Path properties of KPZ models

Sayan Das

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Abstract

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In this thesis we investigate large deviation and path properties of a few models within the Kardar-Parisi-Zhang (KPZ) universality class.

The KPZ equation is the central object in the KPZ universality class. It is a stochastic PDE describing various objects in statistical mechanics such as random interface growth, directed polymers, interacting particle systems. In the first project we study one point upper tail large deviations of the KPZ equation $\mathcal{H}(t,x)$ started from narrow wedge initial data. We obtain precise expression of the upper tail LDP in the long time regime for the KPZ equation. We then extend our techniques and methods to obtain upper tail LDP for the asymmetric exclusion process model, which is a prelimit of the KPZ equation.

In the next direction, we investigate temporal path properties of the KPZ equation. We show that the upper and lower law of iterated logarithms for the rescaled KPZ temporal process occurs at a scale $(\log\log t)^{2/3}$ and $(\log\log t)^{1/3}$ respectively. We also compute the exact Hausdorff dimension of the upper level sets of the solution, i.e., the set of times when the rescaled solution exceeds $\alpha(\log\log t)^{2/3}$. This has relevance from the point of view of fractal geometry of the KPZ equation.

We next study superdiffusivity and localization features of the (1+1)-dimensional continuum directed random polymer whose free energy is given by the KPZ equation. We show that for a point-to-point polymer of length t and any $p \in (0,1)$, the point on the path which is pt distance away from the origin stays within a $O(1)$ stochastic window around a random point $\mathcal{M}_{p,t}$ that

depends on the environment. This provides an affirmative case of the folklore 'favorite region' conjecture. Furthermore, the quenched density of the point when centered around $\mathcal{M}_{p,t}$ converges in law to an explicit random density function as $t \to \infty$ without any scaling. The limiting random density is proportional to $e^{-\mathcal{R}(x)}$ where $\mathcal{R}(x)$ is a two-sided 3D Bessel process with diffusion coefficient 2. Our proof techniques also allow us to prove properties of the KPZ equation such as ergodicity and limiting Bessel behaviors around the maximum. In a follow up project, we show that the annealed law of polymer of length t, upon $t^{2/3}$ superdiffusive scaling, is tight (as $t \to \infty$) in the space of $C([0,1])$ valued random variables. On the other hand, as $t \to 0$, under diffusive scaling, we show that the annealed law of the polymer converges to Brownian bridge.

In the final part of this thesis, we focus on an integrable discrete half-space variant of the CDRP, called half-space log-gamma polymer. We consider the point-to-point log-gamma polymer of length $2N$ in a half-space with i.i.d. $\mathrm{Gamma}^{-1}(2\theta)$ distributed bulk weights and i.i.d. $\mathrm{Gamma}^{-1}(\alpha+\theta)$ distributed boundary weights for $\theta > 0$ and $\alpha > -\theta$. We establish the KPZ exponents (1/3 fluctuation and 2/3 transversal) for this model when $\alpha \geq 0$. In particular, in this regime, we show that after appropriate centering, the free energy process with spatial coordinate scaled by $N^{2/3}$ and fluctuations scaled by $N^{1/3}$ is tight. The primary technical contribution of our work is to construct the half-space log-gamma Gibbsian line ensemble and develop a toolbox for extracting tightness and absolute continuity results from minimal information about the top curve of such half-space line ensembles. This is the first study of half-space line ensembles. The $\alpha \geq 0$ regime correspond to a polymer measure which is not pinned at the boundary. In a companion work, we investigate the $\alpha < 0$ setting. We show that in this case, the endpoint of the point-to-line polymer stays within $O(1)$ window of the diagonal. We also show that the limiting quenched endpoint distribution of the polymer around the diagonal is given by a random probability mass function proportional to the exponential of a random walk with log-gamma type increments.

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Chapter 1: Introduction

The Kardar-Parisi-Zhang (KPZ) equation, a stochastic PDE which is formally written

$$\partial_t \mathcal{H} = \tfrac{1}{2}\partial_{xx}\mathcal{H} + \tfrac{1}{2}(\partial_x \mathcal{H})^2 + \xi, \qquad \mathcal{H} := \mathcal{H}(t,x) \qquad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{1.0.1}$$

Here $\xi = \xi(t,x)$ is the space time white noise. The KPZ equation was introduced in [217] for studying the fluctuation of growing interfaces and since then, it has found links to many systems including directed polymers, last passage percolation, interacting particle systems, and random matrices via its connections to the *KPZ universality class* (see [166, 278, 113, 281]).

The KPZ equation, as given in (1.0.1), is ill-posed as a stochastic PDE due to the presence of the nonlinear term $(\partial_x \mathcal{H})^2$. The physically relevant notion of solution for the KPZ equation is given by the *Cole-Hopf solution* which is defined as

$$\mathcal{H}(t,x) := \log \mathcal{Z}(t,x),$$

where $\mathcal{Z}(t,x)$ is the solution of the stochastic heat equation (SHE):

$$\partial_t \mathcal{Z} = \tfrac{1}{2}\partial_{xx}\mathcal{Z} + \xi\mathcal{Z}, \qquad \mathcal{Z} := \mathcal{Z}(t,x). \tag{1.0.2}$$

Throughout this paper, we work with the fundamental solution $\mathcal{Z}^{\mathbf{nw}}(t,x)$ of (1.0.2) and the associated Cole-Hopf solution $\mathcal{H}^{\mathbf{nw}}(t,x) := \log \mathcal{Z}^{\mathbf{nw}}(t,x)$ which corresponds to the SHE being started from the delta initial measure, i.e., $\mathcal{Z}^{\mathbf{nw}}(0,x) = \delta_{x=0}$. For any $t > 0$, $\mathcal{Z}^{\mathbf{nw}}(t,x)$ is strictly positive [168] which makes the Cole-Hopf solution $\mathcal{H}^{\mathbf{nw}}(t,x)$ well-defined. The corresponding initial data of the KPZ equation is termed as the *narrow wedge* initial data. We shall often drop the 'nw' superscript from the notation and just write $\mathcal{H}$ or $\mathcal{Z}$ for the rest of the text.

As an important model for the random interface growth, it is valuable to understand long time behavior of the KPZ equation. In this direction, [6] proved that as $t \to \infty$

$$t^{-1/3}\left(\mathcal{H}(2t,0)+\tfrac{t}{12}\right) \xrightarrow{d} \text{Tracy-Widom GUE.}$$

This result asserts that, for large t, the height $\mathcal{H}(2t,0)$ concentrates around $-\frac{t}{12}$, has typical deviations of order $t^{1/3}$, and after being scaled by $t^{-1/3}$ the fluctuations converge to the GUE Tracy–Widom distribution [308].

A natural question that follows the fluctuation result is establishing a Large Deviation Principle (LDP), namely questions about *tails* of the distribution of $\mathcal{H}(2t,0)+\frac{t}{12}$. We seek to find the probability of the rare events when the height $\mathcal{H}(2t,0)+\frac{t}{12}$ has a deviation of order t. In a joint work with Li-Cheng Tsai, we derive the upper tail LDP for the KPZ equation. We give an overview of the precise result and proof idea in Section 1.1. Using the same machinery, we also prove an upper-tail LDP for *asymmetric simple exclusion process*, which is a prelimit of the KPZ equation. The details are given in Section 1.1.

Being a non-linear PDE, KPZ equation exhibits remarkable fractal behavior. A systematic way to study the fractal behavior of the peaks of KPZ/SHE or in general any process Ψ_t is to first determine asymptotic heights of the peaks. This can be done by determining a non-random function $g(t)$ such that

$$\limsup_{t\to\infty} \Psi(t)/g(t) = 1 \quad a.s.$$

Then a natural way to study the peaks is to investigate the geometry of the level sets of the process: $\mathcal{P}_\Psi(\alpha) := \{t > 0 \mid \Psi(t)/g(t) \geq \alpha\}$. The geometry of these random sets can be measured by studying its macroscopic Hausdorff dimension [22]. If there are infinitely many possible values of α each producing a different value for $\dim(\mathcal{P}_\Psi(\alpha))$ (macroscopic Hausdorff dimension of $\mathcal{P}_\Psi(\alpha)$), it indicates a rich geometric structure among the level sets. Such a process is known as *multifractal*. Whereas if $\dim(\mathcal{P}_\Psi(\alpha)) = 1$ for all $\alpha \in (0,1)$, we call it *monofractal*. In a joint work with Promit Ghosal, we initiated the study of the peaks of the KPZ equation in the temporal direction.

We show that KPZ temporal process peaks are monofractal. However, upon an exponential time change it becomes multifractal. We give a brief overview of our results in this direction in Section 1.1.

In the second half of the thesis, we study the model of directed polymers in random environments (DPRE). DPRE considers an up-right random walk on the integer lattice, whose paths – considered the 'polymer' – are reweighted according to a random environment that refreshes at each time step. Based on physics predictions, two phenomena are conjectured in this model: **(a)** The n-length polymer path fluctuates in the order of $n^{2/3}$ (superdiffusive). This is in sharp contrast with the usual random walk diffusive behavior where we see $\sqrt{n}$ fluctuations. **(b)** Upon fixing the environment, the polymer exhibits localization phenomena. Large values in the environment tend to attract the random walker and possibly force it to follow a favorite path dictated by the environment. Although there is immense progress in the rigorous understanding of several aspects of the above two phenomena in the last two decades, the full resolution of these conjectures is far from being settled.

In this thesis, we focus on the continuum directed random polymer (CDRP) model which arises as a universal scaling limit of discrete directed polymers in the intermediate disorder regime. In two joint works with Weitao Zhu, we have settled the above two conjectures for CDRP model. We show that the paths of the CDRP are superdiffusive and upon fixing the environment the paths localized within an $O(1)$ window around the favorite point. We refer to Section 1.2 for more details.

The final part of this thesis is focussed on understanding the geometry of *half-space polymers*. Half-space polymers are a variant of DPREs where the polymer interacts with a given surface. Mathematically they are modeled by restricting the paths to stay on or above the diagonal and introducing a different random weight on the diagonal. Half-space models are interesting as they show certain *depinning transition*. When diagonal weights are not too large, the polymers are conjectured to behave like the full-space ones (unbound phase), whereas if the weights are large enough, the polymers are believed to be pinned to the diagonal (bound phase).

In two recent works, we establish the above picture for a very particular integrable half-space polymer called half-space log-gamma polymer. In a joint work with Ivan Corwin and Guillaume Barraquand, we proved superdiffusivity for this particular polymer in the unbound phase. Our proof relies on the novel construction of the half-space line ensemble for the underlying model. In a companion work with Weitao Zhu, we showed that the endpoint of the half-space log-gamma polymer are localized within $O(1)$ window around the diagonal in the bound phase, i.e., the polymer is pinned to the diagonal. We refer to Section 1.2 for detailed overview of the results in this direction.

1.1 Large Deviations and fractal properties of integrable models

1.1.1 LDP for KPZ equation

This subsection serves as a summary for Chapter 2. We focus on the large deviation problem for the KPZ equation (1.0.1). We establish the *first rigorous proof* of the upper-tail LDP of $\mathcal{H}(2t,0)+\frac{t}{12}$ with the rate function $\Phi_+(s)=\frac{4}{3}s^{3/2}$. Our result confirms the existing physics predictions [243] and also [213].

Theorem 1.1.1 ([131])**.** *For all $s>0$ we have*

$$t^{-1}\log \mathbb{P}(\mathcal{H}(2t,0)+\tfrac{t}{12}>st)\to -\tfrac{4}{3}s^{3/2}. \tag{1.1.1}$$

The above result is obtained by computing $t\to\infty$ asymptotic of the p-th moment of $\mathcal{Z}(2t,0)$, for any real $p>0$. Moments of SHE are historically connected to the concept of *intermittency*. Intermittency in random media is an active area of research in both mathematics and physics literature for the last few decades. Mathematically, it is characterized by rapid growth of moments of a process. Formally, we say a process Ψ_t is intermittent if γ_k/k is strictly increasing where $\gamma_k(\Psi):=\lim_{t\to\infty} t^{-1}\log\mathbb{E}[\Psi_t^k]$ are known as the Lyapunov exponents for Ψ. There are heuristics [69] suggesting that this property indicates a presence of a peculiar fractal structure in the Ψ process.

In case of the SHE, it is known since the physics works of Kardar [216] that the SHE is intermittent. Kardar also predicted $\gamma_p(\mathcal{Z}(0,\cdot)) = \frac{p^3-p}{24}$ for all $p > 0$. This formula was later shown to be true in [94, 115] for all positive integers $p \in \mathbb{N}$. We showed Kardar's formula holds for all $p > 0$.

Theorem 1.1.2 ([131])**.** *For all $p > 0$ we have*

$$t^{-1}\log \mathbb{E}(\mathcal{Z}(2t,0)^p) \to -\tfrac{p^3-p}{24}. \tag{1.1.2}$$

In fact, Theorem 1.1.1 is obtained from Theorem 1.1.2 by taking standard Legendre transform.

Integer moment formulas for SHE [179] and explicit distribution formulas for KPZ equation [6] exist in the literature and have been used to derive integer Lyapunov exponents [115] and sub-optimal tail bounds for KPZ equation [122] respectively. However, it is not clear how to extend or improve their analysis to yield the above theorem.

In [131], we take an unconventional route. We start with a Fredholm determinant formula for the laplace transform of $\mathcal{Z}(t,0)e^{t/24}$ from [6]:

$$\mathbb{E}[\exp(-s\mathcal{Z}(t,0)e^{t/24})] = \det(I - K_{s,t}) = 1 - \int_{\mathbb{R}} K_{s,t}(x,x)dx + \text{higher order.}$$

where $K_{s,t}$ is an explicit kernel involving Airy function (see Eq (1.11) in [131]) and 'higher order' term is an infinite series of integrals of determinants of matrices (increasing in size) involving the kernel $K_{s,t}$. We then insert the above formula in the following elementary fractional moment formula:

$$\mathbb{E}\left[U^{n-1+\alpha}\right] = \frac{(-1)^n}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}\frac{d^n}{ds^n}\mathbb{E}[e^{-sU}]ds, \quad n \in \mathbb{N},\ \alpha \in [0,1),$$

with $U := \mathcal{Z}(t,0)e^{t/24}$. It turns out this formula is quite amenable to our analysis, yielding precise Lyapunov exponents. The leading contribution comes from the $\int_{\mathbb{R}} K_{s,t}(x,x)dx$ term, whereas the higher order term can be shown to be sub-dominant.

Using our formulas as an input, [180] has extended Kardar's formula to other initial data. The robustness of our proof approach makes it applicable for other integrable models. [248] successfully carried our approach to solve the same problem for half-space KPZ. In a joint work with Weitao Zhu [135], we carry out this program for asymmetric simple exclusion process (ASEP) to derive its upper-tail LDP.

1.1.2 LDP for Asymmetric Simple Exclusion Process

This subsection serves as a summary for Chapter 3. Asymmetric Simple Exclusion Process (ASEP) is a classical example of interacting particle systems and is one of the pre-limiting model of the KPZ equation. It is a continuous time Markov chain on particle configurations living on integer lattice. Each site $i \in \mathbb{Z}$ can be occupied by at most one particle, which has an independent exponential clock of rate 1. When the clock rings, the particle jumps to the right with probability q or to the left with probability $p = 1 - q$ $(q > p)$. However, the jump is only permissible when the target site is unoccupied. At time 0, all negative integer sites has a particle.

The observable of interest in ASEP is $H(t, 0)$:= the number of particles to the right of zero at time t. It acts as the height function of interface growth for ASEP. It is well known that $t^{-1}H(0, \frac{t}{q-p}) \to \frac{1}{4}$. In a series of works [307, 306, 306], Tracy and Widom exploit the integrability of ASEP and showed $-H(0, \frac{t}{q-p})$, upon centering and appropriate scaling, has Tracy-Widom (TW) GUE fluctuations.

This leads to the natural question of large deviations for ASEP. In a joint work with Weitao Zhu [135], we obtained the following result in relation to the upper-tail LDP.

Theorem 1.1.3 ([135])**.** *Fix* $q \in (\frac{1}{2}, 1)$. *For any* $y \in (0, 1)$ *we have*

$$t^{-1} \log \mathbb{P}(-H(0, \tfrac{t}{q-p}) + \tfrac{t}{4} > \tfrac{yt}{4}) \to -[\sqrt{y} - (1 - y) \tanh^{-1}(\sqrt{y})].$$

Note that for $y > 1$, the above probability is zero as $H(0, \cdot)$ is non-negative. Prior to our work, [125] obtained a one-sided large deviation bound for the upper tail of the ASEP utilizing distributional formulas from [305]. Their rate function coincides with ours for $y < y_0$ for some explicit threshold $y_0 \in (0,1)$. For $y \in (y_0, 1)$, their rate function is sub-optimal. Their proof is based on contour analysis where one deforms the contours to pass through critical points to obtain the leading behavior. This threshold y_0 appears in their arguments as their choice of contours was unattainable beyond this threshold.

In [135] we follow the approach of [131], producing first Lyapunov exponents for $(p/q)^{H_0(t)}$ using certain known Fredholm determinant formulas [72]. However, the underlying kernel in the Fredholm determinant here is asymmetric, exhibits a periodic behavior, and much more intricate that its KPZ counterpart. To extract the precise asymptotics for the leading term here, one needs to perform careful steepest descent analysis on the ASEP kernel, where the periodic nature of the kernel results in infinitely many critical points. A major technical challenge in our proof is to argue how the contribution from only one of the critical points dominates the those from the rest.

1.1.3 Fractal properties of the KPZ equation

This subsection serves as a summary for Chapter 4. While discussing the LDP for KPZ equation, we mentioned that intermittency indicates a peculiar fractal behavior in complex random media. The fractal behavior of a process can also be studied through the lens of law of iterated logarithms (LIL) as explained in the introduction. In the context of the KPZ equation, this LIL framework was successfully carried out for the spatial process in [221]. In a joint work with Promit Ghosal [128], we initiated the study of the peaks of the KPZ equation in the temporal direction.

Theorem 1.1.4 ([128])**.** *Consider the normalized KPZ height function $\mathfrak{h}_t := t^{-1/3}(\mathcal{H}(t,0) + \frac{t}{24})$. We have*

$$\limsup_{t\to\infty} \frac{\mathfrak{h}_t}{(\log\log t)^{2/3}} = \left(\tfrac{3}{4\sqrt{2}}\right)^{2/3} \ a.s. \tag{1.1.3}$$

Furthermore, $\mathfrak{h}_t$ is monofractal. However, upon an exponential transformation, $\mathfrak{h}_{e^t}$ becomes multifractal.

Note that the constant in (1.1.3) also appears in the rate function in (1.1.1). This is not a complete coincidence. Indeed, this constant can be anticipated by the shallow one-point tail behavior of $\mathfrak{h}_t$. In our analysis, we improved upon the existing one point tail estimates in [115] to make it work in our setting. However, to conclude such a LIL type result, one also requires a good understanding of $\mathfrak{h}_t$ at a process level.

Study of the temporal process is arguably more difficult that the spatial one due to lack of rich structure (such as Gibbs resampling property mentioned in Section 2.1). One of the important piece in analyzing the process $\mathfrak{h}_t$, understanding its increments in particular, is short time tail bounds for the KPZ equation. In [128] we provided first such tail bounds. Using this, along with Gibbs property of the KPZ line ensemble, and a multi-point convolution formula coming from properties of SHE, two of our main contributions are:

- **Computing tail probabilities for the difference in fluctuations at two times.** This result was proven when the time points are sufficiently far apart in [117]. We fill the gap in [117] by establishing similar estimates that works when two times are arbitrarily close. This leads to temporal modulus of continuity estimates for the KPZ equation.

- **Proving an 'independence structure' of the process $\mathfrak{h}_t$.** At a two-point level, roughly speaking, we produce a proxy for $\mathfrak{h}_{\alpha t}$ that is independent of $\mathfrak{h}_t$. The proxy is 'close' to $\mathfrak{h}_{\alpha t}$ for α large. A different version of the independence structure in terms of correlation was proven in [117]. A corollary of our independence structure result is that $\mathfrak{h}_t$ decorrelates on a multiplicative scale. On a high level, this is the reason for mono and multifractal behavior. Upon exponential transformation, decorrelation happens on an additive scale which then allows more chaotic behavior of the height function.

1.2 Directed Polymers in random environments

Directed polymers in random environments (DPREs) were first introduced in statistical physics and math literature [202, 206, 61] to study the phase boundary of the Ising model with random impurities. In the $(1+1)$- dimension, they are modeled by up-right paths on $\mathbb{Z}^2$ lattice (see Figure 1.1). The random environment is specified by a collection of i.i.d. random variables $\{\omega_{i,j} \mid (i,j) \in \mathbb{Z}_+ \times \mathbb{Z}\}$.

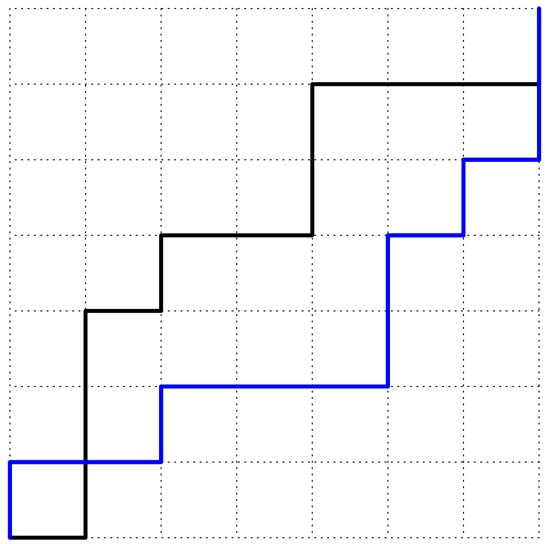

Figure 1.1: DPREs.

The **point-to-point** polymer measure on the set of all up-right paths starting at origin and ending at (n,n) is then defined as

$$\mathbf{P}^{\omega}_{n,\beta}(S) = \frac{1}{Z^{\omega}_{n,\beta}} e^{\beta \sum_{i=1}^{n} \omega_{i,S_i}} \cdot \mathbf{P}(S), \tag{1.2.1}$$

where $\mathbf{P}(S)$ is the uniform measure on set of all up-right paths starting at origin and ending at (n,n), β is the inverse temperature, and $Z^{\omega}_{n,\beta}$ is the partition function. As evident from (1.2.1), in the polymer measure, there is a competition between the *entropy* of paths and the *disorder strength* of the environment. In fact for every $\beta > 0$, the polymers are in *strong disorder* where disorder strength dominates. The following two phenomena are conjectured:

- *Superdiffusivity*: The polymer measure is believed to be in the KPZ universality class and paths have typical fluctuations of the order $n^{2/3}$ (compared to $\sqrt{n}$ order diffusive behavior at $\beta = 0$) (see physics works [202, 203, 218, 235]). This conjectured phenomenon is known as superdiffusion.

- *Localization and the favorite region conjecture*: The polymer exhibits certain localization phenomena. The favorite region conjecture speculates that any point on the path of a point-to-point directed polymer is asymptotically localized in a region of stochastically bounded diameter (see [42, 44]).

Although there is an immense progress in understanding several aspects of the above two phenomena in last two decades (see [274, 253, 294, 44, 29] and the references therein), the full resolution of these conjectures is far from being settled. In a series of two joint works with Weitao Zhu ([132] and [133]), we settle these two questions for continuous polymers. We showed pathwise tightness of continuous polymers under superdiffusive scaling and pointwise localization of continuous polymers. Both of these results are not proven for any discrete polymer model.

1.2.1 Continuum Directed Random Polymer

This subsection serves as a summary of Chapter 5 and 6. In the seminal work, [5] considered *an intermediate disordered regime* where they took $\beta = \beta_n = n^{-1/4}$ with n being the length of the polymer. [5] showed that the partition function Z^{ω}_{n,β_n} has a universal scaling limit given by the solution of the Stochastic Heat Equation (SHE) with multiplicative noise (when ω has finite exponential moments). Furthermore, under the diffusive scaling, the polymer path itself converges to a universal object called the **Continuous Directed Random Polymer** (CDRP) which depends on a continuum random environment given by the space-time white noise.

Following [4], we define the CDRP model using the SHE with multiplicative noise as our building blocks. Namely, we consider a four-parameter random field $\mathcal{Z}(x, s; y, t)$ defined on $\{(x, s; y, t) \in \mathbb{R}^4 : s < t\}$. For each $(x, s) \in \mathbb{R} \times \mathbb{R}$, $(y, t) \mapsto \mathcal{Z}(x, s; y, t)$ is the solution of the SHE starting from location x at time s, i.e., the unique solution of

$$\partial_t \mathcal{Z} = \tfrac{1}{2}\partial_{xx}\mathcal{Z} + \mathcal{Z} \cdot \xi, \qquad (y, t) \in \mathbb{R} \times (s, \infty),$$

with Dirac delta initial data $\lim_{t\downarrow s} \mathcal{Z}(x, s; y, t) = \delta(x - y)$. Here $\xi = \xi(t, x)$ is the space-time white noise.

Definition 1.2.1 (Point-to-point CDRP)**.** Conditioned on the white noise ξ, let $\mathbf{P}^{\xi}$ be a measure on $C([0,t])$ whose finite dimensional distribution is given by

$$\mathbf{P}^{\xi}(X(t_1) \in dx_1, \dots, X(t_k) \in dx_k) = \frac{1}{\mathcal{Z}(0,0;0,t)} \prod_{j=0}^{k} \mathcal{Z}(x_j, t_j, ; x_{j+1}, t_{j+1}) dx_1 \dots dx_k. \quad (1.2.2)$$

for $s = t_0 \le t_1 < \cdots < t_k \le t_{k+1} = t$, with $x_0 = 0$ and $x_{k+1} = 0$. We write $X \sim \mathsf{CDRP}_t$ when $X(\cdot)$ is a random continuous function on $[0,t]$ with $X(0) = X(t) = 0$ and its finite dimensional distributions given by (1.2.2) conditioned on ξ.

The following theorem summarizes our key findings for point-to-point CDRP.

Theorem 1.2.2 ([132, 133])**.** *For each $t > 0$ consider $X \sim \mathsf{CDRP}_t$.*

(a) (Pointwise Localization) For each $t > 0$ and $p \in [0,1]$, there exists a random variable $\mathcal{M}_{p,t}$ dependent only on the environment, such that $|X(pt) - \mathcal{M}_{p,t}| = O(1)$ as $t \to \infty$. Furthermore, the quenched density of $X(pt)$ when centered around $\mathcal{M}_{p,t}$ converges in distribution to an explicit random density proportional to $e^{-\sqrt{2}\mathcal{R}(x)}dx$ where $\mathcal{R}(x)$ is a standard two-sided Bessel process.

(b) (Pathwise Tightness) The annealed law of $(t^{-2/3}X(pt))_{p\in[0,1]}$ when viewed as a random variable in the space of $C[0,1]$ is tight as $t \to \infty$. For each $p \in [0,1]$, $t^{-2/3}X(pt)$ weakly converges to a non-trivial distribution as $t \to \infty$.

A similar localization result for the midpoint of point-to-point stationary log-gamma polymer was established in [101] using Burke property of the model [294]. The Burke property allows one to write the quenched density of the midpoint in terms of exponent of a simple symmetric random walk (SSRW). It then suffices to study the behavior SSRW around maximizer. However, [101] technique does not work for other points besides the midpoint and for the non-stationary log-gamma model.

The principle tool for the proof of (a) is the Gibbs resampling property [110] enjoyed by the KPZ equation $\mathcal{H}(t,x) := \log \mathcal{Z}(0,0;t,x)$. For each fixed $t > 0$, the process $\mathcal{H}(t,\cdot)$ can be viewed as the top curve of the *KPZ line ensemble* [110]. The law of the first k curves restricted to a fixed interval $[a,b]$ (blue part in Figure 1.2 with $k = 3$) conditioned on all the information outside is absolutely continuous w.r.t. k Brownian bridges on $[a,b]$ with appropriate endpoints with an explicit Radon-Nikodym derivative.

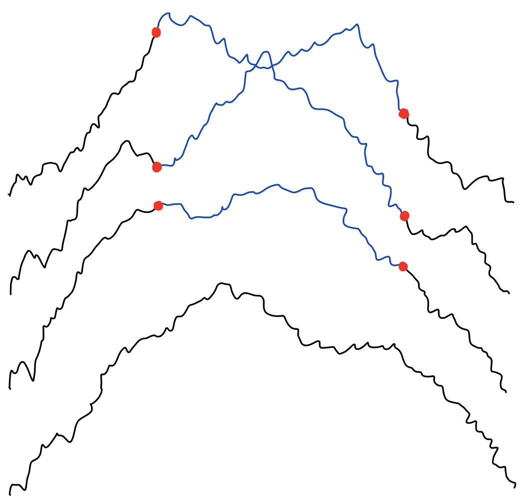

Figure 1.2: KPZ line ensemble.

