathcal{A}_1(2^{-1/3}x)] \ge -\tfrac{5}{4\cdot 2^{1/3}}\delta\right).$$

But by (2.2.1) we know Airy line ensembles are strictly ordered. Thus the above probability can be made arbitrarily small be choose δ small enough. In particular, there exists a $\delta_1 \in (0,1)$ such that for all $\delta \in (0,\delta_1)$ the above probability is always less than $\frac{\varepsilon}{2}$. This forces

$$\liminf_{t\to\infty} \mathbf{P}\left(\mathsf{Gap}_t(\delta)\right) \ge 1 - \tfrac{\varepsilon}{2}. \tag{2.6.17}$$

Recall $\mathsf{Rise}_t(\delta)$ from (2.6.4). Clearly $\mathsf{Rise}_t(\delta) \subset \mathsf{Fluc}_t^{(2)}(\delta)$. Thus for every $\delta > 0$,

$$\liminf_{t\to\infty} \mathbf{P}(\mathsf{Rise}_t(\delta)) = 1. \tag{2.6.18}$$

Finally using Proposition 2.2.8 (a) and (b) we see that $\mathfrak{h}_t^{(1)}(t^{-\alpha}), \mathfrak{h}_t^{(1)}(t^{-\alpha})$ are tight. Thus there exists $\delta_2 \in (0,1)$ such that for all $\delta \in (0,\delta_2)$, we have

$$\liminf_{t\to\infty} \mathbf{P}\left(\mathsf{Tight}_t(\delta)\right) \ge 1 - \tfrac{\varepsilon}{2}. \tag{2.6.19}$$

Combining (2.6.17), (2.6.18), (2.6.19), and recalling the definition of $\mathsf{Fav}_t(\delta)$ from (2.6.6), by union bound we get (2.6.7) for all $\delta \in (0, \min\{\delta_1,\delta_2\})$.

Let us now prove (2.6.16). Recall $\mathsf{Fluc}_t^{(i)}(\delta)$ from (2.6.15). Define the event:

$$\mathsf{Conv}_t(\delta) := \left\{ \sup_{x\in[-1,1]} |\mathfrak{h}_t^{(i)}(x) - 2^{-1/3}\mathcal{A}_i(2^{-1/3}x)| \le \tfrac{1}{16}\delta \right\}.$$

Observe that

$$\left\{\neg\mathsf{Fluc}_t^{(i)}(\delta), \mathsf{Conv}_t(\delta)\right\} \subset \left\{ \sup_{|x|\le 2^{-1/3}t^{-\alpha}} \left[\mathcal{A}_i(x) - \mathcal{A}_i(-2^{-1/3}t^{-\alpha})\right] \ge \tfrac{2^{1/3}}{8}\delta \right\}.$$

Thus by union bound

$$\mathbf{P}\left(\neg\mathsf{Fluc}_t^{(i)}(\delta)\right) \le \mathbf{P}\left(\neg\mathsf{Conv}_t(\delta)\right) + \mathbf{P}\left(\neg\mathsf{Fluc}_t^{(i)}(\delta), \mathsf{Conv}_t(\delta)\right)$$

$$\leq \mathbf{P}\left(\neg \mathsf{Conv}_t(\delta)\right) + \mathbf{P}\left(\sup_{|x|\leq 2^{-1/3}t^{-\alpha}} \left[\mathcal{A}_i(x) - \mathcal{A}_i(-2^{-1/3}t^{-\alpha})\right] \geq \tfrac{2^{1/3}}{8}\delta\right).$$

By (2.6.14), the first term above goes to zero as $t \to \infty$, whereas the second term goes to zero as $t \to \infty$, via modulus of continuity of Airy line ensembles from Proposition 2.2.4. Note that in Proposition 2.2.4 the modulus of continuity is stated for $\mathcal{A}_i(x) + x^2$. However, in the above scenario since we deal with a vanishing interval $[-2^{-1/3}t^{-\alpha}, 2^{-1/3}t^{-\alpha}]$, the parabolic term does not play any role. This establishes (2.6.16).

Step 5. In this step we prove (2.6.12). Let us consider the event

$$\mathsf{Sink}_t(\delta) := \left\{\inf_{x\in I_t} \mathfrak{h}_t^{(1)}(x) \geq -\tfrac{1}{4}\delta + \min\{\mathfrak{h}_t(-t^{-\alpha}), \mathfrak{h}_t(t^{-\alpha})\}\right\}. \tag{2.6.20}$$

See Figure 2.6 and its caption for an interpretation of this event. Recall $\mathsf{Gap}_t(\delta)$ and $\mathsf{Rise}_t(\delta)$ from (2.6.3) and (2.6.4). Observe that on the event $\mathsf{Gap}_t(\delta) \cap \mathsf{Rise}_t(\delta)$, we have $\sup_{x\in I_t} \mathfrak{h}_t^{(2)}(x) \leq \min\{\mathfrak{h}_t(-t^{-\alpha}), \mathfrak{h}_t(t^{-\alpha})\} - \frac{3}{4}\delta$. Thus on $\mathsf{Gap}_t(\delta) \cap \mathsf{Rise}_t(\delta) \cap \mathsf{Sink}_t(\delta)$, we have

$$\inf_{x\in I_t} \left[\mathfrak{h}_t^{(1)}(x) - \mathfrak{h}_t^{(2)}(x)\right] \geq \tfrac{1}{2}\delta.$$

Recall W from (2.6.11). On the event $\{\inf_{x\in I_t} \left[\mathfrak{h}_t^{(1)}(x) - \mathfrak{h}_t^{(2)}(x)\right] \geq \frac{1}{2}\delta\}$ we have the pointwise inequality

$$W > \exp(-2t^{2/3-\alpha}e^{-\frac{1}{2}t^{1/3}\delta}) \geq 1 - \delta,$$

where we choose a $t_1(\delta) > 0$ so that the last inequality is true for all $t \geq t_1$. Thus for all $t \geq t_1$,

$$\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_{\mathrm{free},t}(W \geq 1-\delta) \geq \mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_{\mathrm{free},t}(\mathsf{Sink}_t(\delta)). \tag{2.6.21}$$

Recall that $\mathbf{P}_{\mathrm{free},t}$ denotes the law of a Brownian bridge $B_1(\cdot)$ on I_t starting at $B_1(-t^{-\alpha}) = \mathfrak{h}_t(-t^{-\alpha})$ and ending at $B_1(t^{-\alpha}) = \mathfrak{h}_t(t^{-\alpha})$. Let us consider another Brownian bridge $\widetilde{B}_1(\cdot)$ on I_t starting and ending at $\min\{\mathfrak{h}_t(-t^{-\alpha}), \mathfrak{h}_t(t^{-\alpha})\}$. By standard estimates for Brownian bridge (see Lemma 2.11 in

[74] for example)

$$\mathbf{P}\left(\inf_{x\in I_t} \widetilde{B}_1(x) \ge -\tfrac{1}{4}\delta + \min\{\mathfrak{h}_t(-t^{-\alpha}), \mathfrak{h}_t(t^{-\alpha})\}\right) = 1-\exp\left(-\tfrac{\delta^2}{8|I_t|}\right) = 1-\exp\left(-\tfrac{\delta^2}{16}t^{\alpha}\right).$$

Note that $B_1(\cdot)$ is stochastically larger than $\widetilde{B}_1(\cdot)$. Since the above event is increasing, we thus have $\mathbf{P}_{\text{free},t}\left(\mathsf{Sink}_t(\delta)\right)$ is at least $1-\exp\left(-\frac{\delta^2}{16}t^{\alpha}\right)$. We now choose $t_2(\delta) > 0$, such that $1-\exp\left(-\frac{\delta^2}{16}t^{\alpha}\right) \ge 1-\delta$. Taking $t_0 = \max\{t_1, t_2\}$, we thus get (2.6.12) from (4.5.21).

Step 6. In this step we prove (2.6.13). As before consider the Brownian bridge $B_1(\cdot)$ on I_t starting at $B_1(-t^{-\alpha}) = \mathfrak{h}_t(-t^{-\alpha})$ and ending at $B_1(t^{-\alpha}) = \mathfrak{h}_t(t^{-\alpha})$. We may write B_1 as

$$B_1(x) = \mathfrak{h}_t^{(1)}(-t^{-\alpha}) + \frac{x+t^{-\alpha}}{2t^{-\alpha}}(\mathfrak{h}_t^{(1)}(t^{-\alpha}) - \mathfrak{h}_t^{(1)}(-t^{-\alpha})) + \overline{B}(x).$$

where $\overline{B}$ is a Brownian bridge on I_t starting and ending at zero. Thus,

$$t^{1/3}(B_1(t^{-2/3}x) - B_1(0)) = t^{1/3}\left[\overline{B}(t^{-2/3}x) - \overline{B}(0)\right] + \tfrac{1}{2}t^{\alpha-1/3}x(\mathfrak{h}_t^{(1)}(t^{-\alpha}) - \mathfrak{h}_t^{(1)}(-t^{-\alpha})). \tag{2.6.22}$$

Recall that $\alpha = \frac{1}{6}$. By Brownian scaling, $B_*(x) := t^{1/3}\overline{B}(t^{-2/3}x)$ is a Brownian bridge on the large interval $[-\sqrt{t}, \sqrt{t}]$ starting and ending at zero. By computing the covariances, it is easy to check that as $t\to\infty$, $B_*(x) - B_*(0)$ converges weakly to a two-sided Brownian motion $B(\cdot)$ on $[-a, a]$. This gives us the weak limit for the first term on the r.h.s. of (2.6.22). For the second term, recall the event $\mathsf{Tight}_t(\delta)$ from (2.6.5). As $|x| \le a$, on $\mathsf{Tight}_t(\delta)$, we have

$$\tfrac{1}{2}t^{\alpha-1/3}x(\mathfrak{h}_t^{(1)}(t^{-\alpha}) - \mathfrak{h}_t^{(1)}(-t^{-\alpha})) \le t^{-1/6}a\delta^{-1}.$$

This gives an uniform bound (uniform over the event $\mathsf{Fav}_t(\delta)$) on the second term in (2.6.22). Thus as long as the boundary data is in $\mathsf{Tight}_t(\delta)$, $\mathbf{P}_{\text{free},t}(A) \to \gamma(A)$ where $\gamma(A) = \mathbf{P}(B(\cdot)\in A)$. This proves (2.6.13).

2.6.2 Dyson Behavior around joint maximum

In this subsection we state and prove Proposition 2.6.1.

Proposition 2.6.1 (Dyson behavior around joint maximum)**.** *Fix $p \in (0,1)$. Set $q = 1-p$. Consider 2 independent copies of the KPZ equation $\mathcal{H}_{\uparrow}(x,t)$, and $\mathcal{H}_{\downarrow}(x,t)$, both started from the narrow wedge initial data. Let $\mathcal{M}_{p,t}$ be the almost sure unique maximizer of the process $x \mapsto (\mathcal{H}_{\uparrow}(x,pt)+\mathcal{H}_{\downarrow}(x,qt))$ which exists via Lemma 2.3.1. Set*

$$\begin{aligned} D_1(x,t) &:= \mathcal{H}_{\uparrow}(\mathcal{M}_{p,t},pt) - \mathcal{H}_{\uparrow}(x+\mathcal{M}_{p,t},pt), \\ D_2(x,t) &:= \mathcal{H}_{\downarrow}(x+\mathcal{M}_{p,t},qt) - \mathcal{H}_{\downarrow}(\mathcal{M}_{p,t},qt). \end{aligned} \tag{2.6.23}$$

As $t \to \infty$, we have the following convergence in law

$$(D_1(x,t), D_2(x,t)) \xrightarrow{d} (\mathcal{D}_1(x), \mathcal{D}_2(x)) \tag{2.6.24}$$

in the uniform-on-compact topology. Here $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2) : \mathbb{R} \to \mathbb{R}^2$ is a two-sided DBM*, that is, $\mathcal{D}_+(\cdot) := \mathcal{D}(\cdot) \mid_{[0,\infty)}$ and $\mathcal{D}_-(\cdot) := \mathcal{D}(-\cdot) \mid_{(-\infty,0]}$ are independent copies of* DBM *defined in Definition 2.5.1.*

For clarity, the proof is completed over several subsections (Sections 2.6.2-2.6.2) below and we refer to Figure 2.7 for the structure of the proof.

KPZ line ensemble framework

In this subsection, we convert Proposition 2.6.1 into the language of scaled KPZ line ensemble defined in Proposition 3.5.1. We view $\mathcal{H}_{\uparrow}(x,t) = \mathcal{H}_{t,\uparrow}^{(1)}(x), \mathcal{H}_{\downarrow}(x,t) = \mathcal{H}_{t,\downarrow}^{(1)}(x)$ as the top curves of two (unscaled) KPZ line ensembles: $\{\mathcal{H}_{t,\uparrow}^{(n)}(x), \mathcal{H}_{t,\downarrow}^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$. Following (2.2.5) we define their

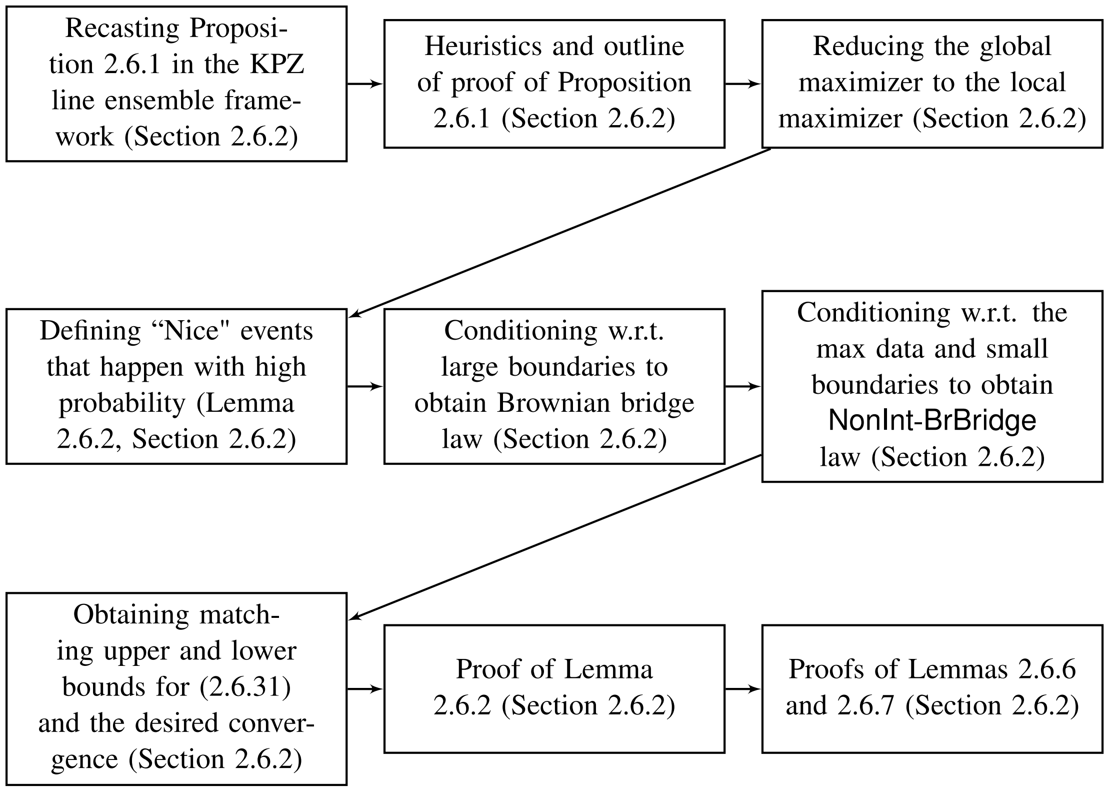


Figure 2.7: Structure of Section 2.6.2.

scaled versions:

$$\mathfrak{h}_{t,\uparrow}^{(n)}(x) := t^{-1/3}\left(\mathcal{H}_{t,\uparrow}^{(n)}(t^{2/3}x) + \tfrac{t}{24}\right), \qquad \mathfrak{h}_{t,\downarrow}^{(n)}(x) := t^{-1/3}\left(\mathcal{H}_{t,\downarrow}^{(n)}(t^{2/3}x) + \tfrac{t}{24}\right).$$

Along with the full maximizer $\mathcal{M}_{p,t}$, we will also consider local maximizer defined by

$$\mathcal{M}_{p,t}^{M} := \underset{x\in[-Mt^{2/3},Mt^{2/3}]}{\operatorname{argmax}} (\mathcal{H}_{pt,\uparrow}^{(1)}(x) + \mathcal{H}_{qt,\downarrow}^{(1)}(x)), \qquad M \in [0,\infty]. \tag{2.6.25}$$

For each $M > 0$, $\mathcal{M}_{p,t}^{M}$ is unique almost surely by $\mathbf{H}_t$-Brownian Gibbs property. We now set

$$\begin{aligned} Y_{M,t,\uparrow}^{(n)}(x) &:= p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(n)}\big((pt)^{-2/3}\mathcal{M}_{p,t}^{M}\big) - p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(n)}\big(p^{-2/3}x\big), \\ Y_{M,t,\downarrow}^{(n)}(x) &:= q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(n)}\big(q^{-2/3}x\big) - q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(n)}\big((qt)^{-2/3}\mathcal{M}_{p,t}^{M}\big). \end{aligned} \tag{2.6.26}$$

Taking into account of (2.6.23) and all the above new notations, it can now be checked that for each $t > 0$,

$$D_1(x,t) \stackrel{d}{=} t^{1/3} Y^{(1)}_{\infty,t,\uparrow}\big(t^{-2/3}(\mathcal{M}^{\infty}_{p,t} + x)\big), \quad D_2(x,t) \stackrel{d}{=} t^{1/3} Y^{(1)}_{\infty,t,\downarrow}\big(t^{-2/3}(\mathcal{M}^{\infty}_{p,t} + x)\big), \tag{2.6.27}$$

both as functions in x. Thus it is equivalent to verify Proposition 2.6.1 for the above $Y^{(1)}_{\infty,t,\uparrow}, Y^{(1)}_{\infty,t,\downarrow}$ expressions. In our proof we will mostly deal with local maximizer version, and so for convenience we define:

$$D_{M,t,\uparrow}(x) := t^{1/3} Y^{(1)}_{M,t,\uparrow}\big(t^{-2/3}(\mathcal{M}^{M}_{p,t} + x)\big), \quad D_{M,t,\downarrow}(x) = t^{1/3} Y^{(1)}_{M,t,\downarrow}\big(t^{-2/3}(\mathcal{M}^{M}_{p,t} + x)\big). \tag{2.6.28}$$

where $Y^{(1)}_{M,t,\uparrow}, Y^{(1)}_{M,t,\downarrow}$ are defined in (2.6.26). We will introduce several other notations and parameters later in the proof. For the moment, the minimal set of notations introduced here facilitate our discussion of ideas and outline of the proof of Proposition 2.6.1 in the next subsection.

Ideas and Outline of Proof of Proposition 2.6.1

Before embarking on a rather lengthy proof, in this subsection we explain the core ideas behind the proof and provide an outline for the remaining subsections.

First we contrast the proof idea with that of Theorem 2.1.11. Indeed, similar to the proof of Theorem 2.1.11, from (2.6.27) we see that at the level $Y^{(1)}_{\infty,t,\uparrow}, Y^{(1)}_{\infty,t,\downarrow}$ we are interested in understanding their laws restricted to a very small symmetric interval of order $O(t^{-2/3})$ around the point $t^{-2/3}\mathcal{M}^{\infty}_{p,t}$. However, the key difference from the conceptual argument presented at the beginning of the proof if Theorem 2.1.11 is that the centered point $t^{-2/3}\mathcal{M}^{\infty}_{p,t}$ is random and it does not go to zero. Rather by Theorem 2.1.8 it converges in distribution to a nontrivial random quantity (namely $\Gamma(p\sqrt{2})$). Hence one must take additional care of this random point. This makes the argument significantly more challenging compared to that of Theorem 2.1.11.

We now give a road-map of our proof. At this point, readers are also invited to look into Figure

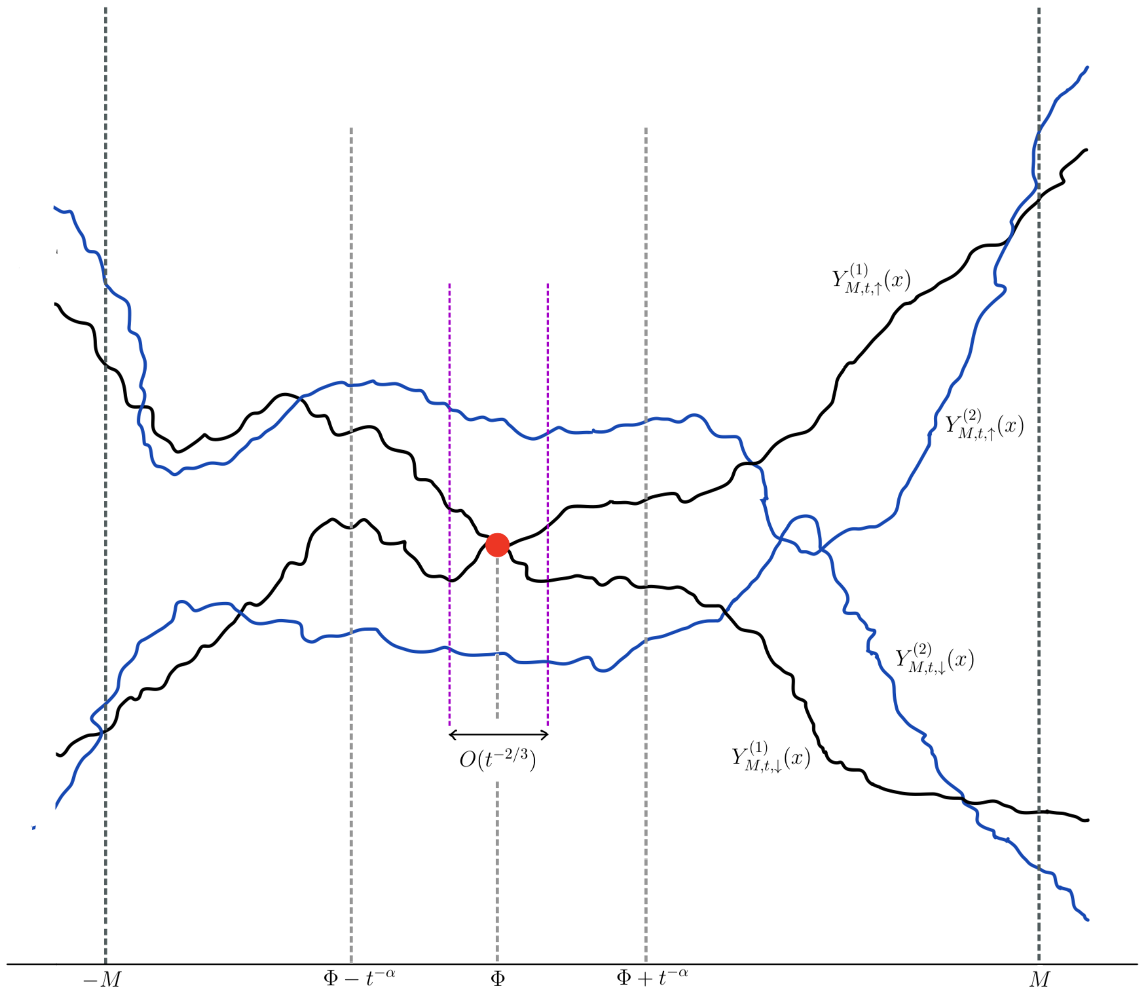


Figure 2.8: An overview of the proof for Proposition 2.6.1. The top and bottom black curves are $Y^{(1)}_{M,t,\uparrow}$ and $Y^{(1)}_{M,t,\downarrow}$ respectively. Note that the way they are defined in (2.6.26), $Y^{(1)}_{M,t,\uparrow}(x) \geq Y^{(1)}_{M,t,\downarrow}(x)$ with equality at $x = \Phi = t^{-2/3}\mathcal{M}^{M}_{p,t}$ labelled as the red dot in the above figure. The blue curves are $Y^{(1)}_{M,t,\uparrow}, Y^{(2)}_{M,t,\downarrow}$. There is no such ordering within blue curves. They may intersect among themselves as well as with the black curves. With $\alpha = \frac{1}{6}$, we consider the interval $K_t = (\Phi - t^{-\alpha}, \Phi + t^{-\alpha})$. In this vanishing interval around Φ, the curves will be ordered with high probability. In fact, with high probability, there will be a uniform separation. For instance, for small enough δ, we will have $Y^{(2)}_{M,t,\uparrow}(x) - Y^{(1)}_{M,t,\uparrow}(x) \geq \frac{1}{4}\delta$, and $Y^{(1)}_{M,t,\downarrow}(x) - Y^{(2)}_{M,t,\downarrow}(x) \geq \frac{1}{4}\delta$, for all $x \in K_t$ wth high probability. This will allow us to conclude black curves are behave approximately like two-sided NonInt-BrBridges on that narrow window. Then upon going into a even smaller window of $O(t^{-2/3})$, the two-sided NonInt-BrBridges turn into a two-sided DBM.

2.8 alongside the explanation offered in its caption.

- As noted in Lemma 2.3.1, the random centering $t^{-2/3}\mathcal{M}^{\infty}_{p,t}$ has decaying properties and can be approximated by $t^{-2/3}\mathcal{M}^{M}_{p,t}$ by taking large enough M. Hence on a heuristic level it suffices to work with the local maximizers instead. In Subsection 2.6.2, this heuristics will be justified rigorously. We will show there how to pass from $Y^{(1)}_{\infty,t,\uparrow}, Y^{(1)}_{\infty,t,\downarrow}$ defined in (2.6.27) to their finite centering analogs: $Y^{(1)}_{M,t,\uparrow}, Y^{(1)}_{M,t,\downarrow}$. The rest of the proof then boils down to analyzing the laws of the latter.

- We now fix a $M > 0$ for the rest of the proof. Our analysis will now operate with $\mathcal{M}^{M}_{p,t}$. For simplicity, let us also use the notation

$$\Phi := t^{-2/3}\mathcal{M}^{M}_{p,t} \tag{2.6.29}$$

for the rest of the proof. We now perform several conditioning on the laws of the curves. Recall that by Proposition 3.5.1, $\{\mathfrak{h}^{(n)}_{pt,\uparrow}(\cdot)\}_{n\in\mathbb{N}}$ $\{\mathfrak{h}^{(n)}_{qt,\downarrow}(\cdot)\}_{n\in\mathbb{N}}$ satisfy the $\mathbf{H}_{pt}$-Brownian Gibbs property and the $\mathbf{H}_{qt}$-Brownian Gibbs property respectively with $\mathbf{H}_t$ given by (2.2.4). Conditioned on the end points of $\mathfrak{h}^{(1)}_{pt,\uparrow}(\pm Mp^{-2/3})$ and $\mathfrak{h}^{(1)}_{qt,\downarrow}(\pm Mq^{-2/3})$ and the second curves $\mathfrak{h}^{(2)}_{pt,\uparrow}(\cdot)$ and $\mathfrak{h}^{(2)}_{qt,\downarrow}(\cdot)$, the laws of $\mathfrak{h}^{(1)}_{pt,\uparrow}(\cdot)$, and $\mathfrak{h}^{(1)}_{pt,\uparrow}(\cdot)$ are absolutely continuous w.r.t. Brownian bridges with appropriate end points. This conditioning is done in Subsection 2.6.2.

- We then condition further on *Max data* : $\mathcal{M}^{M}_{p,t}, \mathfrak{h}^{(1)}_{pt,\uparrow}((pt)^{-2/3}\mathcal{M}^{M}_{p,t}), \mathfrak{h}^{(1)}_{qt,\downarrow}((qt)^{-2/3}\mathcal{M}^{M}_{p,t})$. Under this conditioning, via the decomposition result in Proposition 2.4.9, the underlying Brownian bridges mentioned in the previous point, when viewed from the joint maximizer, becomes two-sided NonInt-BrBridges defined in Definition 2.4.4. This viewpoint from the joint maximizer is given by $Y^{(1)}_{M,t,\uparrow}, Y^{(1)}_{M,t,\downarrow}$. See Figure 2.8 for more details.

- We emphasize the fact that the deduction of NonInt-BrBridges done above is only for the underlying Brownian law. One still needs to analyze the Radon-Nikodym (RN) derivative. As we are interested in the laws of $Y^{(1)}_{M,t,\uparrow}, Y^{(1)}_{M,t,\downarrow}$ on an interval of order $t^{-2/3}$ around Φ, we analyze the RN derivative only on a small interval around Φ. To be precise, we consider a slightly larger yet vanishing interval of length $2t^{-\alpha}$ for $\alpha = \frac{1}{6}$ around the random point Φ. We show that the RN derivative on this small random patch is close to 1. Thus upon further conditioning on the boundary data of this random small interval, the trajectories of $Y_{M,t,\uparrow}^{(1)}$ and $Y_{M,t,\downarrow}^{(1)}$ defined in (2.6.26) around Φ turns out to be close to two-sided NonInt-BrBridge with appropriate (random) endpoints.

- Finally, we zoom further into a tiny interval of order $O(t^{-2/3})$ symmetric around the random point Φ. Utilizing Lemma 2.5.3, we convert the two-sided NonInt-BrBridges to two-sided DBMs.

We now provide an outline of the rest of the subsections. In Subsection 2.6.2 we reduce our proof from understanding laws around global maximizers to that of local maximizers. As explained in the above road-map, the proof follows by performing several successive conditioning. On a technical level, this requires defining several high probability events on which we can carry out our conditional analysis. These events are all defined in Subsection 2.6.2 and are claimed to happen with high probability in Lemma 2.6.2. We then execute the first layer of conditioning in Subsection 2.6.2. The two other layers of conditioning are done in Subsection 2.6.2. Lemma 2.6.6 and Lemma 2.6.7 are the precise technical expressions for the heuristic claims in the last two bullet points of the road-map. Assuming them, we complete the final steps of the proof in Subsection 2.6.2. Proof of Lemma 2.6.2 is then presented in Subsection 2.6.2. Finally, in Subsection 2.6.2, we prove the remaining lemmas: Lemma 2.6.6 and 2.6.7.

Global to Local maximizer

We now fill out the technical details of the road-map presented in the previous subsection. Fix any $a > 0$. Consider any Borel set A of $C([-a, a] \to \mathbb{R}^2)$ which is a continuity set of a two-sided DBM $\mathcal{D}(\cdot)$ restricted to $[-a, a]$. By Portmanteau theorem, it is enough to show that

$$\mathbf{P}((D_1(\cdot, t), D_2(\cdot, t)) \in A) \to \mathbf{P}(\mathcal{D}(\cdot) \in A), \tag{2.6.30}$$

where D_1, D_2 are defined in (2.6.23). In this subsection, we describe how it suffices to check (2.6.30) with $\mathcal{M}_{p,t}^M$. Recall $D_{M,t,\uparrow}(\cdot), D_{M,t,\downarrow}(\cdot)$ from (2.6.28). We claim that for all $M > 0$:

$$\lim_{t\to\infty} \mathbf{P}((D_{M,t,\uparrow}(\cdot), D_{M,t,\downarrow}(\cdot)) \in A) \to \mathbf{P}(\mathcal{D}(\cdot) \in A). \tag{2.6.31}$$

Note that when $\mathcal{M}_{p,t}^\infty = \mathcal{M}_{p,t}^M$, $(D_{M,t,\uparrow}(\cdot), D_{M,t,\downarrow}(\cdot))$ is exactly equal to

$$t^{1/3}Y_{\infty,t,\uparrow}^{(1)}\big(t^{-2/3}(\mathcal{M}_{p,t}^\infty + \cdot)\big)\ ,\ t^{1/3}Y_{\infty,t,\downarrow}^{(1)}\big(t^{-2/3}(\mathcal{M}_{p,t}^\infty + \cdot)\big)$$

which via (2.6.27) is same in distribution as $D_1(\cdot, t), D_2(\cdot, t)$. Thus,

$$\Big|\mathbf{P}((D_1(\cdot, t), D_2(\cdot, t)) \in A) - \mathbf{P}((D_{M,t,\uparrow}(\cdot), D_{M,t,\downarrow}(\cdot)) \in A)\Big| \le 2\mathbf{P}(\mathcal{M}_{p,t} \neq \mathcal{M}_{p,t}^M).$$

Now given any $\varepsilon > 0$, by Lemma 2.3.1, we can take $M = M(\varepsilon) > 0$ large enough so that $2\mathbf{P}(\mathcal{M}_{p,t} \neq \mathcal{M}_{p,t}^M) \le \varepsilon$. Then upon taking $t \to \infty$ in the above equation, in view of (2.6.31), we see that

$$\limsup_{t\to\infty} ||\mathbf{P}((D_1(\cdot, t), D_2(\cdot, t)) \in A) - \mathbf{P}((\mathcal{D}(\cdot) \in A)| \le \varepsilon.$$

As ε is arbitrary, this proves (2.6.30). The rest of the proof is now devoted in proving (2.6.31).

Nice events

In this subsection, we focus on defining several events that are collectively 'nice' in the sense that they happen with high probability. We fix an $M > 0$ for the rest of the proof and work with the local maximizer $\mathcal{M}_{p,t}^M$ defined in (2.6.25). We will also make use of the notation Φ defined in (2.6.29) heavily in this and subsequent subsections. We now proceed to define a few events based on the location and value of the maximizer and values at the endpoints of an appropriate interval. Fix any arbitrary $\delta > 0$. Let us consider the event:

$$\mathsf{ArMx}(\delta) := \{\Phi \in [-M + \delta, M - \delta]\}\,. \tag{2.6.32}$$

The $\mathsf{ArMx}(\delta)$ controls the location of the local maximizer Φ. Set $\alpha = \frac{1}{6}$. We define tightness event that corresponds to the boundary of the interval of length $2t^{-\alpha}$ around Φ :

$$\mathsf{Bd}_{\uparrow}(\delta) := \mathsf{Bd}_{+,\uparrow}(\delta) \cap \mathsf{Bd}_{-,\uparrow}(\delta), \quad \mathsf{Bd}_{\downarrow}(\delta) := \mathsf{Bd}_{+,\downarrow}(\delta) \cap \mathsf{Bd}_{-,\downarrow}(\delta), \tag{2.6.33}$$

where

$$\begin{aligned} \mathsf{Bd}_{\pm,\uparrow}(\delta) &:= \left\{ \left| \mathfrak{h}_{pt,\uparrow}^{(1)}\big(p^{-2/3}(\Phi \pm t^{-\alpha})\big) - \mathfrak{h}_{pt,\uparrow}^{(1)}\big(\Phi p^{-2/3}\big) \right| \le \tfrac{1}{\delta} t^{-\alpha/2} \right\} \\ \mathsf{Bd}_{\pm,\downarrow}(\delta) &:= \left\{ \left| \mathfrak{h}_{qt,\downarrow}^{(1)}\big(q^{-2/3}(\Phi \pm t^{-\alpha})\big) - \mathfrak{h}_{qt,\downarrow}^{(1)}\big(\Phi q^{-2/3}\big) \right| \le \tfrac{1}{\delta} t^{-\alpha/2} \right\}, \end{aligned} \tag{2.6.34}$$

Finally we consider the gap events that provide a gap between the first curve and the second curve for each of the line ensemble:

$$\mathsf{Gap}_{M,\uparrow}(\delta) := \left\{ p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}\big(\Phi p^{-2/3}\big) \ge p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}\big(\Phi p^{-2/3}\big) + \delta \right\}, \tag{2.6.35}$$

$$\mathsf{Gap}_{M,\downarrow}(\delta) := \left\{ q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(1)}\big(\Phi q^{-2/3}\big) \ge q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(2)}\big(\Phi q^{-2/3}\big) + \delta \right\}. \tag{2.6.36}$$

We next define the 'rise' events which roughly says the second curves $\mathfrak{h}_{pt,\uparrow}^{(1)}$ and $\mathfrak{h}_{qt,\downarrow}^{(2)}$ of the line ensembles does not rise too much on a small interval of length $2t^{-\alpha}$ around $\Phi p^{-2/3}$ and $\Phi q^{-2/3}$ respectively.

$$\mathsf{Rise}_{M,\uparrow}(\delta) := \left\{ \sup_{x \in [-t^{-\alpha}, t^{-\alpha}]} p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}\big(\Phi p^{-2/3} + x\big) \le p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}\big(\Phi p^{-2/3}\big) + \tfrac{\delta}{4} \right\}, \tag{2.6.37}$$

$$\mathsf{Rise}_{M,\downarrow}(\delta) := \left\{ \sup_{x \in [-t^{-\alpha}, t^{-\alpha}]} q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(2)}\big(\Phi q^{-2/3} + x\big) \le q^{1/3}\mathfrak{h}_{pt,\downarrow}^{(2)}\big(\Phi q^{-2/3}\big) + \tfrac{\delta}{4} \right\}. \tag{2.6.38}$$

Bd, Gap, Rise type events and their significance are discussed later in Subsection 2.6.2 in greater details. See also Figure 2.9 and its caption for explanation of some of these events. We put all the above events into one final event:

$$\mathsf{Nice}_M(\delta) := \left\{ \mathsf{ArMx}(\delta) \cap \bigcap_{x\in\{\uparrow,\downarrow\}} \mathsf{Bd}_x(\delta) \cap \mathsf{Gap}_{M,x}(\delta) \cap \mathsf{Rise}_{M,x}(\delta) \right\}. \tag{2.6.39}$$

All the above events are dependent on t. But we have suppressed this dependence from the notations. The $\mathsf{Nice}_M(\delta)$ turns out to be a favorable event. We isolate this fact as a lemma below.

Lemma 2.6.2. *For any $M > 0$, under the above setup we have*

$$\liminf_{\delta\downarrow 0} \liminf_{t\to\infty} \mathbf{P}_t\left(\mathsf{Nice}_M(\delta)\right) = 1. \tag{2.6.40}$$

We postpone the proof of this technical lemma to Section 2.6.2 and for the moment we continue with the current proof of Proposition 2.6.1 assuming its validity.

Conditioning with respect to large boundaries

As alluded in Subsection 2.6.2, the proof involves conditioning on different σ-fields successively. We now specify all the different σ-fields that we will use throughout the proof. Set $\alpha = \frac{1}{6}$. We consider the random interval

$$K_t := (\Phi - t^{-\alpha}, \Phi + t^{-\alpha}). \tag{2.6.41}$$

Let us define:

$$\mathcal{F}_1 := \sigma\left(\left\{ \mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x), \mathfrak{h}^{(1)}_{qt,\downarrow}(q^{-2/3}x) \right\}_{x\in(-M,M)^c}, \left\{ \mathfrak{h}^{(2)}_{pt,\uparrow}(x), \mathfrak{h}^{(2)}_{qt,\downarrow}(x) \right\}_{x\in\mathbb{R}} \right) \tag{2.6.42}$$

$$\mathcal{F}_2 := \sigma\left(\Phi, \mathfrak{h}^{(1)}_{pt,\uparrow}(\Phi p^{-2/3}), \mathfrak{h}^{(1)}_{qt,\downarrow}(\Phi q^{-2/3}) \right), \tag{2.6.43}$$

$$\mathcal{F}_3 := \sigma\left(\left\{ \mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x), \mathfrak{h}^{(1)}_{qt,\downarrow}(q^{-2/3}x) \right\}_{x\in K_t^c} \right). \tag{2.6.44}$$

In this step we perform conditioning w.r.t. $\mathcal{F}_1$ for the expression on the l.h.s. of (2.6.31). We denote $\mathbf{P}_t(A) := \mathbf{P}\big((D_{M,t,\uparrow}(\cdot), D_{M,t,\downarrow}(\cdot)) \in A\big)$. Taking the $\mathsf{Nice}_M(\delta)$ event defined in (2.6.39) under consideration, upon conditioning with $\mathcal{F}_1$ we have the following upper and lower bounds:

$$\mathbf{P}_t(A) \geq \mathbf{P}_t(\mathsf{Nice}_M(\delta), A) = \mathbf{E}_t\left[\mathbf{P}_t(\mathsf{Nice}_M(\delta), A \mid \mathcal{F}_1)\right], \tag{2.6.45}$$

$$\mathbf{P}_t(A) \leq \mathbf{P}_t(\mathsf{Nice}_M(\delta), A) + \mathbf{P}_t(\neg\mathsf{Nice}_M(\delta)) = \mathbf{E}_t\left[\mathbf{P}_t(\mathsf{Nice}_M(\delta), A \mid \mathcal{F}_1)\right] + \mathbf{P}_t(\neg\mathsf{Nice}_M(\delta)). \tag{2.6.46}$$

Note that the underlying measure consists of the mutually independent $\mathfrak{h}_{pt,\uparrow}^{(1)}(\cdot)$ and $\mathfrak{h}_{qt,\downarrow}^{(1)}(\cdot)$ which by Proposition 3.5.1 satisfy $\mathbf{H}_{pt}$ and $\mathbf{H}_{qt}$ Brownian Gibbs property respectively. Applying the respectively Brownian Gibbs properties and following (2.2.3) we have

$$\mathbf{P}_t(\mathsf{Nice}_M(\delta), A \mid \mathcal{F}_1) = \frac{\mathbf{E}_{\mathrm{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta),A} W_\uparrow W_\downarrow]}{\mathbf{E}_{\mathrm{free},t}[W_\uparrow W_\downarrow]}. \tag{2.6.47}$$

Here

$$W_\uparrow := \exp\left(-t^{2/3}\int_{-M}^{M} \exp\left(t^{1/3}\left[p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}(p^{-2/3}x) - p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(p^{-2/3}x)\right]\right)\mathrm{d}x\right) \tag{2.6.48}$$

and

$$W_\downarrow := \exp\left(-t^{2/3}\int_{-M}^{M} \exp\left(t^{1/3}\left[q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(2)}(q^{-2/3}x) - q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(1)}(q^{-2/3}x)\right]\right)\mathrm{d}x\right). \tag{2.6.49}$$

In (2.6.47), $\mathbf{P}_{\mathrm{free},t}$ and $\mathbf{E}_{\mathrm{free},t}$ are the probability and the expectation operator respectively corresponding to the joint 'free' law for $(p^{1/3}\mathfrak{h}_{pt,\uparrow}(p^{-2/3}x), q^{1/3}\mathfrak{h}_{qt,\downarrow}(q^{-2/3}x))_{x\in[-M,M]}$ which by Brownian scaling is given by a pair of independent Brownian bridges $(B_1(\cdot), B_2(\cdot))$ on $[-M, M]$ with starting points $(p^{1/3}\mathfrak{h}_{pt,\uparrow}(-Mp^{-2/3}), q^{1/3}\mathfrak{h}_{qt,\downarrow}(-Mq^{-2/3}))$ and endpoints $(q^{1/3}\mathfrak{h}_{pt,\uparrow}(Mp^{-2/3}), q^{1/3}\mathfrak{h}_{qt,\downarrow}(Mq^{-2/3}))$.

Conditioning with respect to maximum data and small boundaries

In this subsection we perform conditioning on the numerator of r.h.s. of (2.6.47) w.r.t. $\mathcal{F}_2$ and $\mathcal{F}_3$ defined in (2.6.43) and (2.6.44). Recall that by Proposition 2.4.9, upon conditioning Brownian bridges on $\mathcal{F}_2$, the conditional laws around the joint local maximizer Φ over $[-M, M]$ is now given by two $\mathsf{NonInt\text{-}BrBridge}$s (defined in Definition 2.4.4) with appropriate lengths and endpoints. Indeed, based on Proposition 2.4.9, given $\mathcal{F}_1, \mathcal{F}_2$, we may construct the conditional laws for the two functions on $[-M, M]$:

Definition 2.6.3 (Nlarge Law)**.** Consider two independent $\mathsf{NonInt\text{-}BrBridge}$ V_ℓ^{large} and V_r^{large} with following description:

1. V_ℓ^{large} is a $\mathsf{NonInt\text{-}BrBridge}$ on $[0, \Phi + M]$ ending at

$$\left(p^{1/3}\left[\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3}) - \mathfrak{h}_{pt,\uparrow}^{(1)}(-Mp^{-2/3})\right], q^{1/3}\left[\mathfrak{h}_{qt,\downarrow}^{(1)}(-Mq^{-2/3}) - \mathfrak{h}_{qt,\downarrow}^{(1)}(\Phi q^{-2/3})\right]\right),$$

2. V_r^{large} is a $\mathsf{NonInt\text{-}BrBridge}$ on $[0, M - \Phi]$ ending at

$$\left(p^{1/3}\left[\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3}) - \mathfrak{h}_{pt,\uparrow}^{(1)}(Mp^{-2/3})\right], q^{1/3}\left[\mathfrak{h}_{qt,\downarrow}^{(1)}(Mq^{-2/3}) - \mathfrak{h}_{qt,\downarrow}^{(1)}(\Phi q^{-2/3})\right]\right).$$

We then define $B^{\mathsf{large}} : [-M, M] \to \mathbb{R}^2$ as follows:

$$B^{\mathsf{large}}(x) = \begin{cases} V_\ell(\Phi - x) & x \in [-M, \Phi] \\ V_r(x - \Phi) & x \in [\Phi, M] \end{cases}.$$

We denote the expectation and probability operator under above law for B^{large} (which depends on $\mathcal{F}_1, \mathcal{F}_2$) as $\mathbf{E}_{\mathsf{Nlarge}|2,1}$ and $\mathbf{P}_{\mathsf{Nlarge}|2,1}$.

Thus we may write

$$\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta),A} W_\uparrow W_\downarrow] = \mathbf{E}_{\text{free},t}[\mathbf{E}_{\mathsf{Nlarge}|2,1}[\mathbf{1}_{\mathsf{Nice}_M(\delta),A} W_\uparrow W_\downarrow]]. \tag{2.6.50}$$

Since $\mathsf{NonInt\text{-}BrBridge}$s are Markovian, we may condition further upon $\mathcal{F}_3$ to get $\mathsf{NonInt\text{-}BrBridge}$s again but on a smaller interval. To precisely define the law, we now give the following definitions:

Definition 2.6.4 (Nsmall law)**.** Consider two independent NonInt-BrBridge V_ℓ^{small} and V_r^{small} with the following descriptions:

1. V_ℓ^{small} is a NonInt-BrBridge on $[0, t^{-\alpha}]$ ending at

$$\left(p^{1/3}\left[\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3}) - \mathfrak{h}_{pt,\uparrow}^{(1)}(p^{-2/3}(\Phi - t^{-\alpha}))\right], q^{1/3}\left[\mathfrak{h}_{qt,\downarrow}^{(1)}(q^{-2/3}(\Phi - t^{-\alpha})) - \mathfrak{h}_{qt,\downarrow}^{(1)}(\Phi q^{-2/3})\right]\right),$$

2. V_r^{small} is a NonInt-BrBridge on $[0, t^{-\alpha}]$ ending at

$$\left(p^{1/3}\left[\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3}) - \mathfrak{h}_{pt,\uparrow}^{(1)}(p^{-2/3}(\Phi + t^{-\alpha}))\right], q^{1/3}\left[\mathfrak{h}_{qt,\downarrow}^{(1)}(q^{-2/3}(\Phi + t^{-\alpha})) - \mathfrak{h}_{qt,\downarrow}^{(1)}(\Phi q^{-2/3})\right]\right).$$

We then define $B^{\mathsf{small}} : [\Phi + t^{-\alpha}, \Phi - t^{-\alpha}] \to \mathbb{R}^2$ as follows:

$$B^{\mathsf{small}}(x) = \begin{cases} V_\ell(\Phi - x) & x \in [\Phi - t^{-\alpha}, \Phi] \\ V_r(x - \Phi) & x \in [\Phi, \Phi + t^{-\alpha}] \end{cases}.$$

We denote the the expectation and probability operators under the above law for B^{small} (which depends on $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$) as $\mathbf{E}_{\mathsf{Nsmall}|3,2,1}$ and $\mathbf{P}_{\mathsf{Nsmall}|3,2,1}$ respectively.

We thus have

$$\text{r.h.s. of (2.6.50)} = \mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}\mathbf{E}_{\mathsf{Nsmall}|3,2,1}[\mathbf{1}_A W_\uparrow W_\downarrow]]. \tag{2.6.51}$$

The $\mathbf{1}_{\mathsf{Nice}_M(\delta)}$ comes of the interior expectation above as $\mathsf{Nice}_M(\delta)$ is measurable w.r.t. $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ (see its definition in (2.6.39)).

Next note that due to the definition of $W_\uparrow, W_\downarrow$ from (2.6.48) and (2.6.49), we may extract certain parts of it which are measurable w.r.t. $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Indeed, we can write $W_\uparrow = W_{\uparrow,1}W_{\uparrow,2}$ and $W_\downarrow = W_{\downarrow,1}W_{\downarrow,2}$ where

$$W_{\uparrow,1} := \exp\left(-t^{2/3}\int_{K_t} \exp\left(t^{1/3}\left[p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}(p^{-2/3}x) - p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(p^{-2/3}x)\right]\right) \mathrm{d}x\right) \tag{2.6.52}$$

$$W_{\uparrow,2} := \exp\left(-t^{2/3}\int_{[-M,M]\cap K_t^c} \exp\left(t^{1/3}\left[p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}(p^{-2/3}x) - p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(p^{-2/3}x)\right]\right)\mathrm{d}x\right),$$

and

$$W_{\downarrow,1} := \exp\left(-t^{2/3}\int_{K_t} \exp\left(t^{1/3}\left[q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(2)}(q^{-2/3}x) - q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(1)}(q^{-2/3}x)\right]\right)\mathrm{d}x\right). \tag{2.6.53}$$

$$W_{\downarrow,2} := \exp\left(-t^{2/3}\int_{[-M,M]\cap K_t^c} \exp\left(t^{1/3}\left[q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(2)}(q^{-2/3}x) - q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(1)}(q^{-2/3}x)\right]\right)\mathrm{d}x\right),$$

where recall K_t from (2.6.41). The key observation is that $W_{\uparrow,2}, W_{\downarrow,2}$ are measurable w.r.t. $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Thus we have

$$\text{r.h.s. of (2.6.51)} = \mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)} W_{\uparrow,2}W_{\downarrow,2} \cdot \mathbf{E}_{\mathsf{Nsmall}|3,2,1}[\mathbf{1}_A W_{\uparrow,1}W_{\downarrow,1}]]. \tag{2.6.54}$$

Remark 2.6.5. It is crucial to note that in (2.6.51) the event $\mathsf{Nice}_M(\delta)$ includes the event $\mathsf{ArMx}(\delta)$ defined in (2.6.32). Indeed, the $\mathsf{ArMx}(\delta)$ event is measurable w.r.t. $\mathcal{F}_1 \cup \mathcal{F}_2$ and ensures that $[\Phi - t^{-\alpha}, \Phi + t^{-\alpha}] \subset [-M, M]$ for all large enough t, which is essential for going from Nlarge law to Nsmall law. Thus such a decomposition is not possible for $\mathbf{E}_{\text{free},t}[W_\uparrow W_\downarrow]$ which appears in the denominator of r.h.s. of (2.6.47). Nonetheless, we may still provide a lower bound for $\mathbf{E}_{\text{free},t}[W_\uparrow W_\downarrow]$ as follows:

$$\mathbf{E}_{\text{free},t}[W_\uparrow W_\downarrow] \geq \mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)} W_\uparrow W_\downarrow] = \mathbf{E}_{\text{free},t}[W_{\uparrow,2}W_{\downarrow,2}\mathbf{1}_{\mathsf{Nice}_M(\delta)} \cdot \mathbf{E}_{\mathsf{Nsmall}|3,2,1}[W_{\uparrow,1}W_{\downarrow,1}]]. \tag{2.6.55}$$

With the deductions in (2.6.54) and (2.6.55), we now come to the task of analyzing $W_{\uparrow,1}W_{\downarrow,1}$ under Nsmall law. The following lemma ensures that on $\mathsf{Nice}_M(\delta)$, $W_{\uparrow,1}W_{\downarrow,1}$ is close to 1 under Nsmall law.

Lemma 2.6.6. *There exist $t_0(\delta) > 0$ such that for all $t \geq t_0$ we have*

$$\mathbf{1}_{\mathsf{Nice}_M(\delta)} \mathbf{P}_{\mathsf{Nsmall}|3,2,1}(W_{\uparrow,1} W_{\downarrow,1} > 1 - \delta) \geq \mathbf{1}_{\mathsf{Nice}_M(\delta)}(1 - \delta). \tag{2.6.56}$$

This allow us to ignore $W_{\uparrow,1} W_{\downarrow,1}$, in $\mathbf{E}_{\mathsf{Nsmall}|3,2,1}[\mathbf{1}_A W_{\uparrow,1} W_{\downarrow,1}]$. Hence it suffices to study $\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A)$. The following lemma then compares this conditional probability with that of DBM.

Lemma 2.6.7. *There exist $t_0(\delta) > 0$ such that for all $t \geq t_0$ we have*

$$\mathbf{1}_{\mathsf{Nice}_M(\delta)} |\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A) - \tau(A)| \leq \mathbf{1}_{\mathsf{Nice}_M(\delta)} \cdot \delta, \tag{2.6.57}$$

where $\tau(A) := \mathbf{P}(\mathcal{D}(\cdot) \in A)$, $\mathcal{D}$ being a two-sided DBM *defined in the statement of Proposition 2.6.1.*

We prove these two lemmas in Section 2.6.2. For now, we proceed with the current proof of (2.6.31) in the next section.

Matching Lower and Upper Bounds

In this subsection, we complete the proof of (2.6.31) by providing matching lower and upper bounds in the two steps below. We assume throughout this subsection that t is large enough, so that (2.6.56) and (2.6.57) holds.

Step 1: Lower Bound. We start with (2.6.45). Following the expression in (2.6.47), and our deductions in (2.6.50), (2.6.51), (2.6.54) we see that

$$\begin{aligned}
\mathbf{P}_t(A) &\geq \mathbf{E}_t\left[\mathbf{P}_t(\mathsf{Nice}_M(\delta), A \mid \mathcal{F}_1)\right] \\
&= \mathbf{E}\left[\frac{\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)} W_{\uparrow,2} W_{\downarrow,2} \cdot \mathbf{E}_{\mathsf{Nsmall}|3,2,1}[\mathbf{1}_A W_{\uparrow,1} W_{\downarrow,1}]]}{\mathbf{E}_{\text{free},t}[W_\uparrow W_\downarrow]}\right] && (2.6.58) \\
&\geq (1-\delta)\mathbf{E}_t\left[\frac{\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)} W_{\uparrow,2} W_{\downarrow,2} \cdot \mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A, W_{\uparrow,1} W_{\downarrow,1} > 1-\delta)]}{\mathbf{E}_{\text{free},t}[W_\uparrow W_\downarrow]}\right] && (2.6.59)
\end{aligned}$$

where in the last inequality we used the fact $W_{\uparrow,1}W_{\downarrow,1} \le 1$. Now applying Lemma 2.6.6 and Lemma 2.6.7 successively we get

$$\begin{aligned}
&\mathbf{1}_{\mathsf{Nice}_M(\delta)}\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A, W_{\uparrow,1}W_{\downarrow,1} > 1-\delta) \\
&\ge \mathbf{1}_{\mathsf{Nice}_M(\delta)}[\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A) - \mathbf{P}_{\mathsf{Nsmall}|3,2,1}(W_{\uparrow,1}W_{\downarrow,1} \le 1-\delta)] \\
&\ge \mathbf{1}_{\mathsf{Nice}_M(\delta)}[\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A) - \delta] \\
&\ge \mathbf{1}_{\mathsf{Nice}_M(\delta)}[\tau(A) - 2\delta]
\end{aligned}$$

where recall $\tau(A) = \mathbf{P}(\mathcal{D}(\cdot) \in A)$. As $W_{\uparrow,1}W_{\downarrow,1} \le 1$ and probabilities are nonnegative, following the above inequalities we have

$$\mathbf{1}_{\mathsf{Nice}_M(\delta)}\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A, W_{\uparrow,1}W_{\downarrow,1} > 1-\delta) \ge \max\{0, \tau(A) - 2\delta\}\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow,1}W_{\downarrow,1}.$$

Substituting the above bound back to (2.6.59) and using the fact that $W_{\uparrow,2}W_{\downarrow,2}W_{\uparrow,1}W_{\downarrow,1} = W_{\uparrow}W_{\downarrow}$, we get

$$\begin{aligned}
\mathbf{P}_t(A) &\ge (1-\delta)\max\{0, \tau(A) - 2\delta\}\mathbf{E}_t\left[\frac{\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow}W_{\downarrow}]}{\mathbf{E}_{\text{free},t}[W_{\uparrow}W_{\downarrow}]}\right] \\
&= (1-\delta)\max\{0, \tau(A) - 2\delta\}\mathbf{P}_t(\mathsf{Nice}_M(\delta)).
\end{aligned}$$

In view of Lemma 2.6.2, taking $\liminf_{t\to\infty}$ followed by $\liminf_{\delta\downarrow 0}$ we get that $\liminf_{t\to\infty}\mathbf{P}_t(A) \ge \tau(A)$. This proves the lower bound.

Step 2: Upper Bound. We start with (2.6.46). Using the equality in (2.6.58) we get

$$\begin{aligned}
\mathbf{P}_t(A) &\le \mathbf{E}\left[\frac{\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow,2}W_{\downarrow,2}\cdot \mathbf{E}_{\mathsf{Nsmall}|3,2,1}[\mathbf{1}_A W_{\uparrow,1}W_{\downarrow,1}]]}{\mathbf{E}_{\text{free},t}[W_{\uparrow}W_{\downarrow}]}\right] + \mathbf{P}_t(\neg\mathsf{Nice}_M(\delta)) \\
&\le \mathbf{E}\left[\frac{\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow,2}W_{\downarrow,2}\cdot \mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A)]}{\mathbf{E}_{\text{free},t}[W_{\uparrow}W_{\downarrow}]}\right] + \mathbf{P}_t(\neg\mathsf{Nice}_M(\delta)) \\
&\le (\tau(A)+\delta)\mathbf{E}\left[\frac{\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow,2}W_{\downarrow,2}]}{\mathbf{E}_{\text{free},t}[W_{\uparrow}W_{\downarrow}]}\right] + \mathbf{P}_t(\neg\mathsf{Nice}_M(\delta)). \qquad (2.6.60)
\end{aligned}$$

Let us briefly justify the inequalities presented above. Going from first line to second line we used the fact $W_{\uparrow,1}W_{\downarrow,1} \leq 1$. The last inequality follows from Lemma 2.6.7 where recall that $\tau(A) = \mathbf{P}(\mathcal{D}(\cdot) \in A)$. Now note that by Lemma 2.6.6, on $\mathsf{Nice}_M(\delta)$,

$$\begin{aligned}\mathbf{E}_{\mathsf{Nsmall}|3,2,1}[W_{\uparrow,1}W_{\downarrow,1}] &\geq \mathbf{E}_{\mathsf{Nsmall}|3,2,1}[\mathbf{1}_{W_{\uparrow,1}W_{\downarrow,1}\geq 1-\delta} \cdot W_{\uparrow,1}W_{\downarrow,1}] \\ &\geq (1-\delta)\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(W_{\uparrow,1}W_{\downarrow,1} \geq 1-\delta) \geq (1-\delta)^2.\end{aligned}$$

Using the expression from (2.6.55) we thus have

$$\begin{aligned}\mathbf{E}_{\text{free},t}[W_{\uparrow}W_{\downarrow}] &\geq \mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow,2}W_{\downarrow,2} \cdot \mathbf{E}_{\mathsf{Nsmall}|3,2,1}[W_{\uparrow,1}W_{\downarrow,1}]] \\ &\geq (1-\delta)^2\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow,2}W_{\downarrow,2}].\end{aligned}$$

Going back to (2.6.60), this forces

$$\text{r.h.s. of (2.6.60)} \leq \frac{\tau(A)+\delta}{(1-\delta)^2} + \mathbf{P}_t(\neg\mathsf{Nice}_M(\delta)).$$

In view of Lemma 2.6.2, taking $\limsup_{t\to\infty}$, followed by $\limsup_{\delta\downarrow 0}$ in above inequality we get that $\limsup_{t\to\infty}\mathbf{P}_t(A) \leq \tau(A)$. Along with the matching lower bound obtained in **Step 1** above, this establishes (2.6.31).

Proof of Lemma 2.6.2

Recall from (2.6.39) that $\mathsf{Nice}_M(\delta)$ event is an intersection of several kinds of events. To show (2.6.40), it suffices to prove the same for each of the events. That is, given an event E which is part of $\mathsf{Nice}_M(\delta)$ we will show

$$\limsup_{\delta\to\infty}\limsup_{t\to\infty}\mathbf{P}(\mathsf{E}) = 1. \tag{2.6.61}$$

Below we analyze each such possible choices for E separately.

$\underline{\mathsf{ArMx}(\delta)\ \textbf{event.}}$ Recall $\mathsf{ArMx}(\delta)$ event from (2.6.32). As noted in (2.3.9),

$$\mathcal{M}_{p,t}^{M} \xrightarrow{d} \operatorname*{argmax}_{x\in[-M,M]} \mathcal{A}(x),$$

where $\mathcal{A}$ is defined in (2.3.8). Since $\mathcal{A}$ restricted to $[-M,M]$ is absolutely continuous with Brownian motion with appropriate diffusion coefficients, $\operatorname{argmax}_{x\in[-M,M]}\mathcal{A}(x) \in (-M,M)$ almost surely. In other words, maximum is not attained on the boundaries almost surely. But then

$$\begin{aligned}\liminf_{\delta\downarrow 0}\liminf_{t\to\infty}\mathbf{P}(\mathsf{ArMx}(\delta)) &= \liminf_{\delta\downarrow 0}\mathbf{P}(\operatorname*{argmax}_{x\in[-M,M]}\mathcal{A}(x)\in[-M+\delta,M-\delta]) \\ &= \mathbf{P}(\operatorname*{argmax}_{x\in[-M,M]}\mathcal{A}(x)\in(-M,M)) = 1.\end{aligned}$$

This proves (2.6.61) with $\mathsf{E}\mapsto\mathsf{ArMx}(\delta)$.

$\underline{\mathsf{Bd}_{\uparrow}(\delta),\mathsf{Bd}_{\downarrow}(\delta)\ \textbf{events.}}$ We first define

$$\begin{aligned}\mathsf{Tight}_{\pm,\uparrow}(\lambda) &:= \left\{p^{1/3}\left|\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3})-\mathfrak{h}_{pt,\uparrow}^{(1)}(\pm Mp^{-2/3})\right|\le\tfrac{1}{\lambda}\right\}, \\ \mathsf{Tight}_{\pm,\downarrow}(\lambda) &:= \left\{q^{1/3}\left|\mathfrak{h}_{qt,\downarrow}^{(1)}(\Phi q^{-2/3})-\mathfrak{h}_{qt,\downarrow}^{(1)}(\pm Mq^{-2/3})\right|\le\tfrac{1}{\lambda}\right\},\end{aligned}$$

and set

$$\mathsf{Sp}(\lambda) := \mathsf{ArMx}(\lambda)\cap\mathsf{Tight}_{+,\uparrow}(\lambda)\cap\mathsf{Tight}_{-,\uparrow}(\lambda)\cap\mathsf{Tight}_{+,\downarrow}(\lambda)\cap\mathsf{Tight}_{-,\downarrow}(\lambda) \tag{2.6.62}$$

where $\mathsf{ArMx}(\lambda)$ is defined in (2.6.32). We claim that

$$\limsup_{\lambda\downarrow 0}\limsup_{t\to\infty}\mathbf{P}(\neg\mathsf{Sp}(\lambda)) = 0. \tag{2.6.63}$$

Let us assume (2.6.63) for the time being and consider the main task of analyzing the probability of the events $\mathsf{Bd}_{\uparrow}(\delta),\mathsf{Bd}_{\downarrow}(\delta)$ defined in (2.6.33). We have $\mathsf{Bd}_{\uparrow}(\delta) = \mathsf{Bd}_{+\uparrow}(\delta)\cap\mathsf{Bd}_{-,\uparrow}(\delta)$ where $\mathsf{Bd}_{\pm,\uparrow}(\delta)$ is defined in (2.6.34). Let us focus on $\mathsf{Bd}_{+,\uparrow}(\delta)$. Recall the σ-fields $\mathcal{F}_1,\mathcal{F}_2$ from

(2.6.42) and (2.6.43). As described in Subsection 2.6.2, upon conditioning on $\mathcal{F}_1 \cup \mathcal{F}_2$, the conditional law on $[-M, M]$ are given by Nlarge defined in Definition 2.6.3, which are made up of $\mathsf{NonInt\text{-}BrBridge}$s $V_\ell^{\mathsf{large}}, V_r^{\mathsf{large}}$ defined in Definition 2.6.3.

Note that applying Markov inequality conditionally we have

$$\begin{aligned}
&\mathbf{1}_{\mathsf{Sp}(\lambda)}\mathbf{P}\left(\mathsf{Bd}_{+,\uparrow}(\delta) \mid \mathcal{F}_1, \mathcal{F}_2\right) \\
&= \mathbf{1}_{\mathsf{Sp}(\lambda)} \cdot \mathbf{P}\left(|\mathfrak{h}_{pt,\uparrow}^{(1)}(p^{-2/3}(\Phi + t^{-\alpha})) - \mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3})| > \tfrac{1}{\delta}t^{-\alpha/2} \mid \mathcal{F}_1, \mathcal{F}_2\right) \\
&\le \mathbf{1}_{\mathsf{Sp}(\lambda)} \cdot \delta^2 t^{2\alpha} \cdot \mathbf{E}_{\mathsf{Nlarge}|2,1}\left[[V_{r,1}^{\mathsf{large}}(p^{-2/3}t^{-\alpha})]^4\right]
\end{aligned}$$

However, on $\mathbf{1}_{\mathsf{Sp}(\lambda)}$, the $\mathsf{NonInt\text{-}BrBridge}$ has length bounded away from zero and the endpoints are tight. Applying (2.5.20) with $K \mapsto 2, t \mapsto 1, s \mapsto 0, n \mapsto p^{2/3}t^{\alpha}, M \mapsto 1/\lambda$, for all large enough t we get $\mathbf{E}_{\mathsf{Nlarge}|2,1}\left[[V_{r,1}^{\mathsf{large}}(p^{-2/3}t^{-\alpha})]^4\right] \le \mathrm{C}_{p,\lambda}t^{-2\alpha}$. Thus,

$$\limsup_{t\to\infty} \mathbf{P}\left(\neg\mathsf{Bd}_{+,\uparrow}(\delta)\right) \le \limsup_{t\to\infty} \mathbf{P}(\neg\mathsf{Sp}(\lambda)) + \delta^2\mathrm{C}_{p,\lambda}.$$

Taking $\delta \downarrow 0$, followed by $\lambda \downarrow 0$, in view of (2.6.63) we get $\limsup_{\delta\downarrow 0}\limsup_{t\to\infty} \mathbf{P}(\neg\mathsf{Bd}_{+,\uparrow}(\delta)) = 0$. Similarly one can conclude $\limsup_{\delta\downarrow 0}\limsup_{t\to\infty} \mathbf{P}(\neg\mathsf{Bd}_{-,\uparrow}(\delta)) = 0$ Thus, this two together yields $\liminf_{\delta\downarrow 0}\liminf_{t\to\infty} \mathbf{P}(\mathsf{Bd}_{\uparrow}(\delta)) = 1$. By exactly the same approach one can derive that $\mathbf{P}(\mathsf{Bd}_{\downarrow}(\delta))$ goes to 1 under the same iterated limit. Thus it remains to show (2.6.63).

Let us recall from (2.6.62) that $\mathsf{Sp}(\lambda)$ event is composed of four tightness events and one event about the argmax. We first claim that $\limsup_{\lambda\downarrow 0}\limsup_{t\to\infty} \mathbf{P}(\mathsf{Tight}_{x,y}(\lambda)) = 1$ for each $x \in \{+, -\}$ and $y \in \{\uparrow, \downarrow\}$. The earlier analysis of $\mathsf{ArMx}(\lambda)$ event in (2.6.62) then enforces (2.6.63). Since all the tightness events are similar, it suffices to prove any one of them say $\mathsf{Tight}_{+,\uparrow}$. By Proposition 3.5.1 we have the distributional convergence of $2^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(2^{1/3}x)$ to $\mathcal{A}_1(x)$ in the uniform-on-compact topology, where $\mathcal{A}_1(\cdot)$ is the parabolic Airy$_2$ process. As $\Phi \in [-M, M]$, we thus have

$$\limsup_{t\to\infty} \mathbf{P}(\mathsf{Tight}_{+,\uparrow}(\lambda)) \le \limsup_{t\to\infty} \mathbf{P}\left(p^{1/3}\sup_{x\in[-M,M]}\left|\mathfrak{h}_{pt,\uparrow}^{(1)}(xp^{-2/3}) - \mathfrak{h}_{pt,\uparrow}^{(1)}(Mp^{-2/3})\right| \le \tfrac{1}{\lambda}\right)$$
$$= \mathbf{P}\left(p^{1/3}\sup_{|x|\le 2^{-1/3}M}\left|\mathcal{A}_1(xp^{-2/3}) - \mathcal{A}_1(2^{-1/3}Mp^{-2/3})\right| \le \tfrac{2^{1/3}}{\lambda}\right).$$

For fixed p, M, by tightness of parabolic Airy_2 process on a compact interval, the last expression goes to one as $\lambda \downarrow 0$, which is precisely what we wanted to show.

<u>$\mathsf{Gap}_{M,\uparrow}(\delta), \mathsf{Gap}_{M,\downarrow}(\delta)$ **events.**</u> Recall the definitions of $\mathsf{Gap}_{M,\uparrow}(\delta)$ and $\mathsf{Gap}_{M,\downarrow}(\delta)$ from(2.6.35) and (2.6.36).We begin with the proof of $\mathsf{Gap}_{M,\uparrow}(\delta)$. Let

$$\mathsf{Diff}_{M,\uparrow}(\delta) := \left\{\inf_{|x|\le M} p^{1/3}\left(\mathfrak{h}_{pt,\uparrow}^{(1)}(p^{-2/3}x) - \mathfrak{h}_{pt,\uparrow}^{(2)}(p^{-2/3}x)\right) \ge \delta\right\}.$$

Note that $\Phi \in [-M, M]$. Thus $\mathsf{Gap}_{M,\uparrow}(\delta) \supset \mathsf{Diff}_{M,\uparrow}(\delta)$. Thus to show (2.6.61) with $\mathsf{E} \mapsto \mathsf{Gap}_{M,\uparrow}(\delta)$ it suffices to prove

$$\liminf_{\delta\downarrow 0}\liminf_{t\to\infty} \mathbf{P}(\mathsf{Diff}_{M,\uparrow}(\delta)) = 1, \tag{2.6.64}$$

We recall from Proposition 2.2.7 the distributional convergence of the KPZ line ensemble to the Airy line ensemble in the uniform-on-compact topology. By Skorokhod representation theorem, we may assume that our probability space is equipped with $\mathcal{A}_1(\cdot)$ and $\mathcal{A}_2(\cdot)$ such that almost surely as $t \to \infty$

$$\max_{i=1,2}\sup_{|x|\le Mp^{-2/3}} |2^{1/3}\mathfrak{h}_{t,\uparrow}^{(i)}(2^{1/3}x) - \mathcal{A}_i(x)| \to 0. \tag{2.6.65}$$

We thus have

$$\liminf_{t\to\infty} \mathbf{P}(\mathsf{Diff}_{M,\uparrow}(\delta)) = \mathbf{P}\left(\inf_{|x|\le M2^{-1/3}p^{-2/3}} p^{1/3}\left(\mathcal{A}_1(x) - \mathcal{A}_2(x)\right) \ge 2^{1/3}\delta\right). \tag{2.6.66}$$

As the Airy line ensemble is absolutely continuous w.r.t. non-intersecting Brownian motions, it is strictly ordered with touching probability zero (see (2.2.1)). Hence r.h.s. of (2.6.66) goes to zero as $\delta \downarrow 0$. This proves (2.6.64). The proof is similar for $\mathsf{Gap}_{M,\downarrow}(\delta)$.

$\mathsf{Rise}_{M,\uparrow}(\delta), \mathsf{Rise}_{M,\uparrow}(\delta)$ events. Recall $\mathsf{Rise}_{M,\uparrow}(\delta), \mathsf{Rise}_{M,\uparrow}(\delta)$ events from (2.6.37) and (2.6.38). Due to their similarities, we only analyze the $\mathsf{Rise}_{M,\uparrow}(\delta)$ event. As with the previous case, we assume that our probability space is equipped with $\mathcal{A}_1(\cdot)$ and $\mathcal{A}_2(\cdot)$ (first two lines of the Airy line ensemble) such that almost surely as $t \to \infty$ (2.6.65) holds. Applying union bound we have

$$\begin{aligned}
\mathbf{P}\left(\neg\mathsf{Rise}_M(\delta)\right) &\le \mathbf{P}\left(\sup_{|x|\le Mp^{-2/3}} p^{1/3}|2^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}(2^{1/3}x) - \mathcal{A}_2(x)| \ge \tfrac{\delta}{16}\right) \\
&\qquad + \mathbf{P}\left(\neg\mathsf{Rise}_M(\delta), \sup_{|x|\le Mp^{-2/3}} p^{1/3}|2^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}(2^{1/3}x) - \mathcal{A}_2(x)| \le \tfrac{\delta}{16}\right) \\
&\le \mathbf{P}\left(\sup_{|x|\le Mp^{-2/3}} p^{1/3}|2^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}(2^{1/3}x) - \mathcal{A}_2(x)| \ge \tfrac{\delta}{16}\right) \\
&\qquad + \mathbf{P}\left(\sup_{\substack{x,y\in[-M,M] \\ |x-y|\le t^{-\alpha}}} p^{1/3}|\mathcal{A}_2(x) - \mathcal{A}_2(y)| \ge \tfrac{\delta}{8}\right).
\end{aligned}$$

In the r.h.s. of above equation, the first term goes to zero as $t \to \infty$ by (2.6.65). The second term on the other hand goes to zero as $t \to \infty$ by modulus of continuity estimates for Airy line ensemble from Proposition 2.2.4. This shows, $\lim_{t\to\infty} \mathbf{P}(\mathsf{Rise}_{M,\uparrow}(\delta)) = 1$. Similarly one has $\lim_{t\to\infty} \mathbf{P}(\mathsf{Rise}_{M,\downarrow}(\delta)) = 1$ as well. This proves (2.6.61) for $\mathsf{E} \mapsto \mathsf{Rise}_{M,\uparrow}(\delta), \mathsf{Rise}_{M,\downarrow}(\delta)$.

We have thus shown (2.6.61) for all the events listed in (2.6.39). This establishes (2.6.40) concluding the proof of Lemma 2.6.2.

Proof of Lemma 2.6.6 and 2.6.7

In this subsection we prove Lemma 2.6.6 and 2.6.7.

Proof of Lemma 2.6.6. Recall $W_{\uparrow,1}$ and $W_{\downarrow,1}$ from (2.6.52) and (2.6.53) respectively. We claim

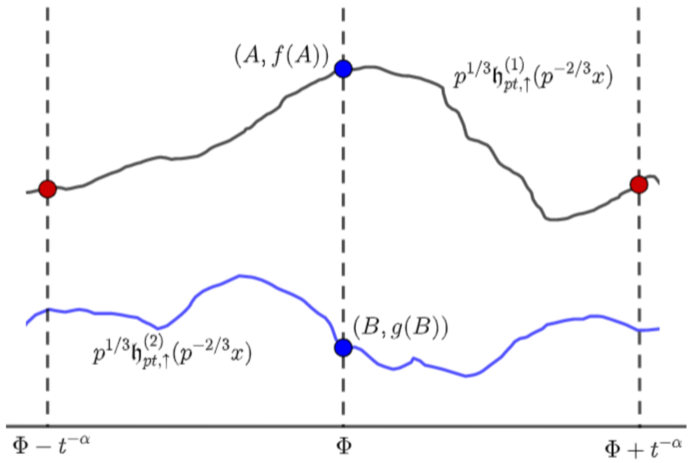


Figure 2.9: In the above figure we have plotted the curves $f(x) := p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x)$ (black) and $g(x) := p^{1/3}\mathfrak{h}^{(2)}_{pt,\uparrow}(p^{-2/3}x)$ (blue) restricted to the interval $K_t := (\Phi - t^{-\alpha}, \Phi + t^{-\alpha})$. For convenience, we have marked two blue points along with their values as $(A, f(A))$, $(B, g(B))$. $\mathsf{Gap}_{M,\uparrow}(\delta)$ defined in (2.6.35) denote the event that the blue points are separated by δ, i.e, $f(A) - g(B) \geq \delta$. The $\mathsf{Rise}_{M,\uparrow}(\delta)$ defined in (2.6.37) ensures *no* point on the blue curve (restricted to K_t) has value larger than $g(B) + \frac{1}{4}\delta$ (that is no significant rise). The $\mathsf{Bd}_{\uparrow}(\delta)$ event defined in (2.6.33) indicates the red points on the black curve are within $[f(A) - \frac{1}{\delta}t^{-\alpha/2}, f(A) + \frac{1}{\delta}t^{-\alpha/2}]$. The $\mathsf{Sink}_{\uparrow}(\delta)$ event defined in (2.6.68) ensures that *all* points on the black curve (restricted to K_t) have values larger than $f(A) - \frac{1}{4}\delta$ (that is no significant sink). Clearly then on $\mathsf{Sink}_{\uparrow}(\delta) \cap \mathsf{Rise}_{M,\uparrow}(\delta) \cap \mathsf{Gap}_{M,\uparrow}(\delta)$ for all $x \in K_t$, we have $f(x) - g(x) \geq f(A) - \frac{1}{4}\delta - g(B) - \frac{1}{4}\delta \geq \frac{1}{2}\delta$.

that for all large enough t, on $\mathsf{Nice}_M(\delta)$ we have

$$\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(W_{\uparrow,1} > \sqrt{1-\delta}) \geq 1 - \tfrac{1}{2}\delta, \quad \mathbf{P}_{\mathsf{Nsmall}|3,2,1}(W_{\downarrow,1} > \sqrt{1-\delta}) \geq 1 - \tfrac{1}{2}\delta \tag{2.6.67}$$

simultaneously. (2.6.56) then follows via union bound. Hence we focus on proving (2.6.67). In the proof below we only focus on first part of (2.6.67) and the second one follows analogously. We now define the 'sink' event:

$$\mathsf{Sink}_{\uparrow}(\delta) := \left\{ \inf_{x \in [-t^{-\alpha}, t^{-\alpha}]} p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(\Phi p^{-2/3} + x) \geq p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(\Phi p^{-2/3}) - \tfrac{\delta}{4} \right\}. \tag{2.6.68}$$

Recall $\mathsf{Rise}_{M,\uparrow}(\delta)$ and $\mathsf{Gap}_{M,\uparrow}(\delta)$ from (2.6.37) and (2.6.35). Note that on $\mathsf{Sink}_{\uparrow}(\delta) \cap \mathsf{Rise}_{M,\uparrow}(\delta) \cap$

$\mathsf{Gap}_{M,\uparrow}(\delta)$ we have uniform separation between $\mathfrak{h}_{pt,\uparrow}^{(1)}$ and $\mathfrak{h}_{pt,\downarrow}^{(2)}$ on the interval $p^{-2/3}K_t$, that is

$$\inf_{x\in[\Phi-t^{-\alpha},\Phi+t^{-\alpha}]}\left[p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(p^{-2/3}x)-p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}(p^{-2/3}x)\right]\geq\tfrac{\delta}{2}. \tag{2.6.69}$$

See Figure 2.9 alongside its caption for further explanation of the above fact. But then (2.6.69) forces $W_{\uparrow,1}\geq\exp(-t^{2/3}2t^{-\alpha}e^{-\frac{1}{4}t^{1/3}\delta})$ which can be made strictly larger than $\sqrt{1-\delta}$ for all large enough t. Thus,

$$\mathbf{1}_{\mathsf{Nice}_M(\delta)}\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(W_{\uparrow,1}>\sqrt{1-\delta})\geq\mathbf{1}_{\mathsf{Nice}_M(\delta)}\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(\mathsf{Sink}_{\uparrow}(\delta)). \tag{2.6.70}$$

Now we divide the sink event into two parts: $\mathsf{Sink}_{\uparrow}(\delta)=\mathsf{Sink}_{+\uparrow}(\delta)\cap\mathsf{Sink}_{-,\uparrow}(\delta)$ where

$$\mathsf{Sink}_{\pm,\uparrow}(\delta):=\left\{\inf_{x\in[0,t^{-\alpha}]}p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3}\pm x)\geq p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3})-\tfrac{\delta}{4}\right\},$$

In view of (2.6.70), to prove first part of (2.6.67), it suffices to show for all large enough t, on $\mathsf{Nice}_M(\delta)$ we have

$$\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(\mathsf{Sink}_{+,\uparrow}(\delta))\geq1-\tfrac{\delta}{4},\quad \mathbf{P}_{\mathsf{Nsmall}|3,2,1}(\mathsf{Sink}_{-,\uparrow}(\delta))\geq1-\tfrac{\delta}{4}. \tag{2.6.71}$$

We only prove first part of (2.6.71) below. Towards this end, recall $Y_{M,t,\uparrow}^{(1)},Y_{M,t,\downarrow}^{(1)}$ from (2.6.26). Observe that

$$Y_{M,t,\uparrow}^{(1)}(\Phi+x)=p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3})-p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3}+x).$$

Recall Nsmall law from Definition 2.6.4. Our discussion in Subsection 2.6.2 implies that under $\mathbf{P}_{\mathsf{Nsmall}|3.2,1}$,

$$(Y_{M,t,\uparrow}^{(1)},Y_{M,t,\downarrow}^{(1)})(\Phi+\cdot)|_{[0,t^{-\alpha}]}\stackrel{d}{=}V_r^{\mathsf{small}}(\cdot),\quad (Y_{M,t,\uparrow}^{(1)},Y_{M,t,\downarrow}^{(1)})(\Phi+\cdot)|_{[-t^{-\alpha},0]}\stackrel{d}{=}V_\ell^{\mathsf{small}}(-\cdot),$$

where recall that V_ℓ^{small} and V_r^{small} are conditionally independent NonInt-BrBridge on $[0, t^{-\alpha}]$ with appropriate end points, defined in Definition 2.6.4. In particular we have,

$$\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(\mathsf{Sink}_{+,\uparrow}(\delta)) = \mathbf{P}_{\mathsf{Nsmall}|3,2,1}\left(\sup_{x\in[0,t^{-\alpha}]} V_{r,1}^{\mathsf{small}}(x) \le \tfrac{1}{4}\delta\right) \tag{2.6.72}$$

where $V_r^{\mathsf{small}} = (V_{r,1}^{\mathsf{small}}, V_{r,2}^{\mathsf{small}})$. Recall $\mathsf{Nice}_M(\delta)$ event from (2.6.39). It contains $\mathsf{Bd}_\uparrow(\delta)$ event defined in (2.6.33). On this event, $-\frac{1}{\delta} \le V_{r,1}^{\mathsf{Small}}(t^{-\alpha}), V_{r,2}^{\mathsf{Small}}(t^{-\alpha}) \le \frac{1}{\delta}t^{-\alpha/2}$. We consider another NonInt-BrBridge $U = (U_1, U_2)$ on $[0, t^{-\alpha}]$ with non-random endpoints $U_1(t^{-\alpha}) = U_2(t^{-\alpha}) = \frac{1}{\delta}t^{-\alpha/2}$. On $\mathsf{Bd}_\uparrow(\delta)$ event, by monotonicity of non-intersecting Brownian bridges (Lemma 2.6 in [73]), one may couple $U = (U_1, U_2)$ and V_r^{small} so that U_i always lies above $V_{r,i}^{\mathsf{small}}$ for $i = 1, 2$. Thus on $\mathsf{Bd}_\uparrow(\delta)$ event,

$$\mathbf{P}_{\mathsf{Nsmall}|3,2,1}\left(\sup_{x\in[0,t^{-\alpha}]} V_{r,1}^{\mathsf{small}}(x) \le \lambda t^{-\alpha/2}\right) \ge \mathbf{P}\left(\sup_{x\in[0,1]} t^{\alpha/2}U_1(xt^{-\alpha}) \le \lambda\right) \ge 1 - \tfrac{\delta}{4},$$

where the last inequality is true by taking λ large enough. This choice of λ is possible as by Brownian scaling, $t^{\alpha/2}U_1(xt^{-\alpha}), t^{\alpha/2}U_2(xt^{-\alpha})$ is NonInt-BrBridge on $[0, 1]$ ending at $(\frac{1}{\delta}, \frac{1}{\delta})$. Taking t large enough one can ensure $\lambda t^{-\alpha/2} \le \frac{\delta}{4}$. Using the equality in (2.6.72) we thus establish the first part of (2.6.71). The second part is analogous. This proves the first part of (2.6.67). The second part of (2.6.67) follows similarly. This completes the proof of Lemma 2.6.6.

Proof of Lemma 2.6.7. The idea behind this proof is Proposition 2.5.8, which states that a NonInt-BrBridge after Brownian rescaling converges in distribution to a DBM. The following fills out the details. Recall that

$$\mathbf{P}_{\mathsf{Nsmall}|3.2,1}(A) = \mathbf{P}_{\mathsf{Nsmall}|3.2,1}(D_{M,t,\uparrow}, D_{M,t,\downarrow}(\cdot) \in A).$$

Recall from (2.6.28) that $D_{M,t,\uparrow}, D_{M,t,\downarrow}$ is a diffusive scaling of $Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)}$ when centering at Φ, where $Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)}$ are defined in (2.6.26). Recall Nsmall law from Definition 2.6.4. Our discussion in Subsection 2.6.2 implies that under $\mathbf{P}_{\mathsf{Nsmall}|3.2,1}$,

$$(Y^{(1)}_{M,t,\uparrow}, Y^{(1)}_{M,t,\downarrow})(\Phi + \cdot)|_{[0,t^{-\alpha}]} \stackrel{d}{=} V_r^{\mathsf{small}}(\cdot), \quad (Y^{(1)}_{M,t,\uparrow}, Y^{(1)}_{M,t,\downarrow})(\Phi + \cdot)|_{[-t^{-\alpha},0]} \stackrel{d}{=} V_\ell^{\mathsf{small}}(-\cdot),$$

where V_ℓ^{small} and V_r^{small} are conditionally independent $\mathsf{NonInt\text{-}BrBridge}$ on $[0, t^{-\alpha}]$ with appropriate end points defined in Definition 2.6.4. Using Brownian scaling, we consider

$$V_\ell^0(x) := t^{\alpha/2} V_\ell^{\mathsf{small}}(xt^{-\alpha}), \quad V_r^0(x) := t^{\alpha/2} V_r^{\mathsf{small}}(xt^{-\alpha}),$$

which are now $\mathsf{NonInt\text{-}BrBridge}$ on $[0,1]$. Note that on $\mathsf{Bd}_\uparrow(\delta), \mathsf{Bd}_\downarrow(\delta)$ (defined in (2.6.33)), we see that endpoints of V_ℓ^0, V_r^0 are in $[-\frac{1}{\delta}, \frac{1}{\delta}]$. Thus as $\alpha = \frac{1}{6}$, performing another diffusive scaling by Proposition 2.5.8 we see that as $t \to \infty$

$$t^{1/4} V_\ell^0(xt^{-1/2}) \ , \ t^{1/4} V_r(xt^{-1/2})$$

converges to two independent copies of DBMs (defined in Definition 2.5.1) in the uniform-on-compact topology. Hence we get two-sided DBM convergence for the pair $(D_{M,t,\uparrow}, D_{M,t,\downarrow})$ under $\mathbf{P}_{\mathsf{Nsmall}|3.2,1}$ as long as $\mathbf{1}\{\mathsf{Nice}_M(\delta)\}$ holds. This proves (2.6.57).

2.6.3 Proof of Theorem 2.1.10

We take $p \mapsto \frac{1}{2}$ and $t \mapsto 2t$ in Proposition 2.6.1. Then by Lemma 2.3.2, $\mathcal{P}_{2,t}$ defined in the statement of Theorem 2.1.10 is same as $\mathcal{M}_{\frac{1}{2},2t}$ considered in Proposition 2.6.1. Its uniqueness is already justified in Lemma 2.3.1. Furthermore,

$$R_2(x,t) \stackrel{d}{=} D_1(x,t) - D_2(x,t),$$

as functions in x, where $R_2(x,t)$ is defined in (2.1.11) and D_1, D_2 are defined in (2.6.24). By Proposition 2.6.1 and Lemma 2.5.3 we get that $D_1(x,t) - D_2(x,t) \xrightarrow{d} \mathcal{R}_2(x)$ in the uniform-on-compact topology. This proves Theorem 2.1.10 for $k = 2$ case.

For $k = 1$ case, by Lemma 2.3.2, $\mathcal{P}_{1,t}$ is same as $\mathcal{M}_{*,t}$ which is unique almost surely by Lemma 2.3.1. This guarantees $\mathcal{P}_{1,t}$ is unique almost surely as well. Thus we are left to show

$$\mathcal{H}(\mathcal{P}_{1,t}, t) - \mathcal{H}(x + \mathcal{P}_{1,t}, t) \xrightarrow{d} \mathcal{R}_1(x). \tag{2.6.73}$$

where $\mathcal{R}_1(x)$ is a two-sided Bessel process with diffusion coefficient 1 defined in Definition 2.5.2. The proof of (2.6.73) is exactly similar to that of Proposition 2.6.1 with few minor alterations listed below.

1. Just as in Subsection 2.6.2, one may put the problem in (2.6.73) under the framework of KPZ line ensemble. Compared to Subsection 2.6.2, in this case, clearly there will be just one set of line ensemble.

2. Given the decay estimates for $\mathcal{M}_{*,t}$ from Lemma 2.3.1, it boils down to show Bessel behavior around local maximizers. The rigorous justification follows from a soft argument analogous to what is done in Subsection 2.6.2.

3. In the spirit of Subsection 2.6.2, one can define a similar $\mathsf{Nice}'_M(\delta)$ event but now for a single line ensemble. $\mathsf{Nice}'_M(\delta)$ will contain similar events, such as:

   - control on the location of local maximizer (analog of $\mathsf{ArMx}(\delta)$ event (2.6.32)),
   - control on the gap between first curve and second curve at the maximizer (analog of $\mathsf{Gap}_{M,\uparrow}(\delta)$ event (2.6.35)),
   - fluctuations of the first curve on a small interval say I around maximizer (analog of $\mathsf{Rise}_{M,\uparrow}(\delta)$ event (2.6.37),
   - and control on the value of the endpoints of I (analog of $\mathsf{Bd}_{\uparrow}(\delta)$ event (2.6.33)).

   On $\mathsf{Nice}'_M(\delta)$ event, the conditional analysis can be performed in the same manner.

4. Next, as in proof of Proposition 2.6.1, we proceed by three layers of conditioning. For first layer, we use the $\mathbf{H}_t$ Brownian Gibbs property of the single line ensemble under consideration. Next, conditioning on the location and values of the maximizer, we similarly apply the same Bessel bridge decomposition result from Proposition 2.4.8 to convert the conditional law to that of the Bessel bridges over a large interval (see Subsection 2.6.2). Finally, analogous to Subsection 2.6.2, the third layer of conditioning reduces large Bessel bridges to smaller ones following the Markovian property of Bessel bridges, see Lemma 2.4.2.

5. Since a Bessel bridge say on $[0,1]$ is a Brownian bridge conditioned to stay positive on $[0,1]$, it has the Brownian scaling property and it admits monotonicity w.r.t. endpoints. These are two crucial tools that went into the Proof of Lemma 2.6.6 in Subsection 2.6.2. Thus the Bessel analogue of Lemma 2.6.6 can be derived using the scaling property and monotonicity stated above in the exact same way. Finally, the Bessel analogue of Lemma 2.6.7 can be obtained from Corollary 2.5.9. Indeed Corollary 2.5.9 ensures that small Bessel bridges converges to Bessel process under appropriate diffusive limits on the $\mathsf{Nice}'_M(\delta)$ event.

Executing all the above steps in an exact same manner as proof of Proposition 2.6.1, (2.6.73) is established. This completes the proof of Theorem 2.1.10.

2.7 Proof of localization theorems

In this section we prove our main results: Theorem 2.1.4 and Theorem 2.1.5. In Section 2.7.1 we study certain tail properties (Lemma 2.7.1 and Proposition 2.7.2) of the quantities that we are interested in and prove Theorem 2.1.4. Proof of Proposition 2.7.2 is then completed in Section 2.7.2 along with proof of Theorem 2.1.5.

2.7.1 Tail Properties and proof of Theorem 2.1.4

We first settle the question of finiteness of the Bessel integral appearing in the statements of Theorems 2.1.4 and 2.1.5 in the following Lemma.

Lemma 2.7.1. *Let $R_\sigma(\cdot)$ be a Bessel process with diffusion coefficient $\sigma > 0$, defined in Definition 2.5.2. Then*

$$\mathbf{P}\left(\int_{\mathbb{R}} e^{-R_\sigma(x)}\mathrm{d}x \in (0,\infty)\right) = 1.$$

Proof. Since $R_\sigma(\cdot)$ has continuous paths, $\sup_{x\in[0,1]} R_\sigma(x)$ is finite almost surely. Thus almost surely we have

$$\int_{\mathbb{R}} e^{-R_\sigma(x)}\mathrm{d}x \geq \int_0^1 e^{-R_\sigma(x)}\mathrm{d}x > 0.$$

On the other hand, by the classical result from [191] it is known that

$$\mathbf{P}(R_\sigma(x) < x^{1/4} \text{ infinitely often}) = 0.$$

Thus, there exists Ω such that $\mathbf{P}(\Omega) = 1$ and for all $\omega \in \Omega$, there exists $x_0(\omega) \in (0,\infty)$ such that

$$R_\sigma(x)(\omega) \geq x^{1/4} \text{ for all } x \geq x_0(\omega).$$

Hence for this ω,

$$\int_0^\infty e^{-R_\sigma(x)(\omega)}\mathrm{d}x = \int_0^{x_0(\omega)} e^{-R_\sigma(x)(\omega)}\mathrm{d}x + \int_{x_0(\omega)}^\infty e^{-R_\sigma(x)(\omega)}\mathrm{d}x < x_0(\omega) + \int_0^\infty e^{-x^{1/4}}\mathrm{d}x < \infty.$$

This establishes that $\int_{\mathbb{R}} e^{-R_\sigma(x)}\mathrm{d}x$ is finite almost surely. ☐

Our next result studies the tail of the integral of the pre-limiting process.

Proposition 2.7.2. *Fix $p \in (0,1)$. Set $q = 1-p$. Consider 2 independent copies of the KPZ equation $\mathcal{H}_\uparrow(x,t)$, and $\mathcal{H}_\downarrow(x,t)$, both started from the narrow wedge initial data. Let $\mathcal{M}_{p,t}$ be the almost sure unique maximizer of the process $x \mapsto (\mathcal{H}_\uparrow(x,pt) + \mathcal{H}_\downarrow(x,qt))$ which exists via Lemma 2.3.1. Set*

$$\begin{aligned} D_1(x,t) &:= \mathcal{H}_\uparrow(\mathcal{M}_{p,t},pt) - \mathcal{H}_\uparrow(x+\mathcal{M}_{p,t},pt), \\ D_2(x,t) &:= \mathcal{H}_\downarrow(x+\mathcal{M}_{p,t},qt) - \mathcal{H}_\downarrow(\mathcal{M}_{p,t},qt). \end{aligned} \tag{2.7.1}$$

For all $\rho > 0$ we have

$$\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{P}\left(\int_{[-K,K]^c} e^{D_2(x,t)-D_1(x,t)}\,\mathrm{d}x \geq \rho\right) = 0. \tag{2.7.2}$$

As a corollary, we derive that for any $p \in (0,1)$ the pt-point density of point-to-point CDRP of length t indeed concentrates in a microscopic region of size $O(1)$ around the favorite point.

Corollary 2.7.3. *Recall the definition of* CDRP *and the notation* $\mathbf{P}^{\xi}$ *from Definition 3.1.1. Fix $p \in (0,1)$. Suppose $X \sim$* $\mathsf{CDRP}(0,0;0,t)$*. Consider $\mathcal{M}_{p,t}$ the almost sure unique mode of $f_{p,t}$, the quenched density of $X(pt)$. We have*

$$\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{P}^{\xi}\left(|X(pt)-\mathcal{M}_{p,t}| \geq K\right) = 0, \text{ in probability.}$$

One also has the analogous version of Proposition 2.7.2 involving one single copy of the KPZ equation viewed around its maximum. This leads to a similar corollary about tightness of the quenched endpoint distribution for point-to-line CDRP (see Definition 3.1.2) when re-centered around its mode. The details are skipped for brevity.

The proof of Proposition 2.7.2 is heavily technical and relies on the tools as well as notations from Proposition 2.6.1. For clarity, we first prove Corollary 2.7.3 and Theorem 2.1.4 assuming the validity of Proposition 2.7.2. The proof of Proposition 2.7.2 is then presented in Section 2.7.2.

Proof of Corollary 2.7.3. We have $\mathcal{Z}(0,0;x,pt) \stackrel{d}{=} e^{\mathcal{H}_{\uparrow}(x,pt)}$ and by time reversal property $\mathcal{Z}(x,pt;0,t) \stackrel{d}{=} e^{\mathcal{H}_{\downarrow}(x,qt)}$ as functions in x, where $\mathcal{H}_{\uparrow},\mathcal{H}_{\downarrow}$ are independent copies of KPZ equation started from narrow wedge initial data. The uniqueness of the mode $\mathcal{M}_{p,t}$ for $f_{p,t}$ is already settled in Lemma 2.3.1. Thus, the quenched density of $X(pt)-\mathcal{M}_{p,t}$ is given by

$$f_{p,t}(x+\mathcal{M}_{p,t}) = \frac{\exp(D_2(x,t)-D_1(x,t))}{\int_{\mathbb{R}}\exp(D_2(y,t)-D_1(y,t))\mathrm{d}y}, \tag{2.7.3}$$

where $D_i(x,t), i=1,2$ are defined in (2.6.23). Thus,

$$\mathbf{P}^{\xi}\left(|X(pt)-\mathcal{M}_{p,t}|\geq K\right)=\frac{\int\limits_{[-K,K]^c} e^{D_2(x,t)-D_1(x,t)}\,\mathrm{d}x}{\int\limits_{\mathbb{R}} e^{D_2(x,t)-D_1(x,t)}\,\mathrm{d}x}\leq\frac{\int\limits_{[-K,K]^c} e^{D_2(x,t)-D_1(x,t)}\,\mathrm{d}x}{\int\limits_{[-K,K]} e^{D_2(x,t)-D_1(x,t)}\,\mathrm{d}x}. \tag{2.7.4}$$

Notice that by by (2.7.2) the numerator of r.h.s. of (2.7.4) goes to zero in probability under the iterated limit $\limsup_{t\to\infty}$ followed by $\limsup_{K\to\infty}$. Whereas due to Proposition 2.6.1, under the iterated limit, the denominator converges in distribution to $\int_{\mathbb{R}} e^{-R_2(x)}\mathrm{d}x$ which is strictly positive by Lemma 2.7.1. Thus overall the r.h.s. of (2.7.4) goes to zero in probability under the iterated limit. This completes the proof. □

Proof of Theorem 2.1.4. Fix any $p\in(0,1)$. Set $q=1-p$. Recall from (2.7.3) that

$$f_{p,t}(x+\mathcal{M}_{p,t})=\frac{\exp(D_2(x,t)-D_1(x,t))}{\int\limits_{\mathbb{R}}\exp(D_2(y,t)-D_1(y,t))\mathrm{d}y} \tag{2.7.5}$$

where $D_i(x,t), i=1,2$ are defined in (2.6.23). Note that by Proposition 2.6.1, a continuous mapping theorem immediately implies that for any $K<\infty$

$$\frac{\exp(D_2(x,t)-D_1(x,t))}{\int_{-K}^{K}\exp(D_2(y,t)-D_1(y,t))\mathrm{d}y}\xrightarrow{d}\frac{e^{-\mathcal{R}_2(x)}}{\int_{-K}^{K}e^{-\mathcal{R}_2(y)}\mathrm{d}y} \tag{2.7.6}$$

in the uniform-on-compact topology. Here $\mathcal{R}_2$ is a 3D Bessel process with diffusion coefficient 2. For simplicity, we denote

$$\mathfrak{g}_t(x):=\exp(D_2(x,t)-D_1(x,t))\text{ and }\mathfrak{g}(x)=\exp(-\mathcal{R}_2(x)).$$

We can then rewrite (2.7.5) as product of four factors:

$$f_{p,t}(x+\mathcal{M}_{p,t})=\frac{\mathfrak{g}_t(x)}{\int_{\mathbb{R}}\mathfrak{g}_t(y)\mathrm{d}y}=\frac{\int_{-K}^{K}\mathfrak{g}_t(y)\mathrm{d}y}{\int_{\mathbb{R}}\mathfrak{g}_t(y)\mathrm{d}y}\cdot\frac{\int_{\mathbb{R}}\mathfrak{g}(y)\mathrm{d}y}{\int_{-K}^{K}\mathfrak{g}(y)\mathrm{d}y}\cdot\frac{\int_{-K}^{K}\mathfrak{g}(y)\mathrm{d}y}{\int_{\mathbb{R}}\mathfrak{g}(y)\mathrm{d}y}\cdot\frac{\mathfrak{g}_t(x)}{\int_{-K}^{K}\mathfrak{g}_t(y)\mathrm{d}y}.$$

Corollary 2.7.3 ensures

$$\frac{\int_{-K}^{K} \mathfrak{g}_t(y)\mathrm{d}y}{\int_{\mathbb{R}} \mathfrak{g}_t(y)\mathrm{d}y} = \mathbf{P}^{\xi}(|X(pt) - \mathcal{M}_{p,t}| \le K) \xrightarrow{p} 1$$

as $t \to \infty$ followed by $K \to \infty$. Lemma 2.7.1 with $\sigma = 2$ yields that $\int_{[-K,K]^c} \mathfrak{g}(y)\mathrm{d}y = \int_{[-K,K]^c} e^{-\mathcal{R}_2(y)}\mathrm{d}y \xrightarrow{p} 0$ as $K \to \infty$. Thus as $K \to \infty$

$$\frac{\int_{\mathbb{R}} \mathfrak{g}(y)\mathrm{d}y}{\int_{-K}^{K} \mathfrak{g}(y)\mathrm{d}y} \xrightarrow{p} 1.$$

Meanwhile, (2.7.6) yields that as $t \to \infty$,

$$\frac{\int_{-K}^{K} \mathfrak{g}(y)\mathrm{d}y}{\int_{\mathbb{R}} \mathfrak{g}(y)\mathrm{d}y} \cdot \frac{\mathfrak{g}_t(x)}{\int_{-K}^{K} \mathfrak{g}_t(y)\mathrm{d}y} \xrightarrow{d} \frac{\int_{-K}^{K} \mathfrak{g}(y)\mathrm{d}y}{\int_{\mathbb{R}} \mathfrak{g}(y)\mathrm{d}y} \cdot \frac{\mathfrak{g}(x)}{\int_{-K}^{K} \mathfrak{g}(y)\mathrm{d}y} = \frac{\mathfrak{g}(x)}{\int_{\mathbb{R}} \mathfrak{g}(y)\mathrm{d}y}.$$

in the uniform-on-compact topology. Thus, overall we get that $f_{p,t}(x + \mathcal{M}_{p,t}) \xrightarrow{d} \frac{\mathfrak{g}(x)}{\int_{\mathbb{R}} \mathfrak{g}(y)\mathrm{d}y}$, in the uniform-on-compact topology. This establishes (2.1.7), completing the proof of Theorem 2.1.4. □

2.7.2 Proof of Proposition 2.7.2 and Theorem 2.1.5

Coming to the proof of Proposition 2.7.2, we note that the setup of Proposition 2.7.2 is same as that of Proposition 2.6.1. Hence all the discussions pertaining to Proposition 2.6.1 are applicable here. In particular, to prove Proposition 2.7.2, we will be using few notations and certain results from the proof of Proposition 2.6.1.

Proof of Proposition 2.7.2. Fix any $M > 0$. The proof of (2.6.24) proceeds by dividing the integral into two parts depending on the range:

$$U_1 := [-t^{2/3}M - \mathcal{M}_{p,t}, t^{2/3}M - \mathcal{M}_{p,t}]^c, \qquad \text{(Deep Tail)}$$

$$U_2 := [K, K]^c \cap [-t^{2/3}M - \mathcal{M}_{p,t}, t^{2/3}M - \mathcal{M}_{p,t}], \qquad \text{(Shallow Tail)}$$

and controlling each of them individually. See Figure 2.10 for details. In the following two steps, we control these two kind of tails respectively.

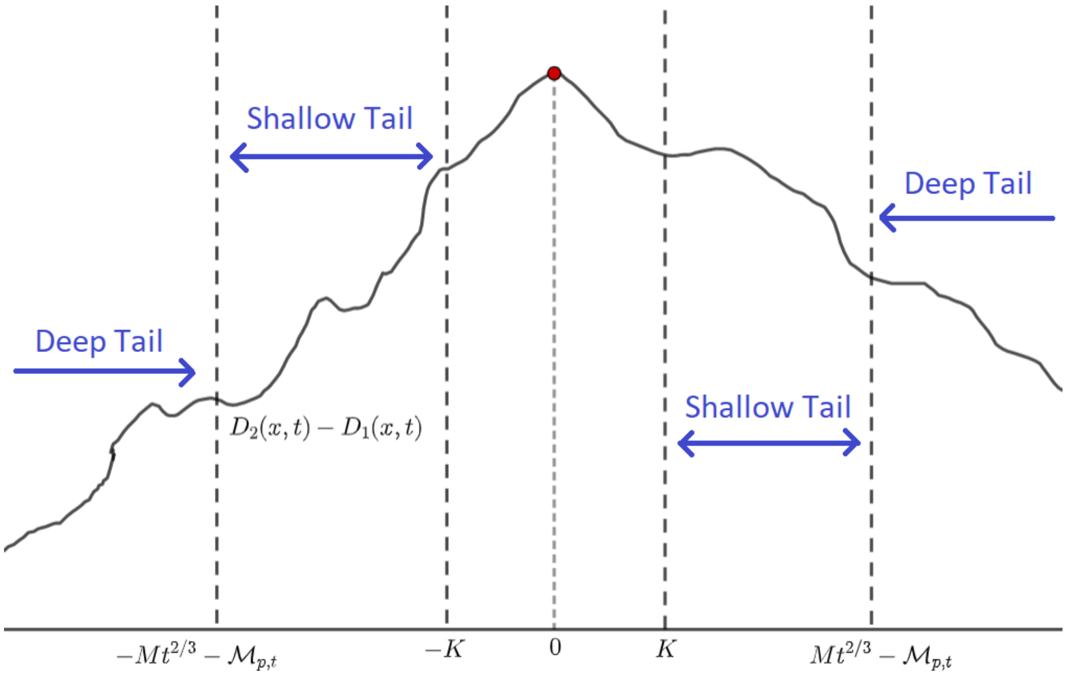


Figure 2.10: Illustration for the proof of Proposition 2.7.2. In Deep Tail region we use parabolic decay of KPZ line ensemble, and in Shallow Tail we use non-intersecting Brownian bridge separation estimates from Proposition 2.5.6.

Step 1. In this step, we control the Deep Tail region: U_1. The goal of this step is to show

$$\limsup_{t\to\infty} \mathbf{P}\left(\int_{U_1} e^{D_2(x,t)-D_1(x,t)}\,\mathrm{d}x \geq \tfrac{\rho}{2}\right) \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}M^3), \tag{2.7.7}$$

for some constant $\mathrm{C} = \mathrm{C}(p) > 0$. We now recall the framework of KPZ line ensemble discussed in Subsection 2.6.2. We define

$$\mathcal{S}_{p,t}(x) := p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x) + q^{1/3}\mathfrak{h}^{(1)}_{qt,\downarrow}(q^{-2/3}x) \tag{2.7.8}$$

where $\mathfrak{h}_{t,\uparrow}, \mathfrak{h}_{t,\downarrow}$ are scaled KPZ line ensembles corresponding to $\mathcal{H}_\uparrow, \mathcal{H}_\downarrow$, see (2.2.6). Observe that

$$D_2(x,t) - D_1(x,t) \stackrel{d}{=} t^{1/3}\left[\mathcal{S}_{p,t}(t^{-2/3}(x+\mathcal{M}_{p,t})) - \sup_{z\in\mathbb{R}} \mathcal{S}_{p,t}(z)\right],$$

where D_1, D_2 are defined in (2.7.1). Thus we have

$$\int_{U_1} \exp(D_2(x,t) - D_1(x,t))\mathrm{d}x \stackrel{d}{=} \int_{|x|\geq M} \exp\left(t^{1/3}\left[\mathcal{S}_{p,t}(x) - \sup_{z\in\mathbb{R}} \mathcal{S}_{p,t}(z)\right]\right)\mathrm{d}x$$

where U_1 is defined in (Deep Tail). Towards this end, we define two events

$$\mathsf{A} := \left\{\sup_{z\in\mathbb{R}} \mathcal{S}_{p,t}(z) \leq -\tfrac{M^2}{4}\right\}, \quad \mathsf{B} := \left\{\sup_{x\in\mathbb{R}} \left(\mathcal{S}_{p,t}(x) + x^2\right) > \tfrac{M^2}{4}\right\},$$

Note that on $\neg A \cap \neg B$, for all $|x| \geq M$, we have

$$\mathcal{S}_{p,t}(x) - \sup_{z\in\mathbb{R}} \mathcal{S}_{p,t}(z) \leq \tfrac{M^2}{4} + \tfrac{M^2}{4} - x^2 \leq \tfrac{M^2}{2} - \tfrac{3M^2}{4} - \tfrac{x^2}{4} \leq -\tfrac{M^2}{4} - \tfrac{x^2}{4}.$$

This forces

$$\int_{|x|\geq M} \exp\left(t^{1/3}\left[\mathcal{S}_{p,t}(x) - \sup_{z\in\mathbb{R}} \mathcal{S}_{p,t}(z)\right]\right)\mathrm{d}x \leq \int_{[-M,M]^c} \exp\left(-t^{1/3}(\tfrac{M^2}{2} + \tfrac{y^2}{4})\right)\mathrm{d}y,$$

which goes to zero as $t \to \infty$. Hence l.h.s. of (2.7.7) $\leq \mathbf{P}(\neg\mathsf{A}) + \mathbf{P}(\neg\mathsf{B})$. Hence it suffices to show

$$\mathbf{P}(\neg\mathsf{A}) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}M^3\right), \quad \mathbf{P}(\neg\mathsf{B}) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}M^3\right). \tag{2.7.9}$$

To prove the first part of (2.7.9), note that

$$\begin{aligned}\mathbf{P}(\neg A) &\leq \mathbf{P}\left(\mathcal{S}_{p,t}(0) \leq -\tfrac{M^2}{4}\right) \\ &\leq \mathbf{P}\left(p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(0) \leq -\tfrac{M^2}{8}\right) + \mathbf{P}\left(q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(1)}(0) \leq -\tfrac{M^2}{8}\right) \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}M^3).\end{aligned}$$

where the last inequality follows by Proposition 2.2.8 (b), for some constant $\mathrm{C} = \mathrm{C}(p) > 0$. This proves the first part of (2.7.9). For the second part of (2.7.9), following the definition of $\mathcal{S}_{p,t}(x)$

from (2.7.8), and using the elementary inequality $\frac{1}{4p}+\frac{1}{4q}\ge 1$ by a union bound we have

$$\mathbf{P}\left(\sup_{x\in\mathbb{R}}\left(\mathcal{S}_{p,t}(x)+x^2\right)>\tfrac{M^2}{4}\right)\le \mathbf{P}\left(\sup_{x\in\mathbb{R}}\left(p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x)+\tfrac{x^2}{4p}\right)>\tfrac{M^2}{8}\right) + \mathbf{P}\left(\sup_{x\in\mathbb{R}}\left(q^{1/3}\mathfrak{h}^{(1)}_{qt,\uparrow}(q^{-2/3}x)+\tfrac{x^2}{4q}\right)>\tfrac{M^2}{8}\right). \tag{2.7.10}$$

Applying Proposition (2.2.8) (c) with $\beta=\frac{1}{2}$, we get that each of the terms on r.h.s. of (2.7.10) are at most $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^3)$ where $\mathrm{C}=\mathrm{C}(p)>0$. This establishes the second part of (2.7.9) completing the proof of (2.7.7).

Step 2. In this step, we control the Shallow Tail region: U_2. We first lay out the heuristic idea behind the Shallow Tail region controls. We recall the nice event $\mathsf{Sp}(\lambda)$ from (2.6.62) which occurs with high probability. Assuming $\mathsf{Sp}(\lambda)$ holds, we apply the the $\mathbf{H}_t$ Brownian Gibbs property of the KPZ line ensembles, and analyze the desired integral

$$\int_{U_2} e^{D_2(x,t)-D_1(x,t)}\mathrm{d}x$$

under the 'free' Brownian bridge law. Further conditioning on the information of the maximizer converts the free law into the law of the $\mathsf{NonInt\text{-}BrBridge}$ (defined in Definition 2.4.4). On $\mathsf{Sp}(\lambda)$, we may apply Proposition 2.5.6 to obtain the desired estimates for the 'free' law. One then obtain the desired estimates for KPZ law using the lower bound for the normalizing constant from Proposition 3.5.1 (b).

We now expand upon the technical details. In what follows we will only work with the right tail:

$$U_{+,2}:=[-t^{2/3}M-\mathcal{M}_{p,t},t^{2/3}M-\mathcal{M}_{p,t}]\cap[K,\infty)=[K,t^{2/3}M-\mathcal{M}_{p,t}]$$

and the argument for the left part of the shallow tail is analogous. Note that we also implicitly assumed $t^{2/3}M-\mathcal{M}_{p,t}\ge K$ above. Otherwise there is nothing to prove. As before we utilize the the notations defined in Subsection 2.6.2. Recall the local maximizer $\mathcal{M}_{p,t}^{M}$ defined in (2.6.25). Recall $Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)}$ from (2.6.26). Set

$$\begin{aligned}\Gamma_{t,M,K} &:= \int_{K}^{Mt^{2/3}-\mathcal{M}_{p,t}} e^{-t^{1/3}\left[Y_{M,t,\uparrow}^{(1)}(t^{-2/3}(\mathcal{M}_{p,t}^{M}+x))-Y_{M,t,\downarrow}^{(1)}(t^{-2/3}(\mathcal{M}_{p,t}^{M}+x))\right]}\mathrm{d}x \\ &= \int_{K}^{Mt^{2/3}-\mathcal{M}_{p,t}} \exp(-D_{M,t,\uparrow}(x)+D_{M,t,\downarrow}(x))\mathrm{d}x,\end{aligned} \tag{2.7.11}$$

where the last equality follows from the definition of $D_{M,t,\uparrow}, D_{M,t,\downarrow}$ from (2.6.28). Recall that the only difference between D_1, D_2 (defined in (2.6.27)) and $D_{M,t,\uparrow}, D_{M,t,\downarrow}$ is that former is defined using the global maximizer $\mathcal{M}_{p,t}$ and the latter by local maximizer $\mathcal{M}_{p,t}^{M}$. However, Lemma 2.3.1 implies that with probability at least $1-\mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^3)$, we have $\mathcal{M}_{p,t}=\mathcal{M}_{p,t}^{M}$. Next, fix $\lambda>0$. Consider $\mathsf{Sp}(\lambda)$ event defined in (2.6.62). We thus have

$$\mathbf{P}\left(\int_{U_{+,2}} e^{D_2(x,t)-D_1(x,t)}\mathrm{d}x \geq \frac{\rho}{4}\right) \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}M^3)+\mathbf{P}(\neg\mathsf{Sp}(\lambda))+\mathbf{P}\left(\Gamma_{t,M,K}\geq\tfrac{\rho}{4},\mathsf{Sp}(\lambda)\right). \tag{2.7.12}$$

We recall the σ-fields $\mathcal{F}_1, \mathcal{F}_2$ defined in (2.6.42) and (2.6.43). We first condition on $\mathcal{F}_1$. As noted in Subsection 2.6.2, since $\mathfrak{h}_{pt,\uparrow}^{(1)}$ and $\mathfrak{h}_{qt,\downarrow}^{(1)}$ are independent, applying $\mathbf{H}_{pt}$ and $\mathbf{H}_{qt}$ Brownian Gibbs property from Proposition 3.5.1 for $\mathfrak{h}_{pt,\uparrow}^{(1)}, \mathfrak{h}_{qt,\downarrow}^{(1)}$ respectively we have

$$\mathbf{P}\left(\Gamma_{t,M,K}\geq\tfrac{\rho}{2},\mathsf{Sp}(\lambda)\right)=\mathbf{E}\left[\frac{\mathbf{E}_{\mathrm{free},t}[\mathbf{1}_{\Gamma_{t,M,K}\geq\frac{\rho}{4},\mathsf{Sp}(\lambda)}W_\uparrow W_\downarrow]}{\mathbf{E}_{\mathrm{free},t}[W_\uparrow W_\downarrow]}\right], \tag{2.7.13}$$

where $W_\uparrow$, $W_\downarrow$ are defined in (2.6.48) and (2.6.49). Here $\mathbf{P}_{\mathrm{free},t}$ and $\mathbf{E}_{\mathrm{free},t}$ are the probability and the expectation operator respectively corresponding to the joint 'free' law for $(p^{1/3}\mathfrak{h}_{pt,\uparrow}(p^{-2/3}x)$, and $q^{1/3}\mathfrak{h}_{qt,\downarrow}(q^{-2/3}x))_{x\in[-M,M]}$ which by Brownian scaling is given by a pair of independent Brownian bridges $(B_1(\cdot),B_2(\cdot))$ on $[-M,M]$ with starting points $(p^{1/3}\mathfrak{h}_{pt,\uparrow}(-Mp^{-2/3}), q^{1/3}\mathfrak{h}_{qt,\downarrow}(-Mq^{-2/3}))$ and endpoints $(q^{1/3}\mathfrak{h}_{pt,\uparrow}(Mp^{-2/3}), q^{1/3}\mathfrak{h}_{qt,\downarrow}(Mq^{-2/3}))$.

In addition, from the last part of Proposition 3.5.1 we know that for any given $\lambda>0$, there exists $\delta(M,p,\lambda) > 0$ such that

$$\mathbf{P}(\mathbf{E}_{\mathrm{free},t}[W_{\uparrow}W_{\downarrow}] > \delta) \geq 1-\lambda. \tag{2.7.14}$$

Since the weight $W_{\uparrow}W_{\downarrow} \in [0,1]$, (2.7.13) and (2.7.14) give us

$$\text{r.h.s. of (2.7.12)} \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}M^3) + \mathbf{P}(\neg\mathsf{Sp}(\lambda)) + \lambda + \frac{1}{\delta}\mathbf{E}\left[\mathbf{P}_{\mathrm{free},t}\left(\Gamma_{t,M,K} \geq \tfrac{\rho}{4}, \mathsf{Sp}(\lambda)\right)\right]. \tag{2.7.15}$$

Next we condition on $\mathcal{F}_2$ defined in (2.6.43). By Proposition 2.4.9, upon conditioning the free measure of two Brownian bridges when viewed around the maximizer are given by two NonInt-BrBridge (defined in Definition 2.4.4). The precise law is given by Nlarge law defined in Definition 2.6.3. Note that $\mathsf{Sp}(\lambda)$ is measurable w.r.t. $\mathcal{F}_1 \cup \mathcal{F}_2$. By Reverse Fatou's Lemma and the tower property of conditional expectations, we obtain that

$$\begin{aligned} &\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{E}\left[\mathbf{P}_{\mathrm{free},t}\left(\Gamma_{t,M,K} \geq \tfrac{\rho}{4}, \mathsf{Sp}(\lambda)\right)\right] \\ &\leq \mathbf{E}\left[\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{1}_{\mathsf{Sp}(\lambda)}\mathbf{P}_{\mathsf{Nlarge}|2,1}\left(\Gamma_{t,M,K} \geq \tfrac{\rho}{4}\right)\right]. \end{aligned} \tag{2.7.16}$$

Following the Definition 2.6.3 and (2.7.11) we see that under Nlarge law,

$$\Gamma_{t,M,K} \stackrel{d}{=} \int_K^{Mt^{2/3}-\mathcal{M}_{p,t}} e^{-t^{1/3}\left[V_{r,1}^{\mathsf{large}}(t^{-2/3}x)-V_{r,2}^{\mathsf{large}}(t^{-2/3}x)\right]}\mathrm{d}x. \tag{2.7.17}$$

where $V_r^{\mathsf{large}} = (V_{r,1}^{\mathsf{large}}, V_{r,2}^{\mathsf{large}})$ is a NonInt-BrBridge defined in Definition 2.6.3. Now notice that by the definition in (2.6.62), on the $\mathsf{Sp}(\lambda)$ event, the length of the Brownian bridges considered are bounded from below and above and the end points are tight. Following the equality in distribution in (2.7.17), the technical result of Proposition 2.5.6 precisely tells us that the term inside the expectation of r.h.s. of (2.7.16) is zero. Thus, going back to (2.7.15) we get that

$$\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{P}\left(\int_{U_{+,2}} e^{D_2(x,t)-D_1(x,t)}\mathrm{d}x \ge \frac{\rho}{4}\right) \le \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}M^3) + \limsup_{t\to\infty}\mathbf{P}(\neg\mathsf{Sp}(\lambda)) + \lambda.$$

Taking $\limsup_{\lambda\downarrow 0}$, in view of (2.6.63), we get that last two terms in r.h.s. of the above equation are zero. Similarly one can show the same bound for the integral under $U_{-,2} := [-t^{2/3}M - \mathcal{M}_{p,t}, t^{2/3}M - \mathcal{M}_{p,t}] \cap (-\infty, -K]$. Together with (2.7.7), we thus have

$$\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{P}\left(\int_{[-K,K]^c} e^{D_2(x,t)-D_1(x,t)}\mathrm{d}x \ge \rho\right) \le \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}M^3).$$

Taking $M \to \infty$ we get (2.7.2) completing the proof. □

Proof of Theorem 2.1.5. Recall from (3.1.6) that

$$f_{*,t}(x) = \frac{\mathcal{Z}(0,0;x,t)}{\mathcal{Z}(0,0;*,t)} = \frac{e^{\mathcal{H}(x,t)}}{\int_{\mathbb{R}} e^{\mathcal{H}(y,t)}\mathrm{d}y}.$$

The uniqueness of the mode $\mathcal{M}_{*,t}$ for $f_{*,t}$ is already proved in Lemma 2.3.1. Thus, we have

$$f_{*,t}(x+\mathcal{M}_{*,t}) = \frac{\exp\left(\mathcal{H}(\mathcal{M}_{*,t}+x,t) - \mathcal{H}(\mathcal{M}_{*,t},t)\right)}{\int_{\mathbb{R}} \exp\left(\mathcal{H}(\mathcal{M}_{*,t}+y,t) - \mathcal{H}(\mathcal{M}_{*,t},t)\right)\mathrm{d}y}.$$

Just like in Proposition 2.7.2, we claim that

$$\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{P}\left(\int_{[-K,K]^c} e^{\mathcal{H}(\mathcal{M}_{*,t}+y,t)-\mathcal{H}(\mathcal{M}_{*,t},t)}\mathrm{d}y \ge \rho\right) = 0. \tag{2.7.18}$$

The proof of (2.7.18) is exactly same as that of (2.7.2), where we divide the integral in (2.7.18) into a deep tail and a shallow tail and bound them individually. To avoid repetition, we just add few pointers for the readers. Indeed the two key steps of proof of Proposition 2.7.2 that bound the deep and shallow tails can be carried out for the (2.7.18) case. The deep tail regime follows an exact similar strategy as Step 1 of the proof of Proposition 2.7.2 and utilizes the same parabolic decay of the KPZ equation from Proposition 2.2.8. The analogous shallow tail regime also follows in a similar manner by using the uniform separation estimate for Bessel bridges from Corollary 2.5.7.

Now note that by Theorem 2.1.10 with $k = 1$, we have

$$\mathcal{H}(\mathcal{M}_{*,t} + x, t) - \mathcal{H}(\mathcal{M}_{*,t}, t) \xrightarrow{d} \mathcal{R}_1(x), \tag{2.7.19}$$

in the uniform-on-compact topology. Here $\mathcal{R}_1$ is a 3D-Bessel process with diffusion coefficient 1. With the tail decay estimate in (2.7.18) and the same for the Bessel process from Proposition 2.7.1, in view of (2.7.19) one can show $f_{*,t}(x + \mathcal{M}_{*,t}) \to \frac{e^{-\mathcal{R}_1(x)}}{\int_{\mathbb{R}} e^{-\mathcal{R}_1(y)} \mathrm{d}y}$ in the uniform-on-compact topology by following the analogous argument from the proof of Theorem 2.1.4. This completes the proof. □

2.8 Appendix: Non-intersecting random walks

In this section we prove Lemma 2.4.7 that investigates the convergence of non-intersecting random walks to non-intersecting brownian motions. We remark that similar types of Theorems are already known in the literature such as [119], where the authors considered random walks to start at different locations. Since our walks starts at the same point, additional care is required.

We now recall Lemma 2.4.7 for readers' convenience.

Lemma 2.8.1. *Let X_j^i be i.i.d. $\mathrm{N}(0,1)$ random variables. Let $S_0^{(i)} = 0$ and $S_k^{(i)} = \sum_{j=1}^k X_j^i$. Consider $Y_n(t) = (Y_{n,1}(t), Y_{n,2}(t)) := (\frac{S_{nt}^{(1)}}{\sqrt{n}}, \frac{S_{nt}^{(2)}}{\sqrt{n}})$ an $\mathbb{R}^2$ valued process on $[0,1]$ where the in-between points are defined by linear interpolation. Then conditioned on the non-intersecting event $\Lambda_n := \cap_{j=1}^n \{S_j^{(1)} > S_j^{(2)}\}$, $Y_n \xrightarrow{d} W$, where $W(t) = (W_1(t), W_2(t))$ is distributed as* NonInt-BM *defined in Definition 2.4.3.*

Proof of Lemma 2.8.1. To show weak convergence, it suffices to show finite dimensional convergence and tightness. Based on the availability of exact joint densities for non-intersecting random walks from Karlin-McGregor formula [161], the verification of weak convergence is straightforward. So, we only highlight major steps of the computations below.

Step 1. One point convergence at $t = 1$. Note that

$$\mathbf{P}\left(|\sqrt{n}Y_{n,i}(t) - S^{(i)}_{\lfloor nt \rfloor}| > \sqrt{n}\varepsilon \mid \Lambda_n\right) \le \frac{1}{\mathbf{P}(\Lambda_n)}\mathbf{P}\left(|X_{\lfloor nt \rfloor+1}| > \sqrt{n}\varepsilon\right) \le \frac{\mathrm{C}}{\varepsilon^2\sqrt{n}}$$

The last inequality above follows by Markov inequality and the classical result that $\mathbf{P}(\Lambda_n) \ge \frac{\mathrm{C}}{\sqrt{n}}$ in Spitzer [222]. Thus it suffices to show finite dimensional convergence for the cadlag process:

$$(Z^{(1)}_{nt}, Z^{(2)}_{nt}) := \frac{1}{\sqrt{n}}(S^{(1)}_{\lfloor nt \rfloor}, S^{(2)}_{\lfloor nt \rfloor}). \tag{2.8.1}$$

We assume that n large enough so that $\frac{n-1}{M\sqrt{n}} \ge 1$ for some $M > 0$ to be chosen later. When $t = 1$, applying the Karlin-McGregor formula, we obtain that

$$\mathbf{P}(Z_n(1) \in \mathrm{d}y_1, Z_n(1) \in \mathrm{d}y_2|\Lambda_n) = \tau_n \cdot f_{n,1}(y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2$$

where

$$f_{n,1}(y_1, y_2) := \int_{a_1 > a_2} p_1(a_1)p_1(a_2)\det(p_{n-1}(a_i - y_j\sqrt{n}))^2_{i,j=1}\mathrm{d}a_1\mathrm{d}a_2,$$

and

$$\tau_n^{-1} := \int_{r_1>r_2}\int_{a_1>a_2} p_1(a_1)p_1(a_2)\det(p_{n-1}(a_i - r_j\sqrt{n}))^2_{i,j=1}\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}r_1\mathrm{d}r_2. \tag{2.8.2}$$

Note that here the Karlin-McGregor formula, after we have conditioned on the first step of the random walks with $X^1_1 = a_1 > X^2_1 = a_2$.