KPZ equation comes up, as via (6.1.5) the quenched density of $X(pt)$ can be written as function of two independent copies of the KPZ equation. The choice of $\mathcal{M}_{p,t}$ is given by the random mode of the quenched density. It is not hard to check that the quenched density upon this random centering is proportional to

$$x \mapsto \left[\mathcal{H}_1(\mathcal{M}_{p,t}, pt) + \mathcal{H}_2(\mathcal{M}_{p,t}, (1-p)t)\right] - \left[\mathcal{H}_1(\mathcal{M}_{p,t}+x, pt) + \mathcal{H}_2(\mathcal{M}_{p,t}+x, (1-p)t)\right] \tag{1.2.3}$$

where $\mathcal{M}_{p,t} := \operatorname{argmax}_{x\in\mathbb{R}}(\mathcal{H}_1(pt,\cdot) + \mathcal{H}_2((1-p)t,\cdot))$ and $\mathcal{H}_1, \mathcal{H}_2$ are two independent copies of the KPZ equation. Although traditional tools associated to the Gibbs property such as stochastic monotonicity or the Gibbs property on an interval (described above informally) has been used extensively in the literature, such tools are inapplicable in the analysis the process in (1.2.3) because of the random centering by $\mathcal{M}_{p,t}$.

Fix any $K > 0$ and let $\mathcal{M}^*_{p,t} := \operatorname{argmax}_{|x|\le Kt^{2/3}}(\mathcal{H}_1(pt,\cdot) + \mathcal{H}_2((1-p)t,\cdot))$. Choosing K large enough, one can ensure $\mathcal{M}_{p,t} = \mathcal{M}^*_{p,t}$ with high probability. Hence it suffices to work with these finite maximizers. Coming back to the line ensemble framework, there are two sets of KPZ line ensemble corresponding to each $\mathcal{H}_i$ for $i = 1, 2$. In [132], we give an explicit way to resample the top curves of both the line ensembles simultaneously over intervals of the form $[\mathcal{M}^*_{p,t} - a_t, \mathcal{M}^*_{p,t} + b_t] \subset [Kt^{2/3}, Kt^{2/3}]$. This is done by first analyzing the behavior of two independent copies of Brownian motions around its joint maximizer and then utilizing the explicit description of the Radon-Nikodym derivative. The above solution has the potential to generalize to other integrable models such as the non-stationary log-gamma polymer model.

Our localization theorem and results on the random mode $\mathcal{M}_{p,t}$ stated in [132] leads to pointwise tightness of $t^{-2/3}X(pt)$ for each $p \in [0,1]$. However, to upgrade the result to pathwise tightness (b), one needs to control the fluctuations of the path on mesoscopic scales. One of the ingredients in establishing such control is the short time tail bounds in the KPZ equation developed in one of my previous paper [128]. Utilizing this, in [133], we produce quantitative modulus of continuity estimates for the polymer paths which eventually leads to pathwise tightness result. We mention that similar pointwise tightness result are known for stationary and non-stationary log-gamma polymer model in [294, 29] but pathwise tightness is not shown for any discrete polymer model due to lack of such short-time estimates. In fact, assuming a conjecture about *KPZ sheet to Airy sheet convergence*, in [133] we show that the process limit of $t^{-2/3}X(pt)$ is given by the *geodesic of directed landscape*, an universal limiting object in the KPZ universality class [138].

1.2.2 Half-space log-gamma polymers

In the final chapters, Chapters 7 and 8, we focus on half-space log-gamma polymers. Half-space directed polymers are variants of DPREs where the polymers interact with a given surface. Mathematically, in this model the paths are restricted to the octant and the weights on the diagonal are of different strength. It has been predicted in physics literature [215] that such polymers undergo a phase transition called 'depinning transition'. When the strength of the diagonal is under a threshold, the model is expected to behave like the full-space ones, whereas for large enough strength of the diagonal, the free energy of the model is conjectured to have Gaussian fluctuations with polymer paths within $O(1)$ window of the diagonal.

The depinning transition has been recently proven in few of the solvable models in terms of the free energy for the diagonal [34, 205]. However, the transversal exponent is not established so far in any of the half-space models. In an ongoing work with Ivan Corwin (my advisor) and Guillaume Barraquand, we take the first step in establishing such transversal exponent in half-space models.

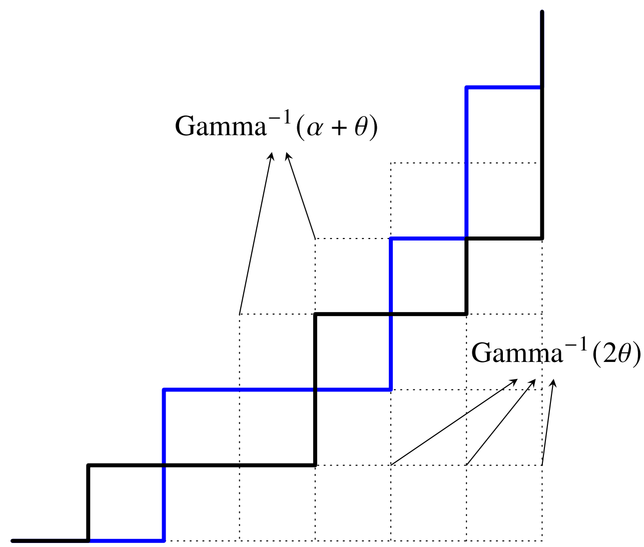

Figure 1.3: HSLG polymers.

We work with the solvable half-space log-gamma (HSLG) polymer model (Figure 1.3) defined via following weights and partition function:

$$W_{i,j} \sim \begin{cases} \mathrm{Gamma}^{-1}(\alpha+\theta) & i=j \\ \mathrm{Gamma}^{-1}(2\theta) & j<i \end{cases}, \quad Z^{\mathrm{half}}(m,n) := \sum_{\pi:(1,1)\to(m,n)} \prod_{(i,j)\in\pi} W_{i,j},$$

where $\theta > 0$, $\alpha > -\theta$, and in the above sum paths are restricted to the octant (see Figure 1.3). This particular choice of weights makes the model solvable via combinatorial techniques [263, 121, 260] and via eigenrelations of Macdonald polynomials [25]. For this model, we prove the following result.

Theorem 1.2.3 ([28]). *Fix $\theta > 0$, $\alpha \geq 0$ and $r > 0$. The law of*

$$(N^{-1/3}[\log Z^{\text{half}}(N + tN^{2/3}, N - tN^{2/3}) - cN])_{t\in[0,r]}$$

when viewed as a random variable in the space of $C[0, r]$ is tight as $N \to \infty$ for some explicit constant $c = c(\theta) \in \mathbb{R}$.

The above result establishes the $2/3$ transversal exponent and the $1/3$ fluctuation exponent away from the diagonal. The main technique in our proof is the novel construction of the HSLG line ensemble using geometric RSK [263, 121, 260]. Although it is now fairly well known how to extract spatial tightness for the line ensemble once one-point uniform tightness of the top-curve along a parabolic curvature is established ([29] and references therein), in our case such tightness result is not available away from the diagonal. To tackle this, we rely on recently proven fluctuations results for half-space point-to-line log-gamma polymer [34] to establish a weaker version of one-point uniform tightness. Utilizing the line ensemble framework, we then establish both the exponents simultaneously.

The above theorem does not covers the case when $\alpha \in (-\theta, 0)$. In fact, the situation in this case is radically different. For $\alpha < 0$, the polymer is pinned to the diagonal and the free energy does not exhibit KPZ fluctuations. In a companion work with Weitao Zhu, we investigate the geometry of the half-space log-gamma polymers for $\alpha < 0$ case (bound phase).

Let Π_N^{half} be the set of all upright lattice paths of length $2N - 2$ starting from $(1, 1)$ that are confined to the half-space $\mathcal{I}^-$ (see Figure 8.2). Given the weights above, the half-space log-gamma ($\mathcal{HSLG}$) polymer is a random measure on Π_N^{half} defined as

$$\mathbf{P}^W(\pi) = \frac{1}{Z(N)} \prod_{(i,j)\in\pi} W_{i,j} \cdot \mathbf{1}_{\pi\in\Pi_N^{\text{half}}}, \tag{1.2.4}$$

where $Z(N)$ is the normalizing constant. Our main result below confirms that in the bound phase, i.e., when $\alpha \in (-\theta, 0)$, the endpoint of the $\mathcal{HSLG}$ polymer is within $O(1)$ window of the diagonal and is the first such result to capture the “pinning" phenomenon of the half-space polymer measure to the diagonal.

Theorem 1.2.4 ([134])**.** *Fix $\theta > 0$ and $\alpha \in (-\theta, 0)$ and consider the random measure $\mathbf{P}^W$ from* (1.2.4)*. For a path $\pi \in \Pi_N^{\mathrm{half}}$, we denote $\pi(2N-2)$ as the height (i.e., y-coordinate) of the endpoint of the polymer. We have*

$$\limsup_{k\to\infty} \limsup_{N\to\infty} \mathbf{P}^W(\pi(2N-2) \le N-k) = 0, \qquad \textit{in probability}. \tag{1.2.5}$$

We also show that the limiting quenched endpoint distribution of the polymer around the diagonal is given by a random probability mass function proportional to the exponential of a random walk with log-gamma type increments. Our proof relies on inputs from the recently developed half-space log-gamma Gibbsian line ensemble Chapter 7, one-point fluctuation results for point-to-(partial)line half-space log-partition functions from [34] and the localization techniques from Chapter 5. At the heart of our argument lies an innovative combinatorial argument that bridges the aforementioned inputs and enables our proof. We refer to Chapter 8 for more details.

1.3 Other Works

During my time as a graduate student, I have worked on other research projects within KPZ area as well as used probability theory to explore other research areas such as network sampling, random quadratic forms, graph coloring and large deviation aspects of random permutations. In this section, I give a brief summary of these works.

1.3.1 Discrete β-ensembles

Note that Theorem 1.1.3 does not cover the special $q = 1$ case which corresponds to totally ASEP (TASEP). TASEP is a much more well understood object. It has connections to Exponential Last Passage Percolation (LPP), log-gases, zero-temperature version of DPREs. In particular, the LDP problem for TASEP was solved by Johansson [211] exploiting the following log-gas connection: $\mathbf{P}(H^{q=1}(t,0) \le zt) = \mathbf{P}(\lambda_{\max}^{(\lfloor zt\rfloor+1} \ge t)$, where $\lambda_{\max}^{(n)}$ is the largest particle of an n-particle system (called continuous log-gas) distributed as

$$\prod_{1\le i\le j\le n}(\lambda_i-\lambda_j)^2\prod_{i=1}^{n}e^{-\lambda_i}1_{\lambda_i>0}d\lambda_i.$$

Using the technology from potential theory, combined with combinatorial and probabilistic arguments, Johansson essentially proved an LDP for the largest particle of the above ensemble. In fact, his argument is quite general and works for discrete log-gases as well where particles now live on $\mathbb{Z}_{>0}$.

Later, [73] proposed a new discrete version of continuous log-gases called discrete β-ensembles (DβE) where the particles now live on different shifted lattices and the interaction term is given by certain ratios of gamma functions. This discretization is of high interest as it bears connections to integrable probability, discrete Selberg integrals, Jack measures and satisfies discrete loop equations. Although loop equations are immensely useful in studying global and edge fluctuations of largest particle in DβE [73, 190], it is not clear how to use them to derive an LDP for the largest particle.

In a joint work with Evgeni Dimitrov [127], we settled this question by proving an LDP (both upper and lower tail) for the largest particle of DβE. To prove this result, we adopt the potential theory based approach followed by Johansson. However, the argument present in [**jo2**] heavily relies on the symmetry of the interaction and the fact that particles live on the same lattice, both of which are absent in DβE. The combinatorial part of the argument in [**jo2**] essentially breaks down in DβE. Thus we had to find a significantly more involved set of arguments involving novel combinatorial constructions as well as detailed estimates of number of particles in appropriate windows.

1.3.2 Short time peaks of the KPZ equation

Another line of related research is the study of short time peaks, i.e., the peaks of the local increments $\Psi_{t+\varepsilon} - \Psi_t$ as $\varepsilon \downarrow 0$. Such behaviors determine local growth of the process and also potential fractal behavior in short time scale. In this line, in [126] I have studied the temporal increments of the KPZ equation. The main result in [126] can be stated as follows.

Theorem 1.3.1 ([126])**.** *We have*

$$\limsup_{\varepsilon\downarrow 0} \frac{\mathcal{H}(0,t+\varepsilon)-\mathcal{H}(t,0)}{\varepsilon^{1/4}\sqrt{\log\log(1/\varepsilon)}} = (8/\pi)^{1/4} \ \ a.s.$$

Furthermore for each $\alpha \in (0,1)$ *almost surely we have*

$$\dim\left\{t \in [1,2] \mid \limsup_{\varepsilon\downarrow 0} \frac{\mathcal{H}(0,t+\varepsilon)-\mathcal{H}(t,0)}{\varepsilon^{1/4}\sqrt{\log(1/\varepsilon)}} \geq \alpha(8/\pi)^{1/4}\right\} = 1-\alpha^2, \tag{1.3.1}$$

where dim *denotes the usual Hausdorff dimension.*

The first result above determines the precise asymptotic height of the short time peaks. Whereas the second result indicates there are exceptional time points with unusually large local growth (log instead of log log in the denominator) and such time points have a rich multifractal structure. The proof of Theorem 1.3.1 uses tools from SPDE theory and is applicable to wide range of initial data. The key idea is to show the multiplicative SHE behaves like additive SHE on small scales. The later quantity is known to be a fractional Brownian Motion of index $\frac{1}{4}$ in temporal direction which essentially yields the above two results.

1.3.3 Long and short-time peaks of the KPZ fixed point

The study of fractal properties can also be undertaken in other models of the KPZ universality class. In this line, recently with Promit Ghosal and Yier Lin [129], we have explored the fractal aspects of the *KPZ fixed point*. The KPZ fixed point $\mathbf{H}_t(x)$ [251, 138] is conjecturally the universal scaling limit of all models in the KPZ universality class. $t \mapsto \mathbf{H}_t$ can be viewed as a Markov process with explicit transition probabilities and initial condition $\mathbf{H}_0$. It has been shown very recently that KPZ equation under 1:2:3 scaling converges to the KPZ fixed point [315, 280]. Being a scaling limit, KPZ fixed point possesses interesting fractal properties which has been the subject of intense recent study (see [170] for a recent survey). Our work [129] follows a host of effort that attempts to unravel such fractal properties of the KPZ fixed point.

Theorem 1.3.2 ([129])**.** *For a large class of initial data $\mathbf{H}_0$, almost surely we have*

$$\limsup_{\varepsilon \downarrow 0} \frac{\mathbf{H}_{1+\varepsilon}(0) - \mathbf{H}_1(0)}{\varepsilon^{\frac{1}{3}} (\log \log \varepsilon^{-1})^{2/3}} = (3/2)^{2/3}.$$

We also proved a long time LIL result for the KPZ fixed point in the spirit of (1.1.3). In fact, similar mono and multifractal behavior can also be established using our proof techniques (not proved explicitly in [129]).

The KPZ fixed point does not have any SPDE description. Neither it satisfies the Gibbs property for general initial data $\mathbf{H}_0$. Thus both of the previous proof ideas are not applicable. Instead, the proof of Theorem 1.3.2 relies on rich geometric structure of the directed landscape (an object related to the KPZ fixed point) and certain basic coupling properties that the KPZ fixed point inherits from TASEP.

1.3.4 Motif estimation

In network analysis, often due to the massive size of network only a sample of the network is observed in practice. The central statistical question then is how to efficiently estimate global features of the parent network, that accounts for the bias and variability induced by the sampling paradigm. In this line, with Bhaswar Bhattacharya and Sumit Mukherjee, we consider the problem of motif estimation, that is, counting the number of copies of a fixed graph H under subgraph sampling model (SSM) [54]. Given a large parent graph G_n, in SSM, each vertex of G_n is sampled independently with probability $p_n \in (0, 1)$ and the subgraph induced by these sampled vertices is observed. Let T_n be the number of copies of H in the sample induced from G_n. Set $Z_n :=$

$[T_n - \mathbb{E}(T_n)]/\sqrt{\mathrm{Var}(T_n)}$. In [54], we derived necessary and sufficient conditions for Z_n to have Gaussian fluctuations. We showed that for $p_n \leq \frac{1}{20}$, the Wasserstein distance between Z_n and $N(0,1)$ is control by the fourth moment of Z_n:

$$\mathrm{Wass}(Z_n, N(0,1)) \lesssim \sqrt{\mathbb{E}(Z_n^4) - 3}.$$

The above result can be viewed as a fourth-moment phenomenon which since the pioneer works of Nualart and Peccati [262] has emerged as a unifying principle governing the central limit theorems for various non-linear functionals of random fields [261]. The proof of the above result relies on Stein's method for normal approximation.

1.3.5 Graph coloring

Given a graph G_n with uniformly random vertex coloring c colors, we consider S_n, number of monochromatic copies of H in G_n. The statistic S_n arises in a variety of contexts. When H is an edge, it counts the number of pairs of friends on friendship network G_n that have the same birthday, thus generalizing the classical birthday paradox. The asymptotic distribution of S_n has been studied extensively in the literature for H being an edge [55] or triangle [56] and for general H under the assumption that G_n are dense [57]. However, there has been little prior work giving precise conditions describing when $S_n(H)$ has a Gaussian limit for a general sequence of G_n. With Zoe Himwich and Nitya Mani, in [130] we provide explicit error rates for asymptotic normality for S_n under general setting. We show that our error rates arise from graph counts of certain joins of H which we called *good-join* (see [130, Definition 1.2]). Furthermore, the statistic S_n exhibits a fourth-moment phenomenon as long as $c \geq 30$.

The proof follows the strategy of [56], which relies on Hoeffding decomposition of S_n. It allows us to write S_n as a sum of terms in a martingale difference sequence. By applying the standard martingale CLT with error bounds, we obtain error rates in terms of moments of the martingale sequence. One of the main contributions of [130] is a delicate understanding of these error rates in terms of graph counts. This is achieved by careful probabilistic and combinatorial type arguments. We show that the nontrivial contributions from the errors precisely come from the good-joins of H that we introduce in the paper.

1.3.6 Random quadratic forms

Suppose X is a $1 \times n$ vector of i.i.d. mean zero entries from some distribution F and A is an adjacency matrix of some graph G_n with n vertices. A quantity of interest is the fluctuations of the quadratic form $R_n := X^\top AX$. The asymptotic normality of R_n has been extensively studied in the literature [47, 286]. Broadly speaking, R_n has Gaussian fluctuations when G_n is somewhat 'sparse'. On the other hand, the fluctuations are given by chi-square type random variable when the graph is 'dense'. A natural question is then *what are the all possible limiting distributions of* R_n?

In a joint work with Bhaswar Bhattacharya, Somabha Mukherjee, and Sumit Mukherjee [53], we gave a comprehensive answer to this question. We showed that the limiting distribution for R_n is a given by a sum of three components: a Gaussian, a (possibly) infinite weighted sum of independent centered chi-square random variables, random variables, and a normal variance mixture, where the random variance is a (possibly) infinite quadratic form in the variables $\{X_i\}_{i\geq 1}$. The proof proceeds by partitioning the vertices of G_n in a clever manner and analyzing each component separately. The proof uses tools from extremal graph theory and several standard probabilistic techniques.

1.3.7 LDP for random permutations

Studying LDP for uniform random permutations is an important question in probabilistic combinatorics and has received attention in [309, 220]. In a joint work with Jacopo Borga, Sumit Mukherjee, and Peter Winkler [63], we initiate the study of LDP for random permutations induced by probability measures of the unit square, called *permutons*. These permutations are called μ-random permutations. We also introduce and study a new general class of models of random permutations, called Gibbs permutation models, which combines and generalizes μ-random permutations and the celebrated Mallows model for permutations. Using the tools that we develop, we prove the existence of at least one phase transition for a generalized version of the Mallows model where the base measure is non-uniform. This is in contrast with the results on the (standard) Mallows model [300, 301] where the absence of phase transition, i.e., phase uniqueness, was proven.

Our results naturally lead us to investigate a new notion of permutons, called *conditionally constant (CC) permutons*. It generalizes both pattern-avoiding and pattern-packing permutons which are extensively studied in the combinatorics community (see [62] and references therein). We describe few measure theoretic properties (such as empty interior) of CC permutons w.r.t. inversions. The tools that we mainly use in our proof is the general LDP theory and our arguments are probabilistic and combinatorial in nature.

Chapter 2: Fractional moments of the Stochastic Heat Equation

2.1 Introduction

In this article we study the Stochastic Heat Equation (SHE) in one spatial dimension

$$\partial_t \mathcal{Z} = \tfrac{1}{2}\partial_{xx}\mathcal{Z} + \xi\mathcal{Z}, \qquad \mathcal{Z} = \mathcal{Z}(t,x), \quad (t,x) \in [0,\infty)\times\mathbb{R}, \tag{2.1.1}$$

where $\xi = \xi(t,x)$ is the Gaussian spacetime white noise. Via the Feynman–Kac formula, solutions of the SHE gives the partition function of the directed polymer in a continuum random environment [203, 98]. On the other hand, the logarithm $\mathcal{H}(t,x) := \log \mathcal{Z}(t,x)$ formally solves the Kardar–Parisi–Zhang (KPZ) equation

$$\partial_t \mathcal{H} = \tfrac{1}{2}\partial_{xx}\mathcal{H} + \tfrac{1}{2}(\partial_x \mathcal{H})^2 + \xi. \tag{2.1.2}$$

Introduced in [218], the KPZ equation is a paradigm for random surface growth. It connects to many physical systems including directed polymers, last passage percolation, random fluids, interacting particle systems, and exhibits statistical behaviors similar to certain random matrices. We refer to [166, 278, 113, 281, 87, 124] and the references therein for the mathematical study of and related to the KPZ equation.

Throughout this paper we will consider the solution $\mathcal{Z}(t,x)$ of the SHE (2.1.1) with the initial data

$$\mathcal{Z}(0,x) = \delta(x), \tag{2.1.3}$$

the Dirac delta function at the origin. The SHE (2.1.1) enjoys a well-developed solution theory based on Itô integral and chaos expansion [316, 48], also [278, 113]. In particular, there exists a unique $C((0,\infty),\mathbb{R})$-valued process $\mathcal{Z}$ that solves (2.1.1) with the delta initial data (2.1.3) in the mild sense, i.e.,

$$\mathcal{Z}(t,x) = p(t,x) + \int_0^t \int_{\mathbb{R}} p(t-s,x-y)\mathcal{Z}(s,y)\xi(s,y)\,\mathrm{d}s\mathrm{d}y,$$

where $p(t,x) := (2\pi t)^{-1/2}\exp(-x^2/(2t))$ denotes the standard heat kernel.

The solution $\mathcal{Z}$ of the SHE can be transformed into a solution of the KPZ equation. For a nonzero initial data $\mathcal{Z}(0,\bullet)$ that is bounded, nonnegative, and has a compact support, [259] showed that almost surely $\mathcal{Z}(t,x) > 0$ for all $(t,x) \in (0,\infty)\times\mathbb{R}$. For the delta initial data (2.1.3) considered here, the same positivity result was established in [168]. The logarithm $\mathcal{H}(t,x) := \log \mathcal{Z}(t,x)$ is defined to be **Hopf–Cole solution** of the KPZ equation. This is the notion of solutions that we will be working with throughout this paper. The motivation is, as mentioned previously, that non-rigorously taking logarithm in (2.1.1) yields the KPZ equation (2.1.2). The KPZ equation (2.1.2) itself is ill-posed due to the roughness of the solution and the presence of the quadratic term. New theories have been developed for making sense of the KPZ equation and constructing the corresponding solution process. This includes regularity structures [192, 191], paracontrolled distributions [186, 188], and energy solutions [184, 187]. The Hopf–Cole formulation bypasses the ill-posedness issue, and arises in several discrete or regularized version of the KPZ equation, e.g., [48, 49]. Further, other notions of solutions from the aforementioned theories have been shown to coincide with the Hopf–Cole solution within the class of initial datas the theory applies.

Of interest is the large time behaviors of $\mathcal{H}(t,x) := \log \mathcal{Z}(t,x)$. Simultaneously and independently, the physics works [80, 156, 291] and mathematics work [6] gave the following large t asymptotic fluctuation result of $\mathcal{H}(t,x)$, and [6] provided a rigorous proof:

$$\tfrac{1}{t^{1/3}}\big(\mathcal{H}(2t,0) + \tfrac{t}{12}\big) \Longrightarrow \text{GUE Tracy–Widom distribution.}$$

This result asserts that, for large t, the height $\mathcal{H}(2t,0)$ concentrates around $-\frac{t}{12}$, has typical deviations of order $t^{1/3}$, and after being scaled by $t^{-1/3}$ the fluctuations converge to the GUE Tracy–Widom distribution [308].

A natural question that follows the fluctuation result is establishing a LDP, namely questions about *tails* of the distribution of $\mathcal{H}(2t, 0) + \frac{t}{12}$. We seek to find the probability of the rare events when the height $\mathcal{H}(2t, 0) + \frac{t}{12}$ has a deviation of order t. Interestingly the lower- and upper-tail LDPs have different speeds. The lower-tail deviations occurs at speed t^2 while the upper-tail deviations occurs at speed t.

$$\mathbf{P}[\mathcal{H}(2t, 0) + \tfrac{t}{12} < ty] \approx e^{-t^2\Phi_-(y)}, \qquad (y < 0) \tag{2.1.4}$$

$$\mathbf{P}[\mathcal{H}(2t, 0) + \tfrac{t}{12} > ty] \approx e^{-t\Phi_+(y)}. \qquad (y > 0) \tag{2.1.5}$$

Such distinct speeds can be heuristically explained by directed polymers. For a *discrete* polymer on an $N \times N$ grid with i.i.d. site weights, we consider the point to point partition function. It can be made anomalously large by increasing the weights along any *single* path. The cost of changing the weights of N such sites amounts to $\exp(-O(N))$. However, smaller partition function can be realized only when the weights along *most* of the paths are decreased jointly. This can occur with probability $\exp(-O(N^2))$ by decreasing the weights of most of the sites, c.f., Remark 2.1.1. For the KPZ equation, recall that the Feynman–Kac formula identifies solution of the SHE as the partition function of the directed polymer in a continuum random environment. This is analogous to discrete polymers, with Brownian motion replacing random walks and space-time white noise replacing site weights. In the continuum setting t plays the analogous role as N, since both t and N parametrize the polymer length. Identifying t with N, we should expect the t^2 vs t speeds in (2.1.4) and (2.1.5). These speeds were predicted in the physics work [243], where the prescribed polymer argument was given.

Remark 2.1.1. The speed of lower-tail deviations is in fact not universal when the random environment is unbounded. Specifically, [46] showed that the lower-tail speed of the directed polymer with a Gaussian environment is $N^2/\log N$ instead of N^2.

Recently there has been much development around the large deviations of the KPZ equation in the mathematics and physics communities. Employing the optimal fluctuation theory, the physics works [226, 225, 252] predicted various tail behaviors of the KPZ equation. These predictions were further supported by the analysis of exact formulae in the physics works [241, 230, 233]. In mathematics terms, the optimal fluctuation theory corresponds to Fredilin–Wentzell type large deviations of stochastic PDEs with a small noise. There has been rigorous treatment [194, 86] of such large deviations for certain nonlinear stochastic PDEs.

Under the same initial data as this paper, the physics works [292, 118, 234] each employed a different method to derive the same explicit rate function for the lower-tail deviations of $\mathcal{H}(2t,0)+\frac{t}{12}$. The work [116] provides detailed, rigorous bounds on tails of $\mathcal{H}(2t,0)+\frac{t}{12}$, which are valid for all $t>0$ and capture a crossover behavior predicted in [225, 252]. The lower-tail LDP with the exact rate function was later proven in [310], and more recently in [79]. The four different routes [292, 118, 234, 310] of deriving the lower-tail LDP were later shown to be closely related [232]. Two new routes have been recently obtained in the rigorous work [79] and physics work [240].

In this paper we focus on the *upper* tail — the complement of the aforementioned results. Since $\mathcal{Z}(t,x)=\exp(\mathcal{H}(t,x))$, the upper tail is closely related to positive moments of $\mathcal{Z}$. The moments of SHE and its connection to intermittency property [175, 176] has been previously studied in [104, 92, 94, 223]. These works established finite time estimates of tails or moments of $\mathcal{Z}(t,x)$ and solutions of related stochastic PDEs. The work [93] studied a class of equations that includes the SHE with the delta initial data considered here. With the aim of establishing the existence of the smooth density, the work obtained finite time tail estimates of the solution.

For the large time regime considered here, the form $\Phi_+(y)=\frac{4}{3}y^{3/2}$ was predicted in [243] by analyzing an exact formula. The analysis also yields subdominant corrections; see [242, Supp. Mat.]. We note that, for the short time regime, [213] predicted the same $\frac{3}{2}$-power law. A priori, the optimal fluctuation theory used therein works only for short time, although the validity in large time was argued therein. For the large time regime, [122] gave a bound on of the upper tail of $\mathcal{Z}(t,x)$ (with a different initial data). The bound exhibits the predicted $\frac{3}{2}$-power for small y but not large y. Extracting information from positive integer moments of $\mathcal{Z}$, [115] provided detail bounds on the upper-tail probability. The upper and lower bounds therein capture the aforementioned $\frac{3}{2}$-power law but do not match as $t \to \infty$.

In this paper we present the *first rigorous proof* of the upper-tail LDP of $\mathcal{H}(2t,0) + \frac{t}{12}$ with the predicted $\Phi_+(y) = \frac{4}{3}y^{3/2}$ rate function. Interestingly, this matches exactly with the upper-tail rate function for the Tracy-Widom distribution [308]. Our main result gives both the $t \to \infty$ asymptotic of the p-th moment of $\mathcal{Z}(2t,0)$, for any real $p > 0$, and the upper-tail LDP of the KPZ equation.

Theorem 2.1.2. *Let $\mathcal{Z}(t,x)$ be the solution of the SHE* (2.1.1) *with the delta initial data* (2.1.3)*, and let $\mathcal{H}(t,x) := \log \mathcal{Z}(t,x)$ be the Hopf–Cole solution of the KPZ equation* (2.1.2).

(a) For any $p \in (0,\infty)$, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}\Big[e^{p(\mathcal{H}(2t,0)+\frac{t}{12})}\Big] = \lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}\Big[\big(\mathcal{Z}(2t,0)e^{\frac{t}{12}}\big)^p\Big] = \frac{p^3}{12}. \tag{2.1.6}$$

(b) For any $y \in (0,\infty)$, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{P}\Big[\mathcal{H}(2t,0) + \tfrac{t}{12} \geq ty\Big] = -\Phi_+(y) := -\frac{4}{3}y^{3/2}. \tag{2.1.7}$$

Remark 2.1.3. The results in Theorem 2.1.2 immediately generalize to $x \neq 0$. This is so because, under the delta initial data (2.1.3), the random variables $\mathcal{Z}(2t,0)$ and $\mathcal{Z}(2t,x)\exp(x^2/4t)$ have the same law. This fact can be verified from either the Feynman–Kac formula or the chaos expansion. Hence, the results in Theorem 2.1.2 hold with $\mathcal{Z}(2t,x)\exp(x^2/4t)$ replacing $\mathcal{Z}(t,0)$ and $\mathcal{H}(2t,x) + \frac{x^2}{4t}$ replacing $\mathcal{H}(2t,0)$.

Our method is based on a perturbative analysis of Fredholm determinants, and the major input is the formula (2.1.10) that expresses the Laplace transform of $\mathcal{Z}(2t,0)$ as a Fredholm determinant. We emphasize that our method *differs* from existing methods used in the same context. The work [243] postulates a form of the upper tail and verifies a posteriori the consistency with the formula (2.1.10); see [242, Supp. Mat.]. There are, however, infinitely many postulated forms that are consistent with (2.1.10). We explain this phenomenon in Section 2.1.1. There we reprint the consistency check as a variational problem (2.1.14), which has infinitely many solutions given in (2.1.15). The work [122] utilizes an formula of the tail probabilty of $\mathcal{H}(2t,0)+\frac{t}{12}$, under the Brownian initial data. Such a formula can be viewed as the inverse Laplace transform of (2.1.10). By analyzing the inverse Laplace transform formula, it was shown [122, Corollary 14] that there exists constants c_1, c_2, c_3 such that for all $y>0$ and large enough t

$$\mathbf{P}\left[\mathcal{H}(2t,0)+\tfrac{t}{12}\geq ty\right]\leq c_1 t^{1/2}e^{-c_2yt}+c_1t^{1/2}e^{-c_3y^{3/2}t}.$$

This bound exhibits the $\frac{3}{2}$-power law for small y but becomes linear in y (in the exponent) for large y. In this paper we employ a new way of utilizing the formula (2.1.10), by applying it for getting the p-moment growth of $\mathcal{Z}(2t,0)$.

The main body of our proof is devoted to proving Theorem 2.1.2(a), or more precisely its refined version Theorem 2.1.2(a)* stated in the following. From Theorem 2.1.2(a) standard argument produces Theorem 2.1.2(b), with the rate function $-\frac{4}{3}y^{3/2}$ being the Legendre transform of $\frac{p^3}{12}$. The first indication of Theorem 2.1.2(a) being true came form the study of positive integer moments of (2.1.6). The mixed joint moment of $\mathcal{Z}$ solves the delta Bose gas, and the delta Bose operator can be diagonalized by the Bethe ansatz. The work [216] carried out such analysis and pointed out that (2.1.6) should hold for positive integers, i.e.,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbf{E}\Big[e^{n(\mathcal{H}(2t,0)+\frac{t}{12})}\Big]=\frac{n^3}{12},\quad \text{for } n\in\mathbb{Z}_{>0}. \tag{2.1.6-int}$$

This assertion (2.1.6-int) was proven in [94] for function-valued, bounded initial data, and in [115, Lemma 4.5] for the delta initial data considered here. It has long been speculated and conjectured that (2.1.6-int) should extend to all positive real p. However, the connection to the delta Bose gas only gave access to integer moments. Here, by utilizing a known formula but in an unconventional way, we bridge the gap between integers. In the same spirit as [115, Lemma 4.5], we provide a quantitative bound on the p-th moment of $\mathcal{Z}$ that holds for all t and p away from 0. This is stated as a refined version of Theorem 2.1.2(a) as

Theorem 2.1.2(a)*. *Let $\mathcal{Z}$ be as in Theorem 2.1.2. We have a decomposition*

$$\mathbf{E}\big[\big(\mathcal{Z}(2t,0)e^{\frac{t}{12}}\big)^p\big] = \mathcal{A}_p(t) + \mathcal{B}_p(t)$$

of the p-th moment of $\mathcal{Z}(2t,0)e^{\frac{t}{12}}$ into a leading term $\mathcal{A}_p(t)$ and remainder term $\mathcal{B}_p(t)$. For any $t_0, p_0 > 0$, there exists a constant $\mathrm{C} = \mathrm{C}(t_0, p_0) > 0$ that depends only on t_0, p_0, such that for all $t \geq t_0$ and $p \geq p_0$,

$$\frac{1}{\mathrm{C}} p^{-\frac{3}{2}} \Gamma(p+1)\, t^{-\frac{1}{2}}\, e^{\frac{p^3 t}{12}} \leq \mathcal{A}_p(t) \leq \mathrm{C} p^{-\frac{3}{2}} \Gamma(p+1)\, t^{-\frac{1}{2}}\, e^{\frac{p^3 t}{12}}, \tag{2.1.8}$$

and for $n := \lfloor p \rfloor + 1 \in \mathbb{Z}_{>0}$ and $\kappa_p := \min\{\frac{1}{6}, \frac{p^3}{16}\}$,

$$|\mathcal{B}_p(t)| \leq n \cdot (n!)^2\, (n\mathrm{C})^n\, t^{\frac{1}{2}}\, e^{\frac{p^3 t}{12} - \kappa_p t}. \tag{2.1.9}$$

From the bounds (2.1.8) and (2.1.9), one see that $\mathcal{A}_p(t)$ dominates as $t \to \infty$, uniformly over any close intervals in $(0, \infty)\mathsf{N}\mathsf{I}p$. Theorem 2.1.2(a)* immediately implies Theorem 2.1.2(a).

The upper tail problem has also been studied for several other models in the class of integrable systems starting from the fluctuation results and LDP for the longest increasing subsequence [224, 293, 148, 14]. There are also analogous results on upper-tail LDP for integrable polymer models [177, 208], and also for last passage percolation in Bernoulli and white noise environments [96, 209] and inhomogeneous corner growth models [162].

The main input of our proof is the known formula (2.1.10) that express the Laplace transform of $\mathcal{Z}(2t,0)$ as a Fredholm determinant. There are multiple equivalent ways to define Fredholm determinants [296]. We will work with the exterior algebra definition: for a trace-class operator T on a Hilbert space, consider $\bigwedge_{i=1}^{L} H$ and the operator $T^{\wedge L}$ defined by $T^{\wedge L}(v_1 \wedge \cdots \wedge v_L) := (Tv_1) \wedge \cdots \wedge (Tv_L)$. The operator $T^{\wedge L}$ is trace-class on $\bigwedge_{i=1}^{L} H$. We then define the Fredholm determant as

$$\det(I - T) := 1 + \sum_{L=1}^{\infty} (-1)^L \mathrm{tr}(T^{\wedge L}).$$

The following formula is known thanks to the integrability of the SHE and related models:

$$\mathbf{E}\big[\exp(-s\mathcal{Z}(2t,0)e^{\frac{t}{12}})\big] = \det(I - K_{s,t}) = 1 + \sum_{L=1}^{\infty} (-1)^L \mathrm{tr}(K_{s,t}^{\wedge L}), \tag{2.1.10}$$

where $K_{s,t}$ is an integral operator $L^2(\mathbb{R}_{\geq 0})$ with the kernel

$$K_{s,t}(x,y) := \int_{\mathbb{R}} \frac{\mathrm{Ai}(x+r)\,\mathrm{Ai}(y+r)}{1 + \frac{1}{s}e^{-t^{1/3}r}} \mathrm{d}r, \tag{2.1.11}$$

and $\mathrm{Ai}(x)$ is the Airy function. It is standard to check that $K_{s,t}$ is a positive operator via the square-root trick, c.f., Lemma 2.2.1. The formula (2.1.10) or its closely related forms was first derived simultaneously and independently in [6, 80, 156, 291], with a rigorous proof given in [6] based on results of [306]. In particular, the formula (2.1.10) can be obtained by taking Laplace transform of [6, Eq. (1.13)]. A direct derivation of (2.1.10) with a rigorous proof can be found in [70]; see Theorem 1.10 (a) and Eq. (1.7) therein.

A standard way to extract tail information from (2.1.10) is to parameterize $s = e^{-ty}$ and substitute in $\mathcal{Z}(2t,0) = \exp(\mathcal{H}(2t,0))$ to get

$$\mathbf{E}\big[\exp(-e^{\mathcal{H}(2t,0)+\frac{t}{12}-ty})\big] = 1 - \mathrm{tr}(K_{s,t}) + \sum_{L=2}^{\infty} (-1)^L \mathrm{tr}(K_{s,t}^{\wedge L}). \tag{2.1.12}$$

The double exponential function $\exp(-e^{\bullet})$ on the l.h.s. of (2.1.12) may be deemed as a good proxy of the indicator function $\mathbf{1}_{(-\infty,0)}$, and hence analyzing the r.h.s. of (2.1.12) could produce information on $\mathbf{P}[\mathcal{H}(2t,0) + \frac{t}{12} < ty]$. This approximation procedure has been successfully implemented in getting the limiting fluctuations and lower-tail LDP (but using different representations of the r.h.s. than the Fredholm determinant).

2.1.1 An issue of nonuniqueness

However, for the upper tail, the preceding procedure would not produce the full LDP. To see this, rewrite (2.1.12) as

$$\mathbf{E}\big[1 - \exp(-e^{\mathcal{H}(2t,0)+\frac{t}{12}-ty})\big] = \sum_{L=1}^{\infty}(-1)^{L-1}\mathrm{tr}(K^{\wedge L}_{e^{-ty},t}). \tag{2.1.13}$$

For $y > 0$, it is possible to show that the r.h.s. of (2.1.13) is dominated by the $L = 1$ term as $t \to \infty$, and analyzing the trace of $K_{s,t}$ from the formula (2.1.11) should yield

$$\lim_{t\to\infty} \frac{1}{t} \log \big(\text{r.h.s. of (2.1.13)}\big) = I(y) := \begin{cases} -\frac{4}{3}y^{3/2}, & y \in (0, \frac{1}{4}], \\ \frac{1}{12} - y\ , & y \in (\frac{1}{4}, \infty). \end{cases}$$

For the left hand side, if we assume the existence of the upper-tail LDP but with an unknown rate function, i.e., $\lim_{t\to\infty} \frac{1}{t} \log \mathbf{P}[\mathcal{H}(2t, 0) + \frac{t}{12} > ty] = -\Phi_+(y)$, for $y \in (0, \infty)$, using the fact that $1 - \exp(-e^{t\xi}) \approx \exp(t\min\{\xi, 0\})$, as $t \to \infty$, we should have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}\big[1 - \exp(-e^{\mathcal{H}(2t,0)+\frac{t}{12}-ty})\big] = \sup_{\xi>0} \big\{ \min\{\xi - y, 0\} - \Phi_+(\xi)\big\}.$$

Putting these two sides together suggests the variational problem

$$\sup_{\xi>0} \big\{ \min\{\xi - y, 0\} - \Phi_+(\xi)\big\} = \begin{cases} -\frac{4}{3}y^{3/2}, & y \in (0, \frac{1}{4}], \\ \frac{1}{12} - y\ , & y \in (\frac{1}{4}, \infty). \end{cases} \tag{2.1.14}$$

The function $\Phi_+(y) = \frac{4}{3}y^{3/2}$ does solve this variational problem. However, the solution is *not* unique. *Any* function that satisfies

$$\Phi_+(y) = -\tfrac{4}{3}y^{3/2}, \text{ for } y \in (0, \tfrac{1}{4}], \qquad \tfrac{1}{12} - y \le \Phi_+(y) \le \tfrac{4}{3}y^{3/2}, \text{ for } y \in (\tfrac{1}{4}, \infty) \tag{2.1.15}$$

solves the preceding variational problem.

The preceding calculations strongly suggest that the conventional scheme (2.1.12) and (2.1.13) of using the Fredholm determinant would not produce the exact rate function.

2.1.2 Our solution

To circumvent the aforementioned issue, we provide a new way of using the formula (2.1.10). The start point is the following elementary identity:

Lemma 2.1.4. *Let U be a nonnegative random variable with a finite n-th moment, where $n \in \mathbb{Z}_{>0}$. Let $\alpha \in [0, 1)$ Then the $(n-1+\alpha)$-th moment of U is given by*

$$\mathbf{E}[U^{n-1+\alpha}] = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} \mathbf{E}[U^n e^{-sU}]\, \mathrm{d}s = \frac{(-1)^n}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}s^n} \mathbf{E}[e^{-sU}]\, \mathrm{d}s. \tag{2.1.16}$$

The proof of this lemma follows by an interchange of measure via Fubini's theorem. We will apply this lemma with $U = \mathcal{Z}(2t, 0)e^{\frac{t}{12}}$ and with $n := \lfloor p \rfloor + 1 \in \mathbb{Z}_{>0}$ and $\alpha := p - \lfloor p \rfloor \in [0, 1)$ so that $p = n - 1 + \alpha$.

Utilizing the formula (2.1.10) for $\mathbf{E}[e^{-sU}] = \mathbf{E}[e^{-s\mathcal{Z}(2t,0)e^{\frac{t}{12}}}]$ in (2.1.16), we will then be able to express the p-th moment of $\mathcal{Z}(2t, 0)e^{\frac{t}{12}}$ as a series. From this series we identify the leading term and higher order terms. This eventually leads to the desired estimate in Theorem 2.1.2(a)*.

It seems possible to directly analyze the inverse Laplace transform formula in [6, Theorem 1.1]. Doing so may provide an alternative proof of Theorem 2.1.2(b).

Outline.

In Section 2.2 we setup the framework of the proof. Namely we introduce an expansion of the p-th moment of $\mathcal{Z}$, identify a trace term as the leading term, and establish several technical lemmas. In Section 2.3, we give precise asymptotics of the leading trace term, and in Section 2.4 we establish bounds on the remaining terms. Finally, in Section 2.5, we collect results from previous sections to give a proof of Theorem 2.1.2 and Theorem 2.1.2(a)*.

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2.2 Basic framework

Throughout this paper we use $C = C(a, b, c, \ldots) > 0$ to denote a generic deterministic positive finite constant that may change from line to line, but dependent on the designated variables $a, b, c, \ldots$.

As mentioned previously, we will utilize Lemma 2.1.4 and (2.1.10) to develop a series expansion for $\mathbf{E}[(\mathcal{Z}(2t,0)e^{\frac{t}{12}})^p]$. This, however, requires a truncation at $s = 1$ first. To see why, referring to (2.1.12), with $s = e^{-ty}$, we see that $s < 1$ corresponds to upper tail while $s > 1$ corresponds to lower tail. While we expect the later to have minor contribution in the regime $p > 0$ we are probing, it is known that for $s \gg 1$ the Fredholm determinant (2.1.12) behaves in an oscillatory fashion as $t \to \infty$. With $n := \lfloor p \rfloor + 1 \in \mathbb{Z}_{>0}$ and $\alpha := p - \lfloor p \rfloor \in [0, 1)$, we truncate

$$\mathbf{E}\big[(\mathcal{Z}(2t,0)e^{\frac{t}{12}})^p\big] = \frac{(-1)^n}{\Gamma(1-\alpha)} \int_0^1 s^{-\alpha}\, \partial_s^n\, \mathbf{E}[e^{-s\mathcal{Z}(2t,0)e^{\frac{t}{12}}}]\, \mathrm{d}s + \mathcal{B}_{p,1}(t), \tag{2.2.1}$$

where

$$\mathcal{B}_{p,1}(t) := \frac{1}{\Gamma(1-\alpha)} \int_1^\infty s^{-\alpha} \mathbf{E}[U^n e^{-sU}]\, \mathrm{d}s, \quad U := \mathcal{Z}(2t,0)e^{\frac{t}{12}}. \tag{2.2.2}$$

For this term $\mathcal{B}_{p,1}(t)$ we bound

$$0 \le \mathcal{B}_{p,1}(t) = \frac{1}{\Gamma(1-\alpha)} \int_1^\infty s^{-n-\alpha} \mathbf{E}[(sU)^n e^{-sU}]\,\mathrm{d}s \le \frac{1}{\Gamma(1-\alpha)} \sup_{x\ge 0} \left\{x^n e^{-x}\right\} \frac{1}{n+\alpha-1}.$$

Recognize $n+\alpha-1 = p$, and apply the bounds $\frac{1}{\Gamma(1-\alpha)} \le \mathrm{C}$, for $\alpha \in [0,1)$, and $\sup_{x\ge0}\{x^n e^{-x}\} \le n^n$.

$$|\mathcal{B}_{p,1}(t)| \le \mathrm{C}\, p^{-1}\, n^n. \tag{2.2.3}$$

The bound (2.2.3) does not grow with t, and hence $\mathcal{B}_{p,1}(t)$ will be a subdominant term.

Next, we wish to take ∂_s^n in the Fredholm determinant expansion (2.1.10) and develop the corresponding series. Assuming (justified later) the derivative can be passed into the sum, we have

$$\begin{aligned}&\frac{(-1)^n}{\Gamma(1-\alpha)} \int_0^1 s^{-\alpha}\, \partial_s^n\, \mathbf{E}[e^{-s\mathcal{Z}(2t,0)e^{\frac{t}{12}}}]\,\mathrm{d}s \\ =&\frac{(-1)^n}{\Gamma(1-\alpha)} \int_0^1 s^{-\alpha}\, \partial_s^n \Big(\sum_{L=1}^\infty (-1)^L \mathrm{tr}(K_{s,t}^{\wedge L})\Big)\mathrm{d}s = \frac{(-1)^n}{\Gamma(1-\alpha)} \int_0^1 s^{-\alpha} \sum_{L=1}^\infty (-1)^L\, \partial_s^n\, \mathrm{tr}(K_{s,t}^{\wedge L})\,\mathrm{d}s.\end{aligned} \tag{2.2.4}$$

The passing of derivatives into sums will be justified in Lemma 2.4.4, and in Sections 2.3 and 2.4.1, we will show that $\mathrm{tr}(K_{s,t}^{\wedge L})$ is infinitely differentiable in s. As it turns out, the $L = 1$ term dominates. We then let

$$\widetilde{\mathcal{A}}_p(t) := \frac{(-1)^{n+1}}{\Gamma(1-\alpha)} \int_0^1 s^{-\alpha}\, \partial_s^n\, \mathrm{tr}(K_{s,t})\,\mathrm{d}s, \tag{2.2.5}$$

$$\mathcal{B}_{p,L}(t) := \frac{(-1)^{n+L}}{\Gamma(1-\alpha)} \int_0^1 s^{-\alpha}\, \partial_s^n\, \mathrm{tr}(K_{s,t}^{\wedge L})\,\mathrm{d}s, \quad L \ge 2 \tag{2.2.6}$$

denote the leading and higher order terms.

In the following we will work with the Schatten norms of operators. Recall that, for $u \in [1,\infty]$

and for a compact operator T on $L^2(\mathbb{R}_{\geq 0})$, the u**-th Schatten norm** of T is defined as

$$\|T\|_u := \big(\operatorname{tr}(T^*T)^{u/2}\big)^{1/u} = \Big(\sum_{i=1}^{\infty} s_i(T)^u\Big)^{1/u},$$

with the convention $\|T\|_\infty := \lim_{u\to\infty} \|T\|_u$, where $s_i(T)$, $i \in \mathbb{Z}_{>0}$, are the singular values of T. In particular, $u = 1$ gives the trace norm, $u = 2$ gives the Hilbert–Schmidt norm, and $u = \infty$ gives the operator norm $\|T\|_{\mathrm{op}} := \sup\{\frac{|Tf|}{|f|} : f \in L^2(\mathbb{R}_{\geq 0}) \setminus \{0\}\}$, where $|f| := (\int_0^\infty |f(x)|^2\, \mathrm{d}x)^{1/2}$ denotes the norm on $L^2(\mathbb{R}_{\geq 0})$. The Schatten norm decreases in u, so the trace norm is the strongest among all $u \in [1, \infty]$. We will use the following 'square-root trick' to evaluate the trace norm of some operators.

Lemma 2.2.1. *Consider a square-integrable kernel $J(r, y)$ with $\int_{\mathbb{R}_+}(\int_{\mathbb{R}} |J(r, y)|^2\, \mathrm{d}r)\mathrm{d}y < \infty$. Then the integral operator T on $L^2(\mathbb{R}_{\geq 0})$ with the kernel*

$$T(x, y) := \int_{\mathbb{R}} \overline{J}(r, x)\, J(r, y)\, \mathrm{d}r$$

is positive and trace-class, with $\operatorname{tr}(T) = \|T\|_1 = \int_{\mathbb{R}_+}(\int_{\mathbb{R}} |J(r, y)|^2\, \mathrm{d}r)\mathrm{d}y$.

Proof. It is more convenient to embed T into operators on $L^2(\mathbb{R})$. We do this by setting the kernel

$$T(x, y) := \mathbf{1}_{\mathbb{R}_{\geq 0}}(x)\mathbf{1}_{\mathbb{R}_{\geq 0}}(y) \int_{\mathbb{R}} \overline{J}(r, x)\, J(r, y)\, \mathrm{d}r$$

to be zero outside $(x, y) \in \mathbb{R}^2_{\geq 0}$. This way we have the factorization $T = J^*J$, where J is an operator on $L^2(\mathbb{R})$ with kernel $\mathbf{1}_{\mathbb{R}_{\geq 0}}(y)J(r, y)$. The square integrability of $J(r, y)$ guarantees that the operator J is Hilbert–Schmidt, and the Cauchy–Schwartz inequality $\|T_1T_2\|_1 \leq \|T_1\|_2\|T_2\|_2$ applied with $T_1 = J^*, T_2 = J$ concludes that T is trace-class, whence $\operatorname{tr}(T) = \int_{\mathbb{R}_+}(\int_{\mathbb{R}} |J(r, y)|^2\, \mathrm{d}r)\mathrm{d}y$ by Theorem 3.1 in [77]. The factorization $T = J^*J$ implies that T is positive, whence $\operatorname{tr}(T) = \|T\|_1$. □

Lemma 2.2.1 applied with $J(r, y) = \mathrm{Ai}(y + r)(1 + \frac{1}{s}e^{-t^{1/3}r})^{-1/2}$ proves that the operator $K_{s,t}$ (defined in (2.1.11)) is positive and trace-class.

Much of our subsequent analysis boils down to estimating integrals involving the Airy function $\mathrm{Ai}(x)$. Here we prepare two technical lemmas that will be frequently used. To setup the notation, set

$$\Phi(y) := \int_y^\infty \mathrm{Ai}^2(x)\,\mathrm{d}x. \tag{2.2.7}$$

Using the Airy differential equation, one can explicitly compute the antiderivative of $\mathrm{Ai}(x)^2$ to get $\Phi(y) = \mathrm{Ai}'(y)^2 - y\,\mathrm{Ai}(y)^2$. Using known expansions of $\mathrm{Ai}(x), \mathrm{Ai}'(x)$ for $|x| \gg 1$, e.g., Equation (1.07), (1.08), and (1.09) in Chapter 11 of [267], we have that, for all $y \geq 0$ and for some universal $\mathrm{C} > 0$,

$$\frac{1}{\mathrm{C}}(\sqrt{|y|}+1) \leq \Phi(-y) \leq \mathrm{C}\,(\sqrt{|y|}+1), \tag{2.2.8}$$

$$\tfrac{1}{\mathrm{C}\,(y+1)} e^{-\frac{4}{3}y^{3/2}} \leq \Phi(y) \leq \tfrac{\mathrm{C}}{y+1} e^{-\frac{4}{3}y^{3/2}}. \tag{2.2.9}$$

Also consider

$$U_q(x) := qx^2 - \tfrac{4}{3}x^3, \tag{2.2.10}$$

which enjoys the property

$$U_q(x) \text{ increases on } x \in [0, \tfrac{q}{2}] \text{ and decreases on } x \in [\tfrac{q}{2}, \infty), \quad U_q(\tfrac{q}{2}) = \tfrac{q^3}{12}. \tag{2.2.11}$$

Lemma 2.2.2. *Fix $t_0, q_0 \in (0,\infty)$. There exists a constant $\mathrm{C}(t_0,q_0) > 0$, such that for all $t \geq t_0$ and $q \geq q_0$,*

$$\frac{1}{\mathrm{C}(t_0,q_0)} t^{-7/6} q^{-3/2} e^{\frac{q^3 t}{12}} \leq \int_{\mathbb{R}} e^{qrt} \Phi(t^{2/3} r)\,\mathrm{d}r \leq \mathrm{C}(t_0,q_0) t^{-7/6} q^{-3/2} e^{\frac{q^3 t}{12}}. \tag{2.2.12}$$

Proof. Let us first give a heuristic of the proof. The idea is to apply Laplace's method. We seek to approximate $\int_{\mathbb{R}} e^{qrt}\Phi(t^{2/3}r)\,\mathrm{d}r$ by $\int_{\mathbb{R}} e^{tg_q(r)}\mathrm{d}r$, for some appropriate function $g_q(r)$, and search the maximum of $g_q(r)$ over $r \in \mathbb{R}$. The bounds of Φ from (2.2.8) and (2.2.9) suggest $\log \Phi(t^{2/3}r) \approx -\frac{4}{3}tr_+^{3/2}$ and $g_q(r) = qr - \frac{4}{3}r_+^{3/2}$. This function achieves a maximum of $q^3/12$ at $r = q^2/4$, which gives the exponential factor $\exp(\frac{q^3 t}{12})$. The prefactor $t^{-7/6}q^{-3/2}$ can then be obtained from localizing the integral around $r = q^2/4$ and using (2.2.9) to approximate the integral as a Gaussian integral.