We will now show that $\frac{(n-1)^2}{\sqrt{n}}\tau_n^{-1}$ and $\frac{(n-1)^2}{\sqrt{n}}f_{n,1}(y_1, y_2)$ converges to a nontrivial limit. Observe that

$$\tfrac{(n-1)^2}{\sqrt{n}}\det(p_{n-1}(a_i - y_j\sqrt{n}))_{i,j=1}^2 = (n-1)p_{n-1}(a_1 - y_2\sqrt{n})p_{n-1}(a_2 - y_1\sqrt{n}) \cdot \tfrac{n-1}{\sqrt{n}}[e^{\frac{\sqrt{n}(a_1-a_2)(y_1-y_2)}{n-1}} - 1]. \tag{2.8.3}$$

Thus, as $n \to \infty$, we have

$$\tfrac{(n-1)^2}{\sqrt{n}}\det(p_{n-1}(a_i - y_j\sqrt{n}))_{i,j=1}^2 \to p_1(y_1)p_1(y_2)(a_1 - a_2)(y_1 - y_2). \tag{2.8.4}$$

Next we proceed to find a uniform bound for the expression in (2.8.3). Not that for $x, r \geq 1$, one has the elementary inequality $x^r \geq x^r - 1 \geq r(x-1)$. Now taking $r = \frac{n-1}{M\sqrt{n}}$ and $x = \exp(\frac{\sqrt{n}}{n-1}(a_1 - a_2)(y_1 - y_2)$ we get

$$\begin{aligned}
\text{r.h.s. of (2.8.3)} &\leq \frac{1}{2\pi}\exp\left(-\tfrac{(a_1-y_2\sqrt{n})^2}{2n-2} - \tfrac{(a_2-y_1\sqrt{n})^2}{2n-2} + \tfrac{1}{M}(a_1 - a_2)(y_1 - y_2)\right) \\
&\leq \frac{1}{2\pi}\exp\left(-\tfrac{y_2^2}{4} - \tfrac{y_1^2}{4} + \tfrac{1}{M}(a_1 - a_2)(y_1 - y_2) + \tfrac{1}{M}(|a_1y_2| + |a_2y_1|)\right) \\
&\leq \frac{1}{2\pi}\exp\left(-\tfrac{y_2^2}{4} - \tfrac{y_1^2}{4} + \tfrac{2(a_1^2+y_1^2+a_2^2+y_2^2)}{M})\right),
\end{aligned} \tag{2.8.5}$$

where the last inequality follows by several application of the elementary inequality $|xy| \leq \frac{1}{2}(x^2 + y^2)$. One can choose M large enough so that the uniform bound in (2.8.5) is integrable w.r.t. the measure $p_1(a_1)p_1(a_2)\mathrm{d}a_1\mathrm{d}a_2$. With the pointwise limit from (2.8.4), by dominated convergence theorem we have

$$\begin{aligned}
\tfrac{(n-1)^2}{\sqrt{n}}f_{n,1}(y_1, y_2) = \tfrac{(n-1)^2}{\sqrt{n}}&\int_{a_1>a_2} p_1(a_1)p_1(a_2)\det(p_{n-1}(a_i - y_j\sqrt{n}))_{i,j=1}^2\mathrm{d}a_1\mathrm{d}a_2 \\
&\to p_1(y_1)p_1(y_2)(y_1 - y_2)\int_{a_1>a_2}(a_1 - a_2)p_1(a_1)p_1(a_2)\mathrm{d}a_1\mathrm{d}a_2.
\end{aligned}$$

Similarly one can compute the pointwise limit for the integrand in τ_n^{-1} (defined in (2.8.2)) and the uniform bound in (2.8.5) works for the denominator as well. We thus have

$$\frac{(n-1)^2}{\sqrt{n}}\tau_n^{-1} \to \int_{a_1>a_2}\int_{r_1>r_2} p_1(r_1)p_1(r_2)(r_1-r_2)(a_1-a_2)p_1(a_1)p_1(a_2)\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}r_1\mathrm{d}r_2. \quad (2.8.6)$$

Plugging these limits back in (2.8.1), we arrive at (2.4.1) (the one point density formula for NonInt-BM) as the limit for (2.8.1).

Step 2. One point convergence at $0 < t < 1$**.** When $0 < t < 1$, with the Karlin-Mcgregor formula, we similarly obtain

$$\mathbf{P}(Z_{nt}^{(1)} \in \mathrm{d}y_1, Z_{nt}^{(2)} \in \mathrm{d}y_2 \mid \Lambda_n) = \tau_n \cdot f_{n,t}(y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2 \quad (2.8.7)$$

where τ_n is defined in (2.8.2) and

$$\begin{aligned} f_{n,t}(y_1,y_2) = \int_{r_1>r_2}\int_{a_1>a_2} p_1(a_1)p_1(a_2)\Big[&\det(p_{\lfloor nt\rfloor -1}(a_i - y_j\sqrt{n}))_{i,j=1}^2 \\ &n\cdot\det(p_{n-\lfloor nt\rfloor}(\sqrt{n}y_i - \sqrt{n}r_j))_{i,j=1}^2\Big]\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}r_1\mathrm{d}r_2. \end{aligned} \quad (2.8.8)$$

One can check that as $n \to \infty$, we have

$$\begin{aligned} n^{3/2}\det(p_{\lfloor nt\rfloor -1}(a_i - y_j\sqrt{n}))_{i,j=1}^2 &\to \tfrac{1}{t}p_t(y_1)p_t(y_2)(a_1-a_2)(y_1-y_2), \\ n\cdot\det(p_{n-\lfloor nt\rfloor}(\sqrt{n}y_i - \sqrt{n}r_j))_{i,j=1}^2 &\to \det(p_{1-t}(y_i-r_j))_{i,j=1}^2. \end{aligned}$$

One can provide uniformly integrable bound for the integrand in $f_{n,t}(y_1,y_2)$ in a similar fashion. Thus by dominated convergence theorem,

$$\begin{aligned} n^{3/2}f_{n,t}(y_1,y_2) \to \tfrac{1}{t}p_t(y_1)p_t(y_2)(y_1-y_2)&\int_{a_1>a_2}p_1(a_1)p_1(a_2)(a_1-a_2)\mathrm{d}a_1\mathrm{d}a_2 \\ &\int_{r_1>r_2}\det(p_{1-t}(y_i-r_j))_{i,j=1}^2\mathrm{d}r_1\mathrm{d}r_2. \end{aligned}$$

Using (2.8.6) we get that $\tau_n \cdot f_{n,t}(y_1,y_2)$ converges to (2.4.2), the one point density formula for

NonInt-BM.

Step 3. Transition density convergence. For the transition densities, let $0 < t_1 < t_2 < 1$, and fix $x_1 > x_2$. Another application of Karlin-McGregor formula tells us

$$\begin{aligned}
&\mathbf{P}(Z_{nt_2}^{(1)} \in \mathrm{d}y_1, Z_{nt_2}^{(2)} \in \mathrm{d}y_2 \mid Z_{nt_1}^{(1)} = x_1, Z_{nt_1}^{(2)} = x_2) \\
&= n \det(p_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}(\sqrt{n}y_i - \sqrt{n}x_j))_{i,j=1}^2 \\
&\quad \cdot \frac{\int_{r_1 > r_2} \det(p_{n-\lfloor nt_2 \rfloor}(\sqrt{n}y_i - \sqrt{n}r_j))_{i,j=1}^2 \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}y_1 \mathrm{d}y_2}{\int_{r_1 > r_2} \det(p_{n-\lfloor nt_1 \rfloor}(\sqrt{n}x_i - \sqrt{n}r_j))_{i,j=1}^2 \mathrm{d}r_1 \mathrm{d}r_2}.
\end{aligned} \tag{2.8.9}$$

One can check as $n \to \infty$

$$\text{r.h.s of (2.8.9)} \to \frac{\det(p_{t_2-t_1}(y_i - x_j))_{i,j=1}^2 \int_{r_1>r_2} \det(p_{1-t_2}(y_i - r_j))_{i,j=1}^2 \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}y_1 \mathrm{d}y_2}{\int_{r_1>r_2} \det(p_{1-t_1}(x_i - r_j))_{i,j=1}^2 \mathrm{d}r_1 \mathrm{d}r_2}$$

which is same as transition densities for NonInt-BM as shown in (2.4.3). This proves finite dimensional convergence.

Step 4. Tightness. To show tightness, by Kolmogorov tightness criterion, it suffices to show there exist $K > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\mathbf{E}\left[|Y_{n,i}(t) - Y_{n,i}(s)|^K \mid \Lambda_n\right] \leq \mathrm{C}_{K,n_0} \cdot (t-s)^2 \tag{2.8.10}$$

holds for all $0 \leq s < t \leq 1$.

Recall that $\mathbf{P}(\Lambda_n) \geq \frac{\mathrm{C}}{\sqrt{n}}$. For $t - s \leq \frac{1}{n}$ with $K \geq 5$ we have

$$\begin{aligned}
\mathbf{E}\left[|Y_{n,i}(t) - Y_{n,i}(s)|^K \mid \Lambda_n\right] &\leq \mathrm{C} \cdot \sqrt{n}\mathbf{E}\left[|Y_{n,i}(t) - Y_{n,i}(s)|^K\right] \\
&\leq \mathrm{C} \cdot \sqrt{n}\frac{(nt-ns)^K}{n^{K/2}}\mathbf{E}[|X_1^1|^K] \leq \mathrm{C}n^{\frac{1-K}{2}}(nt-ns)^2 \leq \mathrm{C}_K(t-s)^2.
\end{aligned}$$

Thus we may assume $t - s \geq 1/n$. Then it is enough to show (2.8.10) for $Z_{nt}^{(i)}$ (defined in (2.8.1)) instead. Note that if $t - s \in [n^{-1}, n^{-1/4}]$, we may take K large enough so $\frac{1}{4}(K-4) \geq 1$. Then we have

$$\begin{aligned}\mathbf{E}\left[|Z_{nt}^{(i)} - Z_{ns}^{(i)}|^K \mid \Lambda_n\right] &\leq \mathrm{C} \cdot \sqrt{n}\mathbf{E}\left[|Z_{nt}^{(i)} - Z_{ns}^{(i)}|^K\right] \\ &\leq \mathrm{C} \cdot \sqrt{n}(t-s)^{K/2} \leq \mathrm{C} \cdot n^{1/2-(K-4)/8}(t-s)^2\end{aligned}$$

where in the last line we used the fact $(t-s)^{(K-4)/2} \leq n^{-(K-4)/8}$. As $\frac{1}{4}(K-4) \geq 1$, we have $\mathbf{E}\left[|Z_{nt}^{(i)} - Z_{ns}^{(i)}|^K \mid \Lambda_n\right] \leq \mathrm{C}(t-s)^2$ in this case. So, we are left with the case $t - s \geq n^{-1/4}$.

Let us assume $t = 0$, $s \geq n^{-\frac{1}{4}}$. As $ns \geq n^{3/4} \to \infty$, we will no longer make the distinction between ns and $\lfloor ns \rfloor$ in our computations. We use the pdf formula from (2.8.7) and (2.8.8) to get

$$\begin{aligned}\mathbf{E}[|Z_{ns}^{(i)}|^5] \leq \tau_n \int_{y_1>y_2} |y_i|^5 \int_{r_1>r_2} \int_{a_1>a_2} & p_1(a_1)p_1(a_2)\det(p_{ns-1}(a_i - y_j\sqrt{n}))_{i,j=1}^2 \\ & n \cdot \det(p_{n-ns}(\sqrt{n}y_i - \sqrt{n}r_j))_{i,j=1}^2\Big] \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}y_1\mathrm{d}y_2.\end{aligned} \tag{2.8.11}$$

For the last determinant we may use

$$\begin{aligned}& n \cdot \det(p_{n-ns}(\sqrt{n}y_i - \sqrt{n}r_j))_{i,j=1}^2 \mathrm{d}r_1\mathrm{d}r_2 \\ & \leq n \cdot p_{n-ns}(\sqrt{n}y_1 - \sqrt{n}r_1)p_{n-ns}(\sqrt{n}y_2 - \sqrt{n}r_2)\mathrm{d}r_1\mathrm{d}r_2\end{aligned}$$

which integrates to 1 irrespective of the value of y_1, y_2. Thus

$$\text{r.h.s. of (2.8.11)} \leq \tau_n \int_{y_1>y_2} |y_i|^5 \int_{a_1,a_2} p_1(a_1)p_1(a_2)\det(p_{ns-1}(a_i - y_j\sqrt{n}))_{i,j=1}^2 \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}y_1\mathrm{d}y_2. \tag{2.8.12}$$

Making the change of variable $y_i = \sqrt{s}z_i$ and setting $m = ns$, we have

$$\text{r.h.s. of (2.8.12)} \leq \tau_n \cdot s^{\frac{5}{2}+1}\mathcal{I}_m,$$

where

$$\mathcal{I}_m := \int_{z_1>z_2} |z_i|^5 \int_{a_1>a_2} p_1(a_1)p_1(a_2)\det(p_{m-1}(a_i - z_j\sqrt{m}))_{i,j=1}^2 \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}z_1\mathrm{d}z_2.$$

We claim that $\frac{(m-1)^2}{\sqrt{m}}\mathcal{I}_m \le \mathrm{C}$ for some universal constant $\mathrm{C} > 0$. Clearly this integral is finite for each m. And by exact same approach in **Step 1**, one can show as $m \to \infty$,

$$\frac{(m-1)^2}{\sqrt{m}}\mathcal{I}_m := \int_{z_1>z_2} |z_i|^5 \int_{a_1>a_2} p_1(z_1)p_1(z_2)p_1(a_1)p_1(a_2)(a_1-a_2)(z_1-z_2)\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}z_1\mathrm{d}z_2.$$

Thus, $\frac{(m-1)^2}{\sqrt{m}}\mathcal{I} \le \mathrm{C}$ for all $m \ge 1$. Thus following (2.8.11), (2.8.12), in view of the above estimate we get

$$\mathbf{E}[|Z_{ns}^{(i)}|^5] \le \mathrm{C}\tau_n \frac{\sqrt{m}}{(m-1)^2} s^{\frac{5}{2}+1}.$$

However, by **Step 1**, $n^{3/2}\tau_n^{-1}$ converges to a finite positive constant. As $m = ns$, we thus get that the above term is at most $\mathrm{C} \cdot s^2$. The case $t \neq 0$ can be checked similarly using the formulas from (2.8.7) and (2.8.8) as well as transition densities formula (2.8.9). This completes the proof. □

Chapter 3: Short- and long-time path tightness of the continuum directed random polymer

3.1 Introduction

3.1.1 Background and motivation

Directed polymers in random environment can be considered as random walks interacting with a random external environment. First introduced and studied in [148], [152] and [44], they have since become a fertile ground for research in orthogonal polynomials, random matrices, stochastic PDEs, and integrable systems (see [65, 129, 33] and the references therein). In the $(1+1)$-dimensional discrete polymer case, the random environment is specified by a collection of zero-mean i.i.d. random variables $\{\omega = \omega(i,j) \mid (i,j) \in \mathbb{Z}_+ \times \mathbb{Z}\}$. Given the environment, the energy of the n-step nearest neighbour random walk $(S_i)_{i=0}^n$ starting at the origin is given by $H_n^\omega(S) := \sum_{i=1}^n \omega(i, S_i)$. The **point-to-line** polymer measure on the set of all such paths is then defined as

$$\mathbf{P}^\omega_{n,\beta}(S) = \frac{1}{Z^\omega_{n,\beta}} e^{\beta H_n^\omega(S)} \mathbf{P}(S),$$

where $\mathbf{P}(S)$ is the simple random walk measure, β is the inverse temperature, and $Z^\omega_{n,\beta}$ is the partition function.

A competition exists between the *entropy* of paths and the *energy* of the environment in this polymer measure. Spurred by this competition, two distinct regimes appear depending on the inverse temperature β. When $\beta = 0$ the polymer measure is the simple random walk; hence it is entropy-dominated and exhibits diffusive behavior. We refer to this scenario as *weak disorder*. For $\beta > 0$, the polymer measure concentrates on paths with high energies and the diffusive behavior ceases to be guaranteed. This type of energy domination is known as *strong disorder*. For the definitions and results on the precise separation between the two regimes as well as results on higher dimensions, we refer the readers to [69, 171, 64].

While the polymer behavior is characterized by diffusivity in weak disorder, the fluctuations of polymers in strong disorder are conjecturally characterized by two scaling exponents ζ and χ ([220], [4]):

$$\text{Fluctuation of the endpoint of the path: } |S_n| \sim n^{\zeta}, \tag{3.1.1}$$
$$\text{Fluctuation of the log partition function: } [\log Z^{\omega}_{n,\beta} - \rho(\beta)n] \sim n^{\chi}.$$

It is believed that directed polymers fall under the "Kardar-Parisi-Zhang (KPZ) universality class" (see [77, 148, 149, 160, 169]) with fluctuation exponent $\chi = \frac{1}{3}$ and transversal exponent $\zeta = \frac{2}{3}$. This instance of the transversal exponent appearing larger than the diffusive scaling exponent $\frac{1}{2}$ is called *superdiffusivity*. Crucially, the conjectured values for χ and ζ satisfy the "KPZ relation":

$$\chi = 2\zeta - 1. \tag{3.1.2}$$

At the moment, rigorous results on either exponent or the KPZ relation have been scarce. For directed polymers, $\zeta = 2/3$ has only been obtained for log-gamma polymers in [220, 23]. Upper and lower bounds on ζ have been established in [202, 188] under additional weight assumptions. For zero-temperature models, $\zeta = \frac{2}{3}$ has been established in [155, 49, 140, 94, 28]. Outside the temperature models, the KPZ relation in (3.1.2) has also been shown in other random growth models such as first passage percolation in [60] and [7] under the assumption that the exponents exist in a certain sense. In strong disorder, the polymer also exhibits certain localization phenomena (see [69, 33, 89] for partial surveys). In particular, the favorite region conjecture speculates that the endpoint of the polymer is asymptotically localized in a region of stochastically bounded diameter (see [66, 33, 32, 17, 89] for related results).

Given the conceptual pictures on the two extreme regimes, in the present paper, we consider polymer fluctuations in the *intermediate disorder regime*. Introduced in [4], the intermediate disorder regime corresponds to scaling the inverse temperature $\beta = \beta_n = n^{-1/4}$ with the length of the polymer n, which captures the transitions between the weak and strong disorders and retains features of both. Within this regime, [3] showed that the partition function for point-to-point directed polymers has a universal scaling limit given by the solution of the Stochastic Heat Equation (SHE) for environment with finite exponential moments. In addition, the polymer path itself converges to a universal object called the Continuum Directed Random Polymer (denoted as CDRP hereafter) under the diffusive scaling.

We consider point-to-point CDRP of length t. The main contribution of this paper can be summarized as follows.

(a) We show that as $t \downarrow 0$, the polymer paths behave diffusively and its annealed law converges in to the law of a Brownian bridge (Theorem 3.1.4).

(b) On the other hand, as $t \uparrow \infty$, the polymers have $t^{2/3}$ pathwise fluctuations. The latter result confirms superdiffusivity and the conjectural $2/3$ exponent for the CDRP (Theorem 3.1.7 (a)). Moreover, the strength of our result exceeds the conjecture in (3.1.1), which only claims endpoint tightness. Instead, in Theorem 3.1.7 (a), we prove that the annealed law of paths of point-to-point CDRP of length t are tight (as $t \uparrow \infty$) upon $t^{2/3}$ scaling. This marks the first result of path tightness among all positive-temperature models.

(c) We also show pointwise weak convergence of the polymer paths under the $t^{2/3}$ scaling to points on the geodesic of the directed landscape (Theorem 3.1.7 (b)). This ensures the $2/3$ scaling exponent is indeed tight. Modulo a conjecture on convergence of the KPZ sheet to the Airy Sheet (Conjecture 3.1.9), we obtain pathwise convergence of the rescaled CDRP to the geodesic of the directed landscape (Theorem 3.1.10).

These results provide a comprehensive picture of fluctuations of CDRP paths under short- and long-time scaling. Our short-time and long-time tightness results also extend to point-to-line CDRP (Theorem 3.1.8). The formal statement of the main results are given in Section 3.1.2.

3.1.2 The model and the main results

We use the stochastic heat equation (SHE) with multiplicative noise to define the CDRP model. To start with, consider a four-parameter random field $\mathcal{Z}(x,s;y,t)$ defined on

$$\mathbb{R}^4_{\uparrow} := \{(x,s;y,t) \in \mathbb{R}^4 : s < t\}.$$

For each $(x,s) \in \mathbb{R}\times\mathbb{R}$, the function $(y,t) \mapsto \mathcal{Z}(x,s;y,t)$ is the solution of the SHE starting from location x at time s, i.e., the unique solution of

$$\partial_t \mathcal{Z} = \tfrac{1}{2}\partial_{xx}\mathcal{Z} + \mathcal{Z}\cdot\xi, \qquad (y,t) \in \mathbb{R}\times(s,\infty),$$

with Dirac delta initial data $\lim_{t\downarrow s}\mathcal{Z}(x,s;y,t) = \delta(x-y)$. Here $\xi = \xi(x,t)$ is the space-time white noise. The SHE itself enjoys a well-developed solution theory based on Itô integral and chaos expansion [34, 237] also [77, 206]. Via the Feynmann-Kac formula ([149, 65]) the four-parameter random field can be written in terms of chaos expansion as

$$\mathcal{Z}(x,s;y,t) = p(y-x,t-s) + \sum_{k=1}^{\infty}\int_{\Delta_{k,s,t}}\int_{\mathbb{R}^k}\prod_{\ell=1}^{k+1} p(y_\ell - y_{\ell-1}, s_\ell - s_{\ell-1})\xi(y_\ell, s_\ell)d\vec{y}\,d\vec{s}, \quad (3.1.3)$$

with $\Delta_{k,s,t} := \{(s_\ell)_{\ell=1}^k : s < s_1 < \cdots < s_k < t\}$, $s_0 = s$, $y_0 = x$, $s_{k+1} = t$, and $y_{k+1} = y$. Here

$$p(x,t) := (2\pi t)^{-1/2}\exp(-x^2/(2t))$$

denotes the standard heat kernel. The field $\mathcal{Z}$ satisfies several other properties including the Chapman-Kolmogorov equations [3, Theorem 3.1]. For all $0 \le s < r < t$, and $x, y \in \mathbb{R}$ we have

$$\mathcal{Z}(x,s;y,t) = \int_{\mathbb{R}} \mathcal{Z}(x,s;z,r)\mathcal{Z}(z,r;y,t)dz. \quad (3.1.4)$$

Definition 3.1.1 (Point-to-point CDRP)**.** Conditioned on the white noise ξ, let $\mathbf{P}^{\xi}$ be a measure on $C([s,t])$ whose finite-dimensional distribution is given by

$$\mathbf{P}^{\xi}(X(t_1) \in dx_1, \ldots, X(t_k) \in dx_k) = \frac{1}{\mathcal{Z}(x,s;y,t)} \prod_{j=0}^{k} \mathcal{Z}(x_j, t_j, ; x_{j+1}, t_{j+1}) dx_1 \cdots dx_k. \quad (3.1.5)$$

for $s = t_0 \leq t_1 < \cdots < t_k \leq t_{k+1} = t$, with $x_0 = x$ and $x_{k+1} = y$. (3.1.4) ensure $\mathbf{P}^{\xi}$ is a valid probability measure. Note that $\mathbf{P}^{\xi}$ also depends on x and y but we suppress it from our notations. We will use the notation $\mathsf{CDRP}(x,s;y,t)$ and write $X \sim \mathsf{CDRP}(x,s;y,t)$ when $X(\cdot)$ is a random continuous function on $[s,t]$ with $X(s) = x$ and $X(t) = y$ and its finite-dimensional distributions given by (3.1.5) conditioned on ξ. We will also use the notation $\mathbf{P}^{\xi}, \mathbf{E}^{\xi}$ to denote the law and expectation conditioned on the noise ξ, and $\mathbf{P}, \mathbf{E}$ for the annealed law and expectation respectively.

Definition 3.1.2 (Point-to-line CDRP)**.** Conditioned on the white noise ξ, we let $\mathbf{P}_*^{\xi}$ be a measure $C([s,t])$ whose finite-dimensional distributions are given by

$$\mathbf{P}_*^{\xi}(X(t_1) \in dx_1, \ldots, X(t_k) \in dx_k) = \frac{1}{\mathcal{Z}(x,s;*,t)} \prod_{j=0}^{k} \mathcal{Z}(x_j, t_j, ; x_{j+1}, t_{j+1}) dx_1 \cdots dx_k. \quad (3.1.6)$$

for $s = t_0 \leq t_1 < \cdots < t_k \leq t_{k+1} = t$, with $x_0 = x$ and $x_{k+1} = *$. Here $\mathcal{Z}(x,s;*,t) := \int_{\mathbb{R}} \mathcal{Z}(x,s;y,t)dy$. Note that the Chapman-Kolmogorov equations (3.1.4) ensure $\mathbf{P}_*^{\xi}$ is a probability measure. The measure $\mathbf{P}_*^{\xi}$ also depends on x but we again suppress it from our notations. We similarly use $\mathsf{CDRP}(x,y;*,t)$ to refer to random variables with $\mathbf{P}_*^{\xi}$ law.

Remark 3.1.3. In both [3] and [65], the authors considered a five-parameter random field $\mathcal{Z}_\beta(x,s;y,t)$ with inverse temperature β, which is the simultaneous solution of the stochastic heat equation

$$\partial_t \mathcal{Z}_\beta = \tfrac{1}{2}\partial_{xx}\mathcal{Z}_\beta + \beta \mathcal{Z}_\beta \xi, \qquad \lim_{t \downarrow s} \mathcal{Z}_\beta(x,s;y,t) = \delta_x(y).$$

and defined corresponding CDRP measures. Observe that when $\beta = 0$, the stochastic heat equation becomes the heat equation and the corresponding CDRP measures reduce to Brownian measures.

Furthermore, for any $\beta > 0$, by the scaling property of the random field $\mathcal{Z}_\beta$, i.e. (iii) of Theorem 3.1 in [3], we have

$$\mathcal{Z}_\beta(x, s; y, t) \stackrel{d}{=} \beta^{-2}\mathcal{Z}_1(\beta^2 x, \beta^4 s; \beta^2 y, \beta^4 t),$$

Thus in this paper, we focus on exclusively on $\beta = 1$.

We now state our first main result which discusses the annealed convergence of the CDRP in the short-time regime to Brownian bridge law.

Theorem 3.1.4 (Annealed short-time convergence)**.** *Fix $\varepsilon > 0$. Let $X \sim \mathsf{CDRP}(0, 0; 0, \varepsilon)$. Consider the random function $Y^{(\varepsilon)} : [0, 1] \to \mathbb{R}$ defined by $Y_t^{(\varepsilon)} := \frac{1}{\sqrt{\varepsilon}} X(\varepsilon t)$. Let $\mathbf{P}^\varepsilon$ denote the annealed law of $Y^{(\varepsilon)}$ on the space of continuous functions on $C([0, 1])$. As $\varepsilon \downarrow 0$, $\mathbf{P}^\varepsilon$ converges weakly to $\mathbf{P}_B$, where $\mathbf{P}_B$ is the measure on $C([0, 1])$ generated by a Brownian bridge on $[0, 1]$ starting and ending at 0.*

Remark 3.1.5. The proof of Theorem 3.1.4 appears in Section 3.4.1. With minor modification in the proof, the above theorem can be extended to include endpoints of the form $x\sqrt{\varepsilon}$. The resulting distributional limit is then a Brownian bridge on $[0, 1]$ starting at 0 and ending at x. We also remark that we expect Theorem 3.1.4 to hold true even in the quenched case. However, some of our arguments, in particular the tightness, do not generalize to the quenched case. We hope to explore this direction in future works.

Our next result concerns the tightness and annealed convergence of the CDRP in the long-time regime and gives a rigorous justification of the $2/3$ scaling exponent discussed in Section 4.2. The limit is given in terms of the directed landscape constructed in [94, 187] which arises as a universal full scaling limit of several zero-temperature models [97]. Below we briefly introduce this limiting model before stating our result.

The directed landscape $\mathcal{L}$ is a random continuous function $\mathbb{R}^4_\uparrow \to \mathbb{R}$ that satisfies the metric composition law

$$\mathcal{L}(x, s; y, t) = \max_{z \in \mathbb{R}} \left[\mathcal{L}(x, s; z, r) + \mathcal{L}(z, r; y, t)\right], \tag{3.1.7}$$

with the property that $\mathcal{L}(\cdot, t_i; \cdot, t_i+s_i^3)$ are independent for any set of disjoint intervals $(t_i, t_i+s_i^3)$. As a function in x, y, $\mathcal{L}(x, t; y, t+s^3) \stackrel{d}{=} s \cdot \mathcal{S}(x/s^2, y/s^2)$, where $\mathcal{S}(\cdot, \cdot)$ is a parabolic Airy Sheet. We omit definitions of the parabolic Airy Sheet (see Definition 1.2 in [94]) except that $\mathcal{S}(0, \cdot) \stackrel{d}{=} \mathcal{A}(\cdot)$ where $\mathcal{A}$ is the parabolic Airy$_2$ process and $\mathcal{A}(x)+x^2$ is the (stationary) Airy$_2$ process constructed in [204].

Definition 3.1.6 (Geodesics of the directed landscape)**.** For $(x, s; y, t) \in \mathbb{R}^4_\uparrow$, a geodesic from (x, s) to (y, t) of the directed landscape is a random continuous function $\Gamma : [s, t] \to \mathbb{R}$ such that $\Gamma(s) = x$ and $\Gamma(t) = y$ and for any $s \le r_1 < r_2 < r_3 \le t$ we have

$$\mathcal{L}(\Gamma(r_1), r_1; \Gamma(r_3), r_3) = \mathcal{L}(\Gamma(r_1), r_1; \Gamma(r_2), r_2) + \mathcal{L}(\Gamma(r_2), r_2; \Gamma(r_3), r_3).$$

Thus the geodesic precisely contain the points where the equality holds in (3.1.7). Given any $(x, s; y, t) \in \mathbb{R}^4_\uparrow$, by Theorem 12.1 in [94], it is known that almost surely there is a unique geodesic Γ from (x, s) to (y, t).

Theorem 3.1.7 (Long-time CDRP path tightness)**.** *Fix $\varepsilon > (0, 1]$. $V \sim \mathsf{CDRP}(0, 0; 0, \varepsilon^{-1})$. Define a random continuous function $L^{(\varepsilon)} : [0, 1] \to \mathbb{R}$ as $L^{(\varepsilon)}_t := \varepsilon^{2/3} V(\varepsilon^{-1} t)$. We have the following:*

(a) Let $\mathbf{P}^\varepsilon$ denote the annealed law of $L^{(\varepsilon)}$, which is viewed as a random variable in the space of continuous functions on $[0, 1]$ equipped with uniform topology and Borel σ-algebra. The sequence $\mathbf{P}^\varepsilon$ is tight w.r.t. ε.

(b) For each $t \in (0, 1)$, $L^{(\varepsilon)}_t$ converges weakly to $\Gamma(t\sqrt{2})$, where $\Gamma(\cdot)$ is the geodesic of directed landscape from $(0, 0)$ to $(0, \sqrt{2})$.

The above path tightness result under $2/3$ scaling is first such result among all positive-temperature models. Part (b) of the above theorem shows that this $2/3$ scaling is indeed correct: upon this scaling, the CDRP paths have pointwise non-trivial weak limit.

In the same spirit, we have the following short- and long-time tightness result for point-to-line CDRP.

Theorem 3.1.8 (Point-to-line CDRP path tightness)**.** *Fix* $\varepsilon \in (0,1]$. *Suppose* $X \sim \mathsf{CDRP}(0,0;*,\varepsilon)$ *and* $V \sim \mathsf{CDRP}(0,0;*,\varepsilon^{-1})$. *Define two random continuous functions* $Y_*^{(\varepsilon)}, L_*^{(\varepsilon)} : [0,1] \to \mathbb{R}$ *as* $Y_*^{(\varepsilon)}(t) := \varepsilon^{-1/2}X(\varepsilon t)$ *and* $L_*^{(\varepsilon)}(t) := \varepsilon^{2/3}V(\varepsilon^{-1}t)$. *We have the following:*

(a) If we let $\mathbf{P}^{\varepsilon}_{*,S}$ *denote the annealed law of* $Y_*^{(\varepsilon)}(\cdot)$*, then as* $\varepsilon \downarrow 0$, $\mathbf{P}^{\varepsilon}_{*,S}$ *converges weakly to* $\mathbf{P}_{B_*}$*, where* $\mathbf{P}_{B_*}$ *is the measure on* $C([0,1])$ *generated by a standard Brownian motion.*

(b) If we let $\mathbf{P}^{\varepsilon}_{*,L}$ *denote the annealed law of* $L_*^{(\varepsilon)}(\cdot)$*, then the sequence* $\mathbf{P}^{\varepsilon}_{*,L}$ *is tight w.r.t.* ε.

(c) $L_*^{(\varepsilon)}(1)$ *converges weakly to* $2^{1/3}\mathcal{M}$*, where* $\mathcal{M}$ *is the almost sure unique maximizer of* Airy_2 *process minus the parabola* x^2.

We now explain how the pointwise weak convergence result in Theorem 3.1.7 (b) can be upgraded to a process-level convergence modulo the following conjecture.

Conjecture 3.1.9 (KPZ sheet to Airy sheet)**.** *Set* $\mathfrak{h}_t(x,y) := t^{-1/3}[\log \mathcal{Z}(t^{2/3}x, 0; t^{2/3}y, t) + \frac{t}{24}]$. *As* $t \to \infty$ *we have the following convergence in law (as functions in* (x,y)*)*

$$2^{1/3}\mathfrak{h}_t(2^{1/3}x, 2^{1/3}y) \xrightarrow{d} \mathcal{S}(x,y)$$

in the uniform-on-compact topology. Here $\mathcal{S}$ *is the parabolic Airy sheet.*

When either x or y is fixed, the above weak convergence as a function in one variable is proven in [208]. For zero-temperature models, such convergence has been shown recently in [97] for a large class of integrable models. It remains to show that their methods can be extended to prove the Airy sheet convergence for positive-temperature models such as above.

Assuming the validity of Conjecture 3.1.9, we can strengthen Theorem 3.1.7 (b) to the following statement.

Theorem 3.1.10 (Process annealed long-time convergence)**.** *Fix* $\varepsilon > 0$. *Let* $V \sim \mathsf{CDRP}(0,0;0,\varepsilon^{-1})$. *Define* $L_t^{(\varepsilon)} := \varepsilon^{2/3}V(\varepsilon^{-1}t)$, $t \in [0,1]$. *This scaling produces a measure on* $C([0,1])$ *for each* $\varepsilon > 0$ *conditioned on* ξ. *Assume Conjecture 3.1.9. For* $t \in (0,1)$, $\varepsilon \downarrow 0$, *the annealed law of* $L_t^{(\varepsilon)}$ *as a process in t converges weakly to* $\Gamma(\sqrt{2}t)$, *where* $\Gamma(\cdot)$ *is the geodesic of the directed landscape* $\mathcal{L}$ *from* $(0,0)$ *to* $(0,\sqrt{2})$.

3.1.3 Proof Ideas

Our main result on short-time and long-time tightness of CDRP (i.e., Theorems 3.1.4, 3.1.7 and 3.1.8) follows a host of efforts that attempts to unravel the geometry of CDRP paths. In [89], the authors showed that the quenched density of point-to-point long-time CDRP exhibit pointwise localization. In particular, they showed any particular point on a point-to-point CDRP of length t lives within a order 1 window of a 'favorite site' (depending only on the environment) and this favorite site varies in a $t^{2/3}$ window upon changing the environment. This suggests that the annealed law of polymers are within $t^{2/3}$ window *pointwise*. Our theorems on long-time tightness extend this result to the *full path* of the polymers.

One of the key ingredients behind our tightness proofs is a detailed probabilistic understanding of the log-partition function of CDRP. The log of the partition function of point-to-point CDRP, i.e.,

$$\mathcal{H}(x,s;y,t) := \log \mathcal{Z}(x,s;y,t) \tag{3.1.8}$$

solves the KPZ equation with narrow wedge initial data. Introduced in [77] as a model for random growth interfaces, KPZ equation has been extensively studied in both the mathematics and the physics communities (see [123, 206, 77, 137, 136, 209, 58, 84] and the references therein). In [5], the authors showed the one-point distribution of the KPZ equation $\mathcal{H}(x,t) := \mathcal{H}(0,0;x,t)$, has limiting Tracy-Widom GUE fluctuations of the order $t^{1/3}$ as $t \uparrow \infty$ (long-time regime), whereas fluctuations are Gaussian of the order $t^{1/4}$ as $t \downarrow 0$ (short-time regime). Detailed information of the one-point tails of $\mathcal{H}(x,t)$ as well as tail for the spatial process $\mathcal{H}(\cdot,t)$ are rigorously proved in the mathematics works [79, 80, 81, 232, 88] for long-time regime and in [87, 183, 172, 233] for short-time regime.

For brevity, we only sketch the proof for our long-time path tightness result. The proof of short-time path tightness uses a relation of annealed law of CDRP with that of Brownian counterparts (Lemma 3.4.1). The finite-dimensional convergence for the short-time case (Theorem 3.1.4) follows from chaos expansion and the same results for the long-time regime (Theorem 3.1.7 (b) and Theorem 3.1.8 (c)) follow from the localization results in [89]. Let us take a long-time polymer $V \sim \mathsf{CDRP}(0,0;0,\varepsilon^{-1})$ and scale it according to long-time scaling $L_t^{(\varepsilon)} = \varepsilon^{-2/3}V(\varepsilon^{-1}t)$ for $t \in [0,1]$. By the definition of the CDRP (Definition 3.1.1), we see that the joint law of $(L_s^{(\varepsilon)}, L_t^{(\varepsilon)})$ (where $0 < s < t < 1$) is proportional to

$$\varepsilon^{-4/3} \exp\left[\Lambda_{(s,t);\varepsilon}(x,y)\right]$$

where

$$\Lambda_{(s,t);\varepsilon}(x,y) := \mathcal{H}\big(0,0;x\varepsilon^{-\frac{2}{3}},\tfrac{s}{\varepsilon}\big) + \mathcal{H}\big(x\varepsilon^{-\frac{2}{3}},\tfrac{s}{\varepsilon};y\varepsilon^{-\frac{2}{3}},\tfrac{t}{\varepsilon}\big) + \mathcal{H}\big(y\varepsilon^{-\frac{2}{3}},\tfrac{t}{\varepsilon};0,\tfrac{1}{\varepsilon}\big) + \mathsf{Err}_{(s,t);\varepsilon}. \quad (3.1.9)$$

Here $\mathsf{Err}_{(s,t);\varepsilon}$ is a correction term free of x, y that one needs to add to extract meaningful fluctuation and tail results for the KPZ equation (see statement of Lemma 3.3.7). This correction term does not affect the joint density as it can be absorbed into the proportionality constant.

We next proceed to understand behaviors of the process $(x,y) \mapsto \Lambda_{(s,t);\varepsilon}(x,y)$. From [5], it is known that for each fixed $s < t$ and $y \in \mathbb{R}$, the process $x \mapsto [\mathcal{H}(x,s;y,t) + \frac{(x-y)^2}{2(t-s)}]$ is stationary. Naively speaking, $x \mapsto \mathcal{H}(x,s;y,t)$ looks like a negative parabola: $-\frac{(x-y)^2}{2(t-s)}$. Thus it is natural to expect

$$\varepsilon^{1/3}\Lambda_{(s,t);\varepsilon}(x,y) \approx -\frac{x^2}{2s} - \frac{(y-x)^2}{2(t-s)} - \frac{y^2}{2(1-t)}. \quad (3.1.10)$$

One of the technical contributions of this paper is to rigorously prove the above approximation holds for all x, y. Given any $\nu > 0$, we show with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^2)$,

$$\varepsilon^{1/3}\Lambda_{(s,t);\varepsilon}(x,y) \le M - (1-\nu)\left[\frac{x^2}{2s} + \frac{(y-x)^2}{2(t-s)} + \frac{y^2}{2(1-t)}\right], \text{ for all } x, y \in \mathbb{R}.$$

The precise statement of the above result appears in Lemma 3.3.7. This multivariate process estimate allows us to conclude the quenched density of $(L_s^{(\varepsilon)}, L_t^{(\varepsilon)})$ at (x, y) is exponentially small, whenever $\frac{|x-y|}{\sqrt{t-s}} \to \infty$. Armed with this understanding of quenched density, in Proposition 3.3.1, we show that given any $\delta > 0$, with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^2)$ we have

$$|L_s^{(\varepsilon)} - L_t^{(\varepsilon)}| \le M|t-s|^{\frac{1}{2}-\delta}.$$

In fact the sharp decay estimates of quenched density (Lemma 3.3.7) allows us to prove a quenched version of the above statement (Proposition 3.3.1). Due to exponentially tight probability bounds of the above two-point differences, Proposition 3.3.1 can be extended to quenched modulus of continuity estimates (Proposition 3.3.3) by standard methods. This leads to the path tightness of long-time CDRP.

Outline

The rest of the paper is organized as follows. Section 3.2 reviews some of the existing results related to the KPZ equation before proving a useful result on the short-time local fluctuations of the KPZ equation (Proposition 3.2.4). We then prove in Section 3.3 a multivariate spatial process tail bound (Lemma 3.3.7) and modulus of continuity results (Propositions 3.3.1 and 3.3.1-(point-to-line)) that culminate in the quenched modulus of continuity estimate in Proposition 3.3.3 and Proposition 3.3.3-(point-to-line). In Section 3.4, we prove Theorems 3.1.4, 3.1.7, and 3.1.8, and Theorem 3.1.10 (modulo Conjecture 3.1.9). Lastly, proof of a technical lemma used in Section 3.2 appears in Appendix 3.5.

3.2 Short- and long-time tail results for KPZ equation

Throughout this paper we use $\mathrm{C} = \mathrm{C}(x, y, z, \ldots) > 0$ to denote a generic deterministic positive finite constant that may change from line to line, but dependent on the designated variables $x, y, z, \ldots$. We use sans serif fonts such as $\mathsf{A}, \mathsf{B}, \ldots$ to denote events and $\neg\mathsf{A}, \neg\mathsf{B}, \ldots$ to denote their complements.

In this section, we collect several estimates related to the short-time and long-time tails of the KPZ equation. We record existing estimates from the literature in Proposition 3.2.2 and Proposition 3.2.3. These estimates form crucial tools to our later proofs. For our analysis, we also require an estimate on the short-time local fluctuations of the KPZ equation which is not available in the literature. We present this new estimate in Proposition 3.2.4. Its proof appears at the end of this section.

Recall the four-parameter stochastic heat equation $\mathcal{Z}(x, s; y, t)$ from (3.1.3). We set

$$\mathcal{H}(x, s; y, t) := \log \mathcal{Z}(x, s; y, t). \tag{3.2.1}$$

When $x = s = 0$, we use the abbreviated notation $\mathcal{H}(y, t) := \mathcal{H}(0, 0; y, t)$. As mentioned in the introduction, fluctuation and scaling of the KPZ equation varies as $t \downarrow 0$ (short-time) and $t \uparrow \infty$ (long-time). For the two separate regimes we consider the following scalings:

$$\begin{aligned} \mathfrak{g}_{s,t}(x, y) &:= \frac{\mathcal{H}\left(\sqrt{\frac{\pi(t-s)}{4}}x, s; \sqrt{\frac{\pi(t-s)}{4}}y, t\right) + \log\sqrt{2\pi(t-s)}}{\left(\frac{\pi(t-s)}{4}\right)^{1/4}} && \text{for the short-time regime,} \\ \mathfrak{h}_{s,t}(x, y) &:= \frac{\mathcal{H}((t-s)^{2/3}x, s; (t-s)^{2/3}y, t) + \frac{t-s}{24}}{(t-s)^{1/3}} && \text{for the long-time regime.} \end{aligned} \tag{3.2.2}$$

We will often refer to the above bivariate functions as short-time and long-time KPZ sheet. In particular, when both $s = 0$ and $x = 0$, we use the shorthands $\mathfrak{g}_t(y) := \mathfrak{g}_{0,t}(0, y)$, and $\mathfrak{h}_t(y) := \mathfrak{h}_{0,t}(0, y)$.

Remark 3.2.1. The above scalings satisfy several distributional identities. For fixed $s < t$ and

$y \in \mathbb{R}$, from chaos representation for SHE it follows that

$$\mathcal{Z}(0,s;x,t) \stackrel{d}{=} \mathcal{Z}(0,s;-x,t), \quad \mathcal{Z}(x,s;y,t) \stackrel{d}{=} \mathcal{Z}(0,0;y-x,t-s).$$

where the equality in distribution holds as processes in x. This leads to $\mathfrak{g}_{s,t}(x,y) \stackrel{d}{=} \mathfrak{g}_{t-s}(x-y)$ and $\mathfrak{h}_{s,t}(x,y) \stackrel{d}{=} \mathfrak{h}_{t-s}(x-y)$, as processes in x.

The following proposition collects several probabilistic facts for the long-time rescaled KPZ equation.

Proposition 3.2.2. *Recall $\mathfrak{h}_t(x)$ from* (3.2.2)*. The following results hold:*

(a) For each $t > 0$, $\mathfrak{h}_t(x) + x^2/2$ is stationary in x.

(b) Fix $t_0 > 0$. There exists a constant $\mathrm{C} = \mathrm{C}(t_0) > 0$ such that for all $t \geq t_0$ and $s > 0$ we have

$$\mathbf{P}\left(|\mathfrak{h}_t(0)| \geq s\right) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^{3/2}\right).$$

(c) Fix $t_0 > 0$. There exists a constant $\mathrm{C} = \mathrm{C}(t_0) > 0$ such that for all $x \in \mathbb{R}$, $s > 0$, $t \geq t_0$, and $\gamma \in (0,1]$, we have

$$\mathbf{P}\left(\sup_{z\in[x,x+\gamma]}\left|\mathfrak{h}_t(z) + \tfrac{z^2}{2} - \mathfrak{h}_t(x) - \tfrac{x^2}{2}\right| \geq s\sqrt{\gamma}\right) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^{3/2}\right).$$

The results in Proposition 3.2.2 are a culmination of results from several papers. Part (a) follows from [5, Corollary 1.3 and Proposition 1.4]. The one-point tail estimates for KPZ equation are obtained in [79, 80]. One can derive part (b) from those results or can combine the statements of Proposition 2.11 and 2.12 in [81] to get the same. Part (c) is Theorem 1.3 from [81].

The study of short-time tails was initiated in [87]. Below we recall some known results from the same paper.

Proposition 3.2.3. *Recall $\mathfrak{g}_t(x)$ from* (3.2.2)*. The following results hold:*

(a) For each $t > 0$, $\mathfrak{g}_t(x) + \frac{(\pi t/4)^{3/4}}{2t} x^2$ is stationary in x.

(b) There exists a constant $\mathrm{C} > 0$ such that for all $t \le 1$ and $s > 0$ we have

$$\mathbf{P}(|\mathfrak{g}_t(0)| > s) \le \mathrm{C} \exp\left(-\tfrac{1}{\mathrm{C}} s^{3/2}\right).$$

Part (a) follows from [87, Lemma 2.11]. The one-point tail estimates for short-time rescaled KPZ equation are obtained in [87, Corollary 1.6, Theorem 1.7], from which one can derive part (b).

For convenience, we write $m_t(x) := (\frac{\pi t}{4})^{3/4} \frac{x^2}{2t}$ to denote the parabolic term associated to the short-time scaling. The following result concerns the short-time analogue of Proposition 3.2.2 (c).

Proposition 3.2.4 (Short-time local fluctuations of the KPZ equation)**.** *There exists a constant $\mathrm{C} > 0$ such that for all $t \in (0, 1)$, $x \in \mathbb{R}$, $\gamma \in (0, \sqrt{t})$ and $s > 0$ we have*

$$\mathbf{P}\left(\sup_{z \in [x, x+\gamma]} |\mathfrak{g}_t(z) + m_t(z) - \mathfrak{g}_t(x) - m_t(x)| \ge s\sqrt{\gamma}\right) \le \mathrm{C} \exp\left(-\tfrac{1}{\mathrm{C}} s^{3/2}\right). \tag{3.2.3}$$

Remark 3.2.5. The parabolic term $m_t(x)$ is steeper (as $t \le 1$) than the usual parabola that appears in the long-time scaling. This is the reason why Proposition 3.2.4 requires $\gamma < \sqrt{t}$, whereas Proposition 3.2.2 (c) holds for all $\gamma \in (0, 1]$.

The proof of Proposition 3.2.4 follows the same strategy as those of Proposition 4.3 and Theorem 1.3 in [81] which employ the Brownian Gibbs property of the KPZ line ensemble (see [74]). The same Brownian Gibbs property continues to hold for short-time $\mathfrak{g}_t(\cdot)$ process (see Lemma 2.5 (4) in [87]). We include the proof of Proposition 3.2.4 below for completeness after first describing its key proof ingredient.

We recall a property of $\mathfrak{g}_t(\cdot)$ under *monotone* events. Given an interval $[a, b]$, we denote $\mathcal{B}(C([a, b]))$ to be the Borel σ-algebra on $C([a, b])$ generated by the uniform norm topology. We call an event $A \in \mathcal{B}(C([a, b]))$ monotone w.r.t. $[a, b]$ if for every pair of functions $f, g \in$

$[a,b] \to \mathbb{R}$ with $f(a) = g(a)$, $f(b) = g(b)$ and $f(x) \ge g(x)$ for all $x \in (a,b)$, we have

$$f(x) \in A \implies g(x) \in A. \tag{3.2.4}$$

We call $(\mathfrak{a}, \mathfrak{b})$ a stopping domain for $\mathfrak{g}_t(\cdot)$ if $\{\mathfrak{a} \le a, \mathfrak{b} \ge b\}$ is measurable w.r.t. σ-algebra generated by $(\mathfrak{g}_t(x))_{x \notin (a,b)}$ for all $a, b \in \mathbb{R}$. A crucial property is the following:

Lemma 3.2.6. *Fix any $t > 0$. For any $[a,b] \subset \mathbb{R}$, and a monotone set $A \in \mathcal{B}(C([a,b]))$ (w.r.t. $[a,b]$), we have*

$$\mathbf{P}\left[\mathfrak{g}_t(\cdot)\mid_{[a,b]} \in A \mid (\mathfrak{g}_t(x))_{x \notin (a,b)}\right] \le \mathbf{P}_{\text{free}}^{(a,b),(\mathfrak{g}_t(a),\mathfrak{g}_t(b))}(A) \tag{3.2.5}$$

where $\mathbf{P}_{\text{free}}^{(a,b),(y,z)}$ denotes the law of Brownian bridge on $[a,b]$ starting at y and ending at z. Furthermore (3.2.5) *continues to hold if (a,b) is a stopping domain for $\mathfrak{g}_t(\cdot)$.*

We will abuse our definition and call $\{\mathfrak{g}_t(\cdot)\mid_{[a,b]} \in A\}$ to be monotone w.r.t. $[a,b]$ if A is monotone w.r.t. $[a,b]$. The proof of the above lemma follows by utilizing the notion of the KPZ line ensemble and its Brownian Gibbs property [74, 87]. We defer its proof and the necessary background on the KPZ line ensemble to Appendix 3.5.

Proof of Proposition 3.2.4. Assume $s \ge 100$. For $s \le 100$, the constant C > 0 can be adjusted so that the proposition holds trivially. We fix a $t_0 \in (0,1)$ such that for all $s \ge 100$, and $t \le t_0$ we have

$$\tfrac{1}{4}s \ge t^{1/4}(s + m_t(2)) = t^{1/4}s + 2(\pi/4)^{3/4}. \tag{3.2.6}$$

Let us first consider $t \in [t_0, 1]$. We use the scalings from (3.2.2) to get

$$\mathfrak{g}_t(x) + m_t(x) = \tfrac{1}{\sqrt{r_t}}\left(\mathfrak{h}_t(r_t x) + \tfrac{r_t^2 x^2}{2}\right) + c_t, \tag{3.2.7}$$

where $r_t := t^{-1/6}\sqrt{\pi/4}$ and $c_t := (\pi t/4)^{-1/4}(\sqrt{2\pi t} - t/24)$. Take any $x \in \mathbb{R}$ and $\gamma \in (0, \sqrt{t})$. We have $r_t\gamma \le 1$. Setting $y := r_t x$ and then applying Proposition 3.2.2 (c) with $x \mapsto y$ and $\gamma \mapsto r_t\gamma$ we get

$$\text{l.h.s. of (3.2.3)} = \mathbf{P}\left(\sup_{z\in[y,y+r_t\gamma]}\left|\mathfrak{h}_t(z)+\tfrac{z^2}{2}-\mathfrak{h}_t(y)-\tfrac{y^2}{2}\right|\ge s\sqrt{\gamma r_t}\right)\le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^{3/2}\right). \quad (3.2.8)$$

Let us now assume $t \le t_0$. By Proposition 3.2.3 (a), we know that the process $\mathfrak{g}_t(x)+m_t(x)$ is stationary in x. Thus it suffices to prove Proposition 3.2.4 with $x = 0$. Consider the following events

$$\mathsf{G}_{\gamma,s} := \bigcap_{x\in\{\gamma-2,0,\gamma,2\}}\left\{-\tfrac{s}{4}\le \mathfrak{g}_t(x)+m_t(x)\le \tfrac{s}{4}\right\},$$

$$\mathsf{Fall}_{\gamma,s} := \left\{\inf_{z\in[0,\gamma]}(\mathfrak{g}_t(z)+m_t(z))\le \mathfrak{g}_t(0)-s\gamma^{1/2}\right\},$$

$$\mathsf{Rise}_{\gamma,s} := \left\{\sup_{z\in[0,\gamma]}(\mathfrak{g}_t(z)+m_t(z))\ge \mathfrak{g}_t(0)+s\gamma^{1/2}\right\}.$$

By one-point tail bounds from Proposition 3.2.3 (b) we have that $\mathbf{P}(\neg\mathsf{G}_{\gamma,s}) \le \mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$. Thus, to show the proposition, it suffices to verify the following two bounds:

$$\mathbf{P}\left(\mathsf{Fall}_{\gamma,s},\mathsf{G}_{\gamma,s}\right)\le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^2\right), \qquad \mathbf{P}\left(\mathsf{Rise}_{\gamma,s},\mathsf{G}_{\gamma,s}\right)\le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^2\right). \quad (3.2.9)$$

We begin with the $\mathsf{Fall}_{\gamma,s}$ bound in (3.2.9). Clearly $\mathsf{Fall}_{\gamma,s}$ event is monotone w.r.t. $[0,2]$, by Lemma 3.2.6 we have

$$\mathbf{P}\left(\mathsf{Fall}_{\gamma,s}\mid (\mathfrak{g}_t(x))_{x\in(0,2)}\right)\le \mathbf{P}_{\text{free}}^{(0,2),(\mathfrak{g}_t(0),\mathfrak{g}_t(2))}\left(\mathsf{Fall}_{\gamma,s}\right)$$

where $\mathbf{P}_{\text{free}}^{(a,b),(y,z)}$ denotes the law of Brownian bridge on $[a,b]$ starting at y and ending at z. Using this we have

$$\begin{aligned}\mathbf{P}\left(\mathsf{Fall}_{\gamma,s},\mathsf{G}_{\gamma,s}\right) &\le \mathbf{P}\left(\mathsf{Fall}_{\gamma,s},\mathfrak{g}_t(0)\le\tfrac{s}{4},\mathfrak{g}_t(2)+m_t(2)\ge-\tfrac{s}{4}\right)\\ &\le \mathbf{E}\left[\mathbf{1}_{\mathfrak{g}_t(0)\le\frac{s}{4}}\cdot\mathbf{1}_{\mathfrak{g}_t(2)+m_t(2)\ge-\frac{s}{4}}\mathbf{P}_{\text{free}}^{(0,2),(\mathfrak{g}_t(0),\mathfrak{g}_t(2))}\left(\mathsf{Fall}_{\gamma,s}\right)\right]\end{aligned}$$

$$\leq \sup \left\{ \mathbf{P}^{(0,2),y,z}_{\mathsf{free}} \left(\mathsf{Fall}_{\gamma,s} \right) : y \leq \tfrac{s}{4}, z + m_t(2) \geq -\tfrac{s}{4} \right\}$$
$$= \mathbf{P}^{(0,2),s/4,-s/4-m_t(2)}_{\mathsf{free}} \left(\mathsf{Fall}_{\gamma,s} \right). \tag{3.2.10}$$

Next, we write the final term in (3.2.10) as

$$\mathbf{P}^{(0,2),s/4,-s/4-m_t(2)}_{\mathsf{free}} \left(\mathsf{Fall}_{\gamma,s} \right) = \mathbf{P} \left(\inf_{z\in[0,\gamma]} \left\{ B'(z) + m_t(z) \right\} \leq -s\gamma^{1/2} \right)$$

where $B' : [0,2] \to \mathbb{R}$ is a Brownian bridge with $B'(0) = 0$ and $B'(2) = -m_t(2) - \frac{s}{2}$. Now, set $B(z) := B'(z) - \frac{z}{2}(-m_t(2) - \frac{s}{2})$. Then B is a Brownian bridge with $B(0) = B(2) = 0$ and we obtain

$$\mathbf{P} \left(\inf_{z\in[0,\gamma]} (B'(z) + m_t(z)) \leq -s\gamma^{1/2} \right) \leq \mathbf{P} \left(\inf_{z\in[0,\gamma]} B(z) \leq -s\gamma^{1/2} - \tfrac{\gamma}{2}(-m_t(2) - \tfrac{s}{2}) \right)$$
$$\leq \mathbf{P} \left(\inf_{z\in[0,\gamma]} B(z) \leq -\tfrac{s}{2}\gamma^{1/2} \right). \tag{3.2.11}$$

The latter inequality is due to $\gamma^{1/2}(m_t(2) + \frac{s}{2}) \leq s$ as $s \geq 100$ and $\gamma \leq \sqrt{t}$. The right-hand probability can be estimated via Brownian calculations, which yields the desired bound of the form $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^2)$.

We next prove the $\mathsf{Rise}_{\gamma,s}$ bound in (3.2.9). Note that $\sup_{z\in[0,\gamma]} m_t(z) \leq \frac{\gamma^2}{t^{1/4}} \leq \frac{1}{2}s\gamma^{1/2}$ (as $\gamma \leq \sqrt{t} \leq 1$ and $s \geq 4$). Thus it suffices to show

$$\mathbf{P} \left(\mathsf{Rise}^{(1)}_{\gamma,s}, \mathsf{G}_{\gamma,s} \right) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^2 \right), \quad \mathsf{Rise}^{(1)}_{\gamma,s} := \left\{ \sup_{z\in[0,\gamma]} \mathfrak{g}_t(z) \geq \mathfrak{g}_t(0) + \tfrac{1}{2}s\sqrt{\gamma} \right\}. \tag{3.2.12}$$

Set

$$\chi := \inf \left\{ x \in (0,\gamma] \mid \mathfrak{g}_t(x) - \mathfrak{g}_t(0) \geq \tfrac{1}{2}s\gamma^{1/2} \right\},$$

and set $\chi = \infty$ if no such points exist. Then we have $\mathbf{P}\left(\mathsf{Rise}^{(1)}_{\gamma,s}, \mathsf{G}_{\gamma,s}\right) = \mathbf{P}\left(\chi \leq \gamma, \mathsf{G}_{\gamma,s}\right)$ and we can write the right-hand probability as

$$\mathbf{P} \left(\chi \leq \gamma, \mathsf{G}_{\gamma,s}, \mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) < \tfrac{1}{4}s\sqrt{\gamma} \right) + \mathbf{P} \left(\chi \leq \gamma, \mathsf{G}_{\gamma,s}, \mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \geq \tfrac{1}{4}s\sqrt{\gamma} \right). \tag{3.2.13}$$

On the event $\{\chi \le \gamma, \mathsf{G}_{\gamma,s}, \mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) < \frac{1}{4}s\sqrt{\gamma}\}$ we have that $\{\mathfrak{g}_t(\gamma) - \mathfrak{g}_t(0) \ge \frac{1}{4}s\sqrt{\gamma}\}$ holds as the continuity of $\mathfrak{g}_t(\cdot)$ implies that $\mathfrak{g}_t(\chi) = \mathfrak{g}_t(0) + \frac{1}{2}s\sqrt{\gamma}$ on $\{\chi \le \gamma\}$ event. Now with the same argument of the $\mathsf{Fall}_{\gamma,s}$ event, we bound the probability of this occurrence by $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^2)$ for some constant $\mathrm{C} > 0$. This is why $\mathsf{G}_{\gamma,s}$ involves $\mathfrak{g}_t(-2+\gamma)$ and $\mathfrak{g}_t(\gamma)$. The parabolic term $m_t(z)$ again can be ignored as $\sup_{z\in[0,\gamma]} m_t(z) \le \gamma^{3/2} \le \frac{1}{8}s\gamma^{1/2}$ for $s \ge 8$.

Let us focus on the second term in (3.2.13). Note that $(\chi, 2)$ is a stopping domain and $\{\mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \ge \frac{1}{4}s\sqrt{\gamma}\}$ is a monotone event w.r.t. $[\chi, 2]$. Applying Lemma 3.2.6 one has

$$\mathbf{P}\left(\mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \ge \tfrac{1}{4}s\sqrt{\gamma} \mid (\mathfrak{g}_t(x))_{x\notin(\chi,2)}\right) \le \mathbf{P}_{\text{free}}^{(\chi,2),(\mathfrak{g}_t(\chi),\mathfrak{g}_t(2))}\left(\mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \ge \tfrac{1}{4}s\sqrt{\gamma}\right).$$

Note that on $\{\chi \le \gamma, \mathsf{G}_{\gamma,s}\}$ we have

$$|\mathfrak{g}_t(\chi) - \mathfrak{g}_t(2)| = |\mathfrak{g}_t(0) + \tfrac{1}{2}s\sqrt{\gamma} - \mathfrak{g}_t(2)| \le s/4 + \tfrac{1}{2}s\sqrt{\gamma} + m_t(2) + s/4 = s + m_t(2). \tag{3.2.14}$$

As $2-\chi \ge 1$ on $\{\chi \le \gamma\}$, we thus get that the absolute value of the slope of the linearly interpolated line joining $(\chi, \mathfrak{g}_t(\chi))$ and $(2, \mathfrak{g}_t(2))$ is at most $s + m_t(2)$. Note that $\frac{1}{4}s\sqrt{\gamma} \ge \gamma(s + m_t(2))$ due to (3.2.6). Thus the event $\{\mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \ge \frac{1}{4}s\sqrt{\gamma}\}$ entails that the $\mathfrak{g}_t(\gamma)$ lies below the linearly interpolated line. Under Brownian law, this has probability $1/2$. Thus,

$$\begin{aligned}
&\mathbf{P}\left(\chi \le \gamma, \mathsf{G}_{\gamma,s}, \mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \ge \tfrac{1}{4}s\sqrt{\gamma}\right) \\
&\le \mathbf{E}\left[\mathbf{1}_{\chi\le\gamma,\mathsf{G}_{\gamma,s}} \mathbf{P}_{\text{free}}^{(\chi,2),(\mathfrak{g}_t(\chi),\mathfrak{g}_t(2))}\left(\mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \ge \tfrac{1}{4}s\sqrt{\gamma}\right)\right] \le \tfrac{1}{2}\mathbf{E}\left[\mathbf{1}_{\chi\le\gamma,\mathsf{G}_{\gamma,s}}\right].
\end{aligned}$$

Hence we have shown that $\mathbf{P}\left(\chi \le \gamma, \mathsf{G}_{\gamma,s}\right) \le \mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^2) + \frac{1}{2}\mathbf{P}\left(\chi \le \gamma, \mathsf{G}_{\gamma,s}\right)$ which implies that $\mathbf{P}\left(\chi \le \gamma, \mathsf{G}_{\gamma,s}\right) \le 2\mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^2)$ which gives us the bound in (3.2.12), completing the proof. □

3.3 Modulus of Continuity for rescaled CDRP measures

The main goal of this section is to establish quenched modulus of continuity estimates: Proposition 3.3.3 and Proposition 3.3.3-(point-to-line), for CDRP measures under long-time scalings.

The proof of these propositions requires detailed study of the tail probabilities of two-point difference when scaled according to long-time. This is conducted in Proposition 3.3.1 and Proposition 3.3.1-(point-to-line) respectively. One of the key technical inputs in the proofs of Propositions 3.3.1 and 3.3.1-(point-to-line) is a parabolic decay estimate of a multivariate spatial process involving several long-time KPZ sheets. This estimate appears in Lemma 3.3.7 and is proved in Section 3.3.1. In the following text, we first state those Propositions 3.3.1 and 3.3.1-(point-to-line) and assuming their validity, we state and prove the modulus of continuity estimates. Proofs of Proposition 3.3.1 and 3.3.1-(point-to-line) are deferred to Section 3.3.2.

Proposition 3.3.1 (Long-time two-point difference)**.** *Fix any* $\varepsilon \in (0,1]$, $\delta \in (0,\frac{1}{2})$, *and* $\tau \geq 1$. *Take* $x \in [-\tau\varepsilon^{-\frac{2}{3}}, \tau\varepsilon^{-\frac{2}{3}}]$. *Let* $V \sim \mathsf{CDRP}(0,0;x,\varepsilon^{-1})$. *For* $t \in [0,1]$, *set* $L_t^{(\varepsilon)} := \varepsilon^{\frac{2}{3}} V(\varepsilon^{-1}t)$. *There exist two absolute constants* $\mathrm{C}_1(\tau,\delta) > 0$ *and* $\mathrm{C}_2(\tau,\delta) > 0$ *such that for all* $m \geq 1$ *and* $t \neq s \in [0,1]$ *we have*

$$\mathbf{P}\left[\mathbf{P}^{\xi}(|L_s^{(\varepsilon)} - L_t^{(\varepsilon)}| \geq m|s-t|^{\frac{1}{2}-\delta}) \geq \mathrm{C}_1 \exp(-\tfrac{1}{\mathrm{C}_1}m^2)\right] \leq \mathrm{C}_2 \exp\left(-\tfrac{1}{\mathrm{C}_2}m^3\right).$$

We have the following point-to-line analogue.

Proposition 3.3.1-(point-to-line). *Fix any* $\varepsilon \in (0,1]$, $\delta \in (0,\frac{1}{2})$. *Let* $V \sim \mathsf{CDRP}(0,0;*,\varepsilon^{-1})$. *For* $t \in [0,1]$, *set* $L_{t,*}^{(\varepsilon)} := \varepsilon^{\frac{2}{3}} V(\varepsilon^{-1}t)$. *There exist two absolute constants* $\mathrm{C}_1(\delta) > 0$ *and* $\mathrm{C}_2(\delta) > 0$ *such that for all* $m \geq 1$ *and* $t \neq s \in [0,1]$ *we have*

$$\mathbf{P}\left[\mathbf{P}_*^{\xi}(|L_{s,*}^{(\varepsilon)} - L_{t,*}^{(\varepsilon)}| \geq m|s-t|^{\frac{1}{2}-\delta}) \geq \mathrm{C}_1 \exp(-\tfrac{1}{\mathrm{C}_1}m^2)\right] \leq \mathrm{C}_2 \exp\left(-\tfrac{1}{\mathrm{C}_2}m^3\right).$$

Remark 3.3.2. In the above propositions, the quenched probability of the tail event of two-point difference of rescaled polymers is viewed as a random variable. The above propositions provide quantitative decay estimates of this random variable being away from zero for point-to-point and point-to-line polymers under long-time regime.

Proposition 3.3.3 (Quenched Modulus of Continuity)**.** *Fix* $\varepsilon \in (0,1]$, $\delta \in (0,\frac{1}{2})$ *and* $\tau \geq 1$. *Take*

$y \in [-\tau\varepsilon^{-\frac{2}{3}}, \tau\varepsilon^{-\frac{2}{3}}]$. Let $V \sim \mathsf{CDRP}(0,0;y,\varepsilon^{-1})$. Set $L_t^{(\varepsilon)} := \varepsilon^{\frac{2}{3}}V(\varepsilon^{-1}t)$ for $t \in [0,1]$. Then there exist two constants $\mathrm{C}_1(\tau,\delta) > 0$ and $\mathrm{C}_2(\tau,\delta) > 0$ such that for all $m \geq 1$ we have

$$\mathbf{P}\left[\mathbf{P}^{\xi}\left(\sup_{t\neq s\in[0,1]} \frac{|L_s^{(\varepsilon)} - L_t^{(\varepsilon)}|}{|t-s|^{\frac{1}{2}-\delta}\log\frac{2}{|t-s|}} \geq m\right) \geq \mathrm{C}_1 \exp\left(-\tfrac{1}{\mathrm{C}_1}m^2\right)\right] \leq \mathrm{C}_2 \exp\left(-\tfrac{1}{\mathrm{C}_2}m^3\right). \tag{3.3.1}$$

Proposition 3.3.3-(point-to-line). *Fix $\varepsilon \in (0,1]$, $\delta \in (0,\frac{1}{2})$. Let $V \sim \mathsf{CDRP}(0,0;*,\varepsilon^{-1})$. For $t \in [0,1]$, set $L_{t,*}^{(\varepsilon)} := \varepsilon^{\frac{2}{3}}V(\varepsilon^{-1}t)$. Then there exist two constants $\mathrm{C}_1(\delta) > 0$ and $\mathrm{C}_2(\delta) > 0$ such that for all $m \geq 1$ we have*

$$\mathbf{P}\left[\mathbf{P}_*^{\xi}\left(\sup_{t\neq s\in[0,1]} \frac{|L_{s,*}^{(\varepsilon)} - L_{t,*}^{(\varepsilon)}|}{|t-s|^{\frac{1}{2}-\delta}\log\frac{2}{|t-s|}} \geq m\right) \geq \mathrm{C}_1 \exp\left(-\tfrac{1}{\mathrm{C}_1}m^2\right)\right] \leq \mathrm{C}_2 \exp\left(-\tfrac{1}{\mathrm{C}_2}m^3\right).$$

Remark 3.3.4. The paths of continuum directed random polymer are known to be Hölder continuous with exponent γ, for every $\gamma < 1/2$ (see [3, Theorem 4.3]). Our Theorem 3.3.3 corroborates this fact by giving quantitative tail bounds to the quenched modulus of continuity.

Before proving Propositions 3.3.3 and 3.3.3-(point-to-line), we present below a few important corollaries for point-to-point long-time polymer. Similar corollaries hold for point-to-line case as well.

Corollary 3.3.5. *Fix $\varepsilon \in (0,1]$, and $\tau \geq 1$. Take $x \in [-\tau\varepsilon^{-\frac{2}{3}}, \tau\varepsilon^{-\frac{2}{3}}]$. Let $V \sim \mathsf{CDRP}(0,0;x,\varepsilon^{-1})$. For $t \in [0,1]$ set $L_t^{(\varepsilon)} := \varepsilon^{\frac{2}{3}}V(\varepsilon^{-1}t)$. Then there exist two constants $\mathrm{C}_1(\tau) > 0$ and $\mathrm{C}_2(\tau) > 0$ such that for all $m \geq 1$ we have*

$$\mathbf{P}\left[\mathbf{P}^{\xi}\left(\sup_{t\in[0,1]} |L_t^{(\varepsilon)}| \geq m\right) \geq \mathrm{C}_1 \exp\left(-\tfrac{1}{\mathrm{C}_1}m^2\right)\right] \leq \mathrm{C}_2 \exp\left(-\tfrac{1}{\mathrm{C}_2}m^3\right). \tag{3.3.2}$$

Proof. Set $s = 0$ and $\rho = 1 + \sup_{t\in(0,1]} t^{1/4}\log\frac{2}{t} \in (1,\infty)$. By Proposition 3.3.3, with $\delta = \frac{1}{4}$ there exist $\mathrm{C}_1(\tau)$ and $\mathrm{C}_2(\tau)$ such that for all $m \geq 1$, (3.3.1) holds with $s = 0$. Replacing m with m/ρ in

(3.3.1) yields that

$$\begin{aligned}
&\mathbf{P}\left[\mathbf{P}^{\xi}\left(\sup_{t\in[0,1]}|L_t^{(\varepsilon)}|\geq m\right)\geq \mathrm{C}_1\exp\left(-\tfrac{1}{\mathrm{C}_1}m^2\right)\right]\\
&\leq \mathbf{P}\left[\mathbf{P}^{\xi}\left(\sup_{t\in[0,1]}\frac{|L_t^{(\varepsilon)}|}{t^{\frac14}\log\frac{2}{t}}\geq\frac{m}{\rho}\right)\geq \mathrm{C}_1\exp\left(-\tfrac{1}{\mathrm{C}_1}(\tfrac{m}{\rho})^2\right)\right]\leq \mathrm{C}_2\exp\left(-\tfrac{1}{\mathrm{C}_2}(\tfrac{m}{\rho})^3\right).
\end{aligned}$$

Adjusting C_2 further we get the desired result. □

From Proposition 3.3.3, we also obtain the annealed modulus of continuity.

Corollary 3.3.6 (Annealed Modulus of Continuity)**.** *Fix $\varepsilon\in(0,1]$, $\delta\in(0,\frac12)$ and $\tau\geq 1$. Take $y\in[-\tau\varepsilon^{-\frac23},\tau\varepsilon^{-\frac23}]$. Let $V\sim \mathsf{CDRP}(0,0;y,\varepsilon^{-1})$. Set $L_t^{(\varepsilon)}:=\varepsilon^{\frac23}V(\varepsilon^{-1}t)$ for $t\in[0,1]$. Then there exists a constant $\mathrm{C}(\tau,\delta)>0$ such that for all $m\geq 1$ we have*

$$\mathbf{P}\left(\sup_{t\neq s\in[0,1]}\frac{|L_s^{(\varepsilon)}-L_t^{(\varepsilon)}|}{|t-s|^{\frac12-\delta}\log\frac{2}{|t-s|}}\geq m\right)\leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^2\right). \tag{3.3.3}$$

Clearly one has similar corollaries for the point-to-line version which follow from Proposition 3.3.3-(point-to-line) instead. For brevity, we do not record them separately. We now assume Proposition 3.3.1 (Proposition 3.3.1-(point-to-line)) and complete the proof of Proposition 3.3.3 (Proposition 3.3.3-(point-to-line)).

Proof of Propositions 3.3.3 and 3.3.3-(point-to-line). Fix $\tau\geq 1$ and $m\geq 16\tau^2+1$. The main idea is to mimic Levy's proof of modulus of continuity of Brownian motion. Since our proposition deals with quenched versions, we keep the proof here for the sake of completeness. We only prove (3.3.1) using Proposition 3.3.1. Proof of Proposition 3.3.3-(point-to-line) follows in a similar manner using Proposition 3.3.1-(point-to-line). To prove (3.3.1), we first control the modulus of continuity on dyadic points of $[0,1]$. Fix $\delta>0$ and set $\gamma=\frac12-\delta$. Define

$$\|L^{(\varepsilon)}\|_n:=\sup_{k=\{1,\ldots,2^n\}}\left|L_{k2^{-n}}^{(\varepsilon)}-L_{(k-1)2^{-n}}^{(\varepsilon)}\right|,\quad \|L^{(\varepsilon)}\|:=\sup_{n\geq 0}\frac{\|L^{(\varepsilon)}\|_n 2^{n\gamma}}{n+1}.$$

Observe that by union bound

$$\mathbf{P}^{\xi}\left(\|L^{(\varepsilon)}\| \geq m\right) \leq \sum_{n=0}^{\infty}\sum_{k=1}^{2^n} \mathbf{P}^{\xi}\left(\left|L^{(\varepsilon)}_{k2^{-n}} - L^{(\varepsilon)}_{(k-1)2^{-n}}\right| \geq m2^{-n\gamma}(n+1)\right).$$

Thus in light of Proposition 3.3.1 we see that with probability at least

$$1 - \sum_{n=0}^{\infty} \mathrm{C}_2 2^n \exp\left(-\tfrac{1}{\mathrm{C}_2} m^3 (n+1)^3\right) \geq 1 - \mathrm{C}_2' \exp\left(-\tfrac{1}{\mathrm{C}_2'} m^3\right)$$

we have

$$\mathbf{P}^{\xi}\left(\|L^{(\varepsilon)}\| \geq m\right) \leq \sum_{n=0}^{\infty} \mathrm{C}_1 2^n \exp\left(-\tfrac{1}{\mathrm{C}}{}_1 m^2 (n+1)^2\right) \leq \mathrm{C}_1' \exp\left(-\tfrac{1}{\mathrm{C}_1'} m^2\right).$$

Finally one can extend the results to all points by continuity of $L^{(\varepsilon)}$ and observing the following string of inequalities that holds deterministically. For any $0 \leq s < t \leq 1$ we have

$$|L^{(\varepsilon)}_t - L^{(\varepsilon)}_s| \leq \sum_{n=1}^{\infty} \left| L^{(\varepsilon)}_{2^{-n}\lfloor 2^n t\rfloor} - L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} t\rfloor} - L^{(\varepsilon)}_{2^{-n}\lfloor 2^n s\rfloor} + L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} s\rfloor} \right|. \tag{3.3.4}$$

Note that we have

$$\begin{aligned} &\left| L^{(\varepsilon)}_{2^{-n}\lfloor 2^n t\rfloor} - L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} t\rfloor} - L^{(\varepsilon)}_{2^{-n}\lfloor 2^n s\rfloor} + L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} s\rfloor} \right| \\ &\leq \left| L^{(\varepsilon)}_{2^{-n}\lfloor 2^n t\rfloor} - L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} t\rfloor} \right| + \left| L^{(\varepsilon)}_{2^{-n}\lfloor 2^n s\rfloor} - L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} s\rfloor} \right| \leq 2\|L^{(\varepsilon)}\|_n, \end{aligned}$$

and

$$\begin{aligned} &\left| L^{(\varepsilon)}_{2^{-n}\lfloor 2^n t\rfloor} - L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} t\rfloor} - L^{(\varepsilon)}_{2^{-n}\lfloor 2^n s\rfloor} + L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} s\rfloor} \right| \\ &\leq \left| L^{(\varepsilon)}_{2^{-n}\lfloor 2^n t\rfloor} - L^{(\varepsilon)}_{2^{-n}\lfloor 2^n s\rfloor} \right| + \left| L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} t\rfloor} - L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} s\rfloor} \right| \leq 2(t-s)2^n\|L^{(\varepsilon)}\|_n. \end{aligned}$$

Combining the above two inequalities we get

$$\text{r.h.s. of (3.3.4)} \le \sum_{n=1}^{\infty} 2\left(|t-s|2^n \wedge 2\right) \|L^{(\varepsilon)}\|_n$$
$$\le \|L^{(\varepsilon)}\| \sum_{n=1}^{\infty} (n+1)2^{-n\gamma}\left(|t-s|2^n \wedge 2\right) \le c_2 \|L^{(\varepsilon)}\| \cdot |t-s|^{\gamma} \log \tfrac{2}{|t-s|}.$$

where $c_2 > 0$ is an absolute constant. Combining this with the bound for $\mathbf{P}^{\xi}(\|L^{(\varepsilon)}\| \ge m)$, completes the proof. $\square$

3.3.1 Tail bounds for multivariate spatial process

Recall the KPZ sheet $\mathcal{H}(\cdot,\cdot;\cdot,\cdot)$ defined in (3.2.1). The core idea behind the proof of Propositions 3.3.1 and 3.3.1-(point-to-line) is to establish parabolic decay estimates of sum of several KPZ sheets scaled according to long-time. We record this parabolic decay estimate in the following Lemma 3.3.7.

Lemma 3.3.7 (Long-time multivariate spatial process tail bound)**.** *Fix any* $k \in \mathbb{Z}_{>0}$ *and* $\nu \in (0,1)$. *Set* $x_0 = 0$, *and* $\vec{x} := (x_1, \ldots, x_k)$. *For any* $\varepsilon \in (0,1)$ *consider* $0 = t_0 < t_1 < \cdots < t_k = 1$. *Set* $\vec{t} := (t_1, \ldots, t_k)$. *Then there exists a constant* $\mathrm{C} = \mathrm{C}(k,\nu)$ *such that for all* $s > 0$ *we have*

$$\mathbf{P}\left(\sup_{\vec{x}\in\mathbb{R}^k}\left[F_{\vec{t};\varepsilon}(\vec{x}) + \sum_{i=0}^{k-1}\frac{(1-\nu)(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}\right] \ge s\right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^{3/2}\right). \tag{3.3.5}$$

where

$$F_{\vec{t};\varepsilon}(\vec{x}) := \varepsilon^{1/3}\sum_{i=0}^{k-1}\Big[\mathcal{H}(x_i\varepsilon^{-2/3}, \varepsilon^{-1}t_i; x_{i+1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{i+1}) + \tfrac{\varepsilon^{-1}(t_{i+1}-t_i)}{24} + \mathbf{1}\{t_{i+1}-t_i \le \varepsilon\}\cdot \log\sqrt{2\pi\varepsilon^{-1}(t_{i+1}-t_i)}\Big]. \tag{3.3.6}$$

Proof. For clarity, we split the proof into three steps.

Step 1. Let us fix any $\varepsilon \in (0,1)$ consider $0 = t_0 < t_1 < \cdots < t_k = 1$. For brevity, we denote

$F(\vec{x}) := F_{\vec{t};\varepsilon}(\vec{x})$ and set

$$\overline{F}(\vec{x}) := F(\vec{x}) + \sum_{i=0}^{k-1} \frac{(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}. \tag{3.3.7}$$

For any $\vec{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$, set $V_{\vec{a}} := [a_1, a_1+1] \times \cdots \times [a_k, a_k+1]$ and set

$$\|\vec{a}\|^2 := a_1^2 + \min_{\vec{x}\in V_{\vec{a}}} \sum_{i=1}^{k-1} (x_{i+1}-x_i)^2. \tag{3.3.8}$$

We claim that for any $\vec{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ and $\nu \in (0,1)$

$$\mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}} \left[F(\vec{x}) + \sum_{i=0}^{k-1} \frac{(1-\nu)(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}\right] \geq s\right) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}(s^{3/2} + \|\vec{a}\|^3)\right) \tag{3.3.9}$$

for some $\mathrm{C} = \mathrm{C}(k,\nu) > 0$. Assuming (3.3.9) by union bound we obtain

$$\begin{aligned}
\text{l.h.s of (3.3.5)} &= \mathbf{P}\left(\sup_{\vec{x}\in\mathbb{R}^k} \left[F(\vec{x}) + \sum_{i=0}^{k-1} \frac{(1-\nu)(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}\right] \geq s\right) \\
&\leq \sum_{\vec{a}\in\mathbb{Z}^k} \mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}} \left[F(\vec{x}) + \sum_{i=0}^{k-1} \frac{(1-\nu)(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}\right] \geq s\right) \\
&\leq \sum_{\vec{a}\in\mathbb{Z}^k} \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}(s^{3/2} + \|\vec{a}\|^3)\right).
\end{aligned}$$

The r.h.s. of the above display is upper bounded by $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$ and proves (3.3.5). Thus it suffices to verify (3.3.9) in the rest of the proof.

Step 2. In this step, we prove the claim in (3.3.9). Note that

$$\mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}} \left[F(\vec{x}) + \sum_{i=0}^{k-1} \frac{(1-\nu)(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}\right] \geq s\right) \leq \mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}} |\overline{F}(\vec{x})| \geq s + \tfrac{\nu}{2}\|\vec{a}\|^2\right)$$

by the definition of $\overline{F}(\cdot)$ in (3.3.7) and the defintion of $\|\vec{a}\|^2$ from (3.3.8). Applying union bound yields

$$\mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}}|\overline{F}(\vec{x})|\ge s+\tfrac{\nu}{2}\|\vec{a}\|^2\right)$$
$$\le \mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}}|\overline{F}(\vec{x})-\overline{F}(\vec{a})|\ge \tfrac{s}{2}+\tfrac{\nu}{4}\|\vec{a}\|^2\right)+\mathbf{P}\left(|\overline{F}(\vec{a})|\ge \tfrac{s}{2}+\tfrac{\nu}{4}\|\vec{a}\|^2\right). \tag{3.3.10}$$

In the rest of the proof, we bound both summands on the r.h.s of (3.3.10) from above by $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}(s^{3/2}+\|\vec{a}\|^3))$ individually. To control the first term, we first need an a priori estimate. We claim that for all $u\in[0,1]$, $i=1,2,\ldots,k$ and $s>0$ we have

$$\mathbf{P}\left(\overline{F}(\vec{a}+e_i\cdot u)-\overline{F}(\vec{a})\ge su^{1/4}\right)\le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^{3/2}\right). \tag{3.3.11}$$

for some absolute constant $\mathrm{C}>0$. We will prove (6.2.2) in the next step. Given (6.2.2), appealing to Lemma 3.3 in [96] with $\alpha=\alpha_i=\frac{1}{4},\beta=\beta_i=\frac{3}{2},r=r_i=1$, we get that for all $m>0$

$$\mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}}|\overline{F}(\vec{x})-\overline{F}(\vec{a})|\ge m\right)\le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^{3/2}\right).$$

Taking $m=\frac{s}{2}+\frac{\nu}{4}\|\vec{a}\|^2$ in above, this yields the desired estimate for the first term in (3.3.10).

For the second term in (3.3.10), via the definition of $\overline{F}$ in (3.3.7) applying union bounds we have

$$\mathbf{P}\left(|\overline{F}(\vec{a})|\ge \tfrac{s}{4}+\tfrac{\nu}{4}\|\vec{a}\|^2\right)$$
$$\le \sum_{i=0}^{k-1}\mathbf{P}\Bigg(\Big|\varepsilon^{1/3}\mathcal{H}(a_i\varepsilon^{-2/3},\varepsilon^{-1}t_i;a_{i+1}\varepsilon^{-2/3},\varepsilon^{-1}t_{i+1})+\tfrac{\varepsilon^{-2/3}(t_{i+1}-t_i)}{24}$$
$$+\tfrac{(a_{i+1}-a_i)^2}{2(t_{i+1}-t_i)}+\varepsilon^{1/3}\mathbf{1}\{t_{i+1}-t_i\le\varepsilon\}\cdot\log\sqrt{2\pi\varepsilon^{-1}(t_{i+1}-t_i)}\Big|\ge \tfrac{s}{4k}+\tfrac{\nu}{4k}\|\vec{a}\|^2\Bigg)$$
$$\le \sum_{i=0}^{k-1}\mathbf{P}\Bigg(\Big|\varepsilon^{1/3}\mathcal{H}(0,\varepsilon^{-1}(t_{i+1}-t_i))+\tfrac{\varepsilon^{-2/3}(t_{i+1}-t_i)}{24}$$
$$+\varepsilon^{1/3}\mathbf{1}\{t_{i+1}-t_i\le\varepsilon\}\cdot\log\sqrt{2\pi\varepsilon^{-1}(t_{i+1}-t_i)}\Big|\ge \tfrac{s}{4k}+\tfrac{\nu}{4k}\|\vec{a}\|^2\Bigg) \tag{3.3.12}$$

where the last line follows from stationarity of the shifted version of $\mathcal{H}$. Now if $\varepsilon^{-1}(t_{i+1}-t_i) > 1$, we may use long-time scaling to get

$$\varepsilon^{1/3}\mathcal{H}(0,\varepsilon^{-1}(t_{i+1}-t_i)) + \frac{\varepsilon^{-2/3}(t_{i+1}-t_i)}{24} = \frac{\mathfrak{h}_{\varepsilon^{-1}(t_{i+1}-t_i)}(0)}{(t_{i+1}-t_i)^{-1/3}}.$$

Using the fact that $\varepsilon < |t_{i+1}-t_i| \leq 1$ along with the one-point long-time tail estimates from Proposition 3.2.2 (b) we get

$$\begin{aligned}\mathbf{P}\left(|\mathfrak{h}_{\varepsilon^{-1}(t_{i+1}-t_i)}(0)| \geq (t_{i+1}-t_i)^{-1/3}(\tfrac{s}{4k}+\tfrac{\nu}{4k}\|\vec{a}\|^2)\right) &\leq \mathbf{P}\left(|\mathfrak{h}_{\varepsilon^{-1}(t_{i+1}-t_i)}(0)| \geq \tfrac{s}{4k}+\tfrac{\nu}{4k}\|\vec{a}\|^2\right) \\ &\leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}(s+\|\vec{a}\|^2)^{3/2}) \\ &\leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}(s^{3/2}+\|\vec{a}\|^3)\right),\end{aligned}$$

for some constant $\mathrm{C} = \mathrm{C}(k,\nu) > 0$. If $\varepsilon^{-1}(t_{i+1}-t_i) \leq 1$, we may use short-time scaling to get

$$\begin{aligned}&\varepsilon^{1/3}\mathcal{H}(0,\varepsilon^{-1}(t_{i+1}-t_i)) + \frac{\varepsilon^{-2/3}(t_{i+1}-t_i)}{24} + \varepsilon^{1/3}\log\sqrt{2\pi\varepsilon^{-1}(t_{i+1}-t_i)} \\ &\quad = \varepsilon^{1/3}\big(\tfrac{\pi\varepsilon^{-1}(t_{i+1}-t_i)}{4}\big)^{1/4}\mathfrak{g}_{\varepsilon^{-1}(t_{i+1}-t_i)}(0) + \frac{\varepsilon^{-2/3}(t_{i+1}-t_i)}{24}.\end{aligned}$$

The linear term above is uniformly bounded in this case. Furthermore,

$$\varepsilon^{1/3}\big(\tfrac{\pi\varepsilon^{-1}(t_{i+1}-t_i)}{4}\big)^{1/4} = \big(\tfrac{\pi(t_{i+1}-t_i)}{4}\big)^{1/4}\varepsilon^{1/12} \leq 1.$$

Thus, in this case, appealing to one-point short-time tail estimates from Proposition 3.2.3 (b), we have

$$\text{r.h.s. of (3.3.12)} \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}(s+\|\vec{a}\|^2)^{3/2}) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}(s^{3/2}+\|\vec{a}\|^3)\right)$$

for some constant $\mathrm{C} = \mathrm{C}(k,\nu) > 0$.

This proves the required bound for the second term in (3.3.10). Combining the bounds for the two terms in (6.2.2), we thus arrive at (3.3.9). Hence, all we are left to show is (6.2.2) which we do in the next step.

Step 3. Fix $\vec{a} \in \mathbb{Z}^k$, fix $i = 1, 2, \dots, k$. The goal of this step is to show (6.2.2). Towards this end, note that for each coordinate vector $e_i, i = 1, \dots, k-1$, and for $u \in [0, 1]$ observe that

$$\begin{aligned}
&\overline{F}(\vec{a} + e_i \cdot u) - \overline{F}(\vec{a}) \\
&= \varepsilon^{1/3} \left[\mathcal{H}(a_{i-1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{i-1}; (a_i + u)\varepsilon^{-2/3}, \varepsilon^{-1}t_i) - \mathcal{H}(a_{i-1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{i-1}; a_i\varepsilon^{-2/3}, \varepsilon^{-1}t_i) \right] \\
&\qquad + \frac{(a_{i-1} - a_i - u)^2 - (a_{i-1} - a_i)^2}{2(t_i - t_{i-1})} \\
&\quad + \varepsilon^{1/3} \left[\mathcal{H}((a_i + u)\varepsilon^{-2/3}, \varepsilon^{-1}t_i; a_{i+1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{i+1}) - \mathcal{H}(a_i\varepsilon^{-2/3}, \varepsilon^{-1}t_i; a_{i+1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{i+1}) \right] \\
&\qquad + \frac{(a_{i+1} - a_i - u)^2 - (a_{i+1} - a_i)^2}{2(t_{i+1} - t_i)}.
\end{aligned}$$

Thus using distributional identities (see Remark 3.2.1) by union bound for all $s > 0$ we get that

$$\begin{aligned}
&\mathbf{P}\left(|\overline{F}(\vec{a} + e_i \cdot u) - \overline{F}(\vec{a})| \geq s u^{\frac{1}{4}} \right) \\
&\leq \mathbf{P}\left(\varepsilon^{\frac{1}{3}} \left| \overline{\mathcal{H}}_{\varepsilon^{-1}(t_i - t_{i-1})}((a_i + u - a_{i-1})\varepsilon^{-2/3}) - \overline{\mathcal{H}}_{\varepsilon^{-1}(t_i - t_{i-1})}((a_i - a_{i-1})\varepsilon^{-2/3}) \right| \geq \tfrac{s}{2} u^{\frac{1}{4}} \right) \quad (3.3.13) \\
&\quad + \mathbf{P}\left(\varepsilon^{\frac{1}{3}} \left| \overline{\mathcal{H}}_{\varepsilon^{-1}(t_{i+1} - t_i)}((a_i + u - a_{i+1})\varepsilon^{-2/3}) - \overline{\mathcal{H}}_{\varepsilon^{-1}(t_{i+1} - t_i)}((a_i - a_{i+1})\varepsilon^{-2/3}) \right| \geq \tfrac{s}{2} u^{\frac{1}{4}} \right),
\end{aligned} \tag{3.3.14}$$

where $\overline{\mathcal{H}}_t(x) := \mathcal{H}(x, t) + \frac{x^2}{2t}$. We now proceed to bound the second term on the r.h.s. of above display (that is the term in (3.3.14)); the bound for the first term follows analogously.

Case 1. $\varepsilon^{-1}(t_{i+1} - t_i) \geq 1$. We then use the long-time scaling to conclude

$$\begin{aligned}
&\varepsilon^{\frac{1}{3}} \left| \overline{\mathcal{H}}_{\varepsilon^{-1}(t_{i+1} - t_i)}((a_i + u - a_{i+1})\varepsilon^{-2/3}) - \overline{\mathcal{H}}_{\varepsilon^{-1}(t_{i+1} - t_i)}((a_i - a_{i+1})\varepsilon^{-2/3}) \right| \\
&\qquad = \frac{\overline{\mathfrak{h}}_{\varepsilon^{-1}(t_{i+1} - t_i)}\left(\frac{a_i + u - a_{i+1}}{(t_{i+1} - t_i)^{2/3}} \right) - \overline{\mathfrak{h}}_{\varepsilon^{-1}(t_{i+1} - t_i)}\left(\frac{a_i - a_{i+1}}{(t_{i+1} - t_i)^{2/3}} \right)}{(t_{i+1} - t_i)^{-1/3}}
\end{aligned}$$

where $\overline{\mathfrak{h}}_s(x) := \mathfrak{h}_s(x) + \frac{x^2}{2}$. We now consider two cases depending on the value of u.

Case 1.1. Suppose $u \in [0,(t_{i+1}-t_i)^{2/3}]$. By Proposition 3.2.2 (c) with $\gamma \mapsto \frac{u}{(t_{i+1}-t_i)^{2/3}}$, and using the fact that $\sqrt{\gamma} \le u^{1/4}(t_{i+1}-t_i)^{-1/3}$, we see that (3.3.14) $\le \mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$ for some $\mathrm{C}>0$ in this case.

Case 1.2. For $u \in [(t_{i+1}-t_i)^{2/3},1]$, we rely on one-point tail bounds. Indeed applying union bound we have

$$
\begin{aligned}
(3.3.13) &\le \mathbf{P}\left(\left|\frac{\overline{\mathfrak{h}}_{\varepsilon^{-1}(t_{i+1}-t_i)}\left(\frac{a_i+u-a_{i+1}}{(t_{i+1}-t_i)^{2/3}}\right)}{(t_{i+1}-t_i)^{-1/3}}\right| \ge \tfrac{s}{8}u^{1/4}\right) + \mathbf{P}\left(\left|\frac{\overline{\mathfrak{h}}_{\varepsilon^{-1}(t_{i+1}-t_i)}\left(\frac{a_i-a_{i+1}}{(t_{i+1}-t_i)^{2/3}}\right)}{(t_{i+1}-t_i)^{-1/3}}\right| \ge \tfrac{s}{8}u^{1/4}\right) \\
&\le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^{3/2}u^{3/8}(t_{i+1}-t_i)^{-1/2}\right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^{3/2}\right).
\end{aligned}
$$

The penultimate inequality above follows from Proposition 3.2.2 (a), (b) and the last one follows from the fact $u \ge (t_{i+1}-t_i)^{2/3}$ and $t_{i+1}-t_i \in (0,1]$.

Case 2. $\varepsilon^{-1}(t_{i+1}-t_i) \le 1$. We here use the short-time scaling to conclude

$$
\begin{aligned}
&\varepsilon^{\frac{1}{3}}\left|\overline{\mathcal{H}}_{\varepsilon^{-1}(t_{i+1}-t_i)}((a_i+u-a_{i+1})\varepsilon^{-2/3}) - \overline{\mathcal{H}}_{\varepsilon^{-1}(t_{i+1}-t_i)}((a_i-a_{i+1})\varepsilon^{-2/3})\right| \\
&\quad = \left(\tfrac{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}{4}\right)^{\frac{1}{4}}\left[\overline{\mathfrak{g}}_{\varepsilon^{-1}(t_{i+1}-t_i)}\left(\frac{2(a_i+u-a_{i+1})}{\sqrt{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}}\right) - \overline{\mathfrak{g}}_{\varepsilon^{-1}(t_{i+1}-t_i)}\left(\frac{2(a_i-a_{i+1})}{\sqrt{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}}\right)\right]
\end{aligned}
$$

where $\overline{\mathfrak{g}}_s(x) := \mathfrak{g}_s(x) + \frac{(\pi s/4)^{3/4}x^2}{2s}$. We again consider two cases depending on the value of u.

Case 2.1. Suppose $u \in (0, \frac{\sqrt{\pi}}{2}\varepsilon^{-1/3}(t_{i+1}-t_i))$. Then $\frac{2u}{\sqrt{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}} < \sqrt{\varepsilon^{-1}(t_{i+1}-t_i)}$. This allows us to apply Proposition 3.2.4 with $\gamma \mapsto \frac{2u}{\sqrt{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}}$ and $t \mapsto \varepsilon^{-1}(t_{i+1}-t_i)$. Using the fact that $u^{1/2} \le u^{1/4}$ for $u \in [0,1]$, we see that (3.3.14) $\le \mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$ for some $\mathrm{C}>0$ in this case.

Case 2.2. For $u \in [\frac{\sqrt{\pi}}{2}\varepsilon^{-1/3}(t_{i+1}-t_i),1]$, we rely on stationarity and one-point tail bounds (Proposition 3.2.3 (a), (b)). Indeed applying union bound we have

$$
\begin{aligned}
(3.3.14) \le\ &\mathbf{P}\left(\left(\tfrac{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}{4}\right)^{\frac{1}{4}}\left|\overline{\mathfrak{g}}_{\varepsilon^{-1}(t_{i+1}-t_i)}\left(\frac{2(a_i+u-a_{i+1})}{\sqrt{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}}\right)\right| \ge \tfrac{s}{8}u^{1/4}\right) \\
&+ \mathbf{P}\left(\left(\tfrac{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}{4}\right)^{\frac{1}{4}}\left|\overline{\mathfrak{g}}_{\varepsilon^{-1}(t_{i+1}-t_i)}\left(\frac{2(a_i-a_{i+1})}{\sqrt{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}}\right)\right| \ge \tfrac{s}{8}u^{1/4}\right)
\end{aligned}
$$

$$\leq \mathrm{C} \exp\left(-\frac{1}{\mathrm{C}}\left[su^{1/4}(t_{i+1}-t_i)^{-1/4}\varepsilon^{-\frac{1}{12}}\right]^{3/2}\right).$$

As $u \geq \frac{\sqrt{\pi}}{2}\varepsilon^{-1/3}(t_{i+1}-t_i)$, and $\varepsilon \in (0,1)$ we have $u^{1/4}(t_{i+1}-t_i)^{-1/4}\varepsilon^{-1/12} \geq \frac{\sqrt{\pi}}{2}$. Thus the last expression above is at most $\mathrm{C}\exp\left(-\frac{1}{\mathrm{C}}s^{3/2}\right)$.

Combining the above two cases we have (3.3.14) $\leq \mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$ uniformly for $u \in [0,1]$. By the same argument one can show the term in (3.3.13) is also upper bounded by $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$. This yields (6.2.2) for $i = 1, 2, \ldots, k-1$.

Finally for $i = k$, observe that

$$\begin{aligned}
&\overline{F}(\vec{a} + e_k \cdot u) - \overline{F}(\vec{a}) \\
&= \varepsilon^{1/3}\left[\mathcal{H}(a_{k-1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{k-1}; (a_k+u)\varepsilon^{-2/3}, \varepsilon^{-1}) - \mathcal{H}(a_{k-1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{k-1}; a_k\varepsilon^{-2/3}, \varepsilon^{-1})\right] \\
&\qquad\qquad + \frac{(a_{k-1}-a_k-u)^2 - (a_{k-1}-a_k)^2}{2(1-t_{k-1})}.
\end{aligned}$$

Then (6.2.2) follows for $i = k$ by the exact same computations as above. This completes the proof of the lemma. □

3.3.2 Proof of Proposition 3.3.1 and 3.3.1-(point-to-line)

We now present the proofs of Proposition 3.3.1 and 3.3.1-(point-to-line).

Proof of Proposition 3.3.1. We assume $m \geq 16\tau^2 + 1$. Otherwise the constant C_1 can be chosen large enough so that the inequality holds trivially. Without loss of generality assume $s < t$. We first consider the case when $s, t \in (0,1)$. Note that

$$\begin{aligned}
&\mathbf{P}^{\xi}(|L_s^{(\varepsilon)} - L_t^{(\varepsilon)}| \geq m|s-t|^{\frac{1}{2}-\delta}) \\
&= \iint\limits_{|u-v|\geq m\varepsilon^{-2/3}|s-t|^{\frac{1}{2}-\delta}} \frac{\mathcal{Z}(0,0;u,\varepsilon^{-1}s)\mathcal{Z}(u,\varepsilon^{-1}s;v,\varepsilon^{-1}t)\mathcal{Z}(v,\varepsilon^{-1}t;x,\varepsilon^{-1})}{\mathcal{Z}(0,0;x,\varepsilon^{-1})}\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

We make a change of variable $u = p\varepsilon^{-2/3}$, $v = q\varepsilon^{-2/3}$ and $x = z\varepsilon^{-2/3}$. Then

$$\begin{aligned}
&\mathbf{P}^{\xi}(|L_s^{(\varepsilon)} - L_t^{(\varepsilon)}| \geq m|s-t|^{\frac{1}{2}-\delta}) \\
=&\varepsilon^{-4/3} \iint\limits_{|p-q|\geq m|s-t|^{\frac{1}{2}-\delta}} \frac{\mathcal{Z}(0,0;p\varepsilon^{-2/3},\frac{s}{\varepsilon})\mathcal{Z}(p\varepsilon^{-2/3},\frac{s}{\varepsilon};q\varepsilon^{-2/3},\frac{t}{\varepsilon})\mathcal{Z}(q\varepsilon^{-2/3},\frac{t}{\varepsilon};z\varepsilon^{-2/3},\frac{1}{\varepsilon})}{\mathcal{Z}(0,0;z\varepsilon^{-2/3},\varepsilon^{-1})}\,\mathrm{d}q\,\mathrm{d}p.
\end{aligned} \tag{3.3.15}$$

Recall the multivariate spatial process $F_{\vec{t};\varepsilon}(\vec{x})$ from (3.3.6). Take $k = 3$ and set $\vec{t} = (s,t,1)$, and $\vec{x} = (p,q,z)$. We also set

$$B(\vec{t}) := \mathbf{1}\{s \leq \varepsilon\} \log\sqrt{2\pi\frac{s}{\varepsilon}} + \mathbf{1}\{t-s \leq \varepsilon\}\log\sqrt{2\pi\frac{t-s}{\varepsilon}} + \mathbf{1}\{1-t\leq\varepsilon\}\log\sqrt{2\pi\frac{1-t}{\varepsilon}}.$$

For the numerator of the integrand in (3.3.15) observe that

$$\mathcal{Z}(0,0;p\varepsilon^{-\frac{2}{3}},\tfrac{s}{\varepsilon})\mathcal{Z}(p\varepsilon^{-\frac{2}{3}},\tfrac{s}{\varepsilon};q\varepsilon^{-\frac{2}{3}},\tfrac{t}{\varepsilon})\mathcal{Z}(q\varepsilon^{-\frac{2}{3}},\tfrac{t}{\varepsilon};z\varepsilon^{-\frac{2}{3}},\tfrac{1}{\varepsilon}) = \exp\left[\varepsilon^{-\frac{1}{3}}F_{\vec{t};\varepsilon}(\vec{x}) - \tfrac{\varepsilon^{-1}}{24} - B(\vec{t})\right]. \tag{3.3.16}$$

Set $M = \frac{m^2}{64}$. Applying Lemma 3.3.7 with $\nu = \frac{1}{2}$ and $s = M$, we see that with probability greater than $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^{3/2})$,

$$\text{r.h.s. of (3.3.16)} \leq \exp\left[\varepsilon^{-1/3}M - \varepsilon^{-1/3}\left(\tfrac{p^2}{4s} + \tfrac{(q-p)^2}{4(t-s)} + \tfrac{(z-q)^2}{4(1-t)}\right) - \tfrac{\varepsilon^{-1}}{24} - B(\vec{t})\right]. \tag{3.3.17}$$

On the other hand, for the denominator of the integrand in (3.3.15) by one-point long-time tail bound from Proposition 3.2.2 with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^{3/2})$ we have

$$\mathcal{Z}(0,0;z\varepsilon^{-2/3},\varepsilon^{-1}) \geq \exp\left(\varepsilon^{-1/3}\mathfrak{h}_{\varepsilon^{-1}}(z) - \tfrac{\varepsilon^{-1}}{24}\right) \geq \exp\left(-\varepsilon^{-1/3}(M + \tfrac{1}{2}\tau^2) - \tfrac{\varepsilon^{-1}}{24}\right).$$

Combining the previous equation with (3.3.17) we get that with probability at least $1-\mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^{3/2})$

we have

$$\text{r.h.s. of (3.3.15)} \leq \varepsilon^{-\frac{4}{3}} \exp\left(\varepsilon^{-1/3}(2M + \tfrac{1}{2}\tau^2) - B(\vec{t})\right) \cdot \iint_{|p-q|\geq m|s-t|^{\frac{1}{2}-\delta}} \exp\left[-\varepsilon^{-1/3}\left(\tfrac{p^2}{4s} + \tfrac{(q-p)^2}{4(t-s)} + \tfrac{(z-q)^2}{4(1-t)}\right)\right] dq\, dp$$

$$\leq \varepsilon^{-\frac{4}{3}} \exp\left(\varepsilon^{-\frac{1}{3}}(2M + \tfrac{1}{2}\tau^2 - \tfrac{m^2}{4|t-s|^{2\delta}}) - B(\vec{t})\right) \iint_{\mathbb{R}^2} \exp\left[-\varepsilon^{-\frac{1}{3}}\left(\tfrac{p^2}{4s} + \tfrac{r^2}{4(1-t)}\right)\right] dr\, dp$$

$$= 4\pi\sqrt{s(1-t)}\varepsilon^{-1} \exp\left(\varepsilon^{-\frac{1}{3}}(2M + \tfrac{1}{2}\tau^2 - \tfrac{m^2}{4|t-s|^{2\delta}}) - B(\vec{t})\right). \tag{3.3.18}$$

Observe that

$$\sqrt{r} \exp\left(-\mathbf{1}\{r \leq \varepsilon\} \log \sqrt{\tfrac{2\pi r}{\varepsilon}}\right) \leq 1. \tag{3.3.19}$$

As $M = \frac{m^2}{64}$ we have $2M - \frac{m^2}{4|t-s|^{2\delta}} \leq -\frac{m^2}{8|t-s|^{2\delta}}$. Furthermore $\frac{1}{2}\tau^2 \leq \frac{m^2}{16|t-s|^{2\delta}}$ under the condition $m \geq 16\tau^2 + 1$. Thus,

$$\text{r.h.s. of (3.3.18)} \leq 4\pi\varepsilon^{-1} \exp\left(-\varepsilon^{-\frac{1}{3}} \frac{m^2}{16|t-s|^{2\delta}} - \mathbf{1}\{t-s \leq \varepsilon\} \log \sqrt{\tfrac{2\pi(t-s)}{\varepsilon}}\right).$$

Clearly the last expression is at most $\mathrm{C}_1 \exp(-\frac{1}{\mathrm{C}_1}m^2)$ for some $\mathrm{C}_1 > 0$ depending on τ, δ. This bound holds uniformly over $t, s \in (0,1)$ with $t \neq s$ and $\varepsilon \in (0,1)$. This concludes the proof for $s, t \in (0,1)$.

Finally, when $s = 0$ we have

$$\mathbf{P}^{\xi}(|L_t^{(\varepsilon)}| \geq m|t|^{\frac{1}{2}-\delta}) = \iint_{|v|\geq m\varepsilon^{-2/3}|t|^{\frac{1}{2}-\delta}} \frac{\mathcal{Z}(0,;v,\varepsilon^{-1}t)\mathcal{Z}(v,\varepsilon^{-1}t;x,\varepsilon^{-1})}{\mathcal{Z}(0,0;x,\varepsilon^{-1})}\, \mathrm{d}v.$$

The proof can now be completed by following the argument for $s, t \in (0,1)$ case. Indeed, the denominator can be bounded by the exact same manner as above, whereas the numerator can be controlled with the $k = 2$ version of Lemma 3.3.7. The case $t = 1$ is analogous to the case $s = 0$. We have thus established Proposition 3.3.1. □

Proof of Proposition 3.3.1-(point-to-line). We now explain how the above proof can be modified to extend it to the point-to-line version. Fix any $m > 0$ and $M > 1$. Indeed observe that for $0 < s < t < 1$, one has

$$\begin{aligned}
&\mathbf{P}^{\xi}_{*}(|L^{(\varepsilon)}_{s,*} - L^{(\varepsilon)}_{t,*}| \geq m|s-t|^{\frac{1}{2}-\delta}) \\
=&\varepsilon^{-\frac{4}{3}} \int_{\mathbb{R}} \iint\limits_{|p-q|\geq m|s-t|^{\frac{1}{2}-\delta}} \frac{\mathcal{Z}(0,0;p\varepsilon^{-\frac{2}{3}},\frac{s}{\varepsilon})\mathcal{Z}(p\varepsilon^{-\frac{2}{3}},\frac{s}{\varepsilon};q\varepsilon^{-\frac{2}{3}},\frac{t}{\varepsilon})\mathcal{Z}(q\varepsilon^{-\frac{2}{3}},\frac{t}{\varepsilon};z\varepsilon^{-\frac{2}{3}},\frac{1}{\varepsilon})}{\int_{\mathbb{R}}\mathcal{Z}(0,0;y\varepsilon^{-\frac{2}{3}},\varepsilon^{-1})\mathrm{d}y} \mathrm{d}q\,\mathrm{d}p\,\mathrm{d}z.
\end{aligned} \tag{3.3.20}$$

Since Lemma 3.3.7 is a process-level estimate that allows even the endpoint to vary, (3.3.17) continues to hold simultaneously for all $p, q, z \in \mathbb{R}$ with same high probability. However for the lower bound on the denominator, one-point lower-tail bound is not sufficient. Instead, for the denominator we use long-time process-level lower bound from Proposition 4.1 in [81] to get that with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^{3/2})$ we have

$$\int_{\mathbb{R}} \mathcal{Z}(0,0;y\varepsilon^{-2/3},\varepsilon^{-1})\mathrm{d}y \geq \int_{\mathbb{R}} \exp\left(-\tfrac{M+y^2}{\varepsilon^{1/3}} - \tfrac{\varepsilon^{-1}}{24}\right)\mathrm{d}y \geq \mathrm{C}\varepsilon^{\frac{1}{6}}\exp\left(-\varepsilon^{-1/3}M - \tfrac{\varepsilon^{-1}}{24}\right).$$

Combining the previous equation with (3.3.17) we get that with probability at least $1-\mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^{3/2})$ we have

$$\begin{aligned}
\text{r.h.s. of (3.3.20)} \leq \varepsilon^{-\frac{3}{2}} &\exp\left(2\varepsilon^{-1/3}M - B(\vec{t})\right)\cdot \\
&\int_{\mathbb{R}}\iint_{|p-q|\geq m|s-t|^{\frac{1}{2}-\delta}} \exp\left[-\varepsilon^{-1/3}\left(\tfrac{p^2}{4s} + \tfrac{(q-p)^2}{4(t-s)} + \tfrac{(z-q)^2}{4(1-t)}\right)\right] dq\,dp\,dz.
\end{aligned} \tag{3.3.21}$$

On $|p-q| \geq m|s-t|^{\frac{1}{2}-\delta}$, we have $(q-p)^2/4(t-s) \geq (q-p)^2/8(t-s) + m^2/8|t-s|^{2\delta}$. Applying this inequality followed by expanding the range of integration we get

$$\begin{aligned}
\text{r.h.s. of (3.3.21)} \leq \varepsilon^{-\frac{3}{2}} &\exp\left(\varepsilon^{-\frac{1}{3}}(2M - \tfrac{m^2}{8|t-s|^{2\delta}}) - B(\vec{t})\right) \\
&\cdot \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left[-\varepsilon^{-\frac{1}{3}}\left(\tfrac{p^2}{4s} + \tfrac{r^2}{8(t-s)} + \tfrac{u^2}{4(1-t)}\right)\right] dq\,dr\,du
\end{aligned}$$

$$= \sqrt{2^7\pi^3 s(1-t)(t-s)} \cdot \varepsilon^{-1} \exp\left(\varepsilon^{-\frac{1}{3}}(2M - \tfrac{m^2}{8|t-s|^{2\delta}}) - B(\vec{t})\right).$$

Just as in the proof of Proposition 3.3.1, setting $M = \frac{m^2}{64}$, and using (3.3.19), the above expression can be shown to be at most $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}m^2)$ uniformly over $\varepsilon \in (0,1)$ and $0 < s < t < 1$. This establishes the proposition. □

3.4 Annealed Convergence for short-time and long-time

In this section we prove our main results. In Section 3.4.1 we prove Theorems 3.1.4, 3.1.7, and 3.1.8. In Section 3.4.2, we show Theorem 3.1.10 assuming Conjecture 3.1.9.

3.4.1 Proof of Theorems 3.1.4, 3.1.7, and 3.1.8

In this section we prove results related to short-time and long-time tightness and related pointwise weak convergence. While the proof of long-time tightness relies on modulus of continuity estimates from Proposition 3.3.1 and Proposition 3.3.1-(point-to-line), the proof of short-time tightness relies on the following Brownian relation of annealed law of CDRP.

Lemma 3.4.1 (Brownian Relation)**.** *Let $X \sim \mathsf{CDRP}(0,0;0,t)$ and $Y \sim \mathsf{CDRP}(0,0;*,t)$. For any continuous functional $\mathcal{L} : C([0,t]) \to \mathbb{R}$ we have*

$$\mathbf{E}\left[\mathcal{Z}(0,0;0,t)\sqrt{2\pi t}\cdot\mathcal{L}(X)\right] = \mathbf{E}(\mathcal{L}(B)), \quad \mathbf{E}\left[\mathcal{Z}(0,0;*,t)\cdot\mathcal{L}(Y)\right] = \mathbf{E}(\mathcal{L}(B_*)) \tag{3.4.1}$$

where B_* and B are standard Brownian motion and standard Brownian bridge on $[0,t]$ respectively.

Remark 3.4.2. Note that $\sqrt{2\pi t} = \frac{1}{p(0,t)}$ where $p(0,t)$ is the heat kernel. Since the Brownian bridge finite-dimensional densities are product of heat kernels divided by $p(0,t)$, this additional factor $\frac{1}{p(0,t)}$ is required in the point-to-point version for appropriate comparison to the the Brownian bridge law (see (3.4.2) below).

Proof. Take $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = t$. The Brownian motion identity appears as Lemma 4.2 in [3]. To show the bridge version note that by Definition 3.1.1, the quantity

$$\mathcal{Z}(0,0;0,t)\mathbf{P}^{\xi}(X(t_1) \in \mathrm{d}x_1, \ldots, X(t_k) \in \mathrm{d}x_k)$$

is product of independent random variables with mean $p(x_{j+1}-x_j, t_{j+1}-t_j)$ where $p(x,t)$ denotes the heat kernel. Noting that $p(0,t) = \frac{1}{\sqrt{2\pi t}}$, and recalling the finite-dimensional distribution of Brownian bridge (Problem 6.11 in [157]) we get that

$$\begin{aligned}\mathbf{E}\left[\mathcal{Z}(0,0;0,t)\sqrt{2\pi t}\cdot \mathbf{P}^{\xi}(X(t_1) \in \mathrm{d}x_1, \ldots, X(t_k) \in \mathrm{d}x_k)\right] &= \frac{1}{p(0,t)}\prod_{j=0}^{k} p(x_{j+1}-x_j, t_{j+1}-t_j) \\ &= \mathbf{P}\left(B(t_1) \in \mathrm{d}x_1, \ldots, B(t_k) \in \mathrm{d}x_k\right).\end{aligned} \tag{3.4.2}$$

(3.4.1) now follows from the above by approximation of $\mathcal{L}$ with simple functions. □

Proof of Theorem 3.1.4. We first show finite-dimensional convergence. Fix $0 = t_0 < t_1 < \cdots < t_{k+1} = 1$. Take $x_1, \ldots, x_k \in \mathbb{R}$. Set $x_0 = 0$ and $x_{k+1} = 0$. Note that the density for $(Y_{t_i}^{(\varepsilon)})_{i=1}^k$ at $(x_i)_{i=1}^k$ is given by

$$f_{\vec{t};\varepsilon}(\vec{x}) := \frac{\varepsilon^{k/2}}{\mathcal{Z}(0,0;0,\varepsilon)}\prod_{j=0}^{k}\mathcal{Z}(\sqrt{\varepsilon}x_j, \varepsilon t_j; \sqrt{\varepsilon}x_{j+1}, \varepsilon t_{j+1}).$$

For a Brownian bridge B on $[0,1]$ starting at 0 and ending at x, the density for $(B_{t_i})_{i=1}^k$ at $(x_i)_{i=1}^k$ is given by

$$g_{\vec{t}}(\vec{x}) := \frac{1}{p(0,1)}\prod_{j=0}^{k} p(x_{j+1}-x_j, t_{j+1}-t_j)$$

where $p(x,t) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}$. Using the distributional identities for $\mathcal{Z}$ (see Remark 3.2.1) and using

Equation (8.11) in [65] and Brownian scaling, we deduce

$$\frac{\mathcal{Z}(\sqrt{\varepsilon}x_j, \varepsilon t_j; \sqrt{\varepsilon}x_{j+1}, \varepsilon t_{j+1})}{p(\sqrt{\varepsilon}(x_{j+1}-x_j), \varepsilon(t_{j+1}-t_j))} \stackrel{d}{=} \mathbf{E}_{0,0}^{t_{j+1}-t_j, x_{j+1}-x_j}\left[: \exp : \left\{\varepsilon^{1/4}\int_0^{t_{j+1}-t_j} \xi(s, B(s))ds\right\}\right]$$

where B is a Brownian bridge conditioned $B(0) = 0$ and $B(t_{j+1} - t_j) = x_{j+1} - x_j$. The expectation above is taken w.r.t. this Brownian bridge only. Here : exp : denotes the Wick exponential (see [65] for details). The right side of the above equation is a random variable (function of the noise ξ). We claim that this random variable converges to 1 in probability. Indeed using chaos expansion, and Lemma 2.4 in [76], it follows that for every fixed t, x we have

$$\mathbf{E}\left[\left\{\mathbf{E}_{0,0}^{t,x}\left[: \exp : \left\{\varepsilon^{1/4}\int_0^t \xi(s, B(s))ds\right\}\right] - 1\right\}^2\right] = \sqrt{\varepsilon}\sum_{k=1}^{\infty} \frac{\varepsilon^{(k-1)/2}t^{k/2}}{(4\pi)^{k/2}}\frac{(\Gamma(1/2)^k)}{\Gamma(k/2)}.$$

The above sum converges. Thus as $\varepsilon \downarrow 0$, the above expression goes to zero, proving the claim. As $p(\sqrt{\varepsilon}x, \varepsilon t) = \varepsilon^{-1/2}p(x,t)$, we thus have $f_{\vec{t};\varepsilon}(\vec{x}) \xrightarrow{p} g_{\vec{t}}(\vec{x})$. Thus the quenched finite-dimensional density of $Y^{(\varepsilon)}$ converges in probability to the finite-dimensional density of the Brownian Bridge. We now show that the same holds for the annealed law. Indeed, note that $\left|g_{\vec{t}}(\vec{x}) - f_{\vec{t};\varepsilon}(\vec{x})\right|^+$ converges to zero in probability and is bounded above by $g_{\vec{t}}(\vec{x})$. Thus by DCT and Jensen's inequality, we obtain

$$\left|g_{\vec{t}}(\vec{x}) - \mathbf{E}[f_{\vec{t};\varepsilon}(\vec{x})]\right|^+ \le \mathbf{E}_{\xi}\left|g_{\vec{t}}(\vec{x}) - f_{\vec{t};\varepsilon}(\vec{x})\right|^+ \to 0$$

as $\varepsilon \downarrow 0$. Now by Scheffe's theorem, it follows that the annealed finite-dimensional distribution of $Y^{(\varepsilon)}$ converges weakly to the finite-dimensional distribution of the Brownian bridge.

Let us now verify tightness. Recall that $X(\varepsilon t) = \sqrt{\varepsilon}Y_t^{(\varepsilon)}$. Observe that by union bound followed by Markov inequality we have

$$\mathbf{P}\left[\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}} |Y_t^{(\varepsilon)} - Y_s^{(\varepsilon)}| \ge \eta\right] \le \mathbf{P}\left[\mathcal{Z}(0,0;0,\varepsilon)\sqrt{2\pi\varepsilon}\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}} |Y_t^{(\varepsilon)} - Y_s^{(\varepsilon)}| \ge \eta\delta^{1/3}\right]$$

$$+\mathbf{P}\left[\mathcal{Z}(0,0;0,\varepsilon)\sqrt{2\pi\varepsilon}\le\delta^{1/3}\right]$$

$$\le\frac{\sqrt{2\pi}}{\eta\delta^{1/3}}\mathbf{E}\left[\mathcal{Z}(0,0;0,\varepsilon)\sup_{\substack{0\le t,s\le 1\\|t-s|\le\delta}}|X(\varepsilon t)-X(\varepsilon s)|\right]$$

$$+\mathbf{P}\left[\mathfrak{g}_\varepsilon(0)\le(4\varepsilon/\pi)^{-1/4}\log(\delta^{1/3})\right].$$

Note that by one-point short-time tail bounds from Proposition 3.2.3 (b), the second expression above goes to zero as $\delta\downarrow 0$ uniformly in $\varepsilon\le 1$. For the first expression, by Lemma 3.4.1 we have

$$\mathbf{E}\left[\mathcal{Z}(0,0;0,\varepsilon)\sup_{\substack{0\le t,s\le 1\\|t-s|\le\delta}}|X(\varepsilon t)-X(\varepsilon s)|\right]=\frac{1}{\sqrt{2\pi\varepsilon}}\mathbf{E}\left[\sup_{\substack{0\le t,s\le 1\\|t-s|\le\delta}}|B'_{\varepsilon t}-B'_{\varepsilon s}|\right],$$

where B' is a Brownian bridge on $[0,\varepsilon]$. By scaling property of Brownian bridges we may write the last expression simply as

$$\tfrac{1}{\sqrt{2\pi}}\mathbf{E}\left[\sup_{\substack{0\le t,s\le 1\\|t-s|\le\delta}}|B_t-B_s-(t-s)B_1|\right]$$

where B is a Brownian motion on $[0,1]$. This expression is free of ε and by [**fis**] this goes to zero with rate $O(\delta^{1/2-\gamma})$ for any $\gamma>0$. Thus we have shown

$$\limsup_{\delta\downarrow 0}\sup_{\varepsilon\in(0,1)}\mathbf{P}\left[\sup_{\substack{0\le t,s\le 1\\|t-s|\le\delta}}|Y_t^{(\varepsilon)}-Y_s^{(\varepsilon)}|\ge\eta\right]=0.$$

Since $Y_0^{(\varepsilon)}=0$, by standard criterion of tightness (see Theorem 4.10 in [157]) combined with finite-dimensional convergence shown before, we have weak convergence to Brownian Bridge. This completes the proof. □

Proof of Theorem 3.1.7. Let us first prove (a) using Corollary 3.3.6. Fix $\gamma\in(0,1)$. We consider $\beta\in(0,1)$ small enough so that $\gamma\ge\rho(\beta)$ where $\rho(\beta):=\sup_{t\in(0,\beta]}t^{\frac14}\log\frac{2}{t}$. Taking $\delta=\frac14$, the estimates in (3.3.3) ensure that for all $\varepsilon \in (0,1)$ we have

$$\mathbf{P}\left(\sup_{t\neq s\in[0,1],|t-s|<\beta}|L_s^{(\varepsilon)}-L_t^{(\varepsilon)}|\geq\gamma\right)\leq\mathbf{P}\left(\sup_{t\neq s\in[0,1],|t-s|<\beta}\frac{|L_s^{(\varepsilon)}-L_t^{(\varepsilon)}|}{|t-s|^{\frac14}\log\frac{2}{|t-s|}}\geq\frac{\gamma}{\rho(\beta)}\right)$$
$$\leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}\tfrac{\gamma^2}{\rho(\beta)^2}\right).$$

Note that as $\beta\downarrow 0$, we have $\rho(\beta)\downarrow 0$. Hence

$$\limsup_{\beta\downarrow 0}\sup_{\varepsilon\in(0,1)}\mathbf{P}\left(\sup_{t\neq s\in[0,1],|t-s|<\beta}|L_s^{(\varepsilon)}-L_t^{(\varepsilon)}|\geq\gamma\right)=0.$$

Since $L_0^{(\varepsilon)}=0$, the above modulus of continuity estimate yields tightness for the process $L_t^{(\varepsilon)}$.

For (b), let us fix $t\in(0,1)$ and consider $V\sim\mathsf{CDRP}(0,0;0,\varepsilon^{-1})$. Let $\mathcal{M}_{t,\varepsilon^{-1}}$ denote the unique mode of the quenched density of $V(\varepsilon^{-1}t)$. By [89, Theorem 1.4], we know $\mathcal{M}_{t,\varepsilon^{-1}}$ exists uniquely almost surely. By [89, Corollary 7.3] we have

$$\limsup_{K\to\infty}\limsup_{\varepsilon\downarrow 0}\mathbf{P}^{\xi}(|V(\varepsilon^{-1}t)-\mathcal{M}_{t,\varepsilon^{-1}}|\geq K)=0,\ \text{in probability}.$$

Applying reverse Fatou's Lemma we have

$$\limsup_{K\to\infty}\limsup_{\varepsilon\downarrow 0}\mathbf{P}(|V(\varepsilon^{-1}t)-\mathcal{M}_{t,\varepsilon^{-1}}|\geq K)=0.$$

Thus in particular, $\varepsilon^{-\frac23}[V(\varepsilon^{-1}t)-\mathcal{M}_{t,\varepsilon^{-1}}]\xrightarrow{p}0$. However, $\varepsilon^{-2/3}\mathcal{M}_{t,\varepsilon^{-1}}\xrightarrow{d}\Gamma(t\sqrt{2})$ due to [89, Theorem 1.8]. This proves (b). □

Proof of Theorem 3.1.8. Let us first prove part (a) which claims short-time process convergence. We first show finite-dimensional convergence. Fix $0=t_0<t_1<\cdots<t_{k+1}=1$. Take $x_1,\ldots,x_k\in$

$\mathbb{R}$. Set $x_0 = 0$ and $x_{k+1} = *$. Note that the density for $(Y_*^{(\varepsilon)}(t_i))_{i=1}^k$ at $(x_i)_{i=1}^k$ is given by

$$f^*_{\vec{t};\varepsilon}(\vec{x}) := \frac{\varepsilon^{k/2}}{\mathcal{Z}(0,0;*,\varepsilon)} \prod_{j=0}^{k} \mathcal{Z}(\sqrt{\varepsilon}x_j, \varepsilon t_j; \sqrt{\varepsilon}x_{j+1}, \varepsilon t_{j+1}).$$

From the finite-dimensional convergence argument in proof of Theorem 3.1.4 we know that

$$\varepsilon^{k/2} \prod_{j=0}^{k-1} \mathcal{Z}(\sqrt{\varepsilon}x_j, \varepsilon t_j; \sqrt{\varepsilon}x_{j+1}, \varepsilon t_{j+1}) \xrightarrow{p} \prod_{j=0}^{k-1} p(x_{j+1} - x_j, t_{j+1} - t_j) =: g^*_{\vec{t}}(\vec{x}). \tag{3.4.3}$$

Note that $g^*_{\vec{t}}(\vec{x})$ is the finite-dimensional density for the standard Brownian motion. We now claim that

$$\mathcal{Z}(0,0;*,\varepsilon) \xrightarrow{p} 1, \quad \mathcal{Z}(\sqrt{\varepsilon}x_{k-1}, \varepsilon t_{k-1}; *, \varepsilon t_k) \xrightarrow{p} 1. \tag{3.4.4}$$

Combining (3.4.3) and (3.4.4) we have that $f^*_{\vec{t};\varepsilon}(\vec{x}) \xrightarrow{p} g^*_{\vec{t}}(\vec{x})$ which implies quenched finite-dimensional density convergence. This convergence can then be upgraded to annealed finite-dimensional density convergence by the same argument of the proof of Theorem 3.1.4.