We now start the proof. Fix $t_0, q_0 > 0$. To simplify notation, throughout this proof we write $\mathrm{C} = \mathrm{C}(t_0, q_0) > 0$, and for positive functions $f_1(a, b, \ldots), f_2(a, b, \ldots)$, we write $f_1 \sim f_2$ if they bound each other by a constant multiple, i.e.,

$$\tfrac{1}{\mathrm{C}} f_2(a, b, \ldots) \leq f_1(a, b, \ldots) \leq \mathrm{C} f_2(a, b, \ldots),$$

within the specified ranges of the variables $a, b, \ldots$. Set $\rho := \frac{q_0}{4}$. Divide $\int_{\mathbb{R}} e^{qrt}\Phi(t^{2/3}r)\,\mathrm{d}r$ into three regions and let $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$ denote the respective integrals:

$$\left(\int_{[(\frac{q}{2}-\rho)^2, (\frac{q}{2}+\rho)^2]} + \int_{\mathbb{R}_-} + \int_{\mathbb{R}_{\geq 0} \setminus [(\frac{q}{2}-\rho)^2, (\frac{q}{2}+\rho)^2]}\right) e^{qrt}\Phi(t^{2/3}r)\mathrm{d}r := \mathcal{I}_1(q,t) + \mathcal{I}_2(q,t) + \mathcal{I}_3(q,t). \tag{2.2.13}$$

As suggested by the preceding heuristics, we anticipate $\mathcal{I}_1(q,t)$ to dominate. We begin with estimating this term. Recall $U_q(x)$ from (2.2.10). The bounds from (2.2.9) gives, for all $r, t \in \mathbb{R}_{\geq 0}$,

$$e^{qrt}\Phi(t^{2/3}r) \sim \frac{e^{tU_q(\sqrt{r})}}{1 + t^{2/3}r}. \tag{2.2.14}$$

The function $U_q(x)$ attains a maximum of $\frac{q^3}{12}$ at $x = \frac{q}{2}$ and $U_q(x) - \frac{q^3}{12} = -(x - \frac{q}{2})^2(\frac{4}{3}(x - \frac{q}{2}) + q)$. Integrate both sides of (2.2.14) over $[(\frac{q}{2}-\rho)^2, (\frac{q}{2}+\rho)^2]$ and make a change of variable $\sqrt{r} - \frac{q}{2} \mapsto x$. We get, for all $q, t \in \mathbb{R}_{\geq 0}$,

$$\mathcal{I}_1(q,t) \sim e^{\frac{q^3 t}{12}} \int_{-\rho}^{\rho} \frac{2(x + \frac{q}{2})e^{-tx^2(\frac{4}{3}x + q)}}{1 + t^{2/3}(x + \frac{q}{2})^2}\mathrm{d}x.$$

The choice $\rho = \frac{q_0}{4}$ guarantees that for all $x \in [-\rho, \rho]$ and for all $q \geq q_0$, we have $\frac{q}{\mathrm{C}} \leq \frac{4}{3}x + q, x + \frac{q}{2} \leq \mathrm{C}q$. Then for all $t \geq t_0$ and $q \geq q_0$, there exists $\mathrm{C} > 0$ such that for $x \in [-\rho, \rho]$,

$$\frac{1}{\mathrm{C}t^{2/3}q}e^{-\mathrm{C}qtx^2} \leq \frac{2(x + \frac{q}{2})e^{-tx^2(\frac{4}{3}x+q)}}{1 + t^{2/3}(x + \frac{q}{2})^2} \leq \frac{\mathrm{C}}{t^{2/3}q}e^{-\frac{1}{\mathrm{C}}qtx^2}. \tag{2.2.15}$$

Integrate (2.2.15) over $[-\rho, \rho]$ and use $\int_{-\rho}^{\rho} e^{-qx^2t}\,\mathrm{d}x \sim (tq)^{-1/2}$, for all $t \geq t_0$ and $q \geq q_0$. We now obtain, for $t \geq t_0$ and $q \geq q_0$,

$$\mathcal{I}_1(q, t) \sim t^{-7/6}q^{-3/2}e^{\frac{q^3t}{12}}. \tag{2.2.16}$$

Having settled the asymptotics of $\mathcal{I}_1(q, t)$, we now turn to $\mathcal{I}_2(q, t), \mathcal{I}_3(q, t)$. For $\mathcal{I}_2(q, t)$, use (2.2.8) to get

$$0 \leq \mathcal{I}_2(q, t) \leq \mathrm{C}\int_{-\infty}^{0} e^{qrt}(\sqrt{t^{2/3}|r|} + 1)\mathrm{d}r \leq \mathrm{C}q^{-3/2}t^{-7/6} + \mathrm{C}q^{-1}t^{-1}. \tag{2.2.17}$$

As for $\mathcal{I}_3(q, t)$, integrate both sides of (2.2.14) over $\mathbb{R}_{\geq 0} \setminus [(\frac{q}{2} - \rho)^2, (\frac{q}{2} + \rho)^2]$ and then make the change of variable $\sqrt{r} \mapsto x$ to get

$$0 \leq \mathcal{I}_3(q, t) \leq \mathrm{C}e^{\frac{q^3t}{12}} \int_{\mathbb{R}_{\geq 0}\setminus[(\frac{q}{2}-\rho),(\frac{q}{2}+\rho)]} \frac{2xe^{-\frac{t}{3}(x-\frac{q}{2})^2(4x+q)}}{1 + t^{2/3}x^2}\mathrm{d}x. \tag{2.2.18}$$

For $x \in \mathbb{R}_{\geq 0} \setminus [(\frac{q}{2} - \rho), (\frac{q}{2} + \rho)]$, we have $(x - \frac{q}{2})^2 \geq \rho^2$. The AM-GM inequality inequality gives $1 + t^{2/3}x^2 \geq 2t^{1/3}x$, and equivalently $\frac{2x}{1+t^{2/3}x^2} \leq t^{-1/3}$. Applying these bounds on the r.h.s. of (2.2.18) and then releasing the region of integration to $\mathbb{R}_{\geq 0}$, we get that

$$\mathcal{I}_3(q, t) \leq \mathrm{C}e^{\frac{q^3t}{12}}t^{-4/3}\rho^{-2}e^{-\frac{qt\rho^2}{3}}. \tag{2.2.19}$$

It is straightforward to check that the r.h.s. of (2.2.17) and (2.2.19) can be further bounded by

$\mathrm{C}\, t^{-7/6} q^{-3/2} e^{\frac{q^3 t}{12}}$, for all $t \geq t_0$ and $q \geq q_0$. Hence

$$0 \leq \mathcal{I}_2(q,t) + \mathcal{I}_3(q,t) \leq \mathrm{C}\, t^{-7/6} q^{-3/2} e^{\frac{q^3 t}{12}}.$$

This together with (2.2.16) gives the desired result (2.2.12). □

Lemma 2.2.3. *Recall U_q from* (2.2.10). *There exists a constant* $\mathrm{C} = \mathrm{C}(t_0, q_0) > 0$ *such that for all* $t \geq t_0$*,* $q \geq q_0$*, and* $y \in [0, \infty]$*,*

$$\int_{-\infty}^{y} e^{qrt} \Phi(t^{2/3} r) \mathrm{d}r \leq \mathrm{C}(t_0, q_0)\, t^{-5/6} \exp\left(t U_q\left(\min\{\sqrt{y}, \tfrac{q}{2}\} \right) \right). \tag{2.2.20}$$

Remark 2.2.4. The prefactor $t^{-5/6}$ in (2.2.12) is likely suboptimal, but suffices for our subsequent analysis.

Proof. When $y \in [\frac{q^2}{4}, \infty]$, we release the range of integration of the l.h.s. of (2.2.20) to $\mathbb{R}$ and use the upper bound in Lemma 2.2.2. Observe that $U_q\left(\min\{\sqrt{y}, \frac{q}{2}\}\right) = \frac{q^3}{12}$ and t and q are bounded below by t_0 and q_0. Absorb $t^{-1/3}$ and $q^{-3/2}$ in the constant $C(t_0, q_0)$ to get the desired bound in (2.2.20).

Moving onto $y \in [0, q^2/4)$, from (2.2.17) we already have a bound on $\int_{-\infty}^{0} e^{qrt} \Phi(t^{2/3} r) \mathrm{d}r$ of the desired form. Hence, it suffices to bound for $\int_0^y e^{qrt} \Phi(t^{2/3} r) \mathrm{d}r$. From (2.2.14), make a change of variable $\sqrt{r} \mapsto x$, and in the result bound $\frac{2x}{1+t^{2/3} x^2} \leq t^{-1/3}$. We have

$$\int_0^y e^{qrt} \Phi(t^{2/3} r) \mathrm{d}r \leq \mathrm{C} \int_0^y \frac{e^{t U_q(\sqrt{r})}}{1 + t^{2/3} r} \mathrm{d}r = \mathrm{C} \int_0^{\sqrt{y}} \frac{2x e^{t U_q(x)}}{1 + t^{2/3} x^2} \mathrm{d}x \leq \mathrm{C} t^{-1/3} \int_0^{\sqrt{y}} e^{t U_q(x)} \mathrm{d}x. \tag{2.2.21}$$

We next bound the last expression in (2.2.21) in two cases.

Case 1. $0 \leq y \leq \frac{q^2}{16}$**.** Since $U_q''(x) = 2q - 8x$ is positive for $x \in [0, \frac{q}{4})$, the derivative $U_q'(x) = 2x(q - 2x)$ is increasing in $x \in [0, \frac{q}{4}]$. Hence, for any $z \in [0, \frac{q}{4}]$, $U_q'(z) \leq U_q'(\frac{q}{4}) = \frac{q^2}{4}$. Thus, for any $x \in [0, \sqrt{y}]$, we have $z_* \in [x, \sqrt{y}]$ for which

$$U_q(\sqrt{y}) - U_q(x) = U_q'(z_*)(\sqrt{y} - x) \le U_q'(\tfrac{q}{4})(\sqrt{y} - x) = \tfrac{q^2}{4}(\sqrt{y} - x). \tag{2.2.22}$$

Using (2.2.22) to bound $\exp(tU_q(x))$ and integrating the result over $x \in [0, \sqrt{y}]$ gives

$$\int_0^{\sqrt{y}} e^{tU_q(x)}\mathrm{d}x \le \int_0^{\sqrt{y}} e^{tU_q(\sqrt{y}) - \frac{1}{4}q^2 t(\sqrt{y}-x)}\mathrm{d}x \le \int_{-\infty}^{\sqrt{y}} e^{tU_q(\sqrt{y}) - \frac{1}{4}q^2 t(\sqrt{y}-x)}\mathrm{d}x \le \frac{4}{q^2 t} e^{tU_q(\sqrt{y})}. \tag{2.2.23}$$

Case 2. $\frac{q^2}{16} \le y \le \frac{q^2}{4}$**.** In this case we have $q \ge 2\sqrt{y}$, which gives

$$U_q(\sqrt{y}) - U_q(x) = q(y - x^2) - \tfrac{4}{3}(y^{3/2} - x^3) \ge 2\sqrt{y}(y - x^2) - \tfrac{4}{3}(y^{3/2} - x^3) = \tfrac{2}{3}(\sqrt{y} - x)^2(\sqrt{y} + 2x).$$

In the last expression, further use $\sqrt{y} + 2x \ge \sqrt{y} \ge \frac{q}{4}$ to get

$$U_q(\sqrt{y}) - U_q(x) \ge \tfrac{q}{6}(\sqrt{y} - x)^2. \tag{2.2.24}$$

Using (2.2.24) to bound $\exp(tU_q(x))$ and integrate the result over $x \in [0, \sqrt{y}]$ gives

$$\int_0^{\sqrt{y}} e^{tU_q(x)}\mathrm{d}x \le \int_0^{\sqrt{y}} e^{tU_q(\sqrt{y}) - \frac{1}{6}qt(\sqrt{y}-x)^2}\mathrm{d}x \le \int_{-\infty}^{\sqrt{y}} e^{tU_q(\sqrt{y}) - \frac{1}{6}qt(\sqrt{y}-x)^2}\mathrm{d}x \le \sqrt{\frac{C}{qt}} e^{tU_q(\sqrt{y})}. \tag{2.2.25}$$

Combining (2.2.23) and (2.2.25) and inserting the bounds into (2.2.21) gives the desired result.

□

2.3 Estimates for the leading term

The goal of this section is to obtain the $t \to \infty$ asymptotics of $\widetilde{\mathcal{A}}_p(t)$ defined in (2.2.5), accurate up to constant multiples.

Let us first settle the differentiability in s of the operator $K_{s,t}$, defined in (2.1.11). Recall $K_{s,t}(x,y)$ from (2.1.11), then perform a change of variable $r \mapsto t^{2/3}r$ to get

$$K_{s,t}(x,y) = t^{2/3}\int_{\mathbb{R}} \mathrm{Ai}(x+t^{2/3}r)\,\mathrm{Ai}(y+t^{2/3}r)v(s,t,r)\mathrm{d}r, \tag{2.3.1}$$

$$v(s,t,r) := \frac{1}{1+\frac{1}{s}e^{-rt}}. \tag{2.3.2}$$

Formally differentiating the kernel $K_{s,t}(x,y)$ in (2.1.11) in s suggests that the n-th derivative of $K_{s,t}$ should have kernel

$$K^{(n)}_{s,t}(x,y) := t^{2/3}\int_{\mathbb{R}} \mathrm{Ai}(x+t^{2/3}r)\,\mathrm{Ai}(y+t^{2/3}r)\partial_s^n v(s,t,r)\,\mathrm{d}r, \tag{2.3.3}$$

with the convention $K^{(0)}_{s,t}(x,y) := K_{s,t}(x,y)$. Differentiating (2.3.2) with respect to s we get

$$\partial_s^n v(s,t,r) = \frac{(-1)^{n-1}n!e^{-rt}}{(s+e^{-rt})^{n+1}}. \tag{2.3.4}$$

Since $(-1)^{n-1}\partial_s^n v(s,t,r) > 0$, Lemma 2.2.1 applied with $J(r,y) = \mathrm{Ai}(y+t^{2/3}r)((-1)^{n-1}\partial_s^n v(s,t,r))^{1/2}$ gives that $(-1)^{n-1}K^{(n)}_{s,t}$ defines a positive trace-class operator on $L^2(\mathbb{R}_{\geq 0})$.

Lemma 2.3.1. *For any $n \in \mathbb{Z}_{\geq 0}$, $u \in [1,\infty]$ and $t > 0$, the operator $K^{(n)}_{s,t}$ is differentiable in s at each $s > 0$ in the u-th Schatten norm, with derivative being equal to $K^{(n+1)}_{s,t}$, i.e.,*

$$\lim_{s'\to s}\left\|\frac{K^{(n)}_{s',t}-K^{(n)}_{s,t}}{s'-s} - K^{(n+1)}_{s,t}\right\|_u = 0.$$

Proof. Since the Schatten norms decreases in u, without lost of generality we assume $u = 1$. Fix $n \in \mathbb{Z}_{\geq 0}$ and $t > 0$, and set $D_{s,s'} := \frac{1}{s'-s}(K^{(n)}_{s',t} - K^{(n)}_{s,t}) - K^{(n+1)}_{s,t}$. Use (2.3.3) to express the kernel of $D_{s,s'}$ as an integral involving $\partial_s^n v$ and $\partial_s^{n+1}v$, and Taylor expand $\partial_\sigma^n v(\sigma,t,r)$ around $\sigma = s$ up to the first order, i.e., $\partial_\sigma^n v(s',r) - \partial_\sigma^n v(s,t,r) - (s'-s)\partial_\sigma^{n+1}v(s,t,r) = \frac{1}{2}\int_s^{s'}(s'-\sigma)\partial_\sigma^{n+2}v(\sigma,t,r)\,\mathrm{d}\sigma$.

We then get

$$
\begin{aligned}
D_{s,s'}(x,y) &= t^{2/3}\int_{\mathbb{R}} \mathrm{Ai}(x+t^{2/3}r)\,\mathrm{Ai}(y+t^{2/3}r)\Big(\frac{1}{2(s'-s)}\int_s^{s'}(s'-\sigma)\partial_\sigma^{n+2}v(\sigma,t,r)\,\mathrm{d}\sigma\Big)\mathrm{d}r\\
&= t^{2/3}\int_{\mathbb{R}_{\geq 0}\times\mathbb{R}} \mathrm{Ai}(x+t^{2/3}r)\,\mathrm{Ai}(y+t^{2/3}r)\mathrm{sign}(s'-s)\mathbf{1}_{|(s,s')|}(\sigma)\frac{(s'-\sigma)}{2(s'-s)}\partial_\sigma^{n+2}v(\sigma,t,r)\,\mathrm{d}\sigma\mathrm{d}r,
\end{aligned}
$$

where $|(s,s')| := (s,s')$ for $s<s'$ and $|(s,s')| := (s',s)$ for $s'<s$.

Our goal is to show that $\|D_{s,s'}\|_1$ converges to zero as $s'\to s$. As seen from (2.3.4), we have $(-1)^{n+1}\partial_\sigma^{n+2}v(\sigma,t,r)>0$. Applying Lemma 2.2.1 with $J(r,y)=\mathrm{Ai}(y+t^{2/3}r)((-1)^{n+1}\partial_\sigma^{n+2}v(\sigma,t,r))^{-1/2}$ gives

$$
\|D_{s,s'}\|_1 = t^{2/3}\int_{\mathbb{R}^2_{\geq 0}\times\mathbb{R}} \mathrm{Ai}^2(x+t^{2/3}r)\mathbf{1}_{|(s,s')|}(\sigma)\Big|\frac{(s'-\sigma)}{2(s'-s)}\partial_\sigma^{n+2}v(\sigma,t,r)\Big|\mathrm{d}\sigma\mathrm{d}x\mathrm{d}r, \tag{2.3.5}
$$

provided that the last integral converges. To check the convergence, recognizing $\int_{\mathbb{R}_{\geq 0}}\mathrm{Ai}(x+t^{2/3}r)^2\mathrm{d}x=\Phi(t^{2/3}r)$ substituting (2.3.4) into (2.3.5), bound $(\sigma+e^{-rt})^{n+3}\geq e^{-(n+3)rt}$, and $|\frac{s'-\sigma}{2(s'-s)}|\leq\frac{1}{2}$,

$$
(\text{r.h.s. of (2.3.5)}) \leq \frac{1}{2}(n+2)!t^{2/3}\int_{\mathbb{R}_{\geq 0}\times\mathbb{R}} e^{(n+2)rt}\Phi(t^{2/3}r)\mathbf{1}_{|(s,s')|}(\sigma)|\mathrm{d}\sigma\mathrm{d}r. \tag{2.3.6}
$$

By Lemma 2.2.3 with $y\mapsto\infty$, the r.h.s. of (2.3.6) is finite for each $s'\in\mathbb{R}_{\geq 0}$. From this and the dominated convergence theorem, we conclude the desired result $\|D_{s,s'}\|_1\leq$ (r.h.s. of (2.3.6)) $\to 0$, as $s'\to s$. □

Applying Lemma 2.3.1 with $u=1$ gives $\partial_s^n\mathrm{tr}(K_{s,t})=\mathrm{tr}(K_{s,t}^{(n)})$. Further, since the operator $K_{s,t}^{(n)}$ has a continuous kernel given in (2.3.3) and is a trace-class operator, the trace can be written as $\mathrm{tr}(K_{s,t}^{(n)})=\int_0^\infty K_{s,t}^{(n)}(x,x)\,\mathrm{d}x$ (see Corollary 3.2 in [77]). To evaluate the last integral, insert (2.3.2) into (2.3.1) and (2.3.4) into (2.3.3) to get

$$
\mathrm{tr}(K_{s,t}) = t^{2/3}\int_{\mathbb{R}}\frac{1}{1+\frac{1}{s}e^{-rt}}\Phi(t^{2/3}r)\,\mathrm{d}r, \tag{2.3.7}
$$

$$\partial_s^n \mathrm{tr}(K_{s,t}) = \mathrm{tr}(K_{s,t}^{(n)}) = t^{2/3} \int_{\mathbb{R}} \frac{(-1)^{n-1}\, n!\, e^{-rt}}{(s+e^{-rt})^{n+1}} \Phi(t^{2/3} r)\, \mathrm{d}r, \quad n \in \mathbb{Z}_{>0}, \tag{2.3.8}$$

where $\Phi(y)$ is defined in (2.2.7). Armed with the expressions (2.3.7) and (2.3.8), we now proceed to establish the desired asymptotics of $\widetilde{\mathcal{A}}_p(t)$. Recall from (2.2.5) $\widetilde{\mathcal{A}}_p(t)$ involves an integral over $s \in [0,1]$. It is convention to write it as the difference of an integral over $s \in [0,\infty)$ and over $s \in [0,1]$:

$$\widetilde{\mathcal{A}}_p(t) = \mathcal{A}_p(t) - \widehat{\mathcal{A}}_p(t), \tag{2.3.9}$$

$$\mathcal{A}_p(t) := \frac{(-1)^{n+1}}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha}\, \partial_s^n\, \mathrm{tr}(K_{s,t})\, \mathrm{d}s, \quad \widehat{\mathcal{A}}_p(t) := \frac{(-1)^{n+1}}{\Gamma(1-\alpha)} \int_1^\infty s^{-\alpha}\, \partial_s^n\, \mathrm{tr}(K_{s,t})\, \mathrm{d}s, \tag{2.3.10}$$

where $n := \lfloor p \rfloor + 1 \in \mathbb{Z}_{>0}$ and $\alpha := p - \lfloor p \rfloor \in [0,1)$.

Proposition 2.3.2. *Fix any $t_0, p_0 > 0$. There exists $\mathrm{C} = \mathrm{C}(t_0, p_0) > 0$ such that for all $t \geq t_0$ and $p \geq p_0$,*

$$\tfrac{1}{\mathrm{C}} p^{-\frac{3}{2}} \Gamma(p+1)\, t^{-\frac{1}{2}}\, e^{\frac{p^3 t}{12}} \leq \mathcal{A}_p(t) \leq \mathrm{C} p^{-\frac{3}{2}} \Gamma(p+1)\, t^{-\frac{1}{2}}\, e^{\frac{p^3 t}{12}}, \tag{2.3.11}$$

$$|\widehat{\mathcal{A}}_p(t)| \leq \Gamma(p+1)\mathrm{C}. \tag{2.3.12}$$

Proof. Fix $t_0, p_0 > 0$. To simplify notation, throughout this proof we assume $t \geq t_0$ and $p \geq p_0$ and write $\mathrm{C} = \mathrm{C}(t_0, p_0)$. Referring to (2.2.5) and (2.3.8), we set

$$\phi_{p,t}(s) := \frac{n! t^{2/3}}{\Gamma(1-\alpha)} s^{-\alpha} \int_{\mathbb{R}} \frac{e^{-rt} \Phi(t^{2/3} r)}{(s+e^{-rt})^{n+1}}\, \mathrm{d}r \tag{2.3.13}$$

so that $\mathcal{A}_p(t) = \int_0^\infty \phi_{p,t}(s)\mathrm{d}s$ and $\widehat{\mathcal{A}}_p(t) = \int_1^\infty \phi_{p,t}(s)\mathrm{d}s$.

To estimate $\mathcal{A}_p(t) = \int_0^\infty \phi_{p,t}(s)\mathrm{d}s$, integrate (2.3.13) over $s \in [0,\infty)$ to get

$$\int_0^\infty \phi_{p,t}(s)\, \mathrm{d}s = \frac{n! t^{2/3}}{\Gamma(1-\alpha)} \int_{\mathbb{R}} e^{-rt} \Phi(t^{2/3} r) \Big(\int_0^\infty \frac{s^{-\alpha} \mathrm{d}s}{(s+e^{-rt})^{n+1}} \Big) \mathrm{d}r.$$

The inner integral on the right hand side can be identified with the Beta integral. Namely the change of variable $v = \frac{s}{s+e^{-rt}}$ yields

$$\int_0^\infty \frac{s^{-\alpha}\mathrm{d}s}{(s+e^{-rt})^{n+1}} = e^{nrt+\alpha rt}\int_0^1 v^{-\alpha}(1-v)^{n-1+\alpha}\mathrm{d}v = e^{nrt+\alpha rt}\frac{\Gamma(1-\alpha)\Gamma(n+\alpha)}{n!}. \tag{2.3.14}$$

This then gives $\int_0^\infty \phi_{p,t}(s)\,\mathrm{d}s = t^{2/3}\Gamma(p+1)\int_{\mathbb{R}} e^{prt}\Phi(t^{2/3}r)\mathrm{d}r$. The asymptotics of last integral is given by Lemma 2.2.2 with $q \mapsto p$. From this we conclude the desired estimate (2.3.11) of $\mathcal{A}_p(t)$.

Next we turn to $\widehat{\mathcal{A}}_p(t) = \int_1^\infty \phi_{p,t}(s)\mathrm{d}s$. Integrate (2.3.13) over $s \in (1,\infty)$, divide the integral over $r \in (-\infty, 0]$ and $r \in [0,\infty)$, and for the former release the integral over s from $s \in (1,\infty)$ to $s \in [0,\infty)$. This gives $0 \le \int_1^\infty \phi_{p,t}(s)\,\mathrm{d}s \le A_1 + A_2$, where

$$A_1 := \frac{n!t^{2/3}}{\Gamma(1-\alpha)}\int_{[0,\infty)\times(-\infty,0]} s^{-\alpha}\frac{e^{-rt}\Phi(t^{2/3}r)}{(s+e^{-rt})^{n+1}}\,\mathrm{d}s\mathrm{d}r, \quad A_2 := \frac{n!t^{2/3}}{\Gamma(1-\alpha)}\int_{(1,\infty)\times[0,\infty)} s^{-\alpha}\frac{e^{-rt}\Phi(t^{2/3}r)}{(s+e^{-rt})^{n+1}}\,\mathrm{d}s\mathrm{d}r.$$

For A_1 use (2.3.14) and then the bound from Lemma 2.2.3 with $q \mapsto p$ and $y \mapsto 0$. We have

$$A_1 = t^{2/3}\Gamma(p+1)\int_{-\infty}^0 e^{prt}\Phi(t^{2/3}r)\,\mathrm{d}r \le t^{-1/6}\Gamma(p+1)\mathrm{C}. \tag{2.3.15}$$

For A_2, use $s \ge 1$ to bound $s^{-\alpha}\frac{1}{(s+e^{-rt})^{n+1}} \le s^{-n-1}$ and use the fact that Φ is decreasing (see (2.2.7)) to bound $\Phi(t^{2/3}r) \le \Phi(0) = \mathrm{C}$. Together with $\frac{1}{\Gamma(1-\alpha)} \le 1$, for $\alpha \in [0,1)$, we have

$$A_2 \le \mathrm{C}\frac{n!t^{2/3}}{\Gamma(1-\alpha)}\int_1^\infty s^{-1-n}\,\mathrm{d}s\int_0^\infty e^{-rt}\mathrm{d}r \le (n-1)!t^{-1/3}\mathrm{C} \le t^{-1/3}\Gamma(p+1)\mathrm{C}. \tag{2.3.16}$$

The last inequality follows from the fact that $\Gamma(y)$ is increasing for $y \ge 1$ to bound $(n-1)! = \Gamma(n) \le \Gamma(p+1)$. Using $t \ge t_0$ to bound $t^{-1/6}, t^{-1/3} \le \mathrm{C}$, the bounds (2.3.15) and (2.3.16) together gives the desired bound for (2.3.12). $\square$

2.4 Bounds for higher order terms

To goal of this section is to establish bounds on the term $\mathcal{B}_{p,L}(t)$ defined in (2.2.6). Along the way we will also justify passing derivatives into sums in (2.2.4).

Recall from (2.3.3) and Lemma 2.3.1 that $K^{(n)}_{s,t}$ is the n-th derivative in s of $K_{s,t}$. To prepare for subsequent analysis, we provide bounds on $\mathrm{tr}(K_{s,t})$ and $\mathrm{tr}(K^{(n)}_{s,t})$.

Lemma 2.4.1. *Recall U_q from* (2.2.10)*. For any $t_0 > 0$, there exists a constant $\mathrm{C}(t_0) > 0$ such that for all $\sigma \in [0, \infty]$, $t > t_0$, and $n \in \mathbb{Z}_{>0}$,*

$$\left|\mathrm{tr}(K_{e^{-t\sigma},t})\right| \le \mathrm{C}(t_0) \exp(tU_1(\min\{\sqrt{\sigma}, \tfrac{1}{2}\}) - t\sigma), \tag{2.4.1}$$

$$\left|\mathrm{tr}(K^{(n)}_{e^{-t\sigma},t})\right| \le n!\,\mathrm{C}(t_0)\ \exp(tU_n(\min\{\sqrt{\sigma}, \tfrac{n}{2}\})). \tag{2.4.2}$$

Proof. The starting point of the proof is the explicit expressions (2.3.7) and (2.3.8) of the traces. In (2.3.7), set $s = e^{-\sigma t}$ and divide the integral into $r < \sigma$ and $r > \sigma$ to get

$$\mathrm{tr}(K_{e^{-t\sigma},t}) = t^{2/3}\Big(\int_{-\infty}^{\sigma} + \int_{\sigma}^{\infty}\Big)\frac{\Phi(t^{2/3}r)\mathrm{d}r}{1 + e^{t\sigma - tr}} := \mathcal{I}_1 + \mathcal{I}_2. \tag{2.4.3}$$

For $\mathcal{I}_1$ use $1 + e^{t\sigma - tr} \ge e^{t\sigma - tr}$ and Lemma 2.2.3 with $q = 1$ and $y = \sigma$. We have, for all $t \ge t_0$,

$$\mathcal{I}_1 \le \mathrm{C}(t_0)\ \exp(tU_1(\min\{\sqrt{\sigma}, \tfrac{1}{2}\}) - t\sigma). \tag{2.4.4}$$

The second integral $\mathcal{I}_2$ can be calculated explicitly by using Airy differential equation, whereby

$$\mathcal{I}_2 = \int_{t^{2/3}\sigma}^{\infty} \Phi(r)\mathrm{d}r := g(t^{2/3}\sigma), \quad g(y) = \tfrac{1}{3}(2y^2 \operatorname{Ai}(y)^2 - 2y \operatorname{Ai}'(y)^2 - \operatorname{Ai}(y) \operatorname{Ai}'(y)). \tag{2.4.5}$$

Using the known $|y| \gg 1$ asymptotics of $\operatorname{Ai}(y)$ and $\operatorname{Ai}'(y)$ (see Equations (1.07), (1.08), and (1.09) in Chapter 11 of [267] for example), we obtain $g(y) \le \mathrm{C}\exp(-\frac{4}{3}y^{3/2})$ for all $y \ge 0$. Using (2.2.11) we further bound the exponent $-\frac{4}{3}y^{3/2} \le U_1(\min\{\sqrt{y}, \frac{1}{2}\}) - y$ for all $y \ge 0$. From this we conclude (2.4.1).