We thus focus on proving (3.4.4). To prove the first part of (3.4.4) we utilize the short-time scaling from (3.2.2) to get

$$\mathcal{Z}(0,0;*,\varepsilon) = \int_{\mathbb{R}} e^{\mathcal{H}(x,t)} dx = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}} \exp\left((\tfrac{\pi\varepsilon}{4})^{1/4} \mathfrak{g}_\varepsilon\big(x\sqrt{\tfrac{4}{\pi\varepsilon}}\big) \right) dx. \tag{3.4.5}$$

Fix any $\nu \in (0,1)$. Applying [87, Proposition 4.4] (with $s = \varepsilon^{-\frac{1}{6}}$) we get that with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}\varepsilon^{-\frac{1}{4}})$

$$-\frac{(\pi\varepsilon/4)^{3/4}(1+\nu)x^2}{2\varepsilon} - \varepsilon^{-1/6} \le \mathfrak{g}_\varepsilon\big(x\big) \le -\frac{(\pi\varepsilon/4)^{3/4}(1-\nu)x^2}{2\varepsilon} + \varepsilon^{-1/6}, \text{ for all } x \in \mathbb{R}, \tag{3.4.6}$$

where the constant C depends on ν. Inserting the above inequality in (3.4.5) we get that with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}\varepsilon^{-\frac{1}{4}})$

$$\exp\left(-(\tfrac{\pi}{4})^{1/4}\varepsilon^{1/12}\right)\frac{1}{\sqrt{2\pi\varepsilon}}\int_{\mathbb{R}} e^{-\frac{(1+\nu)x^2}{2\varepsilon}}dx \leq \mathcal{Z}(0,0;*,\varepsilon) \leq \exp\left((\tfrac{\pi}{4})^{1/4}\varepsilon^{1/12}\right)\frac{1}{\sqrt{2\pi\varepsilon}}\int_{\mathbb{R}} e^{-\frac{(1-\nu)x^2}{2\varepsilon}}dx.$$

Thus

$$\mathbf{P}\left(\frac{\exp\left(-(\frac{\pi}{4})^{1/4}\varepsilon^{1/12}\right)}{\sqrt{1+\nu}} \leq \mathcal{Z}(0,0;*,\varepsilon) \leq \frac{\exp\left((\frac{\pi}{4})^{1/4}\varepsilon^{1/12}\right)}{\sqrt{1-\nu}}\right) \geq 1 - \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}\varepsilon^{-\frac{1}{4}}),$$

which implies

$$\limsup_{\varepsilon\to\infty}\mathbf{P}\left(\tfrac{1}{\sqrt{1+\nu}} \leq \mathcal{Z}(0,0;*,\varepsilon) \leq \tfrac{1}{\sqrt{1-\nu}}\right) = 1.$$

Taking $\nu \downarrow 0$, we get the first part of (3.4.4). The second part follows analogously.

Let us now verify tightness. Observe that by union bound followed by Markov inequality we have

$$\begin{aligned}
\mathbf{P}\left[\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}} |Y_*^{(\varepsilon)}(t) - Y_*^{(\varepsilon)}(t)| \geq \eta\right] &\leq \mathbf{P}\left[\mathcal{Z}(0,0;*,\varepsilon)\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}} |Y_*^{(\varepsilon)}(t) - Y_*^{(\varepsilon)}(s)| \geq \eta\delta^{1/3}\right] \\
&\qquad + \mathbf{P}\left[\mathcal{Z}(0,0;*,\varepsilon) \leq \delta^{1/3}\right] \\
&\leq \frac{1}{\eta\delta^{1/3}}\mathbf{E}\left[\mathcal{Z}(0,0;*,\varepsilon)\frac{1}{\sqrt{\varepsilon}}\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}} |X(\varepsilon t) - X(\varepsilon s)|\right] \\
&\qquad + \mathbf{P}\left[\mathfrak{g}_\varepsilon(*) \leq \varepsilon^{-1/4}\log(\delta^{1/3})\right],
\end{aligned} \tag{3.4.7}$$

where

$$\begin{aligned}
\mathfrak{g}_\varepsilon(*) &:= \varepsilon^{-1/4}\log\mathcal{Z}(0,0;*,\varepsilon) \\
&= \varepsilon^{-1/4}\left[-\log\sqrt{2\pi\varepsilon} + \log\int_{\mathbb{R}}\exp\left((\tfrac{\pi\varepsilon}{4})^{1/4}\mathfrak{g}_\varepsilon(\sqrt{\tfrac{4}{\pi\varepsilon}}x)\right)\mathrm{d}x\right]
\end{aligned}$$

with $\mathfrak{g}_\varepsilon(x)$ defined in (3.2.2). Let us now bound each term in the r.h.s. of (3.4.7) separately. For the second term we claim that

$$\limsup_{\delta\downarrow 0}\sup_{\varepsilon\in(0,1)}\mathbf{P}\left[\mathfrak{g}_\varepsilon(*)\le \varepsilon^{-1/4}\log(\delta^{1/3})\right]=0. \tag{3.4.8}$$

Note that by Proposition 4.4 in [87] (the infimum process bound with $\nu = 1$) we have for any $s > 0$ with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$,

$$(\tfrac{\pi\varepsilon}{4})^{1/4}\mathfrak{g}_\varepsilon(\sqrt{\tfrac{4}{\pi\varepsilon}}x) \ge -(\tfrac{\pi\varepsilon}{4})^{1/4}\left[s+(\tfrac{\pi\varepsilon}{4})^{3/4}\cdot\tfrac{4}{\pi\varepsilon}\tfrac{x^2}{\varepsilon}\right] = -(\tfrac{\pi\varepsilon}{4})^{1/4}s-\tfrac{x^2}{\varepsilon},\ \text{for all } x\in\mathbb{R}.$$

Thus, with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$,

$$\begin{aligned}\mathfrak{g}_\varepsilon(*) &\ge \varepsilon^{-1/4}\left[-\log\sqrt{2\pi\varepsilon}+\log\left(\int_{\mathbb{R}}\exp\left(-(\tfrac{\pi\varepsilon}{4})^{1/4}s-\tfrac{x^2}{\varepsilon}\right)\mathrm{d}x\right)\right]\\ &= \varepsilon^{-1/4}\left[-\log\sqrt{2\pi\varepsilon}+\log\left(\sqrt{\pi\varepsilon}\exp\left(-(\tfrac{\pi\varepsilon}{4})^{1/4}s\right)\right)\right]\\ &= \varepsilon^{-1/4}\left[-\log\sqrt{2}-(\tfrac{\pi\varepsilon}{4})^{1/4}s\right]\ge -s-\varepsilon^{-1/4}\log 2.\end{aligned}$$

Now we take $s = -\varepsilon^{-1/4}\log(2\delta^{1/6})$ which is positive for δ small enough. Then $-s - \varepsilon^{-1/4}\log 2 = \frac{1}{2}\varepsilon^{-1/4}\log(\delta^{1/3}) > \varepsilon^{-1/4}\log(\delta^{1/3})$. Hence uniformly in all $\varepsilon\in(0,1)$, with probability at least $1-\mathrm{C}\exp(-\frac{1}{\mathrm{C}}[-\log(2\delta^{1/6})]^{3/2})$, we have $\mathfrak{g}_\varepsilon(*)\ge\varepsilon^{-1/4}\log(\delta^{1/3})$. This verifies (3.4.8).

Next for the first expression on r.h.s. of (3.4.7), by Lemma 3.4.1 we have

$$\mathbf{E}\left[\mathcal{Z}(0,0;*,\varepsilon)\frac{1}{\sqrt{\varepsilon}}\sup_{\substack{0\le t,s\le 1\\|t-s|\le\delta}}|X(\varepsilon t)-X(\varepsilon s)|\right]=\frac{1}{\sqrt{\varepsilon}}\mathbf{E}\left[\sup_{\substack{0\le t,s\le 1\\|t-s|\le\delta}}|B'_{\varepsilon t}-B'_{\varepsilon s}|\right],$$

where B' is a Brownian motion on $[0,\varepsilon]$. By scaling property of Brownian motion we may write the last expression simply as

$$\mathbf{E}\left[\sup_{\substack{0\le t,s\le 1\\|t-s|\le\delta}}|B_t-B_s|\right]$$

where B is a Brownian motion on $[0,1]$. This expression is free of ε and by [**fis**] this goes to zero with rate $O(\delta^{1/2-\gamma})$ for any $\gamma > 0$. Thus we have shown

$$\limsup_{\delta\downarrow 0} \sup_{\varepsilon\in(0,1)} \mathbf{P}\left[\sup_{\substack{0\le t,s\le 1\\|t-s|\le\delta}} |Y_*^{(\varepsilon)}(t) - Y_*^{(\varepsilon)}(s)| \ge \eta\right] = 0.$$

Since $Y_*^{(\varepsilon)}(0) = 0$, this proves tightness. Along with finite-dimensional convergence, this establishes part (a).

The tightness results in part (b) follows via the same arguments as in the proof of Theorem 3.1.7 (a) utilizing the point-to-line modulus of continuity from Proposition 3.3.3-(point-to-line). For part (c), we rely on localization results from [89]. Indeed, by Theorem 1.5 in [89], we know the quenched density of $V(\varepsilon^{-1})$ (recall $V \sim \mathsf{CDRP}(0,0;*,\varepsilon^{-1})$) has a unique mode $\mathcal{M}_{*,\varepsilon^{-1}}$ almost surely. By the same argument as in the proof of Theorem 3.1.7 (b), the point-to-line version of Corollary 7.3 in [89] leads to the fact that $\varepsilon^{-2/3}[L_*^{(\varepsilon)}(1) - \mathcal{M}_{*,\varepsilon^{-1}}] \xrightarrow{p} 0$. Finally from Theorem 1.8 in [89] we have $\varepsilon^{-2/3}\mathcal{M}_{*,\varepsilon^{-1}} \xrightarrow{d} 2^{1/3}\mathcal{M}$. This establishes (c). □

3.4.2 Proof of Theorem 3.1.10 modulo Conjecture 3.1.9

In this section we prove Theorem 3.1.10 assuming Conjecture 3.1.9. The proof also relies on a technical result which we first state below.

Lemma 3.4.3 (Deterministic convergence)**.** *Let $f(x) : \mathbb{R}^k \to \mathbb{R}$ be a continuous function with a unique maximizer $\vec{a} \in \mathbb{R}^k$ and $f_\varepsilon(x) : \mathbb{R}^k \to \mathbb{R}$ be a sequence of continuous functions that converges to $f(x)$ uniformly over compact subsets. Fix any $\delta > 0$ and take $M > 0$ so that $(a_i - \delta, a_i + \delta) \in [-M, M]$ for all i. For $x \in \mathbb{R}$, set*

$$g_\varepsilon(x) := \frac{\exp(\varepsilon^{-\frac{1}{3}} f_\varepsilon(\vec{x}))}{\int_{[-M,M]^k} \exp(\varepsilon^{-\frac{1}{3}} f_\varepsilon(\vec{y}))\mathrm{d}\vec{y}}.$$

For all $\vec{b} \in [-M, M]^k$, we have:

$$\limsup_{\varepsilon\downarrow 0}\int_{-M}^{b_1}\cdots\int_{-M}^{b_k} g_\varepsilon(\vec{x})\mathrm{d}\vec{x} \le \prod_{i=1}^{k}\mathbf{1}\{a_i \le b_i+\delta\}, \tag{3.4.9}$$

$$\liminf_{\varepsilon\downarrow 0}\int_{-M}^{b_1}\cdots\int_{-M}^{b_k} g_\varepsilon(\vec{x})\mathrm{d}\vec{x} \ge \prod_{i=1}^{k}\mathbf{1}\{a_i \le b_i-\delta\}. \tag{3.4.10}$$

Proof of this lemma follows via standard real analysis and hence we defer its proof to the end of this section. We now proceed to prove Theorem 3.1.10 assuming the above lemma.

Proof of Theorem 3.1.10. For clarity we split the proof into three steps.

Step 1. Fix $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$. For convenience set $\Gamma_{t_i} := \Gamma(t_i\sqrt{2})$ where $\Gamma(\cdot)$ is the geodesic of directed landscape from $(0,0)$ to $(0,\sqrt{2})$. Consider any $\vec{a} = (a_1,\ldots,a_k) \in \mathbb{R}^k$, which is a continuity point for the CDF of $(\Gamma_{t_i})_{i=1}^k$. For any $M \ge \sup_i |a_i| + 1$, define

$$V_{\vec{a}}(M) := [-M, a_i] \times \cdots \times [-M, a_k] \subset \mathbb{R}^k. \tag{3.4.11}$$

To show convergence in finite-dimensional distribution, it suffices to prove that as $\varepsilon \downarrow 0$

$$\mathbf{P}\left(\bigcap_{i=1}^{k}\{L_{t_i}^{(\varepsilon)} \le a_i\}\right) \to \mathbf{P}\left(\bigcap_{i=1}^{k}\{\Gamma_{t_i} \le a_i\}\right). \tag{3.4.12}$$

From Definition 3.1.1 and using the long-time scaling from (3.2.2), we obtain that the joint density of $(L_{t_1}^{(\varepsilon)}, L_{t_2}^{(\varepsilon)}, \ldots, L_{t_k}^{(\varepsilon)})$ at $(x_i)_{i=1}^k$ is given by

$$\frac{g_{\vec{t};\varepsilon}(\vec{x})}{\int_{\mathbb{R}^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}, \qquad g_{\vec{t};\varepsilon}(\vec{x}) := \exp(\varepsilon^{-1/3}U_{\vec{t};\varepsilon}(\vec{x}))$$

where

$$U_{\vec{t};\varepsilon}(\vec{x}) := \sum_{i=1}^{k+1}(t_i - t_{i-1})^{1/3}\mathfrak{h}_{\varepsilon^{-1}t_{i-1},\varepsilon^{-1}t_i}((t_i - t_{i-1})^{-2/3}x_{i-1}, (t_i - t_{i-1})^{-2/3}x_i) \tag{3.4.13}$$

Here $x_0 = x_{k+1} = 1$.

In this step, we reduce our computation to understanding the integral behavior of $g_{\vec{t};\varepsilon}$ on a compact set. More precisely, the goal of this step is to show there there exists a constant $\mathrm{C} > 0$ such that for all M large enough

$$\left|\mathbf{P}\left(\bigcap_{i=1}^{k}\{L_{t_i}^{(\varepsilon)} \le a_i\}\right) - \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right]\right| \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}M^2\right) \tag{3.4.14}$$

where $V_{\vec{a}}(M)$ is defined in (3.4.11). We proceed to prove (3.4.14) by demonstrating appropriate lower and upper bounds. For upper bound observe that by union bound we have

$$\begin{aligned}
\mathbf{P}\left(\bigcap_{i=1}^{k}\{L_{t_i}^{(\varepsilon)} \le a_i\}\right) &\le \mathbf{P}\left(\bigcap_{i=1}^{k}\{L_{t_i}^{(\varepsilon)} \in [-M, a_i]\}\right) + \mathbf{P}\left(\sup_{t\in[0,1]} |L_t^{(\varepsilon)}| \ge M\right) \\
&\le \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right] + \mathbf{P}\left(\sup_{t\in[0,1]} |L_t^{(\varepsilon)}| \ge M\right) \\
&\le \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right] + \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}M^2\right) \qquad (3.4.15)
\end{aligned}$$

where the last inequality follows from Corollary 3.3.5 for some constant $\mathrm{C} > 0$. For the lower bound we have

$$\begin{aligned}
\mathbf{P}\left(\bigcap_{i=1}^{k}\{L_{t_i}^{(\varepsilon)} \le a_i\}\right) &\ge \mathbf{P}\left(\bigcap_{i=1}^{k}\{L_{t_i}^{(\varepsilon)} \in [-M, a_i]\}\right) \\
&= \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}} \cdot \frac{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{\mathbb{R}^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right] \\
&\ge \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}} \cdot \mathbf{P}^{\xi}\left(\sup_{t\in[0,1]} |L_t^{(\varepsilon)}| \le M\right)\right]. \qquad (3.4.16)
\end{aligned}$$

By Corollary 3.3.5 we see that there exist two constants $\mathrm{C}_1, \mathrm{C}_2 > 0$ such that with probability at least $1 - \mathrm{C}_2\exp(-\frac{1}{\mathrm{C}_2}M^3)$, the random variable $\mathbf{P}^{\xi}\left(\sup_{t\in[0,1]} |L_t^{(\varepsilon)}| \le M\right)$ is at least $1 -$

$C_1 \exp(-\frac{1}{C_1}M^2)$. Thus,

$$\begin{aligned}\text{r.h.s. of (6.3.33)} &\geq \left[1 - C_2 \exp\left(-\tfrac{1}{C_2}M^3\right)\right] \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}} \cdot \left[1 - C_1 \exp\left(-\tfrac{1}{C_1}M^2\right)\right]\right] \\ &\geq \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right] - C_1 \exp\left(-\tfrac{1}{C_1}M^2\right). \end{aligned} \tag{3.4.17}$$

In view of (3.4.15) and (3.4.17), we thus arrive at (3.4.14) by adjusting the constants. This completes our work for this step.

Step 2. In this step, we discuss how directed landscape and hence the geodesic appear in the limit. Recall the random function $U_{\vec{t};\varepsilon}(\vec{x})$ from (3.4.13). We exploit Conjecture 3.1.9, to show that as $\varepsilon \downarrow 0$, as $\mathbb{R}^k$-valued processes we have the following convergence in law

$$U_{\vec{t};\varepsilon}(\vec{x}) \xrightarrow{d} \mathbf{U}_{\vec{t}}(\vec{x}) := 2^{-\frac{1}{3}} \sum_{i=1}^{k+1} \mathcal{L}(x_{i-1}, t_{i-1}\sqrt{2}; x_i, t_i\sqrt{2}) \tag{3.4.18}$$

in the uniform-on-compact topology. Here $\mathcal{L}(x, s; y, t)$ denotes the directed landscape. Note that by Definition 3.1.6, $(\Gamma_{t_i})_{i=1}^k$ is precisely the almost sure unique k-point maximizer of $\mathbf{f}_{\vec{t}}(\vec{x})$.

To show (3.4.18), we rely on Conjecture 3.1.9 heavily. Indeed, assuming Conjecture 3.1.9, for each i, as $\varepsilon \downarrow 0$ we have

$$\begin{aligned}&\mathfrak{h}_{\varepsilon^{-1}t_{i-1},\varepsilon^{-1}t_i}((t_i - t_{i-1})^{-2/3}x, (t_i - t_{i-1})^{-2/3}y) \\ &\qquad \xrightarrow{d} 2^{-1/3}\mathcal{S}^{(i)}\big(2^{-1/3}(t_i - t_{i-1})^{-2/3}x, 2^{-1/3}(t_i - t_{i-1})^{-2/3}y\big)\end{aligned}$$

where the convergence holds under the uniform-on-compact topology. Here $\mathcal{S}^{(i)}$ are independent Airy sheets as $\mathfrak{h}_{\varepsilon^{-1}t_{i-1},\varepsilon^{-1}t_i}(\cdot,\cdot)$ are independent. Now by the definition of directed landscape we have

$$2^{-\frac{1}{3}} \sum_{i=1}^{k+1} \mathcal{L}(x_{i-1}, t_{i-1}\sqrt{2}; x_i, t_i\sqrt{2})$$

$$\stackrel{d}{=} 2^{-\frac{1}{3}} \sum_{i=1}^{k+1} (t_{i+1}-t_i)^{1/3} \mathcal{S}^{(i)} \big(2^{-1/3}(t_i-t_{i-1})^{-2/3} x_{i-1}, 2^{-1/3}(t_i-t_{i-1})^{-2/3} x_i\big)$$

with $x_0 = x_{k+1} = 1$. Here the equality in distribution holds as $\mathbb{R}^k$-valued processes in $\vec{x}$. This allow us to conclude the desired convergence for $U_{\vec{t};\varepsilon}(\vec{x})$ in (3.4.18), completing our work for this step.

Step 3. In this step, we complete the proof of (3.4.12) utilizing (3.4.14) and the weak convergence in (3.4.18). Using Skorokhod's representation theorem, given any fixed M, we may assume that we are working on a probability space where

$$\mathbf{P}(\mathsf{A}) = 1, \quad \text{for } \mathsf{A} := \left\{ \sup_{\vec{x}\in[-M,M]^k} \left| U_{\vec{t};\varepsilon}(\vec{x}) - \mathbf{U}_{\vec{t}}(\vec{x}) \right| \to 0 \right\}.$$

Let us define

$$(\Gamma_{t_i}(M))_{i=1}^k := \operatorname*{argmax}_{\vec{x}\in[-M,M]^k} \mathbf{f}_{\vec{t}}(\vec{x}),$$

where in case there are multiple maximizers we take the one whose sum of coordinates is the largest. We next define

$$\mathsf{B} := \left\{ \operatorname*{argmax}_{\vec{x}\in[-M,M]^k} \mathbf{U}_{\vec{t}}(\vec{x}) \text{ exists uniquely and } (\Gamma_{t_i}(M))_{i=1}^k \in [-\tfrac{M}{2}, \tfrac{M}{2}]^k \right\}.$$

Fix any $\delta \in (0, \frac{M}{2})$. By Lemma 3.4.3 we have

$$\begin{aligned}
\limsup_{\varepsilon\downarrow 0} \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}} \right] &\le \mathbf{P}(\neg\mathsf{B}) + \mathbf{E}\left[\limsup_{\varepsilon\downarrow 0} \frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}} \mathbf{1}\{\mathsf{A}\cap\mathsf{B}\} \right] \\
&\le \mathbf{P}(\neg\mathsf{B}) + \mathbf{P}\left(\bigcap_{i=1}^k \{\Gamma_{t_i}(M) \le a_i + \delta\} \right) \\
&\le \mathbf{P}(\neg\mathsf{B}) + \mathbf{P}\left(\bigcap_{i=1}^k \{\Gamma_{t_i} \le a_i + \delta\} \right) + \mathbf{P}\left(\sup_{t\in[0,1]} |\Gamma_t| \ge M \right),
\end{aligned} \tag{3.4.19}$$

where the last inequality follows by observing that $\Gamma_{t_i}(M) = \Gamma_{t_i}$ for all i, whenever $\sup_{t\in[0,1]} |\Gamma_t| \le$

M (and the fact that $\Gamma(\cdot)$ exists uniquely almost surely via Theorem 12.1 in [94]). In the same manner we have

$$\liminf_{\varepsilon\downarrow 0} \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right] \ge \mathbf{E}\left[\liminf_{\varepsilon\downarrow 0} \frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}} \mathbf{1}\{\mathsf{A}\cap\mathsf{B}\}\right]$$
$$\ge \mathbf{P}\left(\bigcap_{i=1}^{k}\{\Gamma_{t_i}(M) \le a_i - \delta\}, \mathsf{A}\cap\mathsf{B}\right)$$
$$\ge \mathbf{P}\left(\bigcap_{i=1}^{k}\{\Gamma_{t_i} \le a_i - \delta\}\right) - \mathbf{P}(\neg\mathsf{B}) - \mathbf{P}\left(\sup_{t\in[0,1]} |\Gamma_t| \ge M\right). \tag{3.4.20}$$

By Proposition 12.3 in [94],

$$\mathbf{P}(\neg\mathsf{B}) \le \mathbf{P}\left(\sup_{t\in[0,1]} |\Gamma_t| \ge M\right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}M^3\right).$$

Thus taking $M \uparrow \infty$, followed by $\delta \downarrow 0$, and using the fact that $\vec{a}$ is a continuity point of the density on both sides of (3.4.19) and (3.4.20) we have

$$\lim_{M\to\infty} \limsup_{\varepsilon\downarrow 0} \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right] = \lim_{M\to\infty} \liminf_{\varepsilon\downarrow 0} \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right] = \mathbf{P}\left(\bigcap_{i=1}^{k}\{\Gamma_{t_i} \le a_i\}\right)$$

Combining this with (3.4.14) we thus arrive at (3.4.12). This completes the proof. ☐

Proof of Lemma 3.4.3. We begin by proving (3.4.9). When $a_i \le b_i + \delta$ for all i, the r.h.s of (3.4.9) is 1 whereas the l.h.s of (3.4.9) is always less than 1. Thus we focus on when $a_j > b_j + \delta$ for some j. In that case $\vec{a} \notin [-M, b_1] \times \cdots \times [-M, b_k]$. As $\vec{a}$ is the unique maximizer of the continuous function $f(\vec{x})$, there exists $\eta > 0$ such that

$$\sup_{y_i\in[-M,b_i], i=1,2,\ldots,k} f(\vec{y}) < f(\vec{a}) - \eta.$$

By uniform convergence over compacts, we can get ε_0 such that

$$\sup_{\varepsilon\le\varepsilon_0}\sup_{\vec{x}\in[-M,M]^k}|f_\varepsilon(\vec{x})-f(\vec{x})|<\tfrac{1}{4}\eta.$$

By continuity of f at $\vec{a}$, we can get $\delta_0<\delta$ such that for all $0\le\rho\le\delta$ we have

$$\sup_{x_i\in[a_i-\rho,a_i+\rho],i=1,\ldots,k}|f(\vec{x})-f(\vec{a})|<\tfrac{1}{4}\eta.$$

Thus for all $\varepsilon\le\varepsilon_0$ and $0\le\rho\le\delta_0$ we have $f_\varepsilon(\vec{x})\ge f(\vec{a})-\frac{1}{2}\eta$ for all $\vec{x}$ with $x_i\in[a_i-\rho,a_i+\rho]$. And for all $\varepsilon\le\varepsilon_0$, $f_\varepsilon(\vec{y})<f(\vec{a})-\frac{3}{4}\eta$ for all $\vec{y}$ with $y_i\in[-M,b_i]$. Thus in conclusion

$$\int_{-M}^{b_1}\cdots\int_{-M}^{b_k}\exp(\varepsilon^{-\frac{1}{3}}f_\varepsilon(\vec{x}))\mathrm{d}\vec{x}\le(2M)^k\exp(\varepsilon^{-1/3}[f(\vec{a})-\tfrac{3}{4}\eta)])$$

and

$$\int_{[-M,M]^k}\exp(\varepsilon^{-\frac{1}{3}}f_\varepsilon(\vec{x}))\mathrm{d}\vec{x}\ge\int_{a_1-\delta_0}^{a_1+\delta_0}\cdots\int_{a_k-\delta_0}^{a_k+\delta_0}\exp(\varepsilon^{-\frac{1}{3}}f_\varepsilon(\vec{x}))\mathrm{d}\vec{x}\ge(2\delta_0)^k\exp(\varepsilon^{-1/3}[f(\vec{a})-\tfrac{1}{2}\eta)]).$$

Combining the above two bounds we have

$$\int_{-M}^{b_1}\cdots\int_{-M}^{b_k}g_\varepsilon(\vec{x})\mathrm{d}\vec{x}\le(\tfrac{M}{\delta_0})^k\exp(-\tfrac{1}{4}\varepsilon^{-1/3}\eta),$$

which goes to zero as $\varepsilon\downarrow 0$. Thus, we conclude the proof of (3.4.9). The proof of (3.4.10) follows analogously. □

3.5 Appendix: Proof of Lemma 3.2.6

In this section, we prove Lemma 3.2.6. The idea is to view short-time scaled KPZ equation $\mathfrak{g}_t(\cdot)$ defined in (3.2.2) as the lowest index curve of an appropriate line ensemble and use certain stochastic monotonicity properties of the same. To make our exposition self-contained, below we briefly introduce the line ensemble machinery.

Fix $t > 0$ throughout this section and consider the convex function

$$\mathbf{G}_t(x) = (\pi t/4)^{1/2} e^{(\pi t/4)^{1/4} x}.$$

Recall the general notion of line ensembles from Section 2 in [73]. Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \ldots)$ be an $\mathbb{N} \times \mathbb{R}$ indexed line ensemble. Fix $k_1 \leq k_2$ with $k_1, k_2 \in \mathbb{N}$ and an interval $(a, b) \in \mathbb{R}$ and two vectors $\vec{x}, \vec{y} \in \mathbb{R}^{k_2 - k_1 + 1}$. Let $\mathbb{P}_{\text{free}}^{k_1, k_2, (a,b), \vec{x}, \vec{y}}$ denote the law of $k_2 - k_1 + 1$ many independent Brownian bridges taking values $\vec{x}$ at time a and $\vec{y}$ at time b. Given two measurable functions $f, g : (a, b) \to \mathbb{R} \cup \{\pm\infty\}$, the law $\mathbb{P}_{\mathbf{G}_t}^{k_1, k_2, (a,b), \vec{x}, \vec{y}, f, g}$ on $\mathcal{L}_{k_1}, \ldots, \mathcal{L}_{k_2} : (a, b) \to \mathbb{R}$ has the following Radon-Nikodym derivative w.r.t. $\mathbb{P}_{\text{free}}^{k_1, k_2, (a,b), \vec{x}, \vec{y}}$:

$$\frac{d\mathbb{P}_{\mathbf{G}_t}^{k_1, k_2, (a,b), \vec{x}, \vec{y}, f, g}}{d\mathbb{P}_{\text{free}}^{k_1, k_2, (a,b), \vec{x}, \vec{y}}}(\mathcal{L}_{k_1}, \ldots, \mathcal{L}_{k_2}) = \frac{\exp\left\{ - \sum_{i=k_1}^{k_2+1} \int \mathbf{G}_t\big(\mathcal{L}_i(x) - \mathcal{L}_{i-1}(x)\big) dx \right\}}{Z_{\mathbf{G}_t}^{k_1, k_2, (a,b), \vec{x}, \vec{y}, f, g}}, \tag{3.5.1}$$

where $\mathcal{L}_{k_1 - 1} = f$, or ∞ if $k_1 = 1$; and $\mathcal{L}_{k_2+1} = g$. Here $Z_{\mathbf{G}_t}^{k_1, k_2, (a,b), \vec{x}, \vec{y}, f, g}$ is the normalizing constant which produces a probability measure. We say $\mathcal{L}$ enjoys the $\mathbf{G}_t$*-Brownian Gibbs property* if, for all $K = \{k_1, \ldots, k_2\} \subset \mathbb{N}$ and $(a, b) \subset \mathbb{R}$, the following distributional equality holds:

$$\operatorname{Law}\Big(\mathcal{L}_{K \times (a,b)} \text{ conditioned on } \mathcal{L}_{\mathbb{N} \times \mathbb{R} \setminus K \times (a,b)}\Big) = \mathbb{P}_{\mathbf{G}_t}^{k_1, k_2, (a,b), \vec{x}, \vec{y}, f, g}, \tag{3.5.2}$$

where $\vec{x} = (\mathcal{L}_{k_1}(a), \ldots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \ldots, \mathcal{L}_{k_2}(b))$, and where again $\mathcal{L}_{k_1 - 1} = f$, or ∞ if $k_1 = 1$; and $\mathcal{L}_{k_2+1} = g$.

Similar to the Markov property, a *strong* version of the $\mathbf{G}_t$-Brownian Gibbs property that is valid with respect to *stopping domains* exists. A pair $(\mathfrak{a}, \mathfrak{b})$ of random variables is called a K-stopping domain if $\big\{\mathfrak{a} \leq a, \mathfrak{b} \geq b\big\} \in \mathfrak{F}_{\text{ext}}\big(K \times (a, b)\big)$, the σ-field generated by $\mathcal{L}_{(\mathbb{N} \times \mathbb{R}) \setminus (K \times (a,b))}$. $\mathcal{L}$ satisfies the strong $\mathbf{G}_t$-Brownian Gibbs property if for all $K = \{k_1, \ldots, k_2\} \subset \mathbb{N}$ and K-stopping domain if $(\mathfrak{a}, \mathfrak{b})$, the conditional distribution of $\mathcal{L}_{K \times (\mathfrak{a}, \mathfrak{b})}$ given $\mathfrak{F}_{\text{ext}}\big(K \times (\mathfrak{a}, \mathfrak{b})\big)$ is $\mathbb{P}_{\mathbf{G}_t}^{k_1, k_2, (l,r), \vec{x}, \vec{y}, f, g}$, where $\ell = \mathfrak{a}$, $r = \mathfrak{b}$, $\vec{x} = (\mathcal{L}_i(\mathfrak{a}))_{i \in K}$, $\vec{y} = (\mathcal{L}_i(\mathfrak{b}))_{i \in K}$, and where again $\mathcal{L}_{k_1 - 1} = f$, or ∞ if $k_1 = 1$;

and $\mathcal{L}_{k_2+1} = g$.

The following lemma shows how the short-time scaled KPZ process $\mathfrak{g}_t(\cdot)$ fits into a line ensemble satisfying the $\mathbf{G}_t$-Brownian Gibbs property.

Lemma 3.5.1 (Lemma 2.5 in [87] and Lemma 2.5 of [74])**.** *For each $t > 0$, there exists an $\mathbb{N}\times\mathbb{R}$-indexed line ensemble $\{\mathfrak{g}_t^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$ satisfying the $\mathbf{G}_t$-Brownian Gibbs property and the lowest indexed curve $\mathfrak{g}_t^{(1)}(x)$ is equal in distribution (as a process in x) to $\mathfrak{g}_t(x)$ defined in* (3.2.2)*. Furthermore, the line ensemble $\{\mathfrak{g}_t^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$ satisfies the strong $\mathbf{G}_t$-Brownian Gibbs property.*

Before beginning the proof of Lemma 3.2.6 we recall one more property of line ensembles, i.e. the stochastic monotonicity, which is indispensable to the study of monotone events in Lemma 3.2.6.

Lemma 3.5.2 (Lemmas 2.6 and 2.7 of [74])**.** *Fix a finite interval $(a,b)\subset\mathbb{R}$ and $x,y\in\mathbb{R}$. For $i\in\{1,2\}$, fix measurable functions $g_i:(a,b)\to\mathbb{R}\cup\{-\infty\}$ such that $g_2(s)\le g_1(s)$ for $s\in(a,b)$. For each $v\in\{1,2\}$, let $\mathbf{P}_v$ denote the law $\mathbb{P}_{\mathbf{G}_t}^{1,1,(a,b),x,y,+\infty,g_v}$, so that a $\mathbf{P}_v$-distributed random variable $\mathcal{R}_i=\{\mathcal{R}_v(s)\}_{s\in(a,b)}$ is a random function on $[a,b]$ with endpoints x and y. Then a common probability space may be constructed on which the two measures are supported such that, almost surely, $\mathcal{R}_1(s)\ge\mathcal{R}_2(s)$ for all $s\in(a,b)$.*

Proof of Lemma 3.2.6. Fix an interval $[a,b]$ and a corresponding monotone set $A\in\mathcal{B}(C([a,b]))$. By Lemma 3.5.1 and tower property of expectation we may write

$$\begin{aligned}\mathbf{P}\left[\mathfrak{g}_t(\cdot)\mid_{[a,b]}\in A\mid(\mathfrak{g}_t(x))_{x\notin(a,b)}\right] &= \mathbf{E}^{(\ge 2)}\left[\mathbf{P}\left[\mathfrak{g}_t^{(1)}(\cdot)\mid_{[a,b]}\in A\mid(\mathfrak{g}_t^{(n)}(\cdot))_{n\ge 2},(\mathfrak{g}_t^{(1)}(x))_{x\notin(a,b)}\right]\right]\\ &= \mathbf{E}^{(\ge 2)}\left[\mathbb{P}_{\mathbf{G}_t}^{1,1,(a,b),\mathfrak{g}_t^{(1)}(a),\mathfrak{g}_t^{(1)}(b),+\infty,\mathfrak{g}_t^{(2)}(\cdot)}\left(\mathfrak{g}_t^{(1)}(\cdot)\mid_{[a,b]}\in A\right)\right]\end{aligned} \tag{3.5.3}$$

where the last equality follows from (3.5.2). Here $\mathbf{E}^{(\ge 2)}$ denotes the expectation operator taken over all lower curves $\{\mathfrak{g}_t^{(n)}(\cdot)\}_{n\ge 2}$. Now by Lemma 3.5.2, decreasing $\mathfrak{g}_t^{(2)}(\cdot)$ pointwise on $[a,b]$ reduces the value of $\mathfrak{g}_t^{(1)}(\cdot)$ pointwise stochastically. But by the definition of monotone set A (see

(3.2.4)), we know decreasing $\mathfrak{g}_t^{(1)}(\cdot)\mid_{[a,b]}$ stochastically pointwise and keeping the endpoint fixed, only increases the conditional probability appearing above. Thus, we may drop $\mathfrak{g}_t^{(2)}(\cdot)$ all the way to $-\infty$, to obtain

$$\text{r.h.s. of (4.6.2)} \le \mathbf{E}^{(\ge 2)}\left[\mathbb{P}_{\mathbf{G}_t}^{1,1,(a,b),\mathfrak{g}_t^{(1)}(a),\mathfrak{g}_t^{(1)}(b),+\infty,-\infty}\left(\mathfrak{g}_t^{(1)}(\cdot)\mid_{[a,b]}\in A\right)\right]. \tag{3.5.4}$$

Under the above situation the Radon-Nikodym derivative appearing in (3.5.1) becomes constant, and thus

$$\mathbb{P}_{\mathbf{G}_t}^{1,1,(a,b),\mathfrak{g}_t^{(1)}(a),\mathfrak{g}_t^{(1)}(b),+\infty,-\infty}[\cdot] = \mathbb{P}_{\text{free}}^{1,1,(a,b),\mathfrak{g}_t^{(1)}(a),\mathfrak{g}_t^{(1)}(b)}[\cdot]\,.$$

The measure on the right side above is a single Brownian bridge measure on $[a,b]$ starting at $\mathfrak{g}_t^{(1)}(a)$ and ending at $\mathfrak{g}_t^{(1)}(b)$ and hence free of $\{\mathfrak{g}_t^{(n)}(\cdot)\}_{n\ge 2}$. Thus r.h.s. of (3.5.4) can be viewed as $\mathbf{P}_{\text{free}}^{(a,b),(\mathfrak{g}_t^{(1)}(a),\mathfrak{g}_t^{(1)}(b))}(A)$. This establishes (3.2.5). The case when $[a,b]$ is a stopping domain follows from the same calculation and the fact that $\{\mathfrak{g}_t^{(n)}(\cdot)\}_{n\ge 1}$ satisfies the strong $\mathbf{G}_t$-Brownian Gibbs property via Lemma 3.5.1. □

Chapter 4: The half-space log-gamma polymer in the bound phase

4.1 Introduction

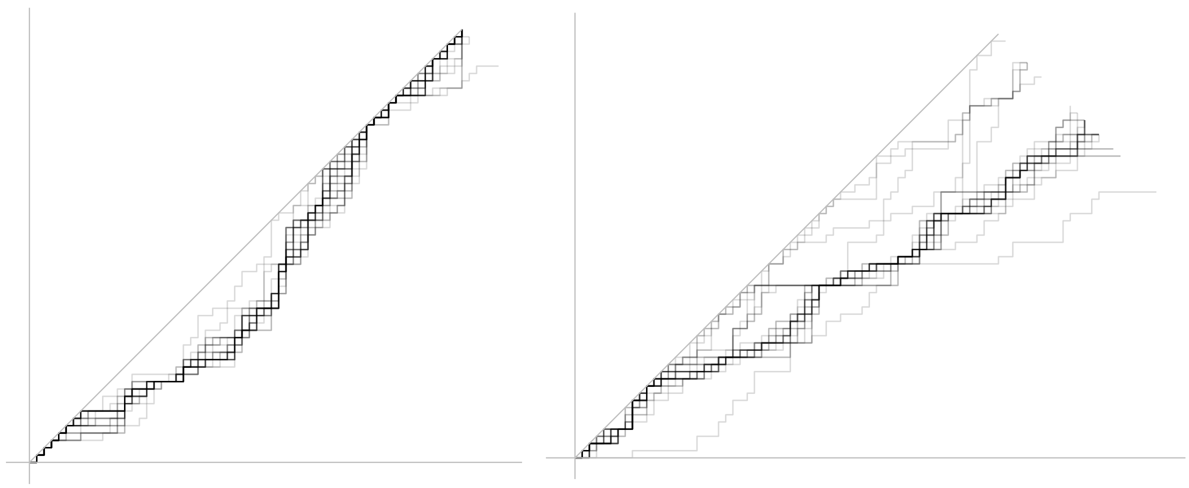


Figure 4.1: The bound and the unbound phase.

Directed polymers in random environments, first appeared in [149, 152, 44], are a rich class of mathematical physics models that have been extensively studied over the last several decades (see books [226, 129, 100, 65] and the references therein). More recently, a particular variant of the polymer models, the half-space polymers, has garnered considerable attention. The structure of the half-space polymers resembles the behavior of an interface in the presence of an attractive wall and their understanding renders importance to the studies of the wetting phenomena ([1, 198, 51]). Depending on the attraction force of the wall, it was conjectured in [158] that these models exhibit a "depinning" phase transition. When the attraction force exceeds a certain critical threshold (colloquially known as the bound phase), [158] conjectured that the endpoint of the polymer stays within a $O(1)$ window around the wall, i.e., it gets pinned to the wall. In this paper, we focus on the half-space polymers with log-gamma weights which make the model integrable and resolve Kardar's $O(1)$ conjecture in the bound phase. Our work is the first rigorous instance that positively solves Kardar's $O(1)$ conjecture.

Presently, we begin with an introduction to the model and the statements of our main results.

4.1.1 The model and the main results

Fix any $\theta > 0$ and $\alpha > -\theta$ and define the half-space index set: $\mathcal{I}^- = \{(i,j) \in \mathbb{Z}^2_{>0} \mid j \le i\}$. We consider a family of independent variables $(W_{i,j})_{(i,j)\in\mathcal{I}^-}$:

$$W_{i,i} \sim \text{Gamma}^{-1}(\alpha+\theta) \qquad W_{i,j} \sim \text{Gamma}^{-1}(2\theta) \;\text{ for } i < j, \tag{4.1.1}$$

where Gamma(β) denotes a random variable with density $\mathbf{1}\{x > 0\}[\Gamma(\beta)]^{-1}x^{\beta-1}e^{-x}$. Let Π_N^{half} be the set of all upright lattice paths of length $2N-2$ starting from $(1,1)$ that are confined to the half-space $\mathcal{I}^-$ (see Figure 4.2). Given the weights in (4.1.1), the half-space log-gamma ($\mathcal{HSLG}$) polymer is a random measure on Π_N^{half} defined as

$$\mathbf{P}^W(\pi) = \frac{1}{Z(N)} \prod_{(i,j)\in\pi} W_{i,j} \cdot \mathbf{1}_{\pi\in\Pi_N^{\text{half}}}, \tag{4.1.2}$$

where $Z(N)$ is the normalizing constant.

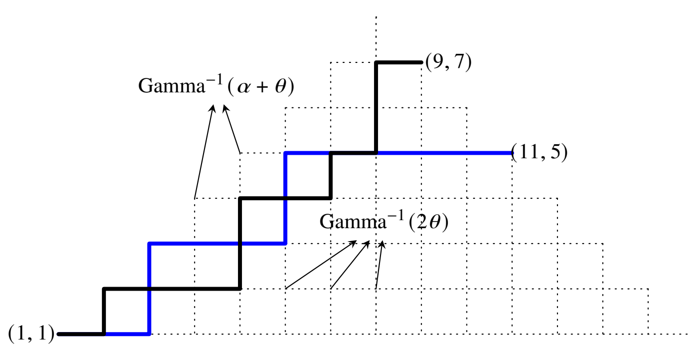


Figure 4.2: Two possible paths of length 14 in Π_8^{half} are shown in the figure.

The parameter α controls the strength of the boundary weights, i.e. the attractiveness of the wall, and a "depinning" phase transition occurs when $\alpha = 0$ (see [158, 204, 33]). When $\alpha \ge 0$, [27, 22] showed that the polymer measure is unpinned and the endpoint lies in a $O(N^{2/3})$ window.

For $\alpha < 0$, the conjecture is that the attraction is strong enough so that the polymer measure is pinned to the diagonal (see Figure 4.1). Indeed, our first main result below confirms that in the bound phase, i.e., when $\alpha \in (-\theta, 0)$, the endpoint of the $\mathcal{HSLG}$ polymer is within $O(1)$ window of the diagonal and is the first such result to capture the “pinning" phenomenon of the half-space polymer measure to the diagonal.

Theorem 4.1.1 (Bounded endpoint)**.** *Fix $\theta > 0$ and $\alpha \in (-\theta, 0)$ and consider the random measure $\mathbf{P}^W$ from* (4.1.2)*. For a path $\pi \in \Pi_N^{\text{half}}$, we denote $\pi(2N-2)$ as the height (i.e., y-coordinate) of the endpoint of the polymer. We have*

$$\limsup_{k\to\infty} \limsup_{N\to\infty} \mathbf{P}^W(\pi(2N-2) \le N-k) = 0, \quad \textit{in probability}. \tag{4.1.3}$$

Theorem 4.1.1 is a quenched result and naturally implies its annealed version. Following the above theorem, our next point of inquiry is the limiting behavior of the quenched distribution of the endpoints around the diagonal. We introduce and clarify a few more notations below before stating our results in this direction. Let $\Pi_{m,n}^{\text{half}}$ is the set of all upright lattice paths starting from $(1,1)$ and ending at (m,n) that reside solely in the half-space $\mathcal{I}^-$. We define the *point-to-point* partition function as

$$Z(m,n) := \sum_{\pi\in\Pi_{m,n}^{\text{half}}} \prod_{(i,j)\in\pi} W_{i,j}. \tag{4.1.4}$$

Under the above definition, the normalizing constant $Z(N)$ in (4.1.2), can also be viewed as the *point-to-line* partition function, i.e.

$$Z(N) = \sum_{p=0}^{N-1} Z(N+p, N-p).$$

The natural logarithm of the partition function is termed as the free energy of the polymer. The aforementioned depinning phase transition can be observed by studying the fluctuations of the free energy of the polymer. In this context, [27] obtained precise one-point fluctuations for the point-to-line free energy $\log Z(N)$ in both the bound and unbound phases and observed the BBP phase transition. A similar fluctuation result and Baik-Rains phase transition were later shown in [151] for the point-to-point free energy $\log Z(N,N)$ on the diagonal. For $\alpha \geq 0$, it was recently proven in [22] that the point-to-point free energy process

$$\left(\log Z(N + pN^{2/3}, N - pN^{2/3})\right)_{p\in[0,r]}$$

after appropriate centering and scaling by $N^{1/3}$ is functionally tight. This result captures the characteristic KPZ $1/3$ fluctuation and $2/3$ transversal scaling exponents. In our present work, we study the point-to-point free energy process under $\alpha < 0$ case. Our second main result below obtains precise fluctuations for the increments of the point-to-point free energy process when $\alpha < 0$. To state the result, we introduce the definition of the *log-gamma random walk*.

Definition 4.1.2. Fix $\theta > 0$ and $\alpha \in (-\theta, 0]$. Let $Y_1 \sim \text{Gamma}(\theta+\alpha)$ and $Y_2 \sim \text{Gamma}(\theta-\alpha)$ be independent random variables. We refer to $X := \log Y_2 - \log Y_1$ as a log-gamma random variable. It has a density given by

$$p(x) := \frac{1}{\Gamma(\theta+\alpha)\Gamma(\theta-\alpha)} \int_{\mathbb{R}} \exp\left((\theta-\alpha)y - e^y + (\theta+\alpha)(y-x) - e^{y-x}\right) dy. \tag{4.1.5}$$

Let $(X_i)_{i\geq 0}$ be a sequence of such iid log-gamma random variables. Set $S_0 = 0$ and $S_k = \sum_{i=1}^k X_i$. We refer to $(S_k)_{k\geq 0}$ as a *log-gamma random walk.*

Our next result states that in the bound phase, the above random walk is an *attractor* for the increments of the half-space log-partition function.

Theorem 4.1.3. *Fix $\theta > 0$ and $\alpha \in (-\theta, 0)$. For each $k \geq 1$, as $N \to \infty$, we have the following multi-point convergence in distribution*

$$\left(\frac{Z(N+r, N-r)}{Z(N,N)}\right)_{r\in[\![0,k]\!]} \xrightarrow{d} \left(e^{-S_r}\right)_{r\in[\![0,k]\!]}, \tag{4.1.6}$$

where $(S_r)_{r\geq 0}$ is a log-gamma random walk from Definition 4.1.2.

From the above result, we deduce the following limiting quenched distribution of the endpoint when viewed around the diagonal.

Theorem 4.1.4. *Fix $\theta > 0$ and $\alpha \in (-\theta, 0)$ and consider the random measure $\mathbf{P}^W$ from* (4.1.2). *Let $(S_k)_{k\geq 0}$ be a log-gamma random walk from Definition 4.1.2. Set $Q := \sum_{p\geq 0} e^{-S_p}$. For a path $\pi \in \Pi_N^{\mathrm{half}}$, we denote $\pi(2N-2)$ as the height (i.e., y-coordinate) of the endpoint of the polymer. Then for each $k \geq 1$, as $N \to \infty$, we have the following multi-point convergence in distribution*

$$\Big(\mathbf{P}^W(\pi(2N-2) = N - r)\Big)_{r\in[\![0,k]\!]} \xrightarrow{d} \Big(Q^{-1} \cdot e^{-S_r}\Big)_{r\in[\![0,k]\!]}. \tag{4.1.7}$$

Beyond proving the $O(1)$ transversal fluctuation around the point (N, N) and pinning down the exact density within this region, our main theorems above also shed light on the attractive properties of half-space log-gamma stationary measures. In [24] a stationary version of the half-space log-gamma polymer was considered for $\alpha \in (-\theta, \theta)$, where the horizontal weights along the first row are assumed to be distributed as $\mathrm{Gamma}^{-1}(\theta - \alpha)$ (i.e., $W_{i,1} \sim \mathrm{Gamma}^{-1}(\theta - \alpha)$). Let us denote $Z^{\mathrm{stat}}(n, m)$ to be the point-to-point $\mathcal{HSLG}$ partition function computed using these weights. It was shown in [24, Proposition 4.5], that this model is stationary in the sense that for all $k \geq 1$, and $N \geq k + 1$

$$(\log Z^{\mathrm{stat}}(N, N) - \log Z^{\mathrm{stat}}(N + r, N - r))_{r\in[\![0,k]\!]} \stackrel{d}{=} (S_r)_{r\in[\![0,k]\!]}.$$

where $(S_r)_{r\geq 0}$ is a log-gamma random walk defined in Definition 4.1.2.

Remark 4.1.5. Using the above stationary weights, one can define an associated polymer measure $\mathbf{P}^W_{\mathrm{stat}}$ in the spirit of (4.1.2). We remark that both Theorem 4.1.1 and Theorem 4.1.4 continue to hold under $\mathbf{P}^W_{\mathrm{stat}}$. This is not hard to check from our log-gamma random walk results presented in Appendix 4.6.

Theorem 4.1.3 shows that for $\alpha < 0$ the above log-gamma random walk measure is an *attractor* for the original polymer model in the sense that the increment of the log-partition function of the original model converges to the same log-gamma random walk measure. We believe that our broad techniques should also lead to a similar convergence result for $\alpha \geq 0$. We leave this for future consideration.

We end this section by mentioning a recent work [21] on the stationary measures for the $\mathcal{HSLG}$ polymer. The point-to-point log-gamma polymer partition function $Z(n,m)$ satisfies a recurrence relation

$$Z(n,m) = W_{n,m} \cdot (Z(n-1,m) + Z(n,m-1)) \text{ for } n > m \geq 1,$$
$$Z(n,n) = W_{n,n} \cdot Z(n,n-1) \text{ for } n \geq 1,$$

We refer to a process $(h(k))_{k\geq 0}$ as horizontal-stationary for the $\mathcal{HSLG}$ polymer if the solution to the above recurrence relation with initial data $z(\cdot,0) = e^{h(\cdot)}$ has stationary horizontal increments. For instance, the distribution of horizontal increments $(\log Z(N+k,N) - \log Z(N,N))_{k\geq 0}$ is same for all $N \geq 0$ (and equal to that of the initial data). Recently, [21] posited a one-parameter family of horizontal-stationary measures for the $\mathcal{HSLG}$ polymer model and conjectured that these stationary measures are attractors for a large class of initial data $(Z(n,0))_{n\geq 0}$ subject to the condition $\lim_{k\to\infty} \log Z(k,0)/k = d \in \mathbb{R}$. However, the initial data relevant to our polymer model corresponds to $Z(k,0) = \mathbf{1}_{k=1}$ and is not covered in [21].

Implications of Gaussian fluctuations on the diagonal

In [151], the authors studied one point fluctuations of the $\mathcal{HSLG}$ log-partition function on the diagonal, $\log Z(N,N)$, in both phases. In bound phase, they showed that

$$\frac{\log Z(N,N) - RN}{\sigma\sqrt{N}} \to G, \tag{4.1.8}$$

where $G \sim \mathcal{N}(0,1)$ and

$$R(\theta,\alpha) := -\Psi(\theta+\alpha) - \Psi(\theta-\alpha), \quad \sigma^2(\theta,\alpha) \;:= \Psi'(\theta+\alpha) - \Psi'(\theta-\alpha).$$

Here $\Psi(\cdot)$ denote the digamma function defined on $\mathbb{R}_{>0}$ by

$$\Psi(z) = \partial_z \log \Gamma(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right), \tag{4.1.9}$$

where γ is the Euler-Mascheroni constant. Combining the above result from [151] with our results, we prove gaussianity away from the diagonal.

Theorem 4.1.6. *Fix any $k \in \mathbb{Z}_{\geq 1}$. For each $N > 0$, fix $(a_{N,1}, \ldots, a_{N,k}) \in \mathbb{Z}^k_{\geq 0}$. Suppose that as $N \to \infty$, $a_{N,i}/\sqrt{N} \to 0$ for each $i \in \{1, \ldots, k\}$. We have*

$$\left(\frac{\log Z(N + a_{N,i}, N - a_{N,i}) - RN}{\sigma\sqrt{N}} \right)_{i=1}^{k} \to (G, G, \ldots, G).$$

where $G \sim \mathcal{N}(0,1)$.

The above theorem establishes gaussianity in the $o(\sqrt{N})$ window around the diagonal with trivial correlations. In fact, we expect the above theorem to hold even if $a_{N,i}/N \to 0$. When $a_{N,i}$ are precisely of the order N, we still expect to see gaussianity but with nontrivial correlations. The above result is proved using a strong coupling result (Proposition 4.5.3) that we prove in Section 4.5. The gaussianity in the above theorem essentially comes from the [151] input. However, we believe that it is possible to establish (4.1.8) using the machinery developed in this paper. We leave this for future work.

4.1.2 Proof Ideas

In this section we sketch the key ideas behind the proofs of our main results. Our proof relies on inputs from the recently developed $\mathcal{HSLG}$ Gibbsian line ensemble in [22], one-point fluctuation results for point-to-(partial)line half-space log-partition functions from [27] and the localization techniques from [89]. At the heart of our argument lies an innovative combinatorial argument that bridges the aforementioned inputs and enables our proof.

The starting point of our analysis is the $\mathcal{HSLG}$ Gibbsian line ensemble in [22], which allows us

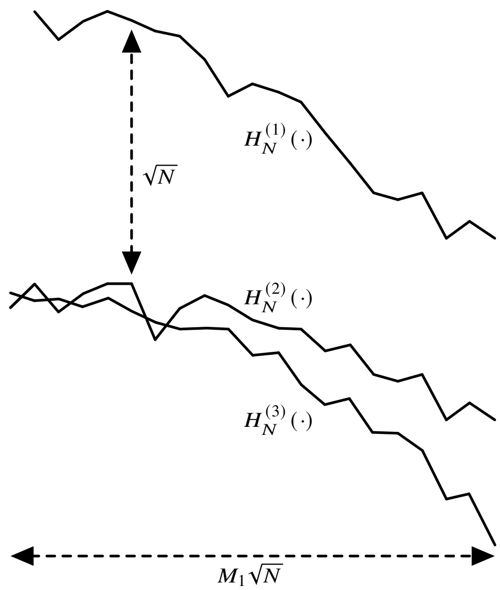


Figure 4.3: First three curves of the $\mathcal{HSLG}$ line ensemble. There is a high probability uniform separation of length $\sqrt{N}$ between the first two curves in the above $M_1\sqrt{N}$ window.

to embed the free energy $\log Z(N+r, N-r)$ of the $\mathcal{HSLG}$ polymer as the top curve of a Gibbsian line ensemble $(H_N^{(k)}(\cdot))_{k\in[\![1,N]\!]}$ of log-gamma increment random walks interacting through a soft version of non-intersection (Theorem 4.2.4) conditioning and subject to an energetic interaction at the left boundary (where $r = 0$) depending on the value of α. This fact is due to the geometric RSK correspondence [75, 195, 193, 40] and the half-space Whittaker process [19]. The key idea of our proof is to show that with high probability, the first and the second curves in our line ensemble ensemble (see Figure 4.3) are sufficiently uniformly separated. Then the separation allows us to conclude that the first curve indeed behaves similarly to a log-gamma random walk by a localization analysis.

The existing literature contains some information about the locations of the top two curves. When $\alpha < 0$, one can deduce from the line ensemble description in [22] that the first and the second curves are repulsed from each other at the left boundary. Results in [27] also supply information about the location for the first curve. However, one cannot deduce that the entire second curve lies uniformly much lower than the first curve from the above two inputs and line ensemble techniques alone.

Intuition behind the separation

Before we proceed to further break down our argument about the separation, it is worth dwelling on the mathematical intuition behind the separation between the first and second curves, which originates from the definition of the line ensemble defined in Section 4.2.1. For simplicity, let us focus only on the left boundary. By Definition 4.2.1, we have $H_N^{(1)}(1) = \log Z(N, N)$, and

$$H_N^{(1)}(1) + H_N^{(2)}(1) := \log \left[2 \sum_{\pi_1, \pi_2} \prod_{(i,j) \in \pi_1 \cup \pi_2} \widetilde{W}_{i,j} \right], \tag{4.1.10}$$

where the above sum is over all pair of non-intersecting upright paths π_1, π_2 from $(1, 1)$ to $(N, N-1)$ and from $(1, 2)$ to (N, N) confined in the entire quadrant $\mathbb{Z}_{\geq 1}^2$ (instead of octant). Here $\widetilde{W}_{i,j}$ is the symmetrized version of the weights defined in (4.1.1) on the entire quadrant as:

$$\widetilde{W}_{i,i} = W_{i,i}/2 \ \text{ for } \ i \geq 1, \qquad \widetilde{W}_{i,j} = \widetilde{W}_{j,i} = W_{i,j} \ \text{ for } \ i > j. \tag{4.1.11}$$

Using point-to-(partial)line log-partition function fluctuation results from [27] and line ensemble techniques, it is not hard to deduce that $\frac{1}{N} H_N^{(1)}(1) \to R := -\Psi(\theta + \alpha) - \Psi(\theta - \alpha)$, where Ψ is the digamma function defined in (4.2.8). However, $H_N^{(2)}(1)$ should follow a different law of large numbers. This can be understood intuitively from (4.1.10) as follows. For α close to $-\theta$, the weights on the diagonal are huge and stochastically dominate all the other weights. The sum in (4.1.10) then concentrates on the pair of paths π_1^*, π_2^* which jointly have the maximal numbers of diagonal points. This occurs when one of the paths carries all the diagonal weights and the other path has no diagonal weights. Thus we expect,

$$\sum_{\pi_1, \pi_2} \prod_{(i,j) \in \pi_1 \cup \pi_2} \widetilde{W}_{i,j} \asymp \left[\sum_{\pi_1} \prod_{(i,j) \in \pi_1} \widetilde{W}_{i,j} \right] \cdot \left[\sum_{\pi_2 | \mathrm{diag}(\pi_2) = \varnothing} \prod_{(i,j) \in \pi_2} \widetilde{W}_{i,j} \right] \tag{4.1.12}$$

Upon taking logarithms and dividing by N, the first term goes to R. However, the second term does not feel the effect of the diagonal and hence should follow the law of large numbers corresponding to the unbound phase, i.e., $\alpha > 0$. The unbound phase law of large numbers is given by $-\Psi(\theta)$ noted in [27, 22]. Thus overall, we expect $\frac{1}{N}(H_N^{(1)}(1) + H_N^{(2)}(1)) \to R - \Psi(\theta)$. As Ψ is concave, the above heuristics suggests $H_N^{(2)}(1)$ follow a lower law of large numbers. While our technical arguments to be presented later do not yield exactly (4.1.12), we utilize the above idea to obtain a large enough separation between the two curves, which turns out to be sufficient for proving our main theorems.

The U map and its consequences

We now describe the key idea that makes the above intuition work. All the statements mentioned in this subsection should be understood as high probability statements. The above idea of having one path having all diagonal weights is made precise in Section 4.3, where we develop a combinatorial map in Lemma 4.3.1, referred to as the U map.

The U map takes every pair of paths π_1, π_2 in the sum in (4.1.10) and returns a pair of non-intersecting paths π_1', π_2' while preserving their shared weights up to reflections (see Figure 4.6). Moreover, the diagonal weights collectively carried by the pair will only rest on one of the paths among π_1', π_2'. The U map is not injective but has at most 2^N many inverses for each pair in its image.

When we apply the U map to a single pair of adjacent paths, we get that

$$\tfrac{1}{N}(H_N^{(1)}(1) + H_N^{(2)}(1)) \le \log 2 + R - \Psi(\theta).$$

The $\log 2$ is an entropy factor that comes from overcounting the number of inverses of our U map. To remove its influence, we rely on the definition of the lower curves of the line ensemble. Indeed, similar to (4.1.10), $\sum_{i=1}^{2k} H_N^{(i)}(1)$ admits a representation in terms of $2k$-many non-intersecting paths. When we apply the U map to k pairs of adjacent paths simultaneously, it leads

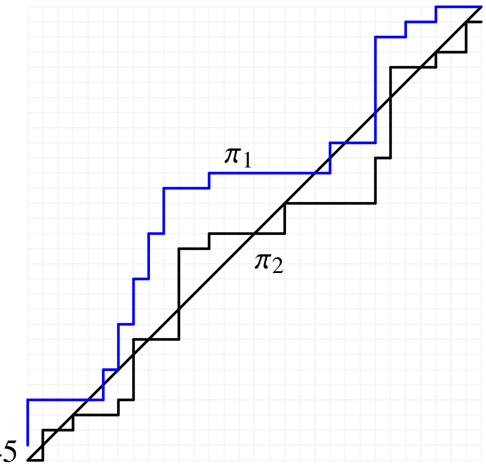
[b]0.45 

Figure 4.4:

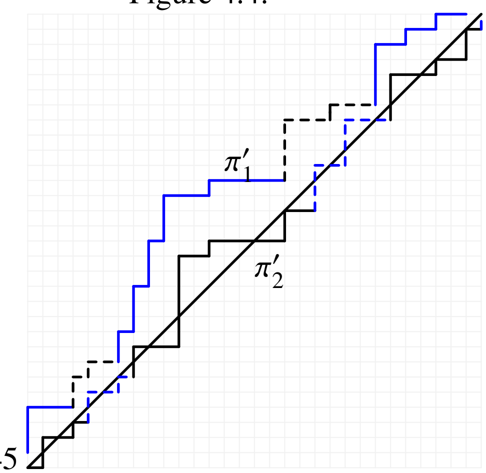
[b]0.45 

Figure 4.5:

Figure 4.6: The U map takes π_1, π_2 from (A) and returns π'_1, π'_2 in (B). The precise description of the map is given in the proof of Lemma 4.3.1

to the following average law of large numbers of the top $2k$ curves:

$$\frac{1}{2kN}\sum_{i=1}^{2k} H_N^{(i)}(1) \leq \tfrac{1}{2k}\log 2 - \tfrac{1}{2}\Psi(\theta) - \tfrac{1}{2}\Psi(\theta+\alpha) - \tfrac{1}{2}\Psi(\theta-\alpha).$$

Taking k large enough, one can ensure the right-hand side constant is strictly less than R. In fact, the above argument can be strengthened to conclude that for large enough k

$$\sup_{p\in[\![1,2N-4k+2]\!]} \frac{1}{2kN}\sum_{i=1}^{2k} H_N^{(i)}(p) \leq R-\delta,$$

for some $\delta > 0$. This is obtained in Proposition 4.3.4.

As a consequence of this result, using soft non-intersection property of the line ensemble (Theorem 4.2.4), we derive that with high probability, the $(2k+2)$-th curve $H_N^{2k+2}(\cdot)$ is uniformly $\mathrm{Const} \cdot N$ below RN over $[\![1, N]\!]$ in Section 4.4. Employing one-point results from [27], one can ensures the point $H_N^{(1)}(M_1\sqrt{N})$ on the top curve is $(M_2+1)\sqrt{N}$ below RN. Combining the last two results and line ensemble techniques we are able to benchmark the second curve from above:

$$\sup_{p \in [\![1, M_1\sqrt{N}]\!]} H_N^{(2)}(p) \leq RN - M_2\sqrt{N} \tag{4.1.13}$$

in Proposition 4.4.2. The details of the argument are presented in Section 4.4. While we are unable to obtain a mismatch in the laws of large numbers for the first two curves following the above procedures, the fact that the second curve is below the diffusive regime of the first curve (since M_2 can be chosen as large as possible) over an interval of length $M_1\sqrt{N}$ is sufficient for our next step of the analysis.

Localization analysis

The remaining piece of our proof of main theorems boils down to a localization analysis of the first curve in Section 4.5. Our proof roughly follows the techniques developed in our paper [89]. First, to prove Theorem 4.1.1 we divide the tail into a deep and a shallow tail depending on the distance away from (N, N), see Figure 4.7.

Our argument in Lemma 4.5.1 uses one-point fluctuations results of point-to-(partial)line log-partition function from [27] as input and shows that the probability of the endpoint living in the deep tail region is exponentially small. To show that the shallow tail contribution is also small and to prove our remaining theorems, we establish the following strong convergence result in Proposition 4.5.3:

(a) the law of the top curve within the $[1, M\sqrt{N}]$ window is arbitrarily close to that of a log-gamma random walk for large enough N (Proposition 4.5.3).

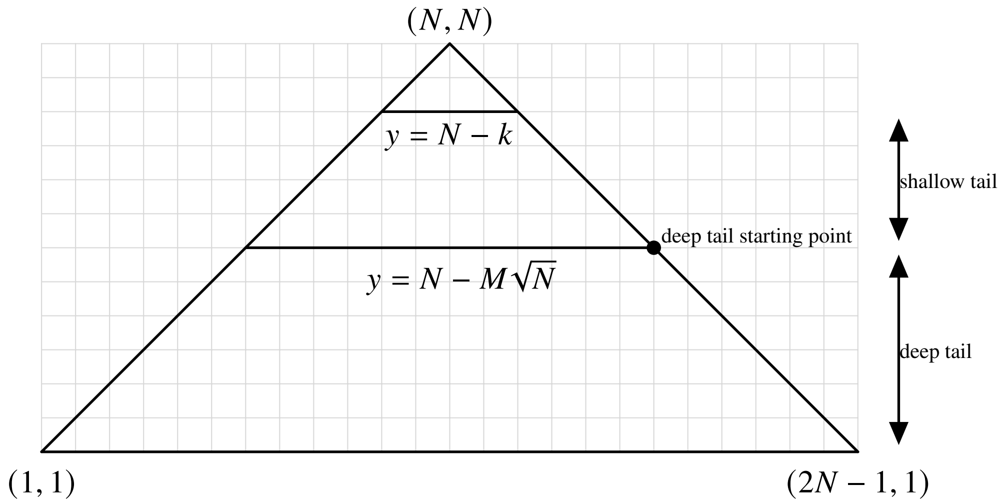


Figure 4.7: If the height of the endpoint of the polymer is less than $N - k$, it either lies in the shallow tail or in the deep tail (illustrated above). Lemma 4.5.1 shows it is exponentially unlikely to lie in the deep tail.

In light of (a), the conclusion that the shallow tail contribution is small follows from estimating the probability of the same event under the log-gamma random walk law. Theorem 4.1.3 is immediate from (a) and Theorem 4.1.4 also follows from (a) after some calculations. The details are presented in Section 4.5.2.

Finally, we briefly explain how we establish (a). A detailed discussion appears in the Step 1 of the proof of Proposition 4.5.3. As $H_N^{(1)}(\cdot)$ is a log-gamma random walk subject to soft non-intersecting condition with $H_N^{(2)}(\cdot)$, it suffices to show that there's sufficient distance between the first and the second curves. Indeed, this will imply $H_N^{(1)}$ behaves like a true log-gamma random walk. As we have already benchmarked the second curve in (4.1.13), it remains to determine a suitable lower bound for the first curve. The key idea here is to find a point $p = O(\sqrt{N})$ on the first curve in the deep tail region such that with high probability

$$H_N^{(1)}(p) \geq RN - M'\sqrt{N}$$

for some M'. This is achieved in Lemma 4.5.2 using fluctuation results from [27]. Then using standard random walk tools such as Kolmogorov's maximal inequality, we derive that with high probability $H_N^{(1)}(q) \geq RN - (M'+1)\sqrt{N}$ for all $q \in [\![1, p]\!]$. Choosing $M_2 = M' + 2$ in (4.1.13) implies that with high probability the first curve is at least $\sqrt{N}$ above the second curve, This completes our deduction and consequently establishes (a).

4.1.3 Related works and future directions

Our study of half-space polymers succeeds an extensive history of endeavors that attempt to unravel their full-space variant. These full-space polymer models have rich connections with symmetric functions, random matrices, stochastic PDEs and integrable systems and are believed to belong to the KPZ universality class (see [**bc20**, 65, 129]). Yet in spite of intense efforts in the past decade, rigorous results proving either the $1/3$ fluctuation exponent or the $2/3$ transversal exponent for general polymers have been scarce outside a few integrable cases (see [**bc20**, 65, 220, 23, 89, 90] and the references therein).

In the half-space geometry, a wealth of literature has focused on the phase diagram for limiting distributions based on the diagonal strength. One of the first mathematical works goes to the series of joint works [13, 15, 14] on the geometric last passage percolation (LPP), i.e. polymers with zero temperature. Their multi-point fluctuations were studied in [217] and similar results were later proved for exponential LPP in [9, 10] using Pfaffian Schur processes. For further recent works on half-space LPP, we refer to [36, 37, 38, 122] and the references therein.

For positive temperature models, i.e., polymers, as they are no longer directly related to the Pfaffian point processes, the first rigorous proof of the depinning transition appeared much later in [27]. Here the authors also included precise fluctuation results such as the BBP phase transition [11] for the point-to-line log-gamma free energy. For the point-to-point log-gamma free energy, the limit theorem as well as the Baik-Rains phase transition were conjectured in [19] based on steepest descent analysis of half-space Macdonald processes. This result has been recently proved in [151] by relating the half- space model to a free boundary version of the Schur process.

Similar to their full-space counterparts, in addition to fluctuations, another dimension of interest to half-space polymers is their localization behaviors, which refer to the concentration of polymers in a very small region given the environment. Figure 4.1 is a simulation of 30 samples of $\mathcal{HSLG}$ polymers of length 120 sampled from the same environment with $\theta = 1$, $\alpha = -0.2$ and $\alpha = +0.2$. The simulation suggests that even in the unbound phase, we expect a localization phenomenon around a favorite site given by the environment. Localization is a unique behavior of the polymer path in the strong disorder regime. In the full space, various levels of localization results have been established for discrete and continuous polymers. The mathematical work began with the *strong* localization result of [56] that confirmed the existence of the favorite sites for the endpoint distributions of point-to-line polymers and has been upgraded to the notion of *atomic* and *geometric* localization for general reference random walks in a series of joint works [**bc20**, 32, 17, 16]. An even stronger notion, the "favorite region conjecture", which conjectures the favorite corridor of a polymer to be stochastically bounded, has been proved for two integrable models: the stationary log-gamma polymer in the discrete case ([67]) and the continuous directed random polymer (CDRP) in the continuous case ([89]). In this direction, building up on [89] work, recently [116] have studied the localization distance of the CDRP.

Investigating the geometry of the half-space CDRP is an interesting question to consider next. Recently, a number of new results have appeared on the half-space KPZ equation, which arises as the free energy of the half-space CDRP [240, 21], in both the mathematics [83, 20, 19, 200, 199, 21, 151] and the physics literature [**bbc16**, 135, 153, 98, 165, 24, 26, 25]. These results on the free energy render the half-space continuous polymers amenable to analysis. However, the challenge with further studying the geometry of the half-space CDRP remains, due to the lack of an analogous half-space KPZ line ensemble.

Outline

The rest of the paper is organized as follows. Section 4.2 reviews some of the existing results related to $\mathcal{HSLG}$ line ensemble and one-point fluctuations of point-to-(partial)line free energy of $\mathcal{HSLG}$ polymer. In Section 4.3 we prove our key combinatorial lemma and use it to control the average law of large numbers for the top curves of the line ensemble. In Section 4.4, we establish control over the second curve of the line ensemble. Finally, in Section 4.5, we complete the proofs of our main theorems. Appendix 4.6 contains basic properties of log-gamma random walks.

Notation

Throughout this paper, we will assume $\theta > 0$ and $\alpha \in (-\theta, 0)$ are fixed parameters. We write $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$ to denote the set of integers between a and b. We will use serif fonts such as $\mathsf{A}, \mathsf{B}, \ldots$ to denote events. The complement of an event A will be denoted as $\neg\mathsf{A}$.

Acknowledgements

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4.2 Basic framework and tools

In this section, we present the necessary background on the half-space log-gamma ($\mathcal{HSLG}$) line ensemble and point-to-(partial) line partition function. From [22] and [27] we gather a few of the known results on these objects that are crucial in our proofs.

4.2.1 The $\mathcal{HSLG}$ line ensemble and its Gibbs property

We begin with the description of the $\mathcal{HSLG}$ line ensemble and its Gibbs property. The definition of the $\mathcal{HSLG}$ line ensemble is based on the point-to-point symmetrized partition function for multiple paths defined in (4.2.1). These are sum over multiple non-intersecting upright paths on the entire quadrant $\mathbb{Z}^2_{>0}$ of products of the symmetrized version defined in (4.1.11) of the weights defined in (4.1.1). Fix $m, n, r \in \mathbb{Z}_{>0}$ with $n \geq r$, let $\Pi^{(r)}_{m,n}$ be the set of all r-tuples of non-intersecting upright paths in $\mathbb{Z}^2_{>0}$ starting from $(1, r), (1, r-1), \ldots, (1, 1)$ and going to $(m, n), (m, n-1), \ldots, (m, n-r+1)$ respectively. We define the point-to-point symmetrized partition function for r paths as

$$Z_{\text{sym}}^{(r)}(m,n) := \sum_{(\pi_1,\ldots,\pi_r)\in\Pi_{m,n}^{(r)}} \prod_{(i,j)\in\pi_1\cup\cdots\cup\pi_r} \widetilde{W}_{i,j}. \tag{4.2.1}$$

where $\widetilde{W}_{i,j}$ are defined in (4.1.11). We write $Z_{\text{sym}}(m,n) := Z_{\text{sym}}^{(1)}(m,n)$ and use the convention that $Z_{\text{sym}}^{(0)}(m,n) \equiv 1$. One can recover $\mathcal{HSLG}$ partition function from symmetrized partition function via the following identity. For each $(m,n) \in \mathcal{I}^-$ we have

$$2Z_{\text{sym}}(m,n) = Z(m,n). \tag{4.2.2}$$

The above identity appears in Section 2.1 of [27] and follows easily due to the symmetry of the weights. We stress that the above relation is an exact equality not just in distribution.

Definition 4.2.1 ($\mathcal{HSLG}$ line ensemble)**.** Fix $N > 1$. For each $k \in [\![1, N-1]\!]$ and $p \in [\![1, 2N - 2k+2]\!]$ set

$$H_N^{(k)}(p) := \log\left(\frac{2Z_{\text{sym}}^{(k)}(N+\lfloor p/2\rfloor, N-\lceil p/2\rceil+1)}{Z_{\text{sym}}^{(k-1)}(N+\lfloor p/2\rfloor, N-\lceil p/2\rceil+1)}\right) \tag{4.2.3}$$

We view the k-th curve $H_N^{(k)}$ as a random continuous function $H_N^{(k)} : [1, 2(N-k+1)] \to \mathbb{R}$ by linearly interpolating its values on integer points. We call the collection of curves $H_N := (H_N^{(1)}, H_N^{(2)}, \ldots, H_N^{(N)})$ as the $\mathcal{HSLG}$ line ensemble.

We remark that in Definition 2.7 in [22], the authors defined the $\mathcal{HSLG}$ line ensemble by defining $\mathcal{L}_i^N(j) = H_N^{(i)}(j) + \text{Const}\cdot N$ where the 'Const' is explicit and encodes the law of large numbers for the $\mathcal{HSLG}$ free energy process (as well as the entire line ensemble) in the unbound phase. Since the law of large numbers for the first curve and the second curve in the bound phase are possibly different (recall our discussion of the proof idea in the introduction), we choose to not add this constant in our definition of line ensemble. All the results from [22] can be easily translated to results in our setting by adding this appropriate constant.

In view of (4.2.2), for all $p \leq 2N$ we have

$$H_N^{(1)}(p) = \log Z(N + \lfloor p/2 \rfloor, N - \lceil p/2 \rceil + 1). \tag{4.2.4}$$

The $\mathcal{HSLG}$ line ensemble enjoys a property that is known as the $\mathcal{HSLG}$ Gibbs property. To state the$\mathcal{HSLG}$ Gibbs property, we introduce the $\mathcal{HSLG}$ Gibbs measures via graphical representation.

We consider a diamond lattice on the lower-right quadrant with vertices $\{(m, -n), (m + \frac{1}{2}, -n + \frac{1}{2}) \mid m, n \in \mathbb{Z}_{>0}^2\}$ and nearest neighbor edges as shown in Figure 4.10. We label the vertices by setting $\phi((m, n)) = (-\lfloor n \rfloor, 2m - 1)$. We shall always use this labeling to identify a vertex in this lattice and will not mention its actual coordinates further.

On the diamond lattice domain, we add potential *directed-colored edges*. A directed-colored edge $\vec{e} = \{v_1 \to v_2\}$ on this lattice is a directed edge from v_1 to v_2 that has three choices of colors: **blue**, **red**, and **black**. Given a directed-colored edge, we associate a weight function based on the color of the edge defined as follows:

$$W_{\vec{e}}(x) = \begin{cases} \exp((\theta - \alpha)x - e^x) & \text{if } \vec{e} \text{ is } \textbf{blue} \\ \exp((\theta + \alpha)x - e^x) & \text{if } \vec{e} \text{ is } \textbf{red} \\ \exp(-e^x) & \text{if } \vec{e} \text{ is } \textbf{black}. \end{cases} \tag{4.2.5}$$

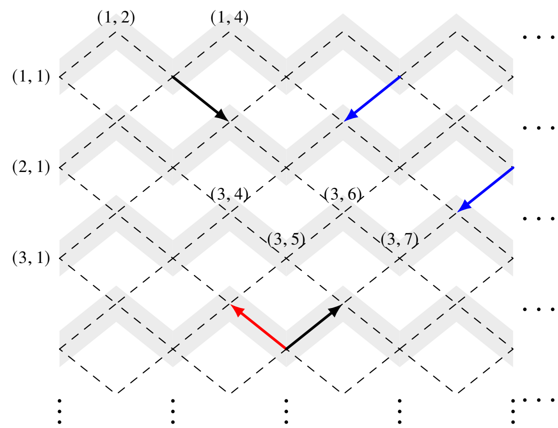
We consider a graph G_N on the diamond lattice with vertex set

$$K_N := \{(i, j) \mid i \in [\![1, N]\!], j \in [\![1, 2N - 2i + 2]\!]\}.$$

with directed-colored edges described below. For each $(p, q) \in K_N$,

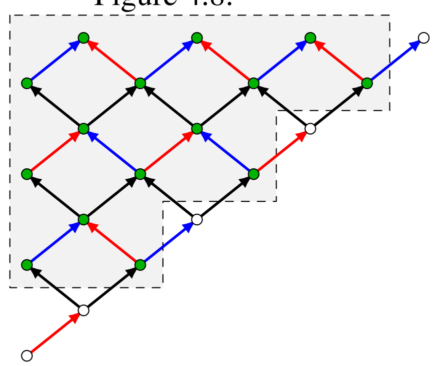
- If q is odd and p is odd (even resp.), we put a **blue** (**red** resp.) edge: $(p, q) \to (p, q + 1)$.
- If $q \geq 3$ is odd and p is odd (even resp.), we put a **red** (**blue** resp.) edge: $(p, q) \to (p, q - 1)$.
- If q is even, we put two **black** edges: $(p, q) \to (p - 1, q)$ and $(p, q) \to (p + 1, q)$.

The corresponding graph is shown in Figure 4.10. We write $E(F)$ for the set of edges of any graph $F \subset G_N$.



[b]0.45

Figure 4.8:



[b]0.45

Figure 4.9:

Figure 4.10: (A) Diamond lattice with a few of the labeling of the vertices shown in the figure. The m-th gray-shaded region have vertices with labels of the form $\{(m, n) \mid n \in \mathbb{Z}^2_{>0}\}$. Thus each such region consists of vertices with the same first coordinate labeling. Potential directed-colored edges on the lattice are also drawn above. (B) K_N with $N = 4$. Λ^*_N consists of all vertices in the shaded region.

The following result from [22] shows how the conditional distribution of the $\mathcal{HSLG}$ line ensemble is given by certain measures called $\mathcal{HSLG}$ Gibbs measures.

Theorem 4.2.2 (Gibbs property)**.** *Consider the directed-colored graph G_N described above. Set*

Λ be a connected subset of the graph G_N on the diamond lattice K_N

$$\Lambda_N^* = \{(i,j) \mid i \in [\![1, N-1]\!], j \in [\![1, 2N-2i+1]\!]\}.$$

Let Λ be a connected subset of Λ_N. Recall the $\mathcal{HSLG}$ line ensemble H^N from Theorem 4.2.1. The law of $\{H_i^N(j) \mid (i,j) \in \Lambda\}$ conditioned on $\{H_i^N(j) \mid (i,j) \in \Lambda^c\}$ is a measure on $\mathbb{R}^{|\Lambda|}$ with density at $(u_{i,j})_{(i,j)\in\Lambda}$ proportional to

$$\prod_{\vec{e}=\{v_1\to v_2\}\in E(\Lambda\cup\partial\Lambda)} W_{\vec{e}}(u_{v_1} - u_{v_2}), \tag{4.2.6}$$

where $u_{i,j} = H_i^N(j)$ for $(i,j) \in \partial\Lambda$.

We call the above conditional law as the $\mathcal{HSLG}$ Gibbs measure with boundary condition $\vec{u} = (u_{i,j})_{(i,j)\in\partial\Lambda}$ and denote this measure as $\mathbf{P}^{\vec{u}}_{\text{gibbs}}(\cdot)$. The above theorem follows directly from the results in [22]). Theorem 1.3 in [22] specifies the Gibbs property for the centered line ensemble $L_i^N(j)$. The same Gibbs property holds for $H_N^{(i)}(j)$ as $\mathcal{HSLG}$ Gibbs measures are translation invariant (Observation 2.1 (b) in [22]. The Gibbs property stated in Theorem 1.3 is different and valid for all $\alpha > -\theta$. When $\alpha \in (-\theta, \theta)$, one can redistribute the edge-weights (see Observation 4.2 in [22]) to obtain the above stated Gibbs property.

The $\mathcal{HSLG}$ Gibbs measures satisfy stochastic monotonicity w.r.t. the boundary data.

Proposition 4.2.3 (Stochastic monotonicity, Proposition 2.6 in [22])**.** *Fix $k_1 \le k_2$, $a_i \le b_i$ for $k_1 \le i \le k_2$ and $\alpha > -\theta$. Let*

$$\Lambda := \{(i,j) \mid k_1 \le i \le k_2, a_i \le j \le b_i\}.$$

There exists a probability space consisting of a collection of random variables

$$\{L(v; (u_w)_{w\in\partial\Lambda}) \mid v \in \Lambda, (u_w)_{w\in\partial\Lambda} \in \mathbb{R}^{|\partial\Lambda|}\}$$

such that

1. *For each* $(u_w)_{w\in\partial\Lambda} \in \mathbb{R}^{|\partial\Lambda|}$*, the law of* $\{L(v;(u_w)_{w\in\partial\Lambda}) \mid v \in \Lambda\}$ *is given by the* $\mathcal{HSLG}$ Gibbs *measure for the domain* Λ *with boundary condition* $(u_w)_{w\in\partial\Lambda} \in \mathbb{R}^{|\partial\Lambda|}$.

2. *With probability* 1*, for all* $v \in \Lambda$ *we have*

$$L(v;(u_w)_{w\in\partial\Lambda}) \le L(v;(u'_w)_{w\in\partial\Lambda}) \text{ whenever } u_w \le u'_w \text{ for all } w \in \partial\Lambda.$$

As mentioned in the introduction, the $\mathcal{HSLG}$ line ensemble enjoys a certain soft non-intersection property. This property is captured in our next theorem.

Theorem 4.2.4 (Ordering of points, Theorem 3.1 in [22])**.** *Fix any* $k \in \mathbb{Z}_{>0}$ *and* $\rho \in (0,1)$*. There exists* $N_0 = N_0(\rho,k) > 0$ *such that for all* $N \ge N_0$*,* $i \in [\![1,k]\!]$ *and* $p \in [\![1,N-i]\!]$ *the following inequalities holds:*

$$\mathbf{P}(H_N^{(i)}(2p+1) \le H_N^{(i)}(2p) + \log^2 N) \ge 1-\rho^N,$$
$$\mathbf{P}(H_N^{(i)}(2p-1) \le H_N^{(i)}(2p) + \log^2 N) \ge 1-\rho^N,$$
$$\mathbf{P}(H_N^{(i+1)}(2p) \le H_N^{(i)}(2p+1) + \log^2 N) \ge 1-\rho^N,$$
$$\mathbf{P}(H_N^{(i+1)}(2p) \le H_N^{(i)}(2p-1) + \log^2 N) \ge 1-\rho^N.$$

We remark that the above theorem is true in the unbound phase as well (i.e., for all $\alpha > -\theta$).

We now introduce the *interacting random walks* which are a specialized version of $\mathcal{HSLG}$ Gibbs measures (see Figure 4.27).

Definition 4.2.5 (Interacting random walk)**.** We say $(L_1, L_2) = (L_1[\![1,2T-2]\!], L_2[\![1,2T-1]\!])$ is an interacting random walk (IRW) of length T with boundary condition (a,b) if its law is a measure on $\mathbb{R}^{4T-3}$ with density at $(u_{1,j})_{j=1}^{2T-2}, (u_{2,j})_{j=1}^{2T-1}$ proportional to

$$\prod_{j=1}^{T-1} \exp(-e^{u_{2,2j}-u_{1,2j-1}} - e^{u_{2,2j}-u_{1,2j+1}}) \prod_{i=1}^{2}\prod_{j=1}^{2T-1} G_{\theta+(-1)^{i+j}\alpha}\big((-1)^{j+1}(u_{i,j}-u_{i,j+1})\big)$$

where $G_\beta(x) = [\Gamma(\beta)]^{-1}\exp(\beta x - e^x)$, $u_{1,2T-1} = a$, $u_{2,2T} = b$, and $u_{1,2T} = 0$ (which forces $G_{\theta+\alpha}(u_{1,2T-1} - u_{1,2T})$ to be a constant). Figure 4.27 provides the graphical representation of IRW.

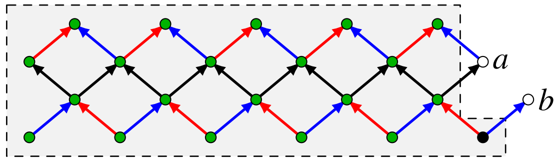


Figure 4.11: IRW of length 6 with boundary condition a and b.

Note that the directed-colored graph associated to IRW can be viewed as a subset of G_N (introduced above). Specifically, for each $i \geq 1$, if we consider the vertex set

$$V_{i,T} := \{(2i, j), (2i+1, j) \mid j \in [\![1, 2T-1]\!]\} \cup \{(2i+1, 2T)\},$$

the subgraph induced by $V_{i,T}, E(V_{i,T})$ corresponds to the graph associated to IRW. Note that the graph associated to IRW can also be viewed as the subgraph induced by $\widehat{V}_T, E(\widehat{V}_T)$ where

$$\widehat{V}_T := \{(1, j), (2, j) \mid j \in [\![1, 2T-1]\!]\} \cup \{(2, 2T)\},$$

provided we switch α to $-\alpha$ in (4.2.5) (i.e., switching red and blue edges). Since we have restricted $\alpha \in (-\theta, 0)$ (bound phase), under this switching IRW can be viewed as certain $\mathcal{HSLG}$ Gibbs measures in the unbound phase. Indeed, after switching α to $-\alpha$, in the language of [22], IRW precisely corresponds to bottom-free measure on the domain $\mathcal{K}_{2,T}$ with boundary condition (a, b) (see Definition 2.3 in [22]). This allows us to use the unbound phase estimates developed in [22]. We end this section by recording one such estimate.

Proposition 4.2.6 (Lemma 5.3 in [22]). *Fix any $T \geq 2$. Let (L_1, L_2) be a* IRW *of length T with boundary condition $(0, -\sqrt{T})$. Fix $\varepsilon \in (0, 1)$. There exists $M_0 = M_0(\varepsilon) > 0$ such that*

$$\mathbf{P}\left(\sup_{p \in [\![1, 2T-1]\!]} |L_1(p)| + \sup_{q \in [\![1, 2T]\!]} |L_2(q)| \geq M_0\sqrt{T}\right) \leq \varepsilon.$$

4.2.2 One-point fluctuations of point-to-(partial)line free energy

In this section, we gather the point-to-(partial)line free energy fluctuation results from [27]. To state the theorem, we introduce a few necessary objects first.

Recall the point-to-point half-space partition function $Z(m,n)$ from (4.1.4). For $k \in [\![0, N-1]\!]$, we define the point-to-(partial)line half-space partition function as

$$\mathcal{Z}_N^{\mathsf{PL}}(m) = \sum_{p=m}^{N-1} Z(N+p, N-p) = \sum_{p=m}^{N-1} e^{H_N^{(1)}(2p+1)}. \tag{4.2.7}$$

For the second equality, note that by (4.2.4) we have $H_N^{(1)}(2p+1) = \log Z(N+p, N-p)$ and thus we can translate the point-to-(partial)line partition function in Definition 1.8 (or equivalently in Definition 1.3) of [27] into sums of $e^{H_N^{(1)}(2p+1)}$ by way of the full-space point-to-point partition function $Z(n+p, n-p)$.

Let $\Psi(\cdot)$ denote the digamma function defined on $\mathbb{R}_{>0}$ by

$$\Psi(z) = \partial_z \log \Gamma(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right), \tag{4.2.8}$$

where γ is the Euler-Mascheroni constant. For any $k \in \mathbb{Z}_{>0}$, we set

$$\begin{aligned} R(\theta,\alpha) &:= -\Psi(\theta+\alpha) - \Psi(\theta-\alpha), \\ \tau(\theta,\alpha) &:= \Psi(\theta-\alpha) - \Psi(\theta+\alpha), \\ \sigma^2(\theta,\alpha) &:= \Psi'(\theta+\alpha) - \Psi'(\theta-\alpha), \\ \Delta_k(\theta,\alpha) &:= \Psi(\theta) - \tfrac{1}{2}[\Psi(\theta+\alpha) + \Psi(\theta-\alpha)] - \tfrac{1}{2k}\log 2. \end{aligned} \tag{4.2.9}$$

For the remainder of the paper, we will make use of the above notation repeatedly. As Ψ is a strictly concave function, for all large enough k (depending on α, θ) we have $\Delta_k > 0$. For the results and proofs in the remainder of the paper, we always choose k large enough such that $\Delta_k > 0$.

We now state the necessary results from [27] about the point-to-(partial)line partition function

$\mathcal{Z}_N^{\mathsf{PL}}(m)$ that we need in our subsequent analysis.

Theorem 4.2.7. *Suppose $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. Suppose further that N is an integer that tends to infinity in such a way that $\frac{g(N)}{N} \to 0$. We have*

$$\frac{1}{N^{1/2}\sigma}\left[\log \mathcal{Z}_N^{\mathsf{PL}}(g(N)) - RN + g(N)\tau\right] \xrightarrow{d} \mathcal{N}(0,1). \tag{4.2.10}$$

where R, τ, σ are defined in (4.2.9). *We have the following law of large numbers*

$$\frac{1}{N}\log\left[\sum_{p=1}^{N-1} Z(N+p, N-p)\right] \xrightarrow{p} R \quad \frac{1}{N}\log\left[\sum_{p=1}^{N} Z(N+p, N-p+1)\right] \xrightarrow{p} R. \tag{4.2.11}$$

Furthermore, the above law of large numbers continues to hold when $\alpha = 0$, i.e., the diagonal weights are assumed to be distributed as $\text{Gamma}^{-1}(\theta)$. *In that case $R(\theta, \alpha)$ is interpreted as $R(\theta, 0) = -2\Psi(\theta)$.*

Proof. Theorem 1.10 in [27] discusses several fluctuation results for point-to-(partial)line partition function for the $\mathcal{HSLG}$ polymer, of which Theorem 1.10(3) applies to the bound phase in this paper. Letting $n = N - g(N)$ and $m = N + g(N)$ in (1.12) of [27] yields

$$\frac{1}{(N - g(N))^{1/2}\sigma_p}\left[\log \mathcal{Z}_N^{\mathsf{PL}}(g(N)) + (N - g(N))\mu_p\right] \xrightarrow{d} \mathcal{N}(0,1).$$

where $\mu_p := \Psi(\theta+\alpha) + p\Psi(\theta-\alpha)$ and $\sigma_p^2 := \Psi'(\theta+\alpha) - p\Psi'(\theta-\alpha)$ with $p = \frac{N+g(N)}{N-g(N)}$. Observe that $(N - g(N))\mu_p = -RN + g(N)\tau$. As $g(N)/N \to 0$, we have that

$$\frac{(N - g(N))^{1/2}\sigma_p}{N^{1/2}\sigma} \to 1.$$

Therefore the above fluctuation result implies (4.2.10). For the law of large numbers, the first one in (4.2.11) follows by taking $g(N) \equiv 1$ and appealing to (4.2.10). The second law of large numbers also follows from Theorem 1.10(3) in [27] as their result also gives fluctuation results for point-to-(partial)line free energy of that form with the same law of large numbers. Finally, the last point of

Theorem 4.2.7 follows by from Theorem 1.10(2) in [27] which deals with the $\alpha = 0$ case. □

4.3 Controlling the average law of large numbers of the top curves

In this section, we control the average law of large numbers of the top $2k$ curves for large enough k (Proposition 4.3.4). As explained in the introduction, the key idea behind this proposition is to show that the contribution of diagonal weights in the $2k$ many non-intersecting paths of $Z_{\text{sym}}^{(2k)}(m,n)$ (defined in (4.2.1)) essentially comes from k many paths. The starting point of this idea is Lemma 4.3.1. Given a pair of non-intersecting paths (π_1, π_2) starting and ending at adjacent locations with the same x-coordinate, Lemma 4.3.1 constructs two new non-intersecting paths (π_1', π_2') from (π_1, π_2) such that the new paths collectively carry the same weight variables but the diagonal weights only rest on the lower path. This combinatorial result proceeds to help us decompose the symmetrized multilayer partition function $Z_{\text{sym}}^{(2k)}(m,n)$ into pairs of single-layer ones in Lemmas 4.3.2 and 4.3.3 before culminating into the final result in Proposition 4.3.4.

Let $\Pi(v_1 \to v_2, u_1 \to u_2)$ denote the set of pairs of non-intersecting upright paths in $\mathbb{Z}_{>0}^2$ starting from u_1, v_1 and ending at u_2, v_2 respectively. Recall that $\mathcal{I}^- = \{(i,j) \in \mathbb{Z}_{>0}^2 | j \le i\}$. Define $\mathcal{I}^+ := \{(i,j) \in \mathbb{Z}_{>0}^2 | j \ge i\}$ which represents the half-space index set that includes points on and above the diagonal. The first lemma constructs the U map.

Lemma 4.3.1 (Construction of U map)**.** *Fix $x \in \mathbb{Z}_{>0}$ and any $(m,n) \in \mathcal{I}^-$ with $n \ge 2$. Then there exists a map $U : \Pi_1 \to \Pi_2$ where*

$$\Pi_1 := \Pi((1, x+1) \to (m,n), (1,x) \to (m, n-1))$$

$$\Pi_2 := \Pi((1, x+1) \to (n-1, m), (1,x) \to (n,m)),$$

such that the following properties hold (let $(\pi_1', \pi_2') := U(\pi_1, \pi_2)$):

(a) π_1' has no diagonal points, i.e., $\{(i,i) \in \mathbb{Z}_{>0}^2\} \cap \pi_1'$ is empty and

$$\{(i,i) \in \mathbb{Z}_{>0}^2\} \cap \pi_2' = \{(i,i) \in \mathbb{Z}_{>0}^2\} \cap \{\pi_1 \cup \pi_2\}.$$

(b) Recall the symmetrized weights $(\widetilde{W}_{i,j})_{(i,j)\in\mathbb{Z}^2_{>0}}$ *from* (4.1.11)*. We have*

$$\prod_{(i,j)\in\pi_1\cup\pi_2} \widetilde{W}_{i,j} \overset{a.s.}{=} \prod_{(i,j)\in\pi_1'\cup\pi_2'} \widetilde{W}_{i,j}.$$

(c) For each $(\pi_1', \pi_2') \in \Pi_2$ *we have*

$$\left|U^{-1}(\{(\pi_1', \pi_2')\})\right| \leq 2^{|\{(i,i)\in\pi_1\cup\pi_2\}|}.$$

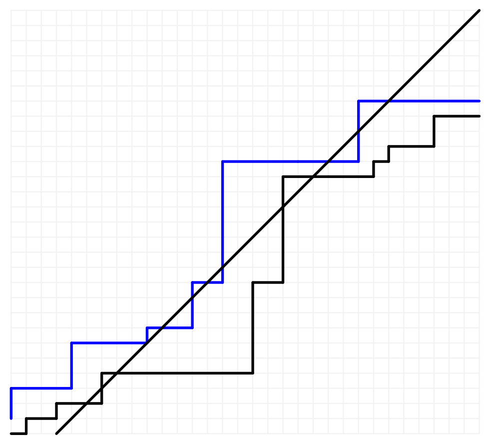
[b]0.45 

Figure 4.12:

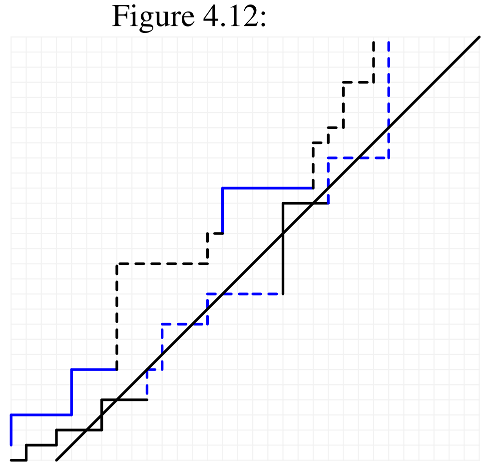
[b]0.45 

Figure 4.13:

Figure 4.14: The U map takes (A) to (B).

We remark that Lemma 4.3.1 is entirely combinatorial and does not use any results about the integrability of the model. Lemma 4.3.1 continues to hold for any collection of symmetrized weights that are not necessarily distributed as inverse-Gamma random variables.

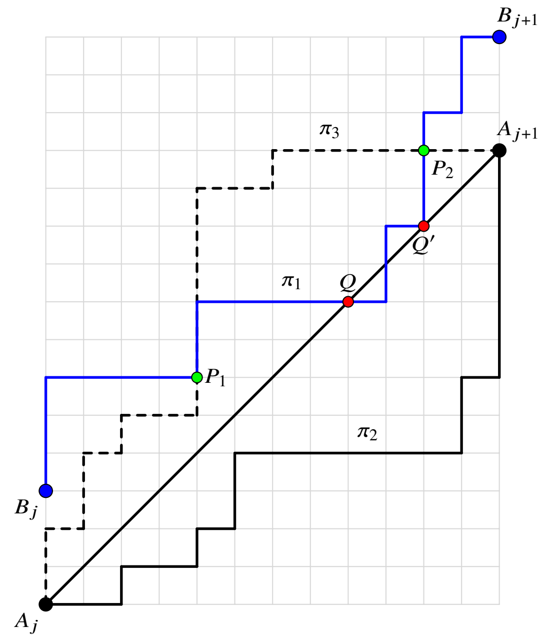
Proof. We define a partial order $<$ on the points $\mathbb{Z}^2_{>0}$ by requiring $P_1 = (a_1, b_1) < P_2 = (a_2, b_2)$ whenever $a_1 + b_1 < a_2 + b_2$. Let π_1 denote the path from $(1, x+1)$ to (m, n) and π_2 the path from $(1, x)$ to $(m, n-1)$. We denote $\mathrm{diag}(\pi_i)$ as the set of points on π_i that lie on the diagonal set $D := \{(i,i) \in \mathbb{Z}^2_{>0}\}$. Recall that $\mathcal{I}^+ = \{(i,j) \in \mathbb{Z}^2_{>0} \mid i \leq j\}$ and $\mathcal{I}^- = \{(i,j) \in \mathbb{Z}^2_{>0} \mid j \leq i\}$.

We first define a special collection of points, SPDiag from $\mathrm{diag}(\pi_2)$. Let $D_1 < D_2 < D_3 < \cdots < D_s$ be all the points in $\mathrm{diag}(\pi_1 \cup \pi_2)$ arranged in the increasing order. We put the point $D_j \in \mathrm{diag}(\pi_2)$ in the set SPDiag if $D_{j-1} \in \mathrm{diag}(\pi_1)$ or $D_{j+1} \in \mathrm{diag}(\pi_1)$. In other words, SPDiag consists of the diagonal points in π_2 that bookend contiguous clusters of $\mathrm{diag}(\pi_1)$ in $\mathrm{diag}(\pi_1 \cup \pi_2)$. We enumerate the points in SPDiag as $A_1 < A_2 < \cdots < A_r$. Let B_j be the first point on π_1 that has the same x-coordinate as A_j. Note that by construction, either only π_1 intersects the diagonal or only π_2 intersects the diagonal between A_j and A_{j+1}, $j = 1, \ldots, r$. Let us denote $A_{r+1} := (m, n-1)$ and $B_{r+1} := (m, n)$.

We now construct new paths π_2' and π_1' from π_2 and π_1 by reconstructing each segment between A_j and A_{j+1} (and B_j and B_{j+1} for π_1 respectively), $j = 1, \ldots, r$. We separate the reconstruction procedures for each segment into the following cases: if only π_2 intersects the diagonal and $j \leq r-1$, if only π_1 intersects the diagonal and $j \leq r-1$, or if $j = r$.

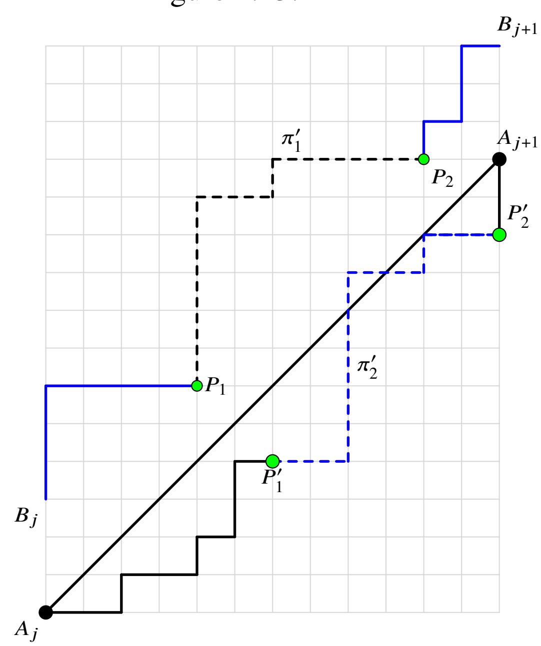
1. **When** $1 \leq j \leq r-1$ **and only** π_2 **intersects the diagonal**, we keep the original paths. We set π_1' and π_2' on these segments to be the same as those on π_1 and π_2 respectively.

2. **When** $1 \leq j \leq r-1$ **and only** π_1 **intersects the diagonal between** A_j **and** A_{j+1} **(see Figure 4.17)**, the portion of the path π_2 from A_j to A_{j+1} (excluding A_j and A_{j+1}) lies in $\mathcal{I}^- \setminus D$. Reflecting the portion of the path π_2 from A_j to A_{j+1} (black path in Figure 4.17) across the diagonal yields a path π_3 (black dashed path in Figure 4.17). Let Q be the first point on $\mathrm{diag}(\pi_1)$ that lies between A_j and A_{j+1} and Q' be the last, which exist by construction of the SPdiag set (Q and Q' may overlap). As the y-coordinate of A_i is strictly smaller than that of



[b]0.45

Figure 4.15:



[b]0.45

Figure 4.16:

Figure 4.17: The second case when $j \leq r - 1$ and only π_1 intersects with the diagonal. π_1 and π_2 are black and blue paths in Figure (A) respectively. π_3 is the black dashed path in Figure (A). π_1' is the path in Figure (B) which is formed by the concatenation of solid blue paths and the black dashed path. π_2' is the path in Figure (B) which is formed by the concatenation of solid black paths and the blue dashed path. The U map takes π_1, π_2 and spits out π_1', π_2'.

B_i and Q, Q' are on the diag(π_1), π_1 and π_3 must intersect on the segments between A_j and Q and Q' and A_{j+1}. Let P_1 be the first point of intersection and P_2 the last point of intersection. Clearly $P_1 \neq P_2$ as the former is between A_j and Q and the latter lies between Q' and A_{j+1}. Replacing the portion of π_1 between P_1 and P_2 with that of π_3 yields a path π_1' from B_j to B_{j+1}. As the part of π_3 between P_1 and P_2 lies in $\mathcal{I}^+ \setminus D$, π_1' lies entirely in $\mathcal{I}^+ \setminus D$. We denote the reflections of P_1 and P_2 across the diagonal as P_1' and P_2', which must lie on the original π_2 by construction. Similarly replacing the portion of π_2 between P_1' and P_2' with the reflection of π_1 between P_1 and P_2 across the diagonal yields a path π_2' from A_j to A_{j+1}. As π_1 and π_2 are non-intersecting, the reflected paths are also non-intersecting. Thus the new paths π_1' and π_2' are non-intersecting.

3. **When** $j = r$**,** consider the portion of the path π_2 from A_r to A_{r+1} (see Figure 4.20). Note that in this segment, all the diagonal points belong to π_1. Reflecting this portion of π_2 across the diagonal gives us π_3 (black dashed path in Figure 4.20). Let Q be the first point on diag(π_1) that lies between A_j and A_{j+1} and Q exists as π_1 ends at $B_{r+1} := (m, n) \in \mathcal{I}^-$. Note that π_3 lies entirely in $\mathcal{I}^+ \setminus D$, excluding A_r. Thus π_1 and π_3 necessarily intersect in $\mathcal{I}^+ \setminus D$. Again, we locate the first point intersection P and replace the portion of π_1 from P to B_{r+1} with the portion of π_3 from P to $A_{r+1}' := (n-1, m)$. Similarly, reflecting the portion of π_1 from P and B_{r+1} across the diagonal and replacing the portions of π_2 between P' and A_{r+1} with the portion of reflection between P and $B_{r+1}' := (n, m)$ yields a path π_2' from A_r to B_{r+1}'. Clearly, the new path π_1' lies in $\mathcal{I}^+ \setminus D$ and the paths π_1' and π_2' are non-intersecting as the reflected portions are non-intersecting.

As A_j and B_j remain unchanged, connecting all the segments between A_j's (and B_j's respectively) for $j \leq r$ and A_r and B_{r+1}' (and B_r and A_{r+1}') yields the new path π_2' from $(1, x)$ to (n, m) and the new path π_1' from $(1, x+1)$ to $(n-1, m)$ (see Figure 4.14). At each step of the above construction, the paths remain non-intersecting. Thus (π_1', π_2') form a non-intersecting pair. We call this explicitly constructed map U. By construction, π_1' lies entirely in $\mathcal{I}^+ \setminus D$ and has no diagonal

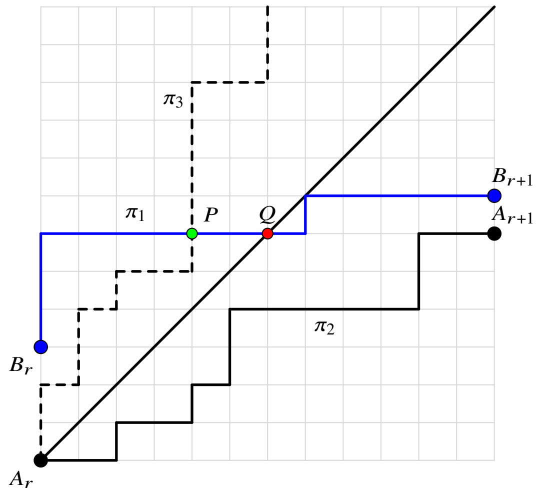
[b]0.45 

Figure 4.18:

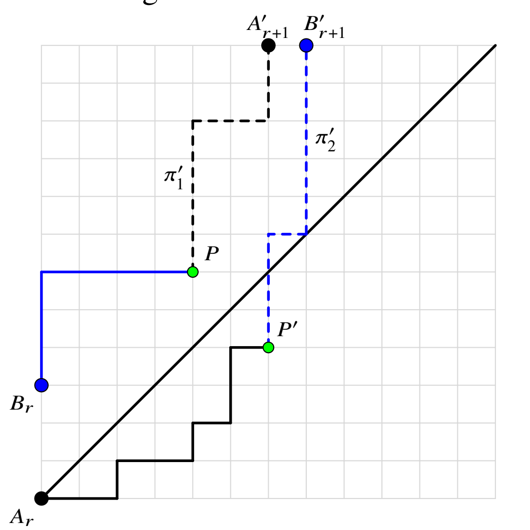
[b]0.45 

Figure 4.19:

Figure 4.20: The $j = r$ case. π_1 and π_2 are black and blue paths in Figure (A) respectively. π_3 is the black dashed path in Figure (A). π'_1 is the path in Figure (B) which is formed by the concatenation of the solid blue path and the black dashed path. π'_2 is the path in Figure (B) which is formed by the concatenation of the solid black path and the blue dashed path. The U map takes π_1, π_2 and spits out π'_1, π'_2.

points. This proves (a). Since the construction involves only exchanges of reflected portions, due to the symmetry of the weights $\widetilde{W}_{i,j}$ across the diagonal, we have (b). Finally to verify (c), note that there are at most $2^{\mathrm{diag}(\pi'_1 \cup \pi'_2)}$ possible choices of $\mathrm{diag}(\pi_1)$ and $\mathrm{diag}(\pi_2)$ for a given pair of two paths (π'_1, π'_2) in the pre-image of U. As $\mathrm{diag}(\pi_1)$ and $\mathrm{diag}(\pi_2)$ uniquely determine SPDiag where reflections are performed, reverting the same operations on π'_1 and π'_2 between consecutive points in SPDiag leads to original π_1 and π_2. Thus the map has at most $2^{\text{diag}(\pi_1'\cup\pi_2')}$ inverses for (π_1', π_2'), which completes the proof. $\square$

Note that the U map in Lemma 4.3.1 gives us a path that does not contain any diagonal vertex. To capture the contribution of this path, we now introduce *diagonal-avoiding symmetrized* partition function. Let $\widetilde{\Pi}^{(1)}_{m,n}$ be the collection of all upright paths from $(1,1)$ to (m,n) that do not touch the diagonal after $(1,1)$. Set

$$\widetilde{Z}_{\text{sym}}(m,n) := \sum_{\pi\in\widetilde{\Pi}^{(1)}_{m,n}} \prod_{(i,j)\in\pi} \widetilde{W}_{i,j}, \quad \widetilde{V}_q := \sum_{(i,j)|i+j=q} \widetilde{Z}_{\text{sym}}(i,j) \tag{4.3.1}$$

where $\widetilde{W}_{i,j}$ is defined in (4.1.11). We call $\widetilde{Z}_{\text{sym}}(m,n)$ the *diagonal-avoiding symmetrized* partition function. Let us recall $Z_{\text{sym}}(m,n)$ from (4.2.1) and we similarly define

$$V_q := \sum_{(i,j)|i+j=q} Z_{\text{sym}}(i,j) \tag{4.3.2}$$

The next lemma establishes a relation between $Z^{(2k)}_{\text{sym}}(m,n)$, V_{m+n} and $\widetilde{V}_{m+n}$.

Lemma 4.3.2. *For all $(m,n)\in\mathcal{I}^-$, almost surely we have*

$$Z^{(2k)}_{\text{sym}}(m,n) \le 2^n \cdot \prod_{i=2}^{2k}\prod_{j=1}^{i-1}(\widetilde{W}_{1,j})^{-1} \cdot \prod_{i=1}^{k}\left[V_{m+n+2-2i}\widetilde{V}_{m+n+1-2i}\right] \tag{4.3.3}$$

where $Z^{(i)}_{\text{sym}}(m,n)$, $V_{m+n+2-2i}$ and $\widetilde{V}_{m+n+1-2i}$ are defined in (4.2.1), (4.3.2) *and* (4.3.1) *respectively.*

Proof. We extend our definition of U map from Lemma 4.3.1 to the domain $\Pi^{(2k)}_{m,n}$ by defining $U(\pi_1,\ldots,\pi_{2k}) := (U(\pi_1,\pi_2),\ldots,U(\pi_{2k-1},\pi_{2k}))$. Let $R^{i,k}_{m,n}$ be the collection of all upright paths from $(1, 2k-2i+1)$ to $(n-2i+2, m)$. Let $\widetilde{R}^{i,k}_{m,n}$ be the collection of all upright paths from $(1, 2k-2i+2)$ to $(n-2i+1, m)$ that avoid the diagonal. Given any $(\pi_1',\ldots,\pi_{2k}')\in U(\Pi^{(2k)}_{m,n})$, by (c), there are at most

$$\prod_{i=1}^{k} 2^{|(j,j)\in\pi'_{2i-1}\cup\pi'_{2i}|}$$

many inverses in the pre-image of the U map. The U map preserves the number of diagonal vertices by (a). Furthermore by non-intersection, a $2k$-tuple of paths in $\Pi_{m,n}^{(2k)}$ has at most n many diagonal vertices. Thus there are at most 2^n many inverses. Hence by (a) we have

$$Z_{\mathrm{sym}}^{(2k)}(m,n) \le 2^n \cdot \prod_{i=1}^{k}\left[\sum_{\pi_1\in\widetilde{R}_{m,n}^{i,k}}\prod_{(i,j)\in\pi_1}\widetilde{W}_{i,j}\right]\cdot\prod_{i=1}^{k}\left[\sum_{\pi_2\in R_{m,n}^{i,k}}\prod_{(i,j)\in\pi_2}\widetilde{W}_{i,j}\right]. \tag{4.3.4}$$

We may elongate each of the path in $R_{m,n}^{i,k}$ and $\widetilde{R}_{m,n}^{i,k}$ by appending an up-path from $(1,1)$ to $(1,2k-2i+2)$ and from $(1,1)$ to $(1,2k-2i+1)$ respectively. This produces elongated paths in $\widetilde{\Pi}_{n-2i+2,m}^{(1)}$ and $\Pi_{n-2i+1,m}$ respectively. In terms of weights, we need to multiply the existing weights in (4.3.4) by $\prod_{j=1}^{2k-2i+1}\widetilde{W}_{1,j}$ and $\prod_{j=1}^{2k-2i}\widetilde{W}_{1,j}$ respectively to get the corresponding weights of elongated paths. After doing precisely the above, we have

$$Z_{\mathrm{sym}}^{(2k)}(m,n) \le 2^n \cdot \prod_{i=2}^{2k}\prod_{j=1}^{i-1}(\widetilde{W}_{1,j})^{-1}\cdot\prod_{i=1}^{k}\left[Z_{\mathrm{sym}}(n-2i+2,m)\widetilde{Z}_{\mathrm{sym}}(n-2i+1,m)\right]. \tag{4.3.5}$$

We get (4.3.3) from the above inequality in (4.3.5) by observing the definition of V_q and $\widetilde{V}_q$ from (4.3.2). This completes the proof. □

The next lemma bounds $\log V_q$ and $\log\widetilde{V}_q$ from above with high probability.

Lemma 4.3.3. *Recall R from* (4.2.9)*. For every $\delta > 0$ and $1 \le p < n$, we have*

$$\lim_{q\to\infty}\mathbf{P}\left(\log V_q \le (R+\delta)\tfrac{q}{2}\right) = 1, \quad \lim_{q\to\infty}\mathbf{P}\left(\log\widetilde{V}_q \le (-2\Psi(\theta)+\delta)\tfrac{q}{2}\right) = 1. \tag{4.3.6}$$

Proof. Fix any $\delta > 0$. By Lemma 4.2.2 we have

$$V_{2N} = \sum_{p=0}^{N-1} Z(N+p, N-p), \quad V_{2N+1} = \sum_{p=1}^{N} Z(N+p, N-p+1).$$

From Theorem 4.2.7 ((4.2.11) in particular) we have that

$$\frac{1}{N}\log\left[\sum_{p=1}^{N} Z(N+p, N-p+1)\right] \xrightarrow{p} R, \quad \frac{1}{N}\log\left[\sum_{p=1}^{N-1} Z(N+p, N-p)\right] \xrightarrow{p} R, \tag{4.3.7}$$

Note that in the above equation, we have excluded $Z(N,N)$ as their result does not contain $Z(N,N)$ in the sum. However, in our case, we may include $Z(N,N)$ by appealing to Theorem 4.2.4. First, in view of the above law of large numbers in (4.3.7), we have

$$\mathbf{P}(\log V_{2N+1} \leq (R + \tfrac{1}{2}\delta)N) \to 1. \tag{4.3.8}$$

On the other hand, by (4.2.4) we have $\sum_{p=1}^{N} e^{H_N^{(1)}(2p)} = \sum_{p=1}^{N} Z(N+p, N-p+1)$. Since $H_N^{(1)}(2) \leq \log \sum_{p=1}^{N} e^{H_N^{(1)}(2p)} = \log V_{2N+1}$, (4.3.8) implies

$$\mathbf{P}(H_N^{(1)}(2) \leq (R + \tfrac{1}{2}\delta)N) \to 1,$$

as $N \to \infty$. In addition, by ordering of points in the line ensemble (Theorem 4.2.4) we know that with probability at least $1 - 2^{-N}$, $H_N^{(1)}(1) \leq H_N^{(2)}(2) + \log^2 N$. Thus we have

$$\mathbf{P}(H_N^{(1)}(1) \leq (R + \delta)N) \to 1, \tag{4.3.9}$$

as $N \to \infty$. Given that $H_N^{(1)}(1) = Z(N,N)$, combining (4.3.9) and the second convergence in (4.3.7) yields $\mathbf{P}(\log V_{2N} \leq (R+\delta)N) \to 1$ and together with (4.3.8) this concludes the proof of the first convergence in (4.3.6).

Next, for the diagonal-avoiding case, let $(W_{i,i}^{\alpha=0})_{i\geq 1}$ be a family of weights distributed as

Gamma(θ) independent of $(W_{i,j})$. We set $W_{i,j}^{\alpha=0} := \widetilde{W}_{i,j}$ for $i \neq j$. This gives us a new collection of symmetrized weights. We denote the corresponding symmetrized partition function and the diagonal-avoiding symmetrized partition function as $Z_{\mathrm{sym}}^{\alpha=0}$ and $\widetilde{Z}_{\mathrm{sym}}^{\alpha=0}$ respectively. Observe that

$$\widetilde{Z}_{\mathrm{sym}}(i,j) \leq \frac{\widetilde{W}_{1,1}}{W_{1,1}^{\alpha=0}} \cdot \widetilde{Z}_{\mathrm{sym}}^{\alpha=0}(i,j) \leq \frac{\widetilde{W}_{1,1}}{W_{1,1}^{\alpha=0}} \cdot Z_{\mathrm{sym}}^{\alpha=0}(i,j). \tag{4.3.10}$$

The first equality in (4.3.10) is due to the fact that the weight corresponding $(1,1)$ is common in all paths and that is the only diagonal weight that appears in the diagonal avoiding symmetrized partition functions. The next inequality is obvious as we have just removed the diagonal avoiding restriction. This leads to

$$\log \widetilde{V}_q \leq \log \widetilde{W}_{1,1} - \log W_{1,1}^{\alpha=0} + \log \left[\sum_{(i,j) \mid i+j=q} Z_{\mathrm{sym}}^{\alpha=0}(i,j) \right].$$

The first two terms on the right-hand side of the above display are tight. An upper bound on the third term can be computed by the exact same analysis as V_q. Indeed, the law of large numbers and Theorem 4.2.4 continue to hold for $\alpha = 0$ when R becomes $-2\Psi(\theta)$ (see the last point in Theorem 4.2.7). This concludes the proof of (4.3.6). □

Finally, with Lemmas 4.3.2 and 4.3.3 in place, we are ready to control the average law of large numbers of the top curves of the $\mathcal{HSLG}$ line ensemble.

Proposition 4.3.4. *Recall Δ_k, R in* (4.2.9)*. Fix any $\varepsilon > 0$ and $k \in \mathbb{Z}_{>0}$ large such that $\Delta_k > 0$. Then there exists $N_0(k,\varepsilon) > 2k+1$ such that for all $N \geq N_0$ we have*

$$\mathbf{P}\left(\sup_{p \in [\![1, 2N-4k+2]\!]} \frac{1}{2k} \sum_{i=1}^{2k} H_N^{(i)}(p) \leq (R - \tfrac{1}{2}\Delta_k)N \right) \geq 1 - \varepsilon.$$

In plain words, Proposition 4.3.4 claims that when k is taken large enough so that $\Delta_k > 0$, the average law of large numbers of top $2k$ curves is strictly less than R, which is the law of large numbers for point-to-(partial)line free energy process (see Theorem 4.2.7).

Proof. Fix any $\varepsilon > 0$. The definition of the $\mathcal{HSLG}$ line ensemble in (4.2.3) and (4.3.3) collectively yield that, for all $p \in [\![1, N-2k+1]\!]$,

$$\sum_{i=1}^{2k} H_N^{(i)}(2p) = 2k\log 2 + \log Z_{\mathrm{sym}}^{(2k)}(N+p, N-p+1)$$
$$\leq 2k\log 2 + N\log 2 - \log\left[\prod_{i=2}^{2k}\prod_{j=1}^{i-1}\widetilde{W}_{1,j}\right] + \log\prod_{i=1}^{k}\left[V_{2N+3-2i}\widetilde{V}_{2N+2-2i}\right],$$

where the r.h.s. is free of p. Hence we may take supremum over $p \in [\![1, N-2k+1]\!]$ over both sides of the above display to get

$$\begin{aligned}\sup_{p\in[\![1,N-2k+1]\!]}\sum_{i=1}^{2k} H_N^{(i)}(2p) \leq (2k+N)\log 2 - \log\left[\prod_{i=2}^{2k}\prod_{j=1}^{i-1}\widetilde{W}_{1,j}\right] \\ + \log\prod_{i=1}^{k}\left[V_{2N+3-2i}\widetilde{V}_{2N+2-2i}\right].\end{aligned} \tag{4.3.11}$$

We now provide high probability upper bounds for each of the terms on the r.h.s. of (4.3.11). Let us take $\delta := \frac{\Delta_k}{4}$. By Lemma 4.3.3, we may choose $N_0(k,\varepsilon) > 2k+1$ large enough such that for all $N \geq N_0$

$$\mathbf{P}(\log V_N \leq (R+\delta)\tfrac{N}{2}) \geq 1 - \tfrac{\varepsilon}{8k}, \quad \mathbf{P}(\log \widetilde{V}_N \leq (-2\Psi(\theta)+\delta)\tfrac{N}{2}) \geq 1 - \tfrac{\varepsilon}{8k}.$$

Thus applying a union bound we see that for all large enough N, with probability $1 - \frac{\varepsilon}{4}$,

$$\log\prod_{i=1}^{k}\left[V_{N+3-2i}\widetilde{V}_{N+2-2i}\right] \leq RkN - 2\Psi(\theta)kN + 2k\delta N. \tag{4.3.12}$$

Note that the random variable $\log\left[\prod_{i=2}^{2k}\prod_{j=1}^{i-1}\widetilde{W}_{1,j}\right]$ is tight and free of N. Hence with probability

$1-\frac{\varepsilon}{4}$ one can ensure that

$$(2k+N)\log 2 - \log\left[\prod_{i=2}^{2k}\prod_{j=1}^{i-1}\widetilde{W}_{1,j}\right] \le N\log 2 + 2k\delta N. \tag{4.3.13}$$

holds for all large enough N. Inserting the above two bounds in (4.3.12) and (4.3.13) back in (4.3.11), we have that with probability at least $1-\frac{\varepsilon}{2}$,

$$\sup_{p\in[\![1,N-2k+1]\!]} \frac{1}{2k}\sum_{i=1}^{2k} H_N^{(i)}(2p) \le \left[\tfrac{1}{2k}\log 2 + 2\delta + \tfrac{R}{2} - \Psi(\theta)\right] N, \tag{4.3.14}$$

for all large enough N. As $\delta = \frac{\Delta_k}{4}$, the r.h.s. of (4.3.14) is precisely $(R-\frac{1}{2}\Delta_k)N$. By the exact same argument, one can check that with probability at least $1-\frac{\varepsilon}{2}$ we have

$$\sup_{p\in[\![0,N-2k]\!]} \frac{1}{2k}\sum_{i=1}^{2k} H_N^{(i)}(2p+1) \le (R-\tfrac{1}{2}\Delta_k)N, \tag{4.3.15}$$

for all large enough N. Taking another union bound of (4.3.14) and (4.3.15), we get the desired result. □

4.4 Controlling the second curve

In this section, we establish the separation between the first and the second curve of our $\mathcal{HSLG}$ line ensemble. Appealing to Proposition 4.3.4, Lemma 4.4.1 first establishes that for large enough k with high probability the $(2k+2)$-th curve $H_N^{(2k+2)}(\cdot)$ is uniformly const $\cdot\, N$ below than RN over an interval of $[\![1,N]\!]$, where R defined in (4.2.9) is the law of large numbers for point-to-(partial)line free energy process (Theorem 4.2.7). This helps us show that with high probability the second curve $H_N^{(2)}(\cdot)$ over an interval of length $O(\sqrt{N})$ is $M\sqrt{N}$ below RN next in Proposition 4.4.2 for any $M>0$.

Lemma 4.4.1. *Recall R in* (4.2.9). *Fix any $\varepsilon>0$ and $k\in\mathbb{Z}_{>0}$ large enough such that $\Delta_k>0$.*

Then there exists $N_0(k,\varepsilon)$ such that for all $N \geq N_0$ we have

$$\mathbf{P}\left(\sup_{p\in[\![1,N]\!]} H_N^{(2k+2)}(p) \leq (R - \tfrac{1}{4}\Delta_k)N\right) \geq 1-\varepsilon. \tag{4.4.1}$$

Proof. Let us consider the following events

$$\begin{aligned}
\mathsf{A} &:= \left\{\sup_{p\in[\![1,N]\!]} H_N^{(2k+2)}(p) \leq (R-\tfrac{1}{4}\Delta_k)N\right\},\\
\mathsf{B} &:= \left\{H_N^{(i+1)}(p) \leq H_N^{(i)}(p) + 2\log^2 N, \text{ for all } i\in[\![1,2k+1]\!], p\in[\![1,N]\!]\right\},\\
\mathsf{C} &:= \left\{\sup_{p\in[\![1,N]\!]} \frac{1}{2k}\sum_{i=1}^{2k} H_N^{(i)}(p) \leq (R-\tfrac{1}{2}\Delta_k)N\right\}.
\end{aligned}$$

We claim that for all large enough N, we have $(\mathsf{B}\cap\neg\mathsf{A})\subset\neg\mathsf{C}$. To see this, note that on $\mathsf{B}\cap\neg\mathsf{A}$, there exists a point $p^*\in[\![1,N]\!]$ such that $H_N^{(2k+2)}(p^*) > (R-\frac{1}{4}\Delta_k)N)$ and hence (as B holds)

$$H_N^{(i)}(p^*) > (R-\tfrac{1}{4}\Delta_k)N - (4k+4)\log^2 N,$$

for all $i\in[\![1,2k+1]\!]$. However, the above display also implies that

$$\sup_{p\in[\![1,N]\!]} \frac{1}{2k}\sum_{i=1}^{2k} H_N^{(i)}(p) > (R-\tfrac{1}{4}\Delta_k)N - (4k+4)\log^2 N$$

which is strictly bigger than $(R-\frac{1}{2}\tau)N$ and implies $\neg\mathsf{C}$. Thus by a union bound, we have

$$\mathbf{P}(\neg\mathsf{A}) \leq \mathbf{P}(\neg\mathsf{B}) + \mathbf{P}(\mathsf{B}\cap\neg\mathsf{A}) \leq \mathbf{P}(\neg\mathsf{B}) + \mathbf{P}(\neg\mathsf{C}). \tag{4.4.2}$$

Note that for fixed k, by Theorem 4.2.4 with $\rho=\frac{1}{2}$ and a union bound, we have $\mathbf{P}(\neg\mathsf{B}) \leq N\cdot(2k+1)\cdot 2^{-N} \leq \frac{\varepsilon}{2}$ for all $N\geq N_1(k,\varepsilon)$. On the other hand, Proposition 4.3.4 yields that for fixed k and ε, $\mathbf{P}(\neg\mathsf{C}) \leq \frac{\varepsilon}{2}$ for all N greater than some $N_2(k,\frac{\varepsilon}{2})$. Letting $N_0(k,\varepsilon)=\max\{N_1,N_2\}$ and inserting these two bounds in (4.4.2) leads to (4.4.1). □

Building on Lemma 4.4.1, the next result demonstrates that on a given interval of length $O(\sqrt{N})$ starting from 1 and any $M_2 > 0$, the second curve $H_N^{(2)}(\cdot)$ is uniformly lower than $RN - M_2\sqrt{N}$ with high probability (see Figure 4.21).