Moving on, similarly to the preceding, in (2.3.7) we set $s = e^{-\sigma t}$ and divide the integral into $r < \sigma$ and $r > \sigma$ to get

$$|\mathrm{tr}(K^{(n)}_{e^{-t\sigma},t})| = n!t^{2/3}\Big(\int_{-\infty}^{\sigma} + \int_{\sigma}^{\infty}\Big)\frac{e^{-rt}\Phi(t^{2/3}r)\mathrm{d}r}{(e^{-\sigma t}+e^{-rt})^{n+1}} := \mathcal{J}_1 + \mathcal{J}_2.$$

For $\mathcal{J}_1$, use $e^{-\sigma t} + e^{-rt} \geq e^{-rt}$ and Lemma 2.2.3 with $q = n$ to get, for $t \geq t_0$,

$$\mathcal{J}_1 \leq \int_{-\infty}^{\sigma} e^{nrt}\Phi(t^{2/3}r)\mathrm{d}r \leq n!\,\mathrm{C}(t_0)\,t^{-5/6}\exp(tU_n(\min\{\sqrt{\sigma},\tfrac{n}{2}\})) \leq n!\,\mathrm{C}(t_0)\,\exp(tU_n(\min\{\sqrt{\sigma},\tfrac{n}{2}\})).$$

This gives the desired bound for showing (2.4.2). As for $\mathcal{J}_2$, use $e^{-\sigma t} + e^{-rt} \geq e^{-\sigma t}$ and the fact that Φ is non-increasing to get

$$\mathcal{J}_2 \leq e^{t(n+1)\sigma}\Phi(t^{2/3}\sigma)\int_{\sigma}^{\infty} e^{-rt}\mathrm{d}r = t^{-1}e^{tn\sigma}\Phi(t^{2/3}\sigma).$$

Further bounding $\Phi(y) \leq \mathrm{C}\exp(-\frac{4}{3}y^{3/2})$ (by (2.2.9)) gives $\mathcal{J}_2 \leq t_0^{-1}\exp(tU_n(\sqrt{\sigma}))$, for all $t \geq t_0$. From (2.2.11) we have $U_q(\sqrt{s}) \leq U_q(\min\{\sqrt{s}, q/2\})$, for all $\sigma, q > 0$. From this we conclude $\mathcal{J}_2 \leq t_0^{-1}\exp(tU_n(\min\{\sqrt{s},\frac{n}{2}\}))$, for all $t \geq t_0$. This completes the proof of (2.4.2). □

2.4.1 Interchange of sum and derivatives

In this subsection, we show that the series

$$\sum_{L=1}^{\infty}(-1)^L\mathrm{tr}(K_{s,t}^{\wedge L}) \tag{2.4.6}$$

is infinitely differentiable in s and the derivative can be obtained by taking term-by-term differentiation. To this end we will use the following standard criterion:

Proposition 2.4.2. *Let $f_k(s)$, $k \in \mathbb{Z}_{>0}$, be $(n+1)$ times continuously differentiable functions on $s \in [0,1]$, where $n \in \mathbb{Z}_{>0}$. If the series $f(s) := \sum_{k=1}^{\infty} f_k(s)$ converges absolutely at each*

$s \in [0,1]$*, and if the absolute derivative series* $\sum_{k=1}^{\infty} |\frac{\mathrm{d}^j}{\mathrm{d}s^j} f_k(s)|$ *converges uniformly over bounded intervals in* $[0,1]$*, for all* $j = 1, \ldots, n+1$*, then* f *is* n*-th differentiable for all* $s \in [0,1]$ *with* $\frac{\mathrm{d}^j}{\mathrm{d}s^j} f(s) = \sum_{k=1}^{\infty} \frac{\mathrm{d}^j}{\mathrm{d}s^j} f_k(s)$*, for all* $j = 1, \ldots, n$.

The proof of this proposition is standard by applying Dini's theorem to the sequence $\sum_{\ell=1}^{k} \int_0^s |\frac{\mathrm{d}^{j+1}}{\mathrm{d}s^{j+1}} f_\ell(s)| \mathrm{d}s$.

Let us consider first the s derivative of $\mathrm{tr}(K_{s,t}^{\wedge L})$. Recall from (2.3.3) and Lemma 2.4.4 that $K_{s,t}^{(n)}$ denotes the n-th s derivative of $K_{s,t}$. Fix any orthonormal basis $\{e_i\}_{i \ge 1}$ for $L^2(\mathbb{R}_{\ge 0})$ and write

$$\mathrm{tr}(K_{s,t}^{\wedge L}) = \sum_{i_1 < \ldots < i_L} \left\langle e_{i_1} \wedge \ldots \wedge e_{i_L}, K_{s,t} e_{i_1} \wedge \ldots \wedge K_{s,t} e_{i_L} \right\rangle = \sum_{i_1 < \ldots < i_L} \det\left(\left\langle e_{i_k}, K_{s,t} e_{i_\ell} \right\rangle \right)_{k,\ell=1}^{L}. \quad (2.4.7)$$

Formally taking ∂_s^n in (2.4.7) and passing (without justification at the moment) the derivatives into the sum and inner product suggest that the following should hold

$$\partial_s^n \mathrm{tr}(K_{s,t}^{\wedge L}) = \sum_{i_1 < \ldots < i_L} \sum_{\vec{m} \in \mathfrak{M}(L,n)} \binom{n}{\vec{m}} \det\left(\left\langle e_{i_k}, K_{s,t}^{(m_\ell)} e_{i_\ell} \right\rangle \right)_{k,\ell=1}^{L},$$

where

$$\mathfrak{M}(L,n) := \left\{ \vec{m} = (m_1, \ldots, m_L) \in (\mathbb{Z}_{\ge 0})^L : m_1 + \cdots + m_L = n \right\}, \quad (2.4.8)$$

$$\binom{n}{\vec{m}} := \frac{n!}{m_1! \cdots m_L!}. \quad (2.4.9)$$

We now proceed to justify this formal calculation. Doing so requires an inequality. Recall that $\|\bullet\|_2$ denotes the Hilbert–Schmidt norm.

Lemma 2.4.3. *Fix any* $k \in \mathbb{Z}_{>0}$ *and any permutation* $\pi \in \mathbb{S}_k$*. Let* $T_1, T_2, \ldots, T_k$ *be self-adjoint Hilbert–Schmidt operators on a separable Hilbert space* H*, and let* $\{e_i\}_{i \ge 1}$ *be any orthonormal basis. Then*

$$\sum_{i_1, \ldots, i_k \in \mathbb{Z}_{>0}} \prod_{\ell=1}^{k} \left| \langle e_{i_\ell}, T_{\pi(\ell)} e_{i_{\pi(\ell)}} \rangle \right| \le \prod_{i=1}^{k} \|T_i\|_2. \quad (2.4.10)$$

Proof. It suffices to prove (2.4.10) for the case when π is a cycle of length k. For general $\pi \in \mathbb{S}_k$, decompose it into cycles of smaller lengths and apply the result within each cycle. Further, since the r.h.s. of (2.4.10) is symmetric in $T_1, \ldots, T_k$, we may assume without loss of generality $\pi = (12 \ldots k)$. Under this assumption the l.h.s. of (2.4.10) becomes

$$\sum_{i_1,\ldots,i_k \in \mathbb{Z}_{>0}} \prod_{\ell=1}^{k} \big|\langle e_{i_\ell}, T_{\ell+1} e_{i_{\ell+1}} \rangle\big|, \tag{2.4.11}$$

with the convention $T_{k+1} := T_1$ and $e_{i_{k+1}} := e_{i_1}$.

Let $|\bullet|_H$ denote the norm of the Hilbert space H. Apply the Cauchy–Schwarz inequality in (2.4.11) over the sum $i_2 \in \mathbb{Z}_{>0}$, and within the result recognize $(\sum_{i_2} |\langle e_{i_1}, T_2 e_{i_2}\rangle|^2)^{1/2} = |T_2 e_{i_1}|_H$ and $(\sum_{i_2} |\langle e_{i_2}, T_3 e_{i_3}\rangle|^2)^{1/2} = |T_3 e_{i_3}|_H$. We have

$$\text{l.h.s. of (2.4.11)} \leq \sum_{i_1,i_3,\ldots,i_k} \big|T_2 e_{i_1}\big|_H \big|T_3 e_{i_3}\big|_H \prod_{\ell=4}^{k} \big|\langle e_{i_\ell}, T_{\ell+1} e_{i_{\ell+1}} \rangle\big|.$$

Next apply the Cauchy-Schwarz inequality over the sum $i_3 \in \mathbb{Z}_{>0}$. Within the result recognize $(\sum_{i_3} |T_3 e_{i_3}|_H^2)^{1/2} = \|T_3\|_2$ and $(\sum_{i_3} |\langle e_{i_3}, T_4 e_{i_4}\rangle|^2)^{1/2} = |T_4 e_{i_4}|_H$. We have

$$\text{l.h.s. of (2.4.11)} \leq \sum_{i_1,i_4,\ldots,i_k} \big|T_2 e_{i_1}\big|_H \big\|T_3 e_{i_3}\big\|_2 \big|T_4 e_{i_4}\big|_H \prod_{\ell=5}^{L} \big|\langle e_{i_\ell}, T_{\ell+1} e_{i_{\ell+1}} \rangle\big|.$$

Continue this procedure through i_j, $j = 4, \ldots, k$. Each application of the the Cauchy-Schwarz inequality turns the preexisting $|T_j e_{i_j}|_H$ into $\|T_j\|_2$ and produces $|T_{j+1} e_{i_{j+1}}|_H$. Finally, after the $j = k$ step, an application of the Cauchy–Schwarz inequality over i_1 turns $|T_2 e_{i_1}|_H$ and $|T_1 e_{i_1}|_H$ into $\|T_2\|_2$ and $\|T_1\|_2$. □

Lemma 2.4.4. *Let $\mathfrak{M}(L, n)$ be in* (2.4.8)*. Fix $L \in \mathbb{Z}_{>0}$, and fix any orthonormal basis $\{e_i\}_{i\geq 1}$ for $L^2(\mathbb{R}_{\geq 0})$. For any $t > 0$, the function $s \mapsto \mathrm{tr}(K_{s,t}^{\wedge L})$ is infinitely differentiable at each $s \in [0, 1]$, with*

$$\partial_s^n \mathrm{tr}(K_{s,t}^{\wedge L}) = \sum_{i_1<\ldots<i_L} \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \det\big(\big\langle e_{i_k}, K_{s,t}^{(m_\ell)} e_{i_\ell} \big\rangle\big)_{k,\ell=1}^{L}, \tag{2.4.12}$$

where the r.h.s. converges absolutely uniformly over $[0,1]$NIs.

Proof. First, by the product rule of calculus we have

$$\partial_s^n \det\left(\left\langle e_{i_k}, K_{s,t} e_{i_\ell}\right\rangle\right)_{k,\ell=1}^L = \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \det\left(\partial_s^{m_\ell}\left\langle e_{i_k}, K_{s,t} e_{i_\ell}\right\rangle\right)_{k,\ell=1}^L.$$

By Lemma 2.3.1 for $u=\infty$, the derivatives on the r.h.s. can be passed into the inner product to give

$$\partial_s^n \det\left(\left\langle e_{i_k}, K_{s,t} e_{i_\ell}\right\rangle\right)_{k,\ell=1}^L = \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \det\left(\left\langle e_{i_k}, K_{s,t}^{(m_\ell)} e_{i_\ell}\right\rangle\right)_{k,\ell=1}^L. \tag{2.4.13}$$

We wish to apply Proposition 2.4.2 with $\{f_k\}_{k=1}^\infty$ being an enumeration of $\{\det(\langle e_{i_k}, K_{s,t} e_{i_\ell}\rangle)_{k,\ell=1}^L\}_{i_1<\ldots<i_L}$. The series in (2.4.7) converges absolutely for each $s\in[0,1]$ (with $t\in(0,\infty)$ fixed) because $K_{s,t}^{\wedge L}$ is trace-class. Given the identity (2.4.13) for the derivative series, it suffices to prove that the r.h.s. of (2.4.12) converges absolutely and uniformly over $[0,1]$NIs. To this end, apply Lemma 2.4.3 with $k=L$ and $T_i = K_{s,t}^{(m_i)}$ to get

$$\sum_{i_1<\ldots<i_L} \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \left|\det\left(\left\langle e_{i_k}, K_{s,t}^{(m_\ell)} e_{i_\ell}\right\rangle\right)_{k,\ell=1}^L\right| \le L! \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \prod_{\ell=1}^L \left\|K_{s,t}^{(m_\ell)}\right\|_2.$$

Recall that $(-1)^{m-1}K_{s,t}^{(m)}$ is a positive trace-class operator, whereby $\|K_{s,t}^{(m)}\|_2 \le \|K_{s,t}^{(m)}\|_1 = |\mathrm{tr}(K_{s,t}^{(m)})|$ and

$$\sum_{i_1<\ldots<i_L} \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \left|\det\left(\left\langle e_{i_k}, K_{s,t}^{(m_\ell)} e_{i_\ell}\right\rangle\right)_{k,\ell=1}^L\right| \le L! \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \prod_{\ell=1}^L \left|\mathrm{tr}(K_{s,t}^{(m_\ell)})\right|. \tag{2.4.14}$$

The bounds from Lemma 2.4.1 guarantee that the r.h.s. of (2.4.14) converges uniformly over $[0,1]$NIs, for fixed $t>0$. □

We now consider the s derivative of the series (2.4.6).

Proposition 2.4.5. *Let $\mathfrak{M}(L,n)$ be in* (2.4.8)*. For $\vec{m} \in \mathfrak{M}(L,n)$, set $\vec{m}_{>0} := \{k : m_k > 0\} \subset \{1,\ldots,L\}$ and let $|\vec{m}_{>0}|$ denotes the cardinality. For any $t > 0$, the series* (2.4.6) *is infinitely differentiable in $s \in [0,1]$, with*

$$\partial_s^n \Big(\sum_{L=1}^{\infty} (-1)^L \mathrm{tr}(K_{s,t}^{\wedge L}) \Big) = \sum_{L=1}^{\infty} (-1)^L \, \partial_s^n \, \mathrm{tr}(K_{s,t}^{\wedge L}), \tag{2.4.15}$$

$$\big|\partial_s^n \mathrm{tr}(K_{s,t}^{\wedge L})\big| \leq \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \frac{(|\vec{m}_{>0}|)!}{(L-|\vec{m}_{>0}|)!} \prod_{k=1}^{L} \big|\mathrm{tr}(K_{s,t}^{(m_k)})\big|. \tag{2.4.16}$$

Proof. We will appeal to Proposition 2.4.2, with the choice $f_L(s) = (-1)^L \mathrm{tr}(K_{s,t}^{\wedge L})$. Doing so requires bounds on the derivatives series, which we achieve by using Lemma 2.4.4. This lemma holds for *any* orthonormal basis, and here, with $K_{s,t}$ being compact and symmetric, we specialize to the eigenbasis of $K_{s,t}$. Let $\{v_i\}_{i\geq 1}$ be an orthonormal basis of $K_{s,t}$, with eigenvalue λ_i. Indeed v_i and λ_i depend on s,t, but we omit such dependence since in the subsequent analysis we will *not* vary s,t. Expand the determinant in (2.4.12) into a sum of permutations, and specialize to $e_i = v_i$:

$$\partial_s^n \mathrm{tr}(K_{s,t}^{\wedge L}) = \sum_{i_1<\ldots<i_L} \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \sum_{\pi\in\mathbb{S}_L} \mathrm{sign}(\pi) \prod_{k=1}^{L} \big\langle v_{i_k}, K_{s,t}^{(m_{\pi(i_k)})} v_{i_{\pi(i_k)}} \big\rangle. \tag{2.4.17}$$

Recall the convention $K_{s,t}^{(0)} := K_{s,t}$. Because of the eigenrelation $K_{s,t} v_i = \lambda_i v_i$, the product in (2.4.17) vanishes unless $\pi(r) = r$ for all $r \in \{k : m_k = 0\}$. Such permutations can be reduced to permutations on the set $\vec{m}_{>0} \subset \{1,\ldots,L\}$, and we let $\mathbb{S}(\vec{m}_{>0})$ denote the subgroup of all such reduced permutations. The preceding discussion brings (2.4.17) to

$$\partial_s^n \mathrm{tr}(K_{s,t}^{\wedge L}) = \sum_{i_1<\ldots<i_L} \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \prod_{k:m_k=0} \lambda_{i_k} \sum_{\pi\in\mathbb{S}(\vec{m}_{>0})} \mathrm{sign}(\pi) \prod_{k\in\vec{m}_{>0}} \big\langle v_{i_k}, K_{s,t}^{(m_{\pi(i_k)})} v_{i_{\pi(i_k)}} \big\rangle.$$

To bound this expression, take absolute value and pass it into the sum and products on the r.h.s., bound the ordered sum $\sum_{i_1<\ldots<i_L}$ by the symmetrized sum $\frac{1}{(L-|\vec{m}_{>0}|)!} \sum_{i_k:m_k=0} \sum_{i_\ell:\ell\in\vec{m}_{>0}}$, and then use $\sum_i |\lambda_i| = \sum_i \lambda_i = \mathrm{tr}(K_{s,t}) = \mathrm{tr}(K_{s,t}^{(0)})$. We have

$$\left|\partial_s^n \mathrm{tr}(K_{s,t}^{\wedge L})\right| \leq \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \frac{1}{(L-|\vec{m}_{>0}|)!} \mathrm{tr}(K_{s,t}^{(0)})^{L-|\vec{m}_{>0}|} \sum_{\pi\in\mathbb{S}(\vec{m}_{>0})} \sum_{i_\ell:\ell\in\vec{m}_{>0}} \prod_{\ell\in\vec{m}_{>0}} \left|\left\langle v_{i_\ell}, K_{s,t}^{(m_{\pi(i_\ell)})} v_{i_{\pi(i_\ell)}}\right\rangle\right|.$$

Now apply Lemma 2.4.3 with $k \mapsto |\vec{m}_{>0}|$ and with the T_i's being the $K_{s,t}^{(m_k)}$'s, and use $\|K_{s,t}^{(m)}\|_2 \leq \|K_{s,t}^{(m)}\|_1 = |\mathrm{tr}(K_{s,t}^{(m)})|$. We further obtain

$$\left|\partial_s^n \mathrm{tr}(K_{s,t}^{\wedge L})\right| \leq \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \frac{(|\vec{m}_{>0}|)!}{(L-|\vec{m}_{>0}|)!} \prod_{k=1}^{L} \left|\mathrm{tr}(K_{s,t}^{(m_k)})\right|. \tag{2.4.18}$$

This is exactly (2.4.16).

The bounds from Lemma 2.4.1 ensure that $\prod_{k=1}^{L} |\mathrm{tr}(K_{s,t}^{(m_k)})| \leq \mathrm{C}(t,n)^L$, for all $s \in [0,1]$. Given this, it is straightforward to verify that, when summed over $L \geq 1$, the r.h.s. of (2.4.18) converges uniformly over $[0,1]\ni s$, for fixed t, n. Proposition 2.4.2 applied with $f_L(s) = (-1)^L \mathrm{tr}(K_{s,t}^{\wedge L})$ completes the proof. $\square$

2.4.2 Bounds.

The goal of this subsection is to bound the term $\mathcal{B}_{p,L}(t)$, defined in (2.2.6). Recall $\mathfrak{M}(L,n)$ from (2.4.8). Referring to (2.2.6) and (2.4.16), we see that

$$\left|\mathcal{B}_{p,L}(t)\right| \leq \frac{1}{\Gamma(1-\alpha)} \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \frac{(|\vec{m}_{>0}|)!}{(L-|\vec{m}_{>0}|)!} \prod_{k=1}^{L} \int_0^1 s^{-\alpha} \left|\mathrm{tr}(K_{s,t}^{(m_k)})\right| \mathrm{d}s. \tag{2.4.19}$$

In view of (2.4.19), we first establish

Proposition 2.4.6. *Fix any $t_0, p_0 > 0$. There exists a constant $\mathrm{C} = \mathrm{C}(t_0, p_0) > 0$ such that for all $t > t_0$, $p \geq p_0$, $L \geq 2$, and $\vec{m} = (m_1, \ldots, m_L) \in \mathfrak{M}(L,n)$,*

$$\frac{1}{\Gamma(1-\alpha)} \int_0^1 s^{-\alpha} \prod_{j=1}^{L} |\mathrm{tr}(K_{s,t}^{(m_j)})| \mathrm{d}s \leq n \cdot n!\, \mathrm{C}^L\, t^{\frac{1}{2}}\, e^{\frac{p^3 t}{12} - \kappa_p t}. \tag{2.4.20}$$

where $n := \lfloor p \rfloor + 1$ *and* $\alpha := p - \lfloor p \rfloor$ *and* $\kappa_p := \min\{\frac{1}{6}, \frac{p^3}{16}\}$.

Proof. Fix $L \geq 2$, $p \geq p_0$, $\vec{m} = (m_1, \ldots, m_L) \in \mathfrak{M}(L, n)$. To simplify notation, throughout this proof we assume $t \geq t_0$ and $p \geq p_0$, and write $\mathrm{C} = \mathrm{C}(t_0, p_0)$. Set

$$\mathcal{I} := \frac{1}{\Gamma(1-\alpha)} \int_0^1 s^{-\alpha} \prod_{j=1}^{L} |\mathrm{tr}(K_{s,t}^{(m_j)})| \mathrm{d}s \tag{2.4.21}$$

and $|\vec{m}_{>0}| := r$. Assume without loss of generality $0 < m_1, \ldots, m_r$ and $m_{r+1} = \cdots = m_L = 0$. Our goal is to bound $\mathcal{I}$. In (2.4.21), perform a change of variable $s = e^{-t\sigma}$, apply the bounds from Lemma 2.4.1, and recall U_q from (2.2.10). We have, for all $t \geq t_0$,

$$\mathcal{I} \leq \frac{\mathrm{C}^L}{\Gamma(1-\alpha)} \int_0^\infty e^{t\sigma\alpha} \Big(\mathrm{C} e^{tU_1(\min\{\sqrt{\sigma}, \frac{1}{2}\} - t\sigma)}\Big)^{L-r} \cdot \prod_{j=1}^{r} (m_j)! e^{tU_{m_j}(\min\{\sqrt{\sigma}, \frac{m_j}{2}\})} \cdot t e^{-t\sigma} \mathrm{d}\sigma, \tag{2.4.22}$$

Given that $m_1 + \ldots + m_L = n$ we have $\prod_{j=1}^{r} (m_j)! \leq n!$. Apply this bound in (2.4.22), and combine the exponential functions in the integrand together to get $\exp(tM(\sigma))$, where

$$M(\sigma) := (\alpha - L + r - 1)\sigma + (L - r)U_1(\min\{\sqrt{\sigma}, \tfrac{1}{2}\}) + \sum_{j=1}^{r} U_{m_j}(\min\{\sqrt{\sigma}, \tfrac{m_j}{2}\}). \tag{2.4.23}$$

We arrive at

$$\mathcal{I} \leq \frac{t\mathrm{C}^L n!}{\Gamma(1-\alpha)} \int_0^\infty e^{tM(\sigma)} \, \mathrm{d}\sigma. \tag{2.4.24}$$

Our next step is to bound the exponent $M(\sigma)$, which we do in several different cases.

1. **When** $\sigma \in [0, \frac{1}{4}]$**.**

   Recall from (2.2.10) that $U_q(x)$ is increasing on $x \in [0, q/2]$. Hence, for $\sigma \leq \frac{1}{4}$, the 'min' operators in (2.4.23) always pick up $\sqrt{\sigma}$, whence $M(\sigma)$ simplifies into $M(\sigma) = p\sigma - \frac{4L}{3}\sigma^{3/2} := g_1(\sigma)$. This function g_1 achieves its maximum $\frac{p^3}{12L^2}$ at $\sigma = \frac{p^2}{4L^2}$. Further, $g_1(\sigma) - \frac{p^3}{12L^2} = -\frac{1}{3}(\sqrt{\sigma} -$

$\frac{p}{2L})^2(p+4L\sqrt{\sigma}) \leq -\frac{p}{3}(\sqrt{\sigma}-\frac{p}{2L})^2$. This gives

$$M(\sigma) \leq \tfrac{p^3}{12L^2} - \tfrac{p}{3}(\sqrt{\sigma}-\tfrac{p}{2L})^2. \tag{2.4.25}$$

2. **When** $r \geq 2$ **and** $\sigma \in (\frac{1}{4}, \infty)$**.**

   In this case, referring to (2.2.10), we see $U_1(\min\{\sqrt{\sigma}, \frac{1}{2}\}) = U_1(\frac{1}{2}) = \frac{1}{12}$. Hence $M(\sigma)$ simplifies into $M(\sigma) = \sigma(\alpha-1) - (L-r)(\sigma - \frac{1}{12}) + \sum_{j=1}^{r} U_{m_j}(\min\{\sqrt{\sigma}, \frac{1}{2}m_j\})$. Forgo the negative term $-(L-r)(\sigma-\frac{1}{12})$ and use (2.2.11) to bound $U_{m_j}(\min\{\sqrt{\sigma}, \frac{1}{2}m_j\}) \leq \frac{1}{12}m_j^3$. We have

$$M(\sigma) \leq \sigma(\alpha-1) + \sum\nolimits_{j=1}^{r} \tfrac{1}{12}m_j^3. \tag{2.4.26}$$

   Recall that $m_1 + \ldots + m_r = n$. The cubic sum in (2.4.26) tends to be larger when mass concentrates on fewer m_i's. Under the current assumption $r \geq 2$, it is conceivable that the cubit sum is at most $(n-1)^3 + 1^3$. To prove this, write $m_1^3 + \ldots + m_n^3 \leq m_1^3 + (m_2 + \cdots + m_n)^3 = m_1^3 + (n-m_1)^3$, and note that the last expression, as a function of $m_1 \in [1, n-1]$, reaches its maximum at $m_1 = 1, (n-1)$. Using this bound on the cubic sum we have

$$M(\sigma) \leq \sigma(\alpha-1) + \tfrac{1}{12}((n-1)^3+1). \tag{2.4.27}$$

3. **When** $r = 1$ **and** $\sigma \in (\frac{n^2}{4}, \infty)$**.**

   Under current assumptions, using (2.2.11) we see that

$$M(\sigma) = \sigma(\alpha - L) + \frac{n^3 + L - 1}{12}. \tag{2.4.28}$$

4. **When** $r = 1$**,** $\sigma \in (\frac{1}{4}, \frac{n^2}{4}]$**, and** $p > L$**.**

When $r = 1$ and $\sigma > \frac{1}{4}$, the exponent $M(\sigma)$ takes the form

$$M(\sigma) = \sigma(\alpha - L) + \tfrac{1}{12}(L-1) + U_n(\sqrt{\sigma}) = \sigma(n + \alpha - L) + \tfrac{1}{12}(L-1) - \tfrac{4}{3}\sigma^{3/2} =: g_2(\sigma). \tag{2.4.29}$$

Differentiating in σ shows that g_2 reaches its maximum $\frac{1}{12}(p - L + 1)^3 + \frac{1}{12}(L-1)$ at $\sigma = (p - L + 1)^2/4$. Further $g_2(\sigma) - \frac{(p-L+1)^3}{12} - \frac{L-1}{12} = -\frac{1}{3}(\sqrt{\sigma} - \frac{p-L+1}{2})^2(p - L + 1 + 4\sqrt{\sigma})$. Using the current assumption $p > L$ to bound $(p - L + 1 + 4\sqrt{\sigma}) \geq 1$ we get

$$M(\sigma) \leq \tfrac{(p-L+1)^3}{12} + \tfrac{L-1}{12} - \tfrac{1}{3}(\sqrt{\sigma} - \tfrac{p-L+1}{2})^2. \tag{2.4.30}$$

5. **When $r = 1$, $\sigma \in (\frac{1}{4}, \frac{n^2}{4}]$, and $p \leq L$.**

   Here we also have the expression (2.4.29) of $M(\sigma)$. Under the current assumption $p \leq L$. Differentiating in σ shows that g_2 is decreasing on $s \in (\frac{1}{4}, \frac{n^2}{4}]$. Further $g_2(\sigma) - g_2(\frac{1}{4}) = (p - L + 1)(\sigma - \frac{1}{4}) - \frac{4}{3}(\sigma^{3/2} - \frac{1}{8})$. Use the current assumptions to bound $(p - L + 1)(\sigma - \frac{1}{4}) \leq (\sigma - \frac{1}{4})$. We get $g_2(\sigma) - g_2(\frac{1}{4}) \leq (\sigma - \frac{1}{4}) - \frac{4}{3}(\sigma^{3/2} - \frac{1}{8}) = -\frac{1}{3}(1 + 4\sqrt{\sigma})(\sqrt{\sigma} - \frac{1}{2})^2$. Further bound $-\frac{1}{3}(1 + 4\sqrt{\sigma}) \leq 1$. Together with $g_2(\frac{1}{4}) = \frac{3p-2L}{12}$, we have

$$M(\sigma) \leq \tfrac{1}{12}(3p - 2L) - (\sqrt{\sigma} - \tfrac{1}{2})^2. \tag{2.4.31}$$

Now, in each of the preceding case, use the respective bound (2.4.25), (2.4.27), (2.4.28), (2.4.30), or (2.4.31) to bound the integral $\int_A e^{tM(\sigma)} \mathrm{d}\sigma$ on the relevant range A. For the resulting integral,

1. perform a change of variable $\sqrt{\sigma} \mapsto u$, which introduces a factor $2u$; bound this factor by $2 \cdot \frac{1}{2}$, release the range of integration from $u \in (0, \frac{1}{2})$ to $u \in \mathbb{R}$, and evaluate the resulting integral.