Proposition 4.4.2. *Recall Δ_k, R in (4.2.9). Fix $\varepsilon \in (0,1)$, $M_1, M_2 \geq 1$ and $k \in \mathbb{Z}_{>0}$ such that $\Delta_k > 0$. Then there exists a constant $N_2(\varepsilon, M_1, M_2) > 0$ such that for all $N \geq N_2$ we have*

$$\mathbf{P}\left(\sup_{p\in[1,2\lfloor M_1\sqrt{N}\rfloor+1]} H_N^{(2)}(p) \leq RN - M_2\sqrt{N}\right) \geq 1 - \tfrac{1}{2}\epsilon. \tag{4.4.3}$$

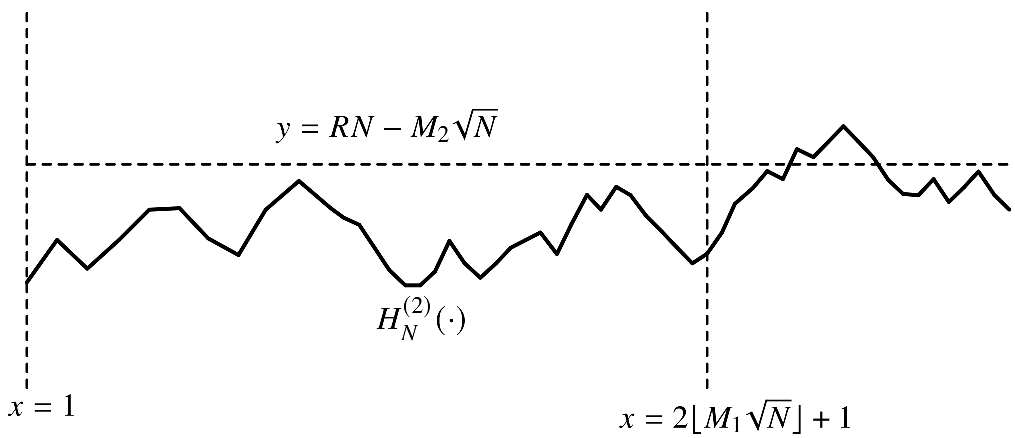


Figure 4.21: The high probability event in Proposition 4.4.2.

Proof. The proof of Proposition 4.4.2 is conducted in the following stages:

- Using Theorem 4.2.7 and Lemma 4.4.1, we determine high probability locations of $H_N^{(1)}(2M\sqrt{N}+1)$ and $H_N^{(2k+2)}(\cdot)$. Using the ordering of points in Theorem 4.2.4, we then bound the endpoints $H_N^{(i)}(2M\sqrt{N}+1)$, $i \in [\![1, 2k+1]\!]$ from above based on the high probability locations of $H_N^{(1)}(2M\sqrt{N}+1)$ and the $(2k+2)$-th curve.

- We next consider the conditional law of $(H_N^{(i)}[\![1, 2M\sqrt{N}]\!])_{i\in[\![1,2k+1]\!]}$ given the above boundary conditions. By Theorem 4.2.2, this law is given by an appropriate $\mathcal{HSLG}$ Gibbs measure. Applying stochastic monotonicity, we may also assume that the $H_N^{(2i-1)}(2M\sqrt{N}+1)$

and $H_N^{(2i)}(2M\sqrt{N}+1)$ are sufficiently far apart. This will allow us to approximate the Gibbs measure as a product of interacting random walks defined in Definition 4.2.5.

- Lastly, we use the associated estimates of interacting random walks from Proposition 4.2.6 to dissect the Gibbs measure and yield a quantitative bound in our favor.

Let us begin by fleshing out the technical details of the above stages. In the following proof, we assume all the multiples of $\sqrt{N}$ appearing below are integers for convenience in notation. The general case follows verbatim by considering the floor function. For clarity, we split our proof into several steps.

Step 1. In this step, we reduce our proof of (4.4.3) to (4.4.7). Let us consider the $\mathcal{HSLG}$ line ensemble $H_N = (H_N^{(1)}, \ldots, H_N^{(N)})$. Fix any $\varepsilon \in (0,1)$, $M_1, M_2 \geq 1$ and any $k \in \mathbb{Z}_{>0}$ such that $\Delta_k > 0$. Let $\Phi(x)$ be the cumulative distribution function of a standard Gaussian random variable. Set $\tau := |\Psi(\theta-\alpha) - \Psi(\theta+\alpha)|$. Let $M \in \mathbb{Z}_{>0}$ whose precise value is to be determined. Taking $g(N) = M\sqrt{N}$ in Theorem 4.2.7 yields

$$\frac{1}{\sigma\sqrt{N}}\left[\log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) - RN + M\tau\sqrt{N}\right] \xrightarrow{d} \mathcal{N}(0,1). \tag{4.4.4}$$

Note that (4.4.4) implies

$$\mathbf{P}\left(\frac{1}{\sigma\sqrt{N}}\left[\log\left[\mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N})\right] - RN + M\tau\sqrt{N}\right] \leq \Phi(1-\frac{\varepsilon}{2})\right) \to 1-\frac{\varepsilon}{2}.$$

Thus for N large enough, we have that with probability greater than $1-\varepsilon$,

$$\log\left[\mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N})\right] \leq RN - \big(M\tau - \Phi(1-\tfrac{\varepsilon}{2})\sigma\big)\sqrt{N}. \tag{4.4.5}$$

Set $M \geq \max\{M_1, \frac{1}{\tau}(M_2+k+1+\Phi(1-\frac{\varepsilon}{2})\sigma)\}$. Note that by definition, $H_N^{(1)}(2M\sqrt{N}+1) \leq \log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N})$ and as $M\tau - \Phi(1-\frac{\varepsilon}{2})\sigma > M_2+k+1$, (4.4.5) yields that

$$\mathbf{P}(\mathsf{A}) \geq 1-\varepsilon, \text{ where } \mathsf{A} := \left\{H_N^{(1)}(2M\sqrt{N}+1) \leq RN - (M_2+k+1)\sqrt{N}\right\} \tag{4.4.6}$$

for all large enough N. Set $T = M\sqrt{N} + 1$. We claim that

$$\mathbf{P}(\neg \mathsf{E}) \le 3\varepsilon + \frac{k\varepsilon}{(1-\varepsilon)^{k+1}}, \text{ where } \mathsf{E} := \left\{ \sup_{p \in [\![1, 2T-1]\!]} H_N^{(2)}(p) \le RN - M_2\sqrt{N} \right\}. \tag{4.4.7}$$

Since $2T - 1 \ge 2M_1\sqrt{N} + 1$, assuming (4.4.7) and adjusting ε yield (4.4.3).

Step 2. In this step we prove (4.4.7). To begin with, we consider several events:

$$\begin{aligned}
\mathsf{B} &:= \bigcap_{i=1}^{2k} \Big\{ H_N^{(i+1)}(2T) \le H_N^{(i)}(2T-1) + \log^2 N, \\
&\qquad\qquad\qquad\qquad H_N^{(i+1)}(2T-1) \le H_N^{(i+1)}(2T) + \log^2 N \Big\}, \\
\mathsf{C} &:= \left\{ \sup_{p \in [\![1, N]\!]} H_N^{(2k+2)}(p) \le (R - \tfrac{1}{4}\Delta_k)N \right\}, \\
\mathsf{D} &:= \bigcap_{i=1}^{k} \Big\{ \max\left\{ H_N^{(2i)}(2T-1), H_N^{(2i)}(2T), H_N^{(2i+1)}(2T) \right\} \\
&\qquad\qquad\qquad\qquad \le RN - (M_2 + k + 1)\sqrt{N} + 2k \log^2 N \Big\}.
\end{aligned}$$

Let us consider the σ-field

$$\begin{aligned}
\mathcal{F} := \sigma \Big\{ & H_N^{(2i)}[\![2T-1, 2N-4i+2]\!], H_N^{(2i+1)}[\![2T, 2N-4i]\!],\ i \in [\![1,k]\!], \\
& H_N^{(1)}[\![1, 2N]\!], H_N^{(j)}[\![1, 2N-2j+2]\!],\ j \in [\![2k+2, N]\!] \Big\}.
\end{aligned}$$

By Theorem 4.2.4 with $\rho = \frac{1}{2}$, we have $\mathbf{P}(\neg \mathsf{B}) \le 4k2^{-N} \le \varepsilon$ for all large enough N. Observe that $\mathsf{A} \cap \mathsf{B} \subset \mathsf{D}$ and recall that $\mathbf{P}(\neg \mathsf{A}) < \varepsilon$ in (4.4.6). Thus via the union bound, we have $\mathbf{P}(\neg \mathsf{D}) \le \mathbf{P}(\neg \mathsf{A}) + \mathbf{P}(\neg \mathsf{B}) \le 2\varepsilon$. Note that $\mathsf{C} \cap \mathsf{D}$ is measurable w.r.t. $\mathcal{F}$. Applying the union bound and tower property of conditional expectation we get

$$\mathbf{P}(\neg \mathsf{E}) \le \mathbf{P}(\neg \mathsf{C}) + \mathbf{P}(\neg \mathsf{D}) + \mathbf{P}(\mathsf{C} \cap \mathsf{D} \cap \neg \mathsf{E}) \le 3\varepsilon + \mathbf{E}\left[\mathbf{1}_{\mathsf{C} \cap \mathsf{D}} \cdot \mathbf{E}\left[\mathbf{1}_{\neg \mathsf{E}} \mid \mathcal{F} \right] \right]. \tag{4.4.8}$$

where in the last inequality we have used Lemma 4.4.1 to get that $\mathbf{P}(\neg \mathsf{C}) \le \varepsilon$ for all large enough

N. We claim that

$$\mathbf{E}\left[\mathbf{1}_{\mathsf{C}\cap\mathsf{D}}\cdot\mathbf{E}\left[\mathbf{1}_{\neg\mathsf{E}}\mid\mathcal{F}\right]\right]\le\frac{k\varepsilon}{(1-\varepsilon)^{k+1}}. \tag{4.4.9}$$

We will demonstrate (4.4.9) in the **Steps 3-4**. Currently, assuming the validity of (4.4.9) and appealing to (4.4.8) prove (4.4.7).

Step 3. In this step we study $\mathbf{1}_{\mathsf{C}\cap\mathsf{D}}\mathbf{E}\left[\mathbf{1}_{\neg\mathsf{E}}\mid\mathcal{F}\right]$ by invoking the Gibbs property (Theorem 4.2.2). Let us consider the domain

$$\Theta_{k,T}:=\{(i,j)\mid i\in[\![2,2k+1]\!],\, j\in[\![1,2T-1-\mathbf{1}_{i=\text{even}}]\!]\}.$$

By Theorem 4.2.2, the distribution of the line ensemble conditioned on $\mathcal{F}$ is given by $\mathbf{P}^{\vec{u}}_{\text{gibbs}}$, i.e.

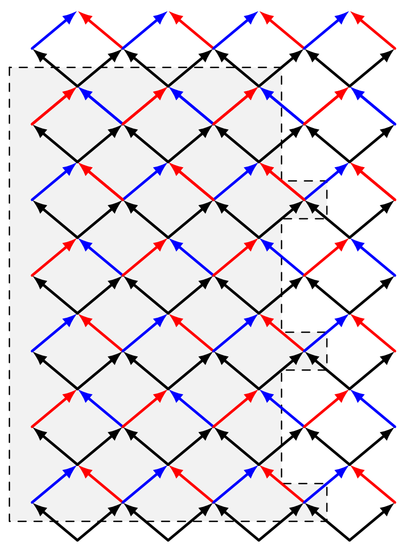


Figure 4.22: $\Theta_{k,T}$ for $k=3$, $T=4$ shown in the shaded region. The $\mathcal{HSLG}$ Gibbs measure on $\Theta_{3,4}$ with boundary condition $(u_{i,j})_{(i,j)\in\partial\Theta_{3,4}}$.

the $\mathcal{HSLG}$ Gibbs measure on the domain $\Theta_{k,T}$ with boundary condition $\vec{u}:=(H_N^{(i)}(j))_{(i,j)\in\partial\Theta_{k,T}}$ and the boundary set of $\Theta_{k,T}$ is given by

$$\partial\Theta_{k,T}:=\{(1,2j-1),(2,2T-1),(3,2T),(2i,2T-1),(2i,2T),(2i+1,2T)\mid i\in[\![2,k]\!],\, j\in[\![1,T]\!]\}.$$

Note that for large enough N, on the event $\mathsf{C} \cap \mathsf{D}$ we have

$$
\begin{aligned}
&H_N^{(1)}(2j-1) \le x_{1,2j-1} := \infty, \quad j \in [\![1,T]\!], \\
&H_N^{(2i)}(2T-1) \le x_{2i,2T-1} = RN - (M_2+i)\sqrt{N}, \quad i \in [\![1,k]\!], \\
&H_N^{(2i)}(2T) \le x_{2i,2T} := RN - (M_2+i)\sqrt{N}, \quad i \in [\![2,k]\!], \\
&H_N^{(2i+1)}(2T) \le x_{2i+1,2T} := RN - (M_2+i)\sqrt{N} - \sqrt{T}, \quad i \in [\![1,k]\!], \\
&H_N^{(2k+2)}(2j) \le x_{2k+2,2j} := RN - (M_2+k+1)\sqrt{N}, \quad j \in [\![1,T]\!].
\end{aligned}
\tag{4.4.10}
$$

where C holds only in the last inequality. Since $\neg\mathsf{E}$ event is increasing with respect to the boundary data, by stochastic monotonicity we have

$$
\mathbf{1}_{\mathsf{C}\cap\mathsf{D}} \cdot \mathbf{E}\left[\mathbf{1}_{\neg\mathsf{E}} \mid \mathcal{F}\right] \le \mathbf{1}_{\mathsf{C}\cap\mathsf{D}} \cdot \mathbf{P}_{\text{gibbs}}^{\vec{u}}(\neg\mathsf{E}) \le \mathbf{P}_{\text{gibbs}}^{\vec{x}}(\neg\mathsf{E}). \tag{4.4.11}
$$

To bound $\mathbf{P}_{\text{gibbs}}^{\vec{x}}(\neg\mathsf{E})$ we seek for a convenient alternative representation for the $\mathbf{P}_{\text{gibbs}}^{\vec{x}}$ measure. Towards this end, by carefully studying the Gibbs measure, we dissect the $\mathbf{P}_{\text{gibbs}}^{\vec{x}}$ measure into blocks of independent interacting random walks (Definition 4.2.5) and the Radon-Nikodym derivatives interleaved between adjacent blocks (see Figure 4.23). Let us now describe this decomposition.

Recall the interacting random walk (IRW) from Definition 4.2.5. Let $(L_{2i}[\![1,2T-2]\!], L_{2i+1}[\![1,2T-1]\!])_{i=1}^{k}$ be k independent IRWs of length T with boundary condition $(x_{2i,2T-1}, x_{2i+1,2T})$. Let us denote the joint law and expectation of L as $\mathbf{P}_{\text{block}}^{\vec{x}}$ and $\mathbf{E}_{\text{block}}^{\vec{x}}$ respectively. Set

$$
W_{\text{br}} := \exp\left(-\sum_{i=1}^{k}\sum_{j=1}^{T}\left[e^{L_{2i+2}(2j)-L_{2i+1}(2j+1)} + e^{L_{2i+2}(2j)-L_{2i+1}(2j-1)}\right]\right) \tag{4.4.12}
$$

with the convention $L_{2i+1}(2T+1) = \infty$ for $i \in [\![1,k]\!]$ and $L_i(j) = x_{i,j}$ for all $(i,j) \in \partial\Theta_{k,T}$. Note that here only $H_N^{(1)}(2j+1)$, $j \in [\![1,T]\!]$ are in the boundary and are set to ∞ in (4.4.10). Thus, their contributions to the Radon-Nikodym derivative W_{br} would be $\prod_{j=1}^{2T-2} \exp(-e^{H_N^{(2)}(j)-\infty}) = 1$. From the description of the $\mathcal{HSLG}$ Gibbs measure, we have

$$\mathbf{P}^{\vec{x}}_{\text{gibbs}}(\neg\mathsf{E}) = \frac{\mathbf{E}^{\vec{x}}_{\text{block}}[W_{\text{br}}\mathbf{1}_{\neg\mathsf{E}}]}{\mathbf{E}^{\vec{x}}_{\text{block}}[W_{\text{br}}]}, \tag{4.4.13}$$

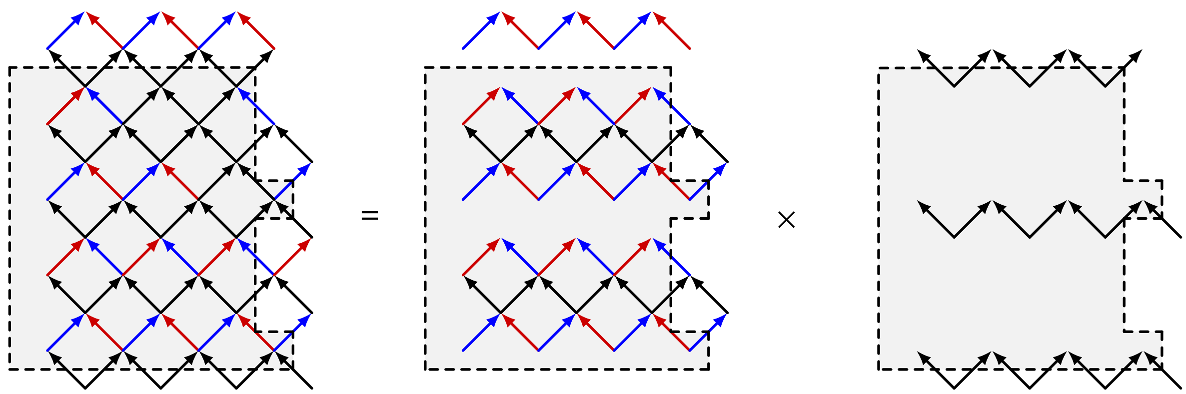


Figure 4.23: Proof Scheme: The Gibbs measure on $\Theta_{2,4}$ domain (left figure) can be decomposed into two parts: One is the combination of the top colored row and 2 IRWs (middle figure) and two are the remaining black weights (right figure) which will be viewed as a Radon-Nikodym derivative. Here note that in the middle figure, the only contribution from the top row comes from the odd points, $H_N^{(1)}(2j-1)$ for $j \in [\![1,T]\!]$, which are set to ∞. Thus, their contribution to (4.4.12) from (4.2.6) would be $\exp(-e^{-\infty}) = 1$.

Step 4. Finally in this step, we provide an upper bound for the right-hand side of (4.4.13) by bounding its numerator and denominator separately. Let us consider the event:

$$\mathsf{G} := \bigcap_{i=1}^{k} \left\{ \sup_{p\in[\![1,2T-1]\!]} |L_{2i}(p) - x_{2i,2T-1}| + \sup_{q\in[\![1,2T]\!]} |L_{2i+1}(q) - x_{2i,2T-1}| \le M_0\sqrt{T} \right\}.$$

where M_0 comes from Proposition 4.2.6. From the description of the Gibbs measure, it is clear that if $(L_{2i}(\cdot), L_{2i+1}(\cdot))$ is an IRW with boundary condition $(x_{2i,2T-1}, x_{2i,2T-1} - \sqrt{T})$, then $(L_{2i}(\cdot) - x_{2i,2T-1}, L_{2i+1}(\cdot) - x_{2i,2T-1})$ is an IRW with boundary condition $(0, -\sqrt{T})$. Thus, appealing to Proposition 4.2.6, we see that

$$\mathbf{P}^{\vec{x}}_{\text{block}}(\mathsf{G}) \ge (1-\varepsilon)^k \ge 1 - k\varepsilon.$$

Let us assume N is large enough so that $\sqrt{N} - 2M_0\sqrt{T} \ge \frac{1}{2}\sqrt{N}$ (recall $T = O(\sqrt{N})$). Observe that under the event G, we have for all $p \le 2T-1$

$$L_{2i}(p) \le x_{2,2T-1} + M_0\sqrt{T} = RN - (M_2+1)\sqrt{N} + M_0\sqrt{T} \le RN - M_2\sqrt{N}.$$

Thus, E defined in (4.4.7) holds. This implies $\neg\mathsf{E} \subset \neg\mathsf{G}$. Hence

$$\mathbf{E}^{\vec{x}}_{\text{block}}[W_{\text{br}}\mathbf{1}_{\neg\mathsf{E}}] \le \mathbf{P}^{\vec{x}}_{\text{block}}(\neg\mathsf{E}) \le \mathbf{P}^{\vec{x}}_{\text{block}}(\neg\mathsf{G}) \le k\varepsilon. \tag{4.4.14}$$

$$\mathbf{1}_{\mathsf{G}} \cdot W_{\text{br}} \ge \mathbf{1}_{\mathsf{G}} \cdot \exp\left(-k(2T-1)e^{\sqrt{N}-2M_0\sqrt{T}}\right) \ge (1-\varepsilon).$$

where the last one follows by taking N large enough (recall $T = O(\sqrt{N})$). Thus, $\mathbf{E}_{\text{block}}[W_{\text{br}}] \ge (1-\varepsilon)\mathbf{P}_{\text{block}}(\mathsf{G}) \ge (1-\varepsilon)^{k+1}$. Inserting this bound and the bound in (4.4.14) back in (4.4.13) we get that $\mathbf{P}^{\vec{x}}_{\text{gibbs}}(\neg\mathsf{E}) \le \frac{k\varepsilon}{(1-\varepsilon)^{k+1}}$. Combining this bound with (4.4.11) yields (4.4.9). This completes the proof. □

4.5 Proof of main theorems

In this section, we prove our main theorems, Theorems 4.1.1, 4.1.3, and 4.1.4. This section is structured as follows: In Section 4.5.1 we first present a few supporting technical results. In Section 4.5.2 we complete the proof of our main theorems by assuming a technical proposition (Proposition 4.5.3) which in turn is proved in Section 6.3.1.

4.5.1 Preparatory lemmas

In this section, we prove two preparatory lemmas that will serve as necessary ingredients in proving our main theorems. Recall the polymer measure $\mathbf{P}^W$ from (4.1.2), the partition function $Z(m,n)$ from (4.1.4), and the $\mathcal{HSLG}$ line ensemble H_N from Definition 4.2.1. Note that the quenched distribution of the endpoint of the polymer is related via

$$\mathbf{P}^W(\pi(2N-2) = N-r) = \frac{Z(N+r, N-r)}{\sum_{p=0}^{N-1} Z(N+p, N-p)} = \frac{e^{H_N^{(1)}(2r+1)}}{\sum_{p=0}^{N-1} e^{H_N^{(1)}(2p+1)}}. \tag{4.5.1}$$

where the second equality follows from the relation (4.2.4). Recalling $\mathcal{Z}_N^{\mathsf{PL}}(k) = \sum_{p=k}^{N-1} e^{H_N^{(1)}(2p+1)}$ from (4.2.7), we obtain

$$\mathbf{P}^W(\pi(2N-2) \le N-k) = \frac{\mathcal{Z}_N^{\mathsf{PL}}(k)}{\mathcal{Z}_N^{\mathsf{PL}}(0)}.$$

Theorem 4.1.1 claims that this quenched probability decays as $N \to \infty$ followed by $k \to \infty$. The following lemma settles a weaker version of Theorem 4.1.1 where we take $k = \lfloor M\sqrt{N} \rfloor$. For notational convenience, we assume all the multiples of $\sqrt{N}$ appearing in the proofs in this section are integers. The general case follows verbatim by considering the floor function.

Lemma 4.5.1. *Fix $\varepsilon > 0$ and recall $\mathcal{Z}_N^{\mathsf{PL}}(\cdot)$ from Theorem 4.2.7. There exist constants $M(\varepsilon) > 0, N_1(\varepsilon) > 0$ such that for all $N \ge N_1$,*

$$\mathbf{P}\left(\frac{\mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N})}{\mathcal{Z}_N^{\mathsf{PL}}(1)} \le e^{-\sqrt{N}} \right) \ge 1 - \tfrac{1}{2}\varepsilon. \tag{4.5.2}$$

Proof. Fix $\varepsilon \in (0,1)$. Recall σ from (4.2.9) Taking $g = 1$ and $g = M\sqrt{N}$ in Theorem 4.2.7 yields

$$\frac{1}{\sigma\sqrt{N}}\left[\log \mathcal{Z}_N^{\mathsf{PL}}(1) - RN\right] \xrightarrow{d} \mathcal{N}(0,1),$$

$$\frac{1}{\sigma\sqrt{N}}\left[\log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) - RN + M\tau\sqrt{N}\right] \xrightarrow{d} \mathcal{N}(0,1) \tag{4.5.3}$$

respectively, where R, σ, τ are defined in (4.2.9). Let us set $P := P(\varepsilon) = \Phi^{-1}(1-\frac{\varepsilon}{8}) + 1$, where $\Phi(\cdot)$ is the cumulative distribution function of $\mathcal{N}(0,1)$. For all large enough N we have

$$\mathbf{P}\left(\log \mathcal{Z}_N^{\mathsf{PL}}(1) \ge RN - P\sigma\sqrt{N}\right) \ge 1 - \tfrac{\varepsilon}{4},$$

$$\mathbf{P}\left(\log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) \leq RN - M\tau\sqrt{N} + P\sigma\sqrt{N}\right) \geq 1 - \tfrac{\varepsilon}{4}.$$

Applying a union bound gives us

$$\mathbf{P}\left(\log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) + (M\tau - 2P\sigma)\sqrt{N} \leq \log \mathcal{Z}_N^{\mathsf{PL}}(1)\right) \geq 1 - \tfrac{\varepsilon}{2},$$

for all large enough N. Taking $M := \frac{1}{\tau}(2P\sigma + 1)$ in above equation leads to (4.5.2). This completes the proof. □

Let us recall our discussion in Section 4.1.2 and Figure 4.7. Let us call the region $[\![N - M\sqrt{N}, N-k]\!]$ and the region $[\![1, N - M\sqrt{N}]\!]$ as shallow tail and deep tail respectively (see Figure 4.7). Lemma 4.5.1 implies that with high probability the quenched probability of $\pi(2N-2)$ living in the deep tail region is exponentially small. Thus the mass accumulates in the window of $M\sqrt{N}$ below the point (N, N). To establish Theorem 4.1.1, we thus have to show the mass in the shallow tail also goes to zero. For convenience, in our proofs below we shall often refer to the point $(N + M\sqrt{N}, N - M\sqrt{N})$ as the deep tail starting point. Given the connection in (4.2.4), the deep tail starting point corresponds to $(2M\sqrt{N}+1)$-th point for the top curve $H_N^{(1)}(\cdot)$ of the $\mathcal{HSLG}$ line ensemble. So, in the coordinates of the $\mathcal{HSLG}$ line ensemble, we shall refer $2M\sqrt{N}+1$ as the deep tail starting point.

Below, we record another important preparatory lemma which claims the existence of a "high point" in $H_N^{(1)}(\cdot)$ not far after the deep tail starting point (see Figure 4.24).

Lemma 4.5.2. *Fix any $\varepsilon > 0$ and recall R, τ from* (4.2.9). *There exists a constant $M_0(\varepsilon) > 0$ such that for all $M \geq M_0$, there exists $N_0(\varepsilon, M)$ such that for all $N \geq N_0$,*

$$\mathbf{P}\left(\sup_{p \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]} H_N^{(1)}(2p+1) \geq RN - \tfrac{5}{2}M\tau\sqrt{N}\right) \geq 1 - \tfrac{1}{2}\varepsilon, \qquad (4.5.4)$$

where $\tau := \Psi(\theta - \alpha) - \Psi(\theta + \alpha)$.

Proof. Let us set $P := P(\varepsilon) = \Phi^{-1}(1 - \frac{\varepsilon}{6}) + 1$, where $\Phi(\cdot)$ is the cumulative distribution function of $\mathcal{N}(0,1)$. In view of (4.5.3), for all large enough N we have

$$\mathbf{P}\left(\log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) \geq RN - M\tau\sqrt{N} - P\sigma\sqrt{N}\right) \geq 1 - \tfrac{\varepsilon}{6}, \tag{4.5.5}$$

$$\mathbf{P}\left(\log \mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) \leq RN - 2M\tau\sqrt{N} + P\sigma\sqrt{N}\right) \geq 1 - \tfrac{\varepsilon}{6}.$$

Applying a union bound gives us

$$\mathbf{P}\left(\log \mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) + (M\tau - 2P\sigma)\sqrt{N} \leq \log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N})\right) \geq 1 - \tfrac{\varepsilon}{3}.$$

Thus for any $M \geq \frac{2P\sigma+1}{\tau}$, we have that with probability at least $1-\frac{\varepsilon}{3}$, $\log \mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) \leq \log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) - \sqrt{N}$, which implies

$$2\mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) \leq \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}).$$

However, by definition of $\mathcal{Z}_N^{\mathsf{PL}}(\cdot)$, the above display implies that with probability at least $1-\frac{\varepsilon}{3}$,

$$\begin{aligned}\sup_{p\in[\![M\sqrt{N},2M\sqrt{N}]\!]} H_N^{(1)}(2p+1) &\geq \log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) - \log \mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) - \log(2M\sqrt{N}) \\ &\geq \log \mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) - \log(2M\sqrt{N}).\end{aligned}$$

Note that by the first entry in (4.5.5) with M substituted by $2M$, with probability at least $1-\frac{\varepsilon}{6}$, we have $\log \mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) \geq RN - 2M\tau\sqrt{N} - P\sigma\sqrt{N}$. Since for all large enough N, we have $RN - (2M\tau + P\sigma)\sqrt{N} - \log(2M\sqrt{N}) \geq RN - \frac{5}{2}M\tau\sqrt{N}$. Thus applying another union bound helps us arrive at (4.5.4) and complete the proof. □

4.5.2 Proof of Theorems 4.1.1, 4.1.3, and 4.1.4

In this section, we prove our main theorems assuming a technical proposition. Let us first begin by describing the proposition. Fix any $M, N \geq 1$ and assume $M\sqrt{N} \in \mathbb{Z}_{>0}$. For any Borel set A of $\mathbb{R}^{M\sqrt{N}}$ we consider the event

$$\mathsf{A} = \left\{ (H_N^{(1)}(1) - H_N^{(1)}(2r+1))_{r=1}^{M\sqrt{N}} \in A \right\}. \tag{4.5.6}$$

for $N > M^2 + 1$. Let $(S_r)_{r=0}^{M\sqrt{N}}$ be the log-gamma random walk defined in Definition 4.1.2. We write

$$\mathbf{P}_{RW}(\mathsf{A}) := \mathbf{P}\left((S_r)_{r=1}^{M\sqrt{N}} \in A \right) \tag{4.5.7}$$

Finally, the last technical proposition below is the main crux of the proof. It claims that $\mathbf{P}$ and $\mathbf{P}_{RW}$ are close to each other when N is large and we postpone its proof to Section 6.3.1.

Proposition 4.5.3. *Fix any $\varepsilon \in (0, \frac{1}{2})$. Set $M(\varepsilon) > 0, N_1(\varepsilon) > 0$ such that Lemma 4.5.1 and Lemma 4.5.2 hold simultaneously for all $N \geq N_1$ for this fixed choice of M. Then there exists $N_0(\varepsilon) > 0$ such that for all $N \geq N_0$,*

$$|\mathbf{P}(\mathsf{A}) - \mathbf{P}_{RW}(\mathsf{A})| \leq 9\varepsilon, \tag{4.5.8}$$

where A and $\mathbf{P}_{RW}(\mathsf{A})$ are defined in (4.5.6) *and* (4.5.7).

In lieu of these results, we are ready to prove our main theorems. Theorems 4.1.3 and 4.1.1 are direct applications of the supporting lemmas. For convenience, we shall assume in the proofs below $M\sqrt{N}$ is an integer. The general case follows verbatim by considering floor functions.

Proof of Theorem 4.1.6. Given (4.1.8), it suffices to check that

$$\tfrac{1}{\sqrt{N}} (\log Z(N,N) - \log Z(N + a_N, N - a_N)) \xrightarrow{p} 0,$$

where $\{a_N\}_{N\geq 1}$ is a sequence of nonnegative integers less than N, with $a_N/\sqrt{N} \to 0$. In light of (4.2.4), it boils down to checking

$$\tfrac{1}{\sqrt{N}}\left(H_N^{(1)}(1) - H_N^{(1)}(2a_N+1)\right) \xrightarrow{p} 0.$$

But thanks to Proposition 4.5.3, it is equivalent to argue that $S_{a_N}/\sqrt{N} \xrightarrow{p} 0$ where $(S_r)_{r\geq 0}$ is the log-gamma random walk defined in Definition 4.1.2. Since the increment of the walk has the finite first moment and $a_N/\sqrt{N} \to 0$, by Markov inequality we deduce that $S_{a_N}/\sqrt{N} \xrightarrow{p} 0$. This establishes Theorem 4.1.6. □

Proof of Theorem 4.1.3. Take the set A as $(-\infty, x_1] \times (-\infty, x_2] \times \cdots \times (-\infty, x_k] \times \mathbb{R}^{M\sqrt{N}-k}$ in (4.5.6). By Proposition 4.5.3,

$$\limsup_{N\to\infty}\left|\mathbf{P}\left(\bigcap_{r=1}^{k}\{H_N^{(1)}(1) - H_N^{(1)}(2r+1) \in (-\infty, x_r]\}\right) - \mathbf{P}_{RW}\left(\bigcap_{r=1}^{k}\{S_r \in (-\infty, x_r]\}\right)\right| \leq 9\varepsilon,$$

where $(S_r)_{r=0}^k$ is defined in Definition (4.1.2). As ε is arbitrary, this implies

$$\left(H_N^{(1)}(1) - H_N^{(1)}(2r+1)\right)_{r=0}^{k} \xrightarrow{d} (S_r)_{r=0}^k.$$

In conjunction with relation (4.2.4), we get the desired convergence in Theorem 4.1.3. □

Proof of Theorem 4.1.1. Fix any $\varepsilon > 0$. Get $M(\varepsilon), N_1(\varepsilon) > 0$ such that Lemma 4.5.1 and Lemma 4.5.2 hold simultaneously for all $N \geq N_1$ for this fixed choice of M. Using this M we split the probability as follows

$$\begin{aligned}
&\mathbf{P}^W(\pi(2N-2) \leq N-k) \\
&\qquad = \mathbf{P}^W\big(\pi(2N-2) \in (N - M\sqrt{N}, N-k]\big) + \mathbf{P}^W\big(\pi(2N-2) \leq N - M\sqrt{N}\big).
\end{aligned}$$

For the first term observe that by (4.5.1)

$$\mathbf{P}^W\big(\pi(2N-2)\in(N-M\sqrt{N},N-k]\big)=\frac{\sum_{p=k}^{M\sqrt{N}-1}e^{H_N^{(1)}(2p+1)}}{\sum_{p=0}^{N-1}e^{H_N^{(1)}(2p+1)}}$$
$$\leq\frac{\sum_{p=k}^{M\sqrt{N}}e^{H_N^{(1)}(2p+1)}}{\sum_{p=1}^{M\sqrt{N}}e^{H_N^{(1)}(2p+1)}}=\frac{\sum_{p=k}^{M\sqrt{N}}e^{H_N^{(1)}(2p+1)-H_N^{(1)}(1)}}{\sum_{p=1}^{M\sqrt{N}}e^{H_N^{(1)}(2p+1)-H_N^{(1)}(1)}}.$$

Fix any $\delta>0$ and consider the set

$$\mathsf{A}_\delta:=\left\{\frac{\sum_{p=K}^{M\sqrt{N}}e^{H_N^{(1)}(2p+1)-H_N^{(1)}(1)}}{\sum_{p=1}^{M\sqrt{N}}e^{H_N^{(1)}(2p+1)-H_N^{(1)}(1)}}\geq\delta\right\}.$$

By Proposition 4.5.3, $\mathbf{P}(\mathsf{A}_\delta)\leq\mathbf{P}_{RW}(\mathsf{A}_\delta)+9\varepsilon$ for all large enough N. On the other hand, by Corollary 4.6.3 we see that $\lim_{k\to\infty}\lim_{N\to\infty}\mathbf{P}_{RW}(\mathsf{A}_\delta)=0$. Thus, as ε is arbitrary,

$$\lim_{k\to\infty}\lim_{N\to\infty}\mathbf{P}^W\big(\pi(2N-2)\in(N-M\sqrt{N},N-k]\big)=0,\ \text{in probability.} \tag{4.5.9}$$

For the second term by Lemma 4.5.1, we see that with probability $1-\frac{\varepsilon}{2}$

$$\mathbf{P}^W\big(\pi(2N-2)\leq N-M\sqrt{N}\big)\leq\frac{\sum_{p=M\sqrt{N}}^{N-1}e^{H_N^{(1)}(2p+1)}}{\sum_{p=1}^{N-1}e^{H_N^{(1)}(2p+1)}}=\frac{\mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N})}{\mathcal{Z}_N^{\mathsf{PL}}(1)}\leq e^{-\sqrt{N}}.$$

Again, as ε is arbitrary, we have that as $N\to\infty$, $\mathbf{P}^W(\pi(2N-2)\leq N-M\sqrt{N})\to0$ in probability. This completes the proof together with (4.5.9). □

Lastly, with Theorems 4.1.1 and 4.1.3 established, we present the proof of the limiting quenched distribution of the endpoint viewed from around the diagonal.

Proof of Theorem 4.1.4. Fixed $\theta>0$ and $\alpha\in(-\theta,0)$. Recall from (4.5.1) that

$$\mathbf{P}_{\theta,\alpha;N}(\pi(2N-2)=N-r)=\frac{e^{H_N^{(1)}(2r+1)}}{\sum_{p=0}^{N-1}e^{H_N^{(1)}(2p+1)}}=\frac{e^{H_N^{(1)}(2r+1)-H_N^{(1)}(1)}}{\sum_{p=0}^{N-1}e^{H_N^{(1)}(2p+1)-H_N^{(1)}(1)}} \tag{4.5.10}$$

where the second equality is derived through (4.2.4). Note that by Theorem 4.1.3, a continuous mapping theorem immediately implies that for a positive integer $k < \infty$,

$$\left(\frac{\exp(H_N^{(1)}(2r+1) - H_N^{(1)}(1))}{\sum_{p=0}^{k} \exp(H_N^{(1)}(2p+1) - H_N^{(1)}(1))} \right)_{r \in [\![0,k]\!]} \xrightarrow{d} \left(\frac{e^{-S_r}}{\sum_{p=0}^{k} e^{-S_p}} \right)_{r \in [\![0,k]\!]} \tag{4.5.11}$$

Here $(S_i)_{i \geq 0}$ denotes a log-gamma random walk. For simplicity, we denote

$$\Lambda_N(p) := \exp(H_N^{(1)}(2p+1) - H_N^{(1)}(1)).$$

We can then rewrite (4.5.10) as

$$\mathbf{P}_{\theta,\alpha;N}(\pi(2N-2) = N-r) = \frac{\Lambda_N(r)}{\sum_{p=0}^{N-1} \Lambda_N(p)} = \frac{\sum_{p=0}^{k} \Lambda_N(p)}{\sum_{p=0}^{N-1} \Lambda_N(p)} \cdot \frac{\sum_{p=0}^{\infty} e^{-S_p}}{\sum_{p=0}^{k} e^{-S_p}} \cdot \frac{\sum_{p=0}^{k} e^{-S_p}}{\sum_{p=0}^{\infty} e^{-S_p}} \cdot \frac{\Lambda_N(r)}{\sum_{p=0}^{k} \Lambda_N(p)}.$$

Theorem 4.1.1 ensures that

$$\frac{\sum_{p=0}^{k} \Lambda_N(p)}{\sum_{p=0}^{N-1} \Lambda_N(p)} = \mathbf{P}_{\theta,\alpha;N}(\pi(2N-2) \geq N-k) = 1 - \mathbf{P}_{\theta,\alpha;N}(\pi(2N-2) < N-k) \xrightarrow{p} 1$$

as $N \to \infty$ followed by $k \to \infty$. By Lemma 4.6.2 we have

$$\lim_{k \to \infty} \frac{\sum_{p=0}^{\infty} e^{-S_p}}{\sum_{p=0}^{k} e^{-S_p}} \xrightarrow{p} 1.$$

Meanwhile, (4.5.11) yields that as $N \to \infty$,

$$\left(\frac{\sum_{p=0}^{k} e^{-S_p}}{\sum_{p=0}^{\infty} e^{-S_p}} \cdot \frac{\Lambda_N(r)}{\sum_{p=0}^{k} \Lambda_N(p)} \right)_{r \in [\![0,k]\!]} \xrightarrow{d} \left(\frac{\sum_{p=0}^{k} e^{-S_p}}{\sum_{p=0}^{\infty} e^{-S_p}} \cdot \frac{\Lambda(r)}{\sum_{p=0}^{k} e^{-S_p}} \right)_{r \in [\![0,k]\!]} = \left(\frac{e^{-S_r}}{\sum_{p=o}^{\infty} e^{-S_p}} \right)_{r \in [\![0,k]\!]}.$$

Thus we establish (4.1.7) and complete the proof of Theorem 4.1.4. □

4.5.3 Proof of Proposition 4.5.3

For clarity, we divide the proof into several steps.

Step 1. In this step we sketch the main ideas behind the proof. At this point, we encourage the readers to consult with Figure 4.24. Recall the event A defined in (4.5.6).

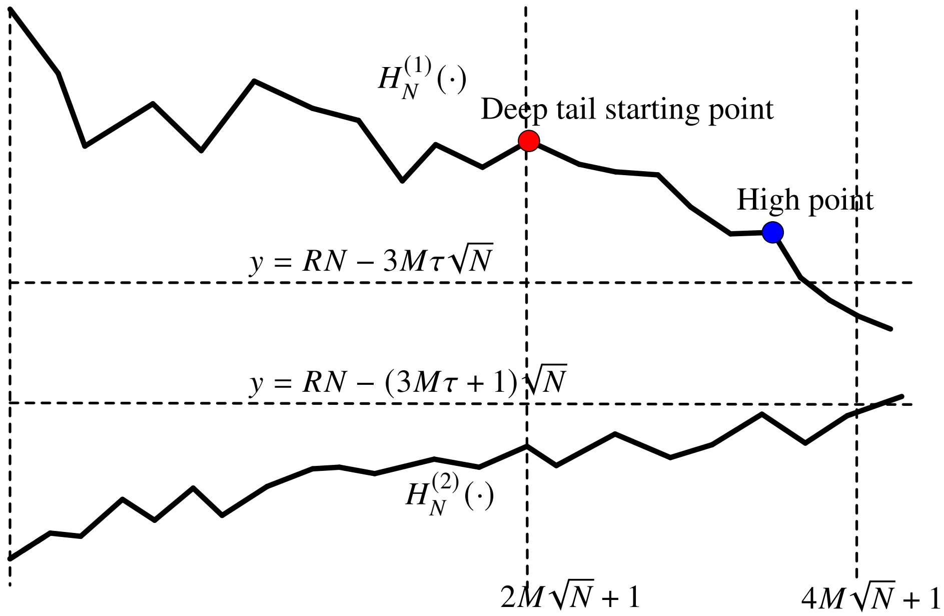


Figure 4.24: Illustration of the proof of Proposition 4.5.3. As claimed by Lemma 4.5.2, there exists a high point in $[\![2M\sqrt{N}+1, 4M\sqrt{N}+1]\!]$ such that $H_N^{(1)}(2p^*+1)$ lies above $RN-\frac{5}{2}M\tau\sqrt{N}$ with high probability. This high point is illustrated as the blue point in the figure. This high point between $[\![2M\sqrt{N} + 1, 4M\sqrt{N} + 1]\!]$ helps us show that $H_N^{(1)}(\cdot) \geq RN - 3M\tau\sqrt{N}$ between $[\![1, 2p^* + 1]\!]$. However, invoking Proposition 4.4.2, we can ensure the second curve stays below the benchmark of $RN - (3M\tau + 1)\sqrt{N}$ on the interval $[\![1, 4M\sqrt{N} + 1]\!]$ with high probability. Thus there is a $\sqrt{N}$ separation (with high probability) between the two curves. By the Gibbs property, this separation ensures that the top curve is close to a log-gamma random walk.

- Let us take M and N_1 as described in the statement of the Proposition 4.5.3. In the language introduced in Figure 4.7 and the text before Lemma 4.5.2, $2M\sqrt{N} + 1$ serves as the *deep tail starting point*. As we have assumed Lemma 4.5.2 holds, we thus have a point in $2p^* + 1 \in [\![2M\sqrt{N} + 1, 4M\sqrt{N} + 1]\!]$ where $H_N^{(1)}(2p^* + 1)$ is 'high' enough (see Figure 4.24). This high point event is denoted as event B in **Step 2** which has a probability of at least $1 - \frac{1}{2}\varepsilon$ by Lemma 4.5.2.

- Invoking Proposition 4.4.2 with high probability we can take the second curve of the line ensemble to be lower than a certain benchmark. More precisely, Proposition 4.4.2 with $M_1 =$

$2M$ and $M_2 = 3M\tau + 1$ implies that

$$\sup_{p\in[\![1,4M\sqrt{N}+1]\!]} H_N^{(2)}(p) \leq RN - (3M\tau+1)\sqrt{N}$$

with probability at least $1 - \frac{\varepsilon}{2}$. We denote this phenomenon as the Fluc event. As B and Fluc are high probability events, to prove our desired estimate in (4.5.8), it suffices to show that $|\mathbf{P}(\mathsf{A}\cap\mathsf{B}\cap\mathsf{Fluc}) - \mathbf{P}_{RW}(\mathsf{A})|$ is small. This is achieved by considering the measure conditioned on the entire second curve and the first curve beyond $2p^*+1$. We remark that in reality, this is not exactly how we do it. But for the sketch of the proof, we present it in this way. We refer to the last bullet point for details.

- From the Gibbs property in Theorem 4.2.2, we deduce a key observation regarding the conditional measure in **Step 3**. In colloquial terms, we note that the conditional measure is absolutely continuous w.r.t. a log-gamma random walk $(S_k)_{k\geq 0}$ from Definition 4.1.2 starting at $H_N^{(1)}(2p^*+1)$ and an explicit Radon-Nikodym derivative W_{p^*}. As the free law is precisely the limiting law we are interested in, it suffices to prove that the Radon-Nikodym derivative W_{p^*} over this interval $[1, 2p^*+1]$ is approximately 1.

- Loosely speaking, W_{p^*} is close to 1 whenever there is a wide enough separation between the two curves. Due to the diffusive nature of the random walk (with positive drift), under the free law, the walk does not become too low. This guarantees that under $\mathsf{B}\cap\mathsf{Fluc}$ event we have a uniform separation of $\sqrt{N}$ between the top two curves between $[\![1, 2p^*+1[\![$. Thus, we deduce that $W_{p^*} \approx 1$ when N is large. The details are presented in **Step 5**. This shows that the law of the $H_N^{(1)}(\cdot)$ is close to the free law of a log-gamma random walk starting at $H_N^{(1)}(2p^*+1)$.

- One issue in carrying out the arguments in the last two bullet points is that p^* is *random*. The Gibbs property cannot be applied at p^*, as the property is formulated for *fixed* boundary points. This issue can be circumvented easily by a graining argument. We write B as $\mathsf{B} = \bigsqcup \mathsf{B}_i$ with B_i being a disjoint collection of events with $\mathsf{B}_i \subset \{H_N^{(1)}(2i+1) \geq RN - \frac{5}{2}M\sqrt{N}\}$ and then apply the Gibbs property for each i.

Step 2. Take $M_1 = 2M$ and $M_2 = 3M\tau + 1$ in Proposition 4.4.2. Taking $N_2(\varepsilon, M_1, M_2) > 0$ (which depends only on ε as M_1, M_2 depends only on ε) from Proposition 4.4.2, we see that

$$\mathbf{P}(\mathsf{Fluc}) \geq 1 - \tfrac{\varepsilon}{2}, \text{ where } \mathsf{Fluc} := \left\{ \sup_{p \in [\![1, 4M\sqrt{N}+1]\!]} H_N^{(2)}(p) \leq RN - (3M\tau + 1)\sqrt{N} \right\} \tag{4.5.12}$$

for all $N \geq N_2$. Next we consider the events

$$\mathsf{G}_i := \left\{ H_N^{(1)}(2i+1) \geq RN - \frac{5}{2} M\tau\sqrt{N} \right\} \text{ and } \mathsf{B}_i := \bigcap_{j=i+1}^{2M\sqrt{N}} \mathsf{G}_j^c \cap \mathsf{G}_i.$$

Note that $(\mathsf{B}_i)_{i \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]}$ forms a disjoint collection of events. Define

$$\begin{aligned} \mathsf{B} &:= \bigsqcup_{i \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]} \mathsf{B}_i \\ &= \bigcup_{i \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]} \mathsf{G}_i = \left\{ \sup_{p \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]} H_N^{(1)}(2p+1) \geq RN - \tfrac{5}{2} M\tau\sqrt{N} \right\}, \end{aligned}$$

where we write $\sqcup$ to stress that the events are disjoint in the union. In particular, as Lemma 4.5.2 holds, we have $\mathbf{P}(\mathsf{B}) \geq 1 - \frac{1}{2}\varepsilon$. Thus for all $N \geq N_1 + N_2$, by a union bound we have

$$|\mathbf{P}(\mathsf{A}) - \mathbf{P}(\mathsf{A} \cap \mathsf{B} \cap \mathsf{Fluc})| \leq \mathbf{P}(\neg\mathsf{B}) + \mathbf{P}(\neg\mathsf{Fluc}) \leq \varepsilon.$$

Hence to prove (4.5.8) it suffices to show

$$|\mathbf{P}(\mathsf{A} \cap \mathsf{B} \cap \mathsf{Fluc}) - \mathbf{P}_{RW}(\mathsf{A})| \leq 8\varepsilon. \tag{4.5.13}$$

Define $\mathcal{F}_i$ as the σ-field $\sigma\big(H_N^{(1)}(x)_{x \geq 2i+1}, H_N^{(j)}(x)_{j \geq 2, x \geq 1}\big)$. Note that $\mathsf{B}_i, \mathsf{Fluc}$ are both measurable w.r.t. $\mathcal{F}_i$. Exploiting the fact that B_i's are disjoint yields

$$\mathbf{P}(\mathsf{A} \cap \mathsf{B} \cap \mathsf{Fluc}) = \sum_{i=M\sqrt{N}}^{2M\sqrt{N}} \mathbf{E}\left[\mathbf{1}_{\mathsf{B}_i \cap \mathsf{Fluc}} \mathbf{E}\left[\mathbf{1}_{\mathsf{A}} \mid \mathcal{F}_i\right]\right] \tag{4.5.14}$$

where the last equality is due to the tower property of the conditional expectation. Thus we are left to estimate $\mathbf{E}\left[\mathbf{1}_{\mathsf{A}} \mid \mathcal{F}_i\right]$ for each i.

Step 3. Gibbs law. To analyze $\mathbf{E}\left[\mathbf{1}_{\mathsf{A}} \mid \mathcal{F}_i\right]$, we invoke the Gibbs property (Theorem 4.2.2) for the $\mathcal{HSLG}$ line ensemble. By Theorem 4.2.2, the distribution of $(H_N^{(1)}(j))_{j=1}^{2i}$ conditioned on $\mathcal{F}_i$ has a density at $(u_j)_{j=1}^{2i}$

$$\exp\left(-\sum_{j=1}^{i}\left[e^{H_N^{(2)}(2j)-u_{2j+1}} + e^{H_N^{(2)}(2j)-u_{2j-1}}\right]\right) \tag{4.5.15}$$

$$\cdot \prod_{j=1}^{i} \exp\left((\theta+\alpha)(u_{2j+1}-u_{2j}) - e^{u_{2j+1}-u_{2j}}\right) \tag{4.5.16}$$

$$\cdot \prod_{j=1}^{i} \exp\left((\theta-\alpha)(u_{2j-1}-u_{2j}) - e^{u_{2j-1}-u_{2j}}\right) \tag{4.5.17}$$

with $u_{2i+1} = H_N^{(1)}(2i+1)$. The above explicit expression is obtained from (4.2.6) and (4.2.5). Note that the terms in (4.5.15), (4.5.16), and (4.5.17) correspond to weights of black, red, and blue edges in the graphical representation (see left figure of Figure 4.25) respectively.

Based on the above decomposition, we define a free law $\mathbf{P}_{\mathrm{free},i}$ that depends only on $H_N^{(1)}(2i+1)$. We define that under the law $\mathbf{P}_{\mathrm{free},i}$, the distribution of $(H_N^{(1)}(j))_{j=1}^{2i}$ has a density at $(u_j)_{j=1}^{2i}$ proportional to

$$\prod_{j=1}^{i} \exp\left((\theta+\alpha)(u_{2j+1}-u_{2j}) - e^{u_{2j+1}-u_{2j}}\right) \cdot \prod_{j=1}^{i} \exp\left((\theta-\alpha)(u_{2j-1}-u_{2j}) - e^{u_{2j-1}-u_{2j}}\right)$$

with $u_{2i+1} = H_N^{(1)}(2i+1)$. Note that free law collects all the blue and red edge weights only. A quick comparison of the above formula with (4.1.5) shows that under the free law, $(H_N^{(1)}(1) -$

$H_N^{(1)}(2r+1))_{r=0}^{i}$ is precisely distributed as log-gamma random walk defined in Definition 4.1.2.

In order to obtain the original conditional distribution from the free law, we may introduce the black weights as a Radon-Nikodym derivative (see the decomposition in Figure 4.25). Indeed, we have

$$\mathbf{E}\left[\mathbf{1}_{\mathsf{A}} \mid \mathcal{F}_i\right] = \frac{\mathbf{E}_{\mathrm{free},i}[W_i \mathbf{1}_{\mathsf{A}}]}{\mathbf{E}_{\mathrm{free},i}[W_i]} \tag{4.5.18}$$

where

$$W_i := \exp\left(-\sum_{j=1}^{i}\left[e^{H_N^{(2)}(2j)-H_N^{(1)}(2j+1)} + e^{H_N^{(2)}(2j)-H_N^{(1)}(2j-1)}\right]\right) \tag{4.5.19}$$

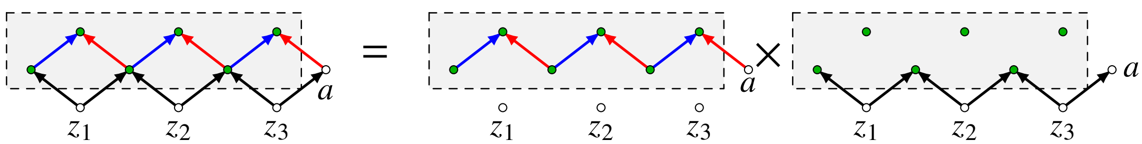


Figure 4.25: Gibbs decomposition. The left figure shows the gibbs measure corresponding to conditioned on $\mathcal{F}_i$ with $i = 3$. Here $a = H_N^{(1)}(2i+1)$, and $z_j := H_N^{(2)}(2j)$ for $j \in [\![1, i]\!]$. The measure has been decomposed into two parts. The free law (middle) and a Radon-Nikodym derivative (right).

We notice that W_i has a trivial upper bound: $W_i \le 1$. For the lower bound, we claim that there exists $N_0(\varepsilon) > 0$ such that for all $N \ge N_0$ we have

$$\mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i}\mathbf{P}_{\mathrm{free},i}(W_i \ge 1-\varepsilon) \ge \mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i} \cdot (1-\varepsilon). \tag{4.5.20}$$

Thus, (4.5.20) implies that W_i is close to 1 with high probability under $\mathsf{Fluc} \cap \mathsf{B}_i$. Thus, going back to (4.5.18), we expect $\mathbf{E}\left[\mathbf{1}_{\mathsf{A}} \mid \mathcal{F}_i\right]$ to be close to $\mathbf{P}_{\mathrm{free},i}(\mathsf{A})$. As under the free law $\mathbf{P}_{\mathrm{free},i}(\mathsf{A}) = \mathbf{P}_{RW}(\mathsf{A})$, for all $i \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]$, (4.5.14) eventually leads to (4.5.13), which we make precise in the next step.

Step 4. Assuming (4.5.20), we complete the proof of (4.5.13) in this step. As $W_i \le 1$, we have

$$\begin{aligned}\mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i}\frac{\mathbf{E}_{\mathrm{free},i}[W_i\mathbf{1}_{\mathsf{A}}]}{\mathbf{E}_{\mathrm{free},i}[W]} &\ge \mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i}\mathbf{E}_{\mathrm{free},i}[W_i\mathbf{1}_{\mathsf{A}}] \ge (1-\varepsilon)\cdot\mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i}\mathbf{P}_{\mathrm{free},i}(\mathsf{A}\cap\{W\ge 1-\varepsilon\}) \\ &\ge (1-\varepsilon)\cdot\mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i}\left[\mathbf{P}_{\mathrm{free},i}(\mathsf{A})-\mathbf{P}_{\mathrm{free},i}(W_i<1-\varepsilon)\right] \\ &\ge (1-\varepsilon)\cdot\mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i}\left[\mathbf{P}_{\mathrm{free},i}(\mathsf{A})-\varepsilon\right]\end{aligned}$$

where we use (4.5.20) in the last inequality. Recall $\mathbf{P}_{\mathrm{free},i}(\mathsf{A}) = \mathbf{P}_{RW}(\mathsf{A})$. Inserting this bound in (4.5.18) and then going back to (4.5.14) yields

$$\begin{aligned}\mathbf{P}(\mathsf{A}\cap\mathsf{B}\cap\mathsf{Fluc}) &\ge (1-\varepsilon)\cdot[\mathbf{P}_{RW}(\mathsf{A})-\varepsilon]\sum_{i=M\sqrt{N}}^{2M\sqrt{N}}\mathbf{P}(\mathsf{B}_i\cap\mathsf{Fluc}) \\ &= (1-\varepsilon)\cdot[\mathbf{P}_{RW}(\mathsf{A})-\varepsilon]\,\mathbf{P}(\mathsf{B}\cap\mathsf{Fluc}) \ge (1-\varepsilon)^2\,[\mathbf{P}_{RW}(\mathsf{A})-\varepsilon]\,.\end{aligned}$$

for all large enough N. The equality in the above equation follows by recalling that B_i's form a disjoint collection of events and the result implies that $\mathbf{P}(\mathsf{A}\cap\mathsf{B}\cap\mathsf{Fluc}) - \mathbf{P}_{RW}(\mathsf{A}) \ge -3\varepsilon$. This proves the lower bound inequality in (4.5.13). Similarly for the upper bound, as $W_i \le 1$, we have

$$\mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i}\cdot\frac{\mathbf{E}_{\mathrm{free},i}[W_i\mathbf{1}_{\mathsf{A}}]}{\mathbf{E}_{\mathrm{free},i}[W_i]} \le \mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i}\cdot\frac{\mathbf{P}_{\mathrm{free},i}(\mathsf{A})}{(1-\varepsilon)\mathbf{P}_{\mathrm{free},i}(W_i\ge 1-\varepsilon)} \le \mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i}\cdot\frac{\mathbf{P}_{\mathrm{free},i}(\mathsf{A})}{(1-\varepsilon)^2}$$

where the last inequality stems from (4.5.20). Again, Inserting this bound in (4.5.18) and then going back to (4.5.14) gives us

$$\mathbf{P}(\mathsf{A}\cap\mathsf{B}\cap\mathsf{Fluc}) \le \frac{\mathbf{P}_{RW}(\mathsf{A})}{(1-\varepsilon)^2}\sum_{i=M\sqrt{N}}^{2M\sqrt{N}}\mathbf{P}(\mathsf{B}_i\cap\mathsf{Fluc}) = \frac{\mathbf{P}_{RW}(\mathsf{A})}{(1-\varepsilon)^2}\mathbf{P}(\mathsf{B}\cap\mathsf{Fluc}) \le \frac{\mathbf{P}_{RW}(\mathsf{A})}{(1-\varepsilon)^2}$$

where again the equality comes from the disjointness of B_i's. As $\varepsilon \le \frac{1}{2}$, this implies

$$\mathbf{P}(\mathsf{A}\cap\mathsf{B}\cap\mathsf{Fluc}) - \mathbf{P}_{RW}(\mathsf{A}) \le \frac{1-(1-\varepsilon)^2}{(1-\varepsilon)^2} \le 8\varepsilon$$

which proves the upper bound in (4.5.13). The proof of Theorem 4.1.3 modulo (4.5.20) is thus complete.

Step 5. Finally in this step we prove (4.5.20). We define the event

$$\mathsf{Sink}(i) := \left\{ \inf_{p\in[\![0,i]\!]} H_N^{(1)}(2p+1) \geq RN - 3M\tau\sqrt{N} \right\}.$$

We claim that there exists $N_0(\varepsilon) > 0$ such that for all $N \geq N_0$, we have

$$\mathbf{1}_{\mathsf{B}_i} \mathbf{P}_{\text{free},i}(\mathsf{Sink}(i)) \geq \mathbf{1}_{\mathsf{B}_i}(1-\varepsilon), \tag{4.5.21}$$

for all $i \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]$.

Recall that the event Fluc in (4.5.12) requires the second curve $H_N^{(2)}(p)$ to lie below certain threshold within the range $p \in [\![1, 4M\sqrt{N}+1]\!]$. Recall the definition of W_j from (4.5.19). Note that on $\mathsf{Sink}(j) \cap \mathsf{Fluc}$ we have

$$W_j \geq \exp(-2je^{-\sqrt{N}}) \geq \exp(-4M\sqrt{N}e^{-\sqrt{N}})$$

as $j \leq 2M\sqrt{N}$. Note that $\exp(-4M\sqrt{N}e^{-\sqrt{N}}) \geq 1-\varepsilon$ for all large enough N. Therefore, in view of (4.5.21) we have

$$\mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i} \mathbf{P}_{\text{free},i}(W_i \geq 1-\varepsilon) \geq \mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i} \mathbf{P}_{\text{free},i}(\mathsf{Sink}(i)) \geq \mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i} \cdot (1-\varepsilon)$$

for all large enough N. This verifies (4.5.20). We are left to show (4.5.21). Towards this end, note that on the event B_i, we have $H_N^{(1)}(2i+1) \geq RN - \frac{5}{2}M\tau\sqrt{N}$. Thus,

$$\mathbf{1}_{\mathsf{B}_i} \mathbf{P}_{\text{free},i}(\mathsf{Sink}(i)) \geq \mathbf{1}_{\mathsf{B}_i} \mathbf{P}_{\text{free},i} \left(\inf_{x\in[\![0,i]\!]} H_N^{(1)}(2x+1) - H_N^{(1)}(2i+1) \geq -\tfrac{1}{2}M\tau\sqrt{N} \right). \tag{4.5.22}$$

Recall from our discussion in **Step 2** that under the law $\mathbf{P}_{\text{free},i}$, $(H_N^{(1)}(1) - H_N^{(1)}(2r+1))_{r=0}^i$ is distributed as a log-gamma random walk. Let us use $(S_k)_{k=0}^i$ to denote a log-gamma random walk.

We have

$$\mathbf{P}_{\text{free},i}\left(\inf_{p\in[\![0,i]\!]} H_N^{(1)}(2p+1) - H_N^{(1)}(2i+1) \geq -\tfrac{1}{2}M\tau\sqrt{N}\right) = \mathbf{P}\left(\inf_{p\in[\![0,i]\!]}(S_i - S_p) \geq -\tfrac{1}{2}M\tau\sqrt{N}\right). \tag{4.5.23}$$

Note that $(S_i - S_p)_{p\geq 0}^{i}$ is again a time-reversed log-gamma random walk. As $i \leq 2M\sqrt{N}$, appealing to Lemma 4.6.1 yields that

$$\mathbf{1}_{\mathsf{B}_i}\mathbf{P}_{\text{free},i}(\mathsf{Sink}(i)) \geq \mathbf{P}\left(\inf_{p\in[\![0,i]\!]}(S_i - S_p) \geq -\tfrac{1}{2}M\tau\sqrt{N}\right) \geq 1 - \frac{8\,\mathrm{Var}(S_1)}{M\tau^2\sqrt{N}} \geq 1 - \varepsilon$$

for all large enough N (uniformly over $i \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]$. Inserting this bound in (4.5.23), in view of the lower bound in (4.5.22), leads to (4.5.21). This completes the proof of Proposition 4.5.3.

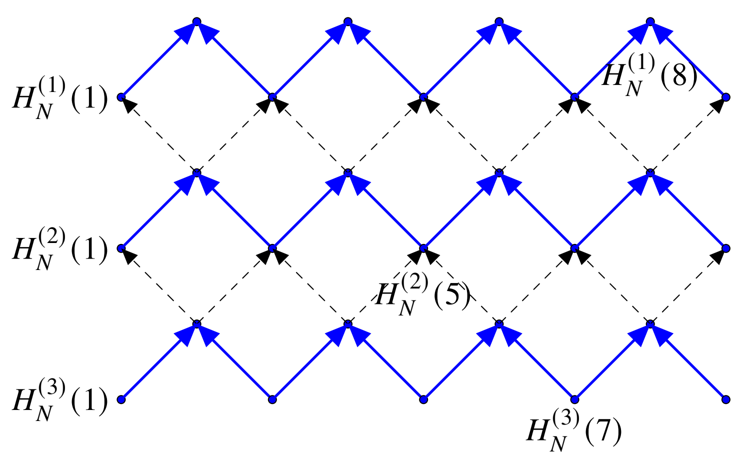


Figure 4.26: Ordering of points within $\mathcal{HSLG}$ line ensemble: The above figure consists of first 3 curves of the line ensemble H_N. An arrow from $a \to b$ signifies $a \leq b - \log^2 N$ with exponential high probability. The blue arrows depict the ordering within a particular indexed curve (inter-ordering). The dashed arrow indicates ordering between the two consecutive curves (intra-ordering).

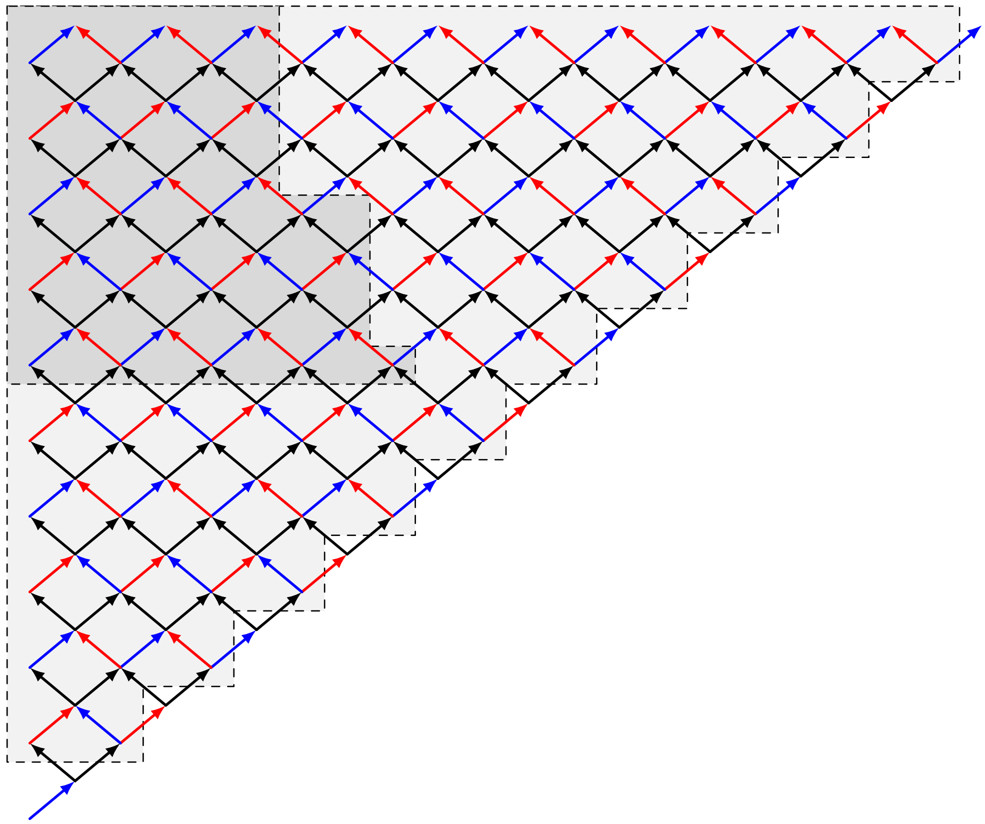


Figure 4.27: $\Theta_{2,2}$

4.6 Appendix: Properties of random walks with positive drift

In this section, we collect some useful properties of random walks with positive drift whose proofs follow by classical analysis. Note that the log-gamma random walk introduced in Definition 4.1.2 is a random walk with positive drift. This is because the density $p(x)$ introduced in (4.1.5) has mean:

$$\int_{\mathbb{R}} x p(x) dx = \Psi(\theta - \alpha) - \Psi(\theta + \alpha),$$

which is positive as the digamma function Ψ is strictly increasing (recall $\alpha < 0$).

Lemma 4.6.1. *Let* $(X_i)_{i\geq 0}$ *be a sequence of iid random variables with* $\mathbf{E}[X_1] = \beta > 0$ *and*

$\mathrm{Var}[X_1] = \gamma < \infty$. *Set* $S_0 = 0$ *and* $S_k = \sum_{i=1}^{k} X_i$. *For all* $M, N, \lambda > 0$ *we have*

$$\mathbf{P}\left(\inf_{k\in[\![1,M\sqrt{N}]\!]} S_k \leq -\lambda\right) \leq \frac{M\sqrt{N}\gamma}{\lambda^2}.$$

Proof. As $\beta > 0$, by Kolmogorov's maximal inequality, we have

$$\mathbf{P}\left(\inf_{k\in[\![1,M\sqrt{N}]\!]} S_k \leq -\lambda\right) = \mathbf{P}\left(\sup_{k\in[\![1,M\sqrt{N}]\!]} |S_k - k\beta| \geq \lambda\right) \leq \frac{1}{\lambda^2}\sum_{i=1}^{M\sqrt{N}} \mathrm{Var}(X_i) = \frac{M\sqrt{N}\gamma}{\lambda^2},$$

which is precisely what we want to show. □

Lemma 4.6.2. *Let* $(X_i)_{i\geq 0}$ *be a sequence of iid random variables with* $\mathbf{E}[X_1] = \beta > 0$ *and* $\mathrm{Var}[X_1] = \gamma < \infty$. *Set* $S_0 = 0$ *and* $S_n = \sum_{i=1}^{n} X_i$. *We have*

$$\mathbf{P}\left(\sum_{r=0}^{\infty} e^{-S_r} < \infty\right) = 1$$

Proof. By Kolmogorov's maximal inequality

$$\mathbf{P}\big(\sup_{1\leq i\leq n^2} |S_i - i\beta| \geq \tfrac{n^2}{2}\beta\big) \leq \frac{4}{n^4\beta^2}\sum_{i=1}^{n^2} \mathrm{Var}(X_i) = \frac{4\gamma}{n^2\beta^2}.$$

The last bound is summable in n. Thus invoking Borel-Cantelli's lemma we have that there exists a random N with $P(7 \leq N < \infty) = 1$ such that

$$S_i \geq i\beta - (N^2/2)\beta \geq -(N^2/2)\beta, \text{ for all } 1 \leq i \leq N^2,$$

and for all $n \geq N + 1$ we have

$$S_i \geq (n-1)^2\beta - (n^2/2)\beta \geq (n^2/4)\beta, \text{ for all } (n-1)^2 + 1 \leq i \leq n^2,$$

where above we used the fact that $n \geq N + 1 \geq 8$. Thus with probability 1, we have

$$\sum_{r=0}^{\infty} e^{-S_r} = \sum_{r=0}^{N^2} e^{-S_r} + \sum_{n=N+1}^{\infty} \sum_{i=(n-1)^2+1}^{n^2} e^{-S_i}$$
$$\leq N^2 e^{(N^2/2)\beta} + \sum_{n=N+1}^{\infty} \sum_{i=(n-1)^2+1}^{n^2} e^{-(n^2/4)\beta} \leq N^2 e^{(N^2/2)\beta} + \sum_{n=N+1}^{\infty} n^2 e^{-(n^2/4)\beta} < \infty.$$

This completes the proof. □

As a corollary, we have the following double-limit result.

Corollary 4.6.3. *Under the setup of Lemma 4.6.2, almost surely we have*

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{\sum_{r=k}^{\infty} e^{-S_r}}{\sum_{r=0}^{n} e^{-S_r}} = 0.$$

Proof. Note that $\sum_{r=0}^{n} e^{S_r}$ is a monotone sequence in n which converges to a random variable that is almost surely finite by Lemma 4.6.2. Thus,

$$\frac{\sum_{r=k}^{\infty} e^{S_r}}{\sum_{r=0}^{n} e^{S_r}} = 1 - \frac{\sum_{r=0}^{k-1} e^{S_r}}{\sum_{r=0}^{n} e^{S_r}} \stackrel{n\to\infty}{\to} 1 - \frac{\sum_{r=0}^{k-1} e^{S_r}}{\sum_{r=0}^{\infty} e^{S_r}}.$$

Taking $k \to \infty$ yields the desired result. □

Chapter 5: Tightness of the Bernoulli Gibbsian line ensemble

5.1 Line ensembles

In this section we introduce various definitions and notation that are used throughout the paper.

5.1.1 Line ensembles and the Brownian Gibbs property

In this section we introduce the notions of a *line ensemble* and the *(partial) Brownian Gibbs property*. Our exposition in this section closely follows that of [105, Section 2] and [73, Section 2].

Given two integers $p \leq q$, we let $[\![p, q]\!]$ denote the set $\{p, p+1, \dots, q\}$. Given an interval $\Lambda \subset \mathbb{R}$ we endow it with the subspace topology of the usual topology on $\mathbb{R}$. We let $(C(\Lambda), \mathcal{C})$ denote the space of continuous functions $f : \Lambda \to \mathbb{R}$ with the topology of uniform convergence over compacts, see [192, Chapter 7, Section 46], and Borel σ-algebra $\mathcal{C}$. Given a set $\Sigma \subset \mathbb{Z}$ we endow it with the discrete topology and denote by $\Sigma \times \Lambda$ the set of all pairs (i, x) with $i \in \Sigma$ and $x \in \Lambda$ with the product topology. We also denote by $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ the space of continuous functions on $\Sigma \times \Lambda$ with the topology of uniform convergence over compact sets and Borel σ-algebra $\mathcal{C}_\Sigma$. Typically, we will take $\Sigma = [\![1, N]\!]$ (we use the convention $\Sigma = \mathbb{N}$ if $N = \infty$) and then we write $\left(C(\Sigma \times \Lambda), \mathcal{C}_{|\Sigma|}\right)$ in place of $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$.

The following defines the notion of a line ensemble.

Definition 5.1.1. Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be an interval. A Σ-*indexed line ensemble* $\mathcal{L}$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that takes values in $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$. Intuitively, $\mathcal{L}$ is a collection of random continuous curves (sometimes referred to as *lines*), indexed by Σ, each of which maps Λ in $\mathbb{R}$. We will often slightly abuse notation and write $\mathcal{L} : \Sigma \times \Lambda \to \mathbb{R}$, even though it is not $\mathcal{L}$ which is such a function, but $\mathcal{L}(\omega)$ for every $\omega \in \Omega$. For $i \in \Sigma$ we write

$\mathcal{L}_i(\omega) = (\mathcal{L}(\omega))(i, \cdot)$ for the curve of index i and note that the latter is a map $\mathcal{L}_i : \Omega \to C(\Lambda)$, which is $(\mathcal{C}, \mathcal{F})-$measurable. If $a, b \in \Lambda$ satisfy $a < b$ we let $\mathcal{L}_i[a, b]$ denote the restriction of $\mathcal{L}_i$ to $[a, b]$.

We will require the following result, whose proof is postponed until Section 5.6.1. In simple terms it states that the space $C(\Sigma \times \Lambda)$ where our random variables $\mathcal{L}$ take value has the structure of a complete, separable metric space.

Lemma 5.1.2. *Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be an interval. Suppose that $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are sequences of real numbers such that $a_n < b_n$, $[a_n, b_n] \subset \Lambda$, $a_{n+1} \leq a_n$, $b_{n+1} \geq b_n$ and $\cup_{n=1}^{\infty}[a_n, b_n] = \Lambda$. For $n \in \mathbb{N}$ we let $K_n = \Sigma_n \times [a_n, b_n]$ where $\Sigma_n = \Sigma \cap [\![-n, n]\!]$. Define $d : C(\Sigma \times \Lambda) \times C(\Sigma \times \Lambda) \to [0, \infty)$ by*

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \min \left\{ \sup_{(i,t) \in K_n} |f(i, t) - g(i, t)|, \ 1 \right\}. \tag{5.1.1}$$

Then d defines a metric on $C(\Sigma \times \Lambda)$ and moreover the metric space topology defined by d is the same as the topology of uniform convergence over compact sets. Furthermore, the metric space $(C(\Sigma \times \Lambda), d)$ is complete and separable.

Definition 5.1.3. Given a sequence $\{\mathcal{L}^n : n \in \mathbb{N}\}$ of random Σ-indexed line ensembles we say that $\mathcal{L}^n$ *converge weakly* to a line ensemble $\mathcal{L}$, and write $\mathcal{L}^n \implies \mathcal{L}$ if for any bounded continuous function $f : C(\Sigma \times \Lambda) \to \mathbb{R}$ we have that

$$\lim_{n \to \infty} \mathbb{E}\left[f(\mathcal{L}^n)\right] = \mathbb{E}\left[f(\mathcal{L})\right].$$

We also say that $\{\mathcal{L}^n : n \in \mathbb{N}\}$ is *tight* if for any $\epsilon > 0$ there exists a compact set $K \subset C(\Sigma \times \Lambda)$ such that $\mathbb{P}(\mathcal{L}^n \in K) \geq 1 - \epsilon$ for all $n \in \mathbb{N}$.

We call a line ensemble *non-intersecting* if $\mathbb{P}$-almost surely $\mathcal{L}_i(r) > \mathcal{L}_j(r)$ for all $i < j$ and $r \in \Lambda$.

We will require the following sufficient condition for tightness of a sequence of line ensembles, which extends [39, Theorem 7.3]. We give a proof in Section 5.6.2.

Lemma 5.1.4. *Let $\Sigma \subset \mathbb{Z}$ and $\Lambda \subset \mathbb{R}$ be an interval. Suppose that $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are sequences of real numbers such that $a_n < b_n$, $[a_n, b_n] \subset \Lambda$, $a_{n+1} \leq a_n$, $b_{n+1} \geq b_n$ and $\cup_{n=1}^{\infty}[a_n, b_n] = \Lambda$. Then $\{\mathcal{L}^n\}$ is tight if and only if for every $i \in \Sigma$ we have*

(i) $\lim_{a\to\infty} \limsup_{n\to\infty} (|\mathcal{L}_i^n(a_0)| \geq a) = 0$*;*

(ii) For all $\epsilon > 0$ and $k \in \mathbb{N}$, $\lim_{\delta\to 0} \limsup_{n\to\infty} \left(\sup_{\substack{x,y\in[a_k,b_k],\\ |x-y|\leq\delta}} |\mathcal{L}_i^n(x) - \mathcal{L}_i^n(y)| \geq \epsilon \right) = 0.$

We next turn to formulating the Brownian Gibbs property – we do this in Definition 5.1.8 after introducing some relevant notation and results. If W_t denotes a standard one-dimensional Brownian motion, then the process

$$\tilde{B}(t) = W_t - tW_1, \quad 0 \leq t \leq 1,$$

is called a *Brownian bridge (from $\tilde{B}(0) = 0$ to $\tilde{B}(1) = 0$) with diffusion parameter* 1. For brevity we call the latter object a *standard Brownian bridge*.

Given $a, b, x, y \in \mathbb{R}$ with $a < b$ we define a random variable on $(C([a, b]), \mathcal{C})$ through

$$B(t) = (b-a)^{1/2} \cdot \tilde{B}\left(\frac{t-a}{b-a}\right) + \left(\frac{b-t}{b-a}\right) \cdot x + \left(\frac{t-a}{b-a}\right) \cdot y, \tag{5.1.2}$$

and refer to the law of this random variable as a *Brownian bridge (from $B(a) = x$ to $B(b) = y$) with diffusion parameter* 1. Given $k \in \mathbb{N}$ and $\vec{x}, \vec{y} \in \mathbb{R}^k$ we let $\mathbb{P}_{free}^{a,b,\vec{x},\vec{y}}$ denote the law of k independent Brownian bridges $\{B_i : [a, b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$ all with diffusion parameter 1.

We next state a couple of results about Brownian bridges from [73] for future use.

Lemma 5.1.5. *[73, Corollary 2.9]. Fix a continuous function $f : [0, 1] \to \mathbb{R}$ such that $f(0) > 0$ and $f(1) > 0$. Let B be a standard Brownian bridge and let $C = \{B(t) > f(t)$ for some $t \in [0, 1]\}$ (crossing) and $T = \{B(t) = f(t)$ for some $t \in [0, 1]\}$ (touching). Then $\mathbb{P}(T \cap C^c) = 0$.*

Lemma 5.1.6. *[73, Corollary 2.10]. Let U be an open subset of $C([0,1])$, which contains a function f such that $f(0) = f(1) = 0$. If $B : [0,1] \to \mathbb{R}$ is a standard Brownian bridge then $\mathbb{P}(B[0,1] \subset U) > 0$.*

The following definition introduces the notion of an (f,g)-avoiding Brownian line ensemble, which in simple terms is a collection of k independent Brownian bridges, conditioned on not-crossing each other and staying above the graph of g and below the graph of f for two continuous functions f and g.

Definition 5.1.7. Let $k \in \mathbb{N}$ and ${}_k$ denote the open Weyl chamber in $\mathbb{R}^k$, i.e.

$$ {}_k = \{\vec{x} = (x_1, \dots, x_k) \in \mathbb{R}^k : x_1 > x_2 > \cdots > x_k\}. $$

(In [73] the notation $\mathbb{R}^k_>$ was used for this set.) Let $\vec{x}, \vec{y} \in_k$, $a, b \in \mathbb{R}$ with $a < b$, and $f : [a,b] \to (-\infty, \infty]$ and $g : [a,b] \to [-\infty, \infty)$ be two continuous functions. The latter condition means that either $f : [a,b] \to \mathbb{R}$ is continuous or $f = \infty$ everywhere, and similarly for g. We also assume that $f(t) > g(t)$ for all $t \in [a,b]$, $f(a) > x_1$, $f(b) > y_1$ and $g(a) < x_k$, $g(b) < y_k$.

With the above data we define the *(f,g)-avoiding Brownian line ensemble on the interval $[a,b]$ with entrance data $\vec{x}$ and exit data $\vec{y}$* to be the Σ-indexed line ensemble $\mathcal{Q}$ with $\Sigma = [\![1,k]\!]$ on $\Lambda = [a,b]$ and with the law of $\mathcal{Q}$ equal to $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}$ (the law of k independent Brownian bridges $\{B_i : [a,b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$) conditioned on the event

$$ E = \{f(r) > B_1(r) > B_2(r) > \cdots > B_k(r) > g(r) \text{ for all } r \in [a,b]\}. $$

It is worth pointing out that E is an open set of positive measure and so we can condition on it in the usual way – we explain this briefly in the following paragraph. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that supports k independent Brownian bridges $\{B_i : [a,b] \to \mathbb{R}\}_{i=1}^k$ from $B_i(a) = x_i$ to $B_i(b) = y_i$ all with diffusion parameter 1. Notice that we can find $\tilde{u}_1, \dots, \tilde{u}_k \in C([0,1])$ and $\epsilon > 0$ (depending on $\vec{x}, \vec{y}, f, g, a, b$) such that $\tilde{u}_i(0) = \tilde{u}_i(1) = 0$ for $i = 1, \dots, k$ and such that if

$\tilde{h}_1, \ldots, \tilde{h}_k \in C([0,1])$ satisfy $\tilde{h}_i(0) = \tilde{h}_i(1) = 0$ for $i = 1, \ldots, k$ and $\sup_{t \in [0,1]} |\tilde{u}_i(t) - \tilde{h}_i(t)| < \epsilon$ then the functions

$$h_i(t) = (b-a)^{1/2} \cdot \tilde{h}_i\left(\frac{t-a}{b-a}\right) + \left(\frac{b-t}{b-a}\right) \cdot x_i + \left(\frac{t-a}{b-a}\right) \cdot y_i,$$

satisfy $f(r) > h_1(r) > \cdots > h_k(r) > g(r)$. It follows from Lemma 5.1.6 that

$$\mathbb{P}(E) \geq \mathbb{P}\left(\max_{1 \leq i \leq k} \sup_{r \in [0,1]} |\tilde{B}_i(r) - \tilde{u}_i(r)| < \epsilon\right) = \prod_{i=1}^{k} \mathbb{P}\left(\sup_{r \in [0,1]} |\tilde{B}_i(r) - \tilde{u}_i(r)| < \epsilon\right) > 0,$$

and so we can condition on the event E.

To construct a realization of Q we proceed as follows. For $\omega \in E$ we define

$$Q(\omega)(i,r) = B_i(r)(\omega) \text{ for } i = 1, \ldots, k \text{ and } r \in [a,b].$$

Observe that for $i \in \{1, \ldots, k\}$ and an open set $U \in C([a,b])$ we have that

$$Q^{-1}(\{i\} \times U) = \{B_i \in U\} \cap E \ \in \ \mathcal{F},$$

and since the sets $\{i\} \times U$ form an open basis of $C([\![1,k]\!] \times [a,b])$ we conclude that Q is $\mathcal{F}$-measurable. This implies that the law Q is indeed well-defined and also it is non-intersecting almost surely. Also, given measurable subsets $A_1, \ldots, A_k$ of $C([a,b])$ we have that

$$\mathbb{P}(Q_i \in A_i \text{ for } i = 1, \ldots, k) = \frac{\mathbb{P}_{free}^{a,b,\vec{x},\vec{y}}\left(\{B_i \in A_i \text{ for } i = 1, \ldots, k\} \cap E\right)}{\mathbb{P}_{free}^{a,b,\vec{x},\vec{y}}(E)}.$$

We denote the probability distribution of Q as $\mathbb{P}_{avoid}^{a,b,\vec{x},\vec{y},f,g}$ and write $\mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}$ for the expectation with respect to this measure.

The following definition introduces the notion of the Brownian Gibbs property from [73].

Definition 5.1.8. Fix a set $\Sigma = [\![1,N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and an interval $\Lambda \subset \mathbb{R}$ and let

$K = \{k_1, k_1+1, \dots, k_2\} \subset \Sigma$ be finite and $a, b \in \Lambda$ with $a < b$. Set $f = \mathcal{L}_{k_1-1}$ and $g = \mathcal{L}_{k_2+1}$ with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$ and $g = -\infty$ if $k_2 + 1 \notin \Sigma$. Write $D_{K,a,b} = K \times (a, b)$ and $D^c_{K,a,b} = (\Sigma \times \Lambda) \setminus D_{K,a,b}$. A Σ-indexed line ensemble $\mathcal{L} : \Sigma \times \Lambda \to \mathbb{R}$ is said to have the *Brownian Gibbs property* if it is non-intersecting and

$$\text{Law}\left(\mathcal{L}|_{K\times[a,b]} \text{ conditional on } \mathcal{L}|_{D^c_{K,a,b}}\right) = \text{Law}\,(Q)\,,$$

where $Q_i = \tilde{Q}_{i-k_1+1}$ and $\tilde{Q}$ is the (f, g)-avoiding Brownian line ensemble on $[a, b]$ with entrance data $(\mathcal{L}_{k_1}(a), \dots, \mathcal{L}_{k_2}(a))$ and exit data $(\mathcal{L}_{k_1}(b), \dots, \mathcal{L}_{k_2}(b))$ from Definition 5.1.7. Note that $\tilde{Q}$ is introduced because, by definition, any such (f, g)-avoiding Brownian line ensemble is indexed from 1 to $k_2 - k_1 + 1$ but we want Q to be indexed from k_1 to k_2.

An equivalent way to express the Brownian Gibbs property is as follows. A Σ-indexed line ensemble $\mathcal{L}$ on Λ satisfies the Brownian Gibbs property if and only if it is non-intersecting and for any finite $K = \{k_1, k_1+1, \dots, k_2\} \subset \Sigma$ and $[a, b] \subset \Lambda$ and any bounded Borel-measurable function $F : C(K \times [a, b]) \to \mathbb{R}$ we have $\mathbb{P}$-almost surely

$$\mathbb{E}\left[F\left(\mathcal{L}|_{K\times[a,b]}\right)\middle|\mathcal{F}_{ext}(K\times(a,b))\right] = \mathbb{E}^{a,b,\vec{x},\vec{y},f,g}_{avoid}\left[F(\tilde{Q})\right], \tag{5.1.3}$$

where

$$\mathcal{F}_{ext}(K\times(a,b)) = \sigma\left\{\mathcal{L}_i(s) : (i,s) \in D^c_{K,a,b}\right\}$$

is the σ-algebra generated by the variables in the brackets above, $\mathcal{L}|_{K\times[a,b]}$ denotes the restriction of $\mathcal{L}$ to the set $K \times [a, b]$, $\vec{x} = (\mathcal{L}_{k_1}(a), \dots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \dots, \mathcal{L}_{k_2}(b))$, $f = \mathcal{L}_{k_1-1}[a, b]$ (the restriction of $\mathcal{L}$ to the set $\{k_1 - 1\} \times [a, b]$) with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$, and $g = \mathcal{L}_{k_2+1}[a, b]$ with the convention that $g = -\infty$ if $k_2 + 1 \notin \Sigma$.

Remark 5.1.9. Let us briefly explain why equation (5.1.3) makes sense. Firstly, since $\Sigma \times \Lambda$ is locally compact, we know by [192, Lemma 46.4] that $\mathcal{L} \to \mathcal{L}|_{K\times[a,b]}$ is a continuous map from $C(\Sigma \times \Lambda)$ to $C(K \times [a, b])$, so that the left side of (5.1.3) is the conditional expectation of a bounded measurable function, and is thus well-defined. A more subtle question is why the right side of (5.1.3) is $\mathcal{F}_{ext}(K \times (a,b))$-measurable. This question was resolved in [105, Lemma 3.4], where it was shown that the right side is measurable with respect to the σ-algebra

$$\sigma \left\{ \mathcal{L}_i(s) : \ i \in K \text{ and } s \in \{a,b\}, \text{ or } i \in \{k_1 - 1, k_2 + 1\} \text{ and } s \in [a,b] \right\},$$

which in particular implies the measurability with respect to $\mathcal{F}_{ext}(K \times (a,b))$.

In the present paper it is convenient for us to use the following modified version of the definition above, which we call the partial Brownian Gibbs property – it was first introduced in [105]. We explain the difference between the two definitions, and why we prefer the second one in Remark 5.1.12.

Definition 5.1.10. Fix a set $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and an interval $\Lambda \subset \mathbb{R}$. A Σ-indexed line ensemble $\mathcal{L}$ on Λ is said to satisfy the *partial Brownian Gibbs property* if and only if it is non-intersecting and for any finite $K = \{k_1, k_1 + 1, \dots, k_2\} \subset \Sigma$ with $k_2 \leq N - 1$ (if $\Sigma \neq \mathbb{N}$), $[a,b] \subset \Lambda$ and any bounded Borel-measurable function $F : C(K \times [a,b]) \to \mathbb{R}$ we have $\mathbb{P}$-almost surely

$$\mathbb{E}\left[F(\mathcal{L}|_{K\times[a,b]}) \middle| \mathcal{F}_{ext}(K \times (a,b)) \right] = \mathbb{E}_{avoid}^{a,b,\vec{x},\vec{y},f,g}\left[F(\tilde{\mathcal{Q}}) \right], \tag{5.1.4}$$

where we recall that $D_{K,a,b} = K \times (a,b)$ and $D_{K,a,b}^c = (\Sigma \times \Lambda) \setminus D_{K,a,b}$, and

$$\mathcal{F}_{ext}(K \times (a,b)) = \sigma \left\{ \mathcal{L}_i(s) : (i,s) \in D_{K,a,b}^c \right\}$$

is the σ-algebra generated by the variables in the brackets above, $\mathcal{L}|_{K\times[a,b]}$ denotes the restriction of $\mathcal{L}$ to the set $K \times [a,b]$, $\vec{x} = (\mathcal{L}_{k_1}(a), \dots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \dots, \mathcal{L}_{k_2}(b))$, $f = \mathcal{L}_{k_1-1}[a,b]$ with the convention that $f = \infty$ if $k_1 - 1 \notin \Sigma$, and $g = \mathcal{L}_{k_2+1}[a,b]$.

Remark 5.1.11. Observe that if $N = 1$ then the conditions in Definition 5.1.10 become void, i.e., any line ensemble with one line satisfies the partial Brownian Gibbs property. Also we mention that (5.1.4) makes sense by the same reason that (5.1.3) makes sense, see Remark 5.1.9.

Remark 5.1.12. Definition 5.1.10 is slightly different from the Brownian Gibbs property of Definition 5.1.8 as we explain here. Assuming that $\Sigma = \mathbb{N}$ the two definitions are equivalent. However, if $\Sigma = \{1, \dots, N\}$ with $1 \leq N < \infty$ then a line ensemble that satisfies the Brownian Gibbs property also satisfies the partial Brownian Gibbs property, but the reverse need not be true. Specifically, the Brownian Gibbs property allows for the possibility that $k_2 = N$ in Definition 5.1.10 and in this case the convention is that $g = -\infty$. As the partial Brownian Gibbs property is more general we prefer to work with it and most of the results later in this paper are formulated in terms of it rather than the usual Brownian Gibbs property.

5.1.2 Bernoulli Gibbsian line ensembles

In this section we introduce the notion of a *Bernoulli line ensemble* and the *Schur Gibbs property*. Our discussion will parallel that of [**CD**], which in turn goes back to [74, Section 2.1].

Definition 5.1.13. Let $\Sigma \subset \mathbb{Z}$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. Consider the set Y of functions $f : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(j, i+1) - f(j, i) \in \{0, 1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$ and let $\mathcal{D}$ denote the discrete topology on Y. We call a function $f : [\![T_0, T_1]\!] \to \mathbb{Z}$ such that $f(i+1) - f(i) \in \{0, 1\}$ when $i \in [\![T_0, T_1 - 1]\!]$ an *up-right path* and elements in Y *collections of up-right paths*.

A Σ*-indexed Bernoulli line ensemble* $\mathfrak{L}$ *on* $[\![T_0, T_1]\!]$ is a random variable defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, taking values in Y such that $\mathfrak{L}$ is a $(\mathcal{B}, \mathcal{D})$-measurable function.

Remark 5.1.14. In [**CD**] Bernoulli line ensembles $\mathfrak{L}$ were called *discrete line ensembles* in order to distinguish them from the continuous line ensembles from Definition 5.1.1. In this paper we have opted to use the term Bernoulli line ensembles to emphasize the fact that the functions $f \in Y$ satisfy the property that $f(j, i+1) - f(j, i) \in \{0, 1\}$ when $j \in \Sigma$ and $i \in [\![T_0, T_1 - 1]\!]$. This condition essentially means that for each $j \in \Sigma$ the function $f(j, \cdot)$ can be thought of as the trajectory of a Bernoulli random walk from time T_0 to time T_1. As other types of discrete line ensembles, see e.g. [239], have appeared in the literature we have decided to modify the notation in [**CD**] so as to avoid any ambiguity.

The way we think of Bernoulli line ensembles is as random collections of up-right paths on the integer lattice, indexed by Σ (see Figure 5.1). Observe that one can view an up-right path L on $[\![T_0, T_1]\!]$ as a continuous curve by linearly interpolating the points $(i, L(i))$. This allows us to define $(\mathfrak{L}(\omega))(i, s)$ for non-integer $s \in [T_0, T_1]$ and to view Bernoulli line ensembles as line ensembles in the sense of Definition 5.1.1. In particular, we can think of $\mathfrak{L}$ as a random variable taking values in $(C(\Sigma \times \Lambda), \mathcal{C}_\Sigma)$ with $\Lambda = [T_0, T_1]$. We will often slightly abuse notation and write

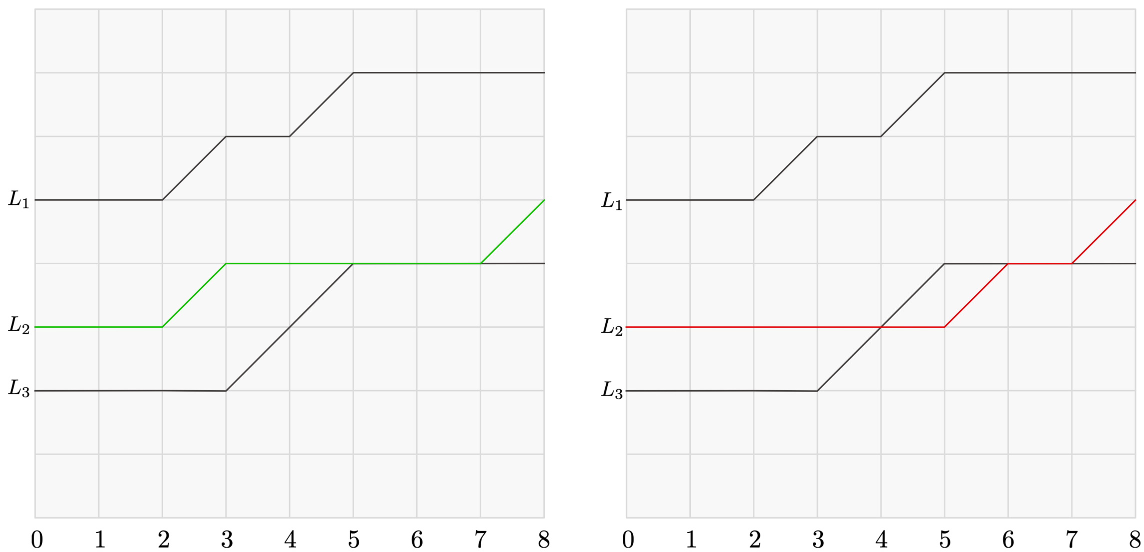


Figure 5.1: Two samples of $[\![1, 3]\!]$-indexed Bernoulli line ensembles with $T_0 = 1$ and $T_1 = 8$, with the left ensemble avoiding and the right ensemble nonavoiding.

$\mathfrak{L} : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$, even though it is not $\mathfrak{L}$ which is such a function, but rather $\mathfrak{L}(\omega)$ for each $\omega \in \Omega$. Furthermore we write $L_i = (\mathfrak{L}(\omega))(i, \cdot)$ for the index $i \in \Sigma$ path. If L is an up-right path on $[\![T_0, T_1]\!]$ and $a, b \in [\![T_0, T_1]\!]$ satisfy $a < b$ we let $L[\![a, b]\!]$ denote the resitrction of L to $[\![a, b]\!]$.

Let $t_i, z_i \in \mathbb{Z}$ for $i = 1, 2$ be given such that $t_1 < t_2$ and $0 \le z_2 - z_1 \le t_2 - t_1$. We denote by $\Omega(t_1, t_2, z_1, z_2)$ the collection of up-right paths that start from (t_1, z_1) and end at (t_2, z_2), by $\mathbb{P}_{Ber}^{t_1, t_2, z_1, z_2}$ the uniform distribution on $\Omega(t_1, t_2, z_1, z_2)$ and write $\mathbb{E}_{Ber}^{t_1, t_2, z_1, z_2}$ for the expectation with respect to this measure. One thinks of the distribution $\mathbb{P}_{Ber}^{t_1, t_2, z_1, z_2}$ as the law of a simple random walk with i.i.d. Bernoulli increments with parameter $p \in (0, 1)$ that starts from z_1 at time t_1 and is conditioned to end in z_2 at time t_2 – this interpretation does not depend on the choice of $p \in (0, 1)$.

Notice that by our assumptions on the parameters the state space $\Omega(t_1, t_2, z_1, z_2)$ is non-empty.

Given $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$ and $\vec{x}, \vec{y} \in \mathbb{Z}^k$ we let $\mathbb{P}_{Ber}^{T_0,T_1,\vec{x},\vec{y}}$ denote the law of k independent Bernoulli bridges $\{B_i : [\![T_0, T_1]\!] \to \mathbb{Z}\}_{i=1}^k$ from $B_i(T_0) = x_i$ to $B_i(T_1) = y_i$. Equivalently, this is just k independent random up-right paths $B_i \in \Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ that are uniformly distributed. This measure is well-defined provided that $\Omega(T_0, T_1, x_i, y_i)$ are non-empty for $i = 1, \dots, k$, which holds if $T_1 - T_0 \geq y_i - x_i \geq 0$ for all $i = 1, \dots, k$.

The following definition introduces the notion of an (f, g)-avoiding Bernoulli line ensemble, which in simple terms is a collection of k independent Bernoulli bridges, conditioned on not-crossing each other and staying above the graph of g and below the graph of f for two functions f and g.

Definition 5.1.15. Let $k \in \mathbb{N}$ and $\mathfrak{W}_k$ denote the set of signatures of length k, i.e.

$$\mathfrak{W}_k = \{\vec{x} = (x_1, \dots, x_k) \in \mathbb{Z}^k : x_1 \geq x_2 \geq \cdots \geq x_k\}.$$

Let $\vec{x}, \vec{y} \in \mathfrak{W}_k$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, $S \subseteq [\![T_0, T_1]\!]$, and $f : [\![T_0, T_1]\!] \to (-\infty, \infty]$ and $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ be two functions.

With the above data we define the *$(f, g; S)$-avoiding Bernoulli line ensemble on the interval $[\![T_0, T_1]\!]$ with entrance data $\vec{x}$ and exit data $\vec{y}$* to be the Σ-indexed Bernoulli line ensemble $\mathfrak{Q}$ with $\Sigma = [\![1, k]\!]$ on $[\![T_0, T_1]\!]$ and with the law of $\mathfrak{Q}$ equal to $\mathbb{P}_{Ber}^{T_0,T_1,\vec{x},\vec{y}}$ (the law of k independent uniform up-right paths $\{B_i : [\![T_0, T_1]\!] \to \mathbb{R}\}_{i=1}^k$ from $B_i(T_0) = x_i$ to $B_i(T_1) = y_i$) conditioned on the event

$$E_S = \{f(r) \geq B_1(r) \geq B_2(r) \geq \cdots \geq B_k(r) \geq g(r) \text{ for all } r \in S\}.$$

The above definition is well-posed if there exist $B_i \in \Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ that satisfy the conditions in E_S (i.e. if the set of such up-right paths is not empty). We will denote by $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g; S)$ the set of collections of k up-right paths that satisfy the conditions in E_S and then the distribution on $\mathfrak{Q}$ is simply the uniform measure on $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g; S)$. We denote the probability distribution of $\mathfrak{Q}$ as $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x},\vec{y},f,g}$ and write $\mathbb{E}_{avoid,Ber;S}^{T_0,T_1,\vec{x},\vec{y},f,g}$ for the expectation with respect to this measure. If $S = [\![T_0, T_1]\!]$, we write $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$, $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$, and $\mathbb{E}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$. If $f = +\infty$ and $g = -\infty$, we write $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y})$, $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y}}_{avoid,Ber}$, and $\mathbb{E}^{T_0,T_1,\vec{x},\vec{y}}_{avoid,Ber}$.

It will be useful to formulate simple conditions under which $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is non-empty and thus $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber}$ well-defined. Note that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g; S) \supseteq \Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ for any $S \subseteq [\![T_0, T_1]\!]$, so $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber;S}$ is also well-defined in this case. We accomplish this in the following lemma, whose proof is postponed until Section 5.6.3.

Lemma 5.1.16. *Suppose that $k \in \mathbb{N}$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. Suppose further that*

1. *$\vec{x}, \vec{y} \in \mathfrak{W}_k$ satisfy $T_1 - T_0 \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$,*
2. *$f : [\![T_0, T_1]\!] \to (-\infty, \infty]$ and $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ satisfy $f(i+1) = f(i)$ or $f(i+1) = f(i) + 1$, and $g(i+1) = g(i)$ or $g(i+1) = g(i) + 1$ for $i = T_0, \dots, T_1 - 1$,*
3. *$f(T_0) \geq x_1, f(T_1) \geq y_1$ and $g(T_0) \leq x_k, g(T_1) \leq y_k$.*

Then the set $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ from Definition 5.1.15 is non-empty.

The following definition introduces the notion of the Schur Gibbs property, which can be thought of a discrete analogue of the partial Brownian Gibbs property the same way that Bernoulli random walks are discrete analogues of Brownian motion.

Definition 5.1.17. Fix a set $\Sigma = [\![1, N]\!]$ with $N \in \mathbb{N}$ or $N = \infty$ and $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$. A Σ-indexed Bernoulli line ensemble $\mathfrak{L} : \Sigma \times [\![T_0, T_1]\!] \to \mathbb{Z}$ is said to satisfy the *Schur Gibbs property* if it is non-crossing, meaning that

$$L_j(i) \geq L_{j+1}(i) \text{ for all } j = 1, \dots, N-1 \text{ and } i \in [\![T_0, T_1]\!],$$

and for any finite $K = \{k_1, k_1 + 1, \dots, k_2\} \subset [\![1, N-1]\!]$ and $a, b \in [\![T_0, T_1]\!]$ with $a < b$ the following holds. Suppose that f, g are two up-right paths drawn in $\{(r, z) \in \mathbb{Z}^2 : a \leq r \leq b\}$ and $\vec{x}, \vec{y} \in \mathfrak{W}_k$ with $k = k_2 - k_1 + 1$ altogether satisfy that $\mathbb{P}(A) > 0$ where A denotes the event

$$A = \{\vec{x} = (L_{k_1}(a), \dots, L_{k_2}(a)), \vec{y} = (L_{k_1}(b), \dots, L_{k_2}(b)), L_{k_1-1}[\![a, b]\!] = f, L_{k_2+1}[\![a, b]\!] = g\},$$

where if $k_1 = 1$ we adopt the convention $f = \infty = L_0$. Then writing $k = k_2 - k_1 + 1$, we have for any $\{B_i \in \Omega(a, b, x_i, y_i)\}_{i=1}^k$ that

$$\mathbb{P}\left(L_{i+k_1-1}[\![a, b]\!] = B_i \text{ for } i = 1, \dots, k \,|A\right) = \mathbb{P}_{avoid,Ber}^{a,b,\vec{x},\vec{y},f,g}\left(\cap_{i=1}^k\{\mathfrak{Q}_i = B_i\}\right). \tag{5.1.5}$$

Remark 5.1.18. In simple words, a Bernoulli line ensemble is said to satisfy the Schur Gibbs property if the distribution of any finite number of consecutive paths, conditioned on their end-points and the paths above and below them is simply the uniform measure on all collection of up-right paths that have the same end-points and do not cross each other or the paths above and below them.

Remark 5.1.19. Observe that in Definition 5.1.17 the index k_2 is assumed to be less than or equal to $N - 1$, so that if $N < \infty$ the N-th path is special and is not conditionally uniform. This is what makes Definition 5.1.17 a discrete analogue of the partial Brownian Gibbs property rather than the usual Brownian Gibbs property. Similarly to the partial Brownian Gibbs property, see Remark 5.1.11, if $N = 1$ then the conditions in Definition 5.1.17 become void, i.e., any Bernoulli line ensemble with one line satisfies the Schur Gibbs property. Also we mention that the well-posedness of $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ in (5.1.5) is a consequence of Lemma 5.1.16 and our assumption that $\mathbb{P}(A) > 0$.

Remark 5.1.20. In [**CD**] the authors studied a generalization of the Gibbs property in Definition 5.1.17 depending on a parameter $t \in (0, 1)$, which was called the *Hall-Littlewood Gibbs property* due to its connection to Hall-Littlewood polynomials [185]. The property in Definition 5.1.17 is the $t \to 0$ limit of the Hall-Littlewood Gibbs property. Since under this $t \to 0$ limit Hall-Littlewood polynomials degenerate to Schur polynomials we have decided to call the Gibbs property in Definition 5.1.17 the Schur Gibbs property.

Remark 5.1.21. An immediate consequence of Definition 5.1.17 is that if $M \leq N$, we have that the induced law on $\{L_i\}_{i=1}^M$ also satisfies the Schur Gibbs property as an $\{1, \dots, M\}$-indexed Bernoulli line ensemble on $[\![T_0, T_1]\!]$.

We end this section with the following definition of the term acceptance probability.

Definition 5.1.22. Assume the same notation as in Definition 5.1.15 and suppose that $T_1 - T_0 \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$. We define the *acceptance probability* $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$ to be the ratio

$$Z(T_0, T_1, \vec{x}, \vec{y}, f, g) = \frac{|\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)|}{\prod_{i=1}^{k} |\Omega(T_0, T_1, x_i, y_i)|}. \tag{5.1.6}$$

Remark 5.1.23. The quantity $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is precisely the probability that if B_i are sampled uniformly from $\Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ then the B_i satisfy the condition

$$E = \{f(r) \geq B_1(r) \geq B_2(r) \geq \cdots \geq B_k(r) \geq g(r) \text{ for all } r \in [\![T_0, T_1]\!]\}.$$

Let us explain briefly why we call this quantity an acceptance probability. One way to sample $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ is as follows. Start by sampling a sequence of i.i.d. up-right paths B_i^N uniformly from $\Omega(T_0, T_1, x_i, y_i)$ for $i = 1, \dots, k$ and $N \in \mathbb{N}$. For each n check if $B_1^n, \dots, B_k^n$ satisfy the condition E and let M denote the smallest index that accomplishes this. If $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is non-empty then M is geometrically distributed with parameter $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$, and in particular M is finite almost surely and $\{B_i^M\}_{i=1}^k$ has distribution $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$. In this sampling procedure we construct a sequence of candidates $\{B_i^N\}_{i=1}^k$ for $N \in \mathbb{N}$ and reject those that fail to satisfy condition E, the first candidate that satisfies it is accepted and has law $\mathbb{P}_{avoid,Ber}^{T_0,T_1,\vec{x},\vec{y},f,g}$ and the probability that a candidate is accepted is precisely $Z(T_0, T_1, \vec{x}, \vec{y}, f, g)$, which is why we call it an acceptance probability.

5.1.3 Main technical result

In this section we present the main technical result of the paper. We start with the following technical definition.

Definition 5.1.24. Fix $k \in \mathbb{N}$, $\alpha, \lambda > 0$ and $p \in (0,1)$. Suppose we are given a sequence $\{T_N\}_{N=1}^{\infty}$ with $T_N \in \mathbb{N}$ and that $\{\mathfrak{L}^N\}_{N=1}^{\infty}$, $\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)$ is a sequence of $[\![1, k]\!]$-indexed Bernoulli line ensembles on $[\![-T_N, T_N]\!]$. We call the sequence (α, p, λ)-*good* if

- for each $N \in \mathbb{N}$ we have that $\mathfrak{L}^N$ satisfies the Schur Gibbs property of Definition 5.1.17;
- there is a function $\psi : \mathbb{N} \to (0, \infty)$ such that $\lim_{N\to\infty} \psi(N) = \infty$ and for each $N \in \mathbb{N}$ we have that $T_N > \psi(N)N^{\alpha}$;
- there is a function $\phi : (0, \infty) \to (0, \infty)$ such that for any $\epsilon > 0$ we have

$$\sup_{n\in\mathbb{Z}} \limsup_{N\to\infty} \mathbb{P}\left(\left|N^{-\alpha/2}(L_1^N(nN^{\alpha}) - pnN^{\alpha} + \lambda n^2 N^{\alpha/2})\right| \geq \phi(\epsilon)\right) \leq \epsilon. \tag{5.1.7}$$

Remark 5.1.25. Let us elaborate on the meaning of Definition 5.1.24. In order for a sequence of $\mathfrak{L}^N$ of $[\![1, k]\!]$-indexed Bernoulli line ensembles on $[\![-T_N, T_N]\!]$ to be (α, p, λ)-*good* we want several conditions to be satisfied. Firstly, we want for each N the Bernoulli line ensemble $\mathfrak{L}^N$ to satisfy the Schur Gibbs property. The second condition is that while the interval of definition of $\mathfrak{L}^N$ is finite for each N and given by $[\![-T_N, T_N]\!]$, we want this interval to grow at least with speed N^{α}. This property is quantified by the function ψ, which can be essentially thought of as an arbitrary unbounded increasing function on $\mathbb{N}$. The third condition is that we want for each $n \in \mathbb{Z}$ the sequence of random variables $N^{-\alpha/2}(L_1^N(nN^{\alpha}) - pnN^{\alpha})$ to be tight but moreover we want globally these random variables to look like the parabola $-\lambda n^2$. This statement is reflected in (5.1.7), which provides a certain uniform tightness of the random variables $N^{-\alpha/2}(L_1^N(nN^{\alpha}) - pnN^{\alpha} + \lambda n^2 N^{\alpha/2})$. A particular case when (5.1.7) is satisfied is for example if we know that for each $n \in \mathbb{Z}$ the random variables $N^{-\alpha/2}(L_1^N(nN^{\alpha}) - pnN^{\alpha} + \lambda n^2 N^{\alpha/2})$ converge to the same random variable X. In the applications that we have in mind these random variables would converge to the 1-point marginals of the Airy$_2$ process that are all given by the same Tracy-Widom distribution (since the Airy$_2$ process is stationary). Equation (5.1.7) is a significant relaxation of the requirement that $N^{-\alpha/2}(L_1^N(nN^{\alpha}) - pnN^{\alpha} + \lambda n^2 N^{\alpha/2})$ all converge weakly to the Tracy-Widom distribution – the convergence requirement is replaced with a mild but uniform control of all subsequential limits.

The main result of the paper is as follows.

Theorem 5.1.26. *Fix $k \in \mathbb{N}$ with $k \geq 2$, $\alpha, \lambda > 0$ and $p \in (0,1)$ and let $\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)$ be an (α, p, λ)-good sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles. Set*

$$f_i^N(s) = N^{-\alpha/2}(L_i^N(sN^\alpha) - psN^\alpha + \lambda s^2 N^{\alpha/2}), \text{ for } s \in [-\psi(N), \psi(N)] \text{ and } i = 1, \dots, k-1,$$

and extend f_i^N to $\mathbb{R}$ by setting for $i = 1, \dots, k-1$

$$f_i^N(s) = f_i^N(-\psi(N)) \text{ for } s \leq -\psi(N) \text{ and } f_i^N(s) = f_N(\psi(N)) \text{ for } s \geq \psi(N).$$

Let $\mathbb{P}_N$ denote the law of $\{f_i^N\}_{i=1}^{k-1}$ as a $[\![1,k-1]\!]$-indexed line ensemble (i.e. as a random variable in $(C([\![1,k-1]\!] \times \mathbb{R}), \mathcal{C})$), and let $\tilde{\mathbb{P}}_N$ denote the law of $\{(f_i^N - \lambda s^2)/\sqrt{p(1-p)}\}_{i=1}^{k-1}$. Then we have

(i) The sequence $\mathbb{P}_N$ is tight;

(ii) Any subsequential limit $\mathcal{L}^\infty = \{f_i^\infty\}_{i=1}^{k-1}$ of $\tilde{\mathbb{P}}_N$ satisfies the partial Brownian Gibbs property of Definition 5.1.10.

Roughly, Theorem 5.1.26 (i) states that if we have a sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles that satisfy the Schur Gibbs property and the top paths of these ensembles under some shift and scaling have tight one-point marginals with a non-trivial parabolic shift, then under the same shift and scaling the top $k-1$ paths of the line ensemble will be tight. The extension of f_i^N to $\mathbb{R}$ is completely arbitrary and irrelevant for the validity of Theorem 5.1.26 since the topology on $C([\![1,k-1]\!] \times \mathbb{R})$ is that of uniform convergence over compacts. Consequently, only the behavior of these functions on compact intervals matters in Theorem 5.1.26 and not what these functions do near infinity, which is where the modification happens as $\lim_{N\to\infty} \psi(N) = \infty$ by assumption. The only reason we perform the extension is to embed all Bernoulli line ensembles into the same space $(C([\![1,k-1]\!] \times \mathbb{R}), \mathcal{C})$.

We mention that the k-th up-right path in the sequence of Bernoulli line ensembles is special and Theorem 5.1.26 provides no tightness result for it. The reason for this stems from the Schur

Gibbs property, see Definition 5.1.17, which assumes less information for the k-th path. In practice, one either has an infinite Bernoulli line ensemble for each N or one has a Bernoulli line ensemble with finite number of paths, which increase with N to infinity. In either of these settings one can use Theorem 5.1.26 to prove tightness of the full line ensemble - we will have more to say about this in Section 7.

The proof of Theorem 5.1.26 is presented in Section 5.3. In the next section we derive various properties for Bernoulli line ensembles.

5.2 Properties of Bernoulli line ensembles

In this section we derive several results for Bernoulli line ensembles, which will be used in the proof of Theorem 5.1.26 in Section 5.3.

5.2.1 Monotone coupling lemmas

In this section we formulate two lemmas that provide couplings of two Bernoulli line ensembles of non-intersecting Bernoulli bridges on the same interval, which depend monotonically on their boundary data. Schematic depictions of the couplings are provided in Figure 5.2. We postpone the proof of these lemmas until Section 5.6.

Lemma 5.2.1. *Assume the same notation as in Definition 5.1.15. Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, $S \subseteq [\![T_0, T_1]\!]$, a function $g : [\![T_0, T_1]\!] \to [-\infty, \infty)$ as well as $\vec{x}, \vec{y}, \vec{x}\,', \vec{y}\,' \in \mathfrak{W}_k$. Assume that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g; S)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}', \vec{y}', \infty, g; S)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which supports two $[\![1, k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0, T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\big($resp. $\mathfrak{L}^b\big)$ under $\mathbb{P}$ is given by $\mathbb{P}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g}_{avoid,Ber;S}$ $\big($resp. $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber;S}\big)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}^t_i(r) \geq \mathfrak{L}^b_i(r)$ for all $i = 1, \dots, k$ and $r \in [\![T_0, T_1]\!]$.*

Lemma 5.2.2. *Assume the same notation as in Definition 5.1.15. Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, $S \subseteq [\![T_0, T_1]\!]$, two functions $g^t, g^b : [\![T_0, T_1]\!] \to [-\infty, \infty)$ and $\vec{x}, \vec{y} \in \mathfrak{W}_k$. We assume that*

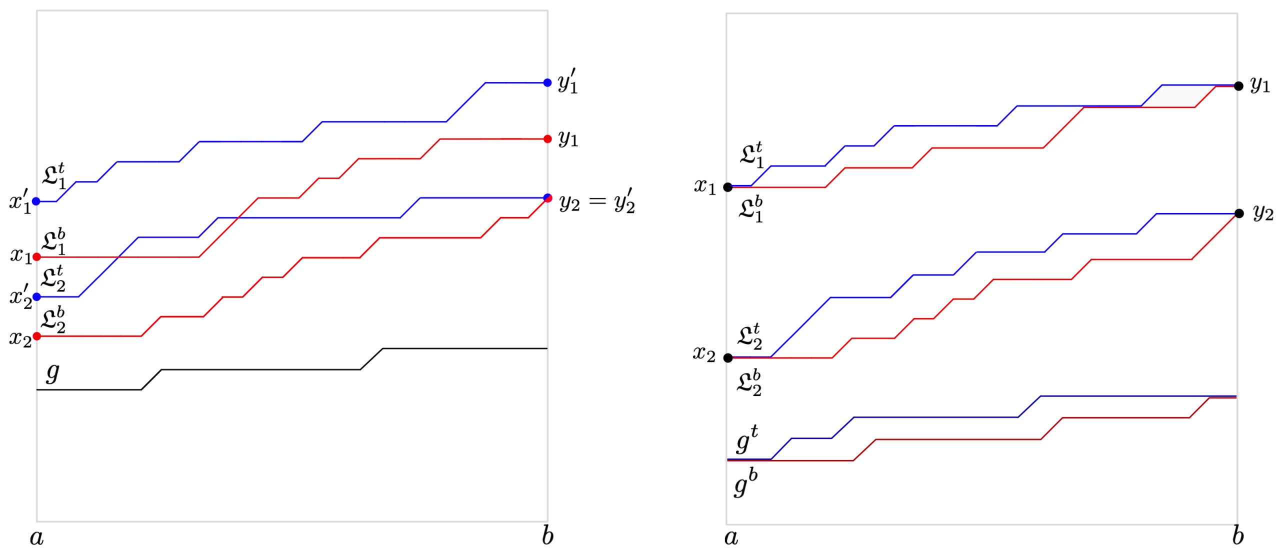


Figure 5.2: Two diagrammatic depictions of the monotone coupling Lemma 5.2.1 (left part) and Lemma 5.2.2 (right part). Red depicts the lower line ensemble and accompanying entry data, exit data, and bottom bounding curve, while blue depicts that of the higher ensemble.

$g^t(r) \geq g^b(r)$ for all $r \in [\![T_0, T_1]\!]$ and that $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^t; S)$ and $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b; S)$ are both non-empty. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which supports two $[\![1, k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0, T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\big($resp. $\mathfrak{L}^b\big)$ under $\mathbb{P}$ is given by $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^t}_{avoid,Ber;S}$ $\big($resp. $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber;S}\big)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}^t_i(r) \geq \mathfrak{L}^b_i(r)$ for all $i = 1, \dots, k$ and $r \in [\![T_0, T_1]\!]$.

In plain words, Lemma 5.2.1 states that one can couple two Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ of non-intersecting Bernoulli bridges, bounded from below by the same function g, in such a way that if all boundary values of $\mathfrak{L}^t$ are above the respective boundary values of $\mathfrak{L}^b$, then all up-right paths of $\mathfrak{L}^t$ are almost surely above the respective up-right paths of $\mathfrak{L}^b$. See the left part of Figure 5.2. Lemma 5.2.2, states that one can couple two Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ that have the same boundary values, but the lower bound g^t of $\mathfrak{L}^t$ is above the lower bound g^b of $\mathfrak{L}^b$, in such a way that all up-right paths of $\mathfrak{L}^t$ are almost surely above the respective up-right paths of $\mathfrak{L}^b$. See the right part of Figure 5.2.

5.2.2 Properties of Bernoulli and Brownian bridges

In this section we derive several results about Bernoulli bridges, which are random up-right paths that have law $\mathbb{P}_{Ber}^{T_0,T_1,x,y}$ as in Section 5.1.2, as well as Brownian bridges with law $\mathbb{P}_{free}^{T_0,T_1,x,y}$ as in Section 5.1.1. Our results will rely on the two monotonicity Lemmas 5.2.1 and 5.2.2 as well as a strong coupling between Bernoulli bridges and Brownian bridges from [**CD**] – recalled here as Theorem 5.2.3.

If W_t denotes a standard one-dimensional Brownian motion and $\sigma > 0$, then the process

$$B_t^\sigma = \sigma(W_t - tW_1), \quad 0 \le t \le 1,$$

is called a *Brownian bridge (conditioned on* $B_0 = 0, B_1 = 0$*) with variance* σ^2. We note that B^σ is the unique a.s. continuous Gaussian process on $[0,1]$ with $B_0 = B_1 = 0$, $[B_t^\sigma] = 0$, and

$$[B_r^\sigma B_s^\sigma] = \sigma^2(r \wedge s - rs - sr + sr) = \sigma^2(r \wedge s - rs). \tag{5.2.1}$$

With the above notation we state the strong coupling result we use.

Theorem 5.2.3. *Let $p \in (0,1)$. There exist constants $0 < C, a, \alpha < \infty$ (depending on p) such that for every positive integer n, there is a probability space on which are defined a Brownian bridge B^σ with variance $\sigma^2 = p(1-p)$ and a family of random paths $\ell^{(n,z)} \in \Omega(0,n,0,z)$ for $z = 0,\dots,n$ such that $\ell^{(n,z)}$ has law $\mathbb{P}_{Ber}^{0,n,0,z}$ and*

$$\mathbb{E}\left[e^{a\Delta(n,z)}\right] \le Ce^{\alpha(\log n)^2}e^{|z-pn|^2/n}, \text{ where } \Delta(n,z) := \sup_{0\le t\le n}\left|\sqrt{n}B^\sigma_{t/n} + \tfrac{t}{n}z - \ell^{(n,z)}(t)\right|. \tag{5.2.2}$$

Remark 5.2.4. When $p = 1/2$ the above theorem follows (after a trivial affine shift) from [173, Theorem 6.3] and the general $p \in (0,1)$ case was done in [**CD**]. We mention that a significant generalization of Theorem 5.2.3 for general random walk bridges has recently been proved in [106, Theorem 2.3].

We will use the following simple corollary of Theorem 5.2.3 to compare Bernoulli bridges with Brownian bridges. We use the same notation as in the theorem.

Corollary 5.2.5. *Fix $p \in (0,1)$, $\beta > 0$, and $A > 0$. Suppose $|z - pn| \leq K\sqrt{n}$ for a constant $K > 0$. Then for any $\epsilon > 0$, there exists N large enough depending on p, ϵ, A, K so that for $n \geq N$,*

$$\mathbb{P}\Big(\Delta(n,z) \geq An^{\beta}\Big) < \epsilon.$$

Proof. Applying Chebyshev's inequality and (5.2.2) gives

$$\begin{aligned}\mathbb{P}\Big(\Delta(n,z) \geq An^{\beta}\Big) &\leq e^{-An^{\beta}}\Big[e^{a\Delta(n,z)}\Big] \leq C\exp\Big[-An^{\beta} + \alpha(\log n)^2 + \frac{|z-pn|^2}{n}\Big] \\ &\leq C\exp\Big[-An^{\beta} + \alpha(\log n)^2 + K\Big].\end{aligned}$$

The conclusion is now immediate. □

We also state the following result regarding the distribution of the maximum of a Brownian bridge, which follows from formulas in [113, Section 12.3].

Lemma 5.2.6. *Fix $p \in (0,1)$, and let B^{σ} be a Brownian bridge of variance $\sigma^2 = p(1-p)$ on $[0,1]$. Then for any $C, T > 0$ we have*

$$\begin{aligned}\mathbb{P}\left(\max_{s\in[0,T]} B^{\sigma}_{s/T} \geq C\right) &= \exp\left(-\frac{2C^2}{p(1-p)}\right), \\ \mathbb{P}\left(\max_{s\in[0,T]} \left|B^{\sigma}_{s/T}\right| \geq C\right) &= 2\sum_{n=1}^{\infty}(-1)^{n-1}\exp\left(-\frac{2n^2C^2}{p(1-p)}\right).\end{aligned} \tag{5.2.3}$$

In particular,

$$\mathbb{P}\left(\max_{s\in[0,T]} \left|B^{\sigma}_{s/T}\right| \geq C\right) \leq 2\exp\left(-\frac{2C^2}{p(1-p)}\right). \tag{5.2.4}$$

Proof. Let B^1 be a Brownian bridge with variance 1 on $[0,1]$. Then B^{σ}_t has the same distribution as σB_t^1. Hence

$$\mathbb{P}\left(\max_{s\in[0,T]} B^{\sigma}_{s/T} \geq C\right) = \mathbb{P}\left(\max_{t\in[0,1]} B^1_t \geq C/\sigma\right) = e^{-2(C/\sigma)^2} = e^{-2C^2/p(1-p)}.$$

The second equality follows from [113, Proposition 12.3.3]. This proves the first equality in (5.2.3). Similarly, using [113, Proposition 12.3.4] we find

$$\mathbb{P}\left(\max_{s\in[0,T]} \left|B^{\sigma}_{s/T}\right| \geq C\right) = \mathbb{P}\left(\max_{t\in[0,1]} \left|B^1_t\right| \geq C/\sigma\right) = 2\sum_{n=1}^{\infty}(-1)^{n-1}e^{-2n^2C^2/\sigma^2},$$

proving the second inequality in (5.2.3).

Lastly to prove (5.2.4), observe that since B^{σ}_t has mean 0, B^{σ}_t and $-B^{\sigma}_t$ have the same distribution. It follows from the first equality above that

$$\mathbb{P}\left(\max_{s\in[0,T]} \left|B^{\sigma}_{s/T}\right| \geq C\right) \leq \mathbb{P}\left(\max_{s\in[0,T]} B^{\sigma}_{s/T} \geq C\right) + \mathbb{P}\left(\max_{s\in[0,T]} \big(-B^{\sigma}_{s/T}\big) \geq C\right) =$$
$$2\,\mathbb{P}\left(\max_{s\in[0,T]} B^{\sigma}_{s/T} \geq C\right) = 2e^{-2C^2/p(1-p)}.$$

□

We state one more lemma about Brownian bridges, which allows us to decompose a bridge on $[0,1]$ into two independent bridges with Gaussian affine shifts meeting at a point in $(0,1)$.