2. evaluate the resulting integral.

3. evaluate the resulting integral.

4. perform a change of variable $\sqrt{\sigma} \mapsto u$, which introduces a factor $2u$; bound this factor by $2u \leq n$, release the range of integration from $u \in (\frac{1}{2}, \frac{n}{2})$ to $u \in \mathbb{R}$, and evaluate the resulting integral.

5. perform a change of variable $\sqrt{\sigma} \mapsto u + \frac{1}{2}$, which introduces a factor $2u + 1$; release the range of integration from $u \in (0, \frac{n-1}{2})$ to $u \in \mathbb{R}_{\geq 0}$, and evaluate the resulting integral.

This gives the following bound on $\int_A e^{tM(\sigma)} \mathrm{d}\sigma$ on the relevant region A:

1. $\mathrm{C}\, p^{-\frac{1}{2}} t^{-\frac{1}{2}} \exp(t \frac{p^3}{12L^2}))$
2. $\mathrm{C}\, t^{-1}(1-\alpha)^{-1} \exp(t(\frac{(n-1)^3+1}{12} - \frac{1-\alpha}{4}))$
3. $\mathrm{C}\, t^{-1}(L-\alpha)^{-1} \exp(t(\frac{(n^3+L-1)}{12} - \frac{n^2(L-\alpha)}{4}))$
4. $\mathrm{C}\, t^{-\frac{1}{2}} n \exp(\frac{t}{12}((p-L+1)^3 + (L-1)))$
5. $\mathrm{C}\, (t^{-1} + t^{-1/2}) \exp(\frac{t}{12}(3p - 2L))$

Our goal is to have the exponent strictly less that $t\frac{p^3}{12}$.

(1) Since $L \geq 2$ we have $\frac{p^3}{12L^2} \leq \frac{p^3 t}{12} - \frac{p^3}{16}$.

(2) Under the current assumption $r \geq 2$ forces $n \geq 2$, and $p \geq 1$ and hence

$$\frac{(n-1)^3+1}{12} - \frac{1-\alpha}{4} = \frac{p^3}{12} - \frac{(p(n-1)-1)\alpha}{4} - \frac{\alpha^3}{12} - \frac{1}{6} \leq \frac{p^3}{12} - \frac{1}{6}.$$

(3) The exponent in (3) therein is decreasing in L. This gives

$$\frac{n^3+L-1}{12} - \frac{n^2(L-\alpha)}{4} \leq \frac{n^3+1}{12} - \frac{n^2(2-\alpha)}{4} = \frac{p^3}{12} - \frac{(1-\alpha)^2(p+2n)}{12} - \frac{3n^2-1}{12} \leq \frac{p^3}{12} - \frac{1}{6}.$$

(4) View the exponent in (4) as a function $g_3(x) := \frac{1}{12}((p-x)^3 + x)$ of $x := L-1$. Under the relevant assumption $p \geq L$ and $2 \leq L$, differentiating g_3 show that g_3 is maximized at $x = 1$. This gives $\frac{1}{12}((p-L+1)^3 + (L-1)) \leq g_3(1) = \frac{1}{12}(p^3 - 3p^2 + 3p) \leq \frac{1}{12}(p^3 - 6)$.

(5) Use $L \geq 2$ to bound $\frac{1}{12}(3p-2L) \leq \frac{1}{12}(3p-4)$. For $p \geq 0$, the last expression is always bounded by $\frac{p^3}{12} - \frac{1}{6}$, which gives $\frac{1}{12}(3p-2L) \leq \frac{p^3}{12} - \frac{1}{6}$.

Collect the preceding discussion and refer back to (2.4.24). We arrive at

$$\mathcal{I} \leq e^{\frac{p^3 t}{12}} \mathrm{C}^L \frac{n!}{\Gamma(1-\alpha)} \Big(p^{-\frac{1}{2}} t^{\frac{1}{2}} e^{-\frac{p^3 t}{16}} + \frac{e^{-\frac{t}{6}}}{(1-\alpha)} + e^{-\frac{t}{6}} + n t^{\frac{1}{2}} e^{-\frac{t}{2}} + (1+t^{\frac{1}{2}}) e^{-\frac{t}{6}} \Big).$$

Further apply the bounds $p^{-\frac{1}{2}} \leq p_0^{-\frac{1}{2}} = \mathrm{C}$, $\frac{1}{\Gamma(1-\alpha)} \leq \mathrm{C}$, and $\frac{1}{(1-\alpha)\Gamma(1-\alpha)} \leq \mathrm{C}$, for all $\alpha \in [0,1)$. We conclude the desired result. □

Proposition 2.4.7. *Fix any $t_0, p_0 > 0$. Recall $\mathcal{B}_{p,L}(t)$ from* (2.2.6). *There exists a constant* $\mathrm{C} = \mathrm{C}(t_0, p_0) > 0$ *such that for all $t > t_0$ and $p \geq p_0$,*

$$\sum_{L \geq 2} |\mathcal{B}_{p,L}(t)| \leq n \cdot (n!)^2 \, (n\mathrm{C})^n \, t^{\frac{1}{2}} \, e^{\frac{p^3 t}{12} - \kappa_p t}, \tag{2.4.32}$$

where $n := \lfloor p \rfloor + 1$ and $\alpha := p - \lfloor p \rfloor$, and $\kappa_p := \min\{\frac{1}{6}, \frac{p^3}{16}\}$.

Proof. Multiply both sides of (2.4.16) by $s^{-\alpha}$, integrate the result over $s \in [0,1]$, and apply the bound (2.4.20). We get, for $\mathrm{C} = \mathrm{C}(t_0, p_0)$,

$$\text{l.h.s. of (2.4.32)} \leq (n+1)! \, t^{\frac{1}{2}} \, e^{\frac{p^3 t}{12} - \kappa_p t} \sum_{L \geq 2} \sum_{\vec{m} \in \mathfrak{M}(L,n)} \binom{n}{\vec{m}} \frac{(|\vec{m}_{>0}|)! \mathrm{C}^L}{(L - |\vec{m}_{>0}|)!}$$

Within the last expression, use $|\vec{m}_{>0}| \leq n$ to bound $\frac{(|\vec{m}_{>0}|)!}{(L-|\vec{m}_{>0}|)!} \leq \frac{n!}{((L-n)_+)!}$, and evaluate the sum $\sum_{\vec{m} \in \mathfrak{M}(L,n)} \binom{n}{\vec{m}} = L^n$. This gives

$$\text{l.h.s. of (2.4.32)} \leq n \cdot \, (n!)^2 \, t^{\frac{1}{2}} \, e^{\frac{p^3 t}{12} - \kappa_p t} \sum_{L \geq 2} \frac{L^n \mathrm{C}^L}{((L-n)_+)!}. \tag{2.4.33}$$

In the sum in (2.4.33), bound $L^n \leq (2n + (L-2n)_+)^n \leq 2^n (2n)^n + 2^n ((L-2n)_+)^n$, use $\frac{((L-2n)_+)^n}{((L-n)_+)!} \leq \frac{1}{((L-2n)_+)!}$, and evaluate the resulting series. The result shows that the sum in (2.4.33) is bounded by $(n\mathrm{C})^n$. This completes the proof. □

2.5 Proof of Theorem 2.1.2 and Theorem 2.1.2(a)*

We begin with the proof of Theorem 2.1.2(a)*. Lemma 2.4.4 justifies the passing of derivatives in (2.2.4). Recall the definition of $\widetilde{\mathcal{A}}_p(t)$, $\mathcal{B}_{p,L}(t)$, $L \geq 2$, and $\mathcal{B}_{p,1}(t)$ from in (2.2.5), (2.2.6), and (2.2.2), we have $\mathbf{E}[(\mathcal{Z}(2t,0)e^{\frac{t}{12}})^p] = \widetilde{\mathcal{A}}_p(t) + \sum_{L\geq 1} \mathcal{B}_{p,L}(t)$. Further, recall from (2.3.9) that $\widetilde{\mathcal{A}}_p(t) = \mathcal{A}_p(t) - \widehat{\mathcal{A}}_p(t)$, so

$$\mathbf{E}\big[(\mathcal{Z}(2t,0)e^{\frac{t}{12}})^p\big] = \mathcal{A}_p(t) - \widehat{\mathcal{A}}_p(t) + \sum_{L\geq 1} \mathcal{B}_{p,L}(t).$$

Given the bound (2.2.3) and the bounds from Propositions 2.3.2 and 2.4.7, Theorem 2.1.2(a)* now follows for $\mathcal{B}_p(t) := -\widehat{\mathcal{A}}_p(t) + \sum_{L\geq 1} \mathcal{B}_{p,L}(t)$.

Next, Theorem 2.1.2(a) follows immediately from Theorem 2.1.2(a)*. It now remains only to show Theorem 2.1.2(b). We will establish the large deviation upper and lower bound separately. To simplify notation set $V_t := \mathcal{H}(2t,0) + \frac{t}{12}$. Fix $y > 0$. Markov's inequality gives $\mathbf{P}[V_t \geq ty] \leq e^{-py}\mathbf{E}[e^{pV_t}]$. Apply Theorem 2.1.2(a), take logarithm, and divide by t. We obtain, for all $p > 0$,

$$\limsup_{t\to\infty} \tfrac{1}{t} \log \mathbf{P}\big[V_t \geq ty\big] \leq -py + \tfrac{1}{12}p^3. \tag{2.5.1}$$

Minimizing the right side of (2.5.1) over $p > 0$, we obtain the desired large deviation upper bound

$$\limsup_{t\to\infty} \tfrac{1}{t} \log \mathbf{P}[V_t \geq ty\big] \leq -\tfrac{4}{3}y^{3/2}.$$

For lower bound we employ the standard change-of-measure argument and utilize the strict convexity of the function $\frac{1}{12}p^3$, $p > 0$. Fix $\varepsilon > 0$, set $q_* := 2(y+\varepsilon)^{1/2}$, and let $\widetilde{V}_t$ denote the random variable with the tilted law $\mathbf{P}[\widetilde{V}_t \in A] = \frac{1}{\mathbf{E}[e^{q_* V_t}]}\mathbf{E}[e^{q_* V_t}\mathbf{1}_{\{A\}}(V_t)]$. We write

$$\mathbf{P}[V_t \geq ty] = \mathbf{E}[e^{-q_*\widetilde{V}_t}\mathbf{1}_{\{\widetilde{V}_t \geq ty\}}] \cdot \mathbf{E}[e^{q_* V_t}] \geq e^{-tq_*(y+2\varepsilon)}\mathbf{E}[e^{q_* V_t}]\,\mathbf{P}\big[\widetilde{V}_t \in [ty, t(y+2\varepsilon))\big]. \tag{2.5.2}$$

Our goal is to show that $\lim_{t\to\infty} \mathbf{P}[\widetilde{V}_t \in [ty, t(y+2\varepsilon))] = 1$. To this end, for $\lambda \in (0, q_*)$ bound the complement probability by Markov's inequality as

$$\mathbf{P}\big[\widetilde{V}_t < ty\big] \le e^{\lambda ty}\mathbf{E}[e^{-\lambda\widetilde{V}_t}] = e^{\lambda ty}\frac{\mathbf{E}[e^{(q_*-\lambda)V_t}]}{\mathbf{E}[e^{q_*V_t}]},$$
$$\mathbf{P}\big[\widetilde{V}_t > t(y+2\varepsilon)\big] \le e^{-\lambda t(y+2\varepsilon)}\mathbf{E}[e^{\lambda\widetilde{V}_t}] = e^{-\lambda t(y+2\varepsilon)}\frac{\mathbf{E}[e^{(q_*+\lambda)V_t}]}{\mathbf{E}[e^{q_*V_t}]}.$$

Take log, divide the result by t, and apply Theorem 2.1.2(a). We obtain

$$\limsup_{t\to\infty}\tfrac{1}{t}\log\mathbf{P}\big[\widetilde{V}_t < ty\big] \le y\lambda + \tfrac{1}{12}(q_*-\lambda)^3 - \tfrac{1}{12}q_*^3, \tag{2.5.3}$$
$$\limsup_{t\to\infty}\tfrac{1}{t}\log\mathbf{P}\big[\widetilde{V}_t > t(y+2\varepsilon)\big] \le -(y+2\varepsilon)\lambda + \tfrac{1}{12}(q_*+\lambda)^3 - \tfrac{1}{12}q_*^3. \tag{2.5.4}$$

Now, view the r.h.s. of (2.5.3) and (2.5.4) as functions of $\lambda \in (-q_*, q_*)$. It is readily checked that these functions are strictly convex, zero at $\lambda = 0$, and has negative derivative at $\lambda = 0$. Hence there exists a small enough $\lambda_* = \lambda_*(\varepsilon, y) > 0$ such that the r.h.s. of (2.5.3) and (2.5.4) are negative for $\lambda = \lambda_*$. This gives $\lim_{t\to\infty}\mathbf{P}[\widetilde{V}_t \in [ty, t(y+2\varepsilon)]] = 1$. Use this in (2.5.2), take log, divide the result by t, and apply Theorem 2.1.2(a) to get

$$\liminf_{t\to\infty}\tfrac{1}{t}\log\mathbf{P}[V_t \ge ty\big] \ge -q_*(y+2\varepsilon) + \tfrac{1}{12}q_*^3 = -\tfrac{4}{3}(y+\varepsilon)^{3/2} - 2\varepsilon(y+\varepsilon)^{1/2}.$$

Since $\varepsilon > 0$ was arbitrary, sending $\varepsilon \to 0$ gives the desired large deviation lower bound.

Chapter 3: Upper-tail large deviation principle for the ASEP

3.1 Introduction

3.1.1 The ASEP and main results

In this paper, we study the upper-tail Large Deviation Principle (LDP) of the *asymmetric simple exclusion process* (ASEP) with step initial data. The ASEP is a continuous-time Markov chain on particle configurations $\mathbf{x} = (x_1 > x_2 > \cdots)$ in $\mathbb{Z}$. The process can be described as follows. Each site $i \in \mathbb{Z}$ can be occupied by at most one particle, which has an independent exponential clock with exponential waiting time of mean 1. When the clock rings, the particle jumps to the right with probability q or to the left with probability $p = 1 - q$. However, the jump is only permissible when the target site is unoccupied. For our purposes, it suffices to consider configurations with a rightmost particle. At any time $t \in \mathbb{R}_{>0}$, the process has the configuration $\mathbf{x}(t) = (x_1(t) > x_2(t) > \cdots)$ in $\mathbb{Z}$, where $x_j(t)$ denotes the location of the j-th rightmost particle at this time. Appearing first in the biology work of Macdonald, Gibbs, and Pipkin [250] and introduced to the mathematics community two years later by [298], the ASEP has since become the "default stochastic model to study transport phenomena", including mass transport, traffic flow, queueing behavior, driven lattices and turbulence. We refer to [72, 247, 246, 299] for the mathematical study of and related to the ASEP.

When $q = 1$, we obtain the *totally asymmetric simple exclusion process* (TASEP), which allows jumps only to the right. It connects to several other physical systems such as the exponential last-passage percolation, zero-temperature directed polymer in a random environment, the corner growth process and is known to possess complete determinantal structure (*free-fermionicity*). We refer the readers to [211, 247, 246, 276] and the references therein for more thorough treatises of the TASEP.

The dynamics of ASEP are uniquely determined once we specify its initial state. In the present paper, we restrict our attention to the ASEP started from the *step* initial configuration, i.e. $x_j(0) = -j$, $j = 1, 2, \ldots$. We set $\gamma = q - p$ and assume $q > \frac{1}{2}$, i.e., ASEP has a drift to the right. An observable of interest in ASEP is $H_0(t)$, the integrated current through 0 which is defined as:

$$H_0(t) := \text{ the number of particles to the right of zero at time } t. \tag{3.1.1}$$

$H_0(t)$ can also be interpreted as the one-dimensional height function of the interface growth of the ASEP and thus carries significance in the broader context of the Kardar-Parisi-Zhang (KPZ) universality class. We will elaborate on the connection to KPZ universality class later in Section 3.1.3. As a well-known random growth model itself, the large-time behaviors of ASEP with step initial conditions have been well-studied. Indeed, it is known [247, Chapter VIII, Theorem 5.12] that the current satisfies the following strong law of large numbers:

$$\tfrac{1}{t} H_0\big(\tfrac{t}{\gamma}\big) \to \tfrac{1}{4}, \text{ almost surely as } t \to \infty.$$

The strong law has been later complemented by fluctuation results in the seminal works by Tracy and Widom. In a series of papers [305], [307] [306], Tracy and Widom exploit the integrability of ASEP with step initial data and establish via contour analysis that $H_0(t)$ when centered by $\frac{t}{4}$ has typical deviations of the order $t^{1/3}$ and has the following asymptotic fluctuations:

$$\tfrac{1}{t^{1/3}} 2^{4/3} \big(- H_0\big(\tfrac{t}{\gamma}\big) + \tfrac{t}{4}\big) \implies \xi_{\mathrm{GUE}}, \tag{3.1.2}$$

where ξ_{GUE} is the GUE Tracy-Widom distribution [308]. When $q = 1$, (3.1.2) recovers the same result on TASEP, which has been proved earlier by [211].

Given the existing fluctuation results on the ASEP with step initial data, it is natural to inquire into its Large Deviation Principle (LDP). Namely, we seek to find the probability of when the event $-H_0(\frac{t}{\gamma}) + \frac{t}{4}$ has deviations of order t. Intriguingly, one expects the lower- and upper-tail LDPs to have different speeds: the upper-tail deviation is expected to occur at speed t whereas the lower-tail has speed t^2:

$$\mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right) + \tfrac{t}{4} < -\tfrac{t}{4}y\right) \approx e^{-t^2\Phi_-(y)}; \qquad \text{(Lower Tail)}$$

$$\mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right) + \tfrac{t}{4} > +\tfrac{t}{4}y\right) \approx e^{-t\Phi_+(y)}. \qquad \text{(Upper Tail)}$$

Thus, the upper tail corresponds to ASEP being “too slow" while the lower tail corresponds to ASEP being “too fast". Heuristically, we can make sense of such speed differentials. Because of the nature of the exclusion process, when a *single* particle is moving slower than the usual, it forces *all* the particles on the left of it to be automatically slower. Hence ASEP becomes slow if *only one* particle is moving slow. This event has probability of the order $\exp(-O(t))$. However, in order to ensure that there are many particles on the right side of origin (this corresponds to ASEP being fast), it requires a large number of particles to move fast *simulatenously*. This event is much more unlikely and happens with probability $\exp(-O(t^2))$.

In this article, we focus on the *upper-tail* deviations of the ASEP with step initial data and present the first proof of the ASEP upper-tail LDP on the *complete* real line. Consider ASEP with $q \in (\frac{1}{2}, 1)$ and set $p = 1 - q$ and $\tau = p/q \in (0, 1)$. Our first theorem computes the sth-*Lyapunov exponent* of $\tau^{H_0(t)}$, which is the limit of the logarithm of $\mathbf{E}[\tau^{sH_0(t)}]$ scaled by time:

Theorem 3.1.1. *For $s \in (0, \infty)$ we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}[\tau^{sH_0(t)}] = -h_q(s) =: -(q-p)\frac{1-\tau^{\frac{s}{2}}}{1+\tau^{\frac{s}{2}}}. \qquad (3.1.3)$$

It is well known (see Proposition 1.12 in [180] for example) that the *upper-tail* large deviation principle of the stochastic process $\log \tau^{H_0(t)}$ is the Legendre-Fenchel dual of the Lyapunov exponent in (3.1.3). Since $\tau < 1$, as a corollary, we obtain the following *upper-tail* large deviation rate function for $-H_0(t)$.

Theorem 3.1.2. *For any $y \in (0,1)$ we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right) + \tfrac{t}{4} > \tfrac{t}{4}y\right) = -[\sqrt{y} - (1-y)\tanh^{-1}(\sqrt{y})] =: -\Phi_+(y), \tag{3.1.4}$$

where $\gamma = 2q-1$. Furthermore, we have the following asymptotics near zero:

$$\lim_{y\to 0^+} y^{-3/2}\Phi_+(y) = \tfrac{2}{3}. \tag{3.1.5}$$

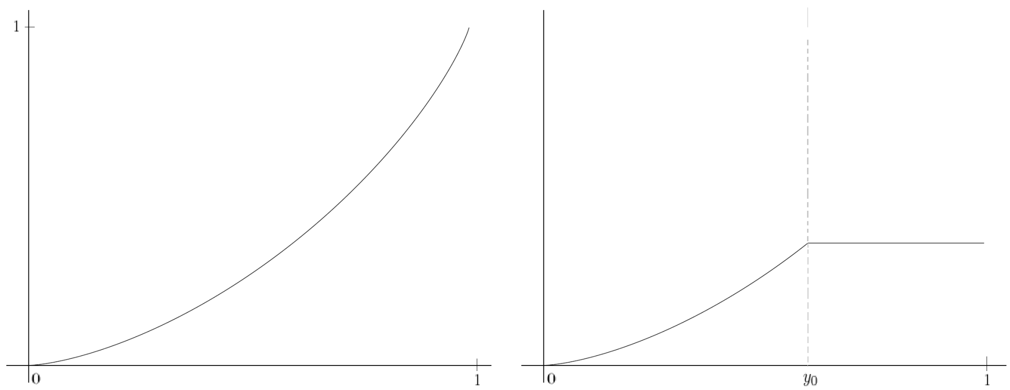

Figure 3.1: The figure on the left is the plot of $\Phi_+(y)$. The right one is the plot of $\widetilde{\Phi}_+(y)$.

Remark 3.1.3. Note that our large deviation result is restricted to $y \in (0,1)$ as $\mathbb{P}(-H_0(\tfrac{t}{\gamma}) + \tfrac{t}{4} > \tfrac{t}{4}y) = 0$ for $y \geq 1$. Furthermore, although Theorem 3.1.2 makes sense when $q = 1$, one cannot recover it from Theorem 3.1.1, which only makes sense for $\tau = (1-q)/q \in (0,1)$. However, as mentioned before, [211] has already settled the $q = 1$ TASEP case and obtained the upper-tail rate function in a variational form. We will later show in Appendix 3.5 that [211] variational formula for TASEP matches with our rate function in (3.1.4).

Remark 3.1.4. Recently, the work [125] has obtained a one-sided large deviation bound for the upper tail of the ASEP. In particular, they showed

$$\mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right) + \tfrac{t}{4} > \tfrac{t}{4}y\right) \leq \mathrm{C}e^{-t\widetilde{\Phi}_+(y)}, \quad y \in (0,1). \tag{3.1.6}$$

The function $\widetilde{\Phi}_+$ coincides with the correct rate function Φ_+ defined in (3.1.4) only for $y \le y_0 := \frac{1-2\sqrt{q(1-q)}}{1+2\sqrt{q(1-q)}}$, as captured by Figure 3.1. We will further compare and contrast our results and method with [125] later in Section 3.1.3.

Remark 3.1.5. For y small enough, following (3.1.2) and upper tail decay of GUE Tracy-Widom distribution [159], one expects

$$\mathbb{P}\left(-H_0(\tfrac{t}{\gamma}) + \tfrac{t}{4} > \tfrac{t}{4}y\right) \approx \mathbb{P}(\xi_{\text{GUE}} > 2^{-2/3}yt^{2/3}) \approx e^{-\frac{2}{3}y^{3/2}t}$$

Thus the asymptotics in (3.1.5) shows that $\Phi_+(y)$ indeed recovers the expected GUE Tracy-Widom tails as $y \to 0^+$.

3.1.2 Sketch of proof

In this section we present a sketch of the proof of our main results. As explained before, Theorem 3.1.2 can be obtained from Theorem 3.1.1 by standard Legendre-Fenchel transform technique. So here we only give a brief account of the proof idea of Theorem 3.1.1. A more detailed overview of the proofs of our main results can be found in Section 3.2.

The main component of our proof is the following τ-Laplace transform formula for $H_0(t)$ that appears in Theorem 5.3 in [72]:

Theorem 3.1.6 (Theorem 5.3 in [72])**.** *Fix any $\delta \in (0,1)$. For $\zeta > 0$ we have*

$$\mathbf{E}\left[F_q(\zeta\tau^{H_0(t)})\right] = \det(I + K_{\zeta,t}), \quad F_q(\zeta) := \prod_{n=0}^{\infty} \frac{1}{1+\zeta\tau^n}. \tag{3.1.7}$$

Here $\det(I + K_{\zeta,t})$ is the Fredholm determinant of $K_{\zeta,t} : L^2(\mathfrak{C}(\tau^{1-\frac{\delta}{2}})) \to L^2(\mathfrak{C}(\tau^{1-\frac{\delta}{2}}))$, and $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$ denotes a positively-oriented circular contour centered at 0 with radius $\tau^{1-\frac{\delta}{2}}$. The operator $K_{\zeta,t}$ is defined through the integral kernel

$$K_{\zeta,t}(w,w') := \frac{1}{2\pi\mathbf{i}} \int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \Gamma(-u)\Gamma(1+u)\zeta^u \frac{g_t(w)}{g_t(\tau^u w)} \frac{\mathrm{d}u}{w' - \tau^u w}, \tag{3.1.8}$$

for $g_t(z) = \exp\left(\frac{(q-p)t}{1+\frac{z}{\tau}}\right)$.

Remark 3.1.7. The original statement of the above theorem in [72] appears in a much more general setup with general conditions on the contours. We will explain the choice of our contours stated above in Section 3.3 and check that it satisfies the general criterion for contours as stated in Theorem 5.3 in [72].

We next recall that the Fredholm determinant is defined as a series as follows.

$$\det(I + K_{\zeta,t}) := 1 + \sum_{L=1}^{\infty} \operatorname{tr}(K_{\zeta,t}^{\wedge L}) \tag{3.1.9}$$

$$:= 1 + \sum_{L=1}^{\infty} \frac{1}{L!} \int_{\mathfrak{C}(\tau^{1-\frac{\delta}{2}})} \cdots \int_{\mathfrak{C}(\tau^{1-\frac{\delta}{2}})} \det(K_{\zeta,t}(w_i, w_j))_{i,j=1}^{L} \prod_{i=1}^{L} \mathrm{d}w_i. \tag{3.1.10}$$

The notation $K_{\zeta,t}^{\wedge L}$ comes from the exterior algebra definition, which we refer to [296] for more details. As a clarifying remark, we use this exterior algebra notation only for the simplicity of its expression and rely essentially on the definition in (3.1.10) throughout the rest of the paper.

To extract information on the fractional moments of $\tau^{H_0(t)}$, we combine the formula in (3.1.7) with the following elementary identity, which is a generalized version of Lemma 1.4 in [131].

Lemma 3.1.8. *Fix $n \in \mathbb{Z}_{>0}$ and $\alpha \in [0, 1)$. Let U be a nonnegative random variable with finite n-th moment. Let $F : [0, \infty) \to [0, 1]$ be a n-times differentiable function such that $\int_0^\infty \zeta^{-\alpha} F^{(n)}(\zeta)\mathrm{d}\zeta$ is finite. Assume further that $\|F^{(k)}\|_\infty < \infty$ for all $1 \le k \le n$. Then the $(n - 1 + \alpha)$-th moment of U is given by*

$$\mathbf{E}[U^{n-1+\alpha}] = \frac{\int_0^\infty \zeta^{-\alpha}\mathbf{E}[U^n F^{(n)}(\zeta U)]\mathrm{d}\zeta}{\int_0^\infty \zeta^{-\alpha} F^{(n)}(\zeta)\mathrm{d}\zeta} = \frac{\int_0^\infty \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\mathbf{E}[F(\zeta U)]\mathrm{d}\zeta}{\int_0^\infty \zeta^{-\alpha} F^{(n)}(\zeta)\mathrm{d}\zeta}.$$

The proof of this lemma follows by an interchange of measure justified by Fubini's theorem and the dominated convergence theorem, as $\mathbf{E}[U^n]$ and $\|F^{(k)}\|_\infty < \infty$ for all $1 \le k \le n$.

For $s > 0$, we apply this lemma with $U = \tau^{H_0(t)}$, $n = \lfloor s \rfloor + 1$ and $\alpha = s - \lfloor s \rfloor$. We take $F(x) = F_q(x)$ defined in (3.1.7) which is shown to be satisfy the hypothesis of Lemma 3.1.8 (see Proposition 3.2.2). As a result, we transform the computation of $\mathbf{E}[\tau^{sH_0(t)}]$ into that of

$$\int_0^\infty \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathbf{E}[F_q(\zeta\tau^{H_0(t)})]\mathrm{d}\zeta. \tag{3.1.11}$$

Utilizing the exact formula from (3.1.7) and the definition of Fredholm determinant from (3.1.10), we can write the above expression as a series where we identify the leading term (corresponding to $L = 1$ term of the series) and a higher-order term (corresponding to $L \geq 2$ terms of the series). We eventually show that the asymptotics of the leading term matches with the exact asymptotics in (3.1.3) while the higher-order term decays much faster. This leads to the proof of Theorem 3.1.1.

The above description of our method is in line with the Lyapunov moment approach adopted in the works of [131], [180] and [248] to obtain upper-tail large deviation results of other integrable models, such as the KPZ equation. Namely, we extract fractional moments from the (τ-)Laplace transform such as (3.1.7) according to Lemma 3.1.8. In particular, our work draws from those of [131] and [248], which studied the fractional moments of the Stochastic Heat Equation (SHE) and the half-line Stochastic Heat Equation, respectively. We will further contextualize the connections of our work to [131], [180] and [248] in Section 3.1.3. In the following text, however, we emphasize a few key differences and technical challenges unique to the ASEP that we have encountered and resolved in our proof.

First, unlike SHE or half-line SHE, the usual Laplace transform is not available in case of the ASEP. Instead, we only have the τ-Laplace transform for our observable of interest. As a result, we have formulated Lemma 3.1.8 in our paper, which is more generalized than its prototype in [131, Lemma 1.4], to feed in the τ-Laplace transform. Consequently, we have worked with τ-exponential functions in our analysis.

Another key difference is that the kernel $K_{\zeta,t}$ in (3.1.8) in our model is much more intricate than its counterpart in the KPZ model and leads to much more involved analysis of the leading term. Indeed, $K_{\zeta,t}$ is asymmetric and as u varies in $(\delta - \mathbf{i}\infty, \delta + \mathbf{i}\infty)$, the function $\frac{g_t(w)}{g_t(\tau^u w)}$ appearing in the kernel $K_{\zeta,t}$, exhibits a periodic behavior, whereas the kernel in the KPZ models involves Airy functions in its integrand which have a unique maximum and are much easier to analyze. Furthermore, our model exhibits exponentially decaying moments of $\tau^{H_0(t)}$ as opposed to the exponentially increasing ones of the KPZ models in [131] and [248] and this demands a more precise understanding of the trace term of our Fredholm determinant expansion. For instance in Section 3.3, to obtain the precise asymptotics for our leading term, we have performed steepest descent analysis on the kernel $K_{\zeta,t}$, where the periodic nature of $\frac{g_t(w)}{g_t(\tau^u w)}$ results in infinitely many critical points. A major technical challenge in our proof is to argue how the contribution from only one of the critical points dominates the those from the rest and this is accomplished in the proof of Proposition 3.2.4. Similarly, the asymmetry of the kernel in the ASEP model has led us to opt for the Hadamard's inequality approach as exemplified in Section 4 of [248], instead of the operator theory argument in [131], to obtain a sufficient upper bound for the higher-order terms in our paper in Section 3.4.

3.1.3 Comparison to Previous Works

In a broader context, our main result on the Lyapunov exponent for the ASEP with step initial data and its upper-tail large deviation belongs to the undertakings of studying the intermittency phenomenon and large deviation problems of integrable models in the KPZ universality class. As we have previously alluded to, the KPZ universality class contains a collection of random growth models that are characterized by scaling exponent of $1/3$ and certain universal non-Gaussian large time fluctuations. We refer to [6, 113, 304] and the references therein for more details. The ASEP is one of the standard one-dimensional models of the KPZ universality class and bears connection to several other integrable models in this class, such as the stochastic six-vertex model [71, 3, 114], KPZ equation [80, 156, 290, 6, 113], and q-TASEP [72].

On the other hand, the intermittency property is a universal phenomenon that captures high population concentrations on small spatial islands over large time. Mathematically, the intermittency of a random field is defined in terms of its Lyapunov exponents. In particular, the connection between integer Lyapunov moments and intermittency has long been an active area of study in the SPDE community in last few decades [**ber**, 175, 85, 169, 201, 104, 92, 21]. For the KPZ equation, [216] predicted the integer Lyapunov exponents for the SHE using replica Bethe anstaz techniques. This result was later first rigorously attempted in [48] and correctly proven in [**che**]. Similar formulas were shown for the moments of the parabolic Anderson model, semi-discrete directed polymers, q-Whittaker process (see [68] and [69]). For the ASEP, integer moments formula for $\tau^{H_0(t)}$ were obtained in [72] using nested contour integral ansatz.

From the perspective of tail events, by studying the asymptotics of integer Lyapunov exponents formulas, one can extract one-sided bounds on the upper tails of integrable models. However, these integer Lyapunov exponents alone are not sufficient to provide the exact large deviation rate function.

Recently, a stream of effort has been devoted to studying large deviations for some KPZ class models by explicitly computing the fractional Lyapunov exponents. The work of [131] set this series of effort in motion by solving the KPZ upper-tail large deviation principle through the fractional Lyapunov exponents of the SHE with delta initial data. [180] soon extended the same result for the SHE for a large class of initial data, including any random bounded positive initial data and the stationary initial data. An exact way to compute every positive Lyapunov exponent of the half-line SHE was also uncovered in [248]. In lieu of these developments, our main result for the ASEP with step initial data and its upper-tail large deviation fits into this broader endeavor of studying large deviation problems of integrable models with the Lyapunov exponent appproach.

Meanwhile, in the direction of the ASEP, as mentioned before, [125] has produced a one-sided large deviation bound for the upper-tail probability appearing in (3.1.4) which coincides with the correct rate function Φ_+ defined in (3.1.4) for $y \le y_0 := \frac{1-2\sqrt{q(1-q)}}{1+2\sqrt{q(1-q)}}$. This result was sufficient for their purpose of establishing a near-exponential fixation time for the coarsening model on $\mathbb{Z}^2$ and [125] obtained it via steepest descent analysis on the exact formula for the probability of $H_0(t/\gamma)$.

More specially, they worked with the following result from [306, Lemma 4] as input:

$$\mathbb{P}\left(-H_0\big(\tfrac{t}{\gamma}\big) + \tfrac{t}{4} > \tfrac{t}{4}y\right) = \frac{1}{2\pi\mathbf{i}} \int_{|\mu|=R} (\mu;\tau)_\infty \det(1+\mu J_{m,t}^{(\mu)}) \frac{\mathrm{d}\mu}{\mu}, \tag{3.1.12}$$

where $m = \lfloor \frac{1}{4}t(1-y) \rfloor$, $R \in (\tau,\infty) \setminus \{1, \tau^{-1}, \tau^{-2}, \ldots\}$ is fixed, $(\mu;\tau)_\infty := (1-\mu)(1-\mu\tau)(1-\mu\tau^2)\ldots$ is the infinite τ-Pochhammer symbol and $J_{m,t}^{(\mu)}$ is the kernel defined in Equation (3.4) of [125]. Analyzing the exact pre-limit Fredholm determinant $\det(1+\mu J_{m,t}^{(\mu)})$, [125] chose appropriate contours for the kernel $J_{m,t}^{(\mu)}$ that pass through its critical points and performed a steepest descent analysis. However, their choice of contours was unattainable beyond the threshold y_0. Namely, if we attempted to deform the same contours for $y > y_0$, we would inevitably cross poles, which rendered the steepest descent analysis much trickier. By adopting the Lyapunov moment approach, we have avoided this problem when looking for the precise large deviation rate function.

In addition to the relavence of our upper-tail LDP result, it is also worthy to remark on the difficulty of obtaining a lower-tail LDP of the ASEP with step initial data. As explained before, the lower-tail $\mathbb{P}(-H_0(\frac{t}{\gamma}) + \frac{t}{4} < -\frac{t}{4}y)$ is expected to go to zero at a much faster rate of $\exp(-t^2\Phi_-(y))$. The existence of the lower-tail rate function has so far only been shown in the case of TASEP in [211] through its connection to continuous log-gases. The functional LDPs for TASEP for both tails have been studied in [210], [312], [282] (upper tail), and [266] (lower-tail). Large deviations for open systems with boundaries in contact with stochastic reservoirs has also been studied in physics literature. We mention [147], [146], [60] and the references therein for works in these directions.

More broadly for integrable models in the KPZ universality class, lower tail of the KPZ equation has been extensively studied in both mathematics and physics communities. In the physics literature, [243] provided the first prediction of the large deviation tails of the KPZ equation for narrow wedge initial data. For the upper tail, their analysis also yields subdominant corrections ([242, Supp. Mat.]). Furthermore, the physics work of [292] first predicted lower-tail rate function of the KPZ equation for narrow wedge initial data in an analytical form, followed by the derivations in [106] and [234] via different methods. The asymptotics of deep lower tail of KPZ equation was later obtained in [233] for a wide class of initial data. From the mathematics front, the work [116] provided detailed, rigorous tail bounds for the lower tail of the KPZ equation for narrow wedge initial data. The precise rate function of its lower-tail LDP was later proved in [310] and [79], which confirmed the prediction of existing physics literature. The four different routes of deriving the lower-tail LDP in [292], [**JointLetter**], [234] and [310] were later shown to be closely related in [232]. A new route has also been recently obtained in the physics work of [240] (see also [277]).

In the short time regime, large deviations for the KPZ equation has been studied extensively in physics literature (see [241], [230], [229] and the references therein for a review). Recently, [249] rigorously derived the large deviation rate function of the KPZ equation in the short-time regime in a variational form and recovered deep lower-tail asymptotics, confirming existing physics predictions. For non-integrable models, large deviations of first-passage percolation were studied in [95] and more recently [39]. For last-passage percolation with general weights, recently, geometry of polymers under lower tail large deviation regime has been studied in [38].

Notation

Throughout the rest of the paper, we use $\mathrm{C} = \mathrm{C}(a, b, c, \ldots) > 0$ to denote a generic deterministic positive finite constant that is dependent on the designated variables $a, b, c, \ldots$. However, its particular content may change from line to line. We also use the notation $\mathfrak{C}(r)$ to denote a positively oriented circle with center at origin and radius $r > 0$.

Outline

The rest of this article is organized as follows. In Section 3.2, we introduce the main ingredients for the proofs of Theorem 3.1.1 and 3.1.2. In particular, we reduce the proof of our main results to Proposition 3.2.4 (asymptotics of the leading order) and Proposition 3.2.5 (estimates for the higher order), which are proved in Sections 3.3 and 3.4 respectively. Finally, in Appendix 3.5 we compare our rate function $\Phi_{+}(y)$, defined in (3.1.4), to that of TASEP.

3.2 Proof of Main Results

In this section, we give a detailed outline of the proofs of Theorems 3.1.1 and 3.1.2. In Section 3.2.1 we collect some useful properties of h_q and F_q functions defined in (3.1.4) and (3.1.7) respectively. In Section 3.2.2 we complete the proof of Theorems 3.1.1 and 3.1.2 assuming technical estimates on the leading order term (Proposition 3.2.4) and higher order term (Proposition 3.2.5).

Throughout this paper, we fix $s > 0$ and set $n = \lfloor s \rfloor + 1 \geq 1$ and $\alpha = s - \lfloor s \rfloor$ so that $s = n - 1 + \alpha$. We also fix $q \in (\frac{1}{2}, 1)$ and set $p = 1 - q$ and $\tau = p/q \in (0, 1)$ for the rest of the article.

3.2.1 Properties of $h_q(x)$ and $F_q(x)$

Recall the Lyapunov exponent $h_q(x)$ defined in (3.1.3) and the $F_q(x)$ function defined in (3.1.7). The following two propositions investigates various properties of these two functions which are necessary for our later proofs.

Proposition 3.2.1 (Properties of h_q)**.** *Consider the function $h_q : (0, \infty) \to \mathbb{R}$ defined by $h_q(x) = (q-p)\frac{1-\tau^{\frac{x}{2}}}{1+\tau^{\frac{x}{2}}}$. Then, the following properties hold true:*

(a) $B_q(x) := \frac{h_q(x)}{x}$ *is strictly positive and strictly decreasing with*

$$\lim_{x \to 0^+} B_q(x) = \tfrac{1}{4}(p-q)\log\tau > 0.$$

(b) h_q *is strictly subadditive in the sense that for any $x, y \in (0, \infty)$ we have*

$$h_q(x+y) < h_q(x) + h_q(y).$$

(c) h_q *is related to Φ_+ defined in* (3.1.4) *via the following Legendre-Fenchel type transformation:*

$$\Phi_+(y) = \sup_{s \in \mathbb{R}_{>0}} \left\{ s\frac{1-y}{4}\log\tau + \frac{1}{q-p}h_q(s) \right\}, \quad y \in (0, 1).$$

Proof. For (a), first, the positivity of $B_q(x)$ follows from the positivity of $h_q(x)$. To see its growth, taking the derivative of $B_q(x)$ we obtain

$$B_q'(x) = \frac{(q-p)(-x\tau^{\frac{x}{2}}\log\tau - 1 + \tau^x)}{(1+\tau^{\frac{x}{2}})^2 x^2}. \tag{3.2.1}$$

Note that the numerator on the r.h.s of (3.2.1) is 0 when $x = 0$ and its derivative against x is $\tau^{\frac{x}{2}}\log\tau(\tau^{\frac{x}{2}} - \frac{x}{2}\log\tau - 1) < 0$ for $x > 0$. Thus $B_q'(x)$ is strictly negative when $x > 0$ and $B_q(x)$ is strictly decreasing for $x > 0$. L'Hôpital's rule yields that $\lim_{x\to 0^+} B_q(x) = h_q'(0) = \frac{1}{4}(q-p)\log\tau$.

For (b), direct computation yields

$$h_q(x+y) - h_q(x) - h_q(y) = -(q-p)\frac{(1-\tau^{\frac{y}{2}})(1-\tau^{\frac{x}{2}})(1-\tau^{\frac{x+y}{2}})}{(1+\tau^{\frac{x+y}{2}})(1+\tau^{\frac{x}{2}})(1+\tau^{\frac{y}{2}})} < 0. \tag{3.2.2}$$

Lastly, for part (c), we fix $y \in (0,1)$ and define

$$g_y(s) := s\frac{1-y}{4}\log\tau + \frac{1}{q-p}h_q(s), \quad s > 0.$$

Direct computation yields $g_y'(s) = (\frac{1-y}{4} - \frac{\tau^{\frac{s}{2}}}{(1+\tau^{\frac{s}{2}})^2})\log\tau$ and $g_y''(s) = \frac{\tau^{\frac{s}{2}}(\tau^{\frac{s}{2}}-1)\log^2\tau}{2(1+\tau^{\frac{s}{2}})^3} < 0$. Thus $g_y(s)$ is concave on $(0,\infty)$ and hence attains its unique maxima when $g_y'(s) = 0$ or equivalently $\frac{1-y}{4} = \frac{\tau^{\frac{s}{2}}}{(1+\tau^{\frac{s}{2}})^2}$. The last equation has $s = 2\log_\tau(\frac{1-\sqrt{y}}{1+\sqrt{y}})$ as the only positive solution and hence it defines the unique maximum. Substituting this s back into $g_y(s)$ generates the final result as $\Phi_+(y)$. $\square$

Proposition 3.2.2 (Properties of $F_q(\zeta)$)**.** *Consider the function $F_q : [0,\infty) \to [0,1]$ defined by $F_q(\zeta) := \prod_{n=0}^{\infty}(1+\zeta\tau^n)^{-1}$. Then, the following properties hold true:*

(a) F_q is an infinitely differentiable function with $(-1)^n F_q^{(n)}(\zeta) \geq 0$ for all $x > 0$. Furthermore, $\|F_q^{(n)}\|_\infty < \infty$ for each n.

(b) For each $n \in \mathbb{Z}_{>0}$, and $\alpha \in [0,1)$, $(-1)^n \int_0^\infty \zeta^{-\alpha} F_q^{(n)}(\zeta)\mathrm{d}\zeta$ is positive and finite.

(c) *All the derivatives of F_q have superpolynomial decay. In other words for any $m, n \in \mathbb{Z}_{\geq 0}$ we have*

$$\sup_{\zeta>0} |\zeta^m F_q^{(n)}(\zeta)| < \infty.$$

Proof. (a) Note that $F_q(\zeta) = \prod_{n=0}^{\infty}(1+\zeta\tau^n)^{-1} = (-\zeta;\tau)_\infty^{-1}$ where we recall that $(-\zeta;\tau)_\infty$ is the τ-Pochhammer symbol. As $(-\zeta;\tau)_\infty$ is analytic [8, Corollary A.1.6.] and nonzero for $\zeta \in [0,\infty)$, its inverse $F_q(\zeta)$ is analytic.

We next rewrite $F_q(\zeta) = \prod_{n=0}^{\infty} f_n(\zeta)$, where $f_n(\zeta) = (1+\zeta\tau^n)^{-1}$. Denote $H(\zeta) := \log F_q(\zeta)$. Since each $f_n(\zeta) \in (0,1)$ is analytic for $\zeta \in [0,\infty)$ and the product $\prod_{n=0}^{\infty} f_n(\zeta) \in (0,1)$ converges locally and uniformly, $H(\zeta)$ is well-defined and $H(\zeta) = \sum_{n=0}^{\infty} \log f_n(\zeta)$. Given that $|\sum_{n=0}^{\infty} \frac{1}{f_n(\zeta)} f_n'(\zeta)| = \sum_{n=0}^{\infty} \frac{\tau^n}{(1+\zeta\tau^n)} < \frac{1}{1-\tau}$, we have

$$H'(\zeta) = \frac{F_q'(\zeta)}{F_q(\zeta)} = \sum_{n=1}^{\infty} \frac{f_n'(\zeta)}{f_n(\zeta)} =: G(\zeta). \tag{3.2.3}$$

Note that $G(\zeta) = -\sum_{j=1}^{\infty} \tau^j f_j(\zeta)$ and $|G(\zeta)| < \infty$. For each $m \in \mathbb{Z}_{>0}$, let us set $G^{(m)}(\zeta) := -\sum_{j=1}^{\infty} \tau^j f_j^{(m)}(\zeta)$. As $f_j^{(m)}(\zeta) = (-1)^m m! \frac{\tau^{mj}}{(1+\xi\tau^j)^{m+1}}$, we obtain $|G^{(m)}(\zeta)| \leq \frac{m!}{1-\tau^{m+1}} < \infty$ converges locally and uniformly. Induction on m gives us that $G(\zeta)$ is infinitely differentiable and the m-th derivative of G is $G^{(m)}$. It follows that $F_q(\zeta)$ is infinitely differentiable too. In particular, for any finite $n \in \mathbb{Z}_{\geq 0}$, by Leibniz's rule on the relation (3.2.3) we obtain

$$F_q^{(n+1)}(\zeta) = \sum_{k=0}^{n} \binom{n}{k} F_q^{(n-k)}(\zeta) G^{(k)}(\zeta). \tag{3.2.4}$$

Observe that $(-1)^{k+1}G^{(k)}$ is positive and finite. As F_q is positive and finite, using (3.2.4), induction gives us that $(-1)^n F_q^{(n)}$ is also positive and finite. As $\|G^{(m)}\|_\infty$ and $\|F_q\|_\infty$ are finite, using (3.2.4), induction gives us that $\|F_q^{(n)}\|_\infty$ is finite for any $n \in \mathbb{Z}_{\geq 0}$.

(b) For $\alpha \in [0,1)$, positivity of the integral $(-1)^n \int_0^\infty \zeta^{-\alpha} F_q^{(n)}(\zeta) \mathrm{d}\zeta$ follows from part (a). To check the integrability, we first verify the $n = 0$ case. Since $\zeta \geq 0$ and $\tau \in (0, 1)$,

$$
\begin{aligned}
0 < \int_0^\infty \zeta^{-\alpha} F_q(\zeta)\mathrm{d}\zeta &= \int_0^\infty \zeta^{-\alpha} \prod_{m=0}^{\infty} \frac{1}{1+\zeta\tau^m}\mathrm{d}\zeta < \int_0^\infty \zeta^{-\alpha} \frac{1}{1+\zeta}\mathrm{d}\zeta \\
&= \int_0^1 \zeta^{-\alpha} \frac{1}{1+\zeta}\mathrm{d}\zeta + \int_1^\infty \frac{\mathrm{d}\zeta}{\zeta^\alpha(1+\zeta)} < \int_0^1 \zeta^{-\alpha}\mathrm{d}\zeta + \int_1^\infty \frac{\mathrm{d}\zeta}{\zeta^{\alpha+1}} < \infty.
\end{aligned}
$$

When $n > 0$, using (3.2.4) and the fact the $|G^{(m)}(\zeta)| < \frac{m!}{1-\tau^{m+1}}$, by induction we deduce the finiteness of $(-1)^n \int_0^\infty \zeta^{-\alpha} F_q^{(n)}(\zeta)\mathrm{d}\zeta$.

(c) Clearly for each m we have $F_q(\zeta) \leq \frac{1}{(1+\zeta\tau^m)^{m+1}}$ forcing superpolynomial decay of F_q. The superpolynomial decay of higher order derivative now follows via induction using (3.2.4). $\square$

3.2.2 Proof of Theorem 3.1.1 and Theorem 3.1.2

Recall $H_0(t)$ from (3.1.1). As explained in Section 3.1.2, the main idea is to use Lemma 3.1.8 with $U = \tau^{H_0(t)}$ and $F = F_q$ defined in (3.1.7). Observe that Proposition 3.2.2 guarantees $F = F_q$ can be chosen in Lemma 3.1.8. In the following proposition, we show that limiting behavior of $\mathbf{E}[\tau^{sH_0(t)}]$ is governed by the integral in (3.1.11) restricted to $[1, \infty)$.

Proposition 3.2.3. *For any $s > 0$, we have*

$$
\lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}[\tau^{sH_0(t)}] = \lim_{t\to\infty} \frac{1}{t} \log \left[(-1)^n \int_1^\infty \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\mathbf{E}[F_q(\zeta\tau^{H_0(t)})]\mathrm{d}\zeta\right], \tag{3.2.5}
$$

where $n = \lfloor s \rfloor + 1 \geq 1$ and $\alpha = s - \lfloor s \rfloor$ so that $s = n - 1 + \alpha$.

Proof. Let $U = \tau^{H_0(t)}$. In this proof, we find an upper and a lower bound of $\mathbf{E}[U^s]$ and show that as $t \to \infty$, after taking logarithm of $\mathbf{E}[U^s]$ and dividing by t, the two bounds give matching results. Note that as $\tau \in (0, 1)$ and $H_0(t) \geq 0$ for any $n \in \mathbb{Z}_{\geq 0}$ and $t > 0$, U has finite n-th moment. By Proposition 3.2.2, F_q is n-times differentiable and $|\int_0^\infty x^{-\alpha} F_q^{(n)}(x)\mathrm{d}x| < \infty$. Denoting $\mathrm{d}\mathbb{P}_U(u)$ as the measure corresponding to the random variable U we have

$$
(-1)^n \int_1^\infty \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\mathbf{E}[F_q(\zeta\tau^{H_0(t)})]\mathrm{d}\zeta = (-1)^n \int_1^\infty \zeta^{-\alpha} \int_0^\infty u^n F_q^{(n)}(\zeta u)\mathrm{d}\mathbb{P}_U(u)\mathrm{d}\zeta. \tag{3.2.6}
$$

The $(-1)^n$ factor ensures that the above quantities are nonnegative via Proposition 3.2.2 (a). By the finiteness of the n-th moment of U, $\|F_q^{(n)}\|_\infty < \infty$ (by Proposition 3.2.2 (a)), and Fubini's theorem, we can interchange the integrals and obtain

$$\begin{aligned}\text{r.h.s of (3.2.6)} &= (-1)^n \int_0^\infty u^{n-1+\alpha} \int_1^\infty (\zeta u)^{-\alpha} F_q^{(n)}(\zeta u) \mathrm{d}(u\zeta) \mathrm{d}\mathbb{P}_U(u) \\ &= (-1)^n \int_0^\infty u^{n-1+\alpha} \int_u^\infty x^{-\alpha} F_q^{(n)}(x) \mathrm{d}x \, \mathrm{d}\mathbb{P}_U(u). \end{aligned} \tag{3.2.7}$$

Since the random variable $U \in [0,1]$, we can lower bound the inner integral on the r.h.s. of (3.2.7) by restricting the x-integral to $[1,\infty)$. Recalling that $s = n - 1 + \alpha$ we have

$$\text{r.h.s. of (3.2.6)} \geq (-1)^n \left(\int_1^\infty x^{-\alpha} F_q^{(n)}(x) \mathrm{d}x \right) \mathbf{E}[\tau^{sH_0(t)}]. \tag{3.2.8}$$

As for the upper bound for r.h.s. of (3.2.6), we may extend the range of integration to $[0,\infty)$. Apply Lemma 3.1.8 with $F \mapsto F_q$ and $U \mapsto \tau^{sH_0(t)}$ to get

$$\begin{aligned}\text{r.h.s. of (3.2.6)} &\leq (-1)^n \int_0^\infty \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathbf{E}\left[F_q(\zeta U)\right] \mathrm{d}\zeta \\ &= \left[(-1)^n \int_0^\infty \zeta^{-\alpha} F_q^{(n)}(\zeta) \mathrm{d}\zeta \right] \mathbf{E}[\tau^{sH_0(t)}]. \end{aligned} \tag{3.2.9}$$

Note that both the prefactors of $\mathbf{E}[\tau^{sH_0(t)}]$ in (3.2.8) and (3.2.9) are positive and free of t. Taking logarithms and dividing by t, we get the desired result. □

Next we truncate the integral in r.h.s. of (3.2.5) further. Recall the function $B_q(x)$ defined in Proposition 3.2.1 (a). We separate the range of integration $[1,\infty)$ into $[1, e^{tB_q(s/2)}]$ and $(e^{tB_q(s/2)}, \infty)$ and make use of the Fredholm determinant formula for $\mathbf{E}[F_q(\zeta\tau^{H_0(t)})]$ from Theorem 3.1.6 to write the integral in r.h.s. of (3.2.5) as follows.

$$(-1)^n \int_1^\infty \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathbf{E}[F_q(\zeta\tau^{H_0(t)})] \mathrm{d}\zeta = (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathbf{E}[F_q(\zeta\tau^{H_0(t)})] \mathrm{d}\zeta + \mathcal{R}_s(t)$$

$$= (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \det(I + K_{\zeta,t}) \mathrm{d}\zeta + \mathcal{R}_s(t), \tag{3.2.10}$$

where

$$\mathcal{R}_s(t) := (-1)^n \int_{e^{tB_q(\frac{s}{2})}}^{\infty} \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathbf{E}[F_q(\zeta\tau^{H_0(t)})] \mathrm{d}\zeta \tag{3.2.11}$$

Recall the definition of Fredholm determinant from (3.1.10). Assuming $\mathrm{tr}(K_{\zeta,t})$ to be differentiable for a moment we may split the first term in (3.2.10) into two parts and write

$$(-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \det(I + K_{\zeta,t}) \mathrm{d}\zeta = \mathcal{A}_s(t) + \mathcal{B}_s(t) \tag{3.2.12}$$

where

$$\mathcal{A}_s(t) := (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} \mathrm{tr}(K_{\zeta,t})\, \mathrm{d}\zeta, \tag{3.2.13}$$

$$\mathcal{B}_s(t) := (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} [\det(I + K_{\zeta,t}) - \mathrm{tr}(K_{\zeta,t})]\, \mathrm{d}\zeta. \tag{3.2.14}$$

The next two propositions verify that both $\mathcal{A}_s(t)$ and $\mathcal{B}_s(t)$ are well-defined and we defer their proofs to Sections 3.3 and 3.4, respectively. The first one guarantees that $\mathrm{tr}(K_{\zeta,t})$ is indeed infinitely differentiable and provides the asymptotics for $\mathrm{Re}[\mathcal{A}_s(t)]$.