Lemma 5.2.7. *Fix $p \in (0,1)$, $T > 0$, $t \in (0,T)$, and let B^{σ} be a Brownian bridge of variance $\sigma^2 = p(1-p)$ on $[0,1]$. Let ξ be a Gaussian random variable with mean 0 and variance*

$$[\xi^2] = \sigma^2\frac{t}{T}\left(1-\frac{t}{T}\right).$$

Let B^1, B^2 be two independent Brownian bridges on $[0,1]$ with variances $\sigma^2 t/T$ and $\sigma^2(T-t)/T$ respectively, also independent from B^{σ}. Define the process

$$\tilde{B}_{s/T} = \frac{s}{t}\,\xi + B^1\Big(\frac{s}{t}\Big), s \leq t, \frac{T-s}{T-t}\,\xi + B^2\Big(\frac{s-t}{T-t}\Big), s \geq t,$$

for $s \in [0,T]$. Then $\tilde{B}$ is a Brownian bridge with variance σ.

Proof. It is clear that the process $\tilde{B}$ is a.s. continuous. Since $\tilde{B}$ is built from three independent zero-centered Gaussian processes, it is itself a zero-centered Gaussian process and thus completely characterized by its covariance. Consequently, to show that $\tilde{B}$ is a Brownian bridge of variance σ^2, it suffices to show by (5.2.1) that if $0 \le r \le s \le T$ we have

$$[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \sigma^2 \frac{r}{T}\Big(1 - \frac{s}{T}\Big). \tag{5.2.5}$$

First assume $s \le t$ Using the fact that ξ and $B^1_\cdot$ are independent with mean 0, we find

$$[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \frac{rs}{t^2} \cdot \sigma^2 \frac{t}{T}\Big(1 - \frac{t}{T}\Big) + \sigma^2 \frac{t}{T} \cdot \frac{r}{t}\Big(1 - \frac{s}{t}\Big) =$$
$$\sigma^2 \frac{r}{T}\Big(\frac{s}{t} - \frac{s}{T} + 1 - \frac{s}{t}\Big) = \sigma^2 \frac{r}{T}\Big(1 - \frac{s}{T}\Big).$$

If $r \ge t$, we compute

$$[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \frac{(T-r)(T-s)}{(T-t)^2} \cdot \sigma^2 \frac{t}{T}\Big(1 - \frac{t}{T}\Big) + \sigma^2 \frac{T-t}{T} \cdot \frac{r-t}{T-t}\Big(1 - \frac{s-t}{T-t}\Big) =$$
$$\frac{\sigma^2(T-s)}{T(T-t)}\left(\frac{t(T-r)}{T} + r - t\right) = \frac{\sigma^2(T-s)}{T(T-t)} \cdot \frac{r(T-t)}{T} = \sigma^2 \frac{r}{T}\Big(1 - \frac{s}{T}\Big).$$

If $r < t < s$, then since ξ, $B^1_\cdot$, and $B^2_\cdot$ are all independent, we have

$$[\tilde{B}_{r/T}\tilde{B}_{s/T}] = \frac{r}{t} \cdot \frac{T-s}{T-t} \cdot \sigma^2 \frac{t(T-t)}{T^2} = \sigma^2 \frac{r(T-s)}{T^2} = \sigma^2 \frac{r}{T}\Big(1 - \frac{s}{T}\Big).$$

This proves (5.2.5) in all cases. □

Below we list four lemmas about Bernoulli bridges. We provide a brief informal explanation of what each result says after it is stated. All six lemmas are proved in a similar fashion. For the first two lemmas one observes that the event whose probability is being estimated is monotone in ℓ. This allows us by Lemmas 5.2.1 and 5.2.2 to replace x, y in the statements of the lemmas with the extreme values of the ranges specified in each. Once the choice of x and y is fixed one can use our strong coupling results, Theorem 5.2.3 and Corollary 5.2.5, to reduce each of the lemmas to an analogous one involving a Brownian bridge with some prescribed variance. The latter statements are then easily confirmed as one has exact formulas for Brownian bridges, such as Lemma 5.2.6.

Lemma 5.2.8. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $T \geq y - x \geq 0$, and suppose that ℓ has distribution $\mathbb{P}_{Ber}^{0,T,x,y}$. Let $M_1, M_2 \in \mathbb{R}$ be given. Then we can find $W_0 = W_0(p, M_2 - M_1) \in \mathbb{N}$ such that for $T \geq W_0$, $x \geq M_1 T^{1/2}$, $y \geq pT + M_2 T^{1/2}$ and $s \in [0,T]$ we have*

$$\mathbb{P}_{Ber}^{0,T,x,y}\left(\ell(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right) \geq \frac{1}{3}. \tag{5.2.6}$$

Remark 5.2.9. If $M_1, M_2 = 0$ then Lemma 5.2.8 states that if a Bernoulli bridge ℓ is started from $(0,x)$ and terminates at (T,y), which are above the straight line of slope p, then at any given time $s \in [0,T]$ the probability that $\ell(s)$ goes a modest distance below the straight line of slope p is upper bounded by $2/3$.

Proof. Define $A = \lfloor M_1 T^{1/2} \rfloor$ and $B = \lfloor pT + M_2 T^{1/2} \rfloor$. Then since $A \leq x$ and $B \leq y$, it follows from Lemma 3.1 that there is a probability space with measure $\mathbb{P}_0$ supporting random variables L_1 and L_2, whose laws under $\mathbb{P}_0$ are $\mathbb{P}_{Ber}^{0,T,A,B}$ and $\mathbb{P}_{Ber}^{0,T,x,y}$ respectively, and $\mathbb{P}_0$-a.s. we have $L_1 \leq L_2$. Thus

$$\begin{gathered}
\mathbb{P}_{Ber}^{0,T,x,y}\left(\ell(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right) = \\
\mathbb{P}_0\left(L_2(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right) \geq \\
\mathbb{P}_0\left(L_1(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right) = \\
\mathbb{P}_{Ber}^{0,T,A,B}\left(\ell(s) \geq \frac{T-s}{T} \cdot M_1 T^{1/2} + \frac{s}{T} \cdot \left(pT + M_2 T^{1/2}\right) - T^{1/4}\right).
\end{gathered} \tag{5.2.7}$$

Since the uniform distribution on upright paths on $[\![0,T]\!] \times [\![A,B]\!]$ is the same as that on upright paths on $[\![0,T]\!] \times [\![0,B-A]\!]$ shifted vertically by A, the last line of (5.2.7) is equal to

$$\mathbb{P}^{0,T,0,B-A}_{Ber}\left(\ell(s) + A \geq \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\left(pT + M_2T^{1/2}\right) - T^{1/4}\right).$$

Now we employ the coupling provided by Theorem 5.2.3. We have another probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random variable $\ell^{(T,B-A)}$ whose law under $\mathbb{P}$ is $\mathbb{P}^{0,T,0,B-A}_{Ber}$ as well as a Brownian bridge B^σ coupled with $\ell^{(T,B-A)}$. We have

$$\begin{aligned}
&\mathbb{P}^{0,T,0,B-A}_{Ber}\left(\ell(s) + A \geq \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\left(pT + M_2T^{1/2}\right) - T^{1/4}\right) = \\
&\mathbb{P}\left(\ell^{(T,B-A)}(s) + A \geq \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\left(pT + M_2T^{1/2}\right) - T^{1/4}\right) = \\
&\mathbb{P}\Bigg(\left[\ell^{(T,B-A)}(s) - \sqrt{T}B^\sigma_{s/T} - \frac{s}{T}\cdot(B-A)\right] + \sqrt{T}B^\sigma_{s/T} \geq \\
&\qquad -A - \frac{s}{T}\cdot(B-A) + \frac{T-s}{T}\cdot M_1T^{1/2} + \frac{s}{T}\cdot\left(pT + M_2T^{1/2}\right) - T^{1/4}\Bigg).
\end{aligned} \tag{5.2.8}$$

Recalling the definitions of A and B, we can rewrite the quantity in the last line of (5.2.8) and bound by

$$\begin{aligned}
&\frac{T-s}{T}\cdot(M_1T^{1/2} - A) + \frac{s}{T}\cdot(pT + M_2T^{1/2} - B) - T^{1/4} \leq \\
&\frac{T-s}{T} + \frac{s}{T} - T^{1/4} = -T^{1/4} + 1.
\end{aligned}$$

Thus the last line of (5.2.7) is bounded below by

$$\begin{aligned}
&\mathbb{P}\left(\left[\ell^{(T,B-A)}(s) - \sqrt{T}B^\sigma_{s/T} - \frac{s}{T}\cdot(B-A)\right] + \sqrt{T}B^\sigma_{s/T} \geq -T^{1/4} + 1\right) \geq \\
&\mathbb{P}\left(\sqrt{T}B^\sigma_{s/T} \geq 0 \quad \text{and} \quad \Delta(T, B-A) < T^{1/4} - 1\right) \geq \\
&\mathbb{P}\left(B^\sigma_{s/T} \geq 0\right) - \mathbb{P}\left(\Delta(T, B-A) \geq T^{1/4} - 1\right) = \\
&\frac{1}{2} - \mathbb{P}\left(\Delta(T, B-A) \geq T^{1/4} - 1\right).
\end{aligned} \tag{5.2.9}$$

For the first inequality, we used the fact that the quantity in brackets is bounded in absolute value by $\Delta(T, B-A)$. The second inequality follows by dividing the event $\{B^{\sigma}_{s/T} \geq 0\}$ into cases and applying subadditivity. Since $|B-A-pT| \leq (M_2-M_1+1)\sqrt{T}$, Corollary 5.2.5 allows us to choose W_0 large enough depending on p and $M_2 - M_1$ so that if $T \geq W_0$, then the last line of (5.2.9) is bounded above by $1/2 - 1/6 = 1/3$. In combination with (5.2.7) this proves (5.2.6). $\square$

Lemma 5.2.10. *Fix $p \in (0,1)$, $T \in \mathbb{N}$ and $y, z \in \mathbb{Z}$ such that $T \geq y, z \geq 0$, and suppose that ℓ_y, ℓ_z have distributions $\mathbb{P}^{0,T,0,y}_{Ber}$, $\mathbb{P}^{0,T,0,z}_{Ber}$ respectively. Let $M > 0$ and $\epsilon > 0$ be given. Then we can find $W_1 = W_1(M, p, \epsilon) \in \mathbb{N}$ and $A = A(M, p, \epsilon) > 0$ such that for $T \geq W_1$, $y \geq pT - MT^{1/2}$, $z \leq pT + MT^{1/2}$ we have*

$$\begin{aligned} &\mathbb{P}^{0,T,0,y}_{Ber}\left(\inf_{s\in[0,T]}\left[\ell_y(s) - ps\right] \leq -AT^{1/2}\right) \leq \epsilon, \\ &\mathbb{P}^{0,T,0,z}_{Ber}\left(\sup_{s\in[0,T]}\left[\ell_z(s) - ps\right] \geq AT^{1/2}\right) \leq \epsilon. \end{aligned} \tag{5.2.10}$$

Remark 5.2.11. Roughly, Lemma 5.2.10 states that if a Bernoulli bridge ℓ is started from $(0,0)$ and terminates at time T not significantly lower (resp. higher) than the straight line of slope p, then the event that ℓ goes significantly below (resp. above) the straight line of slope p is very unlikely.

Proof. The two inequalities are proven in essentially the same way. We begin with the first inequality. If $B = \lfloor pT - MT^{1/2} \rfloor$ then it follows from Lemma 5.2.1 that

$$\mathbb{P}^{0,T,0,y}_{Ber}\left(\inf_{s\in[0,T]}\left[\ell_y(s) - ps\right] \leq -AT^{1/2}\right) \leq \mathbb{P}^{0,T,0,B}_{Ber}\left(\inf_{s\in[0,T]}\left[\ell(s) - ps\right] \leq -AT^{1/2}\right), \tag{5.2.11}$$

where ℓ has law $\mathbb{P}^{0,T,0,B}_{Ber}$. By Theorem 5.2.3, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a random variable $\ell^{(T,B)}$ whose law under $\mathbb{P}$ is also $\mathbb{P}^{0,T,0,B}_{Ber}$, and a Brownian bridge B^{σ} with variance

$\sigma^2 = p(1-p)$. Therefore

$$\mathbb{P}_{Ber}^{0,T,0,B}\left(\inf_{s\in[0,T]}\left[\ell(s)-ps\right]\leq -AT^{1/2}\right) = \mathbb{P}\left(\inf_{s\in[0,T]}\left[\ell^{(T,B)}(s)-ps\right]\leq -AT^{1/2}\right)\leq$$
$$\mathbb{P}\left(\inf_{s\in[0,T]}\sqrt{T}B^{\sigma}_{s/T}\leq -\frac{1}{2}AT^{1/2}\right)+\mathbb{P}\left(\sup_{s\in[0,T]}\left|\sqrt{T}B^{\sigma}_{s/T}+ps-\ell^{(T,B)}(s)\right|\geq \frac{1}{2}AT^{1/2}\right)\leq \tag{5.2.12}$$
$$\mathbb{P}\left(\max_{s\in[0,T]}B^{\sigma}_{s/T}\geq A/2\right)+\mathbb{P}\left(\Delta(T,B)\geq \frac{1}{2}AT^{1/2}-MT^{1/2}-1\right).$$

For the first term in the last line, we used the fact that B^σ and $-B^\sigma$ have the same distribution. For the second term, we used the fact that

$$\sup_{s\in[0,T]}\left|ps-\frac{s}{T}\cdot B\right|\leq \sup_{s\in[0,T]}\left|ps-\frac{pT-MT^{1/2}}{T}\cdot s\right|+1 = MT^{1/2}+1.$$

By Lemma 5.2.6, the first term in the last line of (5.2.12) is equal to $e^{-A^2/2p(1-p)}$. If we choose $A\geq\sqrt{2p(1-p)\log(2/\epsilon)}$, then this is $\leq \epsilon/2$. If we also take $A>2M$, then since $|B-pT|\leq (M+1)\sqrt{T}$, Corollary 5.2.5 gives us a W_1 large enough depending on M,p,ϵ so that the second term in the last line of (5.2.12) is also $<\epsilon/2$ for $T\geq W_1$. Adding the two terms and using (5.2.11) gives the first inequality in (5.2.10).

If we replace B with $\lceil pT+MT^{1/2}\rceil$ and change signs and inequalities where appropriate, then the same argument proves the second inequality in (5.2.10). □

We need the following definition for our next result. For a function $f\in C([a,b])$ we define its *modulus of continuity* for $\delta>0$ by

$$w(f,\delta)=\sup_{\substack{x,y\in[a,b]\\|x-y|\leq\delta}}|f(x)-f(y)|. \tag{5.2.13}$$

Lemma 5.2.12. *Fix $p\in(0,1)$, $T\in\mathbb{N}$ and $y\in\mathbb{Z}$ such that $T\geq y\geq 0$, and suppose that ℓ has distribution $\mathbb{P}_{Ber}^{0,T,0,y}$. For each positive M, ϵ and η, there exist a $\delta(\epsilon,\eta,M)>0$ and $W_2=$*

$W_2(M,p,\epsilon,\eta) \in \mathbb{N}$ *such that for* $T \geq W_2$ *and* $|y - pT| \leq MT^{1/2}$ *we have*

$$\mathbb{P}^{0,T,0,y}_{Ber}\left(w\big(f^{\ell},\delta\big) \geq \epsilon\right) \leq \eta, \tag{5.2.14}$$

where $f^{\ell}(u) = T^{-1/2}\big(\ell(uT) - puT\big)$ *for* $u \in [0,1]$.

Remark 5.2.13. Lemma 5.2.12 states that if ℓ is a Bernoulli bridge that is started from $(0,0)$ and terminates at (T,y) with y close to pT (i.e. with well-behaved endpoints) then the modulus of continuity of ℓ is also well-behaved with high probability.

Proof. By Theorem 5.2.3, we have a probability measure $\mathbb{P}$ supporting a random variable $\ell^{(T,y)}$ with law $\mathbb{P}^{0,T,0,y}_{Ber}$ as well as a Brownian bridge B^{σ} with variance $\sigma^2 = p(1-p)$. We have

$$\mathbb{P}^{0,T,0,y}_{Ber}\Big(w\big(f^{\ell},\delta\big) \geq \epsilon\Big) = \mathbb{P}\Big(w\big(f^{\ell^{(T,y)}},\delta\big) \geq \epsilon\Big), \tag{5.2.15}$$

and

$$\begin{aligned} w\big(f^{\ell^{(T,y)}},\delta\big) = T^{-1/2} \sup_{s,t\in[0,1],\,|s-t|\leq\delta} \left|\ell^{(T,y)}(sT) - psT - \ell^{(T,y)}(tT) + ptT\right| \leq \\ T^{-1/2} \sup_{s,t\in[0,1],\,|s-t|\leq\delta} \Bigg(\left|\sqrt{T}\,B^{\sigma}_s + sy - psT - \sqrt{T}\,B^{\sigma}_t - ty + ptT\right| + \\ \left|\sqrt{T}\,B^{\sigma}_s + sy - \ell^{(T,y)}(sT)\right| + \left|\sqrt{T}\,B^{\sigma}_t + ty - \ell^{(T,y)}(tT)\right|\Bigg) \leq \\ \sup_{s,t\in[0,1],\,|s-t|\leq\delta} \left|B^{\sigma}_s - B^{\sigma}_t + T^{-1/2}(y - pT)(s-t)\right| + 2T^{-1/2}\Delta(T,y) \leq \\ w\big(B^{\sigma},\delta\big) + M\delta + 2T^{-1/2}\Delta(T,y). \end{aligned} \tag{5.2.16}$$

The last line follows from the assumption that $|y - pT| \leq MT^{1/2}$. Now (5.2.15) and (5.2.16) together imply that

$$\begin{aligned} \mathbb{P}^{0,T,0,y}_{Ber}\left(w\big(f^{\ell},\delta\big) \geq \epsilon\right) \leq \mathbb{P}\left(w\big(B^{\sigma},\delta\big) + M\delta + 2T^{-1/2}\Delta(T,y) \geq \epsilon\right) \leq \\ \mathbb{P}\left(w\big(B^{\sigma},\delta\big) + M\delta \geq \epsilon/2\right) + \mathbb{P}\left(\Delta(T,y) \geq \epsilon T^{1/2}/4\right). \end{aligned} \tag{5.2.17}$$

Corollary 5.2.5 gives us a W_2 large enough depending on M, p, ϵ, η so that the second term in the second line of 5.2.17 is $\leq \eta/2$ for $T \geq W_2$. Since B^σ is a.s. uniformly continuous on the compact interval $[0,1]$, $w(B^\sigma, \delta) \to 0$ as $\delta \to 0$. Thus we can find $\delta_0 > 0$ small enough depending on ϵ, η so that $w(B^\sigma, \delta_0) < \epsilon/4$ with probability at least $1 - \eta/2$. Then with $\delta = \min(\delta_0, \epsilon/4M)$, the first term in the second line of (5.2.17) is $\leq \eta/2$ as well. This implies (5.2.14). □

Lemma 5.2.14. *Fix $T \in \mathbb{N}$, $p \in (0,1)$, $C, K > 0$, and $a, b \in \mathbb{Z}$ such that $\Omega(0,T,a,b)$ is nonempty. Let $\ell_{bot} \in \Omega(0,T,a,b)$. Suppose $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$, $k \geq 2$, are such that $T \geq y_i - x_i \geq 0$ for $1 \leq i \leq k-1$. Write $\vec{z} = \vec{y} - \vec{x}$, and suppose that*

(1) $x_{k-1} + (z_{k-1}/T)s - \ell_{bot}(s) \geq C\sqrt{T}$ for all $s \in [0,T]$

(2) $x_i - x_{i+1} \geq C\sqrt{T}$ and $y_i - y_{i+1} \geq C\sqrt{T}$ for $1 \leq i \leq k-2$,

(3) $|z_i - pT| \leq K\sqrt{T}$ for $1 \leq i \leq k-1$, for a constant $K > 0$.

Let $\mathfrak{L} = (L_1, \dots, L_{k-1})$ be a line ensemble with law $\mathbb{P}_{Ber}^{0,T,\vec{x},\vec{y}}$, and let E denote the event

$$E = \{L_1(s) \geq \cdots \geq L_{k-1}(s) \geq \ell_{bot}(s) \text{ for } s \in [0,T]\}.$$

Then we can find $W_3 = W_3(p,C,K)$ so that for $T \geq W_3$,

$$\mathbb{P}_{Ber}^{0,T,\vec{x},\vec{y}}(E) \geq \left(\frac{1}{2} - \sum_{n=1}^{\infty} (-1)^{n-1} e^{-n^2C^2/8p(1-p)}\right)^{k-1}. \tag{5.2.18}$$

Moreover if $C \geq \sqrt{8p(1-p)\log 3}$, then for $T \geq W_3$ we have

$$\mathbb{P}_{Ber}^{0,T,\vec{x},\vec{y}}(E) \geq \left(1 - 3e^{-C^2/8p(1-p)}\right)^{k-1}. \tag{5.2.19}$$

Remark 5.2.15. This lemma states that if k independent Bernoulli bridges are well-separated from each other and ℓ_{bot}, then there is a positive probability that the curves will intersect neither each other nor ℓ_{bot}. We will use this result to compare curves in an avoiding Bernoulli line ensemble with free Bernoulli bridges.

Proof. Observe that condition (1) simply states that ℓ_{bot} lies a distance of at least $C\sqrt{T}$ uniformly below the line segment connecting x_{k-1} and y_{k-1}. Thus (1) and (2) imply that E occurs if each curve L_i remains within a distance of $C\sqrt{T}/2$ from the line segment connecting x_i and y_i. As in Theorem 5.2.3, let $\mathbb{P}_i$ be probability measures supporting random variables $\ell^{(T,z_i)}$ with laws $\mathbb{P}^{0,T,0,z_i}_{Ber}$. Then

$$\begin{gathered}\mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}(E) \geq \mathbb{P}^{0,T,\vec{x},\vec{y}}_{Ber}\left(\sup_{s\in[0,T]}\left|L_i(s) - x_i - (z_i/T)s\right| \leq C\sqrt{T}/2,\ 1 \leq i \leq k-1\right) = \\ \prod_{i=1}^{k-1}\left[\mathbb{P}^{0,T,0,z_i}_{Ber}\left(\sup_{s\in[0,T]}\left|L_i(s+rN^{\alpha}) - (z_i/T)s\right| \leq C\sqrt{T}/2\right)\right] = \\ \prod_{i=1}^{k-1}\left[1 - \mathbb{P}_i\left(\sup_{s\in[0,T]}\left|\ell^{(T,z_i)} - (z_i/T)s\right| > C\sqrt{T}/2\right)\right].\end{gathered} \tag{5.2.20}$$

In the third line, we used the fact that $L_1, \dots, L_{k-1}$ are independent from each other under $\mathbb{P}^{0,T,0,z_i}_{Ber}$. Let $B^{\sigma,i}$ be the Brownian bridge with variance $\sigma^2 = p(1-p)$ coupled with $\ell^{(T,z_i)}$ given by Theorem 5.2.3. Then we have

$$\begin{gathered}\mathbb{P}_i\left(\sup_{s\in[0,T]}\left|\ell^{(T,z_i)}(s) - (z_i/T)s\right| > C\sqrt{T}/2\right) \leq \\ \mathbb{P}_i\left(\sup_{s\in[0,T]}|\sqrt{T}B^{\sigma}_{s/T}| > C\sqrt{T}/4\right) + \mathbb{P}_i\left(\Delta(T,z_i) > C\sqrt{T}/4\right).\end{gathered} \tag{5.2.21}$$

By Lemma 5.2.6, the first term in the second line of (5.2.21) is equal to $2\sum_{n=1}^{\infty}(-1)^{n-1}e^{-n^2C^2/8p(1-p)}$. Moreover, condition (3) in the hypothesis and Corollary 5.2.5 allow us to find W_3 depending on p, C, K but not on i so that the last probability in (5.2.21) is bounded above by $\frac{1}{2}-\sum_{n=1}^{\infty}(-1)^{n-1}e^{-n^2C^2/8p(1-p)}$ for $T \geq W_3$. Adding these two terms and referring to (5.2.20) proves (5.2.18).

Now suppose $C \geq \sqrt{8p(1-p)\log 3}$. By (5.2.4) in Lemma 5.2.6, the first term in the second line of (5.2.21) is bounded above by bounded above by $2e^{-C^2/8p(1-p)}$. After possibly enlarging W_3 from above, the second term is $< e^{-C^2/8p(1-p)}$ for $T \geq W_3$. The assumption on C implies that $1 - 3e^{-C^2/8p(1-p)} \geq 0$, and now combining (5.2.21) and (5.2.20) proves (5.2.19). □

5.2.3 Properties of avoiding Bernoulli line ensembles

In this section we derive two results about avoiding Bernoulli line ensembles, which are Bernoulli line ensembles with law $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},f,g}_{avoid,Ber;S}$ as in Definition 5.1.15. The lemmas we prove only involve the case when $f(r) = \infty$ for all $r \in [\![T_0, T_1]\!]$ and we denote the measure in this case by $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber;S}$. A $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g}_{avoid,Ber;S}$-distributed random variable will be denoted by $\mathfrak{Q} = (Q_1, \dots, Q_k)$ where k is the number of up-right paths in the ensemble. As usual, if $g = -\infty$, we write $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y}}_{avoid,Ber;S}$. Our first result will rely on the two monotonicity Lemmas 5.2.1 and 5.2.2 as well as the strong coupling between Bernoulli bridges and Brownian bridges from Theorem 5.2.3, and the further results make use of the material in Section 5.7.

Lemma 5.2.16. *Fix $p \in (0, 1)$, $k \in \mathbb{N}$. Let $\vec{x}, \vec{y} \in \mathfrak{W}_k$ be such that $T \geq y_i - x_i \geq 0$ for $i = 1, \dots, k$. Then for any $M, M_1 > 0$ we can find $W_4 \in \mathbb{N}$ depending on p, k, M, M_1 such that if $T \geq W_4$, $x_k \geq -M_1\sqrt{T}$, and $y_k \geq pT - M_1\sqrt{T}$, then for any $S \subseteq [\![0, T]\!]$ we have*

$$\mathbb{P}^{0,T,\vec{x},\vec{y}}_{avoid,Ber;S}\left(Q_k(T/2) - pT/2 \geq M\sqrt{T}\right) \geq \frac{2^{k/2}\left(1 - 2e^{-4/p(1-p)}\right)^{2k}}{(\pi p(1-p))^{k/2}} \exp\left(-\frac{2k(M + M_1 + 6)^2}{p(1-p)}\right). \tag{5.2.22}$$

Proof. Define vectors $\vec{x}, \vec{y} \in \mathfrak{W}_k$ by

$$x_i' = \lfloor -M_1\sqrt{T} \rfloor - 10(i-1)\lceil\sqrt{T}\rceil,$$
$$y_i' = \lfloor pT - M_1\sqrt{T} \rfloor - 10(i-1)\lceil\sqrt{T}\rceil.$$

Then $x_i' \leq x_k \leq x_i$ and $y_i' \leq y_k \leq y_i$ for $1 \leq i \leq k-1$. Thus by Lemma 5.2.1, we have

$$\mathbb{P}^{0,T,\vec{x},\vec{y}}_{avoid,Ber;S}\Big(Q_k(T/2) - pT/2 \geq M\sqrt{T}\Big) \geq \mathbb{P}^{0,T,\vec{x}\,',\vec{y}\,'}_{avoid,Ber;S}\Big(Q_k(T/2) - pT/2 \geq M\sqrt{T}\Big).$$

Let us write $K_i = pT/2 + M\sqrt{T} + (10(k-i) - 5)\lceil\sqrt{T}\rceil$ for $1 \leq i \leq k$. Note K_i is the midpoint of $pT/2 + M\sqrt{T} + 10(k-i-1)\lceil\sqrt{T}\rceil$ and $pT/2 + M\sqrt{T} + 10(k-i)\lceil\sqrt{T}\rceil$. Let E denote the event that the following conditions hold for $1 \leq i \leq k$:

(1) $\left|Q_i(T/2) - pT/2 - M\sqrt{T} - (10(k-i)-5)\lceil\sqrt{T}\rceil\right| \leq 2\lceil\sqrt{T}\rceil$,

(2) $\sup_{s\in[0,T/2]}\left|Q_i(s) - x_i' - \frac{K_i - x_i'}{T/2}\, s\right| \leq 3\sqrt{T}$,

(3) $\sup_{s\in[T/2,T]}\left|Q_i(s) - K_i - \frac{y_i' - K_i}{T/2}(s - T/2)\right| \leq 3\sqrt{T}$.

The first condition implies in particular that $Q_k(T/2) - pT/2 \geq M\sqrt{T}$, and also that $Q_i(T/2) - Q_{i+1}(T/2) \geq 6\sqrt{T}$ for each i. The second and third conditions require that each curve Q_i remain within a distance of $3\sqrt{T}$ of the graph of the piecewise linear function on $[0,T]$ passing through the points $(0,x_1')$, $(T/2,K_i)$, and (T,y_i'). We observe that

$$\mathbb{P}^{0,T,\vec{x}\,',\vec{y}\,'}_{avoid,Ber;S}\left(Q_k(T/2) - ptT \geq M\sqrt{T}\right) \geq \mathbb{P}^{0,T,\vec{x}\,',\vec{y}\,'}_{avoid,Ber;S}(E) \geq \mathbb{P}^{0,T,\vec{x}\,',\vec{y}\,'}_{Ber}(E).$$

The second inequality follows since the event E implies that $Q_1(s) \geq \cdots \geq Q_k(s)$ for all $s \in [\![0,T]\!]$. Write $z = y_k' - x_k'$. Then we have

$$\begin{aligned}\mathbb{P}^{0,T,\vec{x}\,',\vec{y}\,'}_{Ber}(E) = \Bigg[\mathbb{P}^{0,T,0,z}_{Ber}\Bigg(&\left|\ell(T/2) - pT/2 - M\sqrt{T} - 5\lceil\sqrt{T}\rceil + x_1'\right| \leq 2\lceil\sqrt{T}\rceil \quad \text{and} \\ &\sup_{s\in[0,T/2]}\left|\ell(s) - \frac{K_1 - x_1'}{T/2}\, s\right| \leq 3\sqrt{T} \quad \text{and} \\ &\sup_{s\in[T/2,T]}\left|\ell(s) - (K_1 - x_1') - \frac{y_1' - K_1}{T/2}(s - T/2)\right| \leq 3\sqrt{T}\Bigg)\Bigg]^k.\end{aligned} \tag{5.2.23}$$

Let $\mathbb{P}$ be a probability space supporting a random variable $\ell^{(T,z)}$ with law $\mathbb{P}^{0,T,0,z}$ coupled with a Brownian bridge B^σ with variance σ^2, as in Theorem 5.2.3. Then the expression in (5.2.23) is

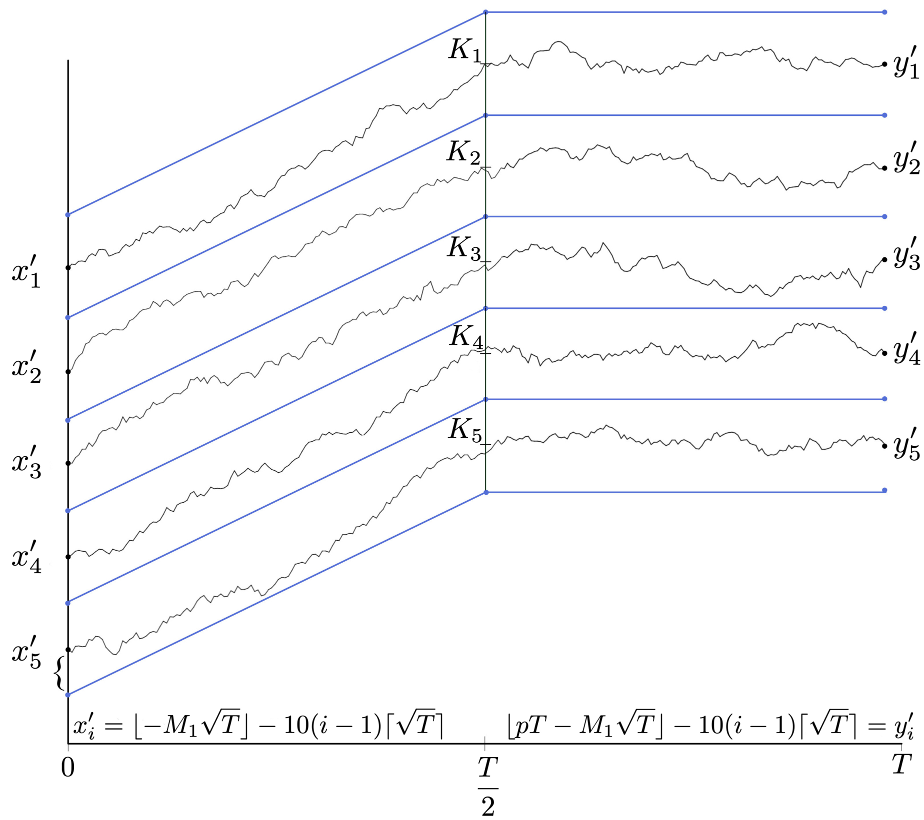


Figure 5.3: Sketch of the argument for Lemma 5.2.16: We use Lemma 5.2.1 to lower the entry and exit data $\vec{x}, \vec{y}$ of the curves to $\vec{x}\,'$ and $\vec{y}\,'$. The event E occurs when each curve lies within the blue bounding lines shown in the figure. We then use strong coupling with Brownian bridges via Theorem 5.2.3 and bound the probability of the bridges remaining within the blue windows.

bounded below by

$$
\begin{aligned}
&\mathbb{P}^{0,T,0,z}_{Ber}\Bigg(\Big|\ell(T/2) - pT/2 - (M+M_1+5)\sqrt{T}\Big| \leq 2\sqrt{T} - 5 \quad \text{and} \\
&\qquad \sup_{s\in[0,T/2]} \left|\ell(s) - ps - \frac{M+M_1+5}{\sqrt{T}/2}\, s\right| \leq 3\sqrt{T} - 1 \quad \text{and} \\
&\qquad \sup_{s\in[T/2,T]} \left|\ell(s) - ps - (M+M_1+5)\sqrt{T} + \frac{M+M_1+5}{\sqrt{T}/2}(s - T/2)\right| \leq 3\sqrt{T} - 1 \Bigg) \geq \\
&\mathbb{P}\Bigg(\Big|\sqrt{T}\, B^{\sigma}_{1/2} - (M+M_1+5)\sqrt{T}\Big| \leq \sqrt{T} \quad \text{and} \\
&\qquad \sup_{s\in[0,T/2]} \left|\sqrt{T}\, B^{\sigma}_{s/T} - (M+M_1+5)\sqrt{T}\cdot\frac{s}{T/2}\right| \leq 2\sqrt{T} \quad \text{and} \\
&\qquad \sup_{s\in[T/2,T]} \left|\sqrt{T}\, B^{\sigma}_{s/T} - (M+M_1+5)\sqrt{T}\cdot\frac{T-s}{T/2}\right| \leq 2\sqrt{T} \Bigg) - \mathbb{P}\left(\Delta(T,z) > \sqrt{T}/2\right).
\end{aligned}
\tag{5.2.24}
$$

Note that $B^{\sigma}_{1/2}$ is a centered Gaussian random variable with variance $p(1-p)/4 = \sigma^2(1/2)(1-1/2)$. Writing $\xi = B^{\sigma}_{1/2}$, it follows from Lemma 5.2.7 that there exist independent Brownian bridges B^1, B^2 with variance $\sigma^2/2$ so that B^{σ}_s has the same law as $\frac{s}{T/2}\xi + B^1_{2s/T}$ for $s \in [0, T/2]$ and $\frac{T-s}{T/2}\xi + B^2_{(2s-T)/T}$ for $s \in [T/2, T]$. The first term in the last expression in (5.2.24) is thus equal to

$$
\begin{aligned}
&\mathbb{P}\Bigg(|\xi - (M+M_1+5)| \leq 1 \quad \text{and} \sup_{s\in[0,T/2]} \left|B^1_{s/T} - (M+M_1+5-\xi)\cdot\frac{s}{T/2}\right| \leq 2 \quad \text{and} \\
&\qquad \sup_{s\in[T/2,T]} \left|B^2_{(2s-T)/T} - (M+M_1+5-\xi)\cdot\frac{T-s}{T/2}\right| \leq 2 \Bigg) \geq \\
&\mathbb{P}\Bigg(|\xi - (M+M_1+5)| \leq 1 \quad \text{and} \sup_{s\in[0,T/2]} \left|B^1_{2s/T}\right| \leq 1 \qquad \text{and} \qquad \sup_{s\in[T/2,T]} \left|B^2_{(2s-T)/T}\right| \leq 1 \Bigg) = \\
&\mathbb{P}\Big(|\xi - (M+M_1+5)| \leq 1 \Big) \cdot \mathbb{P}\Bigg(\sup_{s\in[0,T/2]} \left|B^1_{2s/T}\right| \leq 1 \Bigg) \cdot \mathbb{P}\Bigg(\sup_{s\in[0,T/2]} \left|B^2_{(2s-T)/T}\right| \leq 1 \Bigg) \geq \\
&\left(1 - 2e^{-4/p(1-p)}\right)^2 \int_{M+M_1+4}^{M+M_1+6} \frac{e^{-2\xi^2/p(1-p)}}{\sqrt{\pi p(1-p)/2}}\, d\xi \geq \\
&\frac{2\sqrt{2}\, e^{-2(M+M_1+6)^2/p(1-p)}}{\sqrt{\pi p(1-p)}} \left(1 - 2e^{-4/p(1-p)}\right)^2.
\end{aligned}
\tag{5.2.25}
$$

In the fourth line, we used the fact that ξ, $B^1_\cdot$, and $B^2_\cdot$ are independent, and in the second to last line we used Lemma 5.2.6. Since $|z - pT| \leq (M_1 + 1)\sqrt{T}$, Lemma 5.2.5 allows us to choose T large enough so that $\mathbb{P}(\Delta(T, z) > \sqrt{T}/2)$ is less than 1/2 the expression in the last line of (5.2.25). Then in view of (5.2.23) and (5.2.24), we conclude (5.2.22).

□

We now state an important weak convergence result, whose proof occupies Section 5.7. (See Propositions 5.7.2 and 5.7.3.)

Proposition 5.2.17. *Fix $p, t \in (0,1)$, $k \in \mathbb{N}$, $\vec{a}, \vec{b} \in \mathfrak{W}_k$. Suppose that $\vec{x}^T = (x_1^T, \dots, x_k^T)$ and $\vec{y}^T = (y_1^T, \dots, y_k^T)$ are two sequences in $\mathfrak{W}_k$ such that for $i \in [\![1, k]\!]$,*

$$\lim_{T\to\infty} \frac{x_i^T}{\sqrt{T}} = a_i \quad \textit{and} \quad \lim_{T\to\infty} \frac{y_i^T - pT}{\sqrt{T}} = b_i.$$

Let $(Q_1^T, \dots, Q_k^T)$ have law $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$, and define the sequence $\{Z^T\}$ of random k-dimensional vectors by

$$Z^T = \left(\frac{Q_1^T(tT) - ptT}{\sqrt{T}}, \dots, \frac{Q_k^T(tT) - ptT}{\sqrt{T}}\right).$$

Then as $T \to \infty$, Z^T converges weakly to a random vector $\widehat{Z}$ on $\mathbb{R}^k$ with a probability density ρ supported on W_k°.

The convergence result in Lemma 5.2.17 allows us to prove the following lemma, which roughly states that if the entrance and exit data of a sequence of avoiding Bernoulli line ensembles remain in compact windows, then with high probability the curves of the ensemble will remain separated from one another at each point by some small positive distance on scale $\sqrt{T}$.

Lemma 5.2.18. *Fix $p, t \in (0,1)$ and $k \in \mathbb{N}$. Suppose that $\vec{x}^{\,T} = (x_1^T, \dots, x_k^T)$, $\vec{y}^{\,T} = (y_1^T, \dots, y_k^T)$ are elements of $\mathfrak{W}_k$ such that $T \geq y_i^T - x_i^T \geq 0$ for $i \in [\![1, k]\!]$. Then for any $M_1, M_2 > 0$ and $\epsilon > 0$ there exists $W_5 \in \mathbb{N}$ and $\delta > 0$ depending on p, k, M_1, M_2 such that if $T \geq W_5$, $|x_i^T| \leq M_1\sqrt{T}$ and*

$|y_i^T - pT| \leq M_2\sqrt{T}$, *then*

$$ {}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}\left(\min_{1\leq i\leq k-1} \left[Q_i(tT) - Q_{i+1}(tT) \right] < \delta\sqrt{T} \right) < \epsilon. $$

Proof. We prove the claim by contradiction. Suppose there exist $M_1, M_2, \epsilon > 0$ such that for any $W_5 \in \mathbb{N}$ and $\delta > 0$ there exists some $T \geq W_5$ with

$$ {}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}\left(\min_{1\leq i\leq k-1} \left[Q_i(tT) - Q_{i+1}(tT) \right] < \delta\sqrt{T} \right) \geq \epsilon. $$

Then we can obtain sequences T_n, $\delta_n > 0$, $T_n \nearrow \infty$, $\delta_n \searrow 0$, such that for all n we have

$$ {}^{0,T,\vec{x}^{T_n},\vec{y}^{T_n}}_{avoid,Ber}\left(\min_{1\leq i\leq k-1} \left[\frac{Q_i(tT_n) - Q_{i+1}(tT_n)}{\sqrt{T_n}} \right] < \delta_n \right) \geq \epsilon. $$

Since $|x_i^{T_n}| < M_1\sqrt{T_n}$ and $|y_i^{T_n} - pT_n| \leq M_2\sqrt{T_n}$ for $1 \leq i \leq k$, the sequences $\{\vec{x}^{T_n}/\sqrt{T_n}\}$, $\{(\vec{y}^{T_n} - pT_n)/\sqrt{T_n}\}$ are bounded in $\mathbb{R}^k$. It follows that there exist $\vec{x}, \vec{y} \in \mathbb{R}^n$ and a subsequence $\{T_{n_m}\}$ such that

$$ \frac{\vec{x}^{T_{n_m}}}{\sqrt{T_{n_m}}} \longrightarrow \vec{x}, \quad \frac{\vec{y}^{T_{n_m}} - pT_{n_m}}{\sqrt{T_{n_m}}} \longrightarrow \vec{y} $$

as $m \to \infty$ (see [213, Theorem 3.6]). Denote

$$ Z_i^m = \frac{Q_i(tT_{n_m}) - ptT_{n_m}}{\sqrt{T_{n_m}}}. $$

Fix $\delta > 0$ and choose M large enough so that if $m \geq M$ then $\delta_m < \delta$. Then for $m \geq M$ we have

$$ \epsilon \leq \liminf_{m\to\infty} \left(\min_{1\leq i\leq k-1} \left[Z_i^m - Z_{i+1}^m \right] < \delta_{n_m} \right) \leq \liminf_{m\to\infty} \left(\min_{1\leq i\leq k-1} \left[Z_i^m - Z_{i+1}^m \right] \leq \delta \right). \tag{5.2.26} $$

Now by Lemma 5.2.17, $(Z_1^m, \ldots, Z_k^m)$ converges weakly to a random vector $\widehat{Z}$ on $\mathbb{R}^k$ with a density ρ. It follows from the portmanteau theorem [117, Theorem 3.2.11] applied with the closed set

$K = [0, \delta]$ that

$$\limsup_{m\to\infty} \left(\min_{1\le i\le k-1} \left[Z_i^m - Z_{i+1}^m \right] \in K \right) \le \left(\min_{1\le i\le k-1} \left[\widehat{Z}_i - \widehat{Z}_{i+1} \right] \in K \right). \tag{5.2.27}$$

Combining (5.2.26) and (5.2.27), we obtain

$$\epsilon \le \left(0 \le \min_{1\le i\le k-1} \left[\widehat{Z}_i - \widehat{Z}_{i+1} \right] \le \delta \right) \le \sum_{i=1}^{k-1} \left(0 \le \widehat{Z}_i - \widehat{Z}_{i+1} \le \delta \right). \tag{5.2.28}$$

To conclude the proof, we find a δ for which (6.3.3) cannot hold. For $\tilde{\eta} \ge 0$ put

$$E_i^{\tilde{\eta}} = \{ \vec{z} \in \mathbb{R}^k : 0 \le z_i - z_{i+1} \le \tilde{\eta} \}.$$

For each $i \in [\![1, k-1]\!]$ and $\eta > 0$, we have

$$\mathbb{P}\left(0 \le \widehat{Z}_i - \widehat{Z}_{i+1} \le \eta \right) = \int_{\mathbb{R}^k} \rho \cdot \mathbf{1}_{E_i^\eta} \, dz_1 \cdots dz_k. \tag{5.2.29}$$

Clearly $\rho \cdot \mathbf{1}_{E_i^\eta} \to \rho \cdot \mathbf{1}_{E_i^0}$ pointwise as $\eta \to 0$, and $E_i^0 = \{ \vec{z} \in \mathbb{R}^k : z_i = z_{i+1} \}$ has Lebesgue measure 0. Thus $\rho \cdot \mathbf{1}_{E_i^\eta} \to 0$ a.e. as $\eta \to 0$. Since $|\rho \cdot \mathbf{1}_{E_i^\eta}| \le \rho$ and ρ is integrable, the dominated convergence theorem [214, Theorem 1.34] and (5.2.29) imply that

$$\mathbb{P}\left(0 \le \widehat{Z}_i - \widehat{Z}_{i+1} \le \eta \right) \longrightarrow 0$$

as $\eta \to 0$. Thus for each $i \in [\![1, k-1]\!]$ and $\epsilon > 0$ we can find an $\eta_i > 0$ such that $0 < \eta \le \eta_i$ implies $\mathbb{P}(0 \le \widehat{Z}_i - \widehat{Z}_{i+1} \le \eta) < \epsilon/(k-1)$. Putting $\delta = \min_{1\le i\le k-1} \eta_i$ we find that

$$\sum_{i=1}^{k-1} \left(0 \le \widehat{Z}_i - \widehat{Z}_{i+1} \le \delta \right) < \epsilon,$$

contradicting (6.3.3) for this choice of δ.

□

5.3 Proof of Theorem 5.1.26

The goal of this section is to prove Theorem 5.1.26. Throughout this section, we assume that we have fixed $k \in \mathbb{N}$ with $k \geq 2$, $p \in (0,1)$, $\alpha, \lambda > 0$, and

$$\left\{\mathfrak{L}^N = (L_1^N, L_2^N, \ldots, L_k^N)\right\}_{N=1}^{\infty}$$

an (α, p, λ)-good sequence of $[\![1, k]\!]$-indexed Bernoulli line ensembles as in Definition 5.1.24, all defined on a probability space with measure $\mathbb{P}$. The proof of Theorem 5.1.26 depends on three results – Proposition 5.3.1 and Lemmas 5.3.2 and 5.3.3. In these three statements we establish various properties of the sequence of line ensembles $\mathfrak{L}^N$. The constants in these statements depend implicitly on α, p, λ, k, and the functions ϕ, ψ from Definition 5.1.24, which are fixed throughout. We will not list these dependencies explicitly. The proof of Proposition 5.3.1 is given in Section 5.3.1 while the proofs of Lemmas 5.3.2 and 5.3.3 are in Section 5.4. Theorem 5.1.26 (i) and (ii) are proved in Sections 5.3.2 and 5.3.3 respectively.

5.3.1 Bounds on the acceptance probability

The main result in this section is presented as Proposition 5.3.1 below. In order to formulate it and some of the lemmas below, it will be convenient to adopt the following notation for any $r > 0$ and $m \in \mathbb{N}$:

$$t_m = \lfloor (r + m)N^{\alpha} \rfloor. \tag{5.3.1}$$

Proposition 5.3.1. *Let $\mathbb{P}$ be the measure from the beginning of this section. For any $\epsilon > 0$, $r > 0$ there exist $\delta = \delta(\epsilon, r) > 0$ and $N_1 = N_1(\epsilon, r)$ such that for all $N \geq N_1$ we have*

$$\mathbb{P}\Big(Z\big(-t_1, t_1, \vec{x}, \vec{y}, \infty, L_k^N[\![-t_1, t_1]\!]\big) < \delta\Big) < \epsilon,$$

where $\vec{x} = (L_1^N(-t_1), \ldots, L_{k-1}^N(-t_1))$, $\vec{y} = (L_1^N(t_1), \ldots, L_{k-1}^N(t_1))$, $L_k^N[\![-t_1, t_1]\!]$ is the restriction of L_k^N to the set $[\![-t_1, t_1]\!]$, and Z is the acceptance probability of Definition 5.1.22.

The general strategy we use to prove Proposition 5.3.1 is inspired by the proof of [74, Proposition 6.5]. We begin by stating three key lemmas that will be required. The proofs of Lemmas 5.3.2 and 5.3.3 are postponed to Section 5.4 and Lemma 5.3.4 is proved in Section 5.5.

Lemma 5.3.2. *Let $\mathbb{P}$ be the measure from the beginning of this section. For any $\epsilon > 0$, $r > 0$ there exist $R_1 = R_1(\epsilon, r) > 0$ and $N_2 = N_2(\epsilon, r)$ such that for $N \geq N_2$*

$$\mathbb{P}\left(\sup_{s\in[-t_3,t_3]} \left[L_1^N(s) - ps\right] \geq R_1 N^{\alpha/2}\right) < \epsilon.$$

Lemma 5.3.3. *Let $\mathbb{P}$ be the measure from the beginning of this section. For any $\epsilon > 0$, $r > 0$ there exist $R_2 = R_2(\epsilon, r) > 0$ and $N_3 = N_3(\epsilon, r)$ such that for $N \geq N_3$*

$$\mathbb{P}\left(\inf_{s\in[-t_3,t_3]} \left[L_{k-1}^N(s) - ps\right] \leq -R_2 N^{\alpha/2}\right) < \epsilon.$$

Lemma 5.3.4. *Fix $k \in \mathbb{N}$, $k \geq 2$, $p \in (0,1)$, $r, \alpha, M_1, M_2 > 0$. Suppose that $\ell_{bot} : [\![-t_3, t_3]\!] \to \mathbb{R} \cup \{-\infty\}$, and $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$ are such that $|\Omega_{avoid}(-t_3, t_3, \vec{x}, \vec{y}, \infty, \ell_{bot})| \geq 1$. Suppose further that*

1. $\sup_{s\in[-t_3,t_3]} \left[\ell_{bot}(s) - ps\right] \leq M_2(2t_3)^{1/2}$,
2. $-pt_3 + M_1(2t_3)^{1/2} \geq x_1 \geq x_{k-1} \geq \max\left(\ell_{bot}(-t_3), -pt_3 - M_1(2t_3)^{1/2}\right)$,
3. $pt_3 + M_1(2t_3)^{1/2} \geq y_1 \geq y_{k-1} \geq \max\left(\ell_{bot}(t_3), pt_3 - M_1(2t_3)^{1/2}\right)$.

Then there exist constants g, h and $N_4 \in \mathbb{N}$ all depending on $M_1, M_2, p, k, r, \alpha$ such that for any $\tilde{\epsilon} > 0$ and $N \geq N_4$ we have

$$\mathbb{P}_{avoid,Ber}^{-t_3,t_3,\vec{x},\vec{y},\infty,\ell_{bot}}\Big(Z\big(-t_1, t_1, \mathfrak{Q}(-t_1), \mathfrak{Q}(t_1), \infty, \ell_{bot}[\![-t_1, t_1]\!]\big) \leq gh\tilde{\epsilon}\Big) \leq \tilde{\epsilon}, \tag{5.3.2}$$

where Z is the acceptance probability of Definition 5.1.22, $\ell_{bot}[\![-t_1, t_1]\!]$ is the vector, whose coordinates match those of ℓ_{bot} on $[\![-t_1, t_1]\!]$ and $\mathfrak{Q}(a) = (Q_1(a), \ldots, Q_{k-1}(a))$ is the value of the line ensemble $\mathfrak{Q} = (Q_1, \ldots, Q_{k-1})$ whose law is $\mathbb{P}_{avoid,Ber}^{-t_3,t_3,\vec{x},\vec{y},\infty,\ell_{bot}}$ at location a.

Proof of Proposition 5.3.1. Let $\epsilon > 0$ be given. Define the event

$$E_N = \left\{ L_{k-1}^N(\pm t_3) \mp pt_3 \geq -M_1(2t_3)^{1/2} \right\} \cap \left\{ L_1^N(\pm t_3) \mp pt_3 \leq M_1(2t_3)^{1/2} \right\} \cap \left\{ \sup_{s \in [-t_3, t_3]} [L_k^N(s) - ps] \leq M_2(2t_3)^{1/2} \right\}.$$

In view of Lemmas 5.3.2 and 5.3.3 and the fact that $\mathbb{P}$-almost surely $L_1^N(s) \geq L_k^N(s)$ for all $s \in [-t_3, t_3]$ we can find sufficiently large M_1, M_2 and N_2 such that for $N \geq N_2$ we have

$$\mathbb{P}(E_N^c) < \epsilon/2. \tag{5.3.3}$$

Let g, h, N_4 be as in Lemma 5.3.4 for the values M_1, M_2 as above, the values α, p, k from the beginning of the section and r as in the statement of the proposition. For $\delta = (\epsilon/2) \cdot gh$ we denote

$$V = \left\{ Z\big(-t_1, t_1, \vec{x}, \vec{y}, \infty, L_k^N[\![-t_1, t_1]\!]\big) < \delta \right\}$$

and make the following deduction for $N \geq N_4$

$$\begin{aligned} &\mathbb{P}\big(V \cap E_N\big) = \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{E_N} \cdot \mathbf{1}_V \middle| \sigma\big(\mathfrak{L}^N(-t_3), \mathfrak{L}^N(t_3), L_k^N[\![-t_3, t_3]\!]\big) \right] \right] = \\ &\mathbb{E}\left[\mathbf{1}_{E_N} \cdot \mathbb{E}\left[\mathbf{1}\{Z\big(-t_1, t_1, \vec{x}, \vec{y}, \infty, L_k^N[\![-t_1, t_1]\!]\big) < \delta\} \middle| \sigma\big(\mathfrak{L}^N(-t_3), \mathfrak{L}^N(t_3), L_k^N[\![-t_3, t_3]\!]\big) \right] \right] = \\ &\mathbb{E}\left[\mathbf{1}_{E_N} \cdot \mathbb{E}_{avoid}\left[\mathbf{1}\{Z\big(-t_1, t_1, \mathfrak{L}(-t_1), \mathfrak{L}(t_1), \infty, L_k^N[\![-t_1, t_1]\!]\big) < \delta\} \right] \right] \leq \mathbb{E}\left[\mathbf{1}_{E_N} \cdot \epsilon/2 \right] \leq \epsilon/2. \end{aligned} \tag{5.3.4}$$

In (5.3.4) we have written $\mathbb{E}_{avoid}$ in place of $\mathbb{E}_{avoid,Ber}^{-t_3,t_3,\mathfrak{L}^N(-t_3),\mathfrak{L}^N(t_3),\infty,L_k^N[\![-t_3,t_3]\!]}$ to ease the notation; in addition, we have that $\mathfrak{L}^N(a) = (L_1^N(a), \dots, L_{k-1}^N(a))$ and $\mathfrak{L}$ on the last line is distributed according to $\mathbb{P}_{avoid,Ber}^{-t_3,t_3,\mathfrak{L}^N(-t_3),\mathfrak{L}^N(t_3),\infty,L_k^N[\![-t_3,t_3]\!]}$. We elaborate on (5.3.4) in the paragraph below.

The first equality in (5.3.4) follows from the tower property for conditional expectations. The second equality uses the definition of V and the fact that $\mathbf{1}_{E_N}$ is $\sigma\big(\mathfrak{L}^N(-t_3), \mathfrak{L}^N(t_3), L_k^N[\![-t_3, t_3]\!]\big)$-measurable and can thus be taken outside of the conditional expectation. The third equality uses the

Schur Gibbs property, see Definition 5.1.17. The first inequality on the third line holds if $N \geq N_4$ and uses Lemma 5.3.4 with $\tilde{\epsilon} = \epsilon/2$ as well as the fact that on the event E_N the random variables $\mathfrak{L}^N(-t_3)$, $\mathfrak{L}^N(t_3)$ and $L_k^N[\![-t_3, t_3]\!]$ (that play the roles of $\vec{x}, \vec{y}$ and ℓ_{bot}) satisfy the inequalities

1. $\sup_{s\in[-t_3,t_3]} \left[L_k^N(s) - ps\right] \leq M_2(2t_3)^{1/2}$,
2. $-pt_3 + M_1(2t_3)^{1/2} \geq L_1^N(-t_3) \geq L_{k-1}^N(-t_3) \geq \max\left(L_k^N(-t_3), -pt_3 - M_1(2t_3)^{1/2}\right)$,
3. $pt_3 + M_1(2t_3)^{1/2} \geq L_1^N(t_3) \geq L_{k-1}^N(t_3) \geq \max\left(L_k^N(t_3), pt_3 - M_1(2t_3)^{1/2}\right)$.

The last inequality in (5.3.4) is trivial.

Combining (5.3.4) with (5.3.3), we see that for all $N \geq \max(N_2, N_4)$ we have

$$\mathbb{P}(V) = \mathbb{P}(V \cap E_N) + \mathbb{P}(V \cap E_N^c) \leq \epsilon/2 + \mathbb{P}(E_N^c) < \epsilon,$$

which proves the proposition. □

5.3.2 Proof of Theorem 5.1.26 (i)

By Lemma 5.1.4, it suffices to verify the following two conditions for all $i \in [\![1, k-1]\!]$, $r > 0$, and $\epsilon > 0$:

$$\lim_{a\to\infty} \limsup_{N\to\infty} (|f_i^N(0)| \geq a) = 0 \tag{5.3.5}$$

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} |f_i^N(x) - f_i^N(y)| \geq \epsilon \right) = 0. \tag{5.3.6}$$

For the sake of clarity, we will prove these conditions in several steps.

Step 1. In this step we prove (5.3.5). Let $\epsilon > 0$ be given. Then by Lemmas 5.3.2 and 5.3.3 we can find N_2, N_3 and R_1, R_2 such that for $N \geq \max(N_1, N_2)$

$$\left(\sup_{s\in[-t_3,t_3]} [L_1^N(s) - ps] \geq R_1 N^{\alpha/2} \right) < \epsilon/2,$$

$$\left(\inf_{s \in [-t_3, t_3]} [L_{k-1}^N(s) - ps] \le -R_2 N^{\alpha/2} \right) < \epsilon/2.$$

In particular, if we set $R = \max(R_1, R_2)$ and utilize the fact that $L_1^N(0) \ge \cdots \ge L_{k-1}^N(0)$ we conclude that for any $i \in [\![1, k-1]\!]$ we have

$$\left(|L_i^N(0)| \ge RN^{\alpha/2}\right) \le \left(L_1^N(0) \ge R_1 N^{\alpha/2}\right) + \left(L_{k-1}^N(0) \le -R_2 N^{\alpha/2}\right) < \epsilon,$$

which implies (5.3.5).

Step 2. In this step we prove (5.3.6). In the sequel we fix $r, \epsilon > 0$ and $i \in [\![1, k-1]\!]$. To prove (5.3.6) it suffices to show that for any $\eta > 0$, there exists a $\delta > 0$ and N_0 such that $N \ge N_0$ implies

$$\left(\sup_{x,y \in [-r,r], |x-y| \le \delta} |f_i^N(x) - f_i^N(y)| \ge \epsilon \right) < \eta. \tag{5.3.7}$$

For $\delta > 0$ we define the event

$$A_\delta^N = \left\{ \sup_{x,y \in [-t_1, t_1], |x-y| \le \delta N^\alpha} *L_i^N(x) - L_i^N(y) - p(x-y) \ge \frac{3N^{\alpha/2}\epsilon}{4} \right\}, \tag{5.3.8}$$

where we recall that $t_1 = \lfloor (r+1)N^\alpha \rfloor$ from (5.3.1). We claim that there exist $\delta_0 > 0$ and $N_0 \in \mathbb{N}$ such that for $\delta \in (0, \delta_0]$ and $N \ge N_0$ we have

$$\mathbb{P}(A_\delta^N) < \eta. \tag{5.3.9}$$

We prove (5.3.9) in the steps below. Here we assume its validity and conclude the proof of (5.3.7).

Observe that if $\delta \in \left(0, \min\left(\delta_0, \epsilon \cdot (8\lambda r)^{-1}\right)\right)$, where λ is as in the statement of the theorem, we have the following tower of inequalities

$$\begin{aligned}
&\left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} |f_i^N(x) - f_i^N(y)| \geq \epsilon \right) = \\
&\left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} \ast N^{-\alpha/2} \left(L_i^N(xN^{\alpha}) - L_i^N(yN^{\alpha}) \right) - p(x-y)N^{\alpha/2} + \lambda(x^2 - y^2) \geq \epsilon \right) \leq \\
&\left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} N^{-\alpha/2} \ast L_i^N(xN^{\alpha}) - L_i^N(yN^{\alpha}) - p(x-y)N^{\alpha} + 2\lambda r\delta \geq \epsilon \right) \leq \\
&\left(\sup_{x,y\in[-r,r],|x-y|\leq\delta} \ast L_i^N(xN^{\alpha}) - L_i^N(yN^{\alpha}) - p(x-y)N^{\alpha} \geq \frac{3N^{\alpha/2}\epsilon}{4} \right) \leq \mathbb{P}(A_\delta^N) < \eta.
\end{aligned} \tag{5.3.10}$$

In (5.3.10) the first equality follows from the definition of f_i^N, and the inequality on the second line follows from the inequality $|x^2 - y^2| \leq 2r\delta$, which holds for all $x, y \in [-r, r]$ such that $|x - y| \leq \delta$. The inequality in the third line of (5.3.10) follows from our assumption that $\delta < \epsilon \cdot (8\lambda r)^{-1}$ and the first inequality on the last line follows from the definition of A_δ^N in (5.3.8), and the fact that $t_1 \geq rN^{\alpha}$. The last inequality follows from our assumption that $\delta < \delta_0$ and (5.3.9). In view of (5.3.10) we conclude (5.3.7).

Step 3. In this step we prove (5.3.9) and fix $\eta > 0$ in the sequel. For $\delta_1, M_1 > 0$ and $N \in \mathbb{N}$ we define the events

$$E_1 = \left\{ \max_{1\leq j\leq k-1} \ast L_j^N(\pm t_1) \mp pt_1 \leq M_1 N^{\alpha/2} \right\}, E_2 = \left\{ Z(-t_1, t_1, \vec{x}, \vec{y}, \infty, L_k^N[\![-t_1, t_1]\!]) > \delta_1 \right\}, \tag{5.3.11}$$

where we used the same notation as in Proposition 5.3.1 (in particular $\vec{x} = (L_1^N(-t_1), \dots, L_{k-1}^N(-t_1))$ and $\vec{y} = (L_1^N(t_1), \dots, L_{k-1}^N(t_1)))$. Combining Lemmas 5.3.2, 5.3.3 and Proposition 5.3.1 we know that we can find $\delta_1 > 0$ sufficiently small, M_1 sufficiently large and $\tilde{N} \in \mathbb{N}$ such that for $N \geq \tilde{N}$ we know

$$\mathbb{P}\left(E_1^c \cup E_2^c\right) < \eta/2. \tag{5.3.12}$$

We claim that we can find $\delta_0 > 0$ and $N_0 \geq \tilde{N}$ such that for $N \geq N_0$ and $\delta \in (0, \delta_0)$ we have

$$\mathbb{P}(A_\delta^N \cap E_1 \cap E_2) < \eta/2. \tag{5.3.13}$$

Since

$$(A_\delta^N) = (A_\delta^N \cap E_1 \cap E_2) + (A_\delta^N \cap \left(E_1^c \cup E_2^c\right)) \leq (A_\delta^N \cap E_1 \cap E_2) + \mathbb{P}\left(E_1^c \cup E_2^c\right),$$

we see that (5.3.12) and (5.3.13) together imply (5.3.9).

Step 4. In this step we prove (5.3.13). We define the σ-algebra

$$\mathcal{F} = \sigma\left(L_k^N[\![-t_1, t_1]\!], L_1^N(\pm t_1), L_2^N(\pm t_1), \ldots, L_{k-1}^N(\pm t_1)\right).$$

Clearly $E_1, E_2 \in \mathcal{F}$, so the indicator random variables $_{E_1}$ and $_{E_2}$ are $\mathcal{F}$-measurable. It follows from the tower property of conditional expectation that

$$\left(A_\delta^N \cap E_1 \cap E_2\right) = \left[_{A_\delta^N} {}_{E_1} {}_{E_2}\right] = \left[_{E_1} {}_{E_2} \left[_{A_\delta^N} \mid \mathcal{F}\right]\right]. \tag{5.3.14}$$

By the Schur-Gibbs property (see Definition 5.1.17), we know that $\mathbb{P}$-almost surely

$$\left[_{A_\delta^N} \mid \mathcal{F}\right] =_{avoid,Ber}^{-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]} \left[_{A_\delta^N}\right]. \tag{5.3.15}$$

We now observe that the Radon-Nikodym derivative of ${}_{avoid,Ber}^{-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]}$ with respect to ${}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}$ is given by

$$\frac{d_{avoid,Ber}^{-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]}(Q_1, \ldots, Q_{k-1})}{d_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}} = \frac{_{\{Q_1 \geq \cdots \geq Q_{k-1} \geq Q_k\}}}{Z(-t_1, t_1, \vec{x}, \vec{y}, \infty, L_k^N[\![-t_1, t_1]\!])}, \tag{5.3.16}$$

where $\mathfrak{Q} = (Q_1, \ldots, Q_{k-1})$ is ${}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}$-distributed and $Q_k = L_k^N[\![-t_1, t_1]\!]$. To see this, note that by

Definition 5.1.15 we have for any set $A \subset \prod_{i=1}^{k-1} \Omega(-t_1, t_1, x_i, y_i)$ that

$$
{}_{avoid,Ber}^{-t_1,t_1,\vec{x},\vec{y},\infty,L_k^N[\![-t_1,t_1]\!]}(A) = \frac{{}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}(A \cap \{Q_1 \geq \cdots \geq Q_{k-1} \geq Q_k\})}{{}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}(Q_1 \geq \cdots \geq Q_{k-1} \geq Q_k)} =
$$

$$
\frac{{}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}\left[{}_A \cdot {}_{\{Q_1 \geq \cdots \geq Q_{k-1} \geq Q_k\}}\right]}{Z(-t_1, t_1, \vec{x}, \vec{y}, \infty, L_k^N[\![-t_1, t_1]\!])} = \int_A \frac{{}_{\{Q_1 \geq \cdots \geq Q_{k-1} \geq Q_k\}}}{Z(-t_1, t_1, \vec{x}, \vec{y}, \infty, L_k^N[\![-t_1, t_1]\!])} \, d_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}.
$$

It follows from (5.3.14), (5.3.16), and the definition of E_2 in 5.3.11 that

$$
\begin{aligned}
(A_\delta^N \cap E_1 \cap E_2) &= \left[{}_{E_1} {}_{E_2} {}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}} \left[\frac{{}_{B_\delta^N} \cdot {}_{\{Q_1 \geq \cdots \geq Q_k\}}}{Z(-t_1, t_1, \vec{x}, \vec{y}, L_k^N[\![-t_1, t_1]\!])}\right]\right] \leq \\
&\leq \left[{}_{E_1} {}_{E_2} {}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}} \left[\frac{{}_{B_\delta^N}}{\delta_1}\right]\right] \leq \frac{1}{\delta_1} \left[{}_{E_1} \cdot {}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}(B_\delta^N)\right],
\end{aligned} \tag{5.3.17}
$$

where δ_1 is as in 5.3.11, and

$$
B_\delta^N = \left\{ \sup_{x,y \in [-t_1,t_1], |x-y| \leq \delta N^\alpha} *Q_i(x) - Q_i(y) - p(x-y) \geq \frac{3N^{\alpha/2}\epsilon}{4} \right\}.
$$

Notice that under ${}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}$ the law of Q_i is precisely ${}_{Ber}^{-t_1,t_1,x_i,y_i}$, and so we conclude that

$$
{}_{Ber}^{-t_1,t_1,\vec{x},\vec{y}}(B_\delta^N) = {}_{Ber}^{0,2t_1,0,y_i-x_i} \left(\sup_{x,y \in [0,2t_1], |x-y| \leq \delta N^\alpha} *\ell(x) - \ell(y) - p(x-y) \geq \frac{3N^{\alpha/2}\epsilon}{4} \right), \tag{5.3.18}
$$

where ℓ has law ${}_{Ber}^{0,2t_1,0,y_i-x_i}$ (note that in (5.3.18) we implicitly translated the path ℓ to the right by t_1 and up by $-x_i$, which does not affect the probability in question). Since on the event E_1 we know that $|y_i - x_i - 2pt_1| \leq 2M_1N^\alpha$ we conclude from Lemma 5.2.12 that we can find N_0 and $\delta_0 > 0$ depending on M_1, r, α such that for $N \geq N_0$ and $\delta \in (0, \delta_0)$ we have

$$
{}_{E_1} \cdot {}_{Ber}^{0,2t_1,0,y_i-x_i} \left(\sup_{x,y \in [0,2t_1], |x-y| \leq \delta N^\alpha} *\ell(x) - \ell(y) - p(x-y) \geq \frac{3N^{\alpha/2}\epsilon}{4} \right) < \frac{\delta_1 \eta}{2}. \tag{5.3.19}
$$

Combining (5.3.17), (5.3.18) and (5.3.19) we conclude (5.3.13), and hence statement (i) of the theorem.

5.3.3 Proof of Theorem 5.1.26 (ii)

In this section we fix a subsequential limit $\mathcal{L}^\infty = (f_1^\infty, \dots, f_{k-1}^\infty)$ of the sequence $\tilde{\mathbb{P}}_N$ as in the statement of Theorem 5.1.26, and we prove that $\mathcal{L}^\infty$ possesses the partial Brownian Gibbs property. Our approach is similar to that in [105, Sections 5.1 and 5.2]. We first give a definition of measures on scaled free and avoiding Bernoulli random walks. These measures will appear when we apply the Schur Gibbs property to the scaled line ensembles f^N.

Definition 5.3.5. Let $a, b \in N^{-\alpha}\mathbb{Z}$ with $a < b$ and $x, y \in N^{-\alpha/2}\mathbb{Z}$ satisfy $0 \le y - x \le (b-a)N^{\alpha/2}$. Let $\ell^{(T,z)}$ denote a random variable with law $\mathbb{P}_{Ber}^{0,T,0,z}$ as before Definition 5.1.15. We define $\mathbb{P}_{free,N}^{a,b,x,y}$ to be the law of the $C([a,b])$-valued random variable Y given by

$$Y(t) = \frac{x + N^{-\alpha/2}\left[\ell_{(t-a)N^\alpha}^{((b-a)N^\alpha,\,(y-x)N^{\alpha/2}))} - ptN^\alpha\right]}{\sqrt{p(1-p)}}, \quad t \in [a,b].$$

Now for $i \in [\![1,k]\!]$, let $\ell^{(N,z),i}$ denote iid random variables with laws $\mathbb{P}_{Ber}^{0,N,0,z}$. Let $\vec{x}, \vec{y} \in (N^{-\alpha/2}\mathbb{Z})^k$ satisfy $0 \le y_i - x_i \le (b-a)N^{\alpha/2}$ for $i \in [\![1,k]\!]$. We define the $[\![1,k]\!]$-indexed line ensemble $\mathcal{Y}^N$ by

$$\mathcal{Y}_i^N(t) = \frac{x_i + N^{-\alpha/2}\left[\ell_{(t-a)N^\alpha}^{((b-a)N^\alpha,\,(y_i-x_i)N^{\alpha/2})),i} - ptN^\alpha\right]}{\sqrt{p(1-p)}}, \quad i \in [\![1,k]\!], t \in [a,b].$$

We let $\mathbb{P}_{free,N}^{a,b,\vec{x},\vec{y}}$ denote the law of $\mathcal{Y}^N$. Suppose $\vec{x}, \vec{y} \in (N^{-\alpha/2}\mathbb{Z})^k \cap W_k^\circ$ and $f : [a,b] \to (-\infty, \infty]$, $g : [a,b] \to [-\infty, \infty)$ are continuous functions. We define the probability measure $\mathbb{P}_{avoid,N}^{a,b,\vec{x},\vec{y},f,g}$ to be $\mathbb{P}_{free,N}^{a,b,\vec{x},\vec{y}}$ conditioned on the event

$$E = \{f(r) \ge \mathcal{Y}_1^N(r) \ge \cdots \ge \mathcal{Y}_k^N(r) \ge g(r) \text{ for } r \in [a,b]\}.$$

This measure is well-defined if E is nonempty.

Next, we state two lemmas whose proofs we give in Section 5.6.4. The first lemma proves weak convergence of the scaled avoiding random walk measures in Definition 5.3.5. It states roughly that if the data of these measures converge, then the measures converge weakly to the law of avoiding

Brownian bridges with the limiting data, as in Definition 5.1.7.

Lemma 5.3.6. *Fix $k \in \mathbb{N}$ and $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a-1, b+1] \to (-\infty, \infty]$, $g : [a-1, b+1] \to [-\infty, \infty)$ be continuous functions such that $f(t) > g(t)$ for all $t \in [a-1, b+1]$. Let $\vec{x}, \vec{y} \in W_k^\circ$ be such that $f(a) > x_1$, $f(b) > y_1$, $g(a) < x_k$, and $g(b) < y_k$. Let $a_N = \lfloor aN^\alpha \rfloor N^{-\alpha}$ and $b_N = \lceil bN^\alpha \rceil N^{-\alpha}$, and let $f_N : [a-1, b+1] \to (-\infty, \infty]$ and $g_N : [a-1, b+1] \to [-\infty, \infty)$ be continuous functions such that $f_N \to f$ and $g_N \to g$ uniformly on $[a-1, b+1]$. Lastly, let $\vec{x}^{\,N}, \vec{y}^{\,N} \in (N^{-\alpha/2}\mathbb{Z})^k \cap W_k^\circ$, write $\tilde{x}_i^N = (x_i^N - pa_N N^{\alpha/2})/\sqrt{p(1-p)}$, $\tilde{y}_i^N = (y_i^N - pb_N N^{\alpha/2})/\sqrt{p(1-p)}$, and suppose that $\tilde{x}_i^N \to x_i$ and $\tilde{y}_i^N \to y_i$ as $N \to \infty$ for each $i \in [\![1, k]\!]$. Then there exists $N_0 \in \mathbb{N}$ so that $\mathbb{P}_{avoid,N}^{a_N, b_N, \vec{x}^{\,N}, \vec{y}^{\,N}, f_N, g_N}$ is well-defined for $N \geq N_0$. Moreover, if $\mathcal{Y}^N$ have laws $\mathbb{P}_{avoid,N}^{a_N, b_N, \vec{x}^{\,N}, \vec{y}^{\,N}, f_N, g_N}$ and $\mathcal{Z}^N = \mathcal{Y}^N|_{\Sigma \times [a,b]}$, then the law of $\mathcal{Z}^N$ converges weakly to $\mathbb{P}_{avoid}^{a,b,\vec{x},\vec{y},f,g}$ as $N \to \infty$.*

The next lemma shows that at any given point, the values of the $k-1$ curves in $\mathcal{L}^\infty$ are each distinct, so that Lemma 5.3.6 may be applied.

Lemma 5.3.7. *For any $s \in \mathbb{R}$, we have $\mathcal{L}^\infty(s) = (f_1^\infty(s), \ldots, f_{k-1}^\infty(s)) \in W_{k-1}^\circ$, $\mathbb{P}$-a.s.*

Using these two lemmas whose proofs are postponed, we now give the proof of Theorem 5.1.26 (ii).

Proof. We will write $\Sigma = [\![1, k]\!]$. Let us write $\mathcal{Y}^N = (Y_1^N, \ldots, Y_{k-1}^N)$ with $Y_i^N(s) = N^{-\alpha/2}(L_i^N(sN^\alpha) - psN^\alpha)/\sqrt{p(1-p)}$. We may assume without loss of generality that $\mathcal{Y}^N \implies \mathcal{L}^\infty$ as $N \to \infty$. Fix a set $K = [\![k_1, k_2]\!] \subseteq [\![1, k-2]\!]$ and $a, b \in \mathbb{R}$ with $a < b$. We also fix a bounded Borel-measurable function $F : C(K \times [a,b]) \to \mathbb{R}$. It suffices to prove that $\mathbb{P}$-a.s.,

$$[F(\mathcal{L}^\infty|_{K\times[a,b]}) \,|\, \mathcal{F}_{ext}(K \times (a,b))] =_{avoid}^{a,b,\vec{x},\vec{y},f,g} [F(Q)], \tag{5.3.20}$$

where $\vec{x} = (f_{k_1}^\infty(a), \ldots, f_{k_2}^\infty(a))$, $\vec{y} = (f_{k_1}^\infty(b), \ldots, f_{k_2}^\infty(b))$, $f = f_{k_1-1}^\infty$ (with $f_0^\infty = +\infty$), $g = f_{k_2+1}^\infty$, the σ-algebra $\mathcal{F}_{ext}(K\times(a,b))$ is as in Definition 5.1.8, and Q has law $\mathbb{P}_{avoid}^{a,b,\vec{x},\vec{y},f,g}$. We prove (5.3.20) in two steps.

Step 1. Fix $m \in \mathbb{N}$, $n_1, \dots, n_m \in \Sigma$, $t_1, \dots, t_m \in \mathbb{R}$, and $h_1, \dots, h_m : \mathbb{R} \to \mathbb{R}$ bounded continuous functions. Define $S = \{i \in [\![1, m]\!] : n_i \in K, t_i \in [a, b]\}$. In this step we prove that

$$\left[\prod_{i=1}^{m} h_i(f_{n_i}^{\infty}(t_i))\right] = \left[\prod_{s \notin S} h_s(f_{n_s}^{\infty}(t_s)) \cdot {}_{avoid}^{a,b,\vec{x},\vec{y},f,g} \left[\prod_{s \in S} h_s(Q_{n_s}(t_s))\right]\right], \tag{5.3.21}$$

where Q denotes a random variable with law ${}_{avoid}^{a,b,\vec{x},\vec{y},f,g}$. By assumption, we have

$$\lim_{N \to \infty} \left[\prod_{i=1}^{m} h_i(Y_{n_i}^N(t_i))\right] = \left[\prod_{i=1}^{m} h_i(f_{n_i}^{\infty}(t_i))\right]. \tag{5.3.22}$$

We define the sequences $a_N = \lfloor aN^{\alpha} \rfloor N^{-\alpha}$, $b_N = \lceil bN^{\alpha} \rceil N^{-\alpha}$, $\vec{x}^{\,N} = (L_{k_1}^N(a_N), \dots, L_{k_2}^N(a_N))$, $\vec{y}^{\,N} = (L_{k_1}^N(b_N), \dots, L_{k_2}^N(b_N))$, $f_N = Y_{k_1-1}^N$ (where $Y_0 = +\infty$), $g_N = Y_{k_2+1}^N$. Since $a_N \to a$, $b_N \to b$, we may choose N_0 sufficiently large so that if $N \geq N_0$, then $t_s < a_N$ or $t_s > b_N$ for all $s \notin S$ with $n_s \in K$. Since the line ensemble $(L_1^N, \dots, L_{k-1}^N)$ in the definition of $\mathcal{Y}^N$ satisfies the Schur Gibbs property (see Definition 5.1.17), we see from Definition 5.3.5 that the law of $\mathcal{Y}^N|_{K \times [a,b]}$ conditioned on the σ-algebra $\mathcal{F} = \sigma\left(Y_{k_1-1}^N, Y_{k_2+1}^N, Y_{k_1}^N(a_N), Y_{k_1}^N(b_N), \dots, Y_{k_2}^N(a_N), Y_{k_2}^N(b_N)\right)$ is precisely $\mathbb{P}_{avoid,N}^{a_N,b_N,\vec{x}^{\,N},\vec{y}^{\,N},f_N,g_N}$. Therefore, writing Z^N for a random variable with this law, we have

$$\left[\prod_{i=1}^{m} h_i(Y_{n_i}^N(t_i))\right] = \left[\prod_{s \notin S} h_s(Y_{n_s}^N(t_s)) \cdot {}_{avoid,N}^{a_N,b_N,\vec{x}^{\,N},\vec{y}^{\,N},f_N,g_N} \left[\prod_{s \in S} h_s(Z_{n_s-k_1+1}^N(t_s))\right]\right]. \tag{5.3.23}$$

Now by Lemma 5.3.7, we have $\mathbb{P}$-a.s. that $\vec{x}, \vec{y} \in W_{k_2-k_1+1}^{\circ}$, where we recall that $\vec{x} = \mathcal{L}^{\infty}(a)$, $\vec{y} = \mathcal{L}^{\infty}(b)$. By the Skorohod representation theorem, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting random variables with the laws of $\mathcal{Y}^N$, $\mathcal{L}^{\infty}$ (which we denote by the same symbols), such that $\mathcal{Y}^N \to \mathcal{L}^{\infty}$ uniformly on compact sets at every point of Ω. In particular, $f_N \to f = f_{k_2+1}^{\infty}$ and $g_N \to g = f_{k_1-1}^{\infty}$ uniformly on $[a-1, b+1] \supseteq [a_N, b_N]$, and $({x_i}^N - pa_N N^{\alpha/2})/\sqrt{p(1-p)} \to \vec{x}$, $({y_i}^N - pb_N N^{\alpha/2})/\sqrt{p(1-p)} \to \vec{y}$ for $i \in [\![1, k-1]\!]$. It follows from Lemma 5.3.6 that

$$\lim_{N \to \infty}{}^{a_N,b_N,\vec{x}^{\,N},\vec{y}^{\,N},f_N,g_N}_{avoid,N} \left[\prod_{s \in S} h_s(Z_{n_s-k_1+1}^N(t_s))\right] = {}_{avoid}^{a,b,\vec{x},\vec{y},f,g} \left[\prod_{s \in S} h_s(Q_{n_s}(t_s))\right]. \tag{5.3.24}$$

Lastly, the continuity of the h_i implies that

$$\lim_{N\to\infty} \prod_{s\notin S} h_s(Y^N_{n_s}(t_s)) = \prod_{s\notin S} h_s(f^\infty_{n_s}(t_s)). \tag{5.3.25}$$

Combining (5.3.22), (5.3.23), (5.3.24), and (5.3.25) and applying the bounded convergence theorem proves (5.3.21).

Step 2. In this step we prove (5.3.20) as a consequence of (5.3.21). For $n \in \mathbb{N}$ we define piecewise linear functions

$$\chi_n(x,r) = \begin{cases} 0, & x > r + 1/n, \\ 1 - n(x-r), & x \in [r, r+1/n], \\ 1, & x < r. \end{cases}$$

We fix $m_1, m_2 \in \mathbb{N}$, $n^1_1, \dots, n^1_{m_1}, n^2_1, \dots, n^2_{m_2} \in \Sigma$, $t^1_1, \dots, t^1_{m_1}, t^2_1, \dots, t^2_{m_2} \in \mathbb{R}$, such that $(n^1_i, t^1_i) \notin K \times [a,b]$ and $(n^2_i, t^2_i) \in K \times [a,b]$ for all i. Then (5.3.21) implies that

$$\left[\prod_{i=1}^{m_1} \chi_n(f^\infty_{n^1_i}(t^1_i), a_i) \prod_{i=1}^{m_2} \chi_n(f^\infty_{n^2_i}(t^2_i), b_i)\right] = \left[\prod_{i=1}^{m_1} \chi_n(f^\infty_{n^1_i}(t^1_i), a_i)^{a,b,\vec{x},\vec{y},f,g}_{avoid} \left[\prod_{i=1}^{m_2} \chi_n(Q_{n^2_i}(t^2_i), b_i)\right]\right].$$

Letting $n \to \infty$, we have $\chi_n(x,r) \to \chi(x,r) = \mathbf{1}_{x\le r}$, and the bounded convergence theorem implies that

$$\left[\prod_{i=1}^{m_1} \chi(f^\infty_{n^1_i}(t^1_i), a_i) \prod_{i=1}^{m_2} \chi(f^\infty_{n^2_i}(t^2_i), b_i)\right] = \left[\prod_{i=1}^{m_1} \chi(f^\infty_{n^1_i}(t^1_i), a_i)^{a,b,\vec{x},\vec{y},f,g}_{avoid} \left[\prod_{i=1}^{m_2} \chi(Q_{n^2_i}(t^2_i), b_i)\right]\right].$$

Let $\mathcal{H}$ denote the space of bounded Borel measurable functions $H : C(K \times [a,b]) \to \mathbb{R}$ satisfying

$$\left[\prod_{i=1}^{m_1} \chi(f^\infty_{n^1_i}(t^1_i), a_i) H(\mathcal{L}^\infty|_{K\times[a,b]})\right] = \left[\prod_{i=1}^{m_1} \chi(f^\infty_{n^1_i}(t^1_i), a_i)^{a,b,\vec{x},\vec{y},f,g}_{avoid} \left[H(Q)\right]\right]. \tag{5.3.26}$$

The above shows that $\mathcal{H}$ contains all functions $\mathbf{1}_A$ for sets A contained in the π-system $\mathcal{A}$ consisting of sets of the form

$$\{h \in C(K \times [a,b]) : h(n_i^2, t_i^2) \le b_i \text{ for } i \in [\![1, m_2]\!]\}.$$

We note that $\mathcal{H}$ is closed under linear combinations simply by linearity of expectation, and if $H_n \in \mathcal{H}$ are nonnegative bounded measurable functions converging monotonically to a bounded function H, then $H \in \mathcal{H}$ by the monotone convergence theorem. Thus by the monotone class theorem [117, Theorem 5.2.2], $\mathcal{H}$ contains all bounded $\sigma(\mathcal{A})$-measurable functions. Since the finite dimensional sets in $\mathcal{A}$ generate the full Borel σ-algebra C_K (see for instance [105, Lemma 3.1]), we have in particular that $F \in \mathcal{H}$.

Now let $\mathcal{B}$ denote the collection of sets $B \in \mathcal{F}_{ext}(K \times (a,b))$ such that

$$[\mathbf{1}_B \cdot F(\mathcal{L}^\infty|_{K\times[a,b]})] = [\mathbf{1}_B \cdot_{avoid}^{a,b,\vec{x},\vec{y},f,g} [F(Q)]]. \tag{5.3.27}$$

We observe that $\mathcal{B}$ is a λ-system. Indeed, since (5.3.26) holds for $H = F$, taking $a_i, b_i \to \infty$ and applying the bounded convergence theorem shows that (5.3.27) holds with $\mathbf{1}_B = 1$. Thus if $B \in \mathcal{B}$ then $B^c \in \mathcal{B}$ since $\mathbf{1}_{B^c} = 1 - \mathbf{1}_B$. If $B_i \in \mathcal{B}$, $i \in \mathbb{N}$, are pairwise disjoint and $B = \bigcup_i B_i$, then $\mathbf{1}_B = \sum_i \mathbf{1}_{B_i}$, and it follows from the monotone convergence theorem that $B \in \mathcal{B}$. Moreover, (5.3.26) with $H = F$ implies that $\mathcal{B}$ contains the π-system P of sets of the form

$$\{h \in C(\Sigma \times \mathbb{R}) : h(n_i, t_i) \le a_i \text{ for } i \in [\![1, m_1]\!], \text{ where } (n_i, t_i) \notin K \times (a,b)\}.$$

By the π-λ theorem [117, Theorem 2.1.6] it follows that $\mathcal{B}$ contains $\sigma(P) = \mathcal{F}_{ext}(K \times (a,b))$. Thus (5.3.27) holds for all $B \in \mathcal{F}_{ext}(K \times (a,b))$. It is proven in [105, Lemma 3.4] that ${}_{avoid}^{a,b,\vec{x},\vec{y},f,g}[F(Q)]$ is an $\mathcal{F}_{ext}(K \times (a,b))$-measurable function. Therefore (5.3.20) follows from (5.3.27) by the definition of conditional expectation.

□

5.4 Bounding the max and min

In this section we prove Lemmas 5.3.2 and 5.3.3 and we assume the same notation as in the statements of these lemmas. In particular, we assume that $k \in \mathbb{N}$, $k \geq 2$, $p \in (0,1)$, $\alpha, \lambda > 0$ are all fixed and

$$\left\{\mathfrak{L}^N = (L_1^N, L_2^N, \dots, L_k^N)\right\}_{N=1}^{\infty},$$

is an (α, p, λ)-good sequence of $[\![1,k]\!]$-indexed Bernoulli line ensembles as in Definition 5.1.24 that are all defined on a probability space with measure $\mathbb{P}$. The proof of Lemma 5.3.2 is given in Section 5.4.1 and Lemma 5.3.3 is proved in Section 5.4.2.

5.4.1 Proof of Lemma 5.3.2

Our proof of Lemma 5.3.2 is similar to that of [**CD**]. For clarity we split the proof into three steps. In the first step we introduce some notation that will be required in the proof of the lemma, which is presented in Steps 2 and 3.

Step 1. We write $s_4 = \lceil r+4 \rceil N^{\alpha}$, $s_3 = \lfloor r+3 \rfloor N^{\alpha}$, so that $s_3 \leq t_3 \leq s_4$, and assume that N is large enough so that $\psi(N)N^{\alpha}$ from Definition 5.1.24 is at least s_4. Notice that such a choice is possible by our assumption that $\mathfrak{L}^N$ is an (α, p, λ)-good sequence and in particular, we know that L_i^N are defined at $\pm s_4$ for $i \in [\![1,k]\!]$. We define events

$$E(M) = \left\{\left|L_1^N(-s_4) + ps_4\right| > MN^{\alpha/2}\right\}, \quad F(M) = \left\{L_1^N(-s_3) > -ps_3 + MN^{\alpha/2}\right\},$$

$$G(M) = \left\{\sup_{s \in [0, s_4]} \left[L_1^N(s) - ps\right] \geq (6r+22)(2r+10)^{1/2}(M+1)N^{\alpha/2}\right\}.$$

If $\epsilon > 0$ is as in the statement of the lemma, we note by (5.1.7) that we can find M and $\tilde{N}_1$ sufficiently large so that if $N \geq \tilde{N}_1$ we have

$$\mathbb{P}(E(M)) < \epsilon/4 \text{ and } \mathbb{P}(F(M)) < \epsilon/12. \tag{5.4.1}$$

In the remainder of this step we show that the event $G(M) \setminus E(M)$ can be written as a *countable disjoint union* of certain events, i.e. we show that

$$\bigsqcup_{(a,b,s,\ell_{top},\ell_{bot}) \in D(M)} E(a,b,s,\ell_{top},\ell_{bot}) = G(M) \setminus E(M), \tag{5.4.2}$$

where the sets $E(a,b,s,\ell_{top},\ell_{bot})$ and $D(M)$ are described below.

For $a, b, z_1, z_2, z_3 \in \mathbb{Z}$ with $z_1 \leq a$, $z_2 \leq b$, $s \in [\![0, s_4]\!]$, $\ell_{bot} \in \Omega(-s_4, s, z_1, z_2)$ and $\ell_{top} \in \Omega(s, s_4, b, z_3)$ we define $E(a,b,s,\ell_{top},\ell_{bot})$ to be the event that $L_1^N(-s_4) = a$, $L_1^N(s) = b$, L_1^N agrees with ℓ_{top} on $[\![s, s_4]\!]$, and L_2^N agrees with ℓ_{bot} on $[\![-s_4, s]\!]$. Let $D(M)$ be the set of tuples $(a,b,s,\ell_{top},\ell_{bot})$ satisfying

(1) $0 \leq s \leq s_4$,

(2) $0 \leq b - a \leq s + s_4$, $|a + ps_4| \leq MN^{\alpha/2}$, and $b - ps \geq (6r+22)(2r+10)^{1/2}(M+1)N^{\alpha/2}$,

(3) $z_1 \leq a$, $z_2 \leq b$, and $\ell_{bot} \in \Omega(-s_4, s, z_1, z_2)$,

(4) $b \leq z_3 \leq b + (s_4 - s)$, and $\ell_{top} \in \Omega(s, s_4, b, z_3)$,

(5) if $s < s' \leq s_4$, then $\ell_{top}(s') - ps' < (6r+22)(2r+10)^{1/2}(M+1)N^{\alpha/2}$.

It is clear that $D(M)$ is countable. The five conditions above together imply that

$$\bigcup_{(a,b,s,\ell_{top},\ell_{bot}) \in D(M)} E(a,b,s,\ell_{top},\ell_{bot}) = G(M) \setminus E(M),$$

and what remains to be shown to prove (5.4.2) is that $E(a,b,s,\ell_{top},\ell_{bot})$ are pairwise disjoint.

On the intersection of $E(a,b,s,\ell_{top},\ell_{bot})$ and $E(\tilde{a},\tilde{b},\tilde{s},\tilde{\ell}_{top},\tilde{\ell}_{bot})$ we must have $\tilde{a} = L_1^N(-s_4) = a$ so that $a = \tilde{a}$. Furthermore, we have by properties (2) and (5) that $s \geq \tilde{s}$ and $\tilde{s} \geq s$ from which we conclude that $s = \tilde{s}$ and then we conclude $\tilde{b} = b$, $\ell_{top} = \tilde{\ell}_{top}$, $\ell_{bot} = \tilde{\ell}_{bot}$. In summary, if $E(a,b,s,\ell_{top},\ell_{bot})$ and $E(\tilde{a},\tilde{b},\tilde{s},\tilde{\ell}_{top},\tilde{\ell}_{bot})$ have a non-trivial intersection then $(a,b,s,\ell_{top},\ell_{bot}) = (\tilde{a},\tilde{b},\tilde{s},\tilde{\ell}_{top},\tilde{\ell}_{bot})$, which proves (5.4.2).

Step 2. In this step we prove that we can find an N_2 so that for $N \geq N_2$

$$\mathbb{P}\left(\sup_{s\in[0,t_3]}\left[L_1^N(s) - ps\right] \geq (6r+22)(2r+10)^{1/2}(M+1)N^{\alpha/2}\right) \leq \mathbb{P}(G(M)) < \epsilon/2. \tag{5.4.3}$$

A similar argument, which we omit, proves the same inequality with $[-t_3, 0]$ in place of $[0, t_3]$ and then the statement of the lemma holds for all $N \geq N_2$, with $R_1 = (6r+22)(2r+10)^{1/2}(M+1)$.

We claim that we can find $\tilde{N}_2 \in \mathbb{N}$ sufficiently large so that if $N \geq \tilde{N}_2$ and $(a, b, s, \ell_{top}, \ell_{bot}) \in D(M)$ satisfies $\mathbb{P}(E(a, b, s, \ell_{top}, \ell_{bot})) > 0$ then we have

$$\mathbb{P}^{-s_4,s,a,b,\infty,\ell_{bot}}_{avoid,Ber}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) \geq \frac{1}{3}. \tag{5.4.4}$$

We will prove (5.4.4) in Step 3. For now we assume its validity and conclude the proof of (5.4.3).

Let $(a, b, s, \ell_{top}, \ell_{bot}) \in D(M)$ be such that $\mathbb{P}(E(a, b, s, \ell_{top}, \ell_{bot})) > 0$. By the Schur Gibbs property, see Definition 5.1.17, we have for any $\ell_0 \in \Omega(-s_4, s, a, b)$ that

$$\mathbb{P}\left(L_1^N[\![-s_4, s]\!] = \ell_0 \,|\, E(a, b, s, \ell_{top}, \ell_{bot})\right) = \mathbb{P}^{-s_4,s,a,b,\infty,\ell_{bot}}_{avoid,Ber}(\ell = \ell_0), \tag{5.4.5}$$

where $L_1^N[\![-s_4, s]\!]$ denotes the restriction of L_1^N to the set $[\![-s_4, s]\!]$.

Combining (5.4.4) and (5.4.5) we get for $N \geq \tilde{N}_2$

$$\begin{aligned}&\mathbb{P}\left(L_1^N(-s_3) > -ps_3 + MN^{\alpha/2} | E(a, b, s, \ell_{top}, \ell_{bot})\right) = \\ &\mathbb{P}^{-s_4,s,a,b,\infty,\ell_{bot}}_{avoid,Ber}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) \geq \frac{1}{3}.\end{aligned} \tag{5.4.6}$$

It follows from (5.4.6) that for $N \geq \tilde{N}_2$ we have

$$\epsilon/12 > \mathbb{P}(F(M)) \geq \sum_{\substack{(a,b,s,\ell_{top},\ell_{bot}) \in D(M), \\ \mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot}))>0}} \mathbb{P}\left(F(M) \cap E(a,b,s,\ell_{top},\ell_{bot})\right) =$$
$$\sum_{\substack{(a,b,s,\ell_{top},\ell_{bot}) \in D(M), \\ \mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot}))>0}} \mathbb{P}\left(L_1^N(-s_3) > -ps_3 + MN^{\alpha/2} | E(a,b,s,\ell_{top},\ell_{bot})\right) \mathbb{P}\left(E(a,b,s,\ell_{top},\ell_{bot})\right) \geq$$
$$\sum_{\substack{(a,b,s,\ell_{top},\ell_{bot}) \in D(M), \\ \mathbb{P}(E(a,b,s,\ell_{top},\ell_{bot}))>0}} \frac{1}{3} \cdot \mathbb{P}\left(E(a,b,s,\ell_{top},\ell_{bot})\right) = \frac{1}{3} \cdot \mathbb{P}(G(M) \setminus E(M)), \tag{5.4.7}$$

where in the last equality we used (5.4.2). From (5.4.1) and (5.4.7) we have for $N \geq N_2 = \max(\tilde{N}_1, \tilde{N}_2)$

$$\mathbb{P}(G(M)) \leq \mathbb{P}(E(M)) + \mathbb{P}(G(M) \setminus E(M)) < \epsilon/4 + \epsilon/4,$$

which proves (5.4.3).

Step 3. In this step we prove (5.4.4) and in the sequel we let $(a, b, s, \ell_{top}, \ell_{bot}) \in D(M)$ be such that $\mathbb{P}(E(a, b, s, \ell_{top}, \ell_{bot})) > 0$. We remark that the condition $\mathbb{P}(E(a, b, s, \ell_{top}, \ell_{bot})) > 0$ implies that $\Omega_{avoid}(-s_4, s, a, b, \infty, \ell_{bot})$ is not empty. By Lemma 5.2.2 we know that

$$\mathbb{P}_{avoid,Ber}^{-s_4,s,a,b,\infty,\ell_{bot}} \left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) \geq \mathbb{P}_{Ber}^{-s_4,s,a,b} \left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right),$$

and so it suffices to show that

$$\mathbb{P}_{Ber}^{-s_4,s,a,b} \left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) \geq \frac{1}{3}. \tag{5.4.8}$$

One directly observes that

$$\mathbb{P}_{Ber}^{-s_4,s,a,b}\left(\ell(-s_3) > -ps_3 + MN^{\alpha/2}\right) = \mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\left(\ell(s_4 - s_3) + a \geq -ps_3 + MN^{\alpha/2}\right) \geq \\ \mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\left(\ell(s_4 - s_3) \geq p(s_4 - s_3) + 2MN^{\alpha/2}\right), \tag{5.4.9}$$

where the inequality follows from the assumption in (2) that $a + ps_4 \geq -MN^{\alpha/2}$. Moreover, since $b - ps \geq (6r + 22)(2r + 10)^{1/2}(M + 1)N^{\alpha/2}$ and $a + ps_4 \leq MN^{\alpha/2}$, we have

$$b - a \geq p(s + s_4) + (6r + 21)(2r + 10)^{1/2}(M + 1)N^{\alpha/2} \geq p(s + s_4) + (6r + 21)(M + 1)(s + s_4)^{1/2}.$$

The second inequality follows since $s + s_4 \leq 2s_4 \leq (2r + 10)N^{\alpha}$.

It follows from Lemma 5.2.8 with $M_1 = 0$, $M_2 = (6r + 21)(M + 1)$ that for sufficiently large N

$$\mathbb{P}_{Ber}^{0,s+s_4,0,b-a}\left(\ell(s_4 - s_3) \geq \frac{s_4 - s_3}{s + s_4}[p(s + s_4) + M_2(s + s_4)^{1/2}] - (s + s_4)^{1/4}\right) \geq 1/3. \tag{5.4.10}$$

Note that $\frac{s_4 - s_3}{s + s_4} \geq \frac{N^{\alpha}}{(2r+10)N^{\alpha}} = \frac{1}{2r+10}$ and so for all $N \in \mathbb{N}$ we have

$$\frac{s_4 - s_3}{s + s_4}[p(s + s_4) + M_2(s + s_4)^{1/2}] - (s + s_4)^{1/4} \geq \\ p(s_4 - s_3) + \frac{(6r + 21)(M + 1)(s + s_4)^{1/2}}{2r + 10} - (s + s_4)^{1/4} \geq p(s_4 - s_3) + 2MN^{\alpha/2}. \tag{5.4.11}$$

Combining (5.4.9), (5.4.10) and (5.4.11) we conclude that we can find $\tilde{N}_2 \in \mathbb{N}$ such that if $N \geq \tilde{N}_2$ we have (5.4.8). This suffices for the proof.

5.4.2 Proof of Lemma 5.3.3

We begin by proving the following important lemma, which shows that it is unlikely that the curve L_{k-1}^N falls uniformly very low on a large interval.

Lemma 5.4.1. *Under the same conditions as in Lemma 5.3.3 the following holds. For any $r, \epsilon > 0$*

there exist $R > 0$ and $N_5 \in \mathbb{N}$ such that for all $N \geq N_5$

$$\mathbb{P}\left(\sup_{x\in[r,R]}\left(L_{k-1}^N(xN^\alpha) - pxN^\alpha\right) \leq -(\lambda R^2 + \phi(\epsilon/16) + 1)N^{\alpha/2}\right) < \epsilon, \tag{5.4.12}$$

where λ, ϕ are as in the definition of an (α, p, λ)-good sequence of line ensembles, see Definition 5.1.24. The same statement holds if $[r, R]$ is replaced with $[-R, -r]$ and the constants N_5, R depend on ϵ, r as well as the parameters α, p, λ, k and the functions ϕ, ψ from Definition 5.1.24.

Proof. Before we go into the proof we give an informal description of the main ideas. The key to this lemma is the parabolic shift implicit in the definition of an (α, p, λ)-good sequence. This shift requires that the deviation of the top curve L_1^N from the line of slope p to appear roughly parabolic. On the event in equation (5.4.12) we have that the $(k-1)$-th curve dips very low uniformly on the interval $[r, R]$ and we will argue that on this event the top $k-2$ curves essentially do not feel the presence of the $(k-1)$-th curve. After a careful analysis using the monotone coupling lemmas from Section 5.2.1 we will see that the latter statement implies that the curve L_1^N behaves like a free bridge between its end-points that have been slighly raised. Consequently, we would expect the midpoint $L_1^N(N^\alpha(R+r)/2)$ to be close (on scale $N^{\alpha/2}$) to $[L_1^N(rN^\alpha) + L_1^N(RN^\alpha)]/2$. However, with high probability $[L_1^N(rN^\alpha)+L_1^N(RN^\alpha)]/2$ lies much lower than the inverted parabola $-\lambda(R+r)^2N^{\alpha/2}/4$ (due to the concavity of the latter), and so it is very unlikely for $L_1^N(N^\alpha(R+r)/2)$ to be near it by our assumption. The latter would imply that the event in (5.4.12) is itself unlikely, since conditional on it an unlikely event suddenly became likely.

We proceed to fill in the details of the above sketch of the proof in the following steps. In total there are six steps and we will only prove the statement of the lemma for the interval $[r, R]$, since the argument for $[-R, -r]$ is very similar.

Step 1. We begin by specifying the choice of R in the statement of the lemma, fixing some notation and making a few simplifying assumptions.

Fix $r, \epsilon > 0$ as in the statement of the lemma. Note that for any $R > r$,

$$\sup_{r \le x \le R} \left(L_{k-1}^N(xN^\alpha) - pxN^\alpha\right) \ge \sup_{\lceil r \rceil \le x \le R} \left(L_{k-1}^N(xN^\alpha) - pxN^\alpha\right).$$

Thus by replacing r with $\lceil r \rceil$, we can assume that $r \in \mathbb{Z}$, which we do in the sequel. Notice that by our assumption that $\mathfrak{L}^N$ is (α, p, λ)-good we know that (5.4.12) holds trivially if $k = 2$ (with the right side of (5.4.12) being any number greater than $\epsilon/16$ and in particular ϵ) and so in the sequel we assume that $k \ge 3$.

Define constants

$$C = \sqrt{8p(1-p) \log \frac{3}{1 - (11/12)^{1/(k-2)}}}, \tag{5.4.13}$$

and $R_0 > r$ sufficiently large so that for $R \ge R_0$ and $N \in \mathbb{N}$ we have

$$\frac{\lambda(R-r)^2}{4} \ge 2\phi(\epsilon/16) + 2 + k\lceil C\lceil RN^\alpha \rceil - \lfloor rN^\alpha \rfloor \rceil N^{-\alpha/2}. \tag{5.4.14}$$

We define $R = \lceil R_0 \rceil + \mathbf{1}_{\lceil R_0 \rceil + r \text{ odd}}$, so that $R \ge R_0$ and the midpoint $(R+r)/2$ are integers. This specifies our choice of R and for convenience we denote $m = (R+r)/2$.

In the following, we always assume N is large enough so that $\psi(N) > R$, hence L_i^N are defined at RN^α for $1 \le i \le k$. We may do so by the second condition in the definition of an (α, p, λ)-good sequence (see Definition 5.1.24).

With the choice of R as above we define the events

$$\begin{aligned} A &= \left\{L_1^N(mN^\alpha) - pmN^\alpha + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right\}, \\ B &= \left\{\sup_{x \in [r,R]} \left(L_{k-1}^N(xN^\alpha) - pxN^\alpha\right) \le -[\lambda R^2 + \phi(\epsilon/16) + 1]N^{\alpha/2}\right\}. \end{aligned} \tag{5.4.15}$$

The goal of the lemma is to prove that we can find $N_5 \in \mathbb{N}$ so that for all $N \ge N_5$

$$\mathbb{P}(B) < \epsilon, \tag{5.4.16}$$

which we accomplish in the steps below.

Step 2. In this step we introduce some notation that will be used throughout the next steps. Let $\gamma = \lfloor rN^{\alpha} \rfloor$ and $\Gamma = \lceil RN^{\alpha} \rceil$. We also define the event

$$F = \left\{ \sup_{s \in \{\gamma, \Gamma\}} \left| L_1^N(s) - ps + \lambda s^2 N^{-\alpha/2} \right| < [\phi(\epsilon/16) + 1] N^{\alpha/2} \right\}. \tag{5.4.17}$$

In the remainder of this step we show that $F \cap B$ can be written as a *countable disjoint union*

$$F \cap B = \bigsqcup_{(\vec{x}, \vec{y}, \ell_{bot}) \in D} E(\vec{x}, \vec{y}, \ell_{bot}), \tag{5.4.18}$$

where the sets $E(\vec{x}, \vec{y}, \ell_{bot})$ and D are defined below.

For $\vec{x}, \vec{y} \in \mathfrak{W}_{k-2}$, $z_1, z_2 \in \mathbb{Z}$, and $\ell_{bot} \in \Omega(\gamma, \Gamma, z_1, z_2)$, let $E(\vec{x}, \vec{y}, \ell_{bot})$ denote the event that $L_i^N(\gamma) = x_i$ and $L_i^N(\Gamma) = y_i$ for $1 \leq i \leq k-2$, and L_{k-1}^N agrees with ℓ_{bot} on $[\gamma, \Gamma]$. Let D denote the set of triples $(\vec{x}, \vec{y}, \ell_{bot})$ satisfying

(1) $0 \leq y_i - x_i \leq \Gamma - \gamma$ for $1 \leq i \leq k-2$,

(2) $|x_1 - p\gamma + \lambda \gamma^2 N^{-3\alpha/2}| < \phi(\epsilon/16) N^{\alpha/2}$ and $|y_1 - p\Gamma + \lambda \Gamma^2 N^{-3\alpha/2}| < \phi(\epsilon/16) N^{\alpha/2}$,

(3) $z_1 \leq x_{k-2}$, $z_2 \leq y_{k-2}$, and $\ell_{bot} \in \Omega(\gamma, \Gamma, z_1, z_2)$,

(4) $\sup_{x \in [r,R]} [\ell_{bot}(xN^{\alpha}) - pxN^{\alpha}] \leq -[\lambda R^2 + \phi(\epsilon/16) + 1] N^{\alpha/2}$.

It is clear that D is countable, the events $E(\vec{x}, \vec{y}, \ell_{bot})$ are pairwise disjoint for different elements in D and (5.4.18) is satisfied.

Step 3. We claim that we can find $\tilde{N}_0$ so that for $N \geq \tilde{N}_0$ we have

$$\mathbb{P}(A | E(\vec{x}, \vec{y}, \ell_{bot})) \geq 1/4 \tag{5.4.19}$$

for all $(\vec{x}, \vec{y}, \ell_{bot}) \in D$ such that $\mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot})) > 0$. We will prove (5.4.19) in the steps below. In this step we assume its validity and conclude the proof of (5.4.16).

It follows from (5.4.18) and (5.4.19) that for $N \geq \tilde{N}_0$ and $\mathbb{P}(F \cap B) > 0$ we have

$$\mathbb{P}(A|F \cap B) = \sum_{(\vec{x},\vec{y},\ell_{bot}) \in D, \mathbb{P}(E(\vec{x},\vec{y},\ell_{bot}))) > 0} \frac{\mathbb{P}(A|E(\vec{x}, \vec{y}, \ell_{bot})\mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot}))}{\mathbb{P}(F \cap B)} \geq$$
$$\frac{1}{4} \cdot \frac{\sum_{(\vec{x},\vec{y},\ell_{bot}) \in D, \mathbb{P}(E(\vec{x},\vec{y},\ell_{bot}))) > 0} \mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot}))}{\mathbb{P}(F \cap B)} = \frac{1}{4}.$$

From the third condition in the definition of an (α, p, λ)-good sequence, see Definition 5.1.24, we can find $\tilde{N}_1$ so that $\mathbb{P}(A) < \epsilon/8$ for $N \geq \tilde{N}_1$. Hence if $N \geq \max(\tilde{N}_1, \tilde{N}_2)$ and $\mathbb{P}(F \cap B) > 0$ we have

$$\mathbb{P}(F \cap B) = \frac{\mathbb{P}(A \cap F \cap B)}{\mathbb{P}(A|F \cap B)} \leq 4\mathbb{P}(A) < \epsilon/2. \tag{5.4.20}$$

Lastly, by the same condition in Definition 5.1.24 we can find $\tilde{N}_2$ so that for $N \geq \tilde{N}_2$ we have

$$\mathbb{P}(F^c) = 2 \cdot \epsilon/8 = \epsilon/4. \tag{5.4.21}$$

In deriving (5.4.21) we used the fact that $|L_1^N(\gamma) - L_1^N(rN^\alpha)| \leq 1$, $|L_1^N(\Gamma) - L_1^N(RN^\alpha)| \leq 1$ and $p \in [0, 1]$. Combining (5.4.20) and (5.4.21) we conclude that if $N \geq N_5 = \max(\tilde{N}_0, \tilde{N}_1, \tilde{N}_2)$

$$\mathbb{P}(B) \leq \mathbb{P}(F \cap B) + \mathbb{P}(F^c) \leq \epsilon/2 + \epsilon/4 < \epsilon,$$

which proves (5.4.16).

Step 4. In this step we prove (5.4.19). We define $\vec{x}\,', \vec{y}\,' \in \mathfrak{W}_{k-2}$ through

$$x_i' = \overline{x} + (k-1-i)\lceil C\sqrt{T}\,\rceil, \quad y_i' = \overline{y} + (k-1-i)\lceil C\sqrt{T}\,\rceil \text{ for } i = 1, \dots, k-2 \text{ with}$$
$$\overline{x} = \lceil p\gamma - \lambda\gamma^2 N^{-3\alpha/2} + [\phi(\epsilon/16)+1]N^{\alpha/2}\rceil,\ \overline{y} = \lceil p\Gamma - \lambda\Gamma^2 N^{-3\alpha/2} + [\phi(\epsilon/16)+1]N^{\alpha/2}\rceil, \tag{5.4.22}$$

where C is as in (5.4.13) and $T = \Gamma - \gamma$. Note that for any $(\vec{x}, \vec{y}, \ell_{bot}) \in D$ we have

$$x_i' \geq \overline{x} \geq x_1 \geq x_i \text{ and } y_i' \geq \overline{y} \geq y_1 \geq y_i$$

for each $i = 1, \dots, k-2$. Furthermore,

$$x_i' - x_{i+1}' \geq C\sqrt{T} \text{ and } y_i' - y_{i+1}' \geq C\sqrt{T}$$

for all $i = 1, \dots, k-2$ with the convention $x_{k-1}' = \overline{x}$ and $y_{k-1}' = \overline{y}$.

We claim that we can find $\tilde{N}_0$ so that for all $N \geq \tilde{N}_0$ and $(\vec{x}, \vec{y}, \ell_{bot}) \in D$ such that $\mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot})) > 0$ we have $\prod_{i=1}^{k-2} |\Omega(\gamma, \Gamma, x_i', y_i')| \geq |\Omega_{avoid}(\gamma, \Gamma, \vec{x}', \vec{y}', \infty, \ell_{bot})| \geq 1$ and moreover we have

$$\mathbb{P}_{Ber}^{\gamma,\Gamma,\vec{x}',\vec{y}'}\left(Q_1(mN^\alpha) - pmN^\alpha + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) \geq 1/3, \tag{5.4.23}$$

$$\mathbb{P}_{Ber}^{\gamma,\Gamma,\vec{x}',\vec{y}'}(Q_1 \geq \cdots \geq Q_{k-1}) \geq 11/12, \tag{5.4.24}$$

where $\mathfrak{Q} = (Q_1, \dots, Q_{k-2})$ is $\mathbb{P}_{Ber}^{\gamma,\Gamma,\vec{x}',\vec{y}'}$-distributed and we used the convention that $Q_{k-1} = \ell_{bot}$. We prove (5.4.23) and (5.4.24) in the steps below. In this step we assume their validity and conclude the proof of (5.4.19).

Observe that for any $(\vec{x}, \vec{y}, \ell_{bot}) \in D$ such that $\mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot})) > 0$ we the following tower of inequalities provided that $N \geq \tilde{N}_0$

$$\mathbb{P}(A|E(\vec{x},\vec{y},\ell_{bot})) = \mathbb{P}^{\gamma,\Gamma,\vec{x},\vec{y},\infty,\ell_{bot}}_{avoid,Ber}\left(Q_1\left(mN^{\alpha}\right) - pmN^{\alpha} + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) \geq \mathbb{P}^{\gamma,\Gamma,\vec{x}',\vec{y}',\infty,\ell_{bot}}_{avoid,Ber}\left(Q_1\left(mN^{\alpha}\right) - pmN^{\alpha} + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) = \frac{\mathbb{P}^{\gamma,\Gamma,\vec{x}',\vec{y}'}_{Ber}\left(\{Q_1\left(mN^{\alpha}\right) - pmN^{\alpha} + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\} \cap \{Q_1 \geq \cdots \geq Q_{k-1}\}\right)}{\mathbb{P}^{\gamma,\Gamma,\vec{x}',\vec{y}'}_{Ber}\left(Q_1 \geq \cdots \geq Q_{k-1}\right)}. \tag{5.4.25}$$

Let us elaborate on (5.4.25) briefly. The condition that $\mathbb{P}(E(\vec{x},\vec{y},\ell_{bot})) > 0$ is required to ensure that the probabilities on the first line of (5.4.25) are well-defined and $N \geq \tilde{N}_0$ ensures that all other probabilities are also well-defined. The equality on the first line of (5.4.25) follows from the definition of A and the Schur Gibbs property, see Definition 5.1.17, and $\mathfrak{Q} = (Q_1,\dots,Q_{k-2})$ is $\mathbb{P}^{\gamma,\Gamma,\vec{x},\vec{y},\infty,\ell_{bot}}_{avoid,Ber}$-distributed. The inequality in the first line of (5.4.25) follows from Lemma 5.2.1, while the equality in the second line follows from Definition 5.1.15, and now $\mathfrak{Q} = (Q_1,\dots,Q_{k-2})$ is $\mathbb{P}^{\gamma,\Gamma,\vec{x}',\vec{y}'}_{Ber}$-distributed with the convention that $Q_{k-1} = \ell_{bot}$.

Combining (5.4.23), (5.4.24) and (5.4.25) we conclude that

$$\mathbb{P}(A|E(\vec{x},\vec{y},\ell_{bot})) \geq 1/3 - 1/12 = 1/4,$$

which proves (5.4.19).

Step 5. In this step we prove (5.4.23). We observe that since $\mathbb{P}(E(\vec{x},\vec{y},\ell_{bot})) > 0$ we know that $|\Omega_{avoid}(\gamma,\Gamma,\vec{x},\vec{y},\infty,\ell_{bot})| \geq 1$ and then we conclude from Lemma 5.1.16 that there exist $\widehat{N}_1 \in \mathbb{N}$ such that for $N \geq \widehat{N}_1$ we have $|\Omega_{avoid}(\gamma,\Gamma,\vec{x}',\vec{y}',\infty,\ell_{bot})| \geq 1$.

Below ℓ will be used for a generic random variable with law $\mathbb{P}^{\cdot,\cdot,\cdot,\cdot}_{Ber}$, where the boundary data changes from line to line. With $\overline{x}, \overline{y}$ as in (5.4.22), write $\overline{z} = \overline{y} - \overline{x}$ and recall that $T = \Gamma - \gamma$. Then

$$\begin{aligned}
&\mathbb{P}_{Ber}^{\gamma,\Gamma,x_1',y_1'}\left(\ell\left(mN^{\alpha}\right) - pmN^{\alpha} + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) = \\
&\mathbb{P}_{Ber}^{0,T,x_1',y_1'}\left(\ell(T/2) - pmN^{\alpha} + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) = \\
&\mathbb{P}_{Ber}^{0,T,\overline{x},\overline{y}}\left(\ell(T/2) - pmN^{\alpha} + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2} - (k-2)\lceil C\sqrt{T}\rceil\right) \geq \\
&\mathbb{P}_{Ber}^{0,T,\overline{x},\overline{y}}\left(\ell(T/2) - \frac{\overline{x}+\overline{y}}{2} < \lambda\left(\frac{\gamma^2+\Gamma^2}{2N^{3\alpha/2}}\right) - [2\phi(\epsilon/16) + 1 + \lambda m^2]N^{\alpha/2} - k\lceil C\sqrt{T}\rceil\right) = \\
&\mathbb{P}_{Ber}^{0,T,0,\overline{z}}\left(\ell(T/2) - \overline{z}/2 < \lambda\left(\frac{\gamma^2+\Gamma^2}{2N^{3\alpha/2}}\right) - [2\phi(\epsilon/16) + 1 + \lambda m^2]N^{\alpha/2} - k\lceil C\sqrt{T}\rceil\right).
\end{aligned} \tag{5.4.26}$$

The equalities in (5.4.26) follow from shifting the boundary data of the curve ℓ, while the inequality on the third line follows from the definition of $\overline{x}, \overline{y}$ as in (5.4.22).

From our choice of R in Step 1 and the definition of γ, Γ we know that

$$\lambda\frac{\gamma^2+\Gamma^2}{2N^{2\alpha}} - \lambda m^2 \geq \lambda\frac{(R-r)^2}{4} - \frac{r\lambda}{N^{\alpha}} \geq 2\phi(\epsilon/16) + 2 + k\lceil C\sqrt{T}\rceil N^{-\alpha/2} - \frac{r\lambda}{N^{\alpha}}.$$

.

The last inequality and (5.4.26) imply

$$\begin{aligned}
&\mathbb{P}_{Ber}^{\gamma,\Gamma,x_1',y_1'}\left(\ell\left(mN^{\alpha}\right) - pmN^{\alpha} + \lambda m^2 N^{\alpha/2} < -\phi(\epsilon/16)N^{\alpha/2}\right) \geq \\
&\mathbb{P}_{Ber}^{0,T,0,\overline{z}}\left(\ell(T/2) - \overline{z}/2 < N^{\alpha/2} - r\lambda N^{-\alpha/2}\right).
\end{aligned} \tag{5.4.27}$$

Let $\tilde{\mathbb{P}}$ be the probability measure on the space afforded by Theorem 5.2.3, supporting a random variable $\ell^{(T,\overline{z})}$ with law $\mathbb{P}_{Ber}^{0,T,0,\overline{z}}$ and a Brownian bridge B^{σ} with variance $\sigma^2 = p(1-p)$. Then the probability in the last line of (5.4.26) is equal to

$$\begin{aligned}
&\mathbb{P}_{Ber}^{0,T,0,\overline{z}}\left(\ell(T/2) - \overline{z}/2 < N^{\alpha/2} - r\lambda N^{-\alpha/2}\right) = \tilde{\mathbb{P}}\left(\ell^{(T,\overline{z})}(T/2) - \overline{z}/2 < N^{\alpha/2} - r\lambda N^{-\alpha/2}\right) \geq \\
&\tilde{\mathbb{P}}\left(\sqrt{T}B_{1/2}^{\sigma} < 0 \text{ and } \Delta(T,\overline{z}) < N^{\alpha/2} - r\lambda N^{-\alpha/2}\right) \geq \frac{1}{2} - \tilde{\mathbb{P}}\left(\Delta(T,\overline{z}) \geq N^{\alpha/2} - r\lambda N^{-\alpha/2}\right),
\end{aligned} \tag{5.4.28}$$

where we recall that $\Delta(T, \overline{z})$ is as in (5.2.2). Since as $N \to \infty$ we have

$$T \sim (R - r)N^{\alpha} \text{ and } \frac{|\overline{z} - pT|^2}{T} \sim (R + r),$$

we conclude from Lemma 5.2.5 that there exists $\widehat{N}_2 \in \mathbb{N}$ such that if $N \geq \max(\widehat{N}_1, \widehat{N}_2)$ we have

$$\tilde{\mathbb{P}}\left(\Delta(T, \overline{z}) \geq N^{\alpha/2} - r\lambda N^{-\alpha/2}\right) \leq \frac{1}{6}. \tag{5.4.29}$$

Combining (5.4.27), (5.4.28) and (5.4.29) we obtain (5.4.23).

Step 6. In this last step, we prove (5.4.24). Let $\overline{\ell}_{bot}$ be the straight segment connecting $\overline{x}$ and $\overline{y}$, defined in (5.4.22). By construction, we have that there is $\widehat{N}_3 \in \mathbb{N}$ such that if $N \geq \widehat{N}_3$ we have for any $(\vec{x}, \vec{y}, \ell_{bot}) \in D$ that ℓ_{bot} lies uniformly below the line segment $\overline{\ell}_{bot}$, which in turn lies at least $C\sqrt{T}$ below the straight segment connecting x'_{k-2} and y'_{k-2}. If $\widehat{N}_1$ is as in Step 5 we conclude from Lemma 5.2.14 that there exists $\widehat{N}_4 \in \mathbb{N}$ such that if $N \geq \max(\widehat{N}_1, \widehat{N}_3, \widehat{N}_4)$ and $\mathbb{P}(E(\vec{x}, \vec{y}, \ell_{bot})) > 0$

$$\mathbb{P}_{Ber}^{\gamma,\Gamma,\vec{x}',\vec{y}'}(Q_1 \geq \cdots \geq Q_{k-1}) \geq \left(1 - 3e^{-C^2/8p(1-p)}\right)^{k-2} = \frac{11}{12}. \tag{5.4.30}$$

where the condition that $N \geq \widehat{N}_1$ is included to ensure that the probability $\mathbb{P}_{Ber}^{\gamma,\Gamma,\vec{x}',\vec{y}'}$ is well-defined. In deriving (5.4.30) we also used (5.4.13), which implies

$$C = \sqrt{8p(1-p)\log\frac{3}{1 - (11/12)^{1/(k-2)}}} \geq \sqrt{8p(1-p)\log 3}.$$

We see that (5.4.30) implies (5.4.24), which concludes the proof of the lemma. □

In the remainder of this section we use Lemma 5.4.1 to prove Lemma 5.3.3.

Proof. (Lemma 5.3.3) For clarity we split the proof into five steps.

Step 1. In this step we specify the choice of R_2 in the statement of the lemma and introduce some notation that will be used in the proof of the lemma, which is given in Steps 2,3 and 4 below. Throughout we fix $r, \epsilon > 0$. Define the constant

$$C_1 = \sqrt{16p(1-p)\log\frac{3}{1-2^{-1/(k-1)}}}. \tag{5.4.31}$$

Let $R > r+3$, $M > 0$ and $\tilde{N}_1 \in \mathbb{N}$ be such that for $N \geq \tilde{N}_1$ we have that the event

$$B = \left\{ \sup_{x \in [r+3,R] \cup [-R,-r-3]} \left(L_{k-1}^N (xN^\alpha) - pxN^\alpha \right) \geq -MN^{\alpha/2} \right\} \tag{5.4.32}$$

satisfies

$$\mathbb{P}(B) \geq 1 - \epsilon/2. \tag{5.4.33}$$

Such a choice of $R, M, \tilde{N}_1$ is possible by Lemma 5.4.1.

Let us set

$$s_1^- = \lceil -R \cdot N^\alpha \rceil, \;\; s_2^- = \lfloor -(r+3) \cdot N^\alpha \rfloor, \;\; s_1^+ = \lceil (r+3) \cdot N^\alpha \rceil, \;\; s_2^+ = \lfloor R \cdot N^\alpha \rfloor,$$

and for $a \in [\![s_1^-, s_2^-]\!]$ and $b \in [\![s_1^+, s_2^+]\!]$ we define $\vec{x}\,', \vec{y}\,' \in \mathfrak{W}_{k-1}$ by

$$\begin{aligned} x_i' &= \lfloor pa - MN^{\alpha/2} \rfloor - (i-1)\lceil C_1 N^{\alpha/2} \rceil, \\ y_i' &= \lfloor pb - MN^{\alpha/2} \rfloor - (i-1)\lceil C_1 N^{\alpha/2} \rceil, \end{aligned} \tag{5.4.34}$$

for $i = 1, \dots, k-1$. We will write $\vec{z} = \vec{y}' - \vec{x}'$, and we note that $z_{k-1} \geq p(b-a) - 1$ and also $2RN^\alpha \geq b - a \geq 2(r+3)N^\alpha$. The latter and Lemma 5.2.10 imply that there exists $R_2 > 0$ and $\tilde{N}_2 \in \mathbb{N}$ such that if $N \geq \tilde{N}_2$ we have

$$\mathbb{P}_{Ber}^{0,b-a,0,z_{k-1}} \left(\inf_{s \in [0,b-a]} \big(\ell(s) - ps \big) \leq -(R_2 - M - C_1 k)N^{\alpha/2} \right) < \epsilon/4. \tag{5.4.35}$$

This fixes our choice of R_2 in the statement of the lemma.

With the above choice of R_2 we define the event

$$A = \left\{ \inf_{s \in [-t_3, t_3]} \left[L_{k-1}^N(s) - ps \right] \leq -R_2 N^{\alpha/2} \right\}, \tag{5.4.36}$$

and then to prove the lemma it suffices to show that there exists $N_4 \in \mathbb{N}$ such that for $N \geq N_4$

$$\mathbb{P}(A) < \epsilon \tag{5.4.37}$$

Step 2. In this step, we prove that the event B from (5.4.32) can be written as a *countable disjoint union* of the form

$$B = \bigsqcup_{(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+) \in D} E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+), \tag{5.4.38}$$

where the set D and events $E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)$ are defined below.

For $a \in [\![s_1^-, s_2^-]\!]$ and $b \in [\![s_1^+, s_2^+]\!]$, $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$, $z_1, z_2, z_1^-, z_2^+ \in \mathbb{Z}$, $\ell_{bot} \in \Omega(a,b,z_1,z_2)$, $\ell_{top}^- \in \Omega(s_1^-, a, z_1^-, x_{k-1})$, $\ell_{top}^+ \in \Omega(b, s_2^+, y_{k-1}, z_2^+)$ we define $E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)$ to be the event that $L_i^N(a) = x_i$ and $L_i^N(b) = y_i$ for $1 \leq i \leq k-1$, and L_k^N agrees with ℓ_{bot} on $[\![a,b]\!]$, L_{k-1}^N agrees with ℓ_{top}^- on $[\![s_1^-, a]\!]$ and with ℓ_{top}^+ on $[\![b, s_2^+]\!]$.

We also let D be the collection of tuples $(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^-,\ell_{top}^+)$ satisfying:

(1) $a \in [\![s_1^-, s_2^-]\!]$, $b \in [\![s_1^+, s_2^+]\!]$;

(2) $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$, $0 \leq y_i - x_i \leq b - a$, $x_{k-1} - pa > -MN^{\alpha/2}$, and $y_{k-1} - pb > -MN^{\alpha/2}$;

(3) if $c \in [\![s_1^-, s_2^-]\!] \cap (-\infty, a)$ then $\ell_{top}^-(c) - pc \leq -MN^{\alpha/2}$;

(4) if $d \in [\![s_1^+, s_2^+]\!] \cap (b, \infty)$ then $\ell_{top}^+(d) - pd \geq -MN^{\alpha/2}$;

(5) $z_1 \leq x_{k-1}$, $z_2 \leq y_{k-1}$, and $\ell_{bot} \in \Omega(a,b,z_1,z_2)$.

It is clear that D is countable, and that

$$B = \bigcup_{(a,b,\vec{x},\vec{y},\ell_{bot}) \in D} E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+}),$$

so to prove (5.4.38) it suffices to show that the events $E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+})$ are pairwise disjoint. Observe that on the intersection of $E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+})$ and $E(\tilde{a},\tilde{b},\tilde{\vec{x}},\tilde{\vec{y}},\tilde{\ell}_{bot},\tilde{\ell}_{top}^{-},\tilde{\ell}_{top}^{+})$, conditions (2) and (3) imply that $a = \tilde{a}$, while conditions (2) and (4) that $b = \tilde{b}$. Afterwards, we conclude that $\vec{x} = \tilde{\vec{x}}$, $\vec{y} = \tilde{\vec{y}}$, $\ell_{bot} = \tilde{\ell}_{bot}$, $\ell_{top}^{-} = \tilde{\ell}_{top}^{-}$ and $\ell_{top}^{+} = \tilde{\ell}_{top}^{+}$, confirming (5.4.38).

Step 3. In this step we prove (5.4.37). We claim that we can find $\tilde{N}_3 \in \mathbb{N}$ such that if $N \geq \tilde{N}_3$ and $(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+}) \in D$ is such that $\mathbb{P}\left(E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+})\right) > 0$ we have

$$\mathbb{P}(A \mid E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+})) < \epsilon/2. \tag{5.4.39}$$

We will prove (5.4.39) in the steps below. Here we assume its validity and conclude the proof of (5.4.37).

If $N \geq \max(\tilde{N}_1, \tilde{N}_2, \tilde{N}_3)$ we have in view of (5.4.38) and (5.4.39) that

$$\mathbb{P}(A) \leq \mathbb{P}(A \cap B) + \mathbb{P}(B^c) = \mathbb{P}(B^c) + \sum_{\substack{(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+}) \in D \\ \mathbb{P}\left(E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+})\right) > 0}} \mathbb{P}(A | E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+})) \times$$

$$\mathbb{P}(E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+})) \leq \mathbb{P}(B^c) + \frac{\epsilon}{2} \sum_{\substack{(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+}) \in D \\ \mathbb{P}\left(E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+})\right) > 0}} \mathbb{P}(E(a,b,\vec{x},\vec{y},\ell_{bot},\ell_{top}^{-},\ell_{top}^{+})) =$$

$$\mathbb{P}(B^c) + \frac{\epsilon}{2} \cdot \mathbb{P}(B) < \epsilon,$$

where in the last inequality we used (5.4.33). The above inequality clearly implies (5.4.37).

Step 4. In this step we prove (5.4.39). We claim that there exists $\tilde{N}_4 \in \mathbb{N}$ such that if $N \geq \tilde{N}_4$,

$a \in [\![s_1^-, s_2^-]\!]$ and $b \in [\![s_1^+, s_2^+]\!]$ we have that $\prod_{i=1}^{k-1} |\Omega(a, b, x_i', y_i')| \geq 1$ and

$$\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}(Q_1 \geq \cdots \geq Q_{k-1}) \geq \frac{1}{2}, \tag{5.4.40}$$

where $\mathfrak{Q} = (Q_1, \ldots, Q_{k-1})$ is $\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}$-distributed and we recall that $\vec{x}'$, $\vec{y}'$ were defined in (5.4.34). We will prove (5.4.40) in Step 5 below. Here we assume its validity and conclude the proof of (5.4.39).

Observe that by condition (2) in Step 2, we have that $x_i' \leq pa - MN^{\alpha/2} \leq x_{k-1} \leq x_i$, and similarly $y_i' \leq pb - MN^{\alpha/2} \leq y_{k-1} \leq y_i$ for $i = 1, \ldots, k-1$. From this observation we conclude that if $N \geq \tilde{N}_4$ is sufficiently large and $(a, b, \vec{x}, \vec{y}, \ell_{bot}, \ell_{top}^-, \ell_{top}^+) \in D$ is such that $\mathbb{P}\left(E(a, b, \vec{x}, \vec{y}, \ell_{bot}, \ell_{top}^-, \ell_{top}^+)\right) > 0$ we have

$$\begin{aligned}
&\mathbb{P}(A | E(a, b, \vec{x}, \vec{y}, \ell_{bot}, \ell_{top}^-, \ell_{top}^+)) \leq \\
&\mathbb{P}\left(\inf_{s \in [a,b]} \left(L_{k-1}^N(s) - ps\right) \leq -R_2 N^{\alpha/2} \,\Big|\, E(a, b, \vec{x}, \vec{y}, \ell_{bot}, \ell_{top}^-, \ell_{top}^+)\right) = \\
&\mathbb{P}_{avoid,Ber}^{a,b,\vec{x},\vec{y},\infty,\ell_{bot}}\left(\inf_{s \in [a,b]} \left(Q_{k-1}(s) - ps\right) \leq -R_2 N^{\alpha/2}\right) \leq \\
&\mathbb{P}_{avoid,Ber}^{a,b,\vec{x}',\vec{y}'}\left(\inf_{s \in [a,b]} \left(Q_{k-1}(s) - ps\right) \leq -R_2 N^{\alpha/2}\right) = \\
&\frac{\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}\left(\{\inf_{s \in [a,b]} \left(Q_{k-1}(s) - ps\right) \leq -R_2 N^{\alpha/2}\} \cap \{Q_1 \geq \cdots \geq Q_{k-1}\}\right)}{\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}(Q_1 \geq \cdots \geq Q_{k-1})} \leq \\
&\frac{\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}\left(\inf_{s \in [a,b]} \left(Q_{k-1}(s) - ps\right) \leq -R_2 N^{\alpha/2}\right)}{\mathbb{P}_{Ber}^{a,b,\vec{x}',\vec{y}'}(Q_1 \geq \cdots \geq Q_{k-1})}.
\end{aligned} \tag{5.4.41}$$

Let us elaborate on (5.4.41) briefly. The first inequality in (5.4.41) follows from the definition of A and the fact that $a \leq -t_3$ while $b \geq t_3$ by construction. The condition $\mathbb{P}\left(E(a, b, \vec{x}, \vec{y}, \ell_{bot}, \ell_{top}^-, \ell_{top}^+)\right) > 0$ ensures that the first three probabilities in (5.4.41) are all well-defined. The equality on the second line follows from the Schur Gibbs property and the inequality on the third line follows from Lemmas 5.2.1 and 5.2.2 since $x_i' \leq x_i$ and $y_i' \leq y_i$ by construction. To ensure that the probability in the fourth line is well-defined (and hence Lemmas 5.2.1 and 5.2.2 are applicable) it suffices to assume that $N \geq \tilde{N}_4$, in view of Lemma 5.1.16. The equality on the fourth line follows from the definition of $\mathbb{P}^{a,b,\vec{x}',\vec{y}'}_{avoid,Ber}$, see Definition 5.1.15 and the last inequality is trivial.

By our choice of R_2, see (5.4.35), we know that there is $\tilde{N}_5 \in \mathbb{N}$ such that if $N \geq \tilde{N}_5$

$$\begin{aligned}
&\mathbb{P}^{a,b,\vec{x}',\vec{y}'}_{Ber}\left(\inf_{s\in[a,b]}\big(Q_{k-1}(s) - ps\big) \leq -R_2 N^{\alpha/2}\right) = \\
&\mathbb{P}^{0,b-a,0,z_{k-1}}_{Ber}\left(\inf_{s\in[0,b-a]}\big(\ell(s) - ps\big) \leq -R_2 N^{\alpha/2} - x'_{k-1}\right) \leq \\
&\mathbb{P}^{0,b-a,0,z_{k-1}}_{Ber}\left(\inf_{s\in[0,b-a]}\big(\ell(s) - ps\big) \leq -(R_2 - M - C_1 k) N^{\alpha/2}\right) < \epsilon/4.
\end{aligned} \tag{5.4.42}$$

Combining (5.4.40), (5.4.41) and (5.4.41) we conclude that for $N \geq \tilde{N}_3 = \max(\tilde{N}_4, \tilde{N}_5)$ we have

$$\mathbb{P}(A | E(a, b, \vec{x}, \vec{y}, \ell_{bot}, \ell^-_{top}, \ell^+_{top})) < 2 \cdot \epsilon/4 = \epsilon/2,$$

which implies (5.4.39).

Step 5. In this final step we prove (5.4.40).

Lastly, we prove that we can enlarge $\tilde{N}_1$ so that (5.4.45) holds for $N \geq \tilde{N}_1$. Write $a = a'N^\alpha$, $b = b'N^\alpha$, and $T = a + b = (a' + b')N^\alpha$. Also let $C' = C/\sqrt{a' + b'}$ with C as in (5.4.31), so that $x'_i - x'_{i+1} \geq CN^{\alpha/2} = C'\sqrt{T}$ and likewise for y'_i. Note that $|z_{k-1} - pT| \leq 1$. It follows from Lemma 5.2.14, applied with $\ell_{bot} = -\infty$ and C' in place of C, that for T larger than some T_0,

$$\begin{aligned}
&\mathbb{P}^{-a,b,\vec{x}\,',\vec{y}\,'}_{Ber}(L_1 \geq \cdots \geq L_{k-1}) = \mathbb{P}^{0,a+b,\vec{x}\,',\vec{y}\,'}_{Ber}(L_1 \geq \cdots \geq L_{k-1}) \geq \\
&\left(1 - 3e^{-(C')^2/8p(1-p)}\right)^{k-1} \geq \left(1 - 3e^{-C^2/16p(1-p)R}\right)^{k-1}.
\end{aligned} \tag{5.4.43}$$

Here, we used the fact that $a' + b' \leq 2R$, hence $C' \geq C/\sqrt{2R}$. The constant T_0 depends in particular on C', hence possibly on $a + b$. Referring to the proofs of Lemmas 5.2.14 and 5.2.5, we see that the dependency of T_0 on C' amounts to requiring that $e^{-C'\sqrt{T_0}}$ be sufficiently small. But $C' \geq C/\sqrt{2R}$, so for this it suffices to choose T_0 depending on C and R. Moreover, $T \geq 2rN^{\alpha}$, so as long as $\tilde{N}_1 \geq (T_0/2r)^{1/\alpha}$, we have the bound in (5.4.43) for $N \geq \tilde{N}_1$ independent $a, b, \vec{x}, \vec{y}$. Our choice of C in (5.4.31) ensures that the expression on the right in (5.4.43) is at least $1/2$, proving (5.4.45).

In this step, we fix $R_2 > 0$ and $\tilde{N}_1$ so that for $N \geq \tilde{N}_1$, we have

$$\mathbb{P}_{Ber}^{-a,b,\vec{x}\,',\vec{y}\,'}\left(\inf_{s\in[0,a+b]}\left(L_{k-1}(s) - ps\right) \leq -R_2 N^{\alpha/2}\right) < \epsilon/4, \tag{5.4.44}$$

$$\mathbb{P}_{Ber}^{-a,b,\vec{x}\,',\vec{y}\,'}(L_1 \geq \cdots \geq L_{k-1}) \geq 1/2. \tag{5.4.45}$$

Let us first prove (5.4.44). Writing $\vec{z} = \vec{y}\,' - \vec{x}\,'$, and using the fact that $L_1, \ldots, L_{k-1}$ are independent under $\mathbb{P}_{Ber}^{-a,b,\vec{x}\,',\vec{y}\,'}$, we can rewrite the left hand side of (5.4.44) as

$$\begin{aligned}
&\mathbb{P}_{Ber}^{0,a+b,x'_{k-1},y'_{k-1}}\left(\inf_{s\in[0,a+b]}\left(\ell(s) - p(s-a)\right) \leq -R_2 N^{\alpha/2}\right) = \\
&\mathbb{P}_{Ber}^{0,a+b,0,z_{k-1}}\left(\inf_{s\in[0,a+b]}\left(\ell(s) - ps + pa - \lceil pa + MN^{\alpha/2}\rceil - (k-2)\lceil CN^{\alpha/2}\rceil\right) \leq -R_2 N^{\alpha/2}\right) \leq \\
&\mathbb{P}_{Ber}^{0,a+b,0,z_{k-1}}\left(\inf_{s\in[0,a+b]}\left(\ell(s) - ps\right) \leq -(R_2 - M - Ck)N^{\alpha/2}\right).
\end{aligned} \tag{5.4.46}$$

□

5.5 Lower bounds on the acceptance probability

5.5.1 Proof of Lemma 5.3.4

Throughout this section we assume the same notation as in Lemma 5.3.4, i.e., we assume that we have fixed $k \in \mathbb{N}$, $p \in (0,1)$, $M_1, M_2 > 0$, $\ell_{bot} : [\![-t_3, t_3]\!] \to \mathbb{R} \cup \{-\infty\}$, and $\vec{x}, \vec{y} \in \mathfrak{W}_{k-1}$ such that $|\Omega_{avoid}(-t_3, t_3, \vec{x}, \vec{y}, \infty, \ell_{bot})| \geq 1$. We also assume that

1. $\sup_{s\in[-t_3,t_3]}\left[\ell_{bot}(s) - ps\right] \leq M_2(2t_3)^{1/2}$,
2. $-pt_3 + M_1(2t_3)^{1/2} \geq x_1 \geq x_{k-1} \geq \max\left(\ell_{bot}(-t_3), -pt_3 - M_1(2t_3)^{1/2}\right)$,

3. $pt_3 + M_1(2t_3)^{1/2} \geq y_1 \geq y_{k-1} \geq \max\left(\ell_{bot}(t_3), pt_3 - M_1(2t_3)^{1/2}\right)$.

Definition 5.5.1. We write $S = [\![-t_3, -t_1]\!] \cup [\![t_1, t_3]\!]$, and we denote by $\mathfrak{Q} = (Q_1, \dots, Q_{k-1})$ and $\tilde{\mathfrak{Q}} = (\tilde{Q}_1, \dots, \tilde{Q}_{k-1})$ the $[\![1, k-1]\!]$-indexed line ensembles which are uniformly distributed on $\Omega_{avoid}(-t_3, t_3, \vec{x}, \vec{y}, \infty, \ell_{bot})$ and $\Omega_{avoid}(-t_3, t_3, \vec{x}, \vec{y}, \infty, \ell_{bot}; S)$ respectively. We let $\mathbb{P}_{\mathfrak{Q}}$ and $\mathbb{P}_{\tilde{\mathfrak{Q}}}$ denote these uniform measures.

In other words, $\tilde{\mathfrak{Q}}$ has the law of $k-1$ independent Bernoulli bridges that have been conditioned on not-crossing each other on the set S and also staying above the graph of ℓ_{bot} but only on the intervals $[\![-t_3, -t_1]\!]$ and $[\![t_1, t_3]\!]$. The latter restriction means that the lines are allowed to cross on $[\![-t_1 + 1, t_1 - 1]\!]$, and $\tilde{Q}_{k-1}$ is allowed to dip below ℓ_{bot} on $[\![-t_1 + 1, t_1 - 1]\!]$ as well.

Lemma 5.5.2. *There exists $N_5 \in \mathbb{N}$ and constants g, h such that for $N \geq N_5$ we have*

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left(Z\big(-t_1, t_1, \tilde{\mathfrak{Q}}(-t_1), \tilde{\mathfrak{Q}}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\big) \geq g\right) \geq h. \tag{5.5.1}$$

We will prove Lemma 5.5.2 in Section 5.5.2. In the remainder of this section, we give the proof of Lemma 5.3.4, with the constants g and h given by Lemma 5.5.2. The proof begins by evaluating the Radon-Nidokum derivative between ${}_{\mathfrak{Q}'}$ and ${}_{\tilde{\mathfrak{Q}}'}$. We then use this Radon-Nikodym derivative to transition between $\tilde{\mathfrak{Q}}$ in Lemma 5.5.2 which ignores ℓ_{bot} on $[\![-(t_1 - 1), t_1 - 1]\!]$ and $\mathfrak{Q}$ in Lemma 5.3.4 which avoids ℓ_{bot} everywhere.

Proof of Lemma 5.3.4. Let us denote by ${}_{\mathfrak{Q}'}$ and ${}_{\tilde{\mathfrak{Q}}'}$ the measures on $[\![1, k-1]\!]$-indexed Bernoulli line ensembles $\mathfrak{Q}'$, $\tilde{\mathfrak{Q}}'$ on the set S in Definition 5.5.1 induced by the restrictions of the measures $\mathbb{P}_{\mathfrak{Q}}$, $\mathbb{P}_{\tilde{\mathfrak{Q}}}$ to S. Also let us write $\Omega_a(\cdot)$ for $\Omega_{avoid}(\cdot)$ for simplicity, and denote by $\Omega_a(S)$ the set of elements of $\Omega_{avoid}(-t_3, t_3, \tilde{\mathfrak{Q}}(-t_3), \tilde{\mathfrak{Q}}(t_3))$ restricted to S. We claim that the Radon-Nikodym derivative between these two restricted measures is given on elements $\mathfrak{B}$ of $\Omega_a(S)$ by

$$\frac{d_{\mathfrak{Q}'}}{d_{\tilde{\mathfrak{Q}}'}}(\mathfrak{B}) = \frac{\mathbb{P}_{\mathfrak{Q}'}(\mathfrak{B})}{\mathbb{P}_{\tilde{\mathfrak{Q}}'}(\mathfrak{B})} = (Z')^{-1} Z\left(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right), \tag{5.5.2}$$

with $Z' =_{\tilde{\mathfrak{Q}}'} [Z(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!])]$. The first equality holds simply because the measures are discrete. To prove the second equality, observe that

$$
\begin{aligned}
{}_{\mathfrak{Q}'}(\mathfrak{B}) &= \frac{|\Omega_a(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!])|}{|\Omega_a(-t_3, t_3, \mathfrak{Q}(-t_3), \mathfrak{Q}(t_3), \ell_{bot})|}, \\
{}_{\tilde{\mathfrak{Q}}'}(\mathfrak{B}) &= \frac{\prod_{i=1}^{k-1} |\Omega(-t_1, t_1, B_i(-t_1), B_i(t_1))|}{|\Omega_a(-t_3, t_3, \tilde{\mathfrak{Q}}(-t_3), \tilde{\mathfrak{Q}}(t_3), \ell_{bot}; S)|}
\end{aligned}
\tag{5.5.3}
$$

These identities follow from the restriction, and the fact that the measures are uniform. Then, from Definition 5.1.22,

$$
Z(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}) = \frac{|\Omega_a(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!])|}{\prod_{i=1}^{k-1} |\Omega(-t_1, t_1, B_i(-t_1), B_i(t_1))|}
$$

and hence

$$
\begin{aligned}
Z' = \sum_{\mathfrak{B}\in\Omega_a(S)} \frac{\prod_{i=1}^{k-1} |\Omega(-t_1, t_1, B_i(-t_1), B_i(t_1))|}{|\Omega_a(-t_3, t_3, \tilde{\mathfrak{Q}}(-t_3), \tilde{\mathfrak{Q}}(t_3), \ell_{bot}; S)|} \cdot \frac{|\Omega_a(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot})|}{\prod_{i=1}^{k-1} |\Omega(-t_1, t_1, B_i(-t_1), B_i(t_1))|} = \\
\frac{\sum_{\mathfrak{B}\in\Omega_a(S)} |\Omega_a(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot})|}{|\Omega_a(-t_3, t_3, \tilde{\mathfrak{Q}}(-t_3), \tilde{\mathfrak{Q}}(t_3), \ell_{bot}; S)|} = \frac{|\Omega_a(-t_3, t_3, \mathfrak{Q}(-t_3), \mathfrak{Q}(t_3), \ell_{bot})|}{|\Omega_a(-t_3, t_3, \tilde{\mathfrak{Q}}(-t_3), \tilde{\mathfrak{Q}}(t_3), \ell_{bot}; S)|}.
\end{aligned}
$$

Comparing the above identities proves the second equality in (5.5.2).

Now note that $Z(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!])$ is a deterministic function of $((\mathfrak{B}(-t_1), \mathfrak{B}(t_1))$. In fact, the law of $((\mathfrak{B}(-t_1), \mathfrak{B}(t_1))$ under ${}_{\tilde{\mathfrak{Q}}'}$ is the same as that of $\big(\tilde{\mathfrak{Q}}(-t_1), \tilde{\mathfrak{Q}}(t_1)\big)$ by way of the restriction. It follows from Lemma 5.5.2 that

$$
\begin{aligned}
Z' &=_{\tilde{\mathfrak{Q}}'} [Z(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!])] \\
&=_{\tilde{\mathfrak{Q}}} [Z(-t_1, t_1, \mathfrak{Q}(-t_1), \mathfrak{Q}(t_1), \ell_{bot}[\![-t_1, t_1]\!])] \geq gh,
\end{aligned}
$$

which gives us

$$
(Z')^{-1} \leq \frac{1}{gh}. \tag{5.5.4}
$$

Similarly, the law of $(\mathfrak{B}(-t_1), \mathfrak{B}(t_1))$ under ${}_{\mathfrak{Q}'}$ is the same as that of $(\mathfrak{Q}(-t_1), \mathfrak{Q}(t_1))$ under ${}_{\mathfrak{Q}}$.

Hence

$$
\begin{aligned}
{}_{\mathfrak{Q}}\Big(Z(-t_1, t_1, \mathfrak{Q}(-t_1), \mathfrak{Q}(t_1), \ell_{bot}[\![-t_1, t_1]\!]) \le gh\tilde{\epsilon}\Big) = \\
{}_{\mathfrak{Q}'}\Big(Z\,(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]) \le gh\tilde{\epsilon}\Big).
\end{aligned}
\tag{5.5.5}
$$

Now let us write $E = \{Z\,(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]) \le gh\tilde{\epsilon}\} \subset \Omega_a(S)$. Then according to (5.5.2), we have

$$
{}_{\mathfrak{Q}'}(E) = \int_{\Omega_a(S)} {}_E \, d_{\mathfrak{Q}'} = (Z')^{-1} \int_{\Omega_a(S)} {}_E \cdot Z\,(-t_1, t_1, \mathfrak{B}(-t_1), \mathfrak{B}(t_1), \ell_{bot}[\![-t_1, t_1]\!]) \; d_{\tilde{\mathfrak{Q}}'}(\mathfrak{B}).
$$

From the definition of E, the inequality (5.5.4), and the fact that $\mathbf{1}_E \le 1$, it follows that

$$
{}_{\mathfrak{Q}'}(E) \le (Z')^{-1} \int_{\Omega_a(S)} {}_E \cdot gh\tilde{\epsilon}\, d_{\tilde{\mathfrak{Q}}'} \le \frac{1}{gh} \int_{\Omega_a(S)} gh\tilde{\epsilon}\, d_{\tilde{\mathfrak{Q}}'} \le \tilde{\epsilon}.
$$

In combination with (5.5.5), this proves (5.3.2).

□

5.5.2 Proof of Lemma 5.5.2

In this section, we prove Lemma 5.5.2. We first state and prove two auxiliary lemmas necessary for the proof. The first lemma establishes a set of conditions under which we have the desired lower bound on the acceptance probability.

Lemma 5.5.3. *Let $\epsilon > 0$ and $V^{top} > 0$ be given such that $V^{top} > M_2 + 6(k-1)\epsilon$. Suppose further that $\vec{a}, \vec{b} \in \mathfrak{W}_{k-1}$are such that*

1. $V^{top}(2t_3)^{1/2} \ge a_1 + pt_1 \ge a_{k-1} + pt_1 \ge (M_2 + 2\epsilon)(2t_3)^{1/2}$;
2. $V^{top}(2t_3)^{1/2} \ge b_1 - pt_1 \ge b_{k-1} - pt_1 \ge (M_2 + 2\epsilon)(2t_3)^{1/2}$;
3. $a_i - a_{i+1} \ge 3\epsilon(2t_3)^{1/2}$ *and* $b_i - b_{i+1} \ge 3\epsilon(2t_3)^{1/2}$ *for* $i = 1, \ldots, k-2$.

Then we can find $g = g(\epsilon, V^{top}, M_2) > 0$ and $N_6 \in \mathbb{N}$ such that for all $N \geq N_6$ we have

$$Z\big(- t_1, t_1, \vec{a}, \vec{b}, \ell_{bot}[\![-t_1, t_1]\!]\big) \geq g. \tag{5.5.6}$$

Proof. Observe by the rightmost inequalities in conditions (1) and (2) in the hypothesis, as well as condition (1) in Lemma 5.3.4, that ℓ_{bot} lies a distance of at least $2\epsilon(2t_3)^{1/2} \geq 2\epsilon(2t_1)^{1/2}$ uniformly below the line segment connecting a_{k-1} and b_{k-1}. Also note that (1) and (2) imply $|b_i - a_i - 2pt_1| \leq (V^{top} - M_2 - 2\epsilon)(2t_3)^{1/2}$ for each i. Lastly noting (3), we see that the conditions of Lemma 5.2.14 are satisfied with $C = 2\epsilon$. This implies (5.5.6), with

$$g = \left(\frac{1}{2} - \sum_{n=1}^{\infty} (-1)^{n-1} e^{-\epsilon^2 n^2/2p(1-p)}\right)^{k-1}.$$

□

The next lemma helps us derive the lower bound h in (5.5.1).

Lemma 5.5.4. *For any $R > 0$ we can find $V_1^t, V_1^b \geq M_2 + R$, $h_1 > 0$ and $N_7 \in \mathbb{N}$ (depending on R) such that if $N \geq N_7$ we have*

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left((2t_3)^{1/2} V_1^t \geq \tilde{Q}_1(\pm t_2) \mp pt_2 \geq \tilde{Q}_{k-1}(\pm t_2) \mp pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq h_1. \tag{5.5.7}$$

Proof. We first define the constants V_1^b and h_1, as well as two other constants C and K_1 to be used in the proof. We put

$$\begin{aligned} C &= \sqrt{8p(1-p)\log\frac{3}{1-(11/12)^{1/(k-2)}}}, \\ V_1^b &= M_1 + Ck + M_2 + R, \qquad K_1 = (4r+10)V_1^b, \\ h_1 &= \frac{2^{k/2-5}\big(1-2e^{-4/p(1-p)}\big)^{2k}}{(\pi p(1-p))^{k/2}} \exp\left(-\frac{2k(K_1+M_1+6)^2}{p(1-p)}\right). \end{aligned} \tag{5.5.8}$$

Note in particular that $V_1^b > M_2 + R$. We will fix $V_1^t > V_1^b$ in Step 3 below depending on h_1. We will prove in the following steps that for these choices of V_1^b, V_1^t, h_1, we can find N_7 so that for $N \geq N_7$ we have

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left(\tilde{Q}_{k-1}(\pm t_2) \mp p t_2 \geq (2t_3)^{1/2} V_1^b\right) \geq 2h_1, \tag{5.5.9}$$

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left(\tilde{Q}_1(\pm t_2) \mp p t_2 > (2t_3)^{1/2} V_1^t\right) \leq h_1. \tag{5.5.10}$$

Assuming the validity of the claim, we then observe that the probability in (5.5.7) is bounded below by $2h_1 - h_1 = h_1$, proving the lemma. We will prove (5.5.9) and (5.5.10) in three steps.

Step 1. In this step we prove that there exists N_7 so that (5.5.9) holds for $N \geq N_7$, assuming results from Step 2 below. We condition on the value of $\tilde{\mathfrak{Q}}$ at 0 and divide $\tilde{\mathfrak{Q}}$ into two independent line ensembles on $[-t_3, 0]$ and $[0, t_3]$. Observe by Lemma 5.2.2 that

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left(\tilde{Q}_{k-1}(\pm t_2) \mp p t_2 \geq (2t_3)^{1/2} V_1^b\right) \geq \mathbb{P}^{-t_3, t_3, \vec{x}, \vec{y}}_{avoid, Ber; S}\left(\tilde{Q}_{k-1}(\pm t_2) \mp p t_2 \geq (2t_3)^{1/2} V_1^b\right). \tag{5.5.11}$$

With K_1 as in (5.5.8), we define events

$$E_{\vec{z}} = \left\{\left(\tilde{Q}_1(0), \dots, \tilde{Q}_{k-1}(0)\right) = \vec{z}\right\}, \quad X = \left\{\vec{z} \in \mathfrak{W}_{k-1} : z_{k-1} \geq K_1 (2t_3)^{1/2} \text{ and } \mathbb{P}^{-t_3, t_3, \vec{x}, \vec{y}}_{avoid, Ber; S}(E_{\vec{z}}) > 0\right\},$$

and $E = \bigsqcup_{\vec{z} \in X} E_{\vec{z}}$. By Lemma 5.1.16, we can choose $\tilde{N}_0$ large enough depending on M_1, C, k, M_2, R so that X is non-empty for $N \geq \tilde{N}_0$. By Lemma 5.2.16 we can find $\tilde{N}_1$ so that

$$\mathbb{P}^{-t_3, t_3, \vec{x}, \vec{y}}_{avoid, Ber; S}(E) \geq \mathbb{P}^{-t_3, t_3, \vec{x}, \vec{y}}_{avoid, Ber; S}\left(\tilde{Q}_{k-1}(0) \geq K_1 (2t_3)^{1/2}\right) \geq A \exp\left(-\frac{2k(K_1 + M_1 + 6)^2}{p(1-p)}\right) \tag{5.5.12}$$

for $N \geq \tilde{N}_1$, where $A = A(p, k)$ is a constant given explicitly in (5.2.22).

Now let $\tilde{Q}_i^1$ and $\tilde{Q}_i^2$ denote the restrictions of $\tilde{Q}_i$ to $[-t_3, 0]$ and $[0, t_3]$ respectively for $1 \leq i \leq$

$k-1$, and write $S_1 = S \cap [\![-t_3, 0]\!]$, $S_2 = S \cap [\![0, t_3]\!]$. We observe that if $\vec{z} \in X$, then

$$\mathbb{P}^{-t_3,t_3,\vec{x},\vec{y}}_{avoid,Ber;S}\left(\tilde{Q}^1_{k-1} = \ell_1, \tilde{Q}^2_{k-1} = \ell_2 \mid E_{\vec{z}}\right) = \mathbb{P}^{-t_3,0,\vec{x},\vec{z}}_{avoid,Ber;S_1}(\ell_1) \cdot \mathbb{P}^{0,t_3,\vec{z},\vec{y}}_{avoid,Ber;S_2}(\ell_2). \tag{5.5.13}$$

In Step 2, we will find $\tilde{N}_2$ so that for $N \geq \tilde{N}_2$ we have

$$\begin{aligned}
&\mathbb{P}^{-t_3,0,\vec{x},\vec{z}}_{avoid,Ber;S_1}\left(\tilde{Q}^1_{k-1}(-t_2) + pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq \frac{1}{4},\\
&\mathbb{P}^{0,t_3,\vec{x},\vec{z}}_{avoid,Ber;S_2}\left(\tilde{Q}^2_{k-1}(t_2) - pt_{12} \geq (2t_3)^{1/2} V_1^b\right) \geq \frac{1}{4}.
\end{aligned} \tag{5.5.14}$$

Using (5.5.12), (5.5.13), and (5.5.14), we conclude that

$$\mathbb{P}^{-t_3,t_3,\vec{x},\vec{y}}_{avoid,Ber;S}\left(\tilde{Q}_{k-1}(\pm t_2) \mp pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq \frac{A}{16} \exp\left(-\frac{2k(K_1 + M_1 + 6)^2}{p(1-p)}\right)$$

for $N \geq N_7 = \max(\tilde{N}_0, \tilde{N}_1, \tilde{N}_2)$. In combination with (5.5.11), this proves (5.5.9) with $h_1 = A/16$ as in (5.5.8).

Step 2. In this step, we prove the inequalities in (5.5.14) from Step 1, using Lemma 5.2.8. Let us define vectors $\vec{x}\,', \vec{z}\,', \vec{y}\,'$ by

$$\begin{aligned}
x_i' &= \lfloor -pt_3 - M_1(2t_3)^{1/2} \rfloor - (i-1)\lceil C(2t_3)^{1/2} \rceil,\\
z_i' &= \lfloor K_1(2t_3)^{1/2} \rfloor - (i-1)\lceil C(2t_3)^{1/2} \rceil,\\
y_i' &= \lfloor pt_3 - M_1(2t_3)^{1/2} \rfloor - (i-1)\lceil C(2t_3)^{1/2} \rceil.
\end{aligned}$$

Note that $x_i' \leq x_{k-1} \leq x_i$ and $x_i' - x_{i+1}' \geq C(2t_3)^{1/2}$ for $1 \leq i \leq k-1$, and likewise for z_i', y_i'. By Lemma 5.2.1 we have

$$\begin{aligned}
&\mathbb{P}^{-t_3,0,\vec{x},\vec{z}}_{avoid,Ber;S_1}\left(\tilde{Q}^1_{k-1}(-t_2) + pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq \mathbb{P}^{-t_3,0,\vec{x}\,',\vec{z}\,'}_{avoid,Ber;S_1}\left(\tilde{Q}^1_{k-1}(-t_2) + pt_2 \geq (2t_3)^{1/2} V_1^b\right) \geq\\
&\mathbb{P}^{-t_3,0,x_{k-1}',z_{k-1}'}_{Ber}\left(\ell_1(-t_2) + pt_2 \geq (2t_3)^{1/2} V_1^b\right) - \left(1 - \mathbb{P}^{-t_3,t_3,\vec{x}\,',\vec{z}\,'}_{Ber}\left(\tilde{Q}^1_1 \geq \cdots \geq \tilde{Q}^1_{k-1}\right)\right).
\end{aligned} \tag{5.5.15}$$

To bound the first term on the second line, first note that $x'_{k-1} \geq -pt_3 - (M_1 + C(k-1))(2t_3)^{1/2}$ and $z'_{k-1} \geq K_1(2t_3)^{1/2} - C(k-1)(2t_3)^{1/2}$ for sufficiently large N. Let us write $\tilde{x}, \tilde{z}$ for these two lower bounds. Then by Lemma 5.2.8, we have an $\tilde{N}_3$ so that for $N \geq \tilde{N}_3$,

$$\mathbb{P}_{Ber}^{-t_3,0,x'_{k-1},z'_{k-1}}\left(\ell_1(-t_2) \geq \frac{t_2}{t_3}\tilde{x} + \frac{t_3-t_2}{t_3}\tilde{z} - (2t_3)^{1/4}\right) \geq \frac{1}{3}. \tag{5.5.16}$$

Moreover, as long as $\tilde{N}_3^\alpha > 2$, we have for $N \geq \tilde{N}_3^\alpha$ that

$$\frac{t_3-t_2}{t_3} \geq 1 - \frac{(r+2)N^\alpha}{(r+3)N^\alpha - 1} > 1 - \frac{r+2}{r+5/2} = \frac{1}{2r+5}. \tag{5.5.17}$$

It follows from our choice of V_1^b and $K_1 = 2(2r+5)V_1^b$ in (5.5.8), as well as (5.5.17), that

$$\frac{t_2}{t_3}\tilde{x} + \frac{t_3-t_2}{t_3}\tilde{z} - (2t_3)^{1/4} = -pt_2 - C(k-1)(2t_3)^{1/2} - \frac{t_2}{t_3}M_1(2t_3)^{1/2} + \frac{t_3-t_2}{t_3}K_1(2t_3)^{1/2} - (2t_3)^{1/4} \geq$$
$$-pt_2 - Ck(2t_3)^{1/2} - M_1(2t_3)^{1/2} + \frac{1}{2r+5}K_1(2t_3)^{1/2} = -pt_2 + (M_1 + Ck + 2(M_2+R))(2t_3)^{1/2} >$$
$$-pt_2 + (2t_3)^{1/2}V_1^b.$$

For the first inequality, we used the fact that $t_2/t_3 < 1$, and we assumed that $\tilde{N}_3$ is sufficiently large so that $C(k-1)(2t_3)^{1/2} + (2t_3)^{1/4} \leq Ck(2t_3)^{1/2}$ for $N \geq \tilde{N}_3$. Using (5.5.16), we conclude for $N \geq \tilde{N}_3$ that

$$\mathbb{P}_{Ber}^{-t_3,0,x'_{k-1},z'_{k-1}}\left(\ell_1(-t_2) + pt_2 \geq (2t_3)^{1/2}V_1^b\right) \geq \frac{1}{3}. \tag{5.5.18}$$

Since $|z'_i - x'_i - pt_2| \leq (K_1 + M_1 + 1)(2t_2)^{1/2}$, we have by Lemma 5.2.14 and our choice of C that the second probability in the second line of (5.5.15) is bounded below by

$$\left(1 - 3e^{-C^2/8p(1-p)}\right)^{k-1} \geq 11/12$$

for N larger than some $\tilde{N}_4$. It follows from (5.5.15) and (5.5.18) that for $N \geq \tilde{N}_2 = \max(\tilde{N}_3, \tilde{N}_4)$,

$$\mathbb{P}_{avoid,Ber;S_1}^{-t_3,0,\vec{x},\vec{z}}\left(\tilde{Q}_{k-1}^1(-t_2) + pt_2 \geq (2t_3)^{1/2}V_1^b\right) \geq \frac{1}{3} - \frac{1}{12} = \frac{1}{4},$$

proving the first inequality in (5.5.14). The second inequality is proven similarly.

Step 3. In this last step, we fix V_1^t and prove that we can enlarge N_7 from Step 1 so that (5.5.10) holds for $N \geq N_7$. Let C be as in (5.5.8), and define vectors $\vec{x}\,'', \vec{y}\,'' \in \mathfrak{W}_{k-1}$ by

$$x_i'' = \lceil -pt_3 + M_1(2t_3)^{1/2} \rceil + (k-i)\lceil C(2t_3)^{1/2} \rceil,$$
$$y_i'' = \lceil pt_3 + M_1(2t_3)^{1/2} \rceil + (k-i)\lceil C(2t_3)^{1/2} \rceil.$$

Note that $x_i'' \geq x_1 \geq x_i$ and $x_i'' - x_{i+1}'' \geq C(2t_3)^{1/2}$, and likewise for y_i''. Moreover, ℓ_{bot} lies a distance of at least $C(2t_3)^{1/2}$ uniformly below the line segment connecting x_{k-1}'' and y_{k-1}''. By Lemma 5.2.1 we have

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}\left(\tilde{Q}_1(\pm t_2) \mp pt_2 > (2t_3)^{1/2}V_1^t\right) \leq \mathbb{P}_{avoid,Ber;S}^{-t_3,t_3,\vec{x}\,'',\vec{y}\,'',\infty,\ell_{bot}}\left(\sup_{s\in[-t_3,t_3]}\left[\tilde{Q}_1(s) - ps\right] \geq (2t_3)^{1/2}V_1^t\right) \leq \frac{\mathbb{P}_{Ber}^{-t_3,t_3,x_1'',y_1''}\left(\sup_{s\in[-t_3,t_3]}\left[\tilde{L}_1(s) - ps\right] \geq (2t_3)^{1/2}V_1^t\right)}{\mathbb{P}_{Ber}^{-t_3,t_3,\vec{x}\,'',\vec{y}\,''}\left(\tilde{L}_1 \geq \cdots \geq \tilde{L}_{k-1} \geq \ell_{bot}\right)}.$$

In the numerator in the second line, we used the fact that the curves $\tilde{L}_1, \ldots, \tilde{L}_{k-1}$ are independent under $\mathbb{P}_{Ber}^{-t_3,t_3,x_1'',y_1''}$, and the event in the parentheses depends only on $\tilde{L}_1$. By Lemma 5.2.10, since $\min(x_1'' + pt_3,\ y_1'' - pt_3) \leq (M_1 + C(k-1))(2t_3)^{1/2}$, we can choose $V_1^t > V_1^b$ as well as $\tilde{N}_5$ large enough so that the numerator is bounded above by $h_1/2$ for $N \geq \tilde{N}_5$. Since $|y_i'' - x_i'' - 2pt_3| \leq 1$, our choice of C and Lemma 5.2.14 give a $\tilde{N}_6$ so that the denominator is at least $11/12$ for $N \geq \tilde{N}_6$. This gives an upper bound of $12/11 \cdot h_1/2 < h_1/2$ in the above as long as $N_7 \geq \max(\tilde{N}_5, \tilde{N}_6)$, proving (5.5.10).

□

We are now equipped to prove Lemma 5.5.2. Let us put

$$t_{12} = \left\lfloor \frac{t_1 + t_2}{2} \right\rfloor. \tag{5.5.19}$$

Proof. We first introduce some notation to be used in the proof. Let S be as in Definition 5.5.1. For $\vec{c}, \vec{d} \in \mathfrak{W}_{k-1}$, let us write $\tilde{S} = [\![-t_2, -t_1]\!] \cup [\![t_1, t_2]\!]$, $\tilde{\Omega}(\vec{c}, \vec{d}) = \Omega_{avoid}(-t_2, t_2, \vec{c}, \vec{d}, \infty, \ell_{bot}; \tilde{S})$. For $s \in \tilde{S}$ we define events

$$
\begin{aligned}
&A(\vec{c}, \vec{d}, s) = \left\{ \tilde{\mathfrak{Q}} \in \tilde{\Omega}(\vec{c}, \vec{d}) : \tilde{Q}_{k-1}(\pm s) \mp ps \geq (M_2 + 1)(2t_3)^{1/2} \right\}, \\
&B(\vec{c}, \vec{d}, V^{top}, s) = \left\{ \tilde{\mathfrak{Q}} \in \tilde{\Omega}(\vec{c}, \vec{d}) : \tilde{Q}_1(\pm s) \mp ps \leq V^{top}(2t_3)^{1/2} \right\}, \\
&C(\vec{c}, \vec{d}, \epsilon, s) = \left\{ \tilde{\mathfrak{Q}} \in \tilde{\Omega}(\vec{c}, \vec{d}) : \min_{1 \leq i \leq k-2,\, \varsigma \in \{-1,1\}} \left[\tilde{Q}_i(\varsigma s) - \tilde{Q}_{i+1}(\varsigma s) \right] \geq 3\epsilon (2t_3)^{1/2} \right\}, \\
&D(\vec{c}, \vec{d}, V^{top}, \epsilon, s) = A(\vec{c}, \vec{d}, s) \cap B(\vec{c}, \vec{d}, V^{top}, s) \cap C(\vec{c}, \vec{d}, \epsilon, s).
\end{aligned}
\tag{5.5.20}
$$

Here, ϵ and V^{top} are constants which we will specify later. By Lemma 5.5.3, for all $(\vec{c}, \vec{d})$ and N sufficiently large we have

$$
D(\vec{c}, \vec{d}, V^{top}, \epsilon, s) \subset \{ Z\left(-t_1, t_1, \mathfrak{Q}(-t_1), \mathfrak{Q}(t_1), \ell_{bot}[\![-t_1, t_1]\!]\right) > g \}
$$

for some g depending on ϵ, V^{top}, M_2. Thus we will prove that probability of the event on the left under the uniform measure on $\tilde{\Omega}(\vec{c}, \vec{d})$ is bounded below by $h = h_1/2$, with h_1 as in (5.5.8). We split the proof into several steps.

Step 1. In this step, we show that there exist $R > 0$ and $\overline{N}_0$ sufficiently large so that if $c_{k-1} + pt_2 \geq (2t_3)^{1/2}(M_2 + R)$ and $d_{k-1} - pt_2 \geq (2t_3)^{1/2}(M_2 + R)$, then for all $s \in \tilde{S}$ and $N \geq \overline{N}_0$ we have

$$
\mathbb{P}^{-t_2, t_2, \vec{c}, \vec{d}, \infty, \ell_{bot}}_{avoid, Ber; \tilde{S}} \left(A(\vec{c}, \vec{d}, s) \right) \geq \frac{19}{20} \quad \text{and} \quad \mathbb{P}^{-t_2, t_2, \vec{c}, \vec{d}}_{avoid, Ber; \tilde{S}} \left(Q_{k-1}|_{\tilde{S}} \geq \ell_{bot}|_{\tilde{S}} \right) \geq \frac{99}{100}. \tag{5.5.21}
$$

Let us begin with the first inequality. We observe via Lemma 5.2.2 that

$$
\mathbb{P}^{-t_2, t_2, \vec{c}, \vec{d}, \infty, \ell_{bot}}_{avoid, Ber; \tilde{S}} \left(A(\vec{c}, \vec{d}, s) \right) \geq \mathbb{P}^{-t_2, t_2, \vec{c}, \vec{d}}_{avoid, Ber; \tilde{S}} \left(A(\vec{c}, \vec{d}, s) \right). \tag{5.5.22}
$$

Now define the constant

$$C = \sqrt{8p(1-p)\log\frac{3}{1-(199/200)^{1/(k-1)}}} \tag{5.5.23}$$

and vectors $\vec{c}\,', \vec{d}\,' \in \mathfrak{W}_k$ by

$$c_i' = \lfloor -pt_2 + (M_2+R)(2t_3)^{1/2}\rfloor - (i-1)\lceil C(2t_2)^{1/2}\rceil,$$
$$d_i' = \lfloor pt_2 + (M_2+R)(2t_3)^{1/2}\rfloor - (i-1)\lceil C(2t_2)^{1/2}\rceil.$$

Then by Lemma 5.2.1 we have

$$\begin{aligned}&\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\big(A(\vec{c},\vec{d},s)\big) \geq \mathbb{P}^{-t_2,t_2,\vec{c}\,',\vec{d}\,'}_{avoid,Ber;\tilde{S}}(A(\vec{c}\,',\vec{d}\,',s)) \geq \\ &\mathbb{P}^{-t_2,t_2,c'_{k-1},d'_{k-1}}_{Ber}\left(\inf_{s\in\tilde{S}}\big[\ell(s)-ps\big] \geq (M_2+1)(2t_3)^{1/2}\right) - \\ &\qquad\left(1-\mathbb{P}^{-t_2,t_2,\vec{c}\,',\vec{d}\,'}_{Ber}\,(L_1\geq\cdots\geq L_{k-1})\right).\end{aligned} \tag{5.5.24}$$

By Lemma 5.2.14 and our choice of C, we can find $\tilde{N}_0$ so that $\mathbb{P}^{-t_2,t_2,\vec{c}\,',\vec{d}\,'}_{Ber}(L_1 \geq \cdots \geq L_{k-1}) > 199/200 > 39/40$ for $N \geq \tilde{N}_0$. Writing $z = d'_{k-1} - c'_{k-1}$, the term in the second line of (5.5.24) is equal to

$$\begin{aligned}&\mathbb{P}^{-t_2,t_2,0,z}_{Ber}\left(\inf_{s\in\tilde{S}}\big[\ell(s)+c'_{k-1}-ps\big] \geq (M_2+1)(2t_3)^{1/2}\right) \geq \\ &\mathbb{P}^{0,2t_2,0,z}_{Ber}\left(\inf_{s\in[0,2t_2]}\big[\ell(s)-ps\big] \geq (-R+Ck+1)(2t_3)^{1/2}\right).\end{aligned}$$

In the second line, we used the estimate $c'_{k-1} \geq -pt_2 + (M_2+R-Ck)(2t_3)^{1/2}$. Now by Lemma 5.2.10, we can choose R large enough depending on C, k, M_2, p so that this probability is greater than 39/40 for N greater than some $\tilde{N}_1$. This gives a lower bound in (5.5.24) of $39/40 - 1/40 = 19/20$ for $N \geq \max(\tilde{N}_0, \tilde{N}_1)$, and in combination with (5.5.22) this proves the first inequality in (5.5.21).

We prove the second inequality in (5.5.21) similarly. Note that since $\ell_{bot}(s) \leq ps + M_2(2t_3)^{1/2}$

on $[-t_3, t_3]$ by assumption, we have

$$\begin{aligned}
&\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\left(\tilde{Q}_{k-1}|_{\tilde{S}} \geq \ell_{bot}|_{\tilde{S}}\right) \geq \mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\left(\inf_{s\in[-t_2,t_2]}\left[Q_{k-1}(s) - ps\right] \geq M_2(2t_3)^{1/2}\right) \geq \\
&\mathbb{P}^{-t_2,t_2,\vec{c}\,',\vec{d}\,'}_{avoid,Ber;\tilde{S}}\left(\inf_{s\in[-t_2,t_2]}\left[\tilde{Q}_{k-1}(s) - ps\right] \geq M_2(2t_3)^{1/2}\right) \geq \\
&\mathbb{P}^{0,2t_2,0,z}_{Ber}\left(\inf_{s\in[0,2t_2]}\left[\ell(s) - ps\right] \geq -(R - Ck)(2t_3)^{1/2}\right) - \\
&\qquad \left(1 - \mathbb{P}^{-t_2,t_2,\vec{c}\,',\vec{d}\,'}_{Ber}(\tilde{L}_1 \geq \cdots \geq \tilde{L}_{k-1})\right).
\end{aligned} \tag{5.5.25}$$

We enlarge R if necessary so that the probability in the third line of (5.5.25) is $> 199/200$ for $N \geq \tilde{N}_2$ by Lemma 5.2.10, and 5.2.14 implies as above that the expression in the last line of (5.5.25) is $> -1/200$ for $N \geq \tilde{N}_3$. This gives us a lower bound of $199/200 - 1/200 = 99/100$ for $N \geq \tilde{N}_0 = \max(\tilde{N}_2, \tilde{N}_3)$ as desired. This proves the two inequalities in (5.5.21) for $N \geq \overline{N}_0 = \max(\tilde{N}_0, \tilde{N}_1, \tilde{N}_2, \tilde{N}_3)$.

Step 2. With R fixed from Step 1, let V_1^t, V_1^b, and h_1 be as in Lemma 5.5.4 for this choice of R. Define the event

$$\begin{aligned}
E = \big\{\vec{c},\vec{d} \in \mathfrak{W}_{k-1} : (2t_3)^{1/2}V_1^t &\geq \max(c_1 + pt_2 d_1 - pt_2) \text{ and} \\
&\min(c_{k-1} + pt_2, d_{k-1} - pt_2) \geq (2t_3)^{1/2}V_1^b\big\}.
\end{aligned} \tag{5.5.26}$$

We show in this step that there exists $V^{top} \geq M_2 + 6(k-1)$ and $\overline{N}_1$ such that for all $(\vec{c},\vec{d}) \in E$, $s \in \tilde{S}$, and $N \geq \overline{N}_1$ we have

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\left(B(\vec{c},\vec{d},V^{top},s)\right) \geq \frac{19}{20}. \tag{5.5.27}$$

Let C be as in (5.5.23), and define $\vec{c}\,'', \vec{d}\,'' \in \mathfrak{W}_{k-1}$ by

$$\begin{aligned}
c_i'' &= \lceil -pt_2 + (2t_3)^{1/2}V_1^t \rceil + (k-1-i)\lceil C(2t_2)^{1/2} \rceil, \\
d_i'' &= \lceil pt_2 + (2t_3)^{1/2}V_1^t \rceil + (k-1-i)\lceil C(2t_2)^{1/2} \rceil.
\end{aligned}$$

Then $c_i'' \geq c_1 \geq c_i$ and $c_i'' - c_{i+1}'' \geq C(2t_2)^{1/2}$ for each i, and likewise for d_i''. Furthermore, since $V_1^b \geq M_2 + R$, we see that ℓ_{bot} lies a distance of at least $R(2t_3)^{1/2}$ uniformly below the line segment connecting c_{k-1}'' and d_{k-1}''. By construction, $R > C$. By Lemma 5.2.1, the left hand side of (5.5.27) is bounded below by

$$\begin{aligned}
&\mathbb{P}_{avoid,Ber;\tilde{S}}^{-t_2,t_2,\vec{c}\,'',\vec{d}\,'',\infty,\ell_{bot}}\left(\sup_{s\in\tilde{S}}\left[\tilde{Q}_1(s) - ps\right] \leq V^{top}(2t_3)^{1/2}\right) \geq \\
&\mathbb{P}_{Ber}^{0,2t_2,0,z'}\left(\sup_{s\in[-t_2,t_2]}\left[\ell(s) - ps\right] \leq (V^{top} - V_1^t - Ck)(2t_3)^{1/2}\right) - \\
&\qquad\left(1 - \mathbb{P}_{Ber}^{-t_2,t_2,\vec{c}\,'',\vec{d}\,'',\infty,\ell_{bot}}\left(L_1 \geq \cdots \geq L_{k-1} \geq \ell_{bot}\right)\right).
\end{aligned} \tag{5.5.28}$$

In the last line, we have written $z' = d_1'' - c_1''$, and we used the fact that $c_1'' \leq -pt_2 + (V_1^t + Ck)(2t_3)^{1/2}$. By Lemma 5.2.10, we can find V^{top} large enough depending on V_1^t, C, k, p so that the probability in the third line of (5.5.28) is at least $39/40$ for $N \geq \tilde{N}_4$. On the other hand, the above observations regarding $\vec{c}\,''$, $\vec{d}\,''$, and ℓ_{bot}, as well as the fact that $|d_1'' - c_1'' - 2pt_2| \leq 1$, allow us to conclude from Lemma 5.2.14 that the probability in the last line of (5.5.28) is at least $39/40$ for $N \geq \tilde{N}_5$. This gives a lower bound of $39/40 - 1/40 = 19/20$ in (5.5.28) for $\overline{N}_1 = \max(\tilde{N}_4, \tilde{N}_5)$ as desired.

Step 3. In this step, we show that with E, V_1^t, and V_1^b as in Step 2, there exist $\epsilon > 0$ sufficiently small and $\overline{N}_2$ such that for $(\vec{c}, \vec{d}) \in E$ and $N \geq \overline{N}_2$, we have

$$\mathbb{P}_{avoid,Ber;\tilde{S}}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}\left(D(\vec{c}, \vec{d}, V^{top}, \epsilon, t_{12})\right) \geq \frac{1}{2}. \tag{5.5.29}$$

We claim that this follows if we find $\tilde{N}_6$ so that for $N \geq \tilde{N}_6$,

$$\mathbb{P}_{avoid,Ber;\tilde{S}}^{-t_2,t_2,\vec{c},\vec{d}}\left(C(\vec{c}, \vec{d}, \epsilon, t_{12}) \mid A(\vec{c}, \vec{d}, t_1) \cap B(\vec{c}, \vec{d}, V^{top}, t_1)\right) \geq \frac{9}{10}. \tag{5.5.30}$$

To see this, note that (5.5.21) and (5.5.27) imply that for $N \geq \max(\overline{N}_0, \overline{N}_1)$,

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\big(A(\vec{c},\vec{d},t_1) \cap B(\vec{c},\vec{d},V^{top},t_1)\big) \geq \frac{19}{20} - \frac{1}{20} - \frac{1}{100} > \frac{4}{5},$$

and then (5.5.30) and the second inequality in (5.5.21) imply that for $N \geq \overline{N}_2 = \max(\overline{N}_0, \overline{N}_1, \tilde{N}_6)$,

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\big(A(\vec{c},\vec{d},t_1) \cap B(\vec{c},\vec{d},V^{top},t_1) \cap C(\vec{c},\vec{d},\epsilon,t_{12})\big) > \frac{9}{10}\cdot\frac{4}{5} - \frac{1}{100} > \frac{17}{25}.$$

Then using (5.5.21) and (5.5.27) once again and recalling the definition of $D(\vec{c},\vec{d},V^{top},\epsilon,t_{12})$ gives a lower bound on the probability in (5.5.29) of $17/25 - 1/10 > 14/25 > 1/2$ for $N \geq \overline{N}_2$ as desired.

In the remainder of this step, we verify (5.5.30). Observe that $A(\vec{c},\vec{d},t_1) \cap B(\vec{c},\vec{d},V^{top},t_1)$ can be written as a countable disjoint union:

$$A(\vec{c},\vec{d},t_1) \cap B(\vec{c},\vec{d},V^{top},t_1) = \bigsqcup_{(\vec{a},\vec{b})\in I} F(\vec{a},\vec{b}). \tag{5.5.31}$$

Here, for $\vec{a},\vec{b} \in \mathfrak{W}_{k-1}$, $F(\vec{a},\vec{b})$ is the event that $\mathfrak{Q}(-t_1) = \vec{a}$ and $\mathfrak{Q}(t_1) = \vec{b}$, and I is the collection of pairs $(\vec{a},\vec{b})$ satisfying

(1) $0 \leq \min(a_i - c_i,\, d_i - b_i) \leq t_2 - t_1$ and $0 \leq b_i - a_i \leq 2t_1$ for $1 \leq i \leq k-1$,

(2) $\min(a_{k-1} + pt_1,\, b_{k-1} - pt_1) \geq (M_2+1)(2t_3)^{1/2}$,

(3) $\max(a_1 + pt_1,\, b_1 - pt_1) \leq V^{top}(2t_3)^{1/2}$.

Now let $\mathfrak{Q}^1 = (Q^1_1,\dots,Q^1_{k-1})$ and $\mathfrak{Q}^2 = (Q^2_2,\dots,Q^2_{k-1})$ denote the restrictions of $\tilde{\mathfrak{Q}}$ to $[\![-t_2,-t_1]\!]$ and $[\![t_1,t_2]\!]$ respectively. Then we observe that

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}}\left(\mathfrak{Q}^1 = \mathfrak{B}^1, \mathfrak{Q}^2 = \mathfrak{B}^2 \,\middle|\, F(\vec{a},\vec{b})\right) = \mathbb{P}^{-t_2,-t_1,\vec{c},\vec{a}}_{avoid,Ber}\left(\mathfrak{Q}^1 = \mathfrak{B}^1\right)\cdot \mathbb{P}^{t_1,t_2,\vec{b},\vec{d}}_{avoid,Ber}\left(\mathfrak{Q}^2 = \mathfrak{B}^2\right). \tag{5.5.32}$$

We also let $\tilde{I} = \{(\vec{a},\vec{b}) \in I :^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}} (F(\vec{a},\vec{b})) > 0\}$, and we choose $\tilde{N}_7$ so that $\tilde{I}$ is nonempty for $N \geq \tilde{N}_7$ using Lemma 5.1.16. We now fix $(\vec{a},\vec{b})$ and argue that we can choose $\epsilon > 0$ small enough and $\tilde{N}_8$ so that for $N \geq \tilde{N}_8$,

$$ {}^{-t_2,t_2,\vec{c},\vec{d}}_{avoid,Ber;\tilde{S}} \left(C(\vec{c},\vec{d},\epsilon,t_{12}) \,\middle|\, F(\vec{a},\vec{b}) \right) \geq \frac{9}{10}. \tag{5.5.33} $$

Then using (5.5.33) and (5.5.31) and summing over $\tilde{I}$ proves (5.5.30) for $N \geq \tilde{N}_6 = \max(\tilde{N}_7, \tilde{N}_8)$.

To prove (5.5.33), we first show that we can find $\delta > 0$ and $\tilde{N}_7$ so that

$$ \mathbb{P}^{-t_2,-t_1,\vec{c},\vec{a}}_{avoid,Ber} \left(\max_{1\leq i\leq k-2} \left[Q^1_i(-t_{12}) - Q^1_{i+1}(-t_{12}) \right] \geq \delta (2t_3)^{1/2} \right) \geq \frac{3}{\sqrt{10}} \tag{5.5.34} $$

for $N \geq \tilde{N}_7$. We prove this inequality using Lemma 5.2.18. In order to apply this result, we first observe that since $|-t_{12} + \frac{1}{2}(t_1+t_2)| \leq 1$ by (5.5.19), we have

$$ 0 \leq Q^1_i(-t_{12}) - Q^1_i(-\tfrac{1}{2}(t_1+t_2)) \leq 1. \tag{5.5.35} $$

Now applying Lemma 5.2.18 with $M_1 = V^t_1$, $M_2 = V^{top}$, we obtain $\tilde{N}_7$ and $\delta > 0$ such that if $N \geq \tilde{N}_7$, then

$$ {}^{-t_2,-t_1,\vec{c},\vec{a}}_{avoid,Ber} \left(\min_{1\leq i\leq k-1} \left[Q^1_i(-\tfrac{1}{2}(t_1+t_2)) - Q^1_{i+1}(-\tfrac{1}{2}(t_1+t_2)) \right] < \delta (t_2-t_1)^{1/2} \right) < 1 - \frac{3}{\sqrt{10}}. $$

Together with (5.5.35) and the fact that $t_3/4 < t_2 - t_1$, this implies that

$$ {}^{-t_2,-t_1,\vec{c},\vec{a}}_{avoid,Ber} \left(\min_{1\leq i\leq k-1} \left[Q^1_i(-t_{12}) - Q^1_{i+1}(-t_{12}) \right] < (\delta/2)(2t_3)^{1/2} - 1 \right) < 1 - \frac{3}{\sqrt{10}} \tag{5.5.36} $$

for $N \geq \tilde{N}_7$. Now we observe that as long as $\tilde{N}_7^{\alpha} \geq \frac{1+8/\delta^2}{r+3}$, then $(\delta/4)(2t_3)^{1/2} \leq (\delta/2)(2t_2)^{1/2} - 1$ for $N \geq \tilde{N}_7$. This implies (5.5.34). A similar argument gives us a $\tilde{\delta} > 0$ such that

$$ {}^{-t_2,-t_1,\vec{c},\vec{a}}_{avoid,Ber} \left(\min_{1\leq i\leq k-1} \left[Q_i(-t_{12}) - Q_{i+1}(-t_{12}) \right] < (\tilde{\delta}/4)(2t_3)^{1/2} \right) < 1 - \frac{3}{\sqrt{10}} $$

for $N \geq \tilde{N}_7$. Then putting $\epsilon = \min(\delta, \tilde{\delta})/12$ and using (5.5.32), we obtain (5.5.33) for $N \geq \tilde{N}_7$.

Step 4. In this step, we find $\overline{N}_3$ so that

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\big(D(\vec{c},\vec{d},V^{top},\epsilon,t_1)\big) \geq \frac{1}{2}\left(\frac{1}{2} - \sum_{n=1}^{\infty}(-1)^{n-1}e^{-\epsilon^2 n^2/2p(1-p)}\right)^{k-1} \tag{5.5.37}$$

for $N \geq \overline{N}_3$. We will find $\tilde{N}_9$ so that for $N \geq \tilde{N}_9$,

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}\Big(D(\vec{c},\vec{d},V^{top},\epsilon,t_1)\,\Big|\,D(\vec{c},\vec{d},V^{top},\epsilon,t_{12})\Big) \geq \left(\frac{1}{2} - \sum_{n=1}^{\infty}(-1)^{n-1}e^{-\epsilon^2 n^2/2p(1-p)}\right)^{k-1}. \tag{5.5.38}$$

Then (5.5.29) implies (5.5.37) for $N \geq \overline{N}_3 = \max(\overline{N}_2, \tilde{N}_9)$.

To prove (5.5.38) we first observe that we can write

$$D(\vec{c},\vec{d},V^{top},\epsilon,t_{12}) = \bigsqcup_{(\vec{a},\vec{b})\in J} G(\vec{a},\vec{b}). \tag{5.5.39}$$

Here, for $\vec{a},\vec{b} \in \mathfrak{W}_{k-1}$, $G(\vec{a},\vec{b})$ is the event that $\mathfrak{Q}(-t_{12}) = \vec{a}$ and $\mathfrak{Q}(t_{12}) = \vec{b}$, and J is the collection of $(\vec{a},\vec{b})$ satisfying

(1) $0 \leq \min(a_i - c_i,\, d_i - b_i) \leq t_2 - t_{12}$ and $0 \leq b_i - a_i \leq 2t_{12}$ for $1 \leq i \leq k-1$,

(2) $\min(a_{k-1} + pt_1,\, b_{k-1} - pt_1) \geq (M_2 + 1)(2t_3)^{1/2}$,

(3) $\max(a_1 + pt_1,\, b_1 - pt_1) \leq V^{top}(2t_3)^{1/2}$,

(4) $\min(a_i - a_{i+1},\, b_i - b_{i+1}) \geq 3\epsilon(2t_3)^{1/2}$ for $1 \leq i \leq k-2$.

We let $\tilde{J} = \{(\vec{a},\vec{b}) \in J : \mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}(G(\vec{a},\vec{b})) > 0\}$, and we take $\tilde{N}_9$ large enough by Lemma 5.1.16 so that $\tilde{J} \neq \varnothing$. We also let $\tilde{D}(V^{top},\epsilon,t_1)$ denote the set consisting of elements of $D(\vec{c}, \vec{d}, V^{top}, \epsilon, t_1)$ restricted to $[\![-t_{12}, t_{12}]\!]$. Then for $(\vec{a}, \vec{b}) \in \tilde{J}$ we have

$$\begin{aligned}&\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}} \left(D(\vec{c}, \vec{d}, V^{top}, \epsilon, t_1) \,\middle|\, G(\vec{a}, \vec{b})\right) = \mathbb{P}^{-t_{12},t_{12},\vec{a},\vec{b},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}} \left(\tilde{D}(V^{top}, \epsilon, t_1)\right) \geq \\ &\mathbb{P}^{-t_{12},t_{12},\vec{a},\vec{b}}_{Ber} \left(\tilde{D}(V^{top}, \epsilon, t_1) \cap \{L_1 \geq \cdots \geq L_{k-1} \geq \ell_{bot}\}\right).\end{aligned} \tag{5.5.40}$$

We observe that the event in the second line of (5.5.40) occurs as long as each curve L_i remains within a distance of $\epsilon(2t_3)^{1/2}$ from the straight line segment connecting a_i and b_i on $[-t_{12}, t_{12}]$, for $1 \leq i \leq k-2$. By the argument in the proof of Lemma 5.2.14, we can enlarge $\tilde{N}_9$ so that the probability of this event is bounded below by the expression on the right in (5.5.38) for $N \geq \tilde{N}_9$. Then using (5.5.40) and (5.5.39) and summing over $\tilde{J}$ implies (5.5.38).

Step 5. In this last step, we complete the proof of the lemma, fixing the constants g and h as well as N_5. Let $g = g(\epsilon, V^{top}, M_2)$ be as in Lemma 5.5.3 for the choices of ϵ, V^{top} in Steps 2 and 3, let

$$h = \frac{h_1}{2}\left(\frac{1}{2} - \sum_{n=1}^{\infty}(-1)^{n-1}e^{-\epsilon^2 n^2/2p(1-p)}\right)^{k-1}$$

with h_1 as in Step 2, and let $N_5 = \max(\overline{N}_0, \overline{N}_1, \overline{N}_2, \overline{N}_3, N_7)$, with N_7 as in Lemma 5.5.4. In the following we assume that $N \geq N_7$. By (5.5.37) we have that if $(\vec{c}, \vec{d}) \in E$ and $N \geq N_5$, then

$$\mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}(H) \geq \frac{h}{h_1},$$

where H is the event that

1. $V^{top}(2t_3)^{1/2} \geq \tilde{Q}_1(-t_1) + pt_1 \geq \tilde{Q}_{k-1}(-t_1) + pt_1 \geq (M_2+1)(2t_2)^{1/2}$,
2. $V^{top}(2t_3)^{1/2} \geq \tilde{Q}_1(t_1) - pt_1 \geq \tilde{Q}_{k-1}(t_1) - pt_1 \geq (M_2+1)(2t_3)^{1/2}$,
3. $\tilde{Q}_i(-t_1) - \tilde{Q}_{i+1}(-t_1) \geq 3\epsilon(2t_2)^{1/2}$ and $\tilde{Q}_i(t_1) - \tilde{Q}_{i+1}(t_1) \geq 3\epsilon(2t_2)^{1/2}$ for $i = 1, \ldots, k-2$.

Let Y denote the event appearing in (5.5.7). Then we can write $Y = \bigsqcup_{(\vec{c},\vec{d})\in E} Y(\vec{c}, \vec{d})$, where $Y(\vec{c}, \vec{d})$ is the event that $\tilde{\mathfrak{Q}}(-t_2) = \vec{c}$, $\tilde{\mathfrak{Q}}(t_2) = \vec{d}$, and E is defined in Step 2. If $\tilde{E} = \{(\vec{c}, \vec{d}) \in E :$

$\mathbb{P}_{\tilde{\mathfrak{Q}}}(Y(\vec{c},\vec{d})) > 0\}$, we can assume by Lemma 5.1.16 that N_5 is large enough so that $\tilde{E} \neq \varnothing$. It follows from Lemma 5.5.4 that $\mathbb{P}_{\tilde{\mathfrak{Q}}}(Y) \geq h_1$. We conclude from the definition of $\mathbb{P}_{\tilde{\mathfrak{Q}}}$ that for all $N \geq N_5$,

$$\mathbb{P}_{\tilde{\mathfrak{Q}}}(H) \geq \mathbb{P}_{\tilde{\mathfrak{Q}}}(H \cap Y) = \sum_{(\vec{c},\vec{d})\in\tilde{E}} \mathbb{P}_{\tilde{\mathfrak{Q}}}(Y(\vec{c},\vec{d})) \cdot \mathbb{P}_{\tilde{\mathfrak{Q}}}(H \,|\, Y(\vec{c},\vec{d})) =$$
$$\sum_{(\vec{c},\vec{d})\in\tilde{E}} \mathbb{P}_{\tilde{\mathfrak{Q}}}(Y(\vec{c},\vec{d})) \cdot \mathbb{P}^{-t_2,t_2,\vec{c},\vec{d},\infty,\ell_{bot}}_{avoid,Ber;\tilde{S}}(H) \geq \frac{h}{h_1} \sum_{(\vec{c},\vec{d})\in\tilde{E}} \mathbb{P}_{\tilde{\mathfrak{Q}}}(Y(\vec{c},\vec{d})) = \frac{h}{h_1} \mathbb{P}_{\tilde{\mathfrak{Q}}}(Y) \geq h.$$

Now Lemma 5.5.3 implies (5.5.1), completing the proof. □

5.6 Appendix A

5.6.1 Proof of Lemma 5.1.2

Observe that the sets $K_1 \subset K_2 \subset \cdots \subset \Sigma \times \Lambda$ are compact, they cover $\Sigma \times \Lambda$, and any compact subset K of $\Sigma \times \Lambda$ is contained in all K_n for sufficiently large n. To see this last fact, let π_1, π_2 denote the canonical projection maps of $\Sigma \times \Lambda$ onto Σ and Λ respectively. Since these maps are continuous, $\pi_1(K)$ and $\pi_2(K)$ are compact in Σ and Λ. This implies that $\pi_1(K)$ is finite, so it is contained in $\Sigma_{n_1} = \Sigma \cap [\![-n_1, n_1]\!]$ for some n_1. On the other hand, $\pi_2(K)$ is closed and bounded in $\mathbb{R}$, thus contained in some closed interval $[\alpha, \beta] \subseteq \Lambda$. Since $a_n \searrow a$ and $b_n \nearrow b$, we can choose n_2 large enough so that $\pi_2(K) \subseteq [\alpha, \beta] \subseteq [a_{n_2}, b_{n_2}]$. Then taking $n = \max(n_1, n_2)$, we have $K \subseteq \pi_1(K) \times \pi_2(K) \subseteq \Sigma_n \times [a_n, b_n] = K_n$.

We now split the proof into several steps.

Step 1. In this step, we show that the function d defined in the statement of the lemma is a metric. For each n and $f, g \in C(\Sigma \times \Lambda)$, we define

$$d_n(f,g) = \sup_{(i,t)\in K_n} |f(i,t) - g(i,t)|, \quad d'_n(f,g) = \min\{d_n(f,g), 1\}$$

Then we have

$$d(f,g) = \sum_{n=1}^{\infty} 2^{-n} d'_n(f,g).$$

Clearly each d_n is nonnegative and satisfies the triangle inequality, and it is then easy to see that the same properties hold for d'_n. Furthermore, $d'_n \leq 1$, so d is well-defined. Observe that d is non-negative, and if $f = g$, then each $d'_n(f,g) = 0$, so the sum $d(f,g)$ is 0. Conversely, if $f \neq g$, then since the K_n cover $\Sigma \times \Lambda$, we can choose n large enough so that K_n contains an x with $f(x) \neq g(x)$. Then $d'_n(f,g) \neq 0$, and hence $d(f,g) \neq 0$. Lastly, the triangle inequality holds for d since it holds for each d'_n.

Step 2. Now we prove that the topology τ_d on $C(\Sigma \times \Lambda)$ induced by d is the same as the topology of uniform convergence over compacts, which we denote by τ_c. Recall that τ_c is generated by the basis consisting of sets

$$B_K(f,\epsilon) = \left\{ g \in C(\Sigma \times \Lambda) : \sup_{(i,t)\in K} |f(i,t) - g(i,t)| < \epsilon \right\},$$

for $K \subset \Sigma \times \Lambda$ compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and τ_d is generated by sets of the form $B^d_\epsilon(f) = \{g : d(f,g) < \epsilon\}$.

We first show that $\tau_d \subseteq \tau_c$. It suffices to prove that every set $B^d_\epsilon(f)$ is a union of sets $B_K(f,\epsilon)$. First, choose $\epsilon > 0$ and $f \in C(\Sigma \times \Lambda)$. Let $g \in B^d_\epsilon(f)$. We will find a basis element A_g of τ_c such that $g \in A_g \subset B^d_\epsilon(f)$. Let $\delta = d(f,g) < \epsilon$, and choose n large enough so that $\sum_{k>n} 2^{-k} < \frac{\epsilon-\delta}{2}$. Define $A_g = B_{K_n}(g, \frac{\epsilon-\delta}{n})$, and suppose $h \in A_g$. Then since $K_m \subseteq K_n$ for $m \leq n$, we have

$$d(f,h) \leq d(f,g) + d(g,h) \leq \delta + \sum_{k=1}^{n} 2^{-k} d_n(g,h) + \sum_{k>n} 2^{-k} \leq \delta + \frac{\epsilon-\delta}{2} + \frac{\epsilon-\delta}{2} = \epsilon.$$

Therefore $g \in A_g \subset B^d_\epsilon(f)$. Then we can write

$$B^d_\epsilon(f) = \bigcup_{g \in B^d_\epsilon(f)} A_g,$$

a union of basis elements of τ_c.

We now prove conversely that $\tau_c \subseteq \tau_d$. Let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$. Choose n so that $K \subset K_n$, and let $g \in B_K(f, \epsilon)$ and $\delta = \sup_{x \in K} |f(x) - g(x)| < \epsilon$. If $d(g, h) < 2^{-n}(\epsilon - \delta)$, then $d_n'(g, h) \leq 2^n d(g, h) < \epsilon - \delta$, hence $d_n(g, h) < \epsilon - \delta$, assuming without loss of generality that $\epsilon \leq 1$. It follows that

$$\sup_{x \in K} |f(x) - h(x)| \leq \delta + \sup_{x \in K} |g(x) - h(x)| \leq \delta + d_n(g, h) \leq \delta + \epsilon - \delta = \epsilon.$$

Thus $g \in B^d_{2^{-n}(\epsilon-\delta)}(g) \subset B_K(f, \epsilon)$, proving that $B_K(f, \epsilon) \in \tau_d$ by the same argument as above. We conclude that $\tau_d = \tau_c$.

Step 3. In this step, we show that $(C(\Sigma \times \Lambda), d)$ is a complete metric space. Let $\{f_n\}_{n \geq 1}$ be Cauchy with respect to d. Then we claim that $\{f_n\}$ must be Cauchy with respect to d_n', on each K_n. This follows from the observation that $d_n'(f_\ell, f_m) \leq 2^n d(f_\ell, f_m)$. Thus $\{f_n\}$ is Cauchy with respect to the uniform metric on each K_n, and hence converges uniformly to a continuous limit f^{K_n} on each K_n (see [213, Theorem 7.15]). Since the pointwise limit must be unique at each $x \in \Sigma \times \Lambda$, we have $f^{K_n}(x) = f^{K_m}(x)$ if $x \in K_n \cap K_m$. Since $\bigcup K_n = \Sigma \times \Lambda$, we obtain a well-defined function f on all of $\Sigma \times \Lambda$ given by $f(x) = \lim_{n \to \infty} f^{K_n}(x)$. We have $f \in C(\Sigma \times \Lambda)$ since $f|_{K_n} = f^{K_n}$ is continuous on K_n for all n. Moreover, if $K \subset \Sigma \times \Lambda$ is compact and n is large enough so that $K \subset K_n$, then because $f_n \to f^{K_n} = f|_{K_n}$ uniformly on K_n, we have $f_n \to f^{K_n}|_K = f|_K$ uniformly on K. That is, for any $K \subset \Sigma \times \Lambda$ compact and $\epsilon > 0$, we have $f_n \in B_K(f, \epsilon)$ for all sufficiently large n. Therefore $f_n \to f$ in τ_c, and equivalently in the metric d by Step 2.

Step 4. Lastly, we prove separability, c.f. [39, Example 1.3]. For each pair of positive integers n, k, let $D_{n,k}$ be the subcollection of $C(\Sigma \times \Lambda)$ consisting of polygonal functions that are piecewise linear on $\{j\} \times I_{n,k,i}$ for each $j \in \Sigma_n$ and each subinterval

$$I_{n,k,i} = \left[a_n + \tfrac{i-1}{k}(b_n - a_n),\, a_n + \tfrac{i}{k}(b_n - a_n)\right], \quad 1 \leq i \leq k,$$

taking rational values at the endpoints of these subintervals, and extended linearly to all of $\Lambda = [a,b]$. Then $D = \bigcup_{n,k} D_{n,k}$ is countable, and we claim that it is dense in τ_c. To see this, let $K \subset \Sigma \times \Lambda$ be compact, $f \in C(\Sigma \times \Lambda)$, and $\epsilon > 0$, and choose n so that $K \subset K_n$. Since f is uniformly continuous on K_n, we can choose k large enough so that for $0 \le i \le k$, if $t \in I_{n,k,i}$, then

$$\left|f(j,t) - f(j, a_n + \tfrac{i}{k}(b_n - a_n))\right| < \epsilon/2$$

for all $j \in \Sigma_n$. We then choose $g \in \bigcup_k D_{n,k}$ with $|g(j, a_n+\frac{i}{k}(b_n-a_n)) - f(j, a_n+\frac{i}{k}(b_n-a_n))| < \epsilon/2$. Then we have

$$\left|f(j,t) - g(j, a_n + \tfrac{i-1}{k}(b_n - a_n))\right| < \epsilon \quad \text{and} \quad \left|f(j,t) - g(j, a_n + \tfrac{i}{k}(b_n - a_n))\right| < \epsilon.$$

Since $g(j, a_n + \frac{i-1}{k}(b_n - a_n)) \le g(j,t) \le g(j, a_n + \frac{i}{k}(b_n - a_n))$, it follows that

$$|f(j,t) - g(j,t)| < \epsilon$$

as well. In summary,

$$\sup_{(j,t)\in K} |f(j,t) - g(j,t)| \le \sup_{(j,t)\in K_n} |f(j,t) - g(j,t)| < \epsilon,$$

so $g \in B_K(f, \epsilon)$. This proves that D is a countable dense subset of $C(\Sigma \times \Lambda)$.

5.6.2 Proof of Lemma 5.1.4

We first prove two lemmas that will be used in the proof of Lemma 5.1.4. The first result allows us to identify the space $C(\Sigma \times \Lambda)$ with a product of copies of $C(\Lambda)$. In the following, we assume the notation of Lemma 5.1.4.

Lemma 5.6.1. *Let $\pi_i : C(\Sigma \times \Lambda) \to C(\Lambda)$, $i \in \Sigma$, be the projection maps given by $\pi_i(F)(x) = F(i,x)$ for $x \in \Lambda$. Then the π_i are continuous. Endow the space $\prod_{i\in\Sigma} C(\Lambda)$ with the product topol-*

ogy induced by the topology of uniform convergence over compacts on $C(\Lambda)$. Then the mapping

$$F : C(\Sigma \times \Lambda) \longrightarrow \prod_{i\in\Sigma} C(\Lambda), \quad f \mapsto (\pi_i(f))_{i\in\Sigma}$$

is a homeomorphism.

Proof. We first prove that the π_i are continuous. Since $C(\Sigma \times \Lambda)$ is metrizable by Lemma 5.1.2, and by a similar argument so is $C(\Lambda)$, it suffices to assume that $f_n \to f$ in $C(\Sigma\times\Lambda)$ and show that $\pi_i(f_n) \to \pi_i(f)$ in $C(\Lambda)$. Let K be compact in Λ. Then $\{i\} \times K$ is compact in $\Sigma \times \Lambda$, and $f_n \to f$ on $\{i\} \times K$ by assumption, so we have $\pi_i(f_n)|_K = f_n|_{\{i\}\times K} \to f|_{\{i\}\times K} = \pi_i(f)|_K$ uniformly on K. Since K was arbitrary, we conclude that $\pi_i(f_n) \to \pi_i(f)$ in $C(\Lambda)$ as desired.

We now observe that F is invertible. If $(f_i)_{i\in\Sigma} \in \prod_{i\in\Sigma} C(\Lambda)$, then the function f defined by $f(i,\cdot) = f_i(\cdot)$ is in $C(\Sigma \times \Lambda)$, since Σ has the discrete topology. This gives a well-defined inverse for F. It suffices to prove that F and F^{-1} are open maps.

We first show that F sends each basis element $B_K(f,\epsilon)$ of $C(\Sigma \times \Lambda)$ to a basis element in $\prod_{i\in\Sigma} C(\Lambda)$. Note that a basis for the product topology is given by products $\prod_{i\in\Sigma} B_{K_i}(f_i,\epsilon)$, where at most finitely many of the K_i are nonempty. Here, we use the convention that $B_\varnothing(f_i,\epsilon) = C(\Lambda)$. Let π_Σ, π_Λ denote the canonical projections of $\Sigma \times \Lambda$ onto Σ, Λ. The continuity of π_Σ implies that if $K \subset \Sigma \times \Lambda$ is compact, then $\pi_\Sigma(K)$ is compact in Σ, hence finite. Observe that the set $K \cap (\{i\} \times \Lambda)$ is an intersection of two compacts sets, hence compact in $\Sigma \times \Lambda$. Therefore the sets $K_i = \pi_\Lambda(K \cap (\{i\} \times \Lambda))$ are compact in Λ for each $i \in \Sigma$ since π_Λ is continuous. We observe that $F(B_K(f,\epsilon)) = \prod_{i\in\Sigma} U_i$, where

$$U_i = B_{K_i}(\pi_i(f),\epsilon), \quad \text{if} \quad i \in \pi_\Sigma(K),$$

and $U_i = C(\Lambda)$ otherwise. Since $\pi_\Sigma(K)$ is finite and the K_i are compact, we see that $F(B_K(f,\epsilon))$ is a basis element in the product topology as claimed.

Lastly, we show that F^{-1} sends each basis element $U = \prod_{i\in\Sigma} B_{K_i}(f_i,\epsilon)$ for the product topology to a set of the form $B_K(f,\epsilon)$. We have $K_i = \varnothing$ for all but finitely many i. Write

$f = F^{-1}((f_i)_{i\in\Sigma})$ and $K = \prod_{i\in\Sigma} K_i$. By Tychonoff's theorem, [192, Theorem 37.3], K is compact in $\Sigma \times \Lambda$, and

$$F^{-1}(U) = B_K(f, \epsilon).$$

□

We next prove a lemma which states that a sequence of line ensembles is tight if and only if all individual curves form tight sequences.

Lemma 5.6.2. *Suppose that $\{\mathcal{L}^n\}_{n\geq 1}$ is a sequence of Σ-indexed line ensembles on Λ, and let $X_i^n = \pi_i(\mathcal{L}^n)$. Then the X_i^n are $C(\Lambda)$-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{L}^n\}$ is tight if and only if for each $i \in \Sigma$ the sequence $\{X_i^n\}_{n\geq 1}$ is tight.*

Proof. The fact that the X_i^n are random variables follows from the continuity of the π_i in Lemma 5.6.1 and [117, Theorem 1.3.5]. First suppose the sequence $\{\mathcal{L}^n\}$ is tight. By Lemma 5.1.2, $C(\Sigma\times\Lambda)$ is a Polish space, so it follows from Prohorov's theorem, [39, Theorem 5.1], that $\{\mathcal{L}^n\}$ is relatively compact. That is, every subsequence $\{\mathcal{L}^{n_k}\}$ has a further subsequence $\{\mathcal{L}^{n_{k_\ell}}\}$ converging weakly to some $\mathcal{L}$. Then for each $i \in \Sigma$, since π_i is continuous by the above, the subsequence $\{\pi_i(\mathcal{L}^{n_{k_\ell}})\}$ of $\{\pi_i(\mathcal{L}^{n_k})\}$ converges weakly to $\pi_i(\mathcal{L})$ by the continuous mapping theorem, [117, Theorem 3.2.10]. Thus every subsequence of $\{\pi_i(\mathcal{L}^n)\}$ has a convergent subsequence. Since $C(\Lambda)$ is a Polish space by the same argument as in the proof of Lemma 5.1.2, Prohorov's theorem implies that each $\{\pi_i(\mathcal{L}^n)\}$ is tight.

Conversely, suppose $\{X_i^n\}$ is tight for all $i \in \Sigma$. Then given $\epsilon > 0$, we can find compact sets $K_i \subset C(\Lambda)$ such that

$$\mathbb{P}(X_i^n \notin K_i) \leq \epsilon/2^i$$

for each $i \in \Sigma$. By Tychonoff's theorem, [192, Theorem 37.3], the product $\tilde{K} = \prod_{i\in\Sigma} K_i$ is compact in $\prod_{i\in\Sigma} C(\Lambda)$. We have

$$\mathbb{P}\big((X_i^n)_{i\in\Sigma} \notin \tilde{K}\big) \leq \sum_{i\in\Sigma} \mathbb{P}(X_i^n \notin K_i) \leq \sum_{i=1}^{\infty} \epsilon/2^i = \epsilon. \tag{5.6.1}$$

By Lemma 5.6.1, we have a homeomorphism $G : \prod_{i\in\Sigma} C(\Lambda) \to C(\Sigma \times \Lambda)$. We observe that $G((X_i^n)_{i\in\Sigma}) = \mathcal{L}^n$, and $K = G(\tilde{K})$ is compact in $C(\Sigma\times\Lambda)$. Thus $\mathcal{L}^n \in K$ if and only if $(X_i^n)_{i\in\Sigma} \in \tilde{K}$, and it follows from (5.6.1) that

$$\mathbb{P}(\mathcal{L}^n \in K) \geq 1 - \epsilon.$$

This proves that $\{\mathcal{L}^n\}$ is tight.

□

We are now ready to prove Lemma 5.1.4.

Proof. Fix an $i \in \Sigma$. By Lemma 5.6.2, it suffices to show that the sequence $\{\mathcal{L}_i^n\}_{n\geq 1}$ of $C(\Lambda)$-valued random variables is tight. By [39, Theorem 7.3], a sequence $\{P_n\}$ of probability measures on $C[0,1]$ with the uniform topology is tight if and only if the following conditions hold:

$$\lim_{a\to\infty} \limsup_{n\to\infty} P_n(|x(0)| \geq a) = 0,$$
$$\lim_{\delta\to 0} \limsup_{n\to\infty} P_n\Big(\sup_{|s-t|\leq\delta} |x(s) - x(t)| \geq \epsilon\Big) = 0 \quad \text{for all } \epsilon > 0.$$

By replacing $[0,1]$ with $[a_m, b_m]$ and 0 with a_0, we see that the hypotheses in the lemma imply that the sequence $\{\mathcal{L}_i^n|_{[a_m,b_m]}\}_n$ is tight for every $m \geq 1$. Let $\pi_m : C(\Lambda) \to C([a_m, b_m])$ denote the map $f \mapsto f|_{[a_m,b_m]}$. Then π_m is continuous, since $C(\Lambda)$ and $C([a_m, b_m])$ with the topologies of uniform convergence over compacts are metrizable by Lemma 5.1.2, and if $f_n \to f$ uniformly on compact subsets of Λ, then $f_n|_{[a_m,b_m]} \to f|_{[a_m,b_m]}$ uniformly on compact subsets of $[a_m, b_m]$. It follows from [117, Theorem 1.3.5] that $\pi_m(\mathcal{L}^n) = \mathcal{L}_i^n|_{[a_m,b_m]}$ is a $C([a_m, b_m])$-valued random variable. Tightness of the sequence implies that for any $\epsilon > 0$, we can find compact sets $K_m \subset C([a_m, b_m])$ so that

$$\mathbb{P}\big(\pi_m(\mathcal{L}_i^n) \notin K_m\big) \leq \epsilon/2^m$$

for each $m \geq 1$. Writing $K = \bigcap_{m=1}^{\infty} \pi_m^{-1}(K_m)$, it follows that

$$\mathbb{P}\big(\mathcal{L}_i^n \in K\big) \geq 1 - \sum_{m=1}^{\infty} \epsilon/2^m = 1 - \epsilon.$$

To conclude tightness of $\{\mathcal{L}_i^n\}$, it suffices to prove that $K = \bigcap_{m=1}^{\infty} \pi_m^{-1}(K_m)$ is sequentially compact in $C(\Lambda)$. We argue by diagonalization. Let $\{f_n\}$ be a sequence in K, so that $f_n|_{[a_m,b_m]} \in K_m$ for every m, n. Since K_1 is compact, there is a sequence $\{n_{1,k}\}$ of natural numbers such that the subsequence $\{f_{n_{1,k}}|_{[a_1,b_1]}\}_k$ converges in $C([a_1, b_1])$. Since K_2 is compact, we can take a further subsequence $\{n_{2,k}\}$ of $\{n_{1,k}\}$ so that $\{f_{n_{2,k}}|_{[a_2,b_2]}\}_k$ converges in $C([a_2, b_2])$. Continuing in this manner, we obtain sequences $\{n_{1,k}\} \supseteq \{n_{2,k}\} \supseteq \cdots$ so that $\{f_{n_{m,k}}|_{[a_m,b_m]}\}_k$ converges in $C([a_m, b_m])$ for all m. Writing $n_k = n_{k,k}$, it follows that the sequence $\{f_{n_k}\}$ converges uniformly on each $[a_m, b_m]$. If K is any compact subset of $C(\Lambda)$, then $K \subset [a_m, b_m]$ for some m, and hence $\{f_{n_k}\}$ converges uniformly on K. Therefore $\{f_{n_k}\}$ is a convergent subsequence of $\{f_n\}$.

□

5.6.3 Proof of Lemma 5.1.16

Proof. We will construct a candidate $\mathfrak{B}$ of $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ with the conditions of 5.1.16 assumed. Construct the ensemble $\mathfrak{B}$ in the following manner. Denote $B_0 = f$ and $B_{k+1} = g$ with $x_0 = f(T_0)$ and $y_0 = f(T_1)$. By Condition (3) of Lemma 5.1.16 we know $x_0 \geq x_1$ and $y_0 \geq y_1$. Then let $B_j(T_0) = x_j$ for all $j \in [\![1, k]\!]$ and then for all $i \in [\![T_0, T_1 - 1]\!]$ we have

$$B_j(i+1) = \begin{cases} B_j(i) + 1 & \text{if } B_j(i) + 1 \leq \min\{B_{j-1}(i+1), y_j\} \\ B_j(i) & \text{Else.} \end{cases} \tag{5.6.2}$$

This definition is well-defined, since we may find B_1 depending solely on the predetermined f, and then inductively find B_j since B_{j-1} has been determined by the previous curves in $\mathfrak{B}$.

In order to verify that this candidate ensemble $\mathfrak{B}$ is an element of $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$, three properties must be ensured:

$$\begin{aligned}&\text{(a) } \mathfrak{B}(T_0) = \vec{x} \text{ and } \mathfrak{B}(T_1) = \vec{y}\\&\text{(b) } f(i) \geq B_1(i) \geq \cdots \geq B_k(i) \geq g(i) \text{ for all } i \in [\![T_0, T_1]\!]\\&\text{(c) } B_j(i+1) - B_j(i) \in \{0,1\} \text{ for all } i \in [\![T_0, T_1 - 1]\!] \text{ and } j \in [\![1, k]\!]\end{aligned} \tag{5.6.3}$$

Property (c) follows directly from Definition 5.6.2, since $B_j(i+1) = B_j(i)$ or $B_j(i+1) = B_j(i)+1$, and hence $B_j(i+1) - B_j(i) \in \{0,1\}$. The remainder of the proof will be broken up into two steps, the first step proving property (a), and the second proving Property (b).

Step 1:

We know by definition that $\mathfrak{B}(T_0) = \vec{x}$, and we claim that $\mathfrak{B}(T_1) = \vec{y}$. We will show this claim inductively on j: We trivially know the claim is true for $j = 0$, since $y_0 = f(T_1)$ is given. Then suppose that $B_j(T_1) = x_j$ holds upto $j = n - 1$. First, we know by definition that $B_n(i+1) = B_n(i)$ if either $B_n(i) = y_n$ or $B_n(i) + 1 > B_n(i+1)$. Suppose that for some $i_0 \in [\![T_0, T_1]\!]$ we have $B_n(i_0) = B_n(i_0 + 1)$.

If $B_n(i_0) = B_n(i_0 + 1)$ because $B_n(i_0) = y_n$, then $B_n(T_1) = y_n$ since

$$y_n = B_n(i_0) = B_n(i_0 + 1) = \cdots = B_n(T_1)$$

and then the claim is true, namely that $B_n(T_1) = y_n$, and so induction holds.

Then, for the other case, when $B_n(i_0) = B_n(i_0 + 1)$ because $B_n(i_0) + 1 > B_{n-1}(i_0 + 1)$, we first need to prove that $B_j(i) \leq B_{j-1}(i)$ for $j \in [\![1, k]\!]$ for any $i \in [\![T_0, T_1]\!]$. We know this is true for T_0 since $x_0 \geq x_1 \geq \cdots \geq x_k$. Then, inductively we know that if $B_j(i) \leq B_{j-1}(i)$, $B_j(i+1) = B_j(i)$ or $B_j(i) + 1$. In the first case, $B_j(i+1) = B_j(i) \leq B_{j-1}(i) \leq B_j(i)$ by property (3) of 5.6.3. Then, if $B_j(i+1) = B_j(i) + 1$ implies $B_j(i+1) \leq B_j(i+1)$ by equation 5.6.2. Hence, we know that for $i \in [\![T_0, T_1]\!]$

$$f(i) \geq B_1(i) \geq \cdots \geq B_k(i) \tag{5.6.4}$$

Therefore, we know that $B_n(i_0) = B_n(i_0+1)$ and $B_n(i_0)+1 > B_{n-1}(i_0+1)$, which implies $B_n(i) = B_{n-1}(i)$. This implies that if we denote i_1 as the least i such that $B_{n-1}(i_1) = y_n$ then

$$B_n(i) = B_{n-1}(i) \text{ for all } i \in [\![i_0, i_1]\!]$$

We know that there exists such an $i_1 \in [\![T_0, T_1]\!]$ because where i_1 is the first i such that $B_{n-1}(i_1) = y_n$, since if $B_{n-1}(i+1) = B_{n-1}(i)$ then $B_n(i)+1 = B_{n-1}(i)+1 > B_{n-1}(i+1)$ by 5.6.2, the definition of $\mathfrak{B}$. Therefore we know $B_n(i+1) = B_n(i) = B_{n-1}(i) = B_{n-1}(i+1)$.

If $B_{n-1}(i+1) = B_n(i)+1$ then $B_n(i)+1 \leq B_{n-1}(i+1)$ by 5.6.2 so $B_n(i+1) = B_{n-1}(i+1)$ therefore inductively until B_n cannot increase above y_n, we know $B_n(i) = B_{n-1}(i)$. Because we know that there is some i_1 such that $B_{n-1}(i_1) = y_n$, and hence $B_n(i_1) = y_n$ we get $B_n(T_1) = y_n$ and the claim that $B_n(T_1) = y_n$ is true if there exists some i_0 such that $B_n(i_0) = B_n(i_0+1)$

Finally, assume that there exists no such i_0 that $B_n(i_0) = B_n(i_0+1)$. Then conversely $B_n(i)+1 \leq B_n(i+1)$ for all i, then we know that $B_n(i+1) = B_n(i)+1$ for all i unless $B_n(i) = y_n$ by 5.6.2. Therefore, until $B_n(i) = y_n$, we have $B_n(i+s) = B_n(i)+s$ hence $B_n(T_0+y_n-x_n) = B_n(T_0)+y_n-x_n = y_n$. By the inequality in condition (1) of 5.1.16, we have the following inequalities:

$$\begin{aligned} &T_1 - T_0 \geq y_n - x_n \geq 0 \\ &T_0 \leq T_0 + y_n - x_n \\ &T_1 \geq T_0 + y_n - x_n \end{aligned} \tag{5.6.5}$$

so $T_0+y_n-x_n \in [\![T_0, T_1]\!]$ and so $B_n(T_0+y_n-x_n)B_n(T_1) = y_n$. This means whether or not i_0 exists, the induction holds and therefore we know that for all j we have $B_j(T_1) = y_j$, so we know that $\mathfrak{B}(T_0) = \vec{x}$ and $\mathfrak{B}(T_1) = \vec{y}$ which concludes Step 1, proving Property (a) of 5.6.3.

Step 2: Now all that is left to verify avoidance, or Property (b) of 5.6.3. In equation 5.6.4, we already found that

$$f(i) \geq B_1(i) \geq \cdots \geq B_k(i)$$

so we must only prove that $B_k(i) \geq g(i)$ for all i. Suppose that $g(i) > B_k(i)$ for some $i \in [\![T_0, T_1]\!]$.

Since $g(T_0) < B_k(T_0) = x_k$ by Condition (3) in Lemma 5.1.16, we know that there exists some point i_0 such that $g(i_0) = B_k(i_0)$ and $g(i_0+1) > B_k(i_0+1)$. In particular, since g and B_k can each only increase by 1, this implies $B_k(i_0) = B_k(i_0+1)$. This implies either $B_k(i_0) = y_k$ or $B_k(i_0) + 1 > B_{k-1}(i_0+1)$. If $B_k(i_0) = y_k$ then since $g(i_0+1) \leq y_k$ by Condition (3) of Lemma 5.1.16, there is a contradiction.

Therefore, it must be the case that $B_k(i)+1 > B_{k-1}(i+1)$. Then we find that for any $j \in [\![1,k]\!]$ we know that $B_j(i) + 1 > B_{j-1}(i+1)$ implies $B_j(i_0) = B_{j-1}(i_0)$ since $B_j(i_0) = B_j(i_0+1)$ and $B_{j-1}(i_0) \geq B_j(i_0)$. This can be applied to each j to find that $g(i_0) = f(i_0)$ and $g(i_0+1) > f(i_0+1)$, which we assumed not to be the case. Therefore, we know that $g \leq B_k$ and so we have proven property (3), implying that if the three conditions in the statement of Lemma 5.1.16 are met then we know $\mathfrak{B} \in \Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ and so $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, f, g)$ is non-empty. □

5.6.4 Proof of Lemmas 5.3.6 and 5.3.7

We first prove Lemma 5.3.6. We will use the following lemma, which proves an analogous convergence result for a single rescaled Bernoulli random walk.

Lemma 5.6.3. *Let $x, y, a, b \in \mathbb{R}$ with $a < b$, and let $a_N, b_N \in N^{-\alpha}\mathbb{Z}$, $x^N, y^N \in N^{-\alpha/2}\mathbb{Z}$ be sequences with $a_N \leq a$, $b_N \geq b$, and $|y^N - x^N| \leq (b_N - a_N)N^{\alpha/2}$. Suppose $a_N \to a$, $b_N \to b$. Write $\tilde{x}^N = (x^N - pa_N N^{\alpha/2})/\sqrt{p(1-p)}$, $\tilde{y}^N = (y^N - pb_N N^{\alpha/2})/\sqrt{p(1-p)}$, and assume $\tilde{x}^N \to x$, $\tilde{y}^N \to y$ as $N \to \infty$. Let Y^N be a sequence of random variables with laws $\mathbb{P}^{a_N, b_N, x^N, y^N}_{free,N}$, and let $Z^N = Y^N|_{[a,b]}$. Then the law of Z^N converges weakly to $\mathbb{P}^{a,b,x,y}_{free}$ as $N \to \infty$.*

Proof. Let us write $z^N = (y^N - x^N)N^{\alpha/2}$ and $T_N = (b_N - a_N)N^{\alpha}$. Let $\tilde{B}$ be a standard Brownian bridge on $[0,1]$, and define random variables B^N, B taking values in $C([a_N, b_N])$, $C([a,b])$ respectively via

$$B^N(t) = \sqrt{b_N - a_N} \cdot \tilde{B}\left(\frac{t - a_N}{b_N - a_N}\right) + \frac{t - a_N}{b_N - a_N} \cdot \tilde{y}^N + \frac{b_N - t}{b_N - a_N} \cdot \tilde{x}^N,$$
$$B(t) = \sqrt{b-a} \cdot \tilde{B}\left(\frac{t-a}{b-a}\right) + \frac{t-a}{b-a} \cdot y + \frac{b-t}{b-a} \cdot x.$$

We observe that B has law $\mathbb{P}_{free}^{a,b,x,y}$ and $B^N \implies B$ as $N \to \infty$. By [39, Theorem 3.1], to show that $Z^N \implies B$, it suffices to find a sequence of probability spaces supporting Y^N, B^N so that

$$\rho(B^N, Y^N) = \sup_{t \in [a_N, b_N]} |B^N(t) - Y^N(t)| \implies 0 \quad \text{as} \quad N \to \infty. \tag{5.6.6}$$

It follows from Theorem 5.2.3 that for each $N \in \mathbb{N}$ there is a probability space supporting B^N and Y^N, as well as constants $C, a', \alpha' > 0$, such that

$$\left[e^{a'\Delta(N,x^N,y^N)}\right] \leq Ce^{\alpha' \log N} e^{|z^N - pT_N|^2/N^\alpha}, \tag{5.6.7}$$

where $\Delta(N, x^N, y^N) = \sqrt{p(1-p)}\, N^{\alpha/2} \rho(B^N, Y^N)$. Since $(z^N - pT_N)N^{-\alpha/2} \to \sqrt{p(1-p)}\,(y - x)$ by assumption, there exist $N_0 \in \mathbb{N}$ and $A > 0$ so that $|z - pT_N| \leq AN^{\alpha/2}$ for $N \geq N_0$. Then for $\epsilon > 0$ and $N \geq N_0$, Chebyshev's inequality and (5.6.7) give

$$\mathbb{P}(\rho(B^N, Y^N) > \epsilon) \leq Ce^{-a'\epsilon\sqrt{p(1-p)}\, N^{\alpha/2}} e^{\alpha' \log N} e^{A^2}.$$

The right hand side tends to 0 as $N \to \infty$, implying (5.6.6). □

We now give the proof of Lemma 5.3.6.

Proof. We prove the two statements of the lemma in two steps.

Step 1. In this step we fix $N_0 \in \mathbb{N}$ so that $\mathbb{P}_{avoid,N}^{a_N,b_N,\vec{x}^N,\vec{y}^N,f_N,g_N}$ is well-defined for $N \geq N_0$. Observe that we can choose $\epsilon > 0$ and continuous functions $h_1, \ldots, h_k : [a,b] \to \mathbb{R}$ depending on $a, b, \vec{x}, \vec{y}, f, g$ with $h_i(a) = x_i$, $h_i(b) = y_i$ for $i \in [\![1, k]\!]$, such that if $u_i : [a,b] \to \mathbb{R}$ are continuous functions with $\rho(u_i, h_i) = \sup_{x \in [a,b]} |u_i(x) - h_i(x)| < \epsilon$, then

$$f(x) - \epsilon > u_1(x) + \epsilon > u_1(x) - \epsilon > \cdots > u_k(x) + \epsilon > u_k(x) - \epsilon > g(x) + \epsilon \tag{5.6.8}$$

for all $x \in [a, b]$. By Lemma 5.1.6, we have

$$\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}(\rho(Q_i, h_i) > \epsilon \text{ for } i \in [\![1, k]\!]) > 0. \tag{5.6.9}$$

Since $y_i^N - x_i^N - p(b_N - a_N)N^{\alpha/2} \to \sqrt{p(1-p)}\,(y_i - x_i)$ as $N \to \infty$ for $i \in [\![1, k]\!]$ and $p < 1$, we can find $N_1 \in \mathbb{N}$ so that for $N \geq N_1$, $|y_i^N - x_i^N| \leq (b_N - a_N)N^{\alpha/2}$. It follows from Lemma 5.6.3 that if $\mathcal{Y}^N$ have laws $\mathbb{P}^{a_N,b_N,\vec{x}^N,\vec{y}^N}_{free,N}$ for $N \geq N_1$ and $\mathcal{Z}^N = \mathcal{Y}^N|_{\Sigma\times[a,b]}$, then the law of $\mathcal{Z}^N$ converges weakly to $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{avoid}$. In view of (5.6.9) we can then find N_2 so that if $N \geq \max(N_1, N_2)$ then

$$\mathbb{P}^{a_N,b_N,\vec{x}^N,\vec{y}^N}_{free,N}(\rho(Q_i, h_i) > \epsilon \text{ for } i \in [\![1, k]\!]) > 0.$$

We now choose N_3 so that $\sup_{x\in[a-1,b+1]} |f(x) - f_N(x)| < \epsilon/4$ and $\sup_{x\in[a-1,b+1]} |g(x) - g_N(x)| < \epsilon/4$. If $f = \infty$ (resp. $g = -\infty$), we interpret this to mean that $f_N = \infty$ (resp. $g_N = -\infty$). We take N_4 large enough so that if $N \geq N_4$ and $|x - y| \leq N^{-\alpha/2}$ then $|f(x) - f(y)| < \epsilon/4$ and $|g(x) - g(y)| < \epsilon/4$. Lastly, we choose N_5 so that $N_5^{-\alpha} < \epsilon/4$. Then for $N \geq N_0 = \max(N_1, N_2, N_3, N_4, N_5)$, we have

$$\{\rho(Q_i, h_i) > \epsilon \text{ for } i \in [\![1, k]\!]\} \subset \{f_N \geq \mathcal{Y}_1^N \geq \cdots \geq \mathcal{Y}_k^N \geq g_N \text{ on } [a_N, b_N]\}.$$

By (5.6.9), this implies that $\mathbb{P}^{a_N,b_N,\vec{x}^N,\vec{y}^N,f_N,g_N}_{avoid,N}$ is well-defined.

Step 2. In this step we prove that $\mathcal{Z}^N \implies \mathbb{P}^{a,b,\vec{x},\vec{y},f,g}_{avoid}$, with $\mathcal{Z}^N$ defined in the statement of the lemma. We write $\Sigma = [\![1, k]\!]$, $\Lambda = [a, b]$, and $\Lambda_N = [a_N, b_N]$. It suffices to show that for any bounded continuous function $F : C(\Sigma \times \Lambda) \to \mathbb{R}$ we have

$$\lim_{N\to\infty} [F(\mathcal{Z}^N)] = [F(Q)], \tag{5.6.10}$$

where Q has law $\mathbb{P}^{a,b,\vec{x},\vec{y},f,g}_{avoid}$.

We define the functions $H_{f,g} : C(\Sigma \times \Lambda) \to \mathbb{R}$ and $H^N_{f,g} : C(\Sigma \times \Lambda_N) \to \mathbb{R}$ by

$$H_{f,g}(\mathcal{L}) = \mathbf{1}\{f > \mathcal{L}_1 > \cdots > \mathcal{L}_k > g \text{ on } \Lambda\},$$
$$H^N_{f,g}(\mathcal{L}^N) = \mathbf{1}\{f \geq \mathcal{L}^N_1 \geq \cdots \geq \mathcal{L}^N_k \geq g \text{ on } \Lambda_N\}.$$

Then we observe that for $N \geq N_0$,

$$[F(\mathcal{Z}^N)] = \frac{[F(\mathcal{L}^N|_{\Sigma\times[a,b]})H^N_{f,g}(\mathcal{L}^N)]}{[H^N_{f,g}(\mathcal{L}^N)]}, \tag{5.6.11}$$

where $\mathcal{L}^N$ has law $\mathbb{P}^{a_N,b_N,\vec{x}^N,\vec{y}^N}_{free,N}$. By our choice of N_0 in Step 1, the denominator in (5.6.11) is positive for all $N \geq N_0$. Similarly, we have

$$[F(Q)] = \frac{[F(\mathcal{L})H_{f,g}(\mathcal{L})]}{[H_{f,g}(\mathcal{L})]}, \tag{5.6.12}$$

where $\mathcal{L}$ has law $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}$. From (5.6.11) and (5.6.12), we see that to prove (5.6.10) it suffices to show that for any bounded continuous function $F : C(\Sigma \times \Lambda) \to \mathbb{R}$,

$$\lim_{N\to\infty} [F(\mathcal{L}^N|_{\Sigma\times[a,b]})H^N_{f,g}(\mathcal{L}^N)] = [F(\mathcal{L})H_{f,g}(\mathcal{L})]. \tag{5.6.13}$$

By Lemma 5.6.3, $\mathcal{L}^N|_{\Sigma\times[a,b]} \implies \mathcal{L}$ as $N \to \infty$. Since $C(\Sigma \times \Lambda)$ is separable, the Skorohod representation theorem [39, Theorem 6.7] gives a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting $C(\Sigma \times \Lambda_N)$-valued random variables $\mathcal{L}^N$ with laws $\mathbb{P}^{a_N,b_N,\vec{x}^N,\vec{y}^N}_{free,N}$ and a $C(\Sigma\times\Lambda)$-valued random variable $\mathcal{L}$ with law $\mathbb{P}^{a,b,\vec{x},\vec{y}}_{free}$ such that $\mathcal{L}^N|_{\Sigma\times[a,b]} \to \mathcal{L}$ uniformly on compact sets, pointwise on Ω. Here we rely on the fact that a_N, b_N are respectively the largest element of $N^{-\alpha}\mathbb{Z}$ less than a and the smallest element greater than b, so that $\mathcal{L}^N|_{\Sigma\times[a,b]}$ uniquely determines $\mathcal{L}^N$ on $[a_N, b_N]$.

Define the events

$$E_1 = \{\omega : f > \mathcal{L}_1(\omega) > \cdots > \mathcal{L}_k(\omega) > g \text{ on } [a,b]\},$$

$$E_2 = \{\omega : \mathcal{L}_i(\omega)(r) < \mathcal{L}_{i+1}(\omega)(r) \text{ for some } i \in [\![0, k]\!] \text{ and } r \in [a, b]\},$$

where in the definition of E_2 we use the convention $\mathcal{L}_0 = f$, $\mathcal{L}_{k+1} = g$. The continuity of F implies that $F(\mathcal{L}^N|_{\Sigma\times[a,b]})H^N_{f_N,g_N}(\mathcal{L}^N) \to F(\mathcal{L})$ on the event E_1, and $F(\mathcal{L}^N|_{\Sigma\times[a,b]})H^N_{f_N,g_N}(\mathcal{L}^N) \to 0$ on the event E_2. By Lemma 5.1.5 we have $\mathbb{P}(E_1 \cup E_2) = 1$, so $\mathbb{P}$-a.s. we have $F(\mathcal{L}^N|_{\Sigma\times[a,b]})H^N_{f_N,g_N}(\mathcal{L}^N) \to F(\mathcal{L})H_{f,g}(\mathcal{L})$. The bounded convergence theorem then implies (5.6.13), completing the proof of (5.6.10). □

We now state two lemmas about Brownian bridges which will be used in the proof of Lemma 5.3.7. The first lemma shows that a Brownian bridge started at 0 almost surely becomes negative somewhere on its domain.

Lemma 5.6.4. *Fix any $T > 0$ and $y \in \mathbb{R}$, and let Q denote a random variable with law $\mathbb{P}^{0,T,0,y}_{free}$. Define the event $C = \{\inf_{s\in[0,T]} Q(s) < 0\}$. Then $\mathbb{P}^{0,T,0,y}_{free}(C) = 1$.*

Proof. Let B denote a standard Brownian bridge on $[0, 1]$, and let

$$\tilde{B}_s = B_{s/T} + \frac{sy}{T}, \quad \text{for } s \in [0, T].$$

Then $\tilde{B}$ has the law of Q. Consider the stopping time $\tau = \inf\{s > 0 : \tilde{B}_s < 0\}$. We will argue that $\tau = 0$ a.s, which implies the conclusion of the lemma since $\{\tau = 0\} \subset C$. We observe that since $\tilde{B}$ is a.s. continuous and $\mathbb{Q}$ is dense in $\mathbb{R}$,

$$\{\tau = 0\} = \bigcap_{\epsilon>0} \bigcup_{s\in(0,\epsilon)\cap\mathbb{Q}} \{\tilde{B}_s < 0\} \in \bigcap_{\epsilon>0} \sigma(\tilde{B}_s : s < \epsilon).$$

Here, $\sigma(\tilde{B}_s : s < \epsilon)$ denotes the σ-algebra generated by $\tilde{B}_s$ for $s < \epsilon$. We used the fact that for a fixed ϵ, each set $\{\tilde{B}_s < 0\}$ for $s \in (0, \epsilon) \cap \mathbb{Q}$ is contained in this σ-algebra, and thus so is their countable union. It follows from Blumenthal's 0-1 law [117, Theorem 7.2.3] that $\mathbb{P}(\tau = 0) \in \{0, 1\}$. To complete the proof, it suffices to show that $\mathbb{P}(\tau = 0) > 0$. By (5.2.1), $B_{s/T}$ is distributed normally with mean 0 and variance $\sigma^2 = (s/T)(1 - s/T)$. We observe that for any $s \in (0, T)$,

$$\mathbb{P}(\tau \leq s) \geq \mathbb{P}(B_{s/T} < -sy/T) = \mathbb{P}\left(\sigma \mathcal{N}(0,1) > (s/T)y\right) = \mathbb{P}\left(\mathcal{N}(0,1) > y\sqrt{s/(T-s)}\right).$$

As $s \to 0$, the probability on the right tends to $\mathbb{P}(\mathcal{N}(0,1) > 0) = 1/2$. Since $\{\tau = 0\} = \bigcap_{n=1}^{\infty}\{\tau \leq 1/n\}$ and $\{\tau \leq 1/(n+1)\} \subset \{\tau \leq 1/n\}$, we conclude that

$$\mathbb{P}(\tau = 0) = \lim_{n\to\infty} \mathbb{P}(\tau \leq 1/n) \geq 1/2.$$

Therefore $\mathbb{P}(\tau = 0) = 1$.

□

The second lemma shows that a difference of two independent Brownian bridges is another Brownian bridge.

Lemma 5.6.5. *Let $a, b, x_1, y_1, x_2, y_2, \in \mathbb{R}$ with $a < b$. Let $B_1(t)$, $B_2(t)$ be independent Brownian bridges from on $[a, b]$ from x_1 to y_1 and from x_2 to y_2 respectively, as defined in 5.1.2. If $B(t) = B_1(t) - B_2(t)$ for $t \in [a, b]$, then B is itself a Brownian bridge on $[a, b]$.*

Proof. By definition, for $i = 1, 2$ we have

$$B_i(t) = (b-a)^{1/2} \cdot \tilde{B}_i\left(\frac{t-a}{b-a}\right) + \left(\frac{b-t}{b-a}\right) \cdot x_i + \left(\frac{t-a}{b-a}\right) \cdot y_i,$$

with $\tilde{B}_i(t) = W_t^i - tW_1^i$ for independent Brownian motions W^1 and W^2. We have

$$B_1(t) - B_2(t) = (b-a)^{1/2} \cdot (\tilde{B}_1 - \tilde{B}_2)\left(\frac{t-a}{b-a}\right) + \left(\frac{b-t}{b-a}\right) \cdot (x_1 - x_2) + \left(\frac{t-a}{b-a}\right) \cdot (y_1 - y_2). \quad (5.6.14)$$

Note that the process $\tilde{B}_1 - \tilde{B}_2$ is a linear combination of continuous Gaussian mean 0 processes, so it is a continuous Gaussian mean 0 process, and is thus characterized by its covariance. Since $\tilde{B}_1(\cdot)$ and $\tilde{B}_2(\cdot)$ are both Gaussian with mean 0 and the same covariance, their difference $\tilde{B}_1(\cdot) - \tilde{B}_2(\cdot)$ is also Gaussian with the same mean and covariance. This implies that $\tilde{B}_1 - \tilde{B}_2$ is itself a Brownian bridge $\tilde{B}$ on $[a,b]$, and hence equation 5.6.14 can be rewritten

$$B_1(t) - B_2(t) = (b-a)^{1/2} \cdot \tilde{B}\left(\frac{t-a}{b-a}\right) + \left(\frac{b-t}{b-a}\right) \cdot (x_1 - x_2) + \left(\frac{t-a}{b-a}\right) \cdot (y_1 - y_2).$$

This is a Brownian bridge on $[a,b]$ from $x_1 - x_2$ to $y_1 - y_2$. □

To conclude this section, we prove Lemma 5.3.7.

Proof. Without loss of generality we may assume that $\mathcal{L}^N$ is the weak limit of $(f^N - \lambda s^2)/\sqrt{p(1-p)}$ as $N \to \infty$. By the Skorohod representation theorem, there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting random variables $\mathcal{X}^N$ and $\mathcal{X}$ with the laws of f^N and f^∞ respectively, such that $\mathcal{X}^N \to \mathcal{X}$ uniformly on compact sets as $N \to \infty$, pointwise on all of Ω. In particular, $\mathcal{X}^N(s) \to \mathcal{X}(s)$. We have $f_i^N(s) = N^{-\alpha/2}(L_i^N(sN^\alpha) - psN^\alpha) + \lambda s^2$, so $\mathcal{X}_i^N(s) = N^{-\alpha/2}(\mathcal{L}_i^N(sN^\alpha) - psN^\alpha)/\sqrt{p(1-p)}$, where $\mathcal{L}^N$ has the law of L^N.

Suppose that $\mathcal{X}_i(s) = \mathcal{X}_{i+1}(s)$ for some $i \in [\![1, k-2]\!]$. Then we have $\mathcal{X}_i^N(s) - \mathcal{X}_{i+1}^N(s) \to 0$, i.e., $N^{-\alpha/2}(\mathcal{L}_i^N(sN^\alpha) - \mathcal{L}_{i+1}^N(sN^\alpha)) \to 0$ as $N \to \infty$. Let us write $a = \lfloor sN^\alpha \rfloor N^{-\alpha}$, $b = \lceil (s+2)N^\alpha \rceil N^{-\alpha}$ and $x^N = \mathcal{L}_i^N(aN^\alpha) - \mathcal{L}_{i+1}^N(aN^\alpha)$, $y^N = \mathcal{L}_i^N(bN^\alpha) - \mathcal{L}_{i+1}^N(bN^\alpha)$. Then $N^{-\alpha/2}x^N \to 0$. If Q_i, Q_{i+1} are independent Bernoulli bridges with laws $\mathbb{P}_{Ber}^{a,b,\mathcal{L}_i^N(aN^\alpha),\mathcal{L}_i^N(bN^\alpha)}$ and $\mathbb{P}_{Ber}^{a,b,\mathcal{L}_{i+1}^N(aN^\alpha),\mathcal{L}_{i+1}^N(bN^\alpha)}$, then $\ell = Q_i - Q_{i+1}$ is a random walk bridge taking values in $\{-1,0,1\}$, from (a, x^N) to (b, y^N). Let us denote the law of $N^{-\alpha/2}\ell/\sqrt{p(1-p)}$ by $\mathbb{P}_{diff}^{a,b,x^N,y^N}$.

By Lemma 5.6.3, $(x^N + N^{-\alpha/2}Q_{i+1} - ptN^\alpha)/\sqrt{p(1-p)}$ and $(x^N + N^{-\alpha/2}Q_i - ptN^\alpha)/\sqrt{p(1-p)}$ converge weakly to the law of two Brownian bridges B^1 and B^2 respectively, and hence their difference $N^{-\alpha/2}\ell/\sqrt{p(1-p)}$ converges weakly to the difference of two independent Brownian bridges, $B^1 - B^2$. By Lemma 5.6.5, this difference is itself a Brownian bridge B on $[s, s+2]$ from 0 to y, i.e., B has law $\mathbb{P}_{free}^{s,s+2,0,y}$. Therefore $\mathbb{P}_{diff}^{a,b,x^N,y^N}$ converges weakly to $\mathbb{P}_{free}^{s,s+2,0,y}$. With probability one, $\min_{t\in[s,s+2]} B_t < 0$ by Lemma 5.6.4. Thus given $\delta > 0$, we can choose N large enough so that the probability of $N^{-\alpha/2}\ell/\sqrt{p(1-p)}$, or equivalently ℓ, remaining above 0 on $[a,b]$ is less than

δ. Thus for large enough N we have

$$\mathbb{P}\left(f_i^\infty(s) = f_{i+1}^\infty(s)\right) \leq \mathbb{P}\left(\mathbb{P}_{diff}^{a,b,x^N,y^N}\left(\sup_{s\in[a,b]} \ell(s) \geq 0\right) < \delta\right) \leq \\ \mathbb{P}\left(Z(a,b,\mathcal{L}^N(aN^\alpha),\mathcal{L}^N(bN^\alpha),\infty,\mathcal{L}_k^N) < \delta\right). \tag{5.6.15}$$

Here, Z denotes the acceptance probability of Definition 5.1.22. This is the probability that $k-1$ independent Bernoulli bridges $Q_1,\dots,Q_{k-1}$ on $[a,b]$ with entrance and exit data $\mathcal{L}^N(a)$ and $\mathcal{L}^N(b)$ do not cross one another or $\mathcal{L}_k^N$. The last inequality follows because ℓ has the law of the difference of Q_i and Q_{i+1}, and the acceptance probability is bounded above by the probability that Q_i and Q_{i+1} do not cross, i.e., that $Q_i - Q_{i+1} \geq 0$. By Proposition 5.3.1, given $\epsilon > 0$ we can choose δ so that the probability on the right in (5.6.15) is $< \epsilon$. We conclude that

$$\mathbb{P}\left(f_i^\infty(s) = f_{i+1}^\infty(s)\right) = 0.$$

□

5.6.5 Proof of Lemmas 5.2.1 and 5.2.2

We will prove the following lemma, of which the two lemmas are immediate consequences. In particular, Lemma 5.2.1 is the special case when $g^b = g^t$, and Lemma 5.2.2 is the case when $\vec{x} = \vec{x}\,'$ and $\vec{y} = \vec{y}\,'$. We argue in analogy to [105, Lemma 5.6].

Lemma 5.6.6. *Fix $k \in \mathbb{N}$, $T_0, T_1 \in \mathbb{Z}$ with $T_0 < T_1$, $S \subseteq [\![T_0,T_1]\!]$, and two functions $g^b, g^t : [\![T_0,T_1]\!] \to [-\infty,\infty)$ with $g^b \leq g^t$ on S. Also fix $\vec{x},\vec{y},\vec{x}\,',\vec{y}\,' \in \mathfrak{W}_k$ such that $x_i \leq x_i'$, $y_i \leq y_i'$ for $1 \leq i \leq k$. Assume that $\Omega_{avoid}(T_0,T_1,\vec{x},\vec{y},\infty,g^b;S)$ and $\Omega_{avoid}(T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t;S)$ are both non-empty. Then there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$, which supports two $[\![1,k]\!]$-indexed Bernoulli line ensembles $\mathfrak{L}^t$ and $\mathfrak{L}^b$ on $[\![T_0,T_1]\!]$ such that the law of $\mathfrak{L}^t$ $\left(\text{resp. } \mathfrak{L}^b\right)$ under $\mathbb{P}$ is given by $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}$ $\left(\text{resp. } \mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}\right)$ and such that $\mathbb{P}$-almost surely we have $\mathfrak{L}_i^t(r) \geq \mathfrak{L}_i^b(r)$ for all $i = 1,\dots,k$ and $r \in [\![T_0,T_1]\!]$.*

Proof. Throughout the proof, we will write $\Omega_{a,S}$ to mean $\Omega_{avoid}(T_0, T_1, \vec{x}, \vec{y}, \infty, g^b; S)$ and $\Omega'_{a,S}$ to mean $\Omega_{avoid}(T_0, T_1, \vec{x}\,', \vec{y}\,', \infty, g^t; S)$. We split the proof into two steps.

Step 1. We first aim to construct a Markov chain $(X^n, Y^n)_{n\geq 0}$, with $X^n \in \Omega_{a,S}$, $Y^n \in \Omega'_{a,S}$, with initial distribution given by

$$X_i^0(t) = \min(x_i + t - T_0,\ y_i), \qquad Y_i^0(t) = \min(x_i' + t - T_0,\ y_i'),$$

for $t \in [\![T_0, T_1]\!]$ and $1 \leq i \leq k$. First observe that we do in fact have $X^0 \in \Omega_{a,S}$, since $X_i^0(T_0) = x_i$, $X_i^0(T_1) = y_i$, $X_i^0(t) \leq \min(x_{i-1} + t - T_0,\ y_{i-1}) = X_{i-1}^0(t)$, and $X_k^0(t) \geq x_i + t - T_0 \geq g^b(T_0) + t - T_0 \geq g^b(t)$. We also note here that X^0 is *maximal* on the entire space $\Omega(T_0, T_1, \vec{x}, \vec{y})$, in the sense that for any $Z \in \Omega(T_0, T_1, \vec{x}, \vec{y})$, we have $Z_i(t) \leq X_i^0(t)$ for all $t \in [\![T_0, T_1]\!]$. In particular, X^0 is maximal on $\Omega_{a,S}$. Likewise, we see that Y^0 is maximal on $\Omega'_{a,S}$.

We want the chain (X^n, Y^n) to have the following properties:

(1) $(X^n)_{n\geq 0}$ and $(Y^n)_{n\geq 0}$ are both Markov in their own filtrations,

(2) (X^n) is irreducible and aperiodic, with invariant distribution $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber;S}$,

(3) (Y^n) is irreducible and aperiodic, with invariant distribution $\mathbb{P}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}_{avoid,Ber;S}$,

(4) $X_i^n \leq Y_i^n$ on $[\![T_0, T_1]\!]$ for all $n \geq 0$ and $1 \leq i \leq k$.

This will allow us to conclude convergence of X^n and Y^n to these two uniform measures.

We specify the dynamics of (X^n, Y^n) as follows. At time n, we uniformly sample a triple $(i, t, z) \in [\![1, k]\!] \times [\![T_0, T_1]\!] \times [\![x_k, y_1' - 1]\!]$. We also flip a fair coin, with $\mathbb{P}(\text{heads}) = \mathbb{P}(\text{tails}) = 1/2$. We update X^n and Y^n using the following procedure. If $j \neq i$, we leave X_j, Y_j unchanged, and for all points $s \neq t$, we set $X_i^{n+1}(s) = X_i^n(s)$. If $T_0 < t < T_1$, $X_i^n(t-1) = z$, and $X_i^n(t+1) = z + 1$ (note that this implies $X_i^n(t) \in \{z, z+1\}$), we consider two cases. If $t \in S$, then we set

$$X_i^{n+1}(t) = \begin{cases} z+1, & \text{if heads,} \\ z, & \text{if tails,} \end{cases}$$

assuming this does not cause $X_i^{n+1}(t)$ to fall below $X_{i+1}^n(t)$, with the convention that $X_{k+1}^n = g^b$. If $t \notin S$, we perform the same update regardless of whether it results in a crossing. In all other cases, we leave $X_i^{n+1}(t) = X_i^n(t)$. We update Y^n using the same rule, with g^t in place of g^b.

We first observe that X^n and Y^n are in fact non-intersecting on S for all n. Note X^0 is non-intersecting, and if X^n is non-intersecting, then the only way X^{n+1} could be intersecting on S is if the update were to push $X_i^{n+1}(t)$ below $X_{i+1}^n(t)$ for some i, t with $t \in S$. But any update of this form is suppressed, so it follows by induction that $X^n \in \Omega_{a,S}$ for all n. Similarly, we see that $Y^n \in \Omega'_{a,S}$.

It is easy to see that (X^n, Y^n) is a Markov chain, since at each time n, the value of (X^{n+1}, Y^{n+1}) depends only on the current state (X^n, Y^n), and not on the time n or any of the states prior to time n. Moreover, the value of X^{n+1} depends only on the state X^n, not on Y^n, so (X^n) is a Markov chain in its own filtration. The same applies to (Y^n). This proves the property (1) above.

We now argue that (X^n) and (Y^n) are irreducible. Fix any $Z \in \Omega_{a;S}$. As observed above, we have $Z_i \leq X_i^0$ on $[\![T_0, T_1]\!]$ for all i. We argue that we can reach the state Z starting from X^0 in some finite number of steps with positive probability. Due to the maximality of X^0, we only need to move the paths downward. If we do this starting with the bottom path, then there is no danger of the paths X_i crossing on S, or of X_k crossing g^b on S. To ensure that $X_k^n = Z_k$, we successively sample triples (k, t, z) as follows. We initialize $t = T_0 + 1$. If $X_k^n(t) = Z_k(t)$, we increment t by 1. Otherwise, we have $X_k^n(t) > Z_k(t)$, so we set $z = X_k^n(t) - 1$ and flip tails. This may or may not push $X_k(t)$ downwards by 1. We then increment t and repeat this process. If t reaches $T_1 - 1$, then at the increment we reset $t = T_0 + 1$. After finitely many steps, X_k will agree with Z_k on all of $[\![T_0, T_1]\!]$. We then repeat this process for X_i^n and Z_i, with i descending. Since each of these samples and flips has positive probability, and this process terminates in finitely many steps, the probability of transitioning from X^n to Z after some number of steps is positive. The same reasoning applies to show that (Y^n) is irreducible.

To see that the chains are aperiodic, simply observe that if we sample a triple (i, T_0, z) or (i, T_1, z), then the states of both chains will be unchanged.

To see that the uniform measure $\mathbb{P}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}_{avoid,Ber;S}$ on $\Omega_{a,S}$ is invariant for (X^n), fix any $\omega \in \Omega_{a,S}$. For simplicity, write μ for the uniform measure. Then for all $\tau \in \Omega_{a,S}$, we have $\mu(\tau) = 1/|\Omega_{a,S}|$. Hence

$$\sum_{\tau \in \Omega_{a,S}} \mu(\tau)\mathbb{P}(X^{n+1} = \omega \mid X^n = \tau) = \frac{1}{|\Omega_{a,S}|} \sum_{\tau \in \Omega_{a,S}} \mathbb{P}(X^{n+1} = \omega \mid X^n = \tau) =$$
$$\frac{1}{|\Omega_{a,S}|} \sum_{\tau \in \Omega_{a,S}} \mathbb{P}(X^{n+1} = \tau \mid X^n = \omega) = \frac{1}{|\Omega_{a,S}|} \cdot 1 = \mu(\omega).$$

The second equality is clear if $\tau = \omega$. Otherwise, note that $\mathbb{P}(X_{n+1} = \omega \mid X_n = \tau) \neq 0$ if and only if τ and ω differ only in one indexed path (say the ith) at one point t, where $|\tau_i(t) - \omega_i(t)| = 1$, and this condition is also equivalent to $\mathbb{P}(X^{n+1} = \tau \mid X^n = \omega) \neq 0$. If $X^n = \tau$, there is exactly one choice of triple (i, t, z) and one coin flip which will ensure $X_i^{n+1}(t) = \omega(t)$, i.e., $X^{n+1} = \omega$. Conversely, if $X^n = \omega$, there is one triple and one coin flip which will ensure $X^{n+1} = \tau$. Since the triples are sampled uniformly and the coin flips are fair, these two conditional probabilities are in fact equal. This proves (2), and an analogous argument proves (3).

Lastly, we argue that $X_i^n \leq Y_i^n$ on $[\![T_0, T_1]\!]$ for all $n \geq 0$ and $1 \leq i \leq k$. This is of course true at $n = 0$. Suppose it holds at some $n \geq 0$, and suppose that we sample a triple (i, t, z). Then the update rule can only change the values of the $X_i^n(t)$ and $Y_i^n(t)$. Notice that the values can change by at most 1, and if $Y_i^n(t) - X_i^n(t) = 1$, then the only way the ordering could be violated is if Y_i were lowered and X_i were raised at the next update. But this is impossible, since a coin flip of heads can only raise or leave fixed both curves, and tails can only lower or leave fixed both curves. Thus it suffices to assume $X_i^n(t) = Y_i^n(t)$.

There are two cases to consider that violate the ordering of $X_i^{n+1}(t)$ and $Y_i^{n+1}(t)$. Either (i) $X_i(t)$ is raised but $Y_i(t)$ is left fixed, or (ii) $Y_i(t)$ is lowered yet $X_i(t)$ is left fixed. These can only occur if the curves exhibit one of two specific shapes on $[\![t-1, t+1]\!]$. For $X_i(t)$ to be raised, we must have $X_i^n(t-1) = X_i^n(t) = X_i^n(t+1) - 1$, and for $Y_i(t)$ to be lowered, we must have $Y_i^n(t-1) - 1 = Y_i^n(t) = Y_i^n(t+1)$. From the assumptions that $X_i^n(t) = Y_i^n(t)$, and $X_i^n \leq Y_i^n$, we observe that both of these requirements force the other curve to exhibit the same shape on $[\![t-1, t+1]\!]$. Then the update rule will be the same for both curves for either coin flip, proving that both (i) and (ii) are impossible.

Step 2. It follows from (2) and (3) and [194, Theorem 1.8.3] that $(X^n)_{n\geq 0}$ and $(Y^n)_{n\geq 0}$ converge weakly to $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$ and $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}$ respectively. In particular, (X^n) and (Y^n) are tight, so $(X^n, Y^n)_{n\geq 0}$ is tight as well. By Prohorov's theorem, it follows that (X^n, Y^n) is relatively compact. Let (n_m) be a sequence such that (X^{n_m}, Y^{n_m}) converges weakly. Then by the Skorohod representation theorem [39, Theorem 6.7], it follows that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting random variables $\mathfrak{X}^n$, $\mathfrak{Y}^n$ and $\mathfrak{X}, \mathfrak{Y}$ taking values in $\Omega_{a,S}, \Omega'_{a,S}$ respectively, such that

(1) The law of $(\mathfrak{X}^n, \mathfrak{Y}^n)$ under $\mathbb{P}$ is the same as that of (X^n, Y^n),

(2) $\mathfrak{X}^n(\omega) \longrightarrow \mathfrak{X}(\omega)$ for all $\omega \in \Omega$,

(3) $\mathfrak{Y}^n(\omega) \longrightarrow \mathfrak{Y}(\omega)$ for all $\omega \in \Omega$.

In particular, (1) implies that $\mathfrak{X}^{n_m}$ has the same law as X^{n_m}, which converges weakly to $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$. It follows from (2) and the uniqueness of limits that $\mathfrak{X}$ has law $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x},\vec{y},\infty,g^b}$. Similarly, $\mathfrak{Y}$ has law $\mathbb{P}_{avoid,Ber;S}^{T_0,T_1,\vec{x}\,',\vec{y}\,',\infty,g^t}$. Moreover, condition (4) in Step 1 implies that $\mathfrak{X}_i^n \leq \mathfrak{Y}_i^n$, $\mathbb{P}$-a.s., so $\mathfrak{X}_i \leq \mathfrak{Y}_i$ for $1 \leq i \leq k$, $\mathbb{P}$-a.s. Thus we can take $\mathfrak{L}^b = \mathfrak{X}$ and $\mathfrak{L}^t = \mathfrak{Y}$.

□

5.7 Appendix B

The goal of this section is to establish the weak convergence of scaled avoiding Bernoulli line ensemble. We consider the $[\![1, k]\!]$-indexed line ensembles with distribution given by $\mathbb{P}_{avoid,Ber}^{0,T,\vec{x},\vec{y},\infty,-\infty}$ in the sense of Definition 5.1.15. Recall that this is just the law of k independent Bernoulli random walks that have been conditioned to start from $\vec{x} = (x_1, \ldots, x_k)$ at time 0 and and at $\vec{y} = (y_1, \cdots, y_k)$ at time T and are always ordered. Here $\vec{x}, \vec{y} \in \mathfrak{W}_k$ satisfy $T \geq y_i - x_i \geq 0$ for $i = 1, \ldots, k$. We will drop the infinities and simply write $\mathbb{P}^{0,T,\vec{x},\vec{y}}_{avoid,Ber}$ for the measure.

This section will be divided into 5 subsections. In Section 5.7.1, we introduce some definitions and formulate the precise statements of two main results we want to prove as Proposition 5.7.2 and Proposition 5.7.3. In Section 5.7.2, we introduce some fundamental knowledge about Skew Schur Polynomials and give the distribution of avoiding Bernoulli line ensembles at integer times through Skew Schur Polynomials as Lemma 5.7.8. In Section 5.7.3, we will prove our first main result Proposition 5.7.2. In Section 5.7.4 we introduce some notations and results about multi-indices and multivariate functions which paves the way for proof of Proposition 5.7.3. Section 5.7.5 will prove our second main result Proposition 5.7.3.

5.7.1 Definitions and Main Results

We start by introducing some helpful notations.

Definition 5.7.1. Fix $p, t \in (0,1)$, $k \in \mathbb{N}$, $\vec{a}, \vec{b} \in \mathbb{W}_k$ are two vectors in Weyl chamber defined in Definition 5.1.7. Suppose that $\vec{x}^T = (x_1^T, \cdots, x_k^T)$ and $\vec{y}^T = (y_1^T, \cdots, y_k^T)$ are two sequences of k-dimensional vectors in $\mathfrak{W}_k$ such that

$$\lim_{T\to\infty} \frac{x_i^T}{\sqrt{T}} = a_i \text{ and } \lim_{T\to\infty} \frac{y_i^T - pT}{\sqrt{T}} = b_i$$

for $i = 1, \ldots, k$. Define the sequence of random k-dimensional vectors Z^T by

$$Z^T = \Big(\frac{L_1(tT) - ptT}{\sqrt{T}}, \cdots, \frac{L_k(tT) - ptT}{\sqrt{T}}\Big) \tag{5.7.1}$$

where $(L_1, \cdots, L_k)$ is $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$-distributed.

We also introduce some constants below

$$c_1(p,t) = \frac{1}{p(1-p)t}, \quad c_2(p,t) = \frac{1}{p(1-p)(1-t)}, \quad c_3(p,t) = \frac{1}{2p(1-p)t(1-t)}$$
$$Z = (2\pi)^{\frac{k}{2}}(p(1-p)t(1-t))^{\frac{k}{2}} \cdot e^{\frac{c_1(t,p)}{2}\sum_{i=1}^k a_i^2} \cdot e^{\frac{c_2(t,p)}{2}\sum_{i=1}^k b_i^2} \det\left[e^{-\frac{1}{2p(1-p)}(b_i-a_j)^2}\right]_{i,j=1}^k \tag{5.7.2}$$

and define the function $\rho(z_1,\cdots,z_k) \equiv \rho(\vec{z})$ as the following:

$$\rho(z_1,\cdots,z_k) = \frac{1}{Z} \cdot \mathbf{1}_{\{z_1>\cdots>z_k\}} \cdot \det\left[e^{c_1(t,p)a_iz_j}\right]_{i,j=1}^k \det\left[e^{c_2(t,p)b_iz_j}\right]_{i,j=1}^k \prod_{i=1}^k e^{-c_3(t,p)z_i^2} \tag{5.7.3}$$

We will prove that the function $\rho(z)$ defined in (5.7.3) is a probability density function, meaning that it is non-negative and integrates to 1 over $\mathbb{R}^k$. Since this is an important ingredient of our results, we isolate it as Lemma 5.7.10 and will prove it in Section 5.7.3. For now, we assume that $\rho(\vec{z})$ in (5.7.3) is a density so that we can state our first main result in the following, which gives the limiting distribution of Z^T when vectors $\vec{a}$ and $\vec{b}$ contain distinct values.

Proposition 5.7.2. *Assume the same notation as in the Definition 5.7.1. When $a_1 > \cdots > a_k$ and $b_1 > \cdots > b_k$ are all distinct, the random vector Z^T converges weakly to a continuous distribution with the density in (5.7.3).*

Proposition 5.7.2 states the result when $\vec{a}$ and $\vec{b}$ consist of distinct values. When the values in $\vec{a}$ and $\vec{b}$ start to collide, the three determinants in the density function (5.7.3) will vanish(one in constant Z in equation (5.7.2) and the other two are in the expression of equation (5.7.3)). In the following, we are going to formulate the result under this new situation. We will construct a modified density function and the random vector Z^T will weakly converge to this new density function.

Suppose vectors $\vec{a}$ and $\vec{b}$ cluster as the following:

$$\begin{aligned}\vec{a} &= (a_1, \cdots, a_k) = (\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1}, \cdots, \underbrace{\alpha_p, \cdots, \alpha_p}_{m_p}) \\ \vec{b} &= (b_1, \cdots, b_k) = (\underbrace{\beta_1, \cdots, \beta_1}_{n_1}, \cdots, \underbrace{\beta_q, \cdots, \beta_q}_{n_q})\end{aligned} \tag{5.7.4}$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_p$, $\beta_1 > \beta_2 > \cdots > \beta_q$ and $\sum_{i=1}^{p} m_i = \sum_{i=1}^{q} n_i = k$. Denote $\vec{m} = (m_1, \cdots, m_p)$, $\vec{n} = (n_1, \cdots, n_q)$ and define two determinants $\varphi(\vec{a}, \vec{z}, \vec{m})$ and $\psi(\vec{b}, \vec{z}, \vec{n})$ below:

$$\begin{aligned}\varphi(\vec{a}, \vec{z}, \vec{m}) &= \det \begin{bmatrix} \left((c_1(t,p)z_j)^{i-1} e^{c_1(t,p)\alpha_1 z_j}\right)_{\substack{i=1,\cdots,m_1 \\ j=1,\cdots,k}} \\ \vdots \\ \left((c_1(t,p)z_j)^{i-1} e^{c_1(t,p)\alpha_p z_j}\right)_{\substack{i=1,\cdots,m_p \\ j=1,\cdots,k}} \end{bmatrix} \\ \psi(\vec{b}, \vec{z}, \vec{n}) &= \det \begin{bmatrix} \left((c_2(t,p)z_j)^{i-1} e^{c_2(t,p)\beta_1 z_j}\right)_{\substack{i=1,\cdots,n_1 \\ j=1,\cdots,k}} \\ \vdots \\ ((c_2(t,p)z_j)^{i-1} e^{c_2(t,p)\beta_q z_j})_{\substack{i=1,\cdots,n_q \\ j=1,\cdots,k}} \end{bmatrix}\end{aligned} \tag{5.7.5}$$

Then define the function

$$H(\vec{z}) = \varphi(\vec{a}, \vec{z}, \vec{m})\psi(\vec{b}, \vec{z}, \vec{n}) \prod_{i=1}^{k} e^{-c_3(t,p)z_i^2} \tag{5.7.6}$$

we can prove that $H(\vec{z})$ in (5.7.6) is non-negative and integrable over $\mathbb{R}^k$, so that we can multiply it with the normalizing constant $Z_c = \int_{\mathbb{R}^k} H(z) \cdot \mathbf{1}_{\{z_1 > \cdots > z_k\}} dz < \infty$ (the subscript c is for "collide") and make it a probability density function:

$$\rho_c(z_1, \cdots, z_k) = \frac{1}{Z_c} \cdot \mathbf{1}_{\{z_1 > \cdots > z_k\}} \cdot \varphi(\vec{a}, \vec{z}, \vec{m})\psi(\vec{b}, \vec{z}, \vec{n}) \prod_{i=1}^{k} e^{-c_3(t,p)z_i^2} \tag{5.7.7}$$

Now we are ready to state our second main result, which gives the weak convergence of Z^T

when $\vec{a}$ and $\vec{b}$ have collided values.

Proposition 5.7.3. *Assume the same notation as in the Definition 5.7.1 and suppose vectors $\vec{a}$, $\vec{b}$ has the form in (5.7.4). Then, the random vector Z^T converges weakly to a continuous distribution with density in (5.7.7).*

5.7.2 Skew Schur polynomials and distribution of avoiding Bernoulli line ensembles

First, We give some definitions and elementary results regarding skew Schur polynomials, which are mainly based on [185, Chapter 1].

Definition 5.7.4. *Partition, Interlaced, Conjugate*

1. A *partition* is an infinite sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r, \cdots)$ of non-negative integers in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \cdots$ and containing only finitely many non-zero terms. The non-zero λ_i are called *parts* of λ, the number of parts is called the *length* of the partition λ, denoted by $l(\lambda)$, and the sum of the parts is the *weight* of λ, denoted by $|\lambda|$.

2. Suppose λ and μ are two partitions, we denote $\lambda \supset \mu$ if $\lambda_i \geq \mu_i$ for all $i \in \mathbb{Z}^+$, and we can define a new partition $\lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \cdots)$.

3. Partitions $\lambda = (\lambda_1, \lambda_2, \cdots)$ and $\mu = (\mu_1, \mu_2, \cdots)$ are call *interlaced*, denoted by $\mu \preceq \lambda$, if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$.

4. The *conjugate* of a partition λ is the partition λ' such that

$$\lambda'_i = \max_{j \geq 1}\{j : \lambda_j \geq i\}$$

   In particular, $\lambda'_1 = l(\lambda)$, $\lambda_1 = l(\lambda')$ and notice that $\lambda'' = \lambda$. For example, the conjugate of (5441) is (43331).

According to Definition 5.7.4, we directly get that if $\mu \subset \lambda$ then $l(\lambda) \geq l(\mu)$ and $l(\lambda') \geq l(\mu')$. Also, $\mu \preceq \lambda$ implies $\mu \subset \lambda$. We can also derive the following corollary that is not very immediate.

Corollary 5.7.5. *If $\mu \leq \lambda$ are interlaced, then $\lambda'_i - \mu'_i = 0$ or 1 for every $i \geq 1$,.*

Proof. By definition, $\lambda'_i = \max\{j : \lambda_j \geq i\}$ and $\mu'_i = \max\{j : \mu_j \geq i\}$. Since $\mu \leq \lambda$ are interlaced, we have $\lambda_j \geq \mu_j \geq \lambda_{j+1}$ for every $j \geq 1$, where the first inequality $\lambda_j \geq \mu_j$ directly implies $\lambda'_i \geq \mu'_i$. Suppose there exists an i such that $\lambda'_i - \mu'_i \geq 2$. Then, by definition of μ'_i and λ'_i we have $\lambda_{\lambda'_i} \geq i$ and $\mu_{\mu'_i+1} < i$. When $\lambda'_i - \mu'_i \geq 2$, we have $\lambda_{\mu'_i+2} \geq \lambda_{\lambda'_i} \geq i > \mu_{\mu'_i+1}$, which contradicts the fact that $\mu \leq \lambda$ are interlaced. Therefore, we conclude that $\lambda'_i - \mu'_i$ can only be 0 or 1. □

Definition 5.7.6. *Elementary Symmetric Function*

For each integer $r \geq 0$, the r-th *elementary symmetric function* e_r is the sum of all products of r distinct variables x_i, so that $e_0 = 1$ and

$$e_r = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \tag{5.7.8}$$

for $r \geq 1$. For $r < 0$, we define e_r to be zero. In particular, when $x_1 = x_2 = \cdots = x_n = 1$, $x_{n+1} = x_{n+2} = \cdots = 0$, e_r is just the binomial coefficient when $0 \leq r \leq n$:

$$e_r(1^n) = \binom{n}{r}$$

and $e_r = 0$ when $r > n$.

Next, we introduce Skew Schur Polynomial based on [185, Chapter 1, (5.5), (5.11), (5.12)].

Definition 5.7.7. *Skew Schur Polynomial, Jacob-Trudi Formula*

1. Suppose $\mu \subset \lambda$ are partitions. If $\mu \leq \lambda$ are interlaced, then the *skew Schur polynomial* $s_{\lambda/\mu}$ with single variable x is defined by $s_{\lambda/\mu}(x) = x^{|\lambda-\mu|}$. Otherwise, we define $s_{\lambda/\mu}(x) = 0$.

2. Suppose $\mu \subset \lambda$ are two partitions, define the *skew Schur polynomial* $s_{\lambda/\mu}$ with respect to variables $x_1, x_2, \cdots, x_n$ by

$$s_{\lambda/\mu}(x_1, \cdots, x_n) = \sum_{(\nu)} \prod_{i=1}^{n} s_{\nu^i/\nu^{i-1}}(x_i) = \sum_{(\nu)} \prod_{i=1}^{n} x_i^{|\nu^i - \nu^{i-1}|} \tag{5.7.9}$$

summed over all sequences $(\nu) = (\nu^0, \nu^1, \cdots, \nu^n)$ of partitions such that $\nu^0 = \mu$, $\nu^n = \lambda$ and $\nu^0 \preceq \nu^1 \preceq \cdots \preceq \nu^n$. In particular, when $x_1 = x_2 = \cdots = x_n = 1$, the skew Schur polynomial is just the number of such sequences of interlaced partitions (ν). This definition also implies the following *branching relation* of skew Schur polynomials:

$$s_{\kappa/\mu} = \sum_{\lambda} s_{\kappa/\lambda} \cdot s_{\lambda/\mu} \tag{5.7.10}$$

3. We also have the following *Jacob-Trudi Formula*[185, Chapter 1, (5.5)] for the skew Schur polynomial:

$$s_{\lambda/\mu} = \det\left(e_{\lambda_i'-\mu_j'-i+j}\right)_{1\le i,j\le m} \tag{5.7.11}$$

where $m \ge l(\lambda')$, and e_r is the elementary symmetric function in Definition 5.7.6.

Based on the above preparation, we are ready to state the following lemma giving the distribution of avoiding Bernoulli line ensembles at time $\lfloor tT \rfloor$.

Lemma 5.7.8. *Assume the same notations as in Section 5.7.1, denote $m = \lfloor tT \rfloor$, $n = T - \lfloor tT \rfloor$. Then, the avoiding Bernoulli line ensemble at time m has the following distribution:*

$$\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}(L_1(m) = \lambda_1, \cdots, L_k(m) = \lambda_k) = \frac{s_{\lambda'/\mu'}(1^m) \cdot s_{\kappa'/\lambda'}(1^n)}{s_{\kappa'/\mu'}(1^T)} \tag{5.7.12}$$

where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k$ are positive integers, $s_{\lambda/\mu}$ denote skew Schur polynomials and they are specialized in all parameters equal to 1. *The μ partition is just the vector $\vec{x}^T$ and the κ partition should be $\vec{y}^T$.*

Remark 5.7.9. Here we let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k$ be positive integers, although they could potentially be negative. However, we can shift all the endpoints up such that all possible λ_i are positive. Also, in the proof we treat finite dimensional vectors as partitions because as long as we add infinitely many zeros at their ends we can and make them "partitions" in Definition 5.7.4.

Proof. Let $\Omega(0, T, \vec{x}^T, \vec{y}^T)$ be the set of all avoiding Bernoulli line ensembles from $\vec{x}^T$ to $\vec{y}^T$ and define the set

$$TB^T_{\lambda/\mu} := \{(\lambda^0, ..., \lambda^T) \mid \lambda^0 = \mu, \lambda^T = \lambda, \lambda^i \preceq \lambda^{i+1} \text{ for } i = 0, \cdots, T-1\}$$

From the result regarding the relationship between number of sequences of interlaced partitions and skew Schur polynomials (Definition 5.7.7, (2)), we have $|TB^T_{\lambda/\mu}| = s_{\lambda/\mu}(1^T)$. In the rest of the proof, we want to establish the fact that there is a bijection between $\Omega(0, T, \vec{x}^T, \vec{y}^T)$ and $TB^T_{\kappa'/\mu'}$ so that we have $|\Omega(0, T, \vec{x}^T, \vec{y}^T)| = s_{\kappa'/\mu'}$. Similarly, we get $|\Omega(0, m, \vec{x}^T, \lambda)| = s_{\lambda'/\mu'}$ and $|\Omega(m, T, \lambda, \vec{y}^T)| = s_{\kappa'/\lambda'}$. Then, since $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$ puts uniform measure on the set $\Omega(0, T, \vec{x}^T, \vec{y}^T)$, we conclude

$$\begin{aligned} {}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}(L_1(m) = \lambda_1, \cdots, L_k(m) = \lambda_k) =& \frac{|\Omega(0, m, \vec{x}^T, \lambda)| \cdot |\Omega(m, T, \lambda, \vec{y}^T)|}{|\Omega(0, T, \vec{x}^T, \vec{y}^T)|} \\ =& \frac{s_{\lambda'/\mu'}(1^m) \cdot s_{\kappa'/\lambda'}(1^n)}{s_{\kappa'/\mu'}(1^T)} \end{aligned}$$

where $\lambda = (\lambda_1, \cdots, \lambda_k)$ is a partition such that $\vec{x}^T \subset \lambda \subset \vec{y}^T$, thus finishing the proof.

Now we prove that there exists a bijection $f : \Omega(0, T, \vec{x}^T, \vec{y}^T) \to TB^T_{\kappa'/\mu'}$. For each line ensemble $\mathfrak{L} \in \Omega(0, T, \vec{x}^T, \vec{y}^T)$ with $\mathfrak{L} = (L_1, \cdots, L_k)$, we define a k-dimensional vector $\lambda^i(\mathfrak{L}) := (\lambda^i_1, \lambda^i_2, ..., \lambda^i_k)$, where $0 \leq i \leq T$ is an integer and $\lambda^i_\alpha = L_\alpha(i)$. In the following discussion we drop $\mathfrak{L}$ and briefly write λ^i. We claim that their conjugates $(\lambda^i)'$ form interlaced partitions:

$$(\lambda^0)' \preceq (\lambda^1)' \preceq \cdots \preceq (\lambda^T)' \tag{5.7.13}$$

where $\left(\lambda^0\right)' = \mu'$ and $\left(\lambda^T\right)' = \kappa'$. We first explain that $(\lambda^i)'$ is a partition for every $i = 0, \cdots, T$. Actually, it simply follows from that $(\lambda^i_\alpha)' = \max\{j : \lambda^i_j \geq \alpha\} \geq \max\{j : \lambda^i_j \geq \alpha + 1\} = (\lambda^i_{\alpha+1})'$. Now we prove (5.7.13), which requires us to show

$$(\lambda^{i+1}_\alpha)' \geq (\lambda^i_\alpha)' \geq (\lambda^{i+1}_{\alpha+1})', \text{ for every } i = 0, \cdots, T-1 \text{ and } \alpha = 1, \cdots, k-1$$

By the definition of Bernoulli random walk, we have $\lambda_j^{i+1} \geq \lambda_j^i \geq \lambda_j^{i+1} - 1$. Therefore, we have

$$\max\{j : \lambda_j^{i+1} \geq \alpha\} \geq \max\{j : \lambda_j^i \geq \alpha\} \geq \max\{j : \lambda_j^{i+1} \geq \alpha + 1\}$$

and this is exactly (5.7.13). Therefore, we have defined a function $f : \Omega(0, T, \vec{x}^T, \vec{y}^T) \to TB^T_{\kappa'/\mu'}$ by

$$f(\mathfrak{L}) = \left((\lambda^0)', \cdots, (\lambda^T)'\right) \tag{5.7.14}$$

Next, we prove the function f is in fact a bijection. First, to show injectivity, suppose that there are two Bernoulli line ensembles, $\mathfrak{L}, \widetilde{\mathfrak{L}} \in \Omega(0, T, \vec{x}^T, \vec{y}^T)$ such that $\mathfrak{L} \neq \widetilde{\mathfrak{L}}$. Bernoulli line ensembles are determined by their values at integer times, so this would imply that there exists some (q, r) such that $0 \leq r \leq T$, $0 \leq q \leq k$ and $L_q(r) \neq \widetilde{L}_q(r)$ where L_q and $\widetilde{L}_q$ are components of $\mathfrak{L}$ and $\widetilde{\mathfrak{L}}$ respectively. This implies that $(\lambda^r(\mathfrak{L}))' \neq \left(\lambda^r(\widetilde{\mathfrak{L}})\right)'$, so we have injectivity.

Now, we prove surjectivity. For any sequence of interlaced partitions $\overline{\lambda} = (\lambda^0, \cdots, \lambda^T)$ satisfying $(\lambda^0)' = \vec{x}^T$ and $(\lambda^T)' = \vec{y}^T$, we claim that $(\lambda^0)'$, $(\lambda^1)', \cdots, (\lambda^T)'$ consist of an avoiding Bernoulli line ensemble in $\Omega(0, T, \vec{x}^T, \vec{y}^T)$ by letting $L_\alpha(i) = (\lambda^i_\alpha)'$. Applying Corollary 5.7.5, we have $L_\alpha(i+1) - L_\alpha(i) = (\lambda^{i+1}_\alpha)' - (\lambda^i_\alpha)'$ can only be 0 or 1, thus $L_\alpha(i)$, $0 \leq i \leq T$ is a Bernoulli random walk for every $1 \leq \alpha \leq k$. In addition, by using $\lambda^i_\alpha \geq \lambda^i_{\alpha+1}$ we get $(\lambda^i_\alpha)' \geq (\lambda^i_{\alpha+1})'$, which indicates that k Bernoulli random walks avoid each other. Therefore, we proved the surjectivity and complete the proof. $\square$

By Jacob-Trudi formula (5.7.11) and Lemma 5.7.8, we further get

$$(L_1(m) = \lambda_1, \cdots, L_k(m) = \lambda_k) = \frac{\det\left[e_{\lambda_i - \mu_j + j - i}(1^m)\right]_{i,j=1}^k \cdot \det\left[e_{\kappa_i - \lambda_j + j - i}(1^n)\right]_{i,j=1}^k}{\det\left[e_{\kappa_i - \mu_j + j - i}(1^T)\right]_{i,j=1}^k} \tag{5.7.15}$$

where $\mu_i = \vec{x}_i^T$ and $\kappa_i = \vec{y}_i^T$.

5.7.3 Proof of Proposition 5.7.2

In this section, we first prove prove that the function in (5.7.3) is a density and then prove the weak convergence result in Proposition 5.7.2. The fact that (5.7.3) is a density is formulated in the following lemma.

Lemma 5.7.10. *Assume the same notations as in Section 5.7.1. Denote the function*

$$\widetilde{\rho}(z_1,\cdots,z_k) = \mathbf{1}_{\{z_1>z_2>\cdots>z_k\}} \det\left[e^{c_1(t,p)a_i z_j}\right]_{i,j=1}^{k} \cdot \det\left[e^{c_2(t,p)b_i z_j}\right]_{i,j=1}^{k} \cdot \prod_{i=1}^{k} e^{-c_3(t,p)z_i^2} \quad (5.7.16)$$

Then $\widetilde{\rho}(z_1,\cdots,z_k) \geq 0$ for all $\vec{z} = (z_1,\cdots,z_k) \in \mathbb{R}^k$ and $\widetilde{\rho}(z_1,\cdots,z_k) > 0$ if $z_1 > z_2 > \cdots > z_k$. Moreover, the function $\widetilde{\rho}$ is integrable on $\mathbb{R}^k$ and we have

$$\int_{\mathbb{R}^k} \widetilde{\rho}(z_1,\cdots,z_k)dz_1\cdots dz_k = Z \quad (5.7.17)$$

where the constant Z is defined in (5.7.2), thus implying the function $\rho(\vec{z})$ in (5.7.3) is a density.

To prove Lemma 5.7.10, we are going to find the asymptotic formula of the probability mass function (5.7.15) and its relationship with function $\rho(z)$ in (5.7.3). By Jacob-Trudi formula (5.7.11), we only need to find the asymptotic formula for elementary symmetric functions $e_{\lambda_i - x_j^T + j - i}(1^m)$, $e_{y_i^T - \lambda_j + j - i}(1^n)$ and $e_{y_i^T - x_j^T + j - i}(1^T)$. By the definition of random vector Z^T in (5.7.1), we find that

$$\left\{Z_1^T = z_1,\cdots,Z_k^T = z_k\right\} \equiv \{L_1(tT) = \lambda_1,\cdots,L_k(tT) = \lambda_k\} \quad (5.7.18)$$

where $\lambda_i = z_i\sqrt{T} + ptT$ are integers for $i = 1,\cdots,k$. In addition, $x_i^T = a_i\sqrt{T} + o\left(\sqrt{T}\right)$ and $y_i^T = b_i\sqrt{T} + pT + o\left(\sqrt{T}\right)$ by Definition 5.7.1. Therefore, we have

$$\begin{aligned} \lambda_i - x_j^T + j - i &= pm + (z_i - a_j)\sqrt{T} + o\left(T^{1/2}\right), \\ y_i^T - \lambda_j + j - i &= pn + (b_i - z_j)\sqrt{T} + o\left(T^{1/2}\right), \\ y_i^T - x_j^T + j - i &= pT + (b_i - a_j)\sqrt{T} + o\left(T^{1/2}\right) \end{aligned} \quad (5.7.19)$$

Thus, we only need to consider the elementary symmetric functions in the form $e_N(1^n)$, where $N = pn + x\sqrt{n}$ and $x \in [-R, R]$ is bounded. In this case, we have the following lemma giving the asymptotic behavior of $e_N(1^n)$.

Lemma 5.7.11. *Suppose that $p \in (0,1)$ and $R > 0$ are given. Suppose that $x \in [-R, R]$ and $N = pn + \sqrt{n}x$ is an integer. Then*

$$\begin{aligned} e_N(1^n) = (\sqrt{2\pi})^{-1} \cdot \exp\left(-\frac{x^2}{2(1-p)p}\right) \cdot \exp\left(N \log\left(\frac{1-p}{p}\right)\right) \cdot \exp\left(O(n^{-1/2})\right) \\ \cdot \exp\left(-n\log(1-p) - (1/2)\log n - (1/2)\log\left(p(1-p)\right)\right) \end{aligned} \tag{5.7.20}$$

where the constant in the big O notation depends on p and R alone. Moreover, there exist positive constants $C, c > 0$ depending on p alone such that for all large enough $n \in \mathbb{N}$ and $N \in [0, n]$,

$$e_N(1^n) \leq C \cdot \exp\left(N \log\frac{1-p}{p} - n\log(1-p) - (1/2)\log n\right) \cdot \exp\left(-cn^{-1}(N - pn)^2\right). \tag{5.7.21}$$

Remark 5.7.12. Notice that when $R > 0$ is fixed, $N \in [pn - R\sqrt{n}, pn + R\sqrt{n}]$. However, we specify the range of N by $[0, n]$. First, it is because when $N < 0$ or $N > n$ the elementary symmetric function $e_N(1^n)$ would be zero by Definition 5.7.6 and the situation becomes trivial. Second, when n is sufficiently large, the interval $[0, n]$ will cover $[pn - R\sqrt{n}, pn + R\sqrt{n}]$, so it's sufficient to consider the case when $N \in [0, n]$.

Proof of Lemma 5.7.11. For clarity the proof is split into several steps.

Step 1. In this step we prove (5.7.20). Using the formula for elementary symmetric function (5.7.6), we obtain

$$e_N(1^n) = \frac{n!}{N!(n-N)!} \tag{5.7.22}$$

We have the following Stirling's formula [145] that for $n \geq 1$

$$n! = \sqrt{2\pi n}n^n e^{-n} e^{r_n}, \text{ where } \frac{1}{12n+1} < r_n < \frac{1}{12n} \tag{5.7.23}$$

Applying the Stirling's formula to equation (5.7.22) implies that

$$
\begin{aligned}
e_N(1^n) &= \frac{\exp\left((n+1/2)\log n - (N+1/2)\log N - (n-N+1/2)\log(n-N) + O\left(n^{-1}\right)\right)}{\sqrt{2\pi}} \\
&= (\sqrt{2\pi})^{-1} \cdot \exp\left((n+1/2)\log n - (N+1/2)\log\frac{N}{pn} - (n-N+1/2)\log\frac{n-N}{(1-p)n}\right) \\
&\cdot \exp\left(-(N+1/2)\log(pn) - (n-N+1/2)\log((1-p)n) + O\left(n^{-1}\right)\right).
\end{aligned}
\tag{5.7.24}
$$

Denote $\Delta = \sqrt{n}x = O\left(n^{-1/2}\right)$, and we now use the Taylor expansion of the logarithm and the expression for N to get

$$
\log\frac{N}{pn} = \log\left(1 + \frac{\Delta}{pn}\right) = \frac{\Delta}{pn} - \frac{1}{2}\frac{\Delta^2}{p^2n^2} + O\left(n^{-3/2}\right)
$$

Analogously, we have

$$
\log\frac{n-N}{(1-p)n} = \log\left(1 - \frac{\Delta}{(1-p)n}\right) = -\frac{\Delta}{(1-p)n} - \frac{1}{2}\frac{\Delta^2}{(1-p)^2n^2} + O\left(n^{-3/2}\right)
$$

Plugging the two equations above to equation (5.7.24) we get

$$
\begin{aligned}
e_N(1^n) &= (\sqrt{2\pi})^{-1} \cdot \exp\left(-(N+1/2)\left[\frac{\Delta}{pn} - \frac{1}{2}\frac{\Delta^2}{p^2n^2} + O\left(n^{-3/2}\right)\right]\right) \\
&\cdot \exp\left(-(n-N+1/2)\left[-\frac{\Delta}{(1-p)n} - \frac{1}{2}\frac{\Delta^2}{(1-p)^2n^2} + O\left(n^{-3/2}\right)\right]\right) \\
&\cdot \exp\left((n+1/2)\log n - (N+1/2)\log(pn) - (n-N+1/2)\log((1-p)n) + O\left(n^{-1}\right)\right)
\end{aligned}
\tag{5.7.25}
$$

We next observe that

$$
\begin{aligned}
&-\frac{\Delta(N+1/2)}{pn}+\frac{(n-N+1/2)\Delta}{(1-p)n}=-\frac{\Delta^2}{p(1-p)n}+O\left(n^{-1/2}\right)\\
&\frac{\Delta^2(N+1/2)}{2n^2p^2}+\frac{\Delta^2(n-N+1/2)}{2(1-p)^2n^2}=\frac{\Delta^2}{2p(1-p)n}+O\left(n^{-1/2}\right)\\
&(n+1/2)\log n-(N+1/2)\log(pn)-(n-N+1/2)\log((1-p)n)=\\
&N\log\frac{1-p}{p}-\frac{1}{2}\log p(1-p)-\frac{1}{2}\log n-n\log(1-p)
\end{aligned}
\tag{5.7.26}
$$

Plugging (5.7.26) into (5.7.25) we arrive at (5.7.20).

Step 2. In this step we prove (5.7.21). If $N=0$ or n we know that $e_N(1^n)=1$ and then (5.7.21) is easily seen to hold with $C=1$ and any $c\in(0,\min(-\log p,-\log(1-p)))$. Thus it suffices to consider the case when $N\in[1,n-1]$ and in the sequel we also assume that $n\geq 2$.

Combining (5.7.22) and (5.7.23) we conclude that

$$
e_N(1^n)\leq\exp\left((n+1/2)\log n-(N+1/2)\log N-(n-N+1/2)\log(n-N)\right) \tag{5.7.27}
$$

From (5.7.27) we get for all large enough n that

$$
\begin{aligned}
&\phi_n:=\log\left[e_N(1^n)\cdot\exp\left(-N\log((1-p)/p)+n\log(1-p)+(1/2)\log n\right)\right]\\
&\leq(n+\frac{1}{2})\log n-(N+\frac{1}{2})\log N-(n-N+\frac{1}{2})\log(n-N)-N\log\frac{1-p}{p}+n\log(1-p)+\frac{1}{2}\log n\\
&=(n+1/2)\log n-(N+1/2)\log\frac{N}{pn}-(N+1/2)\log(pn)-(n-N+1/2)\log\frac{n-N}{(1-p)n}\\
&-(n-N+1/2)\log((1-p)n)-N\log\frac{1-p}{p}+n\log(1-p)+(1/2)\log n\\
&=-(N+\frac{1}{2})\log\frac{N}{pn}-(n-N+\frac{1}{2})\log\frac{n-N}{(1-p)n}-\frac{1}{2}\log\left(p(1-p)\right)\\
&=-(pn+\Delta+1/2)\log\left(1+\frac{\Delta}{pn}\right)-((1-p)n-\Delta+1/2)\log\left(1-\frac{\Delta}{(1-p)n}\right)-\frac{1}{2}\log\left(p(1-p)\right)\\
&\leq C_1+\psi_n(\Delta)
\end{aligned}
$$

where $C_1 > 0$ is sufficiently large depending on p alone and

$$\psi_n(s) = -(pn + s + 1/2)\log\left(1 + \frac{s}{pn}\right) - ((1-p)n - s + 1/2)\log\left(1 - \frac{s}{(1-p)n}\right) \tag{5.7.28}$$

where $s \in [-pn+1, (1-p)n-1]$. We claim that we can find positive constants $C_2 > 0$ and $c > 0$ such that for all n sufficiently large and $s \in [-pn+1, (1-p)n-1]$ we have

$$\psi_n(s) \leq C_2 - cn^{-1}s^2 \tag{5.7.29}$$

We prove (5.7.29) in Step 3 below. For now we assume its validity and conclude the proof of (5.7.21). In view of $\phi_n \leq C_1 + \psi_n(s)$ and (5.7.29) we know that

$$e_N(1^n) \leq \exp\left(C_1 + C_2 + N\log((1-p)/p) - n\log(1-p) - (1/2)\log n\right) \cdot \exp(-cn^{-1}(N-pn)^2),$$

which proves (5.7.21) with $C = e^{C_1+C_2}$.

Step 3. In this step we prove (5.7.29) in the case $s \in [0, n]$. A direct computation gives

$$\begin{aligned} \psi_n'(s) &= -\log\left(1 + \frac{t}{pn}\right) + \log\left(1 - \frac{t}{(1-p)n}\right) + \frac{1}{2}\cdot\frac{1}{pn+t} + \frac{1}{2}\cdot\frac{1}{(1-p)n-t} \\ \psi_n''(s) &= \frac{(n+1)\cdot s^2 + (2p-1)n(n+1)\cdot s + p(p-1)n^2(n+1) + (1/2)n^2}{(pn+s)^2((1-p)n-s)^2} \end{aligned} \tag{5.7.30}$$

Notice that the numerator of $\psi_n''(s)$ is a quadratic function and its minimum is at $x_{min} = -\frac{(2p-1)n(n+1)}{2(n+1)} = (-p+1/2)n$, which is the midpoint of the interval $[-pn+1, (1-p)n-1]$. Thus, the numerator reaches its maximum at either of the two endpoints of the interval $[-pn+1, (1-p)n-1]$. The denominator is the square of a parabola that reaches its minimum also at the endpoints of the interval $[-pn+1, (1-p)n-1]$. Therefore, we conclude that

$$\begin{aligned} \psi_n''(s) &\leq \psi_n''(-pn+1) = \psi_n''((1-p)n-1) = \frac{-\frac{1}{2}n^2+1}{(n-1)^2} = -\frac{1}{2} - \frac{1}{n-1} + \frac{1}{2}\cdot\frac{1}{(n-1)^2} \\ &\leq -\frac{1}{2}\cdot\frac{1}{n-1} \leq -\frac{1}{2n} = -2cn^{-1} \end{aligned} \tag{5.7.31}$$

where $c = 1/4$. Next, we prove (5.7.21) under two cases when $s \in [-pn+1, 0]$ and $s \in [0, (1-p)n-1]$, respectively.

1° When $s \in [-pn+1, 0]$, by the fundamental theorem of calculus and (5.7.31) we get

$$\psi_n'(s) = \psi_n'(0) - \int_s^0 \psi_n''(y)dy \geq \psi_n'(0) - (-s)(-2cn^{-1}) = \frac{2p-1}{2p(1-p)n} - 2cn^{-1}s,$$

and a second application of the same argument yields for $s \in [-pn+1, 0]$

$$\psi_n(s) = \psi_n(0) - \int_s^0 \psi_n'(y)dy \leq -\int_s^0 \left(\frac{2p-1}{2p(1-p)n} - 2cn^{-1}y\right) dy = \frac{(2p-1)s}{2p(1-p)n} - cn^{-1}s^2,$$

When $p \leq 1/2$, $\frac{(2p-1)s}{2p(1-p)n} \leq \frac{(2p-1)pn}{2p(1-p)n} = \frac{1-2p}{2(1-p)}$, so (5.7.29) gets proved with $C_2 = \frac{1-2p}{2(1-p)}$. When $p > 1/2$, (5.7.29) gets proved $C_2 = 0$.

2° When $s \in [0, (1-p)n-1]$, similarly using the fundamental theorem of calculus and (5.7.31) we get

$$\psi_n'(s) = \psi_n'(0) + \int_0^s \psi_n''(y)dy \leq= \frac{2p-1}{2p(1-p)n} - 2cn^{-1}s,$$

and a second application of the same argument yields for $s \in [0, (1-p)n-1]$

$$\psi_n(s) = \psi_n(0) + \int_0^s \psi_n'(y)dy \leq \frac{(2p-1)s}{2p(1-p)n} - cn^{-1}s^2,$$

When $p \geq 1/2$, $\frac{(2p-1)s}{2p(1-p)n} \leq \frac{(2p-1)(1-p)n}{2p(1-p)n} = \frac{2p-1}{2p}$, so (5.7.29) gets proved with $C_2 = \frac{2p-1}{2p}$. When $p < 1/2$, (5.7.29) gets proved $C_2 = 0$. Combining cases 1° and 2° we complete the proof. □

Based on Lemma 5.7.11, we introduce the following lemma computing quantities $A_\lambda(T)$ and $B_\lambda(T)$ which help us to find the asymptotic behavior of probability mass function (5.7.15) and its relationship with $\rho(z)$.

Lemma 5.7.13. *Assume the same notation as in Section 5.7.1 and Section 5.7.2. Fix $\vec{z} \in \mathbb{R}^k$ such* *that $z_1 > \cdots > z_k$. Suppose that $T_0 \in \mathbb{N}$ is sufficiently large so that for $T \geq T_0$ we have*

$$z_k\sqrt{T} + ptT \geq a_1\sqrt{T} + k + 1 \text{ and } b_k\sqrt{T} + pT \geq z_1\sqrt{T} + ptT + k + 1,$$

which ensures that $\lambda_i - x_j^T + j - i$ and $y_i^T - \lambda_j + j - i$ in (5.7.19) are positive. Then, for a signature λ of length k we define

$$A_\lambda(T) = s_{\lambda'/\mu'}(1^m) \cdot s_{\kappa'/\lambda'}(1^n), \text{ where } m = \lfloor tT \rfloor,\ n = T - m,\ \mu = \vec{x}^T,\ \kappa = \vec{y}^T \tag{5.7.32}$$

$$\begin{aligned} B_\lambda(T) &= (\sqrt{2\pi})^k \cdot \exp\left(kT\log(1-p) + k\log T + (k/2)\log(p(1-p))\right) \\ &\cdot \exp\left(-\log\left(\frac{1-p}{p}\right)\sum_{i=1}^{k}(y_i^T - x_i^T)\right) \cdot A_\lambda(T) \end{aligned} \tag{5.7.33}$$

We claim that

$$\lim_{T\to\infty} B_\lambda(T) = \widetilde{\rho}(z_1, \cdots, z_k) \cdot (2\pi p(1-p)t(1-t))^{-\frac{k}{2}} \cdot \prod_{i=1}^{k} \exp\left(-\frac{c_1(t,p)a_i^2 + c_2(t,p)b_i^2}{2}\right) \tag{5.7.34}$$

Proof. From the Jacob-Trudi formula for skew Schur polynomials (5.7.11) and Lemma 5.7.11 we have

$$\begin{aligned} s_{\lambda'/\mu'}(1^m) = \det&\left[\exp\left(-\frac{(\lambda_i - x_j^T + j - i - pm)^2}{2(1-p)pm}\right)\exp\left(O\left(T^{-1/2}\right)\right)\right] \cdot (\sqrt{2\pi})^{-k} \cdot \\ \exp&\left(-km\log(1-p) - (k/2)\log m - (k/2)\log(p(1-p)) + \log\left(\frac{1-p}{p}\right)\sum_{i=1}^{k}(\lambda_i - x_i^T)\right) \end{aligned} \tag{5.7.35}$$

$$
\begin{aligned}
s_{\kappa'/\lambda'}(1^n) = \det\left[\exp\left(-\frac{(y_i^T-\lambda_j+j-i-pn)^2}{2(1-p)pn}\right)\exp\left(O\left(T^{-1/2}\right)\right)\right]\cdot(\sqrt{2\pi})^{-k}\cdot \\
\exp\left(-kn\log(1-p)-(k/2)\log n-(k/2)\log(p(1-p))+\log\left(\frac{1-p}{p}\right)\sum_{i=1}^{k}(y_i^T-\lambda_i)\right)
\end{aligned}
\tag{5.7.36}
$$

$$
\begin{aligned}
s_{\kappa'/\mu'}(1^T) = \det\left[\exp\left(-\frac{(y_i^T-x_j^T+j-i-pT)^2}{2(1-p)pT}\right)\exp\left(O\left(T^{-1/2}\right)\right)\right]\cdot(\sqrt{2\pi})^{-k}\cdot \\
\exp\left(-kT\log(1-p)-(k/2)\log T-(k/2)\log(p(1-p))+\log\left(\frac{1-p}{p}\right)\sum_{i=1}^{k}(y_i^T-x_i^T)\right)
\end{aligned}
\tag{5.7.37}
$$

where the constants in the big O notation are uniform as z_i vary over compact subsets of $\mathbb{R}$. Combining (5.7.36), (5.7.35) and (5.7.19) we see that

$$
\begin{aligned}
B_\lambda(T) = (2\pi)^{-k/2}\cdot\exp(-(k/2)\log(p(1-p))-(k/2)\log(t(1-t))+O(T^{-1})) \\
\cdot\det\left[\exp\left(-\frac{(z_i-a_j)^2}{2p(1-p)t}+O(T^{-1/2})\right)\right]\cdot\det\left[\exp\left(-\frac{(b_i-z_j)^2}{2p(1-p)(1-t)}+O(T^{-1/2})\right)\right]
\end{aligned}
\tag{5.7.38}
$$

Taking the limit $T\to\infty$ in (5.7.38), and noticing the identities

$$
\det\left[\exp\left(-\frac{(z_i-a_j)^2}{2p(1-p)t}\right)\right] = \det\left[e^{c_1(t,p)a_iz_j}\right]_{i,j=1}^{k}\cdot\prod_{i=1}^{k}\exp\left(-\frac{c_1(t,p)}{2}(a_i^2+z_i^2)\right),\text{ and}
$$

$$
\det\left[\exp\left(-\frac{(b_i-z_j)^2}{2p(1-p)(1-t)}\right)\right] = \det\left[e^{c_2(t,p)b_iz_j}\right]_{i,j=1}^{k}\cdot\prod_{i=1}^{k}\exp\left(-\frac{c_2(t,p)}{2}(b_i^2+z_i^2)\right)
$$

we get (5.7.34). □

The following corollary of Lemma 5.7.13 gives the connection between the probability mass function in (5.7.8) and the probability density function in (5.7.3).

Corollary 5.7.14. *Assume the same notation as in Lemma 5.7.8. Fix $R>0$, take any $(z_1,\cdots,z_k)\in[-R,R]^k\cap\mathbb{W}_k^o$ such that $\lambda_i = z_i\sqrt{T}+ptT$ are integers for $i=1,\cdots,k$. Define function $h_T(z)$ on*

$\mathbb{R}^k$*:*

$$h_T(z) = \mathbf{1}_{\{[-R,R]^k \cap \mathbb{W}_k^o\}}(z) \cdot (\sqrt{T})^k \cdot \mathbb{P}(L_1(m) = \lambda_1, \cdots, L_k(m) = \lambda_k)$$

Then, we have

$$\lim_{T\to\infty} h_T(z) = \rho(z_1, \cdots, z_k) \tag{5.7.39}$$

where $\rho(z_1, \cdots, z_k)$ *is defined in (5.7.3). Moreover,* $h_T(z)$ *is uniformly bounded on the compact set* $[-R, R]^k$*.*

Proof. Plugging (5.7.35), (5.7.36) and (5.7.37) into (5.7.12) we get

$$\begin{aligned} &T^{k/2} \cdot \mathbb{P}(L_1(m) = \lambda_1, \cdots, L_k(m) = \lambda_k) \\ &= Z_T \cdot \frac{\det\left[\exp\left(-\frac{(z_i-a_j)^2}{2p(1-p)t}\right)\right] \cdot \det\left[\exp\left(-\frac{(b_i-z_j)^2}{2p(1-p)(1-t)}\right)\right]}{\det\left[\exp\left(-\frac{(b_i-a_j)^2}{2p(1-p)}\right)\right]} \cdot \exp(o(1)) \end{aligned} \tag{5.7.40}$$

where

$$\begin{aligned} Z_T &= (\sqrt{2\pi})^{-k} \exp\left(-km\log(1-p) - (k/2)\log m - (k/2)\log(p(1-p))\right) \\ &\cdot \exp\left(-kn\log(1-p) - (k/2)\log n - (k/2)\log(p(1-p))\right) \\ &\cdot \exp\left(kT\log(1-p) + k\log T + (k/2)\log(p(1-p))\right) \\ &= (\sqrt{2\pi})^{-k} \exp\left(-(k/2)\log(p(1-p)t(1-t)\right) = (2\pi p(1-p)t(1-t))^{-k/2} \end{aligned} \tag{5.7.41}$$

Plugging (5.7.41) into (5.7.40) we conclude (5.7.39) and at the meantime, we have

$$h_T(z_1, \cdots, z_k) = \frac{\det\left[\exp\left(-\frac{(z_i-a_j)^2}{2p(1-p)t}\right)\right] \cdot \det\left[\exp\left(-\frac{(b_i-z_j)^2}{2p(1-p)(1-t)}\right)\right]}{(2\pi p(1-p)t(1-t))^{k/2}\det\left[\exp\left(-\frac{(b_i-a_j)^2}{2p(1-p)}\right)\right]} \cdot \exp(o(1)) \tag{5.7.42}$$

Notice that the determinants in (5.7.42) are continuous function of z, so they are all bounded on the compact set $[-R, R]^k$. Plus, $o(1)$ is uniformly bounded on $[-R, R]^k$. Therefore, $h_T(z)$ is bounded over $[-R, R]^k$. □

Before proving Lemma 5.7.10, we need to introduce another result regarding the non-vanishing of determinant, which will be used in the proof of Lemma 5.7.10.

Lemma 5.7.15. *Suppose the vector $\vec{m} = (m_1, \cdots, m_p)$ satisfies $k = \sum_{i=1}^{p} m_i$, and $\alpha_1 > \alpha_2 > \cdots > \alpha_p$. Then the following determinant*

$$U = \det \begin{bmatrix} (z_j^{i-1} e^{\alpha_1 z_j})_{\substack{i=1,\cdots,m_1 \\ j=1,\cdots,k}} \\ \vdots \\ (z_j^{i-1} e^{\alpha_p z_j})_{\substack{i=1,\cdots,m_p \\ j=1,\cdots,k}} \end{bmatrix}$$

is non-zero for any $(z_1, \cdots, z_k)$ whose elements are distinct.

Proof. We claim that, the following equation with respect to $\vec{z}$:

$$(\xi_1 + \xi_2 z + \cdots + \xi_{m_1} z^{m_i - 1}) e^{\alpha_1 z} + \cdots (\xi_{m_1 + \cdots + m_{p-1} + 1} + \cdots + \xi_k z^{m_p - 1}) e^{\alpha_p z} = 0$$

has at most $(k-1)$ distinct roots, where $(\xi_1, \cdots, \xi_k) \in \mathbb{R}^k$ is non-zero.
Denote the above determinant by $\det \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$. If this claim holds, we can conclude that we cannot find non-zero $(\xi_1, \cdots, \xi_k) \in \mathbb{R}^k$ such that $\xi_1 v_1 + \cdots + \xi_k v_k = 0$. Thus, the k row vectors of the determinant are linear independent and the determinant is non-zero. Then we prove the claim by induction on k.
1° If $k = 2$, the equation is $(\xi_1 + \xi_2 z) e^{\alpha_1 z} = 0$ or $\xi_1 e^{\alpha_1 z} + \xi_2 e^{\alpha_2 z} = 0$, where $\xi_1, \xi_2 \in \mathbb{R}$ cannot be zero at the same time. Then, it's easy to see that the equation has at most 1 root in two scenarios.
2° Suppose the claim holds for $k \le n$.
3° When $k = n + 1$, we have the equation

$$(\xi_1 + \xi_2 z + \cdots + \xi_{m_1} z^{m_i - 1}) e^{\alpha_1 z} + \cdots (\xi_{m_1 + \cdots + m_{p-1} + 1} + \cdots + \xi_k z^{m_p - 1}) e^{\alpha_p z} = 0$$

but now $\sum_{i=1}^{p} m_i = n+1$. WLOG, suppose $(\xi_1, \cdots, \xi_{m_1})$ has a non-zero element and ξ_ℓ is the first non-zero element. Notice that the above equation has the same roots as the following one:

$$F(z) = (\xi_\ell z^{\ell-1} + \cdots + \xi_{m_1} z^{m_1-1}) + \cdots + (\xi_{m_1+\cdots+m_{p-1}+1} + \cdots + \xi_k z^{m_p-1}) e^{(\alpha_p-\alpha_1)z} = 0$$

Assume it has at least $(n+1)$ distinct roots $\eta_1 < \eta_2 < \cdots < \eta_{n+1}$. Then $F'(z) = 0$ has at least n distinct roots $\delta_1 < \cdots < \delta_n$ such that $\eta_1 < \delta_1 < \eta_2 < \cdots < \delta_n < \eta_{n+1}$, by Rolle's Theorem. Actually, $F'(z) = (\xi_\ell(\ell-1))z^{\ell-2} + \cdots + \xi_{m_1}(m_1-1)z^{m_1-2}) + \cdots + (\xi'_{m_1+\cdots+m_{p-1}+1} + \cdots + \xi'_k z^{m_p-1})e^{(\alpha_p-\alpha_1)z} = 0$ where ξ'_i, $i = m_1+1, \cdots, k$ are coefficients that can be calculated. This equation has at most $(m_1-1)+m_2+\cdots+m_p-1 = n-1$ roots by 2°, which leads to a contradiction. Therefore, our claim holds and we proved Lemma 5.7.15. □

Now, we are ready to prove Lemma 5.7.10.

Proof of Lemma 5.7.10. For clarity we split the proof into several steps.

Step 1. In this step we show that $\widetilde{\rho}(z_1, \cdots, z_k) \geq 0$ and $\widetilde{\rho}(z_1, \cdots, z_k) > 0$ if $z_1 > z_2 > \cdots > z_k$. Because of the indicator function in $\widetilde{\rho}(z_1, \cdots, z_k)$, we know $\widetilde{\rho}(z_1, \cdots, z_k) = 0$ unless $z_1 > \cdots > z_k$. Therefore, it suffices to show that

$$\widetilde{\rho}(z_1, \cdots, z_k) > 0 \text{ if } z_1 > \cdots > z_k \tag{5.7.43}$$

Choose T_0 as we did in Lemma 5.7.13 and assume $T \geq T_0$. By definition of $B_\lambda(T)$ we know $B_\lambda(T) \geq 0$ for all $T \geq T_0$, which implies $\widetilde{\rho}(z_1, \cdots, z_k) \geq 0$ combined with (5.7.34). Also, by Lemma 5.7.15 we know that $\widetilde{\rho}(z_1, \cdots, z_k) \neq 0$ so (5.7.43) holds.

Step 2. In this step we prove that $\widetilde{\rho}(z_1, \cdots, z_k)$ is integrable. Using the formula

$$\det \left[A_{i,j}\right]_{i,j=1}^{k} = \sum_{\sigma \in S_k} (-1)^\sigma \cdot \prod_{i=1}^{k} A_{i,\sigma(i)}$$

and the triangle inequality we see that

$$\begin{aligned}\left|\det\left[e^{c_1(t,p)a_iz_j}\right]_{i,j=1}^k\right| &\leq \sum_{\sigma\in S_k}\prod_{j=1}^k e^{c_1(t,p)a_{\sigma(j)}z_j} \leq \sum_{\sigma\in S_k}\prod_{j=1}^k e^{c_1(t,p)\left(\sum_{i=1}^k|a_i|\right)\cdot|z_j|} \\ &\leq (k!)\prod_{i=1}^k e^{C_1|z_j|}, \text{ where } C_1 = \sum_{i=1}^k c_1(t,p)|a_i|\end{aligned} \tag{5.7.44}$$

Analogously, define the constant $C_2 = \sum_{i=1}^k c_2(t,p)|b_i|$ and we have

$$\left|\det\left[e^{c_2(t,p)b_iz_j}\right]_{i,j=1}^k\right| \leq (k!)\prod_{i=1}^k e^{C_2|z_j|} \tag{5.7.45}$$

Plugging (5.7.44) and (5.7.45) into the expression of $\widetilde{\rho}$ we have

$$|\widetilde{\rho}(z_1,\cdots,z_k)| \leq (k!)^2\cdot\prod_{i=1}^k e^{C|z_i|-c_3(t,p)z_i^2} \tag{5.7.46}$$

where $C = C_1 + C_2$. Since the right side of (5.7.46) is integrable (because of the square in the exponential) we conclude that $\widetilde{\rho}$ is also integrable by domination.

Step 3. In this step, we prove (5.7.17) and conclude Lemma 5.7.10. Using the branching relations for skew Schur polynomials (5.7.10) we know that

$$\begin{aligned}\sum_{\lambda\in\mathfrak{W}_k}\frac{B_\lambda(T)}{T^{k/2}} &= (\sqrt{2\pi})^k\cdot\exp(kT\log(1-p) + (k/2)\log T + (k/2)\log p(1-p)) \\ &\cdot\exp\left(-\log\left(\frac{1-p}{p}\right)\sum_{i=1}^k(y_i^T - x_i^T)\right)\cdot s_{\kappa'/\mu'}(1^T)\end{aligned} \tag{5.7.47}$$

Plugging (5.7.19) and (5.7.37) into (5.7.47) we conclude

$$\lim_{T\to\infty}\sum_{\lambda\in\mathfrak{W}_k}\frac{B_\lambda(T)}{T^{k/2}} = \det\left[e^{-\frac{1}{2p(1-p)}(b_i-a_j)^2}\right]_{i,j=1}^k \tag{5.7.48}$$

For a signature $\lambda\in\mathfrak{W}_k$ and $T\in\mathbb{N}$ we define $Q_\lambda(T)$ to be the cube $[\lambda_1T^{-1/2} - pt\sqrt{T}, (\lambda_1+1)T^{-1/2} - pt\sqrt{T}]\times\cdots\times[\lambda_kT^{-1/2} - pt\sqrt{T}, (\lambda_k+1)T^{-1/2} - pt\sqrt{T}]$ with Lebesgue measure $T^{-k/2}$. In addition, we define the simple function f_T through

$$f_T(z) = \sum_{\lambda \in \mathfrak{W}_k} B_\lambda(T) \cdot \mathbf{1}_{Q_\lambda(T)}(z) \cdot \mathbf{1}_{\mathbb{W}_k^o}(z) \tag{5.7.49}$$

and observe that

$$\sum_{\lambda \in \mathfrak{W}_k} \frac{B_\lambda(T)}{T^{k/2}} = \int_{\mathbb{R}^k} f_T(z) dz \tag{5.7.50}$$

where dz represents the usual Lebesgue measure on $\mathbb{R}^k$.

In view of (5.7.34) we know that for almost every $z = (z_1, \cdots, z_k) \in \mathbb{R}^k$ we have

$$\lim_{T \to \infty} f_T(z) = \widetilde{\rho}(z) \cdot (2\pi p(1-p)t(1-t))^{-\frac{k}{2}} \cdot \prod_{i=1}^{k} \exp\left(-\frac{c_1(t,p)a_i^2 + c_2(t,p)b_i^2}{2}\right). \tag{5.7.51}$$

We claim that there exists a non-negative integrable function g on $\mathbb{R}^k$ such that if T is large enough

$$|f_T(z_1, \cdots, z_k)| \leq |g(z_1, \cdots, z_k)| \tag{5.7.52}$$

We will prove (5.7.52) in Step 4 below. For now we assume its validity and conclude the proof of (5.7.17).

From (5.7.51) and the dominated convergence theorem with dominating function g as in (5.7.52) we know that

$$\lim_{T \to \infty} \int_{\mathbb{R}^k} f_T(z) dz = \int_{\mathbb{R}^k} \widetilde{\rho}(z) dz \cdot (2\pi p(1-p)t(1-t))^{-\frac{k}{2}} \cdot \prod_{i=1}^{k} \exp\left(-\frac{c_1(t,p)a_i^2 + c_2(t,p)b_i^2}{2}\right) \tag{5.7.53}$$

Combining (5.7.53), (5.7.50) and (5.7.48) we conclude that

$$\det\left[e^{-\frac{1}{2p(1-p)}(b_i-a_j)^2}\right]_{i,j=1}^{k} = \int_{\mathbb{R}^k} \widetilde{\rho}(z)dz \cdot (2\pi p(1-p)t(1-t))^{-\frac{k}{2}} \cdot \prod_{i=1}^{k} e^{-\frac{c_1(t,p)a_i^2+c_2(t,p)b_i^2}{2}}. \tag{5.7.54}$$

which clearly establishes (5.7.17).

Step 4. In this step we demonstrate an integrable function g that satisfies (5.7.52). Let us fix $\lambda \in \mathfrak{W}_k$. If $\lambda_i \geq x_i^T + m + 1$ or $\lambda_i < \mu_i$ for some $i \in \{1, 2, \cdots, k\}$ we know that $s_{\lambda'/\mu'}(1^m) = 0$ because there is no avoiding Bernoulli ensembles starting with μ and ending with λ. Similarly, if $y_i^T \geq \lambda_i + n + 1$ or $y_i^T < \lambda_i$ for some $i \in \{1, 2, \cdots, k\}$, we have $s_{\kappa'/\lambda'}(1^n) = 0$. We conclude that $B_\lambda(T) = 0$ unless

$$m \geq \lambda_i - x_i^T \geq 0 \text{ and } n \geq y_i^T - \lambda_i \geq 0 \text{ for all } i \in \{1, \cdots, k\}$$

which implies that for all large enough T we have

$$B_\lambda(T) = 0, \text{ unless } |\lambda_i - x_j^T + j - i| \leq (1+p)m \text{ and } |y_i^T - \lambda_j + j - i| \leq (1+p)n \tag{5.7.55}$$

for all $i, j \in \{1, \cdots, k\}$. This is because if there exist i, j such that $(1+p)m < |\lambda_i - x_j^T + j - i|$, then we have

$$(1+p)m < |\lambda_i - x_j^T + j - i| \leq \lambda_1 - x_k^T + k - 1 = (\lambda_1 - \lambda_k) + (\lambda_k - x_k^T) + k - 1$$

When T is sufficiently large, the above inequality implies $\lambda_k - x_k^T > m$ so that $B_\lambda(T) = 0$, and similar result holds for $y_i^T - \lambda_j + j - i$, which justifies (5.7.55). Using the definition of $A_\lambda(T)$ and

$B_\lambda(T)$ we know that

$$\begin{aligned} B_\lambda(T) &= C_T \cdot \det[E(\lambda_i - x_j^T + j - i, m)]_{i,j=1}^k \cdot \det[E(y_i^T - \lambda_j + j - i, n)]_{i,j=1}^k, \text{ where} \\ E(N,n) &= e_N(1^n) \cdot \exp\left(-N \log\left(\frac{1-p}{p}\right) + n\log(1-p) + (1/2)\log n\right), \text{ and} \\ C_T &= (\sqrt{2\pi})^k (p(1-p))^{k/2} \cdot \exp(k \log T - (k/2)\log n - (k/2)\log m). \end{aligned} \tag{5.7.56}$$

Notice that C_T is uniformly bounded for all T large enough, because

$$k\log T - \frac{k}{2}\log n - \frac{k}{2}\log m = \frac{k}{2}\log\left(\frac{T^2}{\lfloor tT \rfloor \cdot (T - \lfloor tT \rfloor)}\right) = -\frac{k}{2}\log(t(1-t)) + O\left(T^{-1}\right) \tag{5.7.57}$$

and $O\left(T^{-1}\right)$ is uniformly bounded.

In view of (5.7.21) we know that we can find constants $C_1, c_1 > 0$ such that for all large enough T and $N_1 \in [0, m]$ and $N_2 \in [0, n]$ we have

$$E(N_1, m) \le C_1 \exp(-c_1 m^{-1}(N_1 - pm)^2) \text{ and } E(N_2, n) \le C_1 \exp(-c_1 n^{-1}(N_2 - pn)^2) \tag{5.7.58}$$

Observing that $e_r(1^n) = 0$ for $r > n$ or $r < 0$, we know that (5.7.58) also holds for all $N_1 \in [-(1+p)m, (1+p)m]$ and $N_2 \in [-(1+p)n, (1+p)n]$. Combining (5.7.55), (5.7.56) and (5.7.58) we see that for all $\lambda \in \mathfrak{W}_k$ and T sufficiently large

$$\begin{aligned} 0 \le B_\lambda(T) \le \widetilde{C} \sum_{\sigma \in S_k} \sum_{\tau \in S_k} \mathbf{1}\{|\lambda_i - x_j^T + j - i| \le (1+p)m\} \cdot \mathbf{1}\{|y_i^T - \lambda_j + j - i| \le (1+p)n\} \\ \cdot \exp\left(-\widetilde{c}T^{-1}\left[(\lambda_i - \sqrt{T}a_{\sigma(i)} - ptT)^2 + (\sqrt{T}b_i - \lambda_{\tau(i)} + ptT)^2\right]\right) \end{aligned} \tag{5.7.59}$$

where $\widetilde{c}, \widetilde{C} > 0$ depend on p, t, k but not on T provided that it is sufficiently large.

In particular, we see that if $z \in \mathbb{R}^k$ then either $z \notin Q_\lambda(T)$ for any $\lambda \in \mathfrak{W}_k$ in which case

$f_T(z) = 0$ or $z \in Q_\lambda(T)$ for some $\lambda \in \mathfrak{W}_k$ in which case (5.7.59) and (5.7.19) imply

$$0 \le f_T(z) \le C \sum_{\sigma \in S_k} \sum_{\tau \in S_k} \exp\left(-c((z_i - a_{\sigma(i)})^2 + (b_i - z_{\tau(i)})^2)\right) \tag{5.7.60}$$

where $C, c > 0$ depend on p, t, k but not on T provided that it is sufficiently large. We finally see that (5.7.46) holds with g being equal to the right side of (5.7.60), which is clearly integrable. □

Now we are ready to prove Proposition 5.7.2.

Proof of Proposition 5.7.2. In the following, we prove the weak convergence of the random vector Z^T, when $\vec{a} = (a_1, \cdots, a_k)$ and $\vec{b} = (b_1, \cdots, b_k)$ consist of distinct entries. In order to show weak convergence, it is sufficient to show that for every open set $O \in \mathbb{R}^k$, we have:

$$\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O) \ge \int_O \rho(z_1, \cdots, z_k) dz_1 dz_2 \cdots dz_k$$

according to [117, Theorem 3.2.11]. It is also sufficient to show that for any open set $U \in \mathbb{W}_k^o$, we have:

$$\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in U) \ge \int_U \rho(z_1, \cdots, z_k) dz_1 dz_2 \cdots dz_k \tag{5.7.61}$$

which implies that:

$$\begin{aligned} &\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O) \ge \liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in O \cap \mathbb{W}_k^o) \\ &\ge \int_{\mathbb{W}_k^o \cap O} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k = \int_O \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \end{aligned}$$

The second inequality uses the above result (5.7.61), since $\mathbb{W}_k^o \cap O$ is an open set in $\mathbb{W}_k^o$. The last equality is because $\rho(z)$ is zero outside $\mathbb{W}_k^o$. The rest of the proof will be divided into 2 steps. In Step 1, we prove that weak convergence holds on every closed rectangle. In Step 2, we prove the inequality (5.7.61) by writing open set as countable union of almost disjoint rectangles.

Step 1. In this step, we establish the following result:

For any closed rectangle $R = [u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_k, v_k] \in \mathbb{W}_k^o$,

$$\lim_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_R \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \tag{5.7.62}$$

where $\rho(z)$ is given in Proposition 5.7.2.

Define $m_i^T = \lceil u_i\sqrt{T} + ptT\rceil$ and $M_i^T = \lfloor v_i\sqrt{T} + ptT\rfloor$. Then we have:

$$\begin{aligned}
&\mathbb{P}\left((Z_1^T, \cdots, Z_k^T) \in R\right) = \mathbb{P}\left(u_1 \le Z_1^T \le v_1, \dots, u_k \le Z_k^T \le v_k\right)\\
&= \mathbb{P}\left(u_i\sqrt{T} + ptT \le L_i(\lfloor tT\rfloor) \le v_i\sqrt{T} + ptT, i = 1, \dots, k\right)\\
&= \sum_{\lambda_1=m_1^T}^{M_1^T} \cdots \sum_{\lambda_k=m_k^T}^{M_k^T} \mathbb{P}(L_1(\lfloor tT\rfloor) = \lambda_1, \dots, L_k(\lfloor tT\rfloor) = \lambda_k)\\
&= \sum_{\lambda_1=m_1^T}^{M_1^T} \cdots \sum_{\lambda_k=m_k^T}^{M_k^T} (\sqrt{T})^{-k} \cdot (\sqrt{T})^k \mathbb{P}(L_1(\lfloor tT\rfloor) = \lambda_1, \dots, L_k(\lfloor tT\rfloor) = \lambda_k)
\end{aligned}$$

Find sufficiently large A such that $R \subset [-A, A]^k$, for example, $A = 1 + \max_{1\le i\le k} |a_i| + \max_{1\le i\le k} |b_i|$. Define $h_T(z_1, \cdots, z_k)$ as a simple function on $\mathbb{R}^k$: When $(z_1, \cdots, z_k) \in R$, it takes value $(\sqrt{T})^k \cdot \mathbb{P}(L_1(\lfloor tT\rfloor) = \lambda_1, \cdots, L_k(\lfloor tT\rfloor) = \lambda_k)$ if there exist integers $\lambda_1 \ge \cdots \ge \lambda_k$ such that $\lambda_i \le z_i\sqrt{T} + ptT < \lambda_i + 1$; It takes value 0 otherwise, when $(z_1, \cdots, z_k) \notin R$. Since the Lebesgue measure of the set $\{z : \lambda_i \le z_i\sqrt{T}+ptT < \lambda_i+1, i = 1, \cdots, k\} = \left[\lambda_1 T^{-1/2} - pt\sqrt{T}, (\lambda_1 + 1)T^{-1/2} - pt\sqrt{T}\right) \times \cdots \times \left[\lambda_k T^{-1/2} - pt\sqrt{T}, (\lambda_k + 1)T^{-1/2} - pt\sqrt{T}\right)$ is $(\sqrt{T})^{-k}$, the above probability can be further written as an integral of simple functions $h_T(z_1, \cdots, z_k)$:

$$\mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_{[-A,A]^k} h_T(z_1, \cdots, z_k) dz_1 \cdots dz_k$$

By Corollary 5.7.14, the function $h_T(z_1, \cdots, z_k)$ pointwise converges to $\rho(z)$ and is bounded on the compact set $[-A, A]^k$. Since the Lebesgue measure of $[-A, A]^k$ is finite, by bounded convergence theorem we have:

$$\lim_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R) = \int_R \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \tag{5.7.63}$$

Step 2. In this step, we prove the statement (5.7.61). Take any open set $U \in \mathbb{W}_k^o$ and it can be written as a countable union of closed rectangles with disjoint interiors: $U = \bigcup_{i=1}^{\infty} R_i$, where $R_i = [a_1^i, b_1^i] \times \cdots \times [a_k^i, b_k^i]$([225, Theorem 1.4]). Choose sufficiently small $\epsilon > 0$, and denote $R_i^\epsilon = [a_1^i + \epsilon, b_1^i - \epsilon] \times \cdots \times [a_k^i + \epsilon, b_k^i - \epsilon]$, then R_i^ϵ are disjoint. Therefore,

$$\begin{aligned}
&\liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in U) \geq \liminf_{T\to\infty} \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in \bigcup_{i=1}^n R_i^\epsilon) \\
&= \liminf_{T\to\infty} \sum_{i=1}^n \mathbb{P}((Z_1^T, \cdots, Z_k^T) \in R_i^\epsilon) = \sum_{i=1}^n \int_{R_i^\epsilon} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \\
&= \int_{\bigcup_{i=1}^n R_i^\epsilon} \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k \xrightarrow{\epsilon\downarrow 0,\ n\uparrow\infty} \int_U \rho(z_1, \cdots, z_k) dz_1 \cdots dz_k
\end{aligned}$$

The last line uses the monotone convergence theorem since the when we let $\epsilon \downarrow 0$ and $n \uparrow \infty$ the indicator function $\mathbf{1}_{\bigcup_{i=1}^n R_i^\epsilon}$ is monotonically increasing, and converges to $\mathbf{1}_U$. Thus, we have proved the inequality (5.7.61). By Lemma 5.7.10, $\rho(z)$ is a probability density function, thus implying the weak convergence of Z^T. □

5.7.4 Multi-indices and Multivariate Taylor Expansion

In this section, we introduce some notations and results about multivariate functions and permutations.

Suppose $\sigma = (\sigma_1, \cdots, \sigma_n)$ is a multi-index of *length n*. In our context, we require $\sigma_1, \cdots, \sigma_n$ be all non-negative integers(some of them might be equal). We define $|\sigma| = \sum_{i=1}^n \sigma_i$ as the *order* of σ. Suppose $\tau = (\tau_1, \cdots, \tau_n)$ is another multi-index of length n. We say $\tau \leq \sigma$ if $\tau_i \leq \sigma_i$ for $i = 1, \cdots, n$. We say $\tau < \sigma$ if $\tau \leq \sigma$ and there exists at least one index i such that $\tau_i < \sigma_i$. Then, define the partial derivative with respect to the multi-index σ:

$$D^\sigma f(x_1, \cdots, x_n) = \frac{\partial^{|\sigma|} f(x_1, \cdots, x_n)}{\partial x_1^{\sigma_1} \partial x_2^{\sigma_2} \cdots \partial x_n^{\sigma_n}}$$

We have the general Leibniz rule:

$$D^\sigma (fg) = \sum_{\tau \le \sigma} \binom{\sigma}{\tau} D^\tau f \cdot D^{\sigma - \tau} g$$

where $\binom{\sigma}{\tau} = \frac{\sigma_1! \cdots \sigma_n!}{\tau_1! \cdots \tau_n! (\sigma_1 - \tau_1)! \cdots (\sigma_n - \tau_n)!}$.

We also have the Taylor expansion for multi-variable functions:

$$f(x_1, \cdots, x_n) = \sum_{|\sigma| \le r} \frac{1}{\sigma!} D^\sigma f(\vec{x}_0)(\vec{x} - \vec{x}_0)^\sigma + R_{r+1}(\vec{x}, \vec{x}_0) \tag{5.7.64}$$

In the equation, $\sigma! = \sigma_1! \sigma_2! \cdots \sigma_n!$ is the factorial with respect to the multi-index σ, $\vec{x}_0 = (x_1^0, \cdots, x_n^0)$ is a constant vector at which we expands the function f, $(\vec{x} - \vec{x}_0)^\sigma$ stands for $(x_1 - x_1^0)^{\sigma_1} \cdots (x_n - x_n^0)^{\sigma_n}$, and

$$R_{r+1}(\vec{x}, \vec{x}_0) = \sum_{\sigma: |\sigma| = r+1} \frac{1}{\sigma!} D^\sigma f(\vec{x}_0 + \theta(\vec{x} - \vec{x}_0))(\vec{x} - \vec{x}_0)^\sigma$$

is the remainder, where $\theta \in (0, 1)$([54, Theorem 3.18 & Corollary 3.19]).

We also need some knowledge about *permutation*. Suppose s_n is a permutation of n non-negative integers, for example $\{1, \cdots, n\}$, and $s_n(i)$ represents the i-th element in the permutation s_n. We define *the number of inversions* of s_n by $I(s_n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathbf{1}_{\{s_n(i) > s_n(j)\}}$. For example, the permutation $s_n = (1, \cdots, n)$ has 0 number of inversions, while the permutation $s_5 = (3, 2, 5, 1, 4)$ has number of inversions $5(2 + 1 + 2 + 0 + 0)$. Define the sign of permutation s_n by $sgn(s_n) = (-1)^{I(s_n)}$. For instance, $sgn((1, \cdots, n)) = 1$ and $sgn(s_5) = -1$ in the previous example.

5.7.5 Proof of Proposition 5.7.3

Based on the notation in Section 5.7.4, we are going to prove Proposition 5.7.3 in this section. We assume vectors $\vec{a}_0$, $\vec{b}_0$ cluster in the way described in (5.7.4):

$$\begin{aligned} \vec{a}_0 &= (\underbrace{\alpha_1, \cdots, \alpha_1}_{m_1}, \cdots, \underbrace{\alpha_p, \cdots, \alpha_p}_{m_p}) \\ \vec{b}_0 &= (\underbrace{\beta_1, \cdots, \beta_1}_{n_1}, \cdots, \underbrace{\beta_q, \cdots, \beta_q}_{n_q}) \end{aligned} \tag{5.7.65}$$

and

$$\lim_{T\to\infty} \frac{\vec{x}^T}{\sqrt{T}} = \vec{a}_0, \ \lim_{T\to\infty} \frac{\vec{y}^T - pT1_k}{\sqrt{T}} = \vec{b}_0$$

Let $\vec{a} = (a_1, \cdots, a_k)$ and $\vec{b} = (b_1, \cdots, b_k)$ denote two vectors in $\mathbb{W}_k^o$ so they contain distinct elements. We also denote $\vec{a}^{(1)} = (a_1, \cdots, a_{m_1})$, $\cdots$, $\vec{a}^{(p)} = (a_{m_1+\cdots+m_{p-1}+1}, \cdots, a_{m_1+\cdots+m_p})$ and $\vec{a} = (\vec{a}^{(1)}, \cdots, \vec{a}^{(p)})$. That is, we divide the vector $\vec{a}$ into p blocks according to the shape of $\vec{a}_0$. Similarly, we write $\vec{b} = (b^{(1)}, \cdots, b^{(q)})$ according to the shape of $\vec{b}_0$. We will keep using similar notations in the following discussion, when we need to divide the vector according to the shape of $\vec{a}_0$ and $\vec{b}_0$. Next, denote

$$f(\vec{a}, \vec{z}) = \det[e^{c_1(t,p)a_i z_j}]_{i,j=1}^k, \ g(\vec{b}, \vec{z}) = \det[e^{c_2(t,p)b_i z_j}]_{i,j=1}^k \tag{5.7.66}$$

and it's not difficult to see that they are all smooth multi-variable functions with respect to corresponding vectors because of the exponentials. In addition, $\lim_{\vec{a}\to\vec{a}_0} f(\vec{a}, \vec{z}) = 0$ and $\lim_{\vec{b}\to\vec{b}_0} g(\vec{b}, \vec{z}) = 0$. However, when we taking proper derivatives with respect to $\vec{a}$ and $\vec{b}$, we can get a non-zero derivative. The following lemma gives the minimal order of derivatives such that $D^\sigma f(\vec{a}_0, \vec{z})$ and $D^\sigma g(\vec{b}_0, \vec{z})$ are non-zero, where $f(\vec{a}, \vec{z})$ and $g(\vec{b}, \vec{z})$ are defined in (5.7.66).

Lemma 5.7.16. *Assume the same notations as in (5.7.65) and $\vec{z} \in \mathbb{W}_k^o$. Then, the smallest order of σ_a that makes the derivative $D^{\sigma_a} f(\vec{a}_0, \vec{z})$ non-zero is $u = \sum_{i=1}^p \frac{m_i(m_i-1)}{2}$. Similarly,*

$v = \sum_{j=1}^{q} \frac{n_j(n_j-1)}{2}$ *is the smallest order of* σ_b *that makes* $D^{\sigma_b} f(\vec{b}_0, \vec{z})$ *non-zero.*

Proof. If the order of derivative is less than u, then there exists an $i \in \{1, \cdots, p\}$ such that $\sigma_a^{(i)}$ contains two equal elements $< m_i - 1$, and the determinant $D^{\sigma_a} f(\vec{a}_0, \vec{z})$ would have two equal rows, thus equal to zero. Suppose s_n is the set of all permutations of $\{0, 1, \cdots, n-1\}$. Then, if $\sigma_a = \left(\sigma_a^{(1)}, \cdots, \sigma_a^{(p)}\right)$ and $\sigma_a^{(i)} \in s_{m_i}$, $D^{\sigma_a} f(\vec{a}_0, \vec{z})$ is non-zero by Lemma 5.7.15. In this case, the order of σ_a is $\sum_{i=1}^{p} \sum_{j=1}^{m_i-1} j = \sum_{i=1}^{p} \frac{m_i(m_i-1)}{2} = u$. Analogous result also holds for $D^{\sigma_b} g(\vec{b}_0, \vec{z})$ and we conclude Lemma 5.7.16. □

Remark 5.7.17. Denote the set

$$\begin{aligned} \Lambda_a &= \{\sigma_a = \left(\sigma_a^{(1)}, \cdots, \sigma_a^{(p)}\right) : \sigma_a^{(i)} \in S_{m_i},\ i = 1, \cdots, p\} \\ \Lambda_b &= \{\sigma_b = \left(\sigma_b^{(1)}, \cdots, \sigma_b^{(p)}\right) : \sigma_b^{(i)} \in S_{n_i},\ i = 1, \cdots, q\} \end{aligned} \tag{5.7.67}$$

Then we have that $\sigma_a \in \Lambda_a$ and $\sigma_b \in \Lambda_b$ imply $D^{\sigma_a} f(\vec{a}_0, \vec{z})$ and $D^{\sigma_b} f(\vec{b}_0, \vec{z})$ are non-zero.

Finally, we give the proof for Proposition 5.7.3.

Proof of Proposition 5.7.3. For clarity, the proof will be split into 3 steps. In Step 1, we use multivariate Taylor expansion to find the speed of convergence of $f(\vec{a}, \vec{z})$ and $g(\vec{b}, \vec{z})$ to zero, when $\vec{a} \to \vec{a}_0$ and $\vec{b} \to \vec{b}_0$. In Step 2, we construct a new density function based on Step 1, and we will prove that Z^T weakly converges to the this newly constructed density in Step 3. In Step 3, we use monotone coupling lemma to "squeeze" the probability and prove the weak convergence.

Step 1. In this step, we find the converging speed of $f(\vec{a}, \vec{z})$ when $\vec{a} \to \vec{a}_0$. Take $\epsilon \in (0, k^{-1} \min_{1 \le i \le p-1} (\alpha_i - \alpha_{i+1}))$ and construct the following vectors:

$$\begin{aligned} \vec{A}_{\epsilon,+} &= (\alpha_1 + m_1\epsilon, \alpha_1 + (m_1 - 1)\epsilon, \cdots, \alpha_1 + \epsilon, \cdots, \alpha_p + m_p\epsilon, \cdots, \alpha_p + \epsilon) \\ \vec{A}_{\epsilon,-} &= (\alpha_1 - \epsilon, \alpha_1 - 2\epsilon, \cdots, \alpha_1 - m_1\epsilon, \cdots, \alpha_p - \epsilon, \cdots, \alpha_p - m_p\epsilon) \end{aligned}$$

That is, the vector $\vec{A}_{\epsilon,+}$(resp. $\vec{A}_{\epsilon,-}$) upwardly(resp. downwardly) spreads out the vector $\vec{a}_0$ such that $\vec{A}_{\epsilon,+}$(resp. $\vec{A}_{\epsilon,-}$) has distinct elements. By the choice of ϵ, the elements of $\vec{A}_{\epsilon,+}$ and $\vec{A}_{\epsilon,-}$ are strictly ordered. In addition, when $\epsilon \downarrow 0$, the vector $\vec{A}_{\epsilon,\pm}$ converges to $\vec{a}_0$. The main result of this step is the following:

$$\lim_{\epsilon\downarrow 0} \epsilon^{-u} f(\vec{A}_{\epsilon,\pm}, \vec{z}) = \varphi(\vec{a}_0, \vec{z}, \vec{m}) \tag{5.7.68}$$

where $u = \sum_{i=1}^{p} \frac{m_i(m_i-1)}{2}$ in Lemma 5.7.16, $\vec{m} = (m_1, \cdots, m_p)$, and $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ defined in (5.7.5). Additionally, $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ is non-zero because of Lemma 5.7.15.

To prove this result, we first keep $\vec{z}$ fixed and expand the function $f(\vec{a}, \vec{z})$ to the order of u at $\vec{a}_0$ using multi-variate Taylor expansion (5.7.64):

$$\begin{aligned} f(\vec{a}, \vec{z}) &= \sum_{|\sigma_a| \le u} \frac{D^{\sigma_a} f(\vec{a}_0, \vec{z})}{\sigma_a!} (\vec{a} - \vec{a}_0)^{\sigma_a} + R_{u+1}(\vec{a}, \vec{a}_0, \vec{z}) \\ &= \sum_{\sigma_a \in \Lambda_a} \frac{D^{\sigma_a} f(\vec{a}_0, \vec{z})}{\sigma_a!} (\vec{a} - \vec{a}_0)^{\sigma_a} + R_{u+1}(\vec{a}, \vec{a}_0, \vec{z}) \end{aligned} \tag{5.7.69}$$

where

$$R_{u+1}(\vec{a}, \vec{a}_0, \vec{z}) = \sum_{\sigma_a : |\sigma_a| = u+1} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0 + \theta(\vec{a} - \vec{a}_0), \vec{z})(\vec{a} - \vec{a}_0)^{\sigma_a}, \theta \in (0, 1) \tag{5.7.70}$$

is the remainder and Λ_a is defined in remark 5.7.17. The second equality in (5.7.69) results from Lemma 5.7.16, since it indicates that all the terms of order less than u are zero, and for the terms of order u, they are non-zero only when $\sigma_a \in \Lambda_a$.

Consider the first term in the second line of (5.7.69). Denote $sgn(\sigma_a^{(i)})$ as the sign of the permutation $\sigma_a^{(i)} \in S_{m_i}$, and define the sign of σ_a by: $sgn(\sigma_a) = \prod_{i=1}^{p} sgn(\sigma_a^{(i)})$. Denote $\sigma_a^\star = \left(\sigma_a^{(1)\star}, \cdots, \sigma_a^{(p)\star}\right)$, where $\sigma_a^{(i)\star} = (0, 1, \cdots, m_i - 1)$. Thus, $\sigma_a^\star$ is a special element in Λ_a and $sgn(\sigma_a^\star) = 1$ because all of $\sigma_a^{(1)\star}, \cdots, \sigma_a^{(p)\star}$ have 0 number of inversions. Notice that for any $\sigma_a \in \Lambda_a$, we have $D^{\sigma_a} f(\vec{a}_0, \vec{z}) = sgn(\sigma_a) \cdot D^{\sigma_a^\star} f(\vec{a}_0, \vec{z})$ by the property of determinant. Then we obtain:

$$\sum_{\sigma_a \in \Lambda_a} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0)(\vec{a}-\vec{a}_0)^{\sigma_a} = \frac{D^{\sigma_a^\star} f(\vec{a}_0)}{\prod_{i=1}^{p}(m_i-1)!} \sum_{\sigma_a \in \Lambda_a} (\vec{a}-\vec{a}_0)^{\sigma_a} \cdot sgn(\sigma_a)$$

Notice that

$$\begin{aligned}\sum_{\sigma_a \in \Lambda_a} (\vec{a}-\vec{a}_0)^{\sigma_a} \cdot sgn(\sigma_a) &= \prod_{i=1}^{p} \left(\sum_{\sigma_a^{(i)} \in S_{m_i}} (\vec{a}^{(i)} - \vec{a}_0^{(i)})^{\sigma_a^{(i)}} \cdot sgn(\sigma_a^{(i)}) \right) \\ &= \prod_{i=1}^{p} \Delta_{m_i}(a_1^{(i)} - \alpha_i, a_2^{(i)} - \alpha_i, \cdots, a_{m_i}^{(i)} - \alpha_i) \equiv \prod_{i=1}^{p} \Delta_{m_i}^{a}\end{aligned}$$

where $\Delta_n(x_1, x_2, \cdots, x_n)$ is the Vandermonde Determinant, $a_j^{(i)} = a_{m_1+\cdots+m_{i-1}+j}$ is the j-th element of $\vec{a}^{(i)}$, and the last line holds by the expansion formula of determinant and definition of Vandermonde Determinant. Now replace $\vec{a}$ with $\vec{A}_{\epsilon,+}$, we get the Vandermonde determinant $\Delta_{m_i}^{a}$ is actually $(m_i-1)! \cdot \epsilon^{\frac{1}{2}m_i(m_i-1)}$. Therefore, we have:

$$\sum_{\sigma_a \in \Lambda_a} \frac{1}{\sigma_a!} D^{\sigma_a} f(\vec{a}_0, \vec{z})(\vec{a}-\vec{a}_0)^{\sigma_a} = D^{\sigma_a^\star} f(\vec{a}_0, \vec{z}) \cdot \epsilon^{u} \tag{5.7.71}$$

Since the i-th row of determinant $f(\vec{a}, \vec{z})$ only depends on one variable a_i if we fix $\vec{z}$, taking derivative of $f(\vec{a}, \vec{z})$ with respect to a_i is actually taking derivatives of entries in the i-th row of $f(\vec{a}, \vec{z})$ and let other rows stay unchanged. Therefore, we observe that $D^{\sigma_a^\star} f(\vec{a}_0, \vec{z})$ is exactly the determinant $\varphi(\vec{a}_0, \vec{z}, \vec{m})$ defined in (5.7.5) and by Lemma 5.7.15, $D^{\sigma_a^\star} f(\vec{a}_0, \vec{z})$ is non-zero.

Next, we consider the remainder $R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0, \vec{z})$ in (5.7.70). Suppose σ_a is a permutation of order $u+1$. First notice that the terms in the sum in the remainder is non-zero only when there exist an $i \in \{1, \cdots, p\}$ such that $\sigma_a^{(i)} = (0, 1, \cdots, m_i-2, m_i)$ and for $j \neq i$, $\sigma_a^{(j)} \in S_{m_j}$. Otherwise, the terms are zero because the determinant $D^{\sigma_a} f(\vec{a}_0, \vec{z})$ will have at least two equal lines. Therefore, there are only finitely many non-zero terms in the sum, and we denote the number of non-zero terms by N, which only depends on $\vec{m}$. Second, we observe that $\sigma_a!$ only has finitely many possible outcomes when its order is $u+1$, thus $\frac{1}{\sigma_a!}$ can be bounded by a constant M only depending on $\vec{m}$.

Third, by the construction of $\vec{A}_{\epsilon,+}$ we have

$$|(\vec{A}_{\epsilon,+} - \vec{a}_0)|^{\sigma_a} \leq (\max_{1\leq i\leq p} m_i \cdot \epsilon)^{u+1} \tag{5.7.72}$$

for every σ_a such that $\sigma_a = u+1$. Finally, denote vector $\vec{A}_\theta = \vec{a}_0 + \theta(\vec{A}_{\epsilon,+} - \vec{a}_0) = (A_{1,\theta}, \cdots, A_{k,\theta})$. Following similar approach as in (5.7.44), combined with the form of $D^{\sigma_a} f(\vec{a}_0, \vec{z})$, we have

$$\begin{aligned}\left|D^{\sigma_a} f(\vec{A}_\theta, \vec{z})\right| &\leq (k!)\left((\max_{1\leq i\leq p} |z_i|)^{u+1}\right)\prod_{j=1}^{k} e^{c_1(t,p)\left(\sum_{i=1}^{k} |A_{i,\theta}|\right)|z_j|} \\ &\leq (k!)(|z_1| + |z_k|)^{u+1}\prod_{j=1}^{k} e^{c_1(t,p)\cdot k\cdot(\max_{1\leq i\leq p} m_i)\cdot\epsilon\cdot|z_j|} \\ &\leq (k!)(|z_1| + |z_k|)^{u+1}\prod_{j=1}^{k} e^{C_1|z_j|}\end{aligned} \tag{5.7.73}$$

when $\epsilon < 1$, and the constant $C_1 = c_1(t,p) \cdot k \cdot (\max_{1\leq i\leq p} m_i)$.

Combining (5.7.73), (5.7.72) and the fact that $\frac{1}{\sigma_a!}$ is bounded by a constant $M(\vec{m})$, we have

$$|R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0, \vec{z})| \leq N \cdot M \cdot (k!)(|z_1| + |z_k|)^{u+1} \exp\left(C_1 \sum_{j=1}^{k} |z_j|\right)(\max_{1\leq i\leq p} m_i \cdot \epsilon)^{u+1} \tag{5.7.74}$$

which indicates that $R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0, \vec{z})$ is $O(\epsilon^{u+1})$, where the constant in big O notation only depends on $\vec{a}_0$, $\vec{A}_{\epsilon,+}$ and $\vec{m}$ and does not depend on ϵ. Therefore, we conclude from (5.7.69), (5.7.71) and the fact that $R_{u+1}(\vec{A}_{\epsilon,+}, \vec{a}_0, \vec{z})$ is $o(\epsilon^u)$ that:

$$\lim_{\epsilon\downarrow 0} \epsilon^{-u} f(\vec{A}_{\epsilon,+}, \vec{z}) = D^{\sigma_a^\star} f(\vec{a}_0, \vec{z})$$

Analogously, we can prove $\lim_{\epsilon\downarrow 0} \epsilon^{-u} f(\vec{A}_{\epsilon,-}, \vec{z}) = D^{\sigma_a^\star} f(\vec{a}_0, \vec{z})$ also holds and we complete the proof of (5.7.68). We can construct vectors $\vec{B}_{\epsilon,\pm}$ similarly, which spread out from vector $\vec{b}_0$

upward and downward, and get similar results for $g(\vec{B}_{\epsilon,\pm},\vec{z})$:

$$\lim_{\epsilon\downarrow 0}\epsilon^{-v}f(\vec{B}_{\epsilon,\pm},\vec{z}) = D^{\sigma_b^\star}g(\vec{b}_0,\vec{z}) \equiv \psi(\vec{b}_0,\vec{z},\vec{n}) \tag{5.7.75}$$

where $v = \sum_{i=1}^{q}\frac{n_i(n_i-1)}{2}$ in Lemma 5.7.16, $\vec{n} = (n_1,\cdots,n_q)$ and the non-zero function $\psi(\vec{b}_0,\vec{z},\vec{n})$ is defined in (5.7.5).

Step 2. In this step, we mainly prove the following result:

$$\text{The function } H(\vec{z}) = \varphi(\vec{a}_0,\vec{z},\vec{m})\cdot\psi(\vec{b}_0,\vec{z},\vec{n})\cdot\prod_{i=1}^{k}e^{-c_3(t,p)z_i^2} \text{ is integrable over } \mathbb{R}^k. \tag{5.7.76}$$

Notice that $\varphi(\vec{a}_0,\vec{z},\vec{m})$ and $\psi(\vec{b}_0,\vec{z},\vec{n})$ are two determinants whose expression are given in (5.7.5), and they are positive when $\vec{z}\in\mathbb{W}_k^o$ because of (5.7.68) and (5.7.75). Following similar approach as in (5.7.73) we can find

$$\begin{aligned}\varphi(\vec{a}_0,\vec{z},\vec{m}) &\le (k!)\,(|z_1|+|z_k|)^u\prod_{j=1}^{k}e^{c_1(t,p)\cdot\left(\sum_{i=1}^{p}|\alpha_i|m_i\right)|z_j|},\\ \psi(\vec{b}_0,\vec{z},\vec{n}) &\le (k!)\,(|z_1|+|z_k|)^v\prod_{j=1}^{k}e^{c_2(t,p)\cdot\left(\sum_{i=1}^{q}|\beta_i|n_i\right)|z_j|},\end{aligned} \tag{5.7.77}$$

when $z_1 > z_2 > \cdots > z_k$. Therefore,

$$H(\vec{z}) \le (k!)^2\cdot(|z_1|+|z_k|)^{u+v}\cdot\prod_{i=1}^{k}e^{C|z_i|-c_3(t,p)\cdot z_i^2} \tag{5.7.78}$$

where $C = c_1(t,p)\cdot\sum_{i=1}^{p}|\alpha_i|m_i + c_2(t,p)\cdot\sum_{i=1}^{q}|\beta_i|n_i$. The right hand side is integrable over $\mathbb{R}^k$ because of the quadratic terms in the exponential. Thus, $H(\vec{z})$ is integrable and we can define the constant $Z_c = \int_{\mathbb{R}^k}H(z)\mathbf{1}_{\{z_1>z_2>\cdots>z_k\}}dz < \infty$ and the function

$$\rho_c(z_1,\cdots,z_k) = Z_c^{-1}\cdot\mathbf{1}_{\{z_1>z_2>\cdots>z_k\}}\cdot\varphi(\vec{a}_0,\vec{z},\vec{m})\cdot\psi(\vec{b}_0,\vec{z},\vec{n})\cdot\prod_{i=1}^{k}e^{-c_3(t,p)z_i^2} \tag{5.7.79}$$

is a density because it's non-negative and integrates to 1 over $\mathbb{R}^k$.

Step 3. Denote $Z^T_{\vec{a}_0,\vec{b}_0}$ as the random vector Z^T in Definition 5.7.1 associated with vectors $\vec{a}_0$ and $\vec{b}_0$, and in this step we prove it weakly converges to the continuous distribution with density $\rho_c(z)$ we just constructed in (5.7.79). Suppose $\mathfrak{L}^T_+$ is an avoiding Bernoulli line ensemble starting with $\vec{x}^T_+ = \left(x^T_{+,1}, \cdots, x^T_{+,k}\right)$ and ending with $\vec{y}^T_+ = \left(y^T_{+,1}, \cdots, y^T_{+,k}\right)$ and follows the distribution $\mathbb{P}^{0,T,\vec{x}^T_+,\vec{y}^T_+}_{Avoid,Ber}$. The vectors $\vec{x}^T_+$ and $\vec{y}^T_+$ are two signatures of length k that satisfies the following:

1. Let 1_k denote the vector $(1, 1, \cdots, 1)$ of length k, then

$$\lim_{T\to\infty} \frac{\vec{x}^T_+}{\sqrt{T}} = \vec{A}_{\epsilon,+}, \quad \lim_{T\to\infty} \frac{\vec{y}^T_+ - pT1_k}{\sqrt{T}} = \vec{B}_{\epsilon,+} \tag{5.7.80}$$

2. $x^T_{+,i} \geq x^T_i$, $y^T_{+,i} \geq y^T_i$, for $i = 1, \cdots, k$, which means the endpoints of the newly constructed line ensembles dominate the original ones.

This can be achieved due to the limiting behavior of $\vec{x}^T_+$ and $\vec{y}^T_+$ and the fact that $\vec{A}_{\epsilon,+}$ and $\vec{B}_{\epsilon,+}$ dominate $\vec{a}_0$ and $\vec{b}_0$. Analogously, we construct another avoiding Bernoulli line ensemble $\mathfrak{L}^T_-$ with endpoints $\vec{x}^T_-$ and $\vec{y}^T_-$ and distribution $\mathbb{P}^{0,T,\vec{x}^T_-,\vec{y}^T_-}_{Avoid,Ber}$ such that $\lim_{T\to\infty} \frac{\vec{x}^T_-}{\sqrt{T}} = \vec{A}_{\epsilon,-}$, $\quad \lim_{T\to\infty} \frac{\vec{y}^T_- - pT1_k}{\sqrt{T}} = \vec{B}_{\epsilon,-}$, and $x^T_{-,i} \leq x^T_i$, $y^T_{-,i} \leq y^T_i$ for $i = 1, \cdots, k$.

Since now $\vec{A}_{\epsilon,+}$, $\vec{A}_{\epsilon,-}$, $\vec{B}_{\epsilon,+}$, $\vec{B}_{\epsilon,-}$ have distinct elements, we can apply the results in Proposition 5.7.2 and conclude the weak convergence:

$$Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} \Rightarrow \rho_{\epsilon,+}(z), \quad Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}} \Rightarrow \rho_{\epsilon,-}(z)$$

where $Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}$ and $Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}}$ are obtained by scaling the line ensembles $\mathfrak{L}^T_+$ and $\mathfrak{L}^T_-$ through Definition 5.7.1, $\rho_{\epsilon,+}(z)$ and $\rho_{\epsilon,-}(z)$ are densities which are obtained by plugging $\vec{A}_{\epsilon,+}$, $\vec{B}_{\epsilon,+}$ and $\vec{A}_{\epsilon,-}$, $\vec{B}_{\epsilon,-}$ into the formula of $\rho(z)$ in (5.7.3).

In order to prove the weak convergence of $Z^T_{\vec{a}_0,\vec{b}_0}$, it is sufficient to prove for any $R = (-\infty, u_1] \times$

$(-\infty, u_2] \times \cdots \times (-\infty, u_k]$, where $u_i \in \mathbb{R}$, we have

$$\lim_{T\to\infty} \mathbb{P}(Z^T_{\vec{a}_0,\vec{b}_0} \in R) = \int_R \rho_c(z)dz \tag{5.7.81}$$

Actually, by Lemma 5.2.1, we can construct a sequence of probability spaces $(\Omega_T, \mathcal{F}_T, \mathbb{P}_T)_{T\geq 1}$ such that for each $T \in \mathbb{Z}^+$, we have random variables $\mathfrak{L}^T_+$ and $\mathfrak{L}^T$ having law $\mathbb{P}^{0,T,\vec{x}^T_+,\vec{y}^T_+}_{Avoid,Ber}$, and $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{Avoid,Ber}$ under measure $\mathbb{P}_T$, respectively. Also, we have $\mathfrak{L}^T_+(i,r) \geq \mathfrak{L}^T(i,r)$ with probability 1, where $\mathfrak{L}^T_+(i,r)$(resp., $\mathfrak{L}^T(i,r)$) is the value of the i-th up-right path of $\mathfrak{L}^T_+$(resp., $\mathfrak{L}^T$) at $r \in [\![0,T]\!]$. Similarly, we can construct another sequence of probability spaces $(\Omega'_T, \mathcal{F}'_T, \mathbb{Q}_T)_{T\geq 1}$ such that for each $T \in \mathbb{Z}^+$, we have random variables $\mathfrak{L}^T_-$ and $\mathfrak{L}^T$ have law $\mathbb{P}^{0,T,\vec{x}^T_-,\vec{y}^T_-}_{avoid,Ber}$, and $\mathbb{P}^{0,T,\vec{x}^T,\vec{y}^T}_{avoid,Ber}$ under measure $\mathbb{Q}_T$, respectively, along with $\mathbb{Q}_T\left(\mathfrak{L}^T_-(i,r) \leq \mathfrak{L}^T(i,r), i = 1,\cdots,k, r \in [\![0,T]\!]\right) = 1$.

Therefore, we have that under measure $\mathbb{P}_T$ and $\mathbb{Q}_T$:

$$\mathbb{P}_T(Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} \in R) \leq \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R), \quad \mathbb{Q}_T(Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}} \in R) \geq \mathbb{Q}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) \tag{5.7.82}$$

Take limit-infimum and limit-supremum on both sides of the first and second inequality in (5.7.82) respectively, we get

$$\int_R \rho_{\epsilon,+}(z)dz \leq \liminf_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R), \quad \int_R \rho_{\epsilon,-}(z)dz \geq \limsup_{T\to\infty} \mathbb{Q}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) \tag{5.7.83}$$

because of the weak convergence of $Z^T_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}$ and $Z^T_{\vec{A}_{\epsilon,-},\vec{B}_{\epsilon,-}}$. Since the distributions of $Z^T_{\vec{a}_0,\vec{b}_0}$ under measure $\mathbb{P}_T$ and $\mathbb{Q}_T$ are the same, we can combine the above two inequalities in (5.7.83) and get

$$\int_R \rho_{\epsilon,+}(z)dz \leq \liminf_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) \leq \limsup_{T\to\infty} \mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0} \in R) \leq \int_R \rho_{\epsilon,-}(z)dz \tag{5.7.84}$$

The rest of the proof establishes the following statement:

$$\lim_{\epsilon\downarrow 0} \int_R \rho_{\epsilon,+}(z)dz = \lim_{\epsilon\downarrow 0} \int_R \rho_{\epsilon,-}(z)dz = \int_R \rho_c(z)dz \tag{5.7.85}$$

and thereby concluding

$$\lim_{T\to\infty}\mathbb{P}_T(Z^T_{\vec{a}_0,\vec{b}_0}\in R)=\int_R \rho_c(z)dz$$

by letting $\epsilon\downarrow 0$ in the inequality (5.7.84), and we prove the weak convergence of $Z^T_{\vec{a}_0,\vec{b}_0}$.

The rest of the proof intends to establish (5.7.85). By (5.7.69), (5.7.71) and (5.7.74), we have when $\epsilon<1$:

$$\epsilon^{-u}f(\vec{A}_{\epsilon,+},\vec{z})\le\varphi(\vec{a}_0,\vec{z},\vec{m})+\widetilde{C}_1\cdot(|z_1|+|z_k|)^{u+1}\cdot e^{C_1\cdot\sum_{j=1}^k|z_j|}\equiv F(\vec{z}) \tag{5.7.86}$$

where the constants $\widetilde{C}_1=N(\vec{m})\cdot M(\vec{m})\cdot(k!)\cdot\left(\max_{1\le i\le p}m_i\right)$, $C_1=c_1(t,p)\cdot k\cdot(\max_{1\le i\le p}m_i)$ only depend on $\vec{m}$. Analogously, we can find constants $\widetilde{C}_2$ and C_2 only depending on $\vec{n}$ such that

$$\epsilon^{-v}f(\vec{B}_{\epsilon,+},\vec{z})\le\varphi(\vec{b}_0,\vec{z},\vec{n})+\widetilde{C}_2\cdot(|z_1|+|z_k|)^{v+1}\cdot e^{C_2\cdot\sum_{j=1}^k|z_j|}\equiv G(\vec{z}) \tag{5.7.87}$$

Therefore, we obtain

$$\epsilon^{-(u+v)}f(\vec{A}_{\epsilon,+},\vec{z})g(\vec{B}_{\epsilon,+},\vec{z})\prod_{i=1}^k e^{-c_3(t,p)z_i^2}\le F(\vec{z})G(\vec{z})\prod_{i=1}^k e^{-c_3(t,p)z_i^2} \tag{5.7.88}$$

and the right hand side of (5.7.88) is integrable because of the quadratic terms in the exponential. Let $Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}$ be the normalizing constant in the density (5.7.3) when $\vec{a}$ and $\vec{b}$ equal to $\vec{A}_{\epsilon,+}$ and $\vec{B}_{\epsilon,+}$. Then, we have

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\epsilon^{-(u+v)}Z_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}&=\lim_{\epsilon\downarrow 0}\int_{\mathbb{W}_k^o}\left(\epsilon^{-u}f(\vec{A}_{\epsilon,+},\vec{z})\right)\left(\epsilon^{-v}g(\vec{B}_{\epsilon,+},\vec{z})\right)\prod_{i=1}^k e^{-c_3(t,p)z_i^2}dz\\&=\int_{\mathbb{W}_k^o}\varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n})\prod_{i=1}^k e^{-c_3(t,p)z_i^2}dz=Z_c\end{aligned} \tag{5.7.89}$$

where the second equality uses dominated convergence theorem with the dominating function being the right hand side of (5.7.88) as well as results (5.7.68) and (5.7.75), and the last equality is due to (5.7.79) which gives the definition of Z_c. Therefore, we conclude

$$\lim_{\epsilon\downarrow 0} Z^{-1}_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}} \cdot f(\vec{a},\vec{z}) \cdot g(\vec{b},\vec{z}) = \lim_{\epsilon\downarrow 0} \left(\epsilon^{u+v} Z^{-1}_{\vec{A}_{\epsilon,+},\vec{B}_{\epsilon,+}}\right) \cdot (\epsilon^{-u} f(\vec{a},\vec{z})) \cdot \left(\epsilon^{-v} g(\vec{b},\vec{z})\right)$$
$$= Z_c^{-1} \varphi(\vec{a}_0,\vec{z},\vec{m})\psi(\vec{b}_0,\vec{z},\vec{n}) \tag{5.7.90}$$

which implies $\rho_{\epsilon,+}(z)$ pointwise converges to $\rho_c(z)$ when $\epsilon \downarrow 0$. Since $\rho_{\epsilon,+}(z)\mathbf{1}_R \leq \rho_{\epsilon,+}(z)$ is bounded by an integrable function, by Dominated Convergence Theorem we have:

$$\lim_{\epsilon\downarrow 0} \int_R \rho_{\epsilon,+}(z)dz = \int_R \rho_c(z)dz$$

Analogously, we can get $\lim_{\epsilon\downarrow 0} \int_R \rho_{\epsilon,-}(z)dz = \int_R \rho_c(z)dz$ and we proved the statement (5.7.85), which completes the proof. □

Chapter 6: Large deviation principle of the asymmetric simple exclusion process (ASEP)

6.1 Introduction

6.1.1 The ASEP and main results

In this paper, we study the upper-tail Large Deviation Principle (LDP) of the *asymmetric simple exclusion process* (ASEP) with step initial data. The ASEP is a continuous-time Markov chain on particle configurations $\mathbf{x} = (x_1 > x_2 > \cdots)$ in $\mathbb{Z}$. The process can be described as follows. Each site $i \in \mathbb{Z}$ can be occupied by at most one particle, which has an independent exponential clock with exponential waiting time of mean 1. When the clock rings, the particle jumps to the right with probability q or to the left with probability $p = 1 - q$. However, the jump is only permissible when the target site is unoccupied. For our purposes, it suffices to consider configurations with a rightmost particle. At any time $t \in \mathbb{R}_{>0}$, the process has the configuration $\mathbf{x}(t) = (x_1(t) > x_2(t) > \cdots)$ in $\mathbb{Z}$, where $x_j(t)$ denotes the location of the j-th rightmost particle at this time. Appearing first in the biology work of Macdonald, Gibbs, and Pipkin [184] and introduced to the mathematics community two years later by [223], the ASEP has since become the "default stochastic model to study transport phenomena", including mass transport, traffic flow, queueing behavior, driven lattices and turbulence. We refer to [49, 181, 180, 224] for the mathematical study of and related to the ASEP.

When $q = 1$, we obtain the *totally asymmetric simple exclusion process* (TASEP), which allows jumps only to the right. It connects to several other physical systems such as the exponential last-passage percolation, zero-temperature directed polymer in a random environment, the corner growth process and is known to possess complete determinantal structure (*free-fermionicity*). We refer the readers to [155, 181, 180, 203] and the references therein for more thorough treatises of the TASEP.

The dynamics of ASEP are uniquely determined once we specify its initial state. In the present paper, we restrict our attention to the ASEP started from the *step* initial configuration, i.e. $x_j(0) = -j$, $j = 1, 2, \ldots$. We set $\gamma = q - p$ and assume $q > \frac{1}{2}$, i.e., ASEP has a drift to the right. An observable of interest in ASEP is $H_0(t)$, the integrated current through 0 which is defined as:

$$H_0(t) := \text{ the number of particles to the right of zero at time } t. \tag{6.1.1}$$

$H_0(t)$ can also be interpreted as the one-dimensional height function of the interface growth of the ASEP and thus carries significance in the broader context of the Kardar-Parisi-Zhang (KPZ) universality class. We will elaborate on the connection to KPZ universality class later in Section 6.1.3. As a well-known random growth model itself, the large-time behaviors of ASEP with step initial conditions have been well-studied. Indeed, it is known [181, Chapter VIII, Theorem 5.12] that the current satisfies the following strong law of large numbers:

$$\tfrac{1}{t}H_0\big(\tfrac{t}{\gamma}\big) \to \tfrac{1}{4}, \text{ almost surely as } t \to \infty.$$

The strong law has been later complemented by fluctuation results in the seminal works by Tracy and Widom. In a series of papers [228], [230] [229], Tracy and Widom exploit the integrability of ASEP with step initial data and establish via contour analysis that $H_0(t)$ when centered by $\frac{t}{4}$ has typical deviations of the order $t^{1/3}$ and has the following asymptotic fluctuations:

$$\tfrac{1}{t^{1/3}}2^{4/3}\big(-H_0\big(\tfrac{t}{\gamma}\big) + \tfrac{t}{4}\big) \implies \xi_{\mathrm{GUE}}, \tag{6.1.2}$$

where ξ_{GUE} is the GUE Tracy-Widom distribution [231]. When $q = 1$, (6.1.2) recovers the same result on TASEP, which has been proved earlier by [155].

Given the existing fluctuation results on the ASEP with step initial data, it is natural to inquire into its Large Deviation Principle (LDP). Namely, we seek to find the probability of when the event $-H_0(\frac{t}{\gamma})+\frac{t}{4}$ has deviations of order t. Intriguingly, one expects the lower- and upper-tail LDPs to have different speeds: the upper-tail deviation is expected to occur at speed t whereas the lower-tail has speed t^2:

$$\mathbb{P}\left(-H_0(\tfrac{t}{\gamma})+\tfrac{t}{4}<-\tfrac{t}{4}y\right)\approx e^{-t^2\Phi_-(y)}; \qquad \text{(Lower Tail)}$$

$$\mathbb{P}\left(-H_0(\tfrac{t}{\gamma})+\tfrac{t}{4}>+\tfrac{t}{4}y\right)\approx e^{-t\Phi_+(y)}. \qquad \text{(Upper Tail)}$$

Thus, the upper tail corresponds to ASEP being “too slow" while the lower tail corresponds to ASEP being “too fast". Heuristically, we can make sense of such speed differentials. Because of the nature of the exclusion process, when a *single* particle is moving slower than the usual, it forces *all* the particles on the left of it to be automatically slower. Hence ASEP becomes slow if *only one* particle is moving slow. This event has probability of the order $\exp(-O(t))$. However, in order to ensure that there are many particles on the right side of origin (this corresponds to ASEP being fast), it requires a large number of particles to move fast *simulatenously*. This event is much more unlikely and happens with probability $\exp(-O(t^2))$.

In this article, we focus on the *upper-tail* deviations of the ASEP with step initial data and present the first proof of the ASEP upper-tail LDP on the *complete* real line. Consider ASEP with $q\in(\frac{1}{2},1)$ and set $p=1-q$ and $\tau=p/q\in(0,1)$. Our first theorem computes the sth-*Lyapunov exponent* of $\tau^{H_0(t)}$, which is the limit of the logarithm of $\mathbf{E}[\tau^{sH_0(t)}]$ scaled by time:

Theorem 6.1.1. *For $s\in(0,\infty)$ we have*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbf{E}[\tau^{sH_0(t)}]=-h_q(s)=:-(q-p)\frac{1-\tau^{\frac{s}{2}}}{1+\tau^{\frac{s}{2}}}. \qquad (6.1.3)$$

It is well known (see Proposition 1.12 in [128] for example) that the *upper-tail* large deviation principle of the stochastic process $\log\tau^{H_0(t)}$ is the Legendre-Fenchel dual of the Lyapunov exponent in (6.1.3). Since $\tau < 1$, as a corollary, we obtain the following *upper-tail* large deviation rate function for $-H_0(t)$.

Theorem 6.1.2. *For any $y \in (0,1)$ we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right) + \tfrac{t}{4} > \tfrac{t}{4}y\right) = -[\sqrt{y} - (1-y)\tanh^{-1}(\sqrt{y})] =: -\Phi_+(y), \tag{6.1.4}$$

where $\gamma = 2q-1$. Furthermore, we have the following asymptotics near zero:

$$\lim_{y\to 0^+} y^{-3/2}\Phi_+(y) = \tfrac{2}{3}. \tag{6.1.5}$$

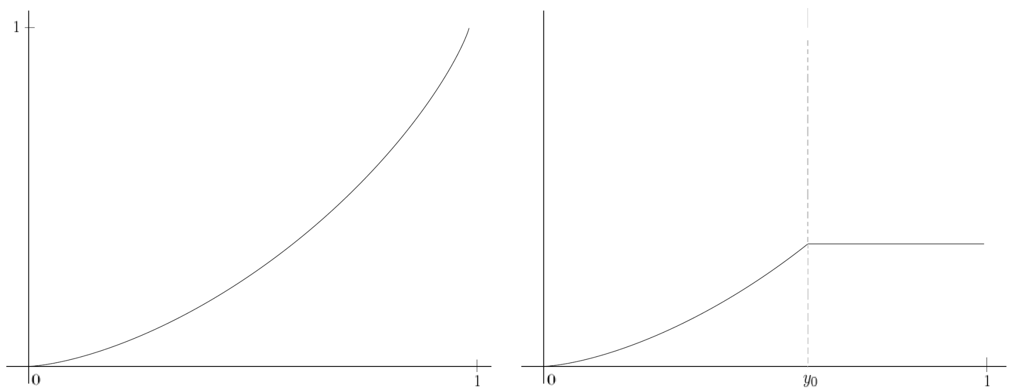


Figure 6.1: The figure on the left is the plot of $\Phi_+(y)$. The right one is the plot of $\widetilde{\Phi}_+(y)$.

Remark 6.1.3. Note that our large deviation result is restricted to $y \in (0,1)$ as $\mathbb{P}(-H_0(\frac{t}{\gamma}) + \frac{t}{4} > \frac{t}{4}y) = 0$ for $y \geq 1$. Furthermore, although Theorem 6.1.2 makes sense when $q = 1$, one cannot recover it from Theorem 6.1.1, which only makes sense for $\tau = (1-q)/q \in (0,1)$. However, as mentioned before, [155] has already settled the $q = 1$ TASEP case and obtained the upper-tail rate function in a variational form. We will later show in Appendix 6.5 that [155] variational formula for TASEP matches with our rate function in (6.1.4).

Remark 6.1.4. Recently, the work [85] has obtained a one-sided large deviation bound for the upper tail of the ASEP. In particular, they showed

$$\mathbb{P}\left(-H_0(\tfrac{t}{\gamma})+\tfrac{t}{4}>\tfrac{t}{4}y\right)\le Ce^{-t\widetilde{\Phi}_+(y)},\quad y\in(0,1). \tag{6.1.6}$$

The function $\widetilde{\Phi}_+$ coincides with the correct rate function Φ_+ defined in (6.1.4) only for $y \le y_0 := \frac{1-2\sqrt{q(1-q)}}{1+2\sqrt{q(1-q)}}$, as captured by Figure 6.1. We will further compare and contrast our results and method with [85] later in Section 6.1.3.

Remark 6.1.5. For y small enough, following (6.1.2) and upper tail decay of GUE Tracy-Widom distribution [114], one expects

$$\mathbb{P}\left(-H_0(\tfrac{t}{\gamma})+\tfrac{t}{4}>\tfrac{t}{4}y\right)\approx\mathbb{P}(\xi_{\mathrm{GUE}}>2^{-2/3}yt^{2/3})\approx e^{-\frac{2}{3}y^{3/2}t}$$

Thus the asymptotics in (6.1.5) shows that $\Phi_+(y)$ indeed recovers the expected GUE Tracy-Widom tails as $y\to 0^+$.

6.1.2 Sketch of proof

In this section we present a sketch of the proof of our main results. As explained before, Theorem 6.1.2 can be obtained from Theorem 6.1.1 by standard Legendre-Fenchel transform technique. So here we only give a brief account of the proof idea of Theorem 6.1.1. A more detailed overview of the proofs of our main results can be found in Section 6.2.

The main component of our proof is the following τ-Laplace transform formula for $H_0(t)$ that appears in Theorem 5.3 in [49]:

Theorem 6.1.6 (Theorem 5.3 in [49])**.** *Fix any $\delta\in(0,1)$. For $\zeta>0$ we have*

$$\mathbf{E}\left[F_q(\zeta\tau^{H_0(t)})\right]=\det(I+K_{\zeta,t}),\quad F_q(\zeta):=\prod_{n=0}^{\infty}\frac{1}{1+\zeta\tau^n}. \tag{6.1.7}$$

Here $\det(I+K_{\zeta,t})$ is the Fredholm determinant of $K_{\zeta,t}:L^2(\mathfrak{C}(\tau^{1-\frac{\delta}{2}}))\to L^2(\mathfrak{C}(\tau^{1-\frac{\delta}{2}}))$, and $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$ denotes a positively-oriented circular contour centered at 0 with radius $\tau^{1-\frac{\delta}{2}}$. The oper-

ator $K_{\zeta,t}$ is defined through the integral kernel

$$K_{\zeta,t}(w,w') := \frac{1}{2\pi\mathbf{i}}\int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty}\Gamma(-u)\Gamma(1+u)\zeta^u\frac{g_t(w)}{g_t(\tau^u w)}\frac{\mathrm{d}u}{w'-\tau^u w}, \tag{6.1.8}$$

for $g_t(z) = \exp\left(\frac{(q-p)t}{1+\frac{z}{\tau}}\right)$.

Remark 6.1.7. The original statement of the above theorem in [49] appears in a much more general setup with general conditions on the contours. We will explain the choice of our contours stated above in Section 6.3 and check that it satisfies the general criterion for contours as stated in Theorem 5.3 in [49].

We next recall that the Fredholm determinant is defined as a series as follows.

$$\det(I+K_{\zeta,t}) := 1+\sum_{L=1}^{\infty}\mathrm{tr}(K_{\zeta,t}^{\wedge L}) \tag{6.1.9}$$

$$:= 1+\sum_{L=1}^{\infty}\frac{1}{L!}\int_{\mathfrak{C}(\tau^{1-\frac{\delta}{2}})}\cdots\int_{\mathfrak{C}(\tau^{1-\frac{\delta}{2}})}\det(K_{\zeta,t}(w_i,w_j))_{i,j=1}^{L}\prod_{i=1}^{L}\mathrm{d}w_i. \tag{6.1.10}$$

The notation $K_{\zeta,t}^{\wedge L}$ comes from the exterior algebra definition, which we refer to [221] for more details. As a clarifying remark, we use this exterior algebra notation only for the simplicity of its expression and rely essentially on the definition in (6.1.10) throughout the rest of the paper.

To extract information on the fractional moments of $\tau^{H_0(t)}$, we combine the formula in (6.1.7) with the following elementary identity, which is a generalized version of Lemma 1.4 in [88].

Lemma 6.1.8. *Fix $n \in \mathbb{Z}_{>0}$ and $\alpha \in [0,1)$. Let U be a nonnegative random variable with finite n-th moment. Let $F : [0,\infty) \to [0,1]$ be a n-times differentiable function such that $\int_0^\infty \zeta^{-\alpha}F^{(n)}(\zeta)\mathrm{d}\zeta$ is finite. Assume further that $\|F^{(k)}\|_\infty < \infty$ for all $1 \le k \le n$. Then the $(n-1+\alpha)$-th moment of U is given by*

$$\mathbf{E}[U^{n-1+\alpha}] = \frac{\int_0^\infty \zeta^{-\alpha}\mathbf{E}[U^nF^{(n)}(\zeta U)]\mathrm{d}\zeta}{\int_0^\infty \zeta^{-\alpha}F^{(n)}(\zeta)\mathrm{d}\zeta} = \frac{\int_0^\infty \zeta^{-\alpha}\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\mathbf{E}[F(\zeta U)]\mathrm{d}\zeta}{\int_0^\infty \zeta^{-\alpha}F^{(n)}(\zeta)\mathrm{d}\zeta}.$$

The proof of this lemma follows by an interchange of measure justified by Fubini's theorem and the dominated convergence theorem, as $\mathbf{E}[U^n]$ and $\|F^{(k)}\|_\infty < \infty$ for all $1 \le k \le n$.

For $s > 0$, we apply this lemma with $U = \tau^{H_0(t)}$, $n = \lfloor s \rfloor + 1$ and $\alpha = s - \lfloor s \rfloor$. We take $F(x) = F_q(x)$ defined in (6.1.7) which is shown to be satisfy the hypothesis of Lemma 6.1.8 (see Proposition 6.2.2). As a result, we transform the computation of $\mathbf{E}[\tau^{sH_0(t)}]$ into that of

$$\int_0^\infty \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathbf{E}[F_q(\zeta \tau^{H_0(t)})] \mathrm{d}\zeta. \tag{6.1.11}$$

Utilizing the exact formula from (6.1.7) and the definition of Fredholm determinant from (6.1.10), we can write the above expression as a series where we identify the leading term (corresponding to $L = 1$ term of the series) and a higher-order term (corresponding to $L \ge 2$ terms of the series). We eventually show that the asymptotics of the leading term matches with the exact asymptotics in (6.1.3) while the higher-order term decays much faster. This leads to the proof of Theorem 6.1.1.

The above description of our method is in line with the Lyapunov moment approach adopted in the works of [88], [128] and [182] to obtain upper-tail large deviation results of other integrable models, such as the KPZ equation. Namely, we extract fractional moments from the (τ-)Laplace transform such as (6.1.7) according to Lemma 6.1.8. In particular, our work draws from those of [88] and [182], which studied the fractional moments of the Stochastic Heat Equation (SHE) and the half-line Stochastic Heat Equation, respectively. We will further contextualize the connections of our work to [88], [128] and [182] in Section 6.1.3. In the following text, however, we emphasize a few key differences and technical challenges unique to the ASEP that we have encountered and resolved in our proof.

First, unlike SHE or half-line SHE, the usual Laplace transform is not available in case of the ASEP. Instead, we only have the τ-Laplace transform for our observable of interest. As a result, we have formulated Lemma 6.1.8 in our paper, which is more generalized than its prototype in [88, Lemma 1.4], to feed in the τ-Laplace transform. Consequently, we have worked with τ-exponential functions in our analysis.

Another key difference is that the kernel $K_{\zeta,t}$ in (6.1.8) in our model is much more intricate than its counterpart in the KPZ model and leads to much more involved analysis of the leading term. Indeed, $K_{\zeta,t}$ is asymmetric and as u varies in $(\delta - \mathbf{i}\infty, \delta + \mathbf{i}\infty)$, the function $\frac{g_t(w)}{g_t(\tau^u w)}$ appearing in the kernel $K_{\zeta,t}$, exhibits a periodic behavior, whereas the kernel in the KPZ models involves Airy functions in its integrand which have a unique maximum and are much easier to analyze. Furthermore, our model exhibits exponentially decaying moments of $\tau^{H_0(t)}$ as opposed to the exponentially increasing ones of the KPZ models in [88] and [182] and this demands a more precise understanding of the trace term of our Fredholm determinant expansion. For instance in Section 6.3, to obtain the precise asymptotics for our leading term, we have performed steepest descent analysis on the kernel $K_{\zeta,t}$, where the periodic nature of $\frac{g_t(w)}{g_t(\tau^u w)}$ results in infinitely many critical points. A major technical challenge in our proof is to argue how the contribution from only one of the critical points dominates the those from the rest and this is accomplished in the proof of Proposition 6.2.4. Similarly, the asymmetry of the kernel in the ASEP model has led us to opt for the Hadamard's inequality approach as exemplified in Section 4 of [182], instead of the operator theory argument in [88], to obtain a sufficient upper bound for the higher-order terms in our paper in Section 6.4.

6.1.3 Comparison to Previous Works

In a broader context, our main result on the Lyapunov exponent for the ASEP with step initial data and its upper-tail large deviation belongs to the undertakings of studying the intermittency phenomenon and large deviation problems of integrable models in the KPZ universality class. As we have previously alluded to, the KPZ universality class contains a collection of random growth models that are characterized by scaling exponent of $1/3$ and certain universal non-Gaussian large time fluctuations. We refer to [5, 77, 227] and the references therein for more details. The ASEP is one of the standard one-dimensional models of the KPZ universality class and bears connection to several other integrable models in this class, such as the stochastic six-vertex model [48, 2, 78], KPZ equation [53, 110, 218, 5, 77], and q-TASEP [49].

On the other hand, the intermittency property is a universal phenomenon that captures high population concentrations on small spatial islands over large time. Mathematically, the intermittency of a random field is defined in terms of its Lyapunov exponents. In particular, the connection between integer Lyapunov moments and intermittency has long been an active area of study in the SPDE community in last few decades [127, 57, 34, 125, 147, 70, 61, 18]. For the KPZ equation, [159] predicted the integer Lyapunov exponents for the SHE using replica Bethe anstaz techniques. This result was later first rigorously attempted in [34] and correctly proven in [62]. Similar formulas were shown for the moments of the parabolic Anderson model, semi-discrete directed polymers, q-Whittaker process (see [46] and [47]). For the ASEP, integer moments formula for $\tau^{H_0(t)}$ were obtained in [49] using nested contour integral ansatz.

From the perspective of tail events, by studying the asymptotics of integer Lyapunov exponents formulas, one can extract one-sided bounds on the upper tails of integrable models. However, these integer Lyapunov exponents alone are not sufficient to provide the exact large deviation rate function.

Recently, a stream of effort has been devoted to studying large deviations for some KPZ class models by explicitly computing the fractional Lyapunov exponents. The work of [88] set this series of effort in motion by solving the KPZ upper-tail large deviation principle through the fractional Lyapunov exponents of the SHE with delta initial data. [128] soon extended the same result for the SHE for a large class of initial data, including any random bounded positive initial data and the stationary initial data. An exact way to compute every positive Lyapunov exponent of the half-line SHE was also uncovered in [182]. In lieu of these developments, our main result for the ASEP with step initial data and its upper-tail large deviation fits into this broader endeavor of studying large deviation problems of integrable models with the Lyapunov exponent appproach.

Meanwhile, in the direction of the ASEP, as mentioned before, [85] has produced a one-sided large deviation bound for the upper-tail probability appearing in (6.1.4) which coincides with the correct rate function Φ_+ defined in (6.1.4) for $y \le y_0 := \frac{1-2\sqrt{q(1-q)}}{1+2\sqrt{q(1-q)}}$. This result was sufficient for their purpose of establishing a near-exponential fixation time for the coarsening model on $\mathbb{Z}^2$ and

[85] obtained it via steepest descent analysis on the exact formula for the probability of $H_0(t/\gamma)$. More specially, they worked with the following result from [229, Lemma 4] as input:

$$\mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right) + \tfrac{t}{4} > \tfrac{t}{4}y\right) = \frac{1}{2\pi\mathbf{i}} \int_{|\mu|=R} (\mu;\tau)_\infty \det(1 + \mu J_{m,t}^{(\mu)}) \frac{\mathrm{d}\mu}{\mu}, \tag{6.1.12}$$

where $m = \lfloor \frac{1}{4}t(1-y) \rfloor$, $R \in (\tau, \infty) \setminus \{1, \tau^{-1}, \tau^{-2}, \ldots\}$ is fixed, $(\mu;\tau)_\infty := (1-\mu)(1-\mu\tau)(1-\mu\tau^2)\ldots$ is the infinite τ-Pochhammer symbol and $J_{m,t}^{(\mu)}$ is the kernel defined in Equation (3.4) of [85]. Analyzing the exact pre-limit Fredholm determinant $\det(1 + \mu J_{m,t}^{(\mu)})$, [85] chose appropriate contours for the kernel $J_{m,t}^{(\mu)}$ that pass through its critical points and performed a steepest descent analysis. However, their choice of contours was unattainable beyond the threshold y_0. Namely, if we attempted to deform the same contours for $y > y_0$, we would inevitably cross poles, which rendered the steepest descent analysis much trickier. By adopting the Lyapunov moment approach, we have avoided this problem when looking for the precise large deviation rate function.

In addition to the relavence of our upper-tail LDP result, it is also worthy to remark on the difficulty of obtaining a lower-tail LDP of the ASEP with step initial data. As explained before, the lower-tail $\mathbb{P}(-H_0(\frac{t}{\gamma}) + \frac{t}{4} < -\frac{t}{4}y)$ is expected to go to zero at a much faster rate of $\exp(-t^2\Phi_-(y))$. The existence of the lower-tail rate function has so far only been shown in the case of TASEP in [155] through its connection to continuous log-gases. The functional LDPs for TASEP for both tails have been studied in [154], [234], [210] (upper tail), and [197] (lower-tail). Large deviations for open systems with boundaries in contact with stochastic reservoirs has also been studied in physics literature. We mention [103], [102], [43] and the references therein for works in these directions.

More broadly for integrable models in the KPZ universality class, lower tail of the KPZ equation has been extensively studied in both mathematics and physics communities. In the physics literature, [175] provided the first prediction of the large deviation tails of the KPZ equation for narrow wedge initial data. For the upper tail, their analysis also yields subdominant corrections ([178, Supp. Mat.]). Furthermore, the physics work of [219] first predicted lower-tail rate function of the KPZ equation for narrow wedge initial data in an analytical form, followed by the derivations in [72] and [168] via different methods. The asymptotics of deep lower tail of KPZ equation was later obtained in [167] for a wide class of initial data. From the mathematics front, the work [80] provided detailed, rigorous tail bounds for the lower tail of the KPZ equation for narrow wedge initial data. The precise rate function of its lower-tail LDP was later proved in [232] and [52], which confirmed the prediction of existing physics literature. The four different routes of deriving the lower-tail LDP in [219], [72], [168] and [232] were later shown to be closely related in [166]. A new route has also been recently obtained in the physics work of [176] (see also [205]).

In the short time regime, large deviations for the KPZ equation has been studied extensively in physics literature (see [177], [164], [163] and the references therein for a review). Recently, [183] rigorously derived the large deviation rate function of the KPZ equation in the short-time regime in a variational form and recovered deep lower-tail asymptotics, confirming existing physics predictions. For non-integrable models, large deviations of first-passage percolation were studied in [63] and more recently [30]. For last-passage percolation with general weights, recently, geometry of polymers under lower tail large deviation regime has been studied in [29].

Notation

Throughout the rest of the paper, we use $\mathrm{C} = \mathrm{C}(a, b, c, \ldots) > 0$ to denote a generic deterministic positive finite constant that is dependent on the designated variables $a, b, c, \ldots$. However, its particular content may change from line to line. We also use the notation $\mathfrak{C}(r)$ to denote a positively oriented circle with center at origin and radius $r > 0$.

Outline

The rest of this article is organized as follows. In Section 6.2, we introduce the main ingredients for the proofs of Theorem 6.1.1 and 6.1.2. In particular, we reduce the proof of our main results to Proposition 6.2.4 (asymptotics of the leading order) and Proposition 6.2.5 (estimates for the higher order), which are proved in Sections 6.3 and 6.4 respectively. Finally, in Appendix 6.5 we compare our rate function $\Phi_+(y)$, defined in (6.1.4), to that of TASEP.

6.2 Proof of Main Results

In this section, we give a detailed outline of the proofs of Theorems 6.1.1 and 6.1.2. In Section 6.2.1 we collect some useful properties of h_q and F_q functions defined in (6.1.4) and (6.1.7) respectively. In Section 6.2.2 we complete the proof of Theorems 6.1.1 and 6.1.2 assuming technical estimates on the leading order term (Proposition 6.2.4) and higher order term (Proposition 6.2.5).

Throughout this paper, we fix $s > 0$ and set $n = \lfloor s \rfloor + 1 \geq 1$ and $\alpha = s - \lfloor s \rfloor$ so that $s = n - 1 + \alpha$. We also fix $q \in (\frac{1}{2}, 1)$ and set $p = 1 - q$ and $\tau = p/q \in (0, 1)$ for the rest of the article.

6.2.1 Properties of $h_q(x)$ and $F_q(x)$

Recall the Lyapunov exponent $h_q(x)$ defined in (6.1.3) and the $F_q(x)$ function defined in (6.1.7). The following two propositions investigates various properties of these two functions which are necessary for our later proofs.

Proposition 6.2.1 (Properties of h_q)**.** *Consider the function $h_q : (0, \infty) \to \mathbb{R}$ defined by $h_q(x) = (q - p)\frac{1-\tau^{\frac{x}{2}}}{1+\tau^{\frac{x}{2}}}$. Then, the following properties hold true:*

(a) $B_q(x) := \frac{h_q(x)}{x}$ is strictly positive and strictly decreasing with

$$\lim_{x \to 0^+} B_q(x) = \tfrac{1}{4}(p - q) \log \tau > 0.$$

(b) h_q is strictly subadditive in the sense that for any $x, y \in (0, \infty)$ we have

$$h_q(x + y) < h_q(x) + h_q(y).$$

(c) h_q is related to Φ_+ defined in (6.1.4) *via the following Legendre-Fenchel type transformation:*

$$\Phi_+(y) = \sup_{s \in \mathbb{R}_{>0}} \left\{ s\frac{1-y}{4} \log \tau + \frac{1}{q-p} h_q(s) \right\}, \quad y \in (0, 1).$$

Proof. For (a), first, the positivity of $B_q(x)$ follows from the positivity of $h_q(x)$. To see its growth, taking the derivative of $B_q(x)$ we obtain

$$B_q'(x) = \frac{(q-p)(-x\tau^{\frac{x}{2}}\log\tau - 1 + \tau^x)}{(1+\tau^{\frac{x}{2}})^2 x^2}. \tag{6.2.1}$$

Note that the numerator on the r.h.s of (6.2.1) is 0 when $x = 0$ and its derivative against x is $\tau^{\frac{x}{2}}\log\tau(\tau^{\frac{x}{2}} - \frac{x}{2}\log\tau - 1) < 0$ for $x > 0$. Thus $B_q'(x)$ is strictly negative when $x > 0$ and $B_q(x)$ is strictly decreasing for $x > 0$. L'Hôpital's rule yields that $\lim_{x\to 0^+} B_q(x) = h_q'(0) = \frac{1}{4}(q-p)\log\tau$.

For (b), direct computation yields

$$h_q(x+y) - h_q(x) - h_q(y) = -(q-p)\frac{(1-\tau^{\frac{y}{2}})(1-\tau^{\frac{x}{2}})(1-\tau^{\frac{x+y}{2}})}{(1+\tau^{\frac{x+y}{2}})(1+\tau^{\frac{x}{2}})(1+\tau^{\frac{y}{2}})} < 0. \tag{6.2.2}$$

Lastly, for part (c), we fix $y \in (0,1)$ and define

$$g_y(s) := s\frac{1-y}{4}\log\tau + \frac{1}{q-p}h_q(s), \quad s > 0.$$

Direct computation yields $g_y'(s) = (\frac{1-y}{4} - \frac{\tau^{\frac{s}{2}}}{(1+\tau^{\frac{s}{2}})^2})\log\tau$ and $g_y''(s) = \frac{\tau^{\frac{s}{2}}(\tau^{\frac{s}{2}}-1)\log^2\tau}{2(1+\tau^{\frac{s}{2}})^3} < 0$. Thus $g_y(s)$ is concave on $(0,\infty)$ and hence attains its unique maxima when $g_y'(s) = 0$ or equivalently $\frac{1-y}{4} = \frac{\tau^{\frac{s}{2}}}{(1+\tau^{\frac{s}{2}})^2}$. The last equation has $s = 2\log_\tau(\frac{1-\sqrt{y}}{1+\sqrt{y}})$ as the only positive solution and hence it defines the unique maximum. Substituting this s back into $g_y(s)$ generates the final result as $\Phi_+(y)$. □

Proposition 6.2.2 (Properties of $F_q(\zeta)$)**.** *Consider the function* $F_q : [0,\infty) \to [0,1]$ *defined by* $F_q(\zeta) := \prod_{n=0}^{\infty}(1+\zeta\tau^n)^{-1}$. *Then, the following properties hold true:*

(a) F_q *is an infinitely differentiable function with* $(-1)^n F_q^{(n)}(\zeta) \geq 0$ *for all* $x > 0$. *Furthermore,* $\|F_q^{(n)}\|_\infty < \infty$ *for each n.*

(b) For each $n \in \mathbb{Z}_{>0}$, *and* $\alpha \in [0,1)$, $(-1)^n \int_0^\infty \zeta^{-\alpha} F_q^{(n)}(\zeta)\mathrm{d}\zeta$ *is positive and finite.*

(c) All the derivatives of F_q have superpolynomial decay. In other words for any $m, n \in \mathbb{Z}_{\geq 0}$ we have

$$\sup_{\zeta>0} |\zeta^m F_q^{(n)}(\zeta)| < \infty.$$

Proof. (a) Note that $F_q(\zeta) = \prod_{n=0}^{\infty}(1+\zeta\tau^n)^{-1} = (-\zeta;\tau)_{\infty}^{-1}$ where we recall that $(-\zeta;\tau)_\infty$ is the τ-Pochhammer symbol. As $(-\zeta;\tau)_\infty$ is analytic [6, Corollary A.1.6.] and nonzero for $\zeta \in [0,\infty)$, its inverse $F_q(\zeta)$ is analytic.

We next rewrite $F_q(\zeta) = \prod_{n=0}^{\infty} f_n(\zeta)$, where $f_n(\zeta) = (1+\zeta\tau^n)^{-1}$. Denote $H(\zeta) := \log F_q(\zeta)$. Since each $f_n(\zeta) \in (0,1)$ is analytic for $\zeta \in [0,\infty)$ and the product $\prod_{n=0}^{\infty} f_n(\zeta) \in (0,1)$ converges locally and uniformly, $H(\zeta)$ is well-defined and $H(\zeta) = \sum_{n=0}^{\infty} \log f_n(\zeta)$. Given that $|\sum_{n=0}^{\infty} \frac{1}{f_n(\zeta)} f_n'(\zeta)| = \sum_{n=0}^{\infty} \frac{\tau^n}{(1+\zeta\tau^n)} < \frac{1}{1-\tau}$, we have

$$H'(\zeta) = \frac{F_q'(\zeta)}{F_q(\zeta)} = \sum_{n=1}^{\infty} \frac{f_n'(\zeta)}{f_n(\zeta)} =: G(\zeta). \tag{6.2.3}$$

Note that $G(\zeta) = -\sum_{j=1}^{\infty} \tau^j f_j(\zeta)$ and $|G(\zeta)| < \infty$. For each $m \in \mathbb{Z}_{>0}$, let us set $G^{(m)}(\zeta) := -\sum_{j=1}^{\infty} \tau^j f_j^{(m)}(\zeta)$. As $f_j^{(m)}(\zeta) = (-1)^m m! \frac{\tau^{mj}}{(1+\xi\tau^j)^{m+1}}$, we obtain $|G^{(m)}(\zeta)| \leq \frac{m!}{1-\tau^{m+1}} < \infty$ converges locally and uniformly. Induction on m gives us that $G(\zeta)$ is infinitely differentiable and the m-th derivative of G is $G^{(m)}$. It follows that $F_q(\zeta)$ is infinitely differentiable too. In particular, for any finite $n \in \mathbb{Z}_{\geq 0}$, by Leibniz's rule on the relation (6.2.3) we obtain

$$F_q^{(n+1)}(\zeta) = \sum_{k=0}^{n} \binom{n}{k} F_q^{(n-k)}(\zeta) G^{(k)}(\zeta). \tag{6.2.4}$$

Observe that $(-1)^{k+1}G^{(k)}$ is positive and finite. As F_q is positive and finite, using (6.2.4), induction gives us that $(-1)^n F_q^{(n)}$ is also positive and finite. As $\|G^{(m)}\|_\infty$ and $\|F_q\|_\infty$ are finite, using (6.2.4), induction gives us that $\|F_q^{(n)}\|_\infty$ is finite for any $n \in \mathbb{Z}_{\geq 0}$.

(b) For $\alpha \in [0,1)$, positivity of the integral $(-1)^n \int_0^\infty \zeta^{-\alpha} F_q^{(n)}(\zeta) d\zeta$ follows from part (a). To check the integrability, we first verify the $n = 0$ case. Since $\zeta \geq 0$ and $\tau \in (0, 1)$,

$$0 < \int_0^\infty \zeta^{-\alpha} F_q(\zeta)\mathrm{d}\zeta = \int_0^\infty \zeta^{-\alpha} \prod_{m=0}^{\infty} \frac{1}{1+\zeta\tau^m}\mathrm{d}\zeta < \int_0^\infty \zeta^{-\alpha}\frac{1}{1+\zeta}\mathrm{d}\zeta$$
$$= \int_0^1 \zeta^{-\alpha}\frac{1}{1+\zeta}\mathrm{d}\zeta + \int_1^\infty \frac{\mathrm{d}\zeta}{\zeta^\alpha(1+\zeta)} < \int_0^1 \zeta^{-\alpha}\mathrm{d}\zeta + \int_1^\infty \frac{\mathrm{d}\zeta}{\zeta^{\alpha+1}} < \infty.$$

When $n > 0$, using (6.2.4) and the fact the $|G^{(m)}(\zeta)| < \frac{m!}{1-\tau^{m+1}}$, by induction we deduce the finiteness of $(-1)^n \int_0^\infty \zeta^{-\alpha} F_q^{(n)}(\zeta)\mathrm{d}\zeta$.

(c) Clearly for each m we have $F_q(\zeta) \leq \frac{1}{(1+\zeta\tau^m)^{m+1}}$ forcing superpolynomial decay of F_q. The superpolynomial decay of higher order derivative now follows via induction using (6.2.4). □

6.2.2 Proof of Theorem 6.1.1 and Theorem 6.1.2

Recall $H_0(t)$ from (6.1.1). As explained in Section 6.1.2, the main idea is to use Lemma 6.1.8 with $U = \tau^{H_0(t)}$ and $F = F_q$ defined in (6.1.7). Observe that Proposition 6.2.2 guarantees $F = F_q$ can be chosen in Lemma 6.1.8. In the following proposition, we show that limiting behavior of $\mathbf{E}[\tau^{sH_0(t)}]$ is governed by the integral in (6.1.11) restricted to $[1, \infty)$.

Proposition 6.2.3. *For any $s > 0$, we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}[\tau^{sH_0(t)}] = \lim_{t\to\infty} \frac{1}{t} \log \left[(-1)^n \int_1^\infty \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathbf{E}[F_q(\zeta\tau^{H_0(t)})]\mathrm{d}\zeta\right], \tag{6.2.5}$$

where $n = \lfloor s \rfloor + 1 \geq 1$ and $\alpha = s - \lfloor s \rfloor$ so that $s = n - 1 + \alpha$.

Proof. Let $U = \tau^{H_0(t)}$. In this proof, we find an upper and a lower bound of $\mathbf{E}[U^s]$ and show that as $t \to \infty$, after taking logarithm of $\mathbf{E}[U^s]$ and dividing by t, the two bounds give matching results. Note that as $\tau \in (0, 1)$ and $H_0(t) \geq 0$ for any $n \in \mathbb{Z}_{\geq 0}$ and $t > 0$, U has finite n-th moment. By Proposition 6.2.2, F_q is n-times differentiable and $|\int_0^\infty x^{-\alpha} F_q^{(n)}(x)\mathrm{d}x| < \infty$. Denoting $\mathrm{d}\mathbb{P}_U(u)$ as the measure corresponding to the random variable U we have

$$(-1)^n \int_1^\infty \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathbf{E}[F_q(\zeta\tau^{H_0(t)})]\mathrm{d}\zeta = (-1)^n \int_1^\infty \zeta^{-\alpha} \int_0^\infty u^n F_q^{(n)}(\zeta u)\mathrm{d}\mathbb{P}_U(u)\mathrm{d}\zeta. \tag{6.2.6}$$

The $(-1)^n$ factor ensures that the above quantities are nonnegative via Proposition 6.2.2 (a). By the finiteness of the n-th moment of U, $\|F_q^{(n)}\|_\infty < \infty$ (by Proposition 6.2.2 (a)), and Fubini's theorem, we can interchange the integrals and obtain

$$\begin{aligned}
\text{r.h.s of (6.2.6)} &= (-1)^n \int_0^\infty u^{n-1+\alpha} \int_1^\infty (\zeta u)^{-\alpha} F_q^{(n)}(\zeta u) \mathrm{d}(u\zeta) \mathrm{d}\mathbb{P}_U(u) \\
&= (-1)^n \int_0^\infty u^{n-1+\alpha} \int_u^\infty x^{-\alpha} F_q^{(n)}(x) \mathrm{d}x \, \mathrm{d}\mathbb{P}_U(u). \qquad (6.2.7)
\end{aligned}$$

Since the random variable $U \in [0,1]$, we can lower bound the inner integral on the r.h.s. of (6.2.7) by restricting the x-integral to $[1,\infty)$. Recalling that $s = n-1+\alpha$ we have

$$\text{r.h.s. of (6.2.6)} \geq (-1)^n \left(\int_1^\infty x^{-\alpha} F_q^{(n)}(x) \mathrm{d}x \right) \mathbf{E}[\tau^{sH_0(t)}]. \qquad (6.2.8)$$

As for the upper bound for r.h.s. of (6.2.6), we may extend the range of integration to $[0,\infty)$. Apply Lemma 6.1.8 with $F \mapsto F_q$ and $U \mapsto \tau^{sH_0(t)}$ to get

$$\begin{aligned}
\text{r.h.s. of (6.2.6)} &\leq (-1)^n \int_0^\infty \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathbf{E}\left[F_q(\zeta U)\right] \mathrm{d}\zeta \\
&= \left[(-1)^n \int_0^\infty \zeta^{-\alpha} F_q^{(n)}(\zeta) \mathrm{d}\zeta \right] \mathbf{E}[\tau^{sH_0(t)}].
\end{aligned} \qquad (6.2.9)$$

Note that both the prefactors of $\mathbf{E}[\tau^{sH_0(t)}]$ in (6.2.8) and (6.2.9) are positive and free of t. Taking logarithms and dividing by t, we get the desired result. □

Next we truncate the integral in r.h.s. of (6.2.5) further. Recall the function $B_q(x)$ defined in Proposition 6.2.1 (a). We separate the range of integration $[1,\infty)$ into $[1, e^{tB_q(s/2)}]$ and $(e^{tB_q(s/2)}, \infty)$ and make use of the Fredholm determinant formula for $\mathbf{E}[F_q(\zeta\tau^{H_0(t)})]$ from Theorem 6.1.6 to write the integral in r.h.s. of (6.2.5) as follows.

$$(-1)^n \int_1^\infty \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathbf{E}[F_q(\zeta\tau^{H_0(t)})] \mathrm{d}\zeta = (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathbf{E}[F_q(\zeta\tau^{H_0(t)})] \mathrm{d}\zeta + \mathcal{R}_s(t)$$

$$= (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \det(I + K_{\zeta,t})\mathrm{d}\zeta + \mathcal{R}_s(t), \tag{6.2.10}$$

where

$$\mathcal{R}_s(t) := (-1)^n \int_{e^{tB_q(\frac{s}{2})}}^{\infty} \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathbf{E}[F_q(\zeta\tau^{H_0(t)})]\mathrm{d}\zeta \tag{6.2.11}$$

Recall the definition of Fredholm determinant from (6.1.10). Assuming $\operatorname{tr}(K_{\zeta,t})$ to be differentiable for a moment we may split the first term in (6.2.10) into two parts and write

$$(-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \det(I + K_{\zeta,t})\mathrm{d}\zeta = \mathcal{A}_s(t) + \mathcal{B}_s(t) \tag{6.2.12}$$

where

$$\mathcal{A}_s(t) := (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \operatorname{tr}(K_{\zeta,t})\,\mathrm{d}\zeta, \tag{6.2.13}$$

$$\mathcal{B}_s(t) := (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} [\det(I + K_{\zeta,t}) - \operatorname{tr}(K_{\zeta,t})]\,\mathrm{d}\zeta. \tag{6.2.14}$$

The next two propositions verify that both $\mathcal{A}_s(t)$ and $\mathcal{B}_s(t)$ are well-defined and we defer their proofs to Sections 6.3 and 6.4, respectively. The first one guarantees that $\operatorname{tr}(K_{\zeta,t})$ is indeed infinitely differentiable and provides the asymptotics for $\mathrm{Re}[\mathcal{A}_s(t)]$.

Proposition 6.2.4. *For each $\zeta > 0$, the function $\zeta \mapsto \operatorname{tr}(K_{\zeta,t})$ is infinitely differentiable and thus $\mathcal{A}_s(t)$ in* (6.2.13) *is well defined. Furthermore, for any $s > 0$, we have*

$$\lim_{t\to\infty} \log\left(\mathrm{Re}[\mathcal{A}_s(t)]\right) = -h_q(s). \tag{6.2.15}$$

From (6.2.10), we know that the Fredholm determinant $\det(I + K_{\zeta,t})$ is infinitely differentiable. Thus, proposition 6.2.4 renders $(\det(I + K_{\zeta,t}) - \operatorname{tr}(K_{\zeta,t}))$ infinitely differentiable as well. Hence $\mathcal{B}_s(t)$ is well-defined. In fact, we have the following asymptotics for $\mathcal{B}_s(t)$.

Proposition 6.2.5. *Fix any $s > 0$ so that $s - \lfloor s \rfloor > 0$. Recall $\mathcal{B}_s(t)$ from* (6.2.14)*. There exists a constant* $\mathrm{C} = \mathrm{C}(q,s) > 0$ *such that for all $t > 0$, we have*

$$|\mathcal{B}_s(t)| \le \mathrm{C}\exp(-th_q(s) - \tfrac{1}{\mathrm{C}}t), \tag{6.2.16}$$

where $h_q(s)$ is defined in (6.1.3)*.*

Note that Proposition 6.2.5 in its current form does not cover integer s. We later explain in Section 6.4 why $s - \lfloor s \rfloor > 0$ is necessary for our proof. However, this does not effect our main results as one can deduce Theorem 6.1.1 for integer s as well via a simple continuity argument, which we present below. Assuming Propositions 6.2.4 and 6.2.5, we now complete the proof of Theorem 6.1.1 and Theorem 6.1.2.

Proof of Theorem 6.1.1. Fix $s > 0$ so that $s - \lfloor s \rfloor > 0$. Appealing to Proposition 6.2.3 and (6.2.10) and (6.2.12) we see that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}[\tau^{sH_0(t)}] = \lim_{t\to\infty} \frac{1}{t} \log\left[\mathcal{A}_s(t) + \mathcal{B}_s(t) + \mathcal{R}_s(t)\right],$$

where $\mathcal{A}_s(t)$, $\mathcal{B}_s(t)$, and $\mathcal{R}_s(t)$ are defined in (6.2.13), (6.2.14) and (6.2.11) respectively. For $\mathcal{R}_s(t)$, setting $V = \zeta\tau^{H_0(t)}$ and noting $s = n - 1 + \alpha$, we see that

$$|\mathcal{R}_s(t)| = \int_{e^{tB_q(\frac{s}{2})}}^{\infty} \zeta^{-\alpha-n}\mathbf{E}\left[|V^n F_q^{(n)}(V)|\right]\mathrm{d}\zeta \le \left[\sup_{v>0}|v^n F_q^{(n)}(v)|\right] s^{-1}\exp(-tsB_q(\tfrac{s}{2})).$$

The fact that $\sup_{v>0}|v^nF_q^{(n)}(v)|$ is finite follows from Proposition 6.2.2 (c). Note that $sB_q(\frac{s}{2})$ is strictly bigger than $h_q(s) = sB_q(s) > 0$ via Proposition 6.2.1 (a). By Proposition 6.2.4, when t is large, we see that $\mathrm{Re}[\mathcal{A}_s(t)]$ grows like $\exp(-th_q(s)) > \exp(-tsB_q(\frac{s}{2}))$. Similarly, Proposition 6.2.5 shows that $\mathrm{Re}[\mathcal{B}_s(t)]$ is bounded from above by $\mathrm{C}\exp(-th_q(s) - \frac{1}{\mathrm{C}}t)$ for some constant $\mathrm{C} = \mathrm{C}(q,s)$, which is strictly less than $\exp(-th_q(s))$ for large enough t. Indeed for all large enough t, we have

$$\frac{1}{2}\operatorname{Re}[\mathcal{A}_s(t)] \leq \operatorname{Re}[\mathcal{A}_s(t)+\mathcal{B}_s(t)+\mathcal{R}_s(t)] \leq \frac{3}{2}\operatorname{Re}[\mathcal{A}_s(t)].$$

Taking logarithms and dividing by t, and noting that $\mathcal{A}_s(t)+\mathcal{B}_s(t)+\mathcal{R}_s(t)$ is always real, we get (6.1.3) for any noninteger positive s.

To prove (6.1.3) for positive integer s, we fix $s \in \mathbb{Z}_{>0}$. For any $K > 2$, observe that as $H_0(t)$ is a non-negative random variable (recall the definition from (6.1.1)) we have

$$\tau^{(s-K^{-1})H_0(t)} \geq \tau^{sH_0(t)} \geq \tau^{(s+K^{-1})H_0(t)}.$$

Taking expectations, then logarithms and dividing by t, in view of noninteger version of (6.1.3) we have

$$-h_q(s-K^{-1}) \geq \limsup_{t\to\infty}\frac{1}{t}\log\mathbf{E}[\tau^{sH_0(t)}] \geq \liminf_{t\to\infty}\frac{1}{t}\log\mathbf{E}[\tau^{sH_0(t)}] \geq -h_q(s+K^{-1}).$$

Taking $K \to \infty$ we get the desired result for integer s. □

Proof of Theorem 6.1.2. For the large deviation result, applying Proposition 1.12 in [128], with $X(t) = H_0(t/\gamma)\cdot\log\tau$, and noting the Legendre-Fenchel type identity for $\Phi_+(y)$ from Proposition 6.2.1 (c), we arrive at (6.1.4). To prove (6.1.5), applying L-Hôpital rule a couple of times we get

$$\lim_{y\to 0^+}\frac{\Phi_+(y)}{y^{3/2}} = \lim_{y\to 0^+}\frac{2}{3}\frac{\Phi_+'(y)}{\sqrt{y}} = \lim_{x\to 0^+}\frac{2}{3}\frac{\tanh^{-1}(x)}{x} = \lim_{x\to 0^+}\frac{2}{3}\cdot\frac{1}{1-x^2} = \frac{2}{3}.$$

This completes the proof of the theorem. □

6.3 Asymptotics of the Leading Term

The goal of this section is to obtain exact asymptotics of $\operatorname{Re}[\mathcal{A}_s(t)]$ defined in (6.2.13) as $t \to \infty$. Recall the definition of the kernel $K_{\zeta,t}$ from (6.1.8). We employ a standard idea that the asymptotic behavior of the kernel $K_{\zeta,t}$ and its 'derivative' (see (6.3.8)) and subsequently that of $\mathrm{Re}[\mathcal{A}_s(t)]$ can be derived by the *steepest descent method*.

Towards this end, we first collect all the technical estimates related to the kernel $K_{\zeta,t}$ in Section 6.3.1 and go on to complete the proof of Proposition 6.2.4 in Section 6.3.2.

6.3.1 Technical estimates of the Kernel

In this section, we analyze the kernel $K_{\zeta,t}$. Much of our subsequent analysis boils down to understanding the function $g_t(z)$, defined in (6.1.8), that appears in the kernel $K_{\zeta,t}$. Towards this end, we consider

$$f(u,z) := \frac{(q-p)}{1+\frac{z}{\tau}} - \frac{(q-p)}{1+\frac{\tau^u z}{\tau}}, \tag{6.3.1}$$

so that the ratio $\frac{g_t(z)}{g_t(\tau^u z)}$ that appears in the kernel $K_{\zeta,t}$ defined in (6.1.8) equals to $\exp(tf(u,z))$. Below we collect some useful properties of this function $f(u,z)$. First note that $\partial_z f(u,z) = 0$ has two solutions $z = \pm\tau^{1-\frac{u}{2}}$, and

$$\begin{aligned}
\partial_z^2 f(u,z)\Big|_{z=-\tau^{1-\frac{u}{2}}} &= -2(q-p)\frac{\tau^{\frac{3u}{2}-2}+\tau^{2u-2}}{(1-\tau^{\frac{u}{2}})^3}, \\
\partial_z^2 f(u,z)\Big|_{z=\tau^{1-\frac{u}{2}}} &= 2(q-p)\frac{\tau^{\frac{3u}{2}-2}-\tau^{2u-2}}{(1+\tau^{\frac{u}{2}})^3}.
\end{aligned} \tag{6.3.2}$$

The following lemma tells us how the maximum of $\mathrm{Re}[f(u,z)]$ behaves.

Lemma 6.3.1. *Fix $\rho > 0$. For any $u \in \mathbb{C}$, with $\mathrm{Re}[u] = \rho$ and $z \in \mathfrak{C}(\tau^{1-\frac{\rho}{2}})$, we have*

$$\mathrm{Re}[f(u,z)] \leq f(\rho, \tau^{1-\frac{\rho}{2}}) = -h_q(\rho) \tag{6.3.3}$$

where $h_q(\rho)$ is defined in (6.1.3) *and $\mathfrak{C}(\tau^{1-\frac{\rho}{2}})$ is the circle with center at the origin and radius $\tau^{1-\frac{\rho}{2}}$. Equality in* (6.3.3) *holds if and only if $\tau^{\mathbf{i}\,\mathrm{Im}\,u} = 1$, and $z = \tau^{1-\frac{\rho}{2}}$ simultaneously. Furthermore,* *for the same range of u and z, we have the following inequality:*

$$f(\mathfrak{g},\tau^{1-\frac{\rho}{2}})-\mathrm{Re}[f(u,z)]\geq\frac{(q-p)(1-\tau^{\frac{\mathfrak{g}}{2}})\tau^{\frac{\rho}{2}}}{4(1+\tau^{\frac{\rho}{2}})^2}(2\tau^{\frac{\rho}{2}-1}|z-\tau^{1-\frac{\rho}{2}}|+|\tau^{\mathbf{i}\,\mathrm{Im}\,u}-1|). \tag{6.3.4}$$

Proof. Set $u=\rho+\mathbf{i}y$ and $z=\tau^{1-\frac{\rho}{2}}e^{\mathbf{i}\theta}$ with $y\in\mathbb{R}$ and $\theta\in[0,2\pi]$. Note that $f(\rho,\tau^{1-\frac{\rho}{2}})=-h_q(\rho)$, where $h_q(x)$ is defined in (6.1.3). Direct computation yields

$$\mathrm{Re}[f(u,z)]=\frac{(q-p)(\tau^{\mathfrak{g}}-1)(|1+\tau^{\frac{\mathfrak{g}}{2}}e^{-\mathbf{i}\theta}|^2+|1+\tau^{\frac{\mathfrak{g}}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|^2)}{2|1+\tau^{\frac{\mathfrak{g}}{2}}e^{-\mathbf{i}\theta}|^2|1+\tau^{\frac{\mathfrak{g}}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|^2}. \tag{6.3.5}$$

Since $\tau<1$, applying the inequality $|1+\tau^{\frac{\mathfrak{g}}{2}}e^{-\mathbf{i}\theta}|^2+|1+\tau^{\frac{\mathfrak{g}}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|^2\geq 2|1+\tau^{\frac{\mathfrak{g}}{2}}e^{-\mathbf{i}\theta}||1+\tau^{\frac{\mathfrak{g}}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|$, and then noting that $|1+\tau^{\frac{\mathfrak{g}}{2}}e^{-\mathbf{i}\theta}||1+\tau^{\frac{\mathfrak{g}}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|\leq(1+\tau^{\frac{\mathfrak{g}}{2}})^2$, we see (r.h.s. of (6.3.5)) $\leq -(q-p)\frac{1-\tau^{\frac{\rho}{2}}}{1+\tau^{\frac{\rho}{2}}}$. Clearly equality holds if and only if $\theta=0$ and $\tau^{\mathbf{i}y}=1$ simultaneously. Furthermore, following the above inequalities, we have $\mathrm{Re}[f(\rho+\mathbf{i}y,z)]\leq-(q-p)\frac{1-\tau^{\frac{\rho}{2}}}{|1+\tau^{\frac{\rho}{2}}e^{\mathbf{i}\theta}|}$ and $\mathrm{Re}[f(\rho+\mathbf{i}y,z)]\leq -(q-p)\frac{1-\tau^{\frac{\rho}{2}}}{|1+\tau^{\frac{\rho}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|}$. This yields

$$\begin{aligned}f(\rho,\tau^{1-\frac{\rho}{2}})-\mathrm{Re}[f(\rho+\mathbf{i}y,z)]&\geq(q-p)\left[\frac{1-\tau^{\frac{\rho}{2}}}{|1+\tau^{\frac{\rho}{2}}e^{\mathbf{i}\theta}|}-\frac{1-\tau^{\frac{\rho}{2}}}{1+\tau^{\frac{\rho}{2}}}\right]\\&\geq\frac{(q-p)(\tau^{\frac{\rho}{2}}-\tau^{\rho})|e^{\mathbf{i}\theta}-1|}{(1+\tau^{\frac{\rho}{2}})^2}\end{aligned} \tag{6.3.6}$$

and

$$\begin{aligned}f(\rho,\tau^{1-\frac{\rho}{2}})-\mathrm{Re}[f(\rho+\mathbf{i}y,z)]&\geq(q-p)\left[\frac{1-\tau^{\frac{\rho}{2}}}{|1+\tau^{\frac{\rho}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|}-\frac{1-\tau^{\frac{\rho}{2}}}{1+\tau^{\frac{\rho}{2}}}\right]\\&\geq\frac{(q-p)(1-\tau^{\frac{\mathfrak{g}}{2}})\tau^{\frac{\rho}{2}}|\tau^{\mathbf{i}y}e^{\mathbf{i}\theta}-1|}{(1+\tau^{\frac{\rho}{2}})^2}.\end{aligned}$$

Adding the above two inequalities we have $f(\rho,\tau^{1-\frac{\rho}{2}})-\mathrm{Re}[f(\rho+\mathbf{i}y,z)]\geq\frac{(q-p)(1-\tau^{\frac{\mathfrak{g}}{2}})\tau^{\frac{\rho}{2}}|\tau^{\mathbf{i}y}-1|}{2(1+\tau^{\frac{\rho}{2}})^2}$. Combining this with (6.3.6) and the substitution $\tau^{1-\frac{\rho}{2}}e^{\mathbf{i}\theta}=z$ we get (6.3.4). This completes the proof. □

Using the above technical lemma we can now explain the proof of Theorem 6.1.6.

Proof of Theorem 6.1.6. Due to Theorem 5.3 in [49], the only thing that we need to verify is

$$\inf_{\substack{w,w'\in\mathfrak{C}(\tau^{1-\frac{\delta}{2}})\\ u\in\delta+\mathbf{i}\mathbb{R}}} |w'-\tau^u w|>0 \quad \text{and} \quad \sup_{\substack{w,w'\in\mathfrak{C}(\tau^{1-\frac{\delta}{2}})\\ u\in\delta+\mathbf{i}\mathbb{R}}} \left|\frac{g_t(w)}{g_t(\tau^u w)}\right|>0. \tag{6.3.7}$$

Indeed, for every $u\in\delta+\mathbf{i}\mathbb{R}$ and $w,w'\in\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$, we have $|w'-\tau^u w|\geq |w'|-|\tau^u w|=\tau^{1-\frac{\delta}{2}}-\tau^{1+\frac{\delta}{2}}>0$. Recall $f(u,z)$ from (6.3.1). Applying Lemma 6.3.1 with $\rho\mapsto\delta$ yields

$$\left|\frac{g_t(w)}{g_t(\tau^u w)}\right| = |\exp(tf(u,w))| = \exp(t\,\mathrm{Re}[f(u,w)]) \leq \exp(tf(\delta,\tau^{1-\frac{\delta}{2}})) = \exp(-th_q(\delta)),$$

where h_q is defined in (6.1.3). This verifies (6.3.7) and completes the proof. □

Remark 6.3.2. We now explain our choice of the contour $K_{\zeta,t}$ defined in (6.1.8), which comes from the method of steepest descent. Suppose $\mathrm{Re}[u]=\delta$. As noted before, directly taking derivative of $f(u,z)=\exp(\frac{g_t(z)}{g_t(\tau^u z)})$, with respect to z suggests that critical points are at $z=\pm\tau^{1-\frac{u}{2}}$, and thus we take our contour to be $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$, so that it passes through the critical points.

Next we turn to the case of differentiability of $\mathrm{tr}(K_{\zeta,t})$ where $K_{\zeta,t}$ is defined in (6.1.8). Using the function f defined in (6.3.1), we rewrite the kernel as follows.

$$K_{\zeta,t}(w,w') = \frac{1}{2\pi\mathbf{i}}\int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty}\Gamma(-u)\Gamma(1+u)\zeta^u e^{tf(u,w)}\frac{\mathrm{d}u}{w'-\tau^u w}.$$

Differentiating the integrand inside the integral in $K_{\zeta,t}(w.w')$ n-times defines a sequence of kernel $\{K_{\zeta,t}^{(n)}\}_{n\geq 1}: L^2(\mathfrak{C}(\tau^{1-\frac{\delta}{2}}))\to L^2(\mathfrak{C}(\tau^{1-\frac{\delta}{2}}))$ given by the kernel:

$$K_{\zeta,t}^{(n)}(w,w') := \frac{1}{2\pi\mathbf{i}}\int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty}\Gamma(-u)\Gamma(1+u)(u)_n\zeta^{u-n} e^{tf(u,w)}\frac{\mathrm{d}u}{w'-\tau^u w}, \tag{6.3.8}$$

where $(a)_n:=\prod_{i=0}^{n-1}(a-i)$ for $n\in\mathbb{Z}_{>0}$ and $(a)_0=1$ is the Pochhammmer symbol and $\delta\in(0,1)$. We also set $K_{\zeta,t}^{(0)}:=K_{\zeta,t}$.

Remark 6.3.3. We remark that unlike Lemma 3.1 in [88], we do not aim to show that $K_{\zeta,t}$ is differentiable as an operator, or its higher order derivatives are equal to the operator $K^{(n)}_{\zeta,t}$. Indeed, showing convergence in the trace class norm is more involved because of the lack of symmetry and positivity of the operator $K_{\zeta,t}$. However, since we are dealing with the Fredholm determinant series only, for our analysis it is enough to investigate how each term of the series are differentiable and how their derivatives are related to $K^{(n)}_{\zeta,t}$.

Remark 6.3.4. Note that when viewing $K^{(n)}_{\zeta,t}$ as a complex integral, we can deform its u-contour to $\mathfrak{g} + \mathbf{i}\mathbb{R}$ for any $\mathfrak{g} \in (0, n \vee 1)$. This is due to the analytic continuity of the integrand as the factor $(u)_n$ removes the poles at $1, \dots, n-1$ of $\Gamma(-u)$.

The following lemma provides estimates of $K^{(n)}_{\zeta,t}$ that is useful for the subsequent analysis in Sections 6.3 and 6.4.

Lemma 6.3.5. *Fix $n \in \mathbb{Z}_{\geq 0}, t > 0, \delta, \rho \in (0, n \vee 1)$, and consider any borel set $A \subset \mathbb{R}$. Recall $h_q(x)$ and $B_q(x)$ from Proposition 6.2.1 and $K^{(n)}_{\zeta,t}$ from* (6.3.8)*. For any $w \in \mathfrak{C}(\tau^{1-\frac{\delta}{2}})$ and $w' \in \mathbb{C}$ and $\zeta \in [1, e^{tB_q(\frac{s}{2})}]$, there exists a constant $\mathrm{C} = \mathrm{C}(n, \delta, q) > 0$ such that whenever $|w'| \neq \tau^{1+\frac{\delta}{2}}$ we have*

$$\begin{aligned}\int_A \left|\frac{(\delta + \mathbf{i}y)_n \zeta^{\rho-n+\mathbf{i}y}}{\sin(-\pi(\delta+\mathbf{i}y))} e^{tf(\delta+\mathbf{i}y,w)}\right| \frac{\mathrm{d}y}{|w' - \tau^{\delta+\mathbf{i}y}w|} &\leq \frac{\mathrm{C}\zeta^{\rho-n}}{\left||w'| - \tau^{1+\frac{\delta}{2}}\right|} e^{t\cdot \sup_{y\in A} \mathrm{Re}[f(\delta+\mathbf{i}y,w)]} \\ &\leq \frac{\mathrm{C}\zeta^{\rho-n}}{\left||w'| - \tau^{1+\frac{\delta}{2}}\right|} e^{-th_q(\delta)}.\end{aligned} \tag{6.3.9}$$

In particular when $w' \in \mathfrak{C}(\tau^{1-\frac{\delta}{2}})$ we have

$$|K^{(n)}_{\zeta,t}(w, w')| \leq \mathrm{C}\zeta^{\delta-n} \exp(-th_q(\delta)). \tag{6.3.10}$$

Consequently, $K^{(n)}_{\zeta,t}(w, w')$ is continuous in the ζ-variable.

Proof. Fix $n \in \mathbb{Z}_{\geq 0}, t > 0$, $\delta, \rho \in (0, n \vee 1)$ and $w \in \mathfrak{C}(\tau^{1-\frac{\delta}{2}})$ and $w' \in \mathbb{C}$ such that $|w'| \neq \tau^{1+\frac{\delta}{2}}$. Throughout the proof the constant $\mathrm{C} > 0$ depends on n, δ, and q – we will not mention it further.

Consider the integral on the r.h.s. of (6.3.9). Observe that when $\delta \notin \mathbb{Z}$, $|(\delta+\mathbf{i}y)_n| \le \mathrm{C}|y|^n$ and $\frac{1}{|\sin(-\pi(\delta+\mathbf{i}y))|} \le \mathrm{C}e^{-|y|/\mathrm{C}}$. For $n \ge 2$, and $\delta \in \mathbb{Z}_{>0} \cap (0,n)$, we observe that the product $(\delta+\mathbf{i}y)_n$ contains the term $\mathbf{i}y$. Hence $|\frac{\mathbf{i}y}{\sin(-\pi(\delta+\mathbf{i}y))}| = |\frac{\mathbf{i}y}{\sin(-\pi(\mathbf{i}y))}| \le \mathrm{C}e^{-|y|/\mathrm{C}}$ for such an integer δ. Whereas, $|\frac{\delta+\mathbf{i}y}{\mathbf{i}y}| \le \mathrm{C}|y|^{n-1}$ for such an integer δ. Finally, $|w' - \tau^{\delta+\mathbf{i}y}w| \ge ||w'| - |\tau^{\delta}w|| = ||w'| - \tau^{1+\frac{\delta}{2}}|$. Combining the aforementioned estimates, we obtain that

$$\text{r.h.s. of (6.3.9)} \le \int_A \mathrm{C}|y|^n e^{-|y|/\mathrm{C}}\zeta^{\rho-n}|e^{tf(\delta+\mathbf{i}y,w)}|\frac{\mathrm{d}y}{||w'| - \tau^{1+\frac{\delta}{2}}|}.$$

Since $\int_{\mathbb{R}} |y|^n e^{-|y|/\mathrm{C}}\mathrm{d}y$ converges applying $|e^{tf(\delta+\mathbf{i}y,w)}| \le e^{t\,\mathrm{Re}[f(\delta+\mathbf{i}y,w)]}$ we arrive at the first inequality in (6.3.9). The second inequality follows by observing $\mathrm{Re}[f(\delta+\mathbf{i}y,w)] \le -h_q(\delta)$ by Lemma 6.3.1.

Recall $K^{(n)}_{\zeta,t}$ from (6.3.8). Recall from Remark 6.3.4 that the δ appearing in (6.3.8) can be chosen in $(0, n \vee 1)$. Pushing the absolute value sign inside the explicit formula in (6.3.8) and applying Euler's reflection principle with change of variables $u = \delta + \mathbf{i}y$ yield

$$|K^{(n)}_{\zeta,t}(w,w')| \le \frac{1}{2\pi}\int_{\mathbb{R}} \left|\frac{(\delta+\mathbf{i}y)_n\zeta^{\delta-n+\mathbf{i}y}}{\sin(-\pi(\delta+\mathbf{i}y))}e^{tf(\delta+\mathbf{i}y,w)}\right|\frac{\mathrm{d}y}{|w' - \tau^{\delta+\mathbf{i}y}w|}.$$

(6.3.10) now follows from (6.3.9) by taking $\rho = \delta$. To see the continuity of $K^{(n)}_{\zeta,t}(w,w')$ in ζ, we fix $\zeta_1 < \zeta_2 < \zeta_1 + 1$. By repeating the same set of arguments as above we arrive at

$$|K^{(n)}_{\zeta_2,t}(w,w') - K^{(n)}_{\zeta_1,t}(w,w')| \le C|\zeta_2^{\delta-n} - \zeta_1^{\delta-n}|\exp(-th_q(\delta)) \tag{6.3.11}$$

with the same constant C in (6.3.10). Clearly l.h.s. of (6.3.11) converges to 0 when $\zeta_2 \to \zeta_1$, which confirms the kernel's ζ-continuity. □

6.3.2 Proof of Proposition 6.2.4

The goal of this section is to prove Proposition 6.2.4. Before diving into the proof, we first settle the infinite differentiability separately in the next proposition.

Proposition 6.3.6. *For any $n \in \mathbb{Z}_{\geq 0}$ and $t > 0$, the operator $K_{\zeta,t}^{(n)}$ defined in* (6.3.8) *is a trace-class operator with*

$$\operatorname{tr}(K_{\zeta,t}^{(n)}) = \frac{1}{2\pi\mathbf{i}} \int_{\mathfrak{C}(\tau^{1-\frac{\delta}{2}})} K_{\zeta,t}^{(n)}(w,w)\mathrm{d}w. \tag{6.3.12}$$

Furthermore, $\operatorname{tr}(K_{\zeta,t}^{(n)})$ *is differentiable in ζ at each $\zeta > 0$ and we have* $\partial_\zeta \operatorname{tr}(K_{\zeta,t}^{(n)}) = \operatorname{tr}(K_{\zeta,t}^{(n+1)})$.

Proof. Fix $n \in \mathbb{Z}_{\geq 0}, t > 0$, and $\zeta > 0$. $K_{\zeta,t}^{(n)}(w,w')$ is simultaneously continuous in both w and w' and $\partial_{w'} K_{\zeta,t}^{(n)}(w,w')$ is continuous in w'. By Lemma 3.2.7 in [46] (also see [174, page 345] or [45]) we see that $K_{\zeta,t}^{(n)}$ is indeed trace-class, and thus (6.3.12) follows from Theorem 12 in [174, Chapter 30]. To show differentiability of $\operatorname{tr}(K_{\zeta,t}^{(n)})$ in variable ζ, we fix $\zeta_1, \zeta_2 > 0$. Without loss of generality we may assume $\zeta_1 + 1 > \zeta_2 > \zeta_1$. Let us define

$$\begin{aligned} D_{\zeta_1,\zeta_2} &:= \frac{\operatorname{tr}(K_{\zeta_2,t}^{(n)}) - \operatorname{tr}(K_{\zeta_1,t}^{(n)})}{\zeta_2 - \zeta_1} - \operatorname{tr}(K_{\zeta_1,t}^{(n+1)}) \\ &= \frac{1}{(2\pi\mathbf{i})^2} \int_{\mathfrak{C}(\tau^{1-\frac{\delta}{2}})} \int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \Gamma(-u)\Gamma(1+u) R_{\zeta_1,\zeta_2;n}(u) e^{tf(u,w)} \frac{\mathrm{d}u}{w - \tau^u w} \mathrm{d}w, \end{aligned}$$

where

$$\begin{aligned} R_{\zeta_1,\zeta_2;n}(u) &:= (u)_n \left[\frac{\zeta_2^{u-n} - \zeta_1^{u-n}}{\zeta_2 - \zeta_1} - (u-n)\zeta_1^{u-n-1} \right] \\ &= \int_{\zeta_1}^{\zeta_2} \frac{(\zeta_2 - \sigma)}{\zeta_2 - \zeta_1} (u)_{n+2} \sigma^{u-n-2} \mathrm{d}\sigma. \end{aligned} \tag{6.3.13}$$

Taking absolute value and appealing to Euler's reflection principle, we obtain

$$\begin{aligned} |D_{\zeta_1,\zeta_2}| &\leq \left| \frac{1}{(2\pi\mathbf{i})^2} \int_{\mathfrak{C}(\tau^{1-\frac{\delta}{2}})} \int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \int_{\zeta_1}^{\zeta_2} \frac{(u)_{n+2}}{\sin(-\pi u)} \frac{(\zeta_2 - \sigma)}{\zeta_2 - \zeta_1} \sigma^{u-n-2} e^{tf(u,w)} \frac{\mathrm{d}\sigma \mathrm{d}u}{w - \tau^u w} \mathrm{d}w \right| \\ &\leq \frac{\tau^{1-\frac{\delta}{2}}}{2\pi} \int_{\zeta_1}^{\zeta_2} |\sigma^{\delta+\mathbf{i}y-n-2}| \mathrm{d}\sigma \cdot \max_{w \in \mathfrak{C}(\tau^{1-\frac{\delta}{2}})} \int_{\mathbb{R}} \frac{(\delta + \mathbf{i}y)_{n+2}}{\sin(-\pi(\delta + \mathbf{i}y))} |e^{tf(\delta+\mathbf{i}y,w)}| \frac{\mathrm{d}y}{|w - \tau^{\delta+\mathbf{i}y} w|}. \end{aligned} \tag{6.3.14}$$

Note that Lemma 6.3.5 ((6.3.9) specifically) we see that the above maximum is bounded by $\mathrm{C}\exp(-th_q(\delta))$ where the constant C is same as in (6.3.9). Since $|\sigma^{u-n-2}| = |\sigma^{\delta-n-2}| \leq |\zeta_1^{\delta-n-2}|$

over the interval $[\zeta_1, \zeta_2]$ for $\delta \in (0, n \vee 1)$, we obtain

$$|D_{\zeta_1,\zeta_2}| \leq \mathrm{C} \exp(-h_q(\delta)) \int_{\zeta_1}^{\zeta_2} |\sigma^{u-n-2}| \mathrm{d}\sigma \leq \mathrm{C} \exp(-t h_q(\delta))(\zeta_2 - \zeta_1)|\zeta_1^{\delta-n-2}|.$$

Thus, taking the limit as $\zeta_2 - \zeta_1 \to 0$ yields $|D_{\zeta_1,\zeta_2}| \to 0$ and completes the proof. □

Remark 6.3.7. We prove a higher order version of Proposition 6.3.6 later in Section 6.4 as Proposition 6.4.1 which includes the statement of the above Proposition when $L = 1$. However, we keep the above simple version for reader's convenience, which will serve as a guide in proving Proposition 6.4.1.

With the above results in place, we can now turn towards the main technical component of the proof of Proposition 6.2.4.

Proof of Proposition 6.2.4. Before proceeding with the proof, we fix some notations. Fix $s > 0$, and set $n = \lfloor s \rfloor + 1 \geq 1$ and $\alpha = s - \lfloor s \rfloor \in [0, 1)$ so that $s = n - 1 + \alpha$. Throughout the proof, we will denote C to be positive constant depending only on s, q – we will not mention this further. We will also use the big O notation. For two complex-valued functions $f_1(t)$ and $f_2(t)$ and $\beta \in \mathbb{R}$, the equations $f_1(t) = (1 + O(t^\beta)) f_2(t)$ and $f_1(t) = f_2(t) + O(t^\beta)$ have the following meaning: there exists a constant $\mathrm{C} > 0$ such that for all large enough t,

$$\left| \frac{f_1(t)}{f_2(t)} - 1 \right| \leq \mathrm{C} \cdot t^\beta, \text{ and } |f_1(t) - f_2(t)| \leq \mathrm{C} \cdot t^\beta,$$

respectively. The constant $\mathrm{C} > 0$ value may change from line to line.

For clarity we divide the proof into seven steps. In Steps 1 and 2, we provide the upper and lower bounds for $|\mathcal{A}_s(t)|$ and $\mathrm{Re}[\mathcal{A}_s(t)]$ respectively and complete the proof of (6.2.15); in Steps 3–7, we verify the technical estimates assumed in the previous steps.

Step 1. Recall $\mathcal{A}_s(t)$ from (6.2.13). The goal of this step is to provide a different expression for $\mathcal{A}_s(t)$, which will be much more amenable to our analysis, as well as an upper bound for $|\mathcal{A}_s(t)|$. By Proposition 6.3.6, we have $\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathrm{tr}(K_{\zeta,t}) = \mathrm{tr}(K_{\zeta,t}^{(n)})$ and consequently using the expression in

(6.3.8) we have

$$\mathcal{A}_s(t) := (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \frac{\zeta^{-\alpha}}{(2\pi\mathbf{i})^2} \int_{\mathfrak{C}(\tau^{1-\frac{\delta}{2}})} \int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \Gamma(-u)\Gamma(1+u)(u)_n \zeta^{u-n} \frac{e^{tf(u,w)}\mathrm{d}u}{w-\tau^u w} \mathrm{d}w\mathrm{d}\zeta.$$

where $\delta \in (0,1)$ is chosen to be less than s. We now proceed to deform the u-contour and w-contour sequentially. As we explained in Remark 6.3.4, the integrand has no poles when $u = 1, 2, \ldots, n-1$. Hence u-contour can be deformed to $(s - \mathbf{i}\infty, s + \mathbf{i}\infty)$ as $s = n - 1 + \alpha \in (0, n)$.

Next, for the w-contour, we wish to deform it from $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$ to $\mathfrak{C}(\tau^{1-\frac{s}{2}})$. In order to do so, we need to ensure that we do not cross any poles. We observe that the potential sources of poles lie in the exponent $f(u,w) := \frac{(q-p)}{1+w\tau^{-1}} - \frac{(q-p)}{1+\tau^{u-1}w}$ (recalled from (6.3.1)) and in the denominator $w - \tau^u w$. Since for any $w \in \mathfrak{C}(\tau^{1-\frac{\delta'}{2}})$, where $\delta' \in (\delta, s)$, and $u \in (s - \mathbf{i}\infty, s + \mathbf{i}\infty)$, we have

$$|w - \tau^u w| \ge |w| - |\tau^u w| = \tau^{1-\frac{\delta'}{2}}(1 - \tau^s) > 0, \quad |1 + w\tau^{-1}| \ge |w\tau^{-1}| - 1 = \tau^{-\frac{\delta'}{2}} - 1 > 0,$$

$$\text{and } |1 + \tau^{u-1}w| \ge 1 - |\tau^{u-1}w| = 1 - \tau^{s-\frac{\delta'}{2}} > 0.$$

Thus, we can deform the w-contour to $\mathfrak{C}(\tau^{1-\frac{s}{2}})$ as well without crossing any poles. With the change of variable $u = s + \mathbf{i}y$, $w = \tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta}$, and Euler's reflection formula we have

$$\mathcal{A}_s(t) = (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \frac{\zeta^{-1}}{4\pi^2} \int_{-\pi}^{\pi} \int_{\mathbb{R}} \frac{(s+\mathbf{i}y)_n \zeta^{\mathbf{i}y}}{\sin(-\pi(s+\mathbf{i}y))} e^{tf(s+\mathbf{i}y, \tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})} \frac{\mathrm{d}y}{1-\tau^{s+\mathbf{i}y}} \mathrm{d}\theta\mathrm{d}\zeta. \tag{6.3.15}$$

With this expression in hand, upper bound is immediate. By Lemma 6.3.5 ((6.3.9) specifically with $\rho \mapsto n-1$, $\delta \mapsto s$) pushing the absolute value inside the integrals we see that

$$|\mathcal{A}_s(t)| \le \mathrm{C}\exp(-th_q(s)) \int_1^{e^{tB_q(\frac{s}{2})}} \frac{1}{\zeta}\mathrm{d}\zeta = \mathrm{C} \cdot tB_q(\tfrac{s}{2})\exp(-th_q(s)) \tag{6.3.16}$$

for some constant $\mathrm{C} = \mathrm{C}(q,s) > 0$. Hence taking logarithm and dividing by t, we get

$$\limsup_{t\to\infty} |\mathcal{A}_s(t)| \le -h_q(s) = -(q-p)\frac{1-\tau^{\frac{s}{2}}}{1+\tau^{\frac{s}{2}}}. \tag{6.3.17}$$

Step 2. In this step, we provide a lower bound for $\mathrm{Re}[\mathcal{A}_s(t)]$. Set $\varepsilon = t^{-2/5} > 0$. For each $k \in \mathbb{Z}$, set $v_k = -\frac{2\pi}{\log\tau}k$ and consider the interval $V_k := [v_k - \varepsilon^2, v_k + \varepsilon^2]$. Also set $A_\varepsilon := \{\theta \in [-\pi, \pi] : |e^{\mathbf{i}\theta} - 1| \le \varepsilon|\log\tau|\}$. We divide the triple integral in (6.3.15) into following parts

$$\mathcal{A}_s(t) = \sum_{k\in\mathbb{Z}} (\mathbf{I})_k + (\mathbf{II}) + (\mathbf{III}), \tag{6.3.18}$$

where

$$(\mathbf{I})_k := \int_1^{e^{tB_q(\frac{s}{2})}} \int_{A_\varepsilon} \int_{V_k} \frac{(-1)^n}{4\pi^2\zeta} \frac{(s+\mathbf{i}y)_n \zeta^{\mathbf{i}y}}{\sin(-\pi(s+\mathbf{i}y))} \frac{e^{tf(s+\mathbf{i}y,\tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})}\mathrm{d}y}{1-\tau^{s+\mathbf{i}y}} \mathrm{d}\theta\mathrm{d}\zeta, \tag{6.3.19}$$

$$(\mathbf{II}) := \int_1^{e^{tB_q(\frac{s}{2})}} \int_{A_\varepsilon} \int_{\mathbb{R}\setminus\cup_k V_k} \frac{(-1)^n}{4\pi^2\zeta} \frac{(s+\mathbf{i}y)_n \zeta^{\mathbf{i}y}}{\sin(-\pi(s+\mathbf{i}y))} \frac{e^{tf(s+\mathbf{i}y,\tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})}\mathrm{d}y}{1-\tau^{s+\mathbf{i}y}} \mathrm{d}\theta\mathrm{d}\zeta, \tag{6.3.20}$$

$$(\mathbf{III}) := \int_1^{e^{tB_q(\frac{s}{2})}} \int_{[-\pi,\pi]\cap A_\varepsilon^c} \int_{\mathbb{R}} \frac{(-1)^n}{4\pi^2\zeta} \frac{(s+\mathbf{i}y)_n \zeta^{\mathbf{i}y}}{\sin(-\pi(s+\mathbf{i}y))} \frac{e^{tf(s+\mathbf{i}y,\tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})}\mathrm{d}y}{1-\tau^{s+\mathbf{i}y}} \mathrm{d}\theta\mathrm{d}\zeta. \tag{6.3.21}$$

In subsequent steps we obtain the following estimates for each integral. We claim that we have

$$(\mathbf{I})_0 = (1 + O(t^{-\frac{1}{5}})) \frac{\mathrm{C}_0}{\sqrt{t}} \exp(-th_q(s)), \tag{6.3.22}$$

where $h_q(s)$ is defined in (6.1.3) and

$$\mathrm{C}_0 := \sqrt{\frac{(1+\tau^{\frac{s}{2}})^3}{4\pi(q-p)(\tau^{\frac{3s}{2}-2} - \tau^{2s-2})}} \frac{(-1)^n (s)_n}{\sin(-\pi s)(1-\tau^s)} > 0. \tag{6.3.23}$$

When s is an integer the above constant is defined in a limiting sense. Note that C_0 is indeed positive as $n = \lfloor s \rfloor + 1$. Furthermore, we claim that we have the following upper bounds for the other integrals:

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} |(\mathbf{I})_k| \le \mathrm{C}t^{-\frac{13}{10}} \exp(-th_q(s)). \tag{6.3.24}$$

where $v_k = -\frac{2\pi}{\log\tau}k$ and

$$|(\mathbf{II})|, |(\mathbf{III})| \le \mathrm{C}t \exp\left(-th_q(s)\right) \exp(-\tfrac{1}{\mathrm{C}} t^{\frac{1}{5}}). \tag{6.3.25}$$

Assuming the validity of (6.3.22), (6.3.24) and (6.3.25) we can complete the proof of lower bound for (6.2.15). Following the decomposition in (6.3.18) we see that for all large enough t,

$$\begin{aligned}\mathrm{Re}[\mathcal{A}_s(t)] &\ge \mathrm{Re}[(\mathbf{I})_0] - \sum_{k\in Z\setminus\{0\}} |(\mathbf{I})_k| - |(\mathbf{II})| - |(\mathbf{III})| \\ &\ge \tfrac{1}{\sqrt{t}} \exp(-th_q(s)) \left[\tfrac{1}{2}\mathrm{C}_0 - \mathrm{C}t^{-\frac{4}{5}} - \mathrm{C}t^{\frac{3}{2}} \exp(-\tfrac{1}{\mathrm{C}} t^{\frac{3}{5}})\right] \ge \tfrac{\mathrm{C}_0}{4\sqrt{t}} \exp(-th_q(s)).\end{aligned}$$

Taking logarithms and dividing by t we get that $\liminf_{t\to\infty} \mathrm{Re}[\mathcal{A}_s(t)] \ge -h_q(s)$. Combining with (6.3.17) we arrive at (6.2.15).

Step 3. From this step on, we dedicate the proof to justifying the various equations and claims that appeared in **Step 2**. First in this step, we prove (6.3.25). Recall (**II**) and (**III**) defined in (6.3.20) and (6.3.21). For each of them, we push the absolute value around each term of the integrand. We use (6.3.9) from Lemma 6.3.5 to get

$$|(\mathbf{II})| \le \mathrm{C}\exp\Bigg(t \sup_{\substack{y\in\mathbb{R}\setminus\cup_k V_k \\ |e^{\mathbf{i}\theta}-1|\le\varepsilon|\log\tau|}} \mathrm{Re}[f(s+\mathbf{i}y, \tau^{1-\frac{s}{2}} e^{\mathbf{i}\theta})]\Bigg) \int_1^{e^{tB_q(\frac{s}{2})}} \frac{\mathrm{d}\zeta}{\zeta}, \tag{6.3.26}$$

$$|(\mathbf{III})| \le \mathrm{C}\exp\Bigg(t \sup_{\substack{y\in\mathbb{R} \\ |e^{\mathbf{i}\theta}-1|>\varepsilon|\log\tau|}} \mathrm{Re}[f(s+\mathbf{i}y, \tau^{1-\frac{s}{2}} e^{\mathbf{i}\theta})]\Bigg) \int_1^{e^{tB_q(\frac{s}{2})}} \frac{\mathrm{d}\zeta}{\zeta}. \tag{6.3.27}$$

Note that in (6.3.26), we have $|\tau^{\mathbf{i}y}-1| \ge |\tau^{\mathbf{i}t^{-\frac{4}{5}}}-1| \ge \frac{1}{2}|\log\tau| t^{-\frac{4}{5}}$ for all large enough t. Meanwhile in (6.3.27), $|\tau^{1-\frac{s}{2}}(e^{\mathbf{i}\theta}-1)| \ge \tau^{1-\frac{s}{2}}\varepsilon|\log\tau| = \tau^{1-\frac{s}{2}}|\log\tau| t^{-\frac{2}{5}}$. In either case, appealing to (6.3.4) in Lemma 6.3.1 with $\rho \mapsto s$ gives us that

$$f(s, \tau^{1-\frac{s}{2}}) - \mathrm{Re}[f(s+\mathbf{i}y, \tau^{1-\frac{s}{2}} e^{\mathbf{i}\theta})] \ge \tfrac{1}{\mathrm{C}} \cdot t^{-\frac{4}{5}}.$$

Substituting $f(s,\tau^{1-\frac{s}{2}})$ with $-h_q(s)$ and evaluating the integrals in (6.3.26) and (6.3.27) gives us (6.3.25).

Step 4. In this step and subsequent steps we prove (6.3.22) and (6.3.24). Recall that $v_k = -\frac{2\pi}{\log\tau}k$ and $\varepsilon = t^{-\frac{2}{5}}$. We first focus on the $(\mathbf{I})_k$ integral defined in (6.3.30). Our goal in this and next step is to show

$$(\mathbf{I})_k = (1+O(t^{-\frac{1}{5}}))\frac{\mathrm{C}_0(k)}{2\pi\sqrt{t}}\int_1^{e^{tB_q(\frac{s}{2})}}\frac{\zeta^{\mathbf{i}v_k}}{\zeta}\int_{-\varepsilon^2}^{\varepsilon^2}\zeta^{\mathbf{i}y}\exp(-th_q(s+\mathbf{i}y))\mathrm{d}y\mathrm{d}\zeta. \tag{6.3.28}$$

where

$$\mathrm{C}_0(k) := \sqrt{\frac{(1+\tau^{\frac{s}{2}})^3}{4\pi(q-p)(\tau^{\frac{3s}{2}-2}-\tau^{2s-2})}}\frac{(-1)^n(s+\mathbf{i}v_k)_n}{\sin(-\pi(s+\mathbf{i}v_k))(1-\tau^s)} \tag{6.3.29}$$

Towards this end, note that in the argument for (6.3.16), we push the absolute value around each term of the integrand. Thus, the upper bound achieved in (6.3.16) guarantees that the triple integral in $(\mathbf{I})_k$ is absolutely convergent. Thereafter, Fubini's theorem allows us to switch the order of integration inside $(\mathbf{I})_k$. By a change-of-variables, we see that

$$(\mathbf{I})_k = (-1)^n\int_1^{e^{tB_q(\frac{s}{2})}}\frac{\zeta^{\mathbf{i}v_k-1}}{4\pi^2}\int_{-\varepsilon^2}^{\varepsilon^2}\frac{(s+\mathbf{i}y+\mathbf{i}v_k)_n\zeta^{\mathbf{i}y}}{\sin(-\pi(s+\mathbf{i}y+\mathbf{i}v_k))}\int_{A_\varepsilon}\frac{e^{tf(s+\mathbf{i}y,\tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})}\mathrm{d}\theta}{1-\tau^{s+\mathbf{i}y}}\mathrm{d}y\mathrm{d}\zeta,$$

where recall $A_\varepsilon = \{\theta\in[-\pi,\pi] : |e^{\mathbf{i}\theta}-1|\le\varepsilon|\log\tau|\}$. Note that in this case range of y lies in a small window of $[-t^{-\frac{4}{5}}, t^{-\frac{4}{5}}]$. As s is fixed, one can replace $(s+\mathbf{i}y+\mathbf{i}v_k)_n$, $\sin(-\pi(s+\mathbf{i}y+\mathbf{i}v_k))$, and $1-\tau^{s+\mathbf{i}y}$ by $(s+\mathbf{i}v_k)_n$, $\sin(-\pi(s+\mathbf{i}v_k))$, and $1-\tau^s$ with an expense of $O(t^{-\frac{4}{5}})$ term (which can be chosen independent of k). We thus obtain

$$(\mathbf{I})_k = \frac{(-1)^n(s+\mathbf{i}v_k)_n(1+O(t^{-\frac{4}{5}}))}{\sin(-\pi(s+\mathbf{i}v_k))(1-\tau^s)}\int_1^{e^{tB_q(\frac{s}{2})}}\frac{\zeta^{\mathbf{i}v_k}}{4\pi^2\zeta}\int_{-\varepsilon^2}^{\varepsilon^2}\zeta^{\mathbf{i}y}\int_{A_\varepsilon}e^{tf(s+\mathbf{i}y,\tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})}\mathrm{d}\theta\mathrm{d}y\mathrm{d}\zeta. \tag{6.3.30}$$

We now evaluate the θ-integral in the above expression. We claim that

$$\int_{A_\varepsilon} e^{tf(s+\mathbf{i}y,\tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})}\mathrm{d}\theta = (1+O(t^{-\frac{1}{5}}))\sqrt{\frac{\pi(1+\tau^{\frac{s}{2}})^3}{t(q-p)(\tau^{\frac{3s}{2}-2}-\tau^{2s-2})}}\exp(-th_q(s+\mathbf{i}y)) \tag{6.3.31}$$

Note that (6.3.28) follows from (6.3.31). Hence we focus on proving (6.3.31) in next step.

Step 5. In this step we prove (6.3.31). For simplicity we let $u = s+\mathbf{i}y$ temporarily. Taylor expanding the exponent appearing in l.h.s. of (6.3.31) around $\theta = -\frac{y}{2}\log\tau$ and using the fact $\partial_z f(u,z)|_{z=\tau^{1-\frac{u}{2}}} = 0$, we get

$$\begin{aligned}\text{l.h.s. of (6.3.31)} &= \int_{A_\varepsilon} e^{tf(u,\tau^{1-\frac{u}{2}}e^{\mathbf{i}(\theta+\frac{y}{2}\log\tau)})}\mathrm{d}\theta \\ &= \exp(tf(u,\tau^{1-\frac{u}{2}}))\int_{A_\varepsilon}\exp\left(-\frac{t}{2}\partial_z^2 f(u,\tau^{1-\frac{u}{2}})(\theta+\tfrac{y}{2}\log\tau)^2+O(t^{-\frac{1}{5}})\right)\mathrm{d}\theta.\end{aligned} \tag{6.3.32}$$

Note that we have replaced the higher order terms by $O(t^{-\frac{1}{5}})$ in the exponent above as θ, y are at most of the order $O(t^{-\frac{2}{5}})$. Furthermore, for all t large enough,

$$\begin{aligned}A_\varepsilon &= \{\theta\in[-\pi,\pi] : |e^{\mathbf{i}\theta}-1|\le\varepsilon|\log\tau|\} \\ &= \{\theta\in[-\pi,\pi] : |\sin\tfrac{\theta}{2}|\le\tfrac{1}{2}\varepsilon|\log\tau|\}\supset\{\theta\in[-\pi,\pi] : |\theta|\le\varepsilon|\log\tau|\}\end{aligned}$$

As $y\in[-\varepsilon^2,\varepsilon^2]$, we see that $A_\varepsilon\supset\{\theta\in[-\pi,\pi] : |\theta+\frac{y}{2}\log\tau|\le\frac{1}{2}\varepsilon|\log\tau|\}$ for all large enough t. Thus on A_ε^c we have $|\theta+\frac{y}{2}\log\tau|\ge\frac{1}{2}t^{-\frac{2}{5}}|\log\tau|$. Furthermore for small enough y, by (6.3.2), we have $\mathrm{Re}[\partial_z^2 f(u,\tau^{1-\frac{u}{2}})]>0$. Hence the above integral can be approximated by Gaussian integral. In particular, we have

$$\text{r.h.s. of (6.3.32)} = (1+O(t^{-\frac{1}{5}}))\exp(tf(u,\tau^{1-\frac{u}{2}}))\sqrt{\frac{2\pi}{t\partial_z^2 f(u,\tau^{1-\frac{u}{2}})}} \tag{6.3.33}$$

Observe that as $u = s+\mathbf{i}y$ and y is at most $O(t^{-\frac{4}{5}})$, $\partial_z^2 f(u,\tau^{1-\frac{u}{2}})$ in r.h.s. of (6.3.33) can be replaced by $\partial_z^2 f(s,\tau^{1-\frac{s}{2}})$ by adjusting the order term. Recall the expression for $\partial_z^2 f(s,\tau^{1-\frac{s}{2}})$ from (6.3.2) and observe that from the definition of f and h_q from (6.3.1) and (6.1.3) we have $f(u,\tau^{1-\frac{u}{2}}) = h_q(s+\mathbf{i}y)$. We thus arrive at (6.3.31).

Step 6. With the expression of $(\mathbf{I})_k$ obtained in (6.3.28), in this step we prove (6.3.22) and (6.3.24). As y varies in the window of $y \in [-t^{-\frac{4}{5}}, t^{-\frac{4}{5}}]$, by Taylor expansion we may replace $th_q(s+\mathbf{i}y)$ appearing in the r.h.s. of (6.3.28) by $t(h_q(s)+\mathbf{i}yh_q'(s))$ at the expense of an $O(t^{-\frac{3}{5}})$ term. Upon making a change of variable $r = \log\zeta - th_q'(s)$ we thus have

$$\begin{aligned}(\mathbf{I})_k &= (1+O(t^{-\frac{1}{5}}))\frac{\mathrm{C}_0(k)}{2\pi\sqrt{t}}e^{-th_q(s)}\int_{-th_q'(s)}^{tB_q(\frac{s}{2})-th_q'(s)} e^{\mathbf{i}v_k(r+th_q'(s))}\int_{-\varepsilon^2}^{\varepsilon^2} e^{\mathbf{i}yr}\mathrm{d}y\mathrm{d}r \\ &= (1+O(t^{-\frac{1}{5}}))\frac{\mathrm{C}_0(k)}{2\pi\sqrt{t}}e^{-th_q(s)}\int_{-th_q'(s)}^{tB_q(\frac{s}{2})-th_q'(s)} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r}-e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r. \end{aligned} \tag{6.3.34}$$

We claim that for $k=0$, (which implies $v_k = 0$) we have

$$\int_{-th_q'(s)}^{tB_q(\frac{s}{2})-th_q'(s)} \frac{e^{\mathbf{i}\varepsilon^2 r}-e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r = 2\pi(1+O(t^{-\frac{1}{5}})) \tag{6.3.35}$$

For $k \neq 0$, we have

$$\left|\int_{-th_q'(s)}^{tB_q(\frac{s}{2})-th_q'(s)} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r}-e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r\right| \le \mathrm{C}t^{-\frac{4}{5}} \tag{6.3.36}$$

where $\mathrm{C} > 0$ can be chosen free of k. Assuming (6.3.35) and (6.3.36) we may now complete the proof of (6.3.22) and (6.3.24). Indeed, for $k=0$ upon observing that $\mathrm{C}_0 = \mathrm{C}_0(0)$ (recall (6.3.23) and (6.3.29)), in view of (6.3.34) and (6.3.35) we get (6.3.22). Whearas for $k \neq 0$, thanks to the estimate in (6.3.36), in view of (6.3.34), we have

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} |(\mathbf{I})_k| \le \mathrm{C}t^{-\frac{13}{10}}\exp(-th_q(s))\sum_{k\in\mathbb{Z}\setminus\{0\}}|\mathrm{C}_0(k)|. \tag{6.3.37}$$

For $y \neq 0$, $|\frac{(s+\mathbf{i}y)_n}{\sin(-\pi(s+\mathbf{i}y))}| \leq \mathrm{C}|y|^n e^{-|y|/\mathrm{C}}$ forces r.h.s. of (6.3.37) to be summable proving (6.3.24).

Step 7. In this step we prove (6.3.35) and (6.3.36). Recalling that $\varepsilon^2 = t^{-\frac{4}{5}}$, we see that

$$\int_{-th_q'(s)}^{tB_q(\frac{s}{2})-th_q'(s)} \frac{e^{\mathbf{i}\varepsilon^2 r} - e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r} \mathrm{d}r = \int_{-t^{1/5}h_q'(s)}^{t^{1/5}B_q(\frac{s}{2})-t^{1/5}h_q'(s)} \frac{2\sin r}{r} \mathrm{d}r. \tag{6.3.38}$$

Following the definition of h_q and B_q in Proposition 6.2.1 we observe that $-h_q'(s) = \frac{\tau^{\frac{s}{2}}\log\tau}{(1+\tau^{\frac{s}{2}})^2} < 0$ and

$$B_q(s) - h_q'(s) = \frac{1-\tau^s + \tau^{\frac{s}{2}} s\log\tau}{s(1+\tau^{\frac{s}{2}})} = -sB_q'(s) > 0,$$

where $B_q'(s) < 0$ follows from (6.2.1). Thus as B_q is strictly decreasing (Proposition 6.2.1 (a)) we have $B_q(\frac{s}{2}) > B_q(s) > h_q'(s)$. Thus the integral on r.h.s. of (6.3.38) can be approximated by $(1+O(t^{-1/5}))\int_{\mathbb{R}} \frac{2\sin r}{r}\mathrm{d}r = 2\pi(1+O(t^{-1/5}))$. This proves (6.3.35). We now focus on proving (6.3.36). Towards this end, we divide the integral appearing in (6.3.36) into three regions as follows

$$\begin{aligned}\text{l.h.s. of (6.3.36)} \leq &\left|\int_{-th_q'(s)}^{-1} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r} - e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r\right| + \left|\int_{-1}^{1} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r} - e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r\right| \\ &+ \left|\int_{1}^{tB_q(\frac{s}{2})-th_q'(s)} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r} - e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r\right|.\end{aligned} \tag{6.3.39}$$

Note that for the second term appearing in r.h.s. of (6.3.39) can be bounded by $4t^{-\frac{4}{5}}$ using

$$\left|\int_{-1}^{1} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{2\sin(\varepsilon^2 r)}{r}\mathrm{d}r\right| \leq \int_{-1}^{1}\left|\frac{2\sin(\varepsilon^2 r)}{r}\right|\mathrm{d}r \leq 4\varepsilon^2 = 4t^{-\frac{4}{5}}.$$

For the first term appearing in r.h.s. of (6.3.39), by making a change of variable $r \mapsto r\frac{v_k-\varepsilon^2}{v_k+\varepsilon^2}$ we observe the following identity:

$$\int_{-th_q'(s)}^{-1} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r = \int_{-th_q'(s)\frac{v_k+\varepsilon^2}{v_k-\varepsilon^2}}^{-\frac{v_k+\varepsilon^2}{v_k-\varepsilon^2}} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r.$$

This leads to

$$\int_{-th_q'(s)}^{-1} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r}-e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r = \int_{-th_q'(s)}^{-th_q'(s)\frac{v_k+\varepsilon^2}{v_k-\varepsilon^2}} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r + \int_{-\frac{v_k+\varepsilon^2}{v_k-\varepsilon^2}}^{-1} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r.$$

In the first integral the length of the interval is $O(t^{1/5})$. However, the integrand itself is $O(t^{-1})$. For the second integral, the length of the interval is $O(t^{-4/5})$, and the integrand itself is $O(1)$. Note that this is only possible when $k \neq 0$ (forcing $v_k \neq 0$). And indeed all the O terms can be taken to be free of v_k (and hence of k). Combining this we get that the first term appearing in r.h.s of (6.3.39) can be bounded by $\mathrm{C}t^{-\frac{4}{5}}$. An exact analogous argument provides the same bound for the third term in r.h.s. of (6.3.39) as well. This proves (6.3.36) completing the proof. □

6.4 Bounds for the Higher order terms

The goal of this section is to establish bounds for the higher-order term $\mathcal{B}_s(t)$ defined in (6.2.14). First, recall the Fredholm determinant formula from (6.1.10). Using the $\mathrm{tr}(K_{\zeta,t}^{\wedge L})$ notation from (6.1.9) we may rewrite $\mathcal{B}_s(t)$ as follows.

$$\mathcal{B}_s(t) = (-1)^n \int_1^{e^{tBq(\frac{s}{2})}} \zeta^{-\alpha}\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\left[1+\sum_{L=2}^{\infty}\mathrm{tr}(K_{\zeta.t}^{\wedge L})\right]\mathrm{d}\zeta. \tag{6.4.1}$$

We claim that we could exchange the various integrals, derivatives and sums appearring in the r.h.s. of (6.4.1) and obtain $\mathcal{B}_s(t)$ through term-by-term differentiation, i.e.

$$\mathcal{B}_s(t) = (-1)^n \sum_{L=2}^{\infty}\int_1^{e^{tBq(\frac{s}{2})}} \zeta^{-\alpha}\partial_\zeta^n(\mathrm{tr}(K_{\zeta,t}^{\wedge L}))\mathrm{d}\zeta. \tag{6.4.2}$$

Towards this end, we devote Section 6.4.1 to its justification. Following the technical lemmas in Section 6.4.1, we proceed to prove Proposition 6.2.5 in Section 6.4.2.

6.4.1 Interchanging sums, integrals and derivatives

Recall from (6.3.8) the definition of $K_{\zeta,t}^{(n)}$. As a starting point of our analysis, we introduce the following notations before providing the bounds on $|\partial_\zeta^n \mathrm{tr}(K_{\zeta,t}^{\wedge L})|$. For any $n, L \in \mathbb{Z}_{>0}$, define

$$\mathfrak{M}(L,n) := \{\vec{m} = (m_1, \ldots, m_L) \in (\mathbb{Z}_{\geq 0})^L : m_1 + \cdots + m_L = n\}, \tag{6.4.3}$$

and $\binom{n}{\vec{m}} := \frac{n!}{m_1! \cdots m_L!}$. Furthermore, for any $L \in \mathbb{Z}_{>0}$, $\zeta \in \mathbb{R}_{>0}$ and $\vec{m} \in \mathfrak{M}(L,n)$, let

$$I_\zeta(\vec{m}) := \int \ldots \int \det(K_{\zeta,t}^{(m_i)}(w_i, w_j))_{i,j=1}^L \prod_{i=1}^L \mathrm{d}w_i \tag{6.4.4}$$

where w_i-contour lies on $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$. We also set $|\vec{m}|_{>0} := |\{i \mid i \in \mathbb{Z} \cap [1, L], m_i > 0\}|$, i.e. the number of positive m_i in $\vec{m}$.

To begin with, the next two lemma investigate the term-by-term n-th derivatives of $\mathrm{tr}(K_{\zeta,t}^{\wedge L})$ that appear on the r.h.s. of (6.4.2). The following should be regarded as a higher order version of Proposition 6.3.6.

Proposition 6.4.1. *Fix $n, L \in \mathbb{Z}_{>0}$ and let $\mathfrak{M}(L,n)$ be defined as in* (6.4.3)*. Recall the function $B_q(x)$ from Proposition 6.2.1. For any $t > 0$, the function $\zeta \mapsto \mathrm{tr}(K_{\zeta,t}^{\wedge L})$ is infinitely differentiable at each $\zeta \in [1, e^{tB_q(\frac{\delta}{2})}]$, with*

$$\partial_\zeta^n \mathrm{tr}(K_{\zeta,t}^{\wedge L}) = \frac{1}{L!} \sum_{\vec{m} \in \mathfrak{M}(L,n)} \binom{n}{\vec{m}} I_\zeta(\vec{m}), \tag{6.4.5}$$

where the r.h.s of (6.4.5) *converges absolutely uniformly. Furthermore, there exists a constant* $\mathrm{C} = \mathrm{C}(n, \delta, q) > 0$ *such that for all $\vec{m} \in \mathfrak{M}(L,n)$ we have*

$$|I_\zeta(\vec{m})| \leq \mathrm{C}^L L^{\frac{L}{2}} \zeta^{L\delta - n} e^{-th_q(\delta)}, \quad |\partial_\zeta^n \mathrm{tr}(K_{\zeta,t}^{\wedge L})| \leq \frac{\mathrm{C}^L}{L!} L^n L^{\frac{L}{2}} \zeta^{L\delta - n} e^{-th_q(\delta)}. \tag{6.4.6}$$

Proof. The proof idea is same as that of Proposition 6.3.6, but it's more cumbersome notationally. For clarity we split the proof into four steps. In the first step, we introduce some necessary notations. In Steps 2-3, we prove (6.4.5) and in the final step, we prove (6.4.6).

Step 1. In this step we summarize the notation we will require in the proof of (6.4.5). We fix $L \in \mathbb{Z}_{>0}, \delta \in (0,1), t > 0$, and $\zeta_1, \zeta_2 > 0$ and recall $B_q(x)$ from Proposition 6.2.1.

We define $\vec{\xi}_k \in [1, e^{tB_q(\frac{\delta}{2})}]^L$ to be the vector whose first k entries are ζ_2 and the rest $L-k$ entries are ζ_1:

$$\vec{\xi}_k := (\xi_{k,1}, \xi_{k,2}, \dots, \xi_{k,L}) := (\underbrace{\zeta_2\ ,\ \zeta_2\ ,\ \dots\ ,\ \zeta_2}_{k \text{ times}}\ ,\ \underbrace{\zeta_1\ ,\ \zeta_1\ ,\ \dots\ ,\ \zeta_1}_{L-k \text{ times}}), \quad k = 0, 1, \dots, L.$$

For any $\vec{m} = (m_1, m_2, \dots, m_L) \in (\mathbb{Z}_{\geq 0})^L$ we define the following integral of mixed parameters

$$I^{(k)}_{\zeta_1,\zeta_2}(\vec{m}) := \int \dots \int \det(K^{(m_i)}_{\xi_{k,i},t}(w_i, w_j))^L_{i,j=1} \prod_{i=1}^{L} \mathrm{d}w_i. \tag{6.4.7}$$

where w_i-contour lies on $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$. $I^{(k)}_{\zeta_1,\zeta_2}(\vec{m})$ serves as an interpolation between $I_{\zeta_1}(\vec{m})$ and $I_{\zeta_2}(\vec{m})$ defined in (6.4.4) as k increases from 0 to L where the parameters ζ are now allowed to be different for different rows in the determinant.

We next define $\vec{e}_k = (e_{k,1}, e_{k,2}, \dots, e_{k,L})$ to be the unit vector with 1 in the k-th position and 0 elsewhere. With the above notations in place, for each $j, k \in \{1, 2, \dots, L\}$ and $\vec{m} \in (\mathbb{Z}_{\geq 0})^L$ we set

$$\mathfrak{L}^{(1)}_{\zeta_1,\zeta_2}(\vec{m}; k) := \frac{1}{\zeta_2 - \zeta_1}\left[I^{(k)}_{\zeta_1,\zeta_2}(\vec{m}) - I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m}) - (\zeta_2 - \zeta_1) I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k)\right], \tag{6.4.8}$$

$$\mathfrak{L}^{(2)}_{\zeta_1,\zeta_2}(\vec{m}; j, k) := I^{(j)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k) - I^{(j-1)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k). \tag{6.4.9}$$

Note that we define (6.4.8) modelling after D_{ζ_1,ζ_2} in the proof of Proposition 6.3.6. Here, the only differences between the three determinants of the respective $I^{(k)}_{\zeta_1,\zeta_2}(\vec{m})$'s lie in the k-th row, i.e. $K^{(m_k)}_{\zeta_2,t}$ v.s. $K^{(m_k)}_{\zeta_1,t}$ v.s. $K^{(m_k+1)}_{\zeta_1,t}$. So we have isolated the differences and tried to reduce the question of differentiability to row-wise in (6.4.8). Meanwhile, (6.4.9) "measures" the distance between $I^{(k)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k)$ and $I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k)$ where they differ only in $\xi_{k,k} = \zeta_2$ or ζ_1 for $K^{(m_k)}_{\xi_{k,k},t}$ on the k-th row of the determinant.

We finally remark that all the w_i-contours in the integrals appearing throughout the proof are on $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$ – we will not mention this further. We would also drop (w_i, w_j) from $K_{\bullet,t}^{(m_i)}(w_i, w_j)$ when it is clear from the context.

Step 2. We show the infinite differentiability of $\mathrm{tr}(K_{\zeta,t}^{\wedge L})$ by proving (6.4.5) in this step. The proof proceeds via induction on n. When $n = 0$, observe that (6.4.5) recovers the formula of $\mathrm{tr}(K_{\zeta,t}^{\wedge L})$. This constitutes the base case. To prove the induction step, suppose (6.4.5) holds for $n = N$. Then for $n = N + 1$, we fix $\zeta_1, \zeta_2 > 0$. Without loss of generality, we assume $\zeta_1 + 1 > \zeta_2 > \zeta_1$ and consider

$$D_{\zeta_1,\zeta_2} := \frac{\partial_\zeta^N \mathrm{tr}(K_{\zeta_2,t}^{\wedge L}) - \partial_\zeta^N \mathrm{tr}(K_{\zeta_1,t}^{\wedge L})}{\zeta_2 - \zeta_1} - \frac{1}{L!} \sum_{\vec{m} \in \mathfrak{M}(L,N+1)} \binom{N+1}{\vec{m}} I_{\zeta_1}(\vec{m}). \tag{6.4.10}$$

To prove (6.4.5), it suffices to show $|D_{\zeta_1,\zeta_2}| \to 0$ as $\zeta_2 \to \zeta_1$. Towards this end, we first claim that for all $\vec{m} \in \mathfrak{M}(L, N)$ and for all $j, k \in \{1, 2, \ldots, L\}$ we have

$$\left|\mathfrak{L}_{\zeta_1,\zeta_2}^{(1)}(\vec{m}; k)\right| \to 0, \text{ and } \left|\mathfrak{L}_{\zeta_1,\zeta_2}^{(2)}(\vec{m}; j, k)\right| \to 0, \text{ as } \zeta_2 \to \zeta_1, \tag{6.4.11}$$

where $\mathfrak{L}_{\zeta_1,\zeta_2}^{(1)}(\vec{m}; k)$ and $\mathfrak{L}_{\zeta_1,\zeta_2}^{(2)}(\vec{m}; j, k)$ are defined in (6.4.8) and (6.4.9) respectively. We postpone the proof of (6.4.11) to the next step. Assuming its validity, we now proceed to complete the induction step.

Towards this end, we first manipulate the expression appearing in r.h.s. of (6.4.10). A simple combinatorial fact shows

$$\sum_{\vec{m} \in \mathfrak{M}(L,N+1)} \binom{N+1}{\vec{m}} I_{\zeta_1}(\vec{m}) = \sum_{k=1}^{L} \sum_{\vec{m} \in \mathfrak{M}(L,N)} \binom{N}{\vec{m}} I_{\zeta_1}(\vec{m} + \vec{e}_k),$$

where $\vec{e}_k$ is defined in Step 1. Substituting this combinatorics back into the r.h.s. of (6.4.10) and using the induction step for $n = N$, allows us to rewrite D_{ζ_1,ζ_2} as follows:

$$\text{r.h.s. of (6.4.10)} = \frac{1}{L!} \sum_{\vec{m}\in\mathfrak{M}(L,N)} \binom{N}{\vec{m}} \left[\frac{I_{\zeta_2}(\vec{m}) - I_{\zeta_1}(\vec{m})}{\zeta_2 - \zeta_1} - \sum_{k=1}^{L} I_{\zeta_1}(\vec{m} + \vec{e}_k) \right]. \tag{6.4.12}$$

Recalling the definition of $I_\zeta(\vec{m})$ in (6.4.4) and that of $I^{(k)}_{\zeta_1,\zeta_2}(\vec{m})$ in (6.4.7), we see that $\sum_{k=1}^{L}[I^{(k)}_{\zeta_1,\zeta_2}(\vec{m}) - I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m})]$ telescopes to $I_{\zeta_2}(\vec{m}) - I_{\zeta_1}(\vec{m})$. Furthermore, if we recall $\mathfrak{L}^{(1)}_{\zeta_1,\zeta_2}(\vec{m};k)$ and $\mathfrak{L}^{(2)}_{\zeta_1,\zeta_2}(\vec{m};j,k)$ from (6.4.8) and (6.4.9) respectively, we observe that

$$I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k) - I_{\zeta_1}(\vec{m} + \vec{e}_k) = I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k) - I^{(0)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k) = \sum_{j=1}^{k} \mathfrak{L}^{(2)}_{\zeta_1,\zeta_2}(\vec{m};j,k).$$

Combining these observations, we have

$$\begin{aligned}
\text{r.h.s. of (6.4.12)} &= \frac{1}{L!} \sum_{\vec{m}\in\mathfrak{M}(L,N)} \binom{N}{\vec{m}} \sum_{k=1}^{L} \frac{\left[I^{(k)}_{\zeta_1,\zeta_2}(\vec{m}) - I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m}) - (\zeta_2 - \zeta_1) I_{\zeta_1}(\vec{m} + \vec{e}_k)\right]}{\zeta_2 - \zeta_1} \\
&= \frac{1}{L!} \sum_{\vec{m}\in\mathfrak{M}(L,N)} \binom{N}{\vec{m}} \sum_{k=1}^{L} \left[\mathfrak{L}^{(1)}_{\zeta_1,\zeta_2}(\vec{m};k) + \sum_{j=1}^{k-1} \mathfrak{L}^{(2)}_{\zeta_1,\zeta_2}(\vec{m};j,k) \right].
\end{aligned} \tag{6.4.13}$$

Clearly r.h.s. of (6.4.13) goes to zero as $\zeta_2 \to \zeta_1$ whenever (6.4.11) is true. Thus by induction we have (6.4.5).

Step 3. In this step we prove (6.4.11). Recall $\mathfrak{L}^{(1)}_{\zeta_1,\zeta_2}(\vec{m};k)$ from (6.4.8). Following the definition of $I^{(k)}_{\zeta_1,\zeta_2}(\vec{m})$ from (6.4.7) we have

$$\begin{aligned}
\left|\mathfrak{L}^{(1)}_{\zeta_1,\zeta_2}(\vec{m};k)\right| \le \int \cdots \int \frac{1}{\zeta_2 - \zeta_1} \Big| &\det(K^{(m_i)}_{\xi_{k,i},t})_{i,j=1}^{L} - \det(K^{(m_i)}_{\xi_{k-1,i},t})_{i,j=1}^{L} \\
&- (\zeta_2 - \zeta_1) \det(K^{(m_i+e_{k,i})}_{\xi_{k-1,i},t})_{i,j=1}^{L} \Big| \prod_{i=1}^{L} \mathrm{d}w_i.
\end{aligned}$$

Recall that in the above expression, up to a constant, the three determinants differ only in the k-th row. Hence the above expression can be written as $\int \cdots \int |\det(A)| \prod_{i=1}^{L} \mathrm{d}w_i$, where the entries of

A are given as follows:

$$\begin{aligned}
A_{i,j} &= K^{(m_i)}_{\zeta_2,t}(w_i,w_j), \quad i<k, \quad A_{i,j} = K^{(m_i)}_{\zeta_1,t}(w_i,w_j), \quad i>k,\\
A_{k,j} &= \frac{1}{\zeta_2-\zeta_1}[K^{(m_k)}_{\zeta_2,t}(w_k,w_j) - K^{(m_k)}_{\zeta_1,t}(w_k,w_j) - (\zeta_2-\zeta_1)K^{(m_k+1)}_{\zeta_1,t}(w_k,w_j)]\\
&= \frac{1}{2\pi\mathbf{i}}\int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty}\Gamma(-u)\Gamma(1+u)R_{\zeta_1,\zeta_2;m_k}(u)e^{tf(u,w_k)}\frac{\mathrm{d}u}{w_j-\tau^u w_k},
\end{aligned}$$

where $R_{\zeta_1,\zeta_2;m_k}(u)$ is same as in (6.3.13). As m_i's are at most n, by Lemma 6.3.5 ((6.3.10) specifically), we can get a constant $\mathrm{C}>0$ depending only on n,δ, and q, so that

$$|A_{i,j}| \le \mathrm{C}(\zeta_1^{\delta-m_k}+\zeta_2^{\delta-m_k})\exp(-th_q(\delta)) \le \mathrm{C}(1+\zeta_2^{\delta})\exp(-th_q(\delta))$$

for all $i\neq k$. For $A_{k,j}$, we follow the same argument as in Proposition 6.3.6 (along the lines of (6.3.14)) to get

$$\begin{aligned}
|A_{k,j}| \le \frac{\tau^{1-\frac{\delta}{2}}}{2\pi}\int_{\zeta_1}^{\zeta_2}\left|\sigma^{\delta+\mathbf{i}y-m_k-2}\right|\mathrm{d}\sigma\\
\cdot \max_{w_j,w_k\in\mathfrak{C}(\tau^{1-\frac{\delta}{2}})}\int_{\mathbb{R}}\left|\frac{(\delta+\mathbf{i}y)_{m_k+2}}{\sin(-\pi(\delta+\mathbf{i}y))}e^{tf(\delta+\mathbf{i}y,w_k)}\right|\frac{\mathrm{d}y}{|w_j-\tau^{\delta+\mathbf{i}y}w_k|}.
\end{aligned}$$

Note that by Lemma 6.3.5 ((6.3.9) specifically) we see that the above maximum is bounded by $\mathrm{C}\exp(-th_q(\delta))$ where again as m_i's are at most n, the constant C can be chosen dependent only on n,δ, and q. Since $|\sigma^{u-n-2}| = |\sigma^{\delta-m_k-2}| \le |\zeta_1^{\delta-m_k-2}| \le |\zeta_1^{\delta-2}|$ over the interval $[\zeta_1,\zeta_2]$ for $\delta\in(0,1)$, we obtain

$$|A_{k,j}| \le \mathrm{C}\exp(-th_q(\delta))\int_{\zeta_1}^{\zeta_2}|\sigma|^{\delta-m_k-2}\mathrm{d}\sigma \le \mathrm{C}\exp(-th_q(\delta))\zeta_1^{\delta-2}(\zeta_2-\zeta_1).$$

As all the above estimates on $|A_{i,j}|$ are uniform in w_i's, using Hadamard inequality we have

$$\int\cdots\int|\det(A)|\prod_{i=1}^{L}\mathrm{d}w_i \le \mathrm{C}^L L^{\frac{L}{2}}\exp(-tLh_q(\delta))(1+\zeta_2^{\delta})^{L-1}\zeta_1^{\delta-2}(\zeta_2-\zeta_1)$$

Taking $\zeta_2 \to \zeta_1$ above, we get the first part of (6.4.11). The proof of the second part of (6.4.11) follows similarly by observing that the corresponding determinants also differ only in one row. One can then deduce the second part of (6.4.11) using the uniform estimates of the kernel and difference of kernels given in (6.3.10) and (6.3.11) respectively. As the proof follows exactly in the lines of above arguments, we omit the technical details.

Step 4. In this step we prove (6.4.6).

Recall the definition of $I_\zeta(\vec{m})$ from (6.4.4). By Hadamard's inequality and Lemma 6.3.5 we have

$$\begin{aligned} |\det(K_{\zeta,t}^{(m_i)})_{i,j=1}^L| &\le L^{\frac{L}{2}} \prod_{i=1}^{L} \max_{w_i,w_j\in\mathfrak{C}(\tau^{1-\frac{\delta}{2}})} |K_{\zeta,t}^{(m_i)}(w_i,w_j)| \\ &\le L^{\frac{L}{2}} \prod_{i=1}^{L} \mathrm{C}\zeta^{\delta-m_i}\exp(-th_q(\delta)) = \mathrm{C}^L L^{\frac{L}{2}}\zeta^{L\delta-n}\exp(-th_q(\delta)), \end{aligned} \tag{6.4.14}$$

where the last equality follows as $\sum_{i=1}^{L} m_i = n$. Note that here also $\mathrm{C} > 0$ can be chosen to be dependent only on n, δ, and q as m_i's are at most n. Recall that w_i-contour in $I_\zeta(\vec{m})$ lies on $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$. Thus in view of (6.4.14) adjusting the constant C we obtain first inequality of (6.4.6).

For the second inequality, We observe the following recurrence relation:

$$|\mathfrak{M}(L,n)| = |\{\vec{m} = (m_1,\ldots,m_L) \in \mathbb{Z}_{\ge 0}^L, \sum_{i=1}^{L} m_i = n\}| \le L\cdot|\mathfrak{M}(L,n-1)|. \tag{6.4.15}$$

It follows immediately that $|\mathfrak{M}(L,n)| \le L^n$. Observe that for each $\vec{m} \in \mathfrak{M}(L,n)$, $\binom{n}{\vec{m}}$ is bounded from above by $n!$. Thus collectively with (6.4.5) we have

$$|\partial_\zeta^n \mathrm{tr}(K_{\zeta,t}^{\wedge L})| \le \frac{n!L^n}{L!} \max_{\vec{m}\in\mathfrak{M}(L,n)} |I_\zeta(\vec{m})|.$$

Applying the first inequality of (6.4.6) above leads to the second inequality of (6.4.6) completing the proof.

□

Lemma 6.4.2. *Fix $n \in \mathbb{Z}_{>0}$, $\zeta \in [1, e^{tB_q(\frac{s}{2})}]$, and $t > 0$. Then*

$$\partial_\zeta^n \left(\sum_{L=1}^{\infty} \mathrm{tr}(K_{\zeta,t}^{\wedge L}) \right) = \sum_{L=1}^{\infty} \partial_\zeta^n (\mathrm{tr}(K_{\zeta,t}^{\wedge L})).$$

Proof. On account of [88, Proposition 4.2]), it suffices to verify the following conditions:

1. $\sum_{L=1}^{\infty} \mathrm{tr}(K_{\zeta,t}^{\wedge L})$ converges absolutely pointwise for $\zeta \in [1, e^{tB_q(\frac{s}{2})}]$;
2. the absolute derivative series $\sum_{L=1}^{\infty} \partial_\zeta^n (\mathrm{tr}(K_{\zeta,t}^{\wedge L}))$ converges uniformly for $\zeta \in [1, e^{tB_q(\frac{s}{2})}]$.

By Proposition 6.4.1, we can pass the derivative inside the trace in (2). Both (1) and (2) follow from (6.4.6) in Proposition 6.4.1 as $\sum_{L=1}^{\infty} \frac{1}{L!} \mathrm{C}^L L^n L^{\frac{L}{2}} \zeta^{L\delta - n} \exp(-tLh_q(\delta)) < \infty$ for each $\zeta \in [1, e^{tB_q(\frac{s}{2})}]$. □

Now, with the results from Lemmas 6.4.1 and 6.4.2, we are poised to justify the interchanges of operations leading to (6.4.2).

Proposition 6.4.3. *For fixed $n, L \in \mathbb{Z}_{\geq 0}$, $\zeta \in [1, e^{tB_q(\frac{s}{2})}]$ and $t > 0$,*

$$\int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} \partial_\zeta^n \left[1 + \sum_{L=2}^{\infty} \mathrm{tr}(K_{\zeta,t}^{\wedge L}) \right] \mathrm{d}\zeta = \sum_{L=2}^{\infty} \sum_{\vec{m} \in \mathfrak{M}(L,n)} \binom{n}{\vec{m}} \frac{1}{L!} \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} I_\zeta(\vec{m}) \mathrm{d}\zeta. \tag{6.4.16}$$

Proof. Thanks to Lemma 6.4.2 we can switch the order of derivative and sum to get

$$\text{l.h.s. of (6.4.16)} = \int_1^{e^{tB_q(\frac{s}{2})}} \sum_{L=2}^{\infty} \zeta^{-\alpha} \partial_\zeta^n (\mathrm{tr}(K_{\zeta,t}^{\wedge L})) \mathrm{d}\zeta.$$

We next justify the interchange of the integral and the sum in above expression. Note that via the estimate in (6.4.6) we have

$$\int_1^{e^{tB_q(\frac{s}{2})}} \sum_{L=2}^{\infty} \zeta^{-\alpha} |\partial_\zeta^n (\mathrm{tr}(K_{\zeta,t}^{\wedge L}))| \mathrm{d}\zeta \leq \sum_{L=2}^{\infty} \frac{1}{L!} \mathrm{C}^L L^n L^{\frac{L}{2}} \exp(-th_q(\delta)) \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{L\delta - n - \alpha} \mathrm{d}\zeta < \infty.$$

Hence Fubini's theorem justifies the exchange of summation and integration. Finally we arrive at r.h.s. of (6.4.16) by using the higher order derivative identity (see (6.4.5)) from Proposition 6.4.1.

$\square$

6.4.2 Proof of Proposition 6.2.5

Finally, in this subsection we present the proof of Proposition 6.2.5 via obtaining an upper-bound for $|\mathcal{B}_s(t)|$, defined in (6.2.14).

Recall $I_\zeta(\vec{m})$ from (6.4.4). We first introduce the following technical lemma that upper bounds the absolute value of the integral $\int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} I_\zeta(\vec{m}) \mathrm{d}\zeta$ and will be an important ingredient in the proof of Proposition 6.2.5.

Lemma 6.4.4. *Fix* $s > 0$ *so that* $\alpha := s - \lfloor s \rfloor > 0$. *Set* $n = \lfloor s \rfloor + 1$. *Fix* $L \in \mathbb{Z}_{>0}$ *with* $L \geq 2$ *and* $\vec{m} \in \mathfrak{M}(L, n)$, *where* $\mathfrak{M}(L, n)$ *is defined in* (6.4.3). *There exists a constant* $\mathrm{C} = \mathrm{C}(q, s) > 0$ *such that*

$$\int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} |I_\zeta(\vec{m})| \mathrm{d}\zeta \leq \mathrm{C}^L L^{\frac{L}{2}} \exp(-th_q(s) - \tfrac{1}{\mathrm{C}} t). \tag{6.4.17}$$

where $I_\zeta(\vec{m})$ *is defined in* (6.4.4) *and the functions* B_q *and* h_q *are defined in Proposition 6.2.1.*

Proof. As we obtain upper bounds for the LHS of (6.4.17) differently depending on the value of L, we split the proof into two steps as follows. Fix $L_0 = 2(n+1)$. In Step 1, we prove the inequality for when $2 \leq L \leq L_0$ and in Step 2, we consider the case when $L > L_0$. In both steps, we deform the w-contours in $I_\zeta(\vec{m})$ appropriately to achieve its upper bound.

Step 1. In this step, we prove (6.4.17) for when $2 \leq L \leq L_0$. Fix $\vec{m} = (m_1, \ldots, m_L) \in \mathfrak{M}(L, n)$, where $\mathfrak{M}(L, n)$ is defined in (6.4.3) and set

$$\rho_i := \begin{cases} m_i + \frac{\alpha}{L} - \frac{1}{|\vec{m}|_{>0}} & \text{if } m_i > 0 \\ \frac{\alpha}{L} & \text{if } m_i = 0. \end{cases} \tag{6.4.18}$$

where we recall that $|\vec{m}|_{>0} = |\{i \mid i \in \mathbb{Z}, m_i > 0\}|$.

Recall the definition of $I_\zeta(\vec{m})$ in (6.4.4). Note that each $K_{\zeta,t}^{(m_i)}(w_i, w_j)$ (see (6.3.8)) are themselves complex integral over $\delta + \mathbf{i}\mathbb{R}$. As $\alpha > 0$ and $L \leq L_0 = 2(n+1)$ we may take the δ appearing in the kernel in $K_{\zeta,t}^{(m_i)}$ less than all the ρ_i's. Note that this is only possible when $\alpha > 0$. This is why we assumed this in the hypothesis here and as well as in the statement of Proposition 6.2.5.

In what follows we show that the contours of $K_{\zeta,t}^{(m_i)}(w_i, w_j)$ followed by w_i-contours can be deformed appropriately without crossing any pole in $I_\zeta(\vec{m})$. Indeed for each $K_{\zeta,t}^{(m_i)}$ in $I_\zeta(\vec{m})$ we can write

$$K_{\zeta,t}^{(m_i)}(w_i, w_j) = \frac{1}{2\pi\mathbf{i}} \int_{\rho_i - \mathbf{i}\infty}^{\rho_i + \mathbf{i}\infty} \Gamma(-u_i)\Gamma(1+u_i)(u_i)_n \zeta^{u_i - n} e^{f(u_i, w_i)} \frac{\mathrm{d}u_i}{w_j - \tau^{u_i} w_i}.$$

As each $\rho_i \in (0, m_i \vee 1)$ (see (6.4.18)), by Remark 6.3.4, the above equality is true as we do not cross any poles in the integrand. Ensuing this change, we claim that we can deform the w_i-contour to $\mathfrak{C}(\tau^{1-\frac{\rho_i}{2}})$ one by one without crossing any pole in $I_\zeta(\vec{m})$. Similar to the argument given in the beginning of the proof of Proposition 6.2.4, we note that as we deform the w_i-contours potential sources of poles in $I_\zeta(\vec{m})$ lie in the exponent $f(u_i, w_i) := \frac{(q-p)}{1+w_i\tau^{-1}} - \frac{(q-p)}{1+\tau^{u_i-1}w_i}$ (recalled from (6.3.1)) and in the denominator $w_j - \tau^{u_i} w_i$.

Take $w_i \in \mathfrak{C}(\tau^{1-\frac{\delta_i}{2}})$, $\delta_i, \in [\delta, \rho_i]$, and $u_i \in \rho_i + \mathbf{i}\mathbb{R}$. Observe that

$$|w_j - \tau^{u_i} w_i| \geq |w_j| - |\tau^{u_i} w_i| \geq \tau^{1-\frac{\delta_j}{2}} - \tau^{1+\rho_i - \frac{\delta_i}{2}} > 0,$$

$$|1 + w_i \tau^{-1}| \geq |w_i \tau^{-1}| - 1 \geq \tau^{-\frac{\delta_i}{2}} - 1, \quad |1 + \tau^{u_i - 1} w_i| \geq 1 - |\tau^{u_i - 1} w_i| \geq 1 - \tau^{\rho_i - \frac{\delta_i}{2}}.$$

This ensures that each w_i-contour can be taken as $\mathfrak{C}(\tau^{1-\frac{\rho_i}{2}})$ without crossing any pole.

Permitting these contour deformations, we wish to apply Lemma 6.3.5, (6.3.9) specifically. Indeed we apply (6.3.9) with $\rho, \delta \mapsto \rho_i$, $w \mapsto w'$, $w' \mapsto w_j$. Note that we indeed have $|w_j| \neq \tau^{1+\frac{\rho_i}{2}}$ here. We thus obtain

$$|K_{\zeta,t}^{(m_i)}(w_i, w_j)| \leq \mathrm{C}\zeta^{\rho_i - m_i} \exp(-t h_q(\rho_i)). \tag{6.4.19}$$

Here, C is supposed to be dependent on m_i, ρ_i, and q. Note that ρ_i are in turn dependent on m_i, s and L. Since L is at most $L_0 = 2(n+1)$, there are at most finitely many choices of m_i's which in turn produced finitely many choices of ρ_i's. As s is fixed, all of the ρ_i's are uniformly bounded away from 0. Hence we can choose the constant C to be dependent only s and q (recall that n is also dependent on s).

Observe that as $\vec{m} \in \mathfrak{M}(L, n)$ defined in (6.4.3), we have $\sum m_i = n$ and consequently $\sum \rho_i = n - 1 + \alpha = s$. In view of the estimate in (6.4.19) and the definition of $I_\zeta(\vec{m})$ from (6.4.4), by Hadamard's inequality, we obtain

$$|I_\zeta(\vec{m})| \le \mathrm{C}^L L^{\frac{L}{2}} \zeta^{s-n} \exp\left(-t \sum_{i=1}^{L} h_q(\rho_i)\right) = \mathrm{C}^L L^{\frac{L}{2}} \zeta^{-1+\alpha} \exp\left(-t \sum_{i=1}^{L} h_q(\rho_i)\right).$$

Thus

$$\int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} |I_\zeta(\vec{m})| \mathrm{d}\zeta \le \mathrm{C}^L L^{\frac{L}{2}} \exp\left(-t \sum_{i=1}^{L} h_q(\rho_i)\right) \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-1} \mathrm{d}\zeta. \tag{6.4.20}$$

Observe that $\int_x^y \zeta^{-1} d\zeta = \log \frac{y}{x}$. We appeal to the subadditivity $h_q(x) + h_q(y) > h_q(x+y)$ in Proposition 6.2.1 to get that $\sum_{i=1}^{L} h_q(\rho_i) \ge h_q(s - \rho_1) + h_q(\rho_1)$. Note that here we used the fact that $L \ge 2$. This leads to

$$\text{r.h.s. of (6.4.20)} \le \mathrm{C}^L L^{\frac{L}{2}} t B_q(\tfrac{s}{2}) \exp(-t h_q(s)) \exp(-t(h_q(s-\rho_1) + h_q(\rho_1) - h_q(s))) \tag{6.4.21}$$

Note that from (6.4.18), $\rho_i \ge \frac{\alpha}{L} \ge \frac{\alpha}{L_0}$, this forces $\frac{\alpha}{L_0} \le s - \rho_1, \rho_1 \le s - \frac{\alpha}{L_0}$. Appealing to the strict subadditivity in (6.2.2) gives us that $h_q(s-\rho_1) + h_q(\rho_1) - h_q(s)$ can be lower bounded by a constant $\frac{1}{\mathrm{C}} > 0$ depending only on s and q. Adjusting the constant C we can absorb $tB_q(\frac{s}{2})$ appearing in r.h.s. of (6.4.21), to get (6.4.17), completing our work for this step.

Step 2. In this step, we prove (6.4.17) for the rest of the cases when $L > L_0$. Fix $\vec{m} = (m_1, \ldots, m_L) \in \mathfrak{M}(L, n)$. Recall the definition of $I_\zeta(\vec{m})$ in (6.4.4). Note that each $K_{\zeta,t}^{(m_i)}(w_i, w_j)$ (see (6.3.8)) is a complex integral over $\delta + \mathbf{i}\mathbb{R}$. Here we set $\delta = \min(\frac{1}{2}, \frac{s}{2})$. Thanks to (6.4.6) we have

$$|I_\zeta(\vec{m})| \le \mathrm{C}^L L^{\frac{L}{2}} \zeta^{L\delta - n} \exp(-t L h_q(\delta)),$$

where the constant C depends only on n, δ, and q and thus only on s and q. This leads to

$$\int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha}|I_\zeta(\vec{m})|\mathrm{d}\zeta \le \mathrm{C}^L L^{\frac{L}{2}} \exp(-tLh_q(\delta)) \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha-n+L\delta}\mathrm{d}\zeta. \tag{6.4.22}$$

Recall that $s = n-1+\alpha$. As $L \ge 2(n+1)$ and $\delta = \min(\frac{1}{2}, \frac{s}{2})$ we have $L\delta - n - \alpha > 0$ in this case. Thus, we can upper bound the integral in (6.4.22) to get

$$\text{r.h.s. of (6.4.22)} \le \mathrm{C}^L L^{\frac{L}{2}} \exp(-tLh_q(\delta)) \frac{\exp(tB_q(\frac{s}{2})(-s+L\delta))}{-s+L\delta}. \tag{6.4.23}$$

We incorporate $\frac{1}{-s+L\delta}$ into the constant C, Recall the definition of $B_q(x)$ from Proposition (6.2.1). We have $xB_q(x) = h_q(x)$. As $B_q(x)$ is strictly decreasing for $x > 0$, (Proposition 6.2.1 (a), (b)) we have

$$\begin{aligned}\text{r.h.s. of (6.4.23)} &\le \mathrm{C}^L L^{\frac{L}{2}} \exp(-2th_q(\tfrac{s}{2}) - tL\delta(B_q(\delta) - B_q(\tfrac{s}{2}))) \\ &\le \mathrm{C}^L L^{\frac{L}{2}} \exp(-2th_q(\tfrac{s}{2})) \le \mathrm{C}^L L^{\frac{L}{2}} \exp(-th_q(s) - \tfrac{1}{\mathrm{C}}t),\end{aligned}$$

where the last inequality above follows from (6.2.2) by observing that by subadditivity we can get a constant $\mathrm{C} = \mathrm{C}(q,s) > 0$ such that $2h_q(\frac{s}{2}) - h_q(s) \ge \frac{1}{\mathrm{C}}$. This completes the proof. □

With Lemma 6.4.4, we are now ready to prove Proposition 6.2.5.

Proof of Proposition 6.2.5. Recall the definition of $\mathcal{B}_s(t)$ as defined in (6.2.14). Appealing to (6.4.1) and Proposition (6.4.3) we get that

$$|\mathcal{B}_s(t)| = \sum_{L=2}^{\infty} \frac{1}{L!} \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha}|I_\zeta(\vec{m})|\mathrm{d}\zeta \tag{6.4.24}$$

Note that $\binom{n}{\vec{m}}$ is bounded from above by $n!$, and by (6.4.15) we have $|\mathfrak{M}(L,n)| \le L^n$. Applying these inequalities along with the estimate in Lemma 6.4.4 we have that

$$\text{r.h.s. of (6.4.24)} \le \exp(-th_q(s) - \tfrac{1}{\mathrm{C}}t) \sum_{L=2}^{\infty} \frac{1}{L!} \mathrm{C}^L L^{\frac{L}{2}} L^n$$

for some constant $C = C(q, s) > 0$. By Stirling's formula, $\sum_{L=2}^{\infty} \frac{1}{L!} C^L L^{\frac{L}{2}} L^n$ converges and hence adjusting the constant C, we obtain (6.2.16) completing the proof of the proposition. □

6.5 Appendix: Comparison to TASEP

In this section, we compute explicit expression for the upper tail rate function for TASEP (ASEP with $q = 1$) with step initial data and show that it matches with general ASEP rate function Φ_+ defined in (6.1.4).

Indeed, the large deviation problem for TASEP is already solved in [155] and is formulated in terms of Exponential Last Passage Percolation (LPP) model (Theorem 1.6 in [155]).

In order to state the connection between TASEP and Exponential LPP, we briefly recall the Exponential LPP model. Let Π_N be the set of all upright paths π in $\mathbb{Z}^2_{>0}$ from $(1, 1)$ to (N, N). Let $w(i, j), (i, j) \in \mathbb{Z}^2_{>0}$ be independent exponential distributed random variables with parameter 1. The last passage value for (N, N) is defined to be

$$\mathcal{H}(N) := \max \Big\{ \sum_{(i,j)\in\pi} w(i, j); \pi \in \Pi_N \Big\}.$$

As with the ASEP, for TASEP, we also set $H_0^{q=1}(t)$ to be the number of particles to the right of origin at time t. It is well known (see [155] for example) that $H_0^{q=1}(t)$ is related to the last passage value $\mathcal{H}(N)$ in the following way

$$\mathbb{P}\left(-H_0^{q=1}(t) + \tfrac{t}{4} \geq \tfrac{t}{4} y\right) = \mathbb{P}(\mathcal{H}(M_t) \geq t), \quad \text{where } M_t = \lfloor \tfrac{t}{4}(1 - y) \rfloor + 1. \tag{6.5.1}$$

Theorem 6.5.1. *For $y \in (0, 1)$ we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(-H_0^{q=1}(t) + \tfrac{t}{4} \geq \tfrac{t}{4} y\right) = -\Phi_+(y). \tag{6.5.2}$$

where Φ_+ is defined in (6.1.4).

The idea of the proof of Theorem 6.5.1 is to use large deviation principle for $\mathcal{H}(N)$ which appears in Theorem 1.6 in [155] followed by an application of the relation (6.5.1). The only impediment is that the Johansson result appears in a variational form.

Let us recall Theorem 1.6 in [155]. According to Eq (1.21) in [155] (with $\gamma = 1$), the upper tail of $\mathcal{H}(N)$ satisfy the following large deviation principle

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}(\mathcal{H}(N) \geq Nz) = -J(z), \quad z \geq 4. \tag{6.5.3}$$

where the rate function J is given by

$$\begin{aligned} J(t) &:= \inf_{x\geq t}[G_V(x) - G_V(4)],\ t \geq 4,\ \text{where} \\ G_V(x) &:= -2\int_{\mathbb{R}} \log|x - r|\mathrm{d}\mu_V(r) + V(r),\ x \geq 4. \end{aligned} \tag{6.5.4}$$

Here $V(x) = x$ is defined on $[0, \infty)$, and the measure μ_V is the unique minimizer of $I_V(\mu)$ over $\mathcal{M}(\mathbb{R}_{\geq 0})$, the set of probability measures on $[0, \infty)$. $I_V(\cdot)$ is known as the *logarithmic entropy in presence of the external field V* and is given by

$$I_V(\mu) := -\iint_{\mathbb{R}^2} \log|x_1 - x_2|\mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2) + \int_{\mathbb{R}} V(x)\mathrm{d}\mu(x), \quad \mu \in \mathcal{M}(\mathbb{R}_{\geq 0}).$$

The logarithmic entropy $I_V(\mu)$ is well studied in both mathematical and physics literature and has several applications to random matrix theory and related models. We refer to [215] and [146] and the references there in for more details.

The form of the rate function defined in (6.5.4) is not exactly same as in [155]. However, one can show the rate function J defined in (6.5.4) is same as Eq (2.15) in [155] using the properties of minimizing measure (see Theorem 1.3 in [215] or Eq (1.6) in [111]). Such an expression for the rate function is derived using Coulomb gas theory. We refer to [155], [121], and [86] for treatment on the LDP problems of such nature.

Proof of Theorem 6.5.1. For clarity we split the proof into two steps.

Step 1. We claim that J defined in (6.5.4) has the following explicit expression.

$$J(t) = \sqrt{t^2 - 4t} - 2\log\frac{t - 2 + \sqrt{t^2 - 4t}}{2}, \quad t \geq 4. \tag{6.5.5}$$

We will prove (6.5.5) in Step 2. Here we assume its validity and conclude the proof of (6.5.2).

Towards this end, fix $y \in (0, 1)$ and K large enough such that $[y - \frac{1}{K}, y + \frac{1}{K}] \subset (0, 1)$. Recall the definition of M_t from (6.5.1). Note that for all large enough t, we have $\frac{4}{1-y+K^{-1}}M_t \leq t \leq \frac{4}{1-y-K^{-1}}M_t$. Thus

$$\mathbb{P}\big(\mathcal{H}(M_t) \geq \tfrac{4}{1-y+K^{-1}}M_t\big) \geq \mathbb{P}\left(-H_0^{q=1}(t) + \tfrac{t}{4} \geq \tfrac{t}{4}y\right) \geq \mathbb{P}\big(\mathcal{H}(M_t) \geq \tfrac{4}{1-y-K^{-1}}M_t\big).$$

Taking logarithms on each side, dividing by M_t and then taking $t \to \infty$ we get

$$\begin{aligned} -J\big(\tfrac{4}{1-y+K^{-1}}\big) &\geq \limsup_{t\to\infty} \frac{1}{M_t}\mathbb{P}\left(-H_0^{q=1}(t) + \tfrac{t}{4} \geq \tfrac{t}{4}y\right) \\ &\geq \liminf_{t\to\infty} \frac{1}{M_t}\mathbb{P}\left(-H_0^{q=1}(t) + \tfrac{t}{4} \geq \tfrac{t}{4}y\right) \geq -J\big(\tfrac{4}{1-y-K^{-1}}\big). \end{aligned} \tag{6.5.6}$$

where we used the upper tail large deviation principle for $\mathcal{H}(N)$ from (6.5.3). Observe that $\frac{M_t}{t} \to \frac{1-y}{4}$, and using (6.5.5) we see that

$$\frac{1-y}{4}J\big(\frac{4}{1-y}\big) = \frac{1-y}{4}\left(\frac{4\sqrt{y}}{1-y} - 2\log\frac{2(1+y) - 4\sqrt{y}}{2(1-y)}\right) = \Phi_+(y),$$

where Φ_+ is defined in (6.1.4). Thus taking $K \to \infty$ in (6.5.6) we arrive at (6.5.2).

Step 2. We now turn our attention to prove (6.5.5). It is well known that for $V(x) = x$, the minimizer μ_V is given by the *Marchenko-Pastur* measure (see Equation 3.3.2 and Proposition 5.3.7 in [146] with $\lambda = 1$):

$$\mathrm{d}\mu_V(x) = \frac{\sqrt{4x - x^2}}{2\pi x}\mathbf{1}_{x\in[0,4]}\mathrm{d}x.$$

Recall $G_V(x)$ defined in (6.5.4). Using the Cauchy Transform for μ_V (see the last unnumbered equation in Page 200 of [146]) we get that for $x > 4$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int \log|x-r|\mathrm{d}\mu_V(r) = \frac{1}{2} - \frac{\sqrt{x^2-4x}}{2x},$$

which implies $G_V(z) - G_V(4) = \int_4^z \frac{\sqrt{x^2-4x}}{x}\mathrm{d}x$. Thus $G_V(z) - G_V(4)$ is strictly increasing in y and whence by (6.5.4) we have

$$J(t) = \int_4^t \frac{\sqrt{x^2-4x}}{x}\mathrm{d}x.$$

To compute the above integral, we make the change of variable $x \mapsto \frac{(z+1)^2}{z}$ so that $\mathrm{d}x = (1 - \frac{1}{z^2})\mathrm{d}z$ and $x^2 - 4x = \frac{(z^2-1)^2}{z^2}$. Set $a = \frac{t-2}{2} + \frac{\sqrt{t^2-4t}}{2}$ to get

$$\int_4^t \frac{\sqrt{x^2-4x}}{x}\mathrm{d}x = \int_1^a \frac{(z-1)^2}{z^2}\mathrm{d}z = \left[z - \frac{1}{z} - 2\log z\right]_1^a = a - \frac{1}{a} - 2\log a.$$

Plugging the value of a we get (6.5.5) completing the proof. □

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