Proposition 3.2.4. *For each $\zeta > 0$, the function $\zeta \mapsto \mathrm{tr}(K_{\zeta,t})$ is infinitely differentiable and thus $\mathcal{A}_s(t)$ in* (3.2.13) *is well defined. Furthermore, for any $s > 0$, we have*

$$\lim_{t\to\infty} \log\left(\mathrm{Re}[\mathcal{A}_s(t)]\right) = -h_q(s). \tag{3.2.15}$$

From (3.2.10), we know that the Fredholm determinant $\det(I + K_{\zeta,t})$ is infinitely differentiable. Thus, proposition 3.2.4 renders $(\det(I + K_{\zeta,t}) - \mathrm{tr}(K_{\zeta,t}))$ infinitely differentiable as well. Hence $\mathcal{B}_s(t)$ is well-defined. In fact, we have the following asymptotics for $\mathcal{B}_s(t)$.

Proposition 3.2.5. *Fix any $s > 0$ so that $s - \lfloor s \rfloor > 0$. Recall $\mathcal{B}_s(t)$ from* (3.2.14)*. There exists a constant* $\mathrm{C} = \mathrm{C}(q, s) > 0$ *such that for all $t > 0$, we have*

$$|\mathcal{B}_s(t)| \le \mathrm{C} \exp(-th_q(s) - \tfrac{1}{\mathrm{C}}t), \tag{3.2.16}$$

where $h_q(s)$ is defined in (3.1.3)*.*

Note that Proposition 3.2.5 in its current form does not cover integer s. We later explain in Section 3.4 why $s - \lfloor s \rfloor > 0$ is necessary for our proof. However, this does not effect our main results as one can deduce Theorem 3.1.1 for integer s as well via a simple continuity argument, which we present below. Assuming Propositions 3.2.4 and 3.2.5, we now complete the proof of Theorem 3.1.1 and Theorem 3.1.2.

Proof of Theorem 3.1.1. Fix $s > 0$ so that $s - \lfloor s \rfloor > 0$. Appealing to Proposition 3.2.3 and (3.2.10) and (3.2.12) we see that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{E}[\tau^{sH_0(t)}] = \lim_{t\to\infty} \frac{1}{t} \log \left[\mathcal{A}_s(t) + \mathcal{B}_s(t) + \mathcal{R}_s(t)\right],$$

where $\mathcal{A}_s(t)$, $\mathcal{B}_s(t)$, and $\mathcal{R}_s(t)$ are defined in (3.2.13), (3.2.14) and (3.2.11) respectively. For $\mathcal{R}_s(t)$, setting $V = \zeta\tau^{H_0(t)}$ and noting $s = n - 1 + \alpha$, we see that

$$|\mathcal{R}_s(t)| = \int_{e^{tB_q(\frac{s}{2})}}^{\infty} \zeta^{-\alpha-n}\mathbf{E}\left[|V^n F_q^{(n)}(V)|\right] \mathrm{d}\zeta \le \left[\sup_{v>0} |v^n F_q^{(n)}(v)|\right] s^{-1} \exp(-tsB_q(\tfrac{s}{2})).$$

The fact that $\sup_{v>0} |v^n F_q^{(n)}(v)|$ is finite follows from Proposition 3.2.2 (c). Note that $sB_q(\frac{s}{2})$ is strictly bigger than $h_q(s) = sB_q(s) > 0$ via Proposition 3.2.1 (a). By Proposition 3.2.4, when t is large, we see that $\mathrm{Re}[\mathcal{A}_s(t)]$ grows like $\exp(-th_q(s)) > \exp(-tsB_q(\frac{s}{2}))$. Similarly, Proposition 3.2.5 shows that $\mathrm{Re}[\mathcal{B}_s(t)]$ is bounded from above by $\mathrm{C}\exp(-th_q(s) - \frac{1}{\mathrm{C}}t)$ for some constant $\mathrm{C} = \mathrm{C}(q, s)$, which is strictly less than $\exp(-th_q(s))$ for large enough t. Indeed for all large enough t, we have

$$\frac{1}{2}\operatorname{Re}[\mathcal{A}_s(t)] \leq \operatorname{Re}[\mathcal{A}_s(t)+\mathcal{B}_s(t)+\mathcal{R}_s(t)] \leq \frac{3}{2}\operatorname{Re}[\mathcal{A}_s(t)].$$

Taking logarithms and dividing by t, and noting that $\mathcal{A}_s(t)+\mathcal{B}_s(t)+\mathcal{R}_s(t)$ is always real, we get (3.1.3) for any noninteger positive s.

To prove (3.1.3) for positive integer s, we fix $s \in \mathbb{Z}_{>0}$. For any $K > 2$, observe that as $H_0(t)$ is a non-negative random variable (recall the definition from (3.1.1)) we have

$$\tau^{(s-K^{-1})H_0(t)} \geq \tau^{sH_0(t)} \geq \tau^{(s+K^{-1})H_0(t)}.$$

Taking expectations, then logarithms and dividing by t, in view of noninteger version of (3.1.3) we have

$$-h_q(s-K^{-1}) \geq \limsup_{t\to\infty}\frac{1}{t}\log\mathbf{E}[\tau^{sH_0(t)}] \geq \liminf_{t\to\infty}\frac{1}{t}\log\mathbf{E}[\tau^{sH_0(t)}] \geq -h_q(s+K^{-1}).$$

Taking $K \to \infty$ we get the desired result for integer s. □

Proof of Theorem 3.1.2. For the large deviation result, applying Proposition 1.12 in [180], with $X(t) = H_0(t/\gamma)\cdot\log\tau$, and noting the Legendre-Fenchel type identity for $\Phi_+(y)$ from Proposition 3.2.1 (c), we arrive at (3.1.4). To prove (3.1.5), applying L-Hôpital rule a couple of times we get

$$\lim_{y\to0^+}\frac{\Phi_+(y)}{y^{3/2}} = \lim_{y\to0^+}\frac{2}{3}\frac{\Phi_+'(y)}{\sqrt{y}} = \lim_{x\to0^+}\frac{2}{3}\frac{\tanh^{-1}(x)}{x} = \lim_{x\to0^+}\frac{2}{3}\cdot\frac{1}{1-x^2} = \frac{2}{3}.$$

This completes the proof of the theorem. □

3.3 Asymptotics of the Leading Term

The goal of this section is to obtain exact asymptotics of $\operatorname{Re}[\mathcal{A}_s(t)]$ defined in (3.2.13) as $t \to \infty$. Recall the definition of the kernel $K_{\zeta,t}$ from (3.1.8). We employ a standard idea that the asymptotic behavior of the kernel $K_{\zeta,t}$ and its 'derivative' (see (3.3.8)) and subsequently that of $\mathrm{Re}[\mathcal{A}_s(t)]$ can be derived by the *steepest descent method*.

Towards this end, we first collect all the technical estimates related to the kernel $K_{\zeta,t}$ in Section 8.5.3 and go on to complete the proof of Proposition 3.2.4 in Section 3.3.2.

3.3.1 Technical estimates of the Kernel

In this section, we analyze the kernel $K_{\zeta,t}$. Much of our subsequent analysis boils down to understanding the function $g_t(z)$, defined in (3.1.8), that appears in the kernel $K_{\zeta,t}$. Towards this end, we consider

$$f(u,z) := \frac{(q-p)}{1+\frac{z}{\tau}} - \frac{(q-p)}{1+\frac{\tau^u z}{\tau}}, \tag{3.3.1}$$

so that the ratio $\frac{g_t(z)}{g_t(\tau^u z)}$ that appears in the kernel $K_{\zeta,t}$ defined in (3.1.8) equals to $\exp(tf(u,z))$. Below we collect some useful properties of this function $f(u,z)$. First note that $\partial_z f(u,z) = 0$ has two solutions $z = \pm\tau^{1-\frac{u}{2}}$, and

$$\begin{aligned} \partial_z^2 f(u,z)\Big|_{z=-\tau^{1-\frac{u}{2}}} &= -2(q-p)\frac{\tau^{\frac{3u}{2}-2}+\tau^{2u-2}}{(1-\tau^{\frac{u}{2}})^3}, \\ \partial_z^2 f(u,z)\Big|_{z=\tau^{1-\frac{u}{2}}} &= 2(q-p)\frac{\tau^{\frac{3u}{2}-2}-\tau^{2u-2}}{(1+\tau^{\frac{u}{2}})^3}. \end{aligned} \tag{3.3.2}$$

The following lemma tells us how the maximum of $\mathrm{Re}[f(u,z)]$ behaves.

Lemma 3.3.1. *Fix $\rho > 0$. For any $u \in \mathbb{C}$, with $\mathrm{Re}[u] = \rho$ and $z \in \mathfrak{C}(\tau^{1-\frac{\rho}{2}})$, we have*

$$\mathrm{Re}[f(u,z)] \le f(\rho, \tau^{1-\frac{\rho}{2}}) = -h_q(\rho) \tag{3.3.3}$$

where $h_q(\rho)$ is defined in (3.1.3) *and $\mathfrak{C}(\tau^{1-\frac{\rho}{2}})$ is the circle with center at the origin and radius $\tau^{1-\frac{\rho}{2}}$. Equality in* (3.3.3) *holds if and only if $\tau^{\mathbf{i}\,\mathrm{Im}\,u} = 1$, and $z = \tau^{1-\frac{\rho}{2}}$ simultaneously. Furthermore,* *for the same range of u and z, we have the following inequality:*

$$f(\rho,\tau^{1-\frac{\rho}{2}})-\mathrm{Re}[f(u,z)]\geq\frac{(q-p)(1-\tau^{\frac{\rho}{2}})\tau^{\frac{\rho}{2}}}{4(1+\tau^{\frac{\rho}{2}})^2}(2\tau^{\frac{\rho}{2}-1}|z-\tau^{1-\frac{\rho}{2}}|+|\tau^{\mathbf{i}\,\mathrm{Im}\,u}-1|). \tag{3.3.4}$$

Proof. Set $u=\rho+\mathbf{i}y$ and $z=\tau^{1-\frac{\rho}{2}}e^{\mathbf{i}\theta}$ with $y\in\mathbb{R}$ and $\theta\in[0,2\pi]$. Note that $f(\rho,\tau^{1-\frac{\rho}{2}})=-h_q(\rho)$, where $h_q(x)$ is defined in (3.1.3). Direct computation yields

$$\mathrm{Re}[f(u,z)]=\frac{(q-p)(\tau^{\rho}-1)(|1+\tau^{\frac{\rho}{2}}e^{-\mathbf{i}\theta}|^2+|1+\tau^{\frac{\rho}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|^2)}{2|1+\tau^{\frac{\rho}{2}}e^{-\mathbf{i}\theta}|^2|1+\tau^{\frac{\rho}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|^2}. \tag{3.3.5}$$

Since $\tau<1$, applying the inequality $|1+\tau^{\frac{\rho}{2}}e^{-\mathbf{i}\theta}|^2+|1+\tau^{\frac{\rho}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|^2\geq 2|1+\tau^{\frac{\rho}{2}}e^{-\mathbf{i}\theta}||1+\tau^{\frac{\rho}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|$, and then noting that $|1+\tau^{\frac{\rho}{2}}e^{-\mathbf{i}\theta}||1+\tau^{\frac{\rho}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|\leq(1+\tau^{\frac{\rho}{2}})^2$, we see (r.h.s. of (3.3.5)) $\leq-(q-p)\frac{1-\tau^{\frac{\rho}{2}}}{1+\tau^{\frac{\rho}{2}}}$. Clearly equality holds if and only if $\theta=0$ and $\tau^{\mathbf{i}y}=1$ simultaneously. Furthermore, following the above inequalities, we have $\mathrm{Re}[f(\rho+\mathbf{i}y,z)]\leq-(q-p)\frac{1-\tau^{\frac{\rho}{2}}}{|1+\tau^{\frac{\rho}{2}}e^{\mathbf{i}\theta}|}$ and $\mathrm{Re}[f(\rho+\mathbf{i}y,z)]\leq-(q-p)\frac{1-\tau^{\frac{\rho}{2}}}{|1+\tau^{\frac{\rho}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|}$. This yields

$$\begin{aligned}f(\rho,\tau^{1-\frac{\rho}{2}})-\mathrm{Re}[f(\rho+\mathbf{i}y,z)]&\geq(q-p)\left[\frac{1-\tau^{\frac{\rho}{2}}}{|1+\tau^{\frac{\rho}{2}}e^{\mathbf{i}\theta}|}-\frac{1-\tau^{\frac{\rho}{2}}}{1+\tau^{\frac{\rho}{2}}}\right]\\&\geq\frac{(q-p)(\tau^{\frac{\rho}{2}}-\tau^{\rho})|e^{\mathbf{i}\theta}-1|}{(1+\tau^{\frac{\rho}{2}})^2}\end{aligned} \tag{3.3.6}$$

and

$$\begin{aligned}f(\rho,\tau^{1-\frac{\rho}{2}})-\mathrm{Re}[f(\rho+\mathbf{i}y,z)]&\geq(q-p)\left[\frac{1-\tau^{\frac{\rho}{2}}}{|1+\tau^{\frac{\rho}{2}+\mathbf{i}y}e^{\mathbf{i}\theta}|}-\frac{1-\tau^{\frac{\rho}{2}}}{1+\tau^{\frac{\rho}{2}}}\right]\\&\geq\frac{(q-p)(1-\tau^{\frac{\rho}{2}})\tau^{\frac{\rho}{2}}|\tau^{\mathbf{i}y}e^{\mathbf{i}\theta}-1|}{(1+\tau^{\frac{\rho}{2}})^2}.\end{aligned}$$

Adding the above two inequalities we have $f(\rho,\tau^{1-\frac{\rho}{2}})-\mathrm{Re}[f(\rho+\mathbf{i}y,z)]\geq\frac{(q-p)(1-\tau^{\frac{\rho}{2}})\tau^{\frac{\rho}{2}}|\tau^{\mathbf{i}y}-1|}{2(1+\tau^{\frac{\rho}{2}})^2}$. Combining this with (3.3.6) and the substitution $\tau^{1-\frac{\rho}{2}}e^{\mathbf{i}\theta}=z$ we get (3.3.4). This completes the proof. □

Using the above technical lemma we can now explain the proof of Theorem 3.1.6.

Proof of Theorem 3.1.6. Due to Theorem 5.3 in [72], the only thing that we need to verify is

$$\inf_{\substack{w,w'\in\mathfrak{C}(\tau^{1-\frac{\delta}{2}})\\ u\in\delta+\mathbf{i}\mathbb{R}}} |w'-\tau^u w|>0 \quad \text{and} \quad \sup_{\substack{w,w'\in\mathfrak{C}(\tau^{1-\frac{\delta}{2}})\\ u\in\delta+\mathbf{i}\mathbb{R}}} \left|\frac{g_t(w)}{g_t(\tau^u w)}\right|>0. \tag{3.3.7}$$

Indeed, for every $u\in\delta+\mathbf{i}\mathbb{R}$ and $w,w'\in\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$, we have $|w'-\tau^u w|\geq|w'|-|\tau^u w|=\tau^{1-\frac{\delta}{2}}-\tau^{1+\frac{\delta}{2}}>0$. Recall $f(u,z)$ from (3.3.1). Applying Lemma 3.3.1 with $\rho\mapsto\delta$ yields

$$\left|\frac{g_t(w)}{g_t(\tau^u w)}\right| = |\exp(tf(u,w))| = \exp(t\,\mathrm{Re}[f(u,w)]) \leq \exp(tf(\delta,\tau^{1-\frac{\delta}{2}})) = \exp(-th_q(\delta)),$$

where h_q is defined in (3.1.3). This verifies (3.3.7) and completes the proof. $\square$

Remark 3.3.2. We now explain our choice of the contour $K_{\zeta,t}$ defined in (3.1.8), which comes from the method of steepest descent. Suppose $\mathrm{Re}[u]=\delta$. As noted before, directly taking derivative of $f(u,z)=\exp(\frac{g_t(z)}{g_t(\tau^u z)})$, with respect to z suggests that critical points are at $z=\pm\tau^{1-\frac{u}{2}}$, and thus we take our contour to be $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$, so that it passes through the critical points.

Next we turn to the case of differentiability of $\mathrm{tr}(K_{\zeta,t})$ where $K_{\zeta,t}$ is defined in (3.1.8). Using the function f defined in (3.3.1), we rewrite the kernel as follows.

$$K_{\zeta,t}(w,w') = \frac{1}{2\pi\mathbf{i}}\int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \Gamma(-u)\Gamma(1+u)\zeta^u e^{tf(u,w)}\frac{\mathrm{d}u}{w'-\tau^u w}.$$

Differentiating the integrand inside the integral in $K_{\zeta,t}(w.w')$ n-times defines a sequence of kernel $\{K_{\zeta,t}^{(n)}\}_{n\geq1}: L^2(\mathfrak{C}(\tau^{1-\frac{\delta}{2}}))\to L^2(\mathfrak{C}(\tau^{1-\frac{\delta}{2}}))$ given by the kernel:

$$K_{\zeta,t}^{(n)}(w,w') := \frac{1}{2\pi\mathbf{i}}\int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \Gamma(-u)\Gamma(1+u)(u)_n\zeta^{u-n} e^{tf(u,w)}\frac{\mathrm{d}u}{w'-\tau^u w}, \tag{3.3.8}$$

where $(a)_n:=\prod_{i=0}^{n-1}(a-i)$ for $n\in\mathbb{Z}_{>0}$ and $(a)_0=1$ is the Pochhammmer symbol and $\delta\in(0,1)$. We also set $K_{\zeta,t}^{(0)}:=K_{\zeta,t}$.

Remark 3.3.3. We remark that unlike Lemma 3.1 in [131], we do not aim to show that $K_{\zeta,t}$ is differentiable as an operator, or its higher order derivatives are equal to the operator $K^{(n)}_{\zeta,t}$. Indeed, showing convergence in the trace class norm is more involved because of the lack of symmetry and positivity of the operator $K_{\zeta,t}$. However, since we are dealing with the Fredholm determinant series only, for our analysis it is enough to investigate how each term of the series are differentiable and how their derivatives are related to $K^{(n)}_{\zeta,t}$.

Remark 3.3.4. Note that when viewing $K^{(n)}_{\zeta,t}$ as a complex integral, we can deform its u-contour to $\rho + \mathbf{i}\mathbb{R}$ for any $\rho \in (0, n \vee 1)$. This is due to the analytic continuity of the integrand as the factor $(u)_n$ removes the poles at $1, \dots, n-1$ of $\Gamma(-u)$.

The following lemma provides estimates of $K^{(n)}_{\zeta,t}$ that is useful for the subsequent analysis in Sections 3.3 and 3.4.

Lemma 3.3.5. *Fix $n \in \mathbb{Z}_{\geq 0}, t > 0, \delta, \rho \in (0, n \vee 1)$, and consider any borel set $A \subset \mathbb{R}$. Recall $h_q(x)$ and $B_q(x)$ from Proposition 3.2.1 and $K^{(n)}_{\zeta,t}$ from* (3.3.8). *For any $w \in \mathfrak{C}(\tau^{1-\frac{\delta}{2}})$ and $w' \in \mathbb{C}$ and $\zeta \in [1, e^{tB_q(\frac{s}{2})}]$, there exists a constant $\mathrm{C} = \mathrm{C}(n, \delta, q) > 0$ such that whenever $|w'| \neq \tau^{1+\frac{\delta}{2}}$ we have*

$$\begin{aligned}\int_A \left| \frac{(\delta + \mathbf{i}y)_n \zeta^{\rho - n + \mathbf{i}y}}{\sin(-\pi(\delta + \mathbf{i}y))} e^{tf(\delta + \mathbf{i}y, w)} \right| \frac{\mathrm{d}y}{|w' - \tau^{\delta + \mathbf{i}y} w|} &\leq \frac{\mathrm{C}\zeta^{\rho - n}}{||w'| - \tau^{1+\frac{\delta}{2}}|} e^{t \cdot \sup_{y \in A} \mathrm{Re}[f(\delta + \mathbf{i}y, w)]} \\ &\leq \frac{\mathrm{C}\zeta^{\rho - n}}{||w'| - \tau^{1+\frac{\delta}{2}}|} e^{-th_q(\delta)}.\end{aligned} \tag{3.3.9}$$

In particular when $w' \in \mathfrak{C}(\tau^{1-\frac{\delta}{2}})$ we have

$$|K^{(n)}_{\zeta,t}(w, w')| \leq \mathrm{C}\zeta^{\delta - n} \exp(-th_q(\delta)). \tag{3.3.10}$$

Consequently, $K^{(n)}_{\zeta,t}(w, w')$ is continuous in the ζ-variable.

Proof. Fix $n \in \mathbb{Z}_{\geq 0}, t > 0$, $\delta, \rho \in (0, n \vee 1)$ and $w \in \mathfrak{C}(\tau^{1-\frac{\delta}{2}})$ and $w' \in \mathbb{C}$ such that $|w'| \neq \tau^{1+\frac{\delta}{2}}$. Throughout the proof the constant $\mathrm{C} > 0$ depends on n, δ, and q – we will not mention it further.

Consider the integral on the r.h.s. of (3.3.9). Observe that when $\delta \notin \mathbb{Z}$, $|(\delta+\mathbf{i}y)_n| \le \mathrm{C}|y|^n$ and $\frac{1}{|\sin(-\pi(\delta+\mathbf{i}y))|} \le \mathrm{C}e^{-|y|/\mathrm{C}}$. For $n \ge 2$, and $\delta \in \mathbb{Z}_{>0} \cap (0,n)$, we observe that the product $(\delta+\mathbf{i}y)_n$ contains the term $\mathbf{i}y$. Hence $|\frac{\mathbf{i}y}{\sin(-\pi(\delta+\mathbf{i}y))}| = |\frac{\mathbf{i}y}{\sin(-\pi(\mathbf{i}y))}| \le \mathrm{C}e^{-|y|/\mathrm{C}}$ for such an integer δ. Whereas, $|\frac{\delta+\mathbf{i}y}{\mathbf{i}y}| \le \mathrm{C}|y|^{n-1}$ for such an integer δ. Finally, $|w' - \tau^{\delta+\mathbf{i}y}w| \ge ||w'| - |\tau^{\delta}w|| = ||w'| - \tau^{1+\frac{\delta}{2}}|$. Combining the aforementioned estimates, we obtain that

$$\text{r.h.s. of (3.3.9)} \le \int_A \mathrm{C}|y|^n e^{-|y|/\mathrm{C}}\zeta^{\rho-n}|e^{tf(\delta+\mathbf{i}y,w)}|\frac{\mathrm{d}y}{||w'|-\tau^{1+\frac{\delta}{2}}|}.$$

Since $\int_{\mathbb{R}}|y|^n e^{-|y|/\mathrm{C}}\mathrm{d}y$ converges applying $|e^{tf(\delta+\mathbf{i}y,w)}| \le e^{t\,\mathrm{Re}[f(\delta+\mathbf{i}y,w)]}$ we arrive at the first inequality in (3.3.9). The second inequality follows by observing $\mathrm{Re}[f(\delta+\mathbf{i}y,w)] \le -h_q(\delta)$ by Lemma 3.3.1.

Recall $K^{(n)}_{\zeta,t}$ from (3.3.8). Recall from Remark 3.3.4 that the δ appearing in (3.3.8) can be chosen in $(0, n \vee 1)$. Pushing the absolute value sign inside the explicit formula in (3.3.8) and applying Euler's reflection principle with change of variables $u = \delta + \mathbf{i}y$ yield

$$|K^{(n)}_{\zeta,t}(w,w')| \le \frac{1}{2\pi}\int_{\mathbb{R}}\left|\frac{(\delta+\mathbf{i}y)_n\zeta^{\delta-n+\mathbf{i}y}}{\sin(-\pi(\delta+\mathbf{i}y))}e^{tf(\delta+\mathbf{i}y,w)}\right|\frac{\mathrm{d}y}{|w'-\tau^{\delta+\mathbf{i}y}w|}.$$

(3.3.10) now follows from (3.3.9) by taking $\rho = \delta$. To see the continuity of $K^{(n)}_{\zeta,t}(w,w')$ in ζ, we fix $\zeta_1 < \zeta_2 < \zeta_1 + 1$. By repeating the same set of arguments as above we arrive at

$$|K^{(n)}_{\zeta_2,t}(w,w') - K^{(n)}_{\zeta_1,t}(w,w')| \le C|\zeta_2^{\delta-n} - \zeta_1^{\delta-n}|\exp(-th_q(\delta)) \tag{3.3.11}$$

with the same constant C in (3.3.10). Clearly l.h.s. of (3.3.11) converges to 0 when $\zeta_2 \to \zeta_1$, which confirms the kernel's ζ-continuity. □

3.3.2 Proof of Proposition 3.2.4

The goal of this section is to prove Proposition 3.2.4. Before diving into the proof, we first settle the infinite differentiability separately in the next proposition.

Proposition 3.3.6. *For any $n \in \mathbb{Z}_{\geq 0}$ and $t > 0$, the operator $K^{(n)}_{\zeta,t}$ defined in* (3.3.8) *is a trace-class operator with*

$$\operatorname{tr}(K^{(n)}_{\zeta,t}) = \frac{1}{2\pi\mathbf{i}} \int_{\mathfrak{C}(\tau^{1-\frac{\delta}{2}})} K^{(n)}_{\zeta,t}(w,w)\mathrm{d}w. \tag{3.3.12}$$

Furthermore, $\operatorname{tr}(K^{(n)}_{\zeta,t})$ *is differentiable in ζ at each $\zeta > 0$ and we have* $\partial_\zeta \operatorname{tr}(K^{(n)}_{\zeta,t}) = \operatorname{tr}(K^{(n+1)}_{\zeta,t})$.

Proof. Fix $n \in \mathbb{Z}_{\geq 0}, t > 0$, and $\zeta > 0$. $K^{(n)}_{\zeta,t}(w,w')$ is simultaneously continuous in both w and w' and $\partial_{w'} K^{(n)}_{\zeta,t}(w,w')$ is continuous in w'. By Lemma 3.2.7 in [68] (also see [239, page 345] or [64]) we see that $K^{(n)}_{\zeta,t}$ is indeed trace-class, and thus (3.3.12) follows from Theorem 12 in [239, Chapter 30]. To show differentiability of $\operatorname{tr}(K^{(n)}_{\zeta,t})$ in variable ζ, we fix $\zeta_1, \zeta_2 > 0$. Without loss of generality we may assume $\zeta_1 + 1 > \zeta_2 > \zeta_1$. Let us define

$$\begin{aligned} D_{\zeta_1,\zeta_2} &:= \frac{\operatorname{tr}(K^{(n)}_{\zeta_2,t}) - \operatorname{tr}(K^{(n)}_{\zeta_1,t})}{\zeta_2 - \zeta_1} - \operatorname{tr}(K^{(n+1)}_{\zeta_1,t}) \\ &= \frac{1}{(2\pi\mathbf{i})^2} \int_{\mathfrak{C}(\tau^{1-\frac{\delta}{2}})} \int_{\delta - \mathbf{i}\infty}^{\delta + \mathbf{i}\infty} \Gamma(-u)\Gamma(1+u) R_{\zeta_1,\zeta_2;n}(u) e^{tf(u,w)} \frac{\mathrm{d}u}{w - \tau^u w} \mathrm{d}w, \end{aligned}$$

where

$$\begin{aligned} R_{\zeta_1,\zeta_2;n}(u) &:= (u)_n \left[\frac{\zeta_2^{u-n} - \zeta_1^{u-n}}{\zeta_2 - \zeta_1} - (u-n)\zeta_1^{u-n-1} \right] \\ &= \int_{\zeta_1}^{\zeta_2} \frac{(\zeta_2 - \sigma)}{\zeta_2 - \zeta_1} (u)_{n+2} \sigma^{u-n-2} \mathrm{d}\sigma. \end{aligned} \tag{3.3.13}$$

Taking absolute value and appealing to Euler's reflection principle, we obtain

$$\begin{aligned} |D_{\zeta_1,\zeta_2}| &\leq \left| \frac{1}{(2\pi\mathbf{i})^2} \int_{\mathfrak{C}(\tau^{1-\frac{\delta}{2}})} \int_{\delta - \mathbf{i}\infty}^{\delta + \mathbf{i}\infty} \int_{\zeta_1}^{\zeta_2} \frac{(u)_{n+2}}{\sin(-\pi u)} \frac{(\zeta_2 - \sigma)}{\zeta_2 - \zeta_1} \sigma^{u-n-2} e^{tf(u,w)} \frac{\mathrm{d}\sigma \mathrm{d}u}{w - \tau^u w} \mathrm{d}w \right| \\ &\leq \frac{\tau^{1-\frac{\delta}{2}}}{2\pi} \int_{\zeta_1}^{\zeta_2} |\sigma^{\delta + \mathbf{i}y - n-2}| \mathrm{d}\sigma \cdot \max_{w \in \mathfrak{C}(\tau^{1-\frac{\delta}{2}})} \int_{\mathbb{R}} \frac{(\delta + \mathbf{i}y)_{n+2}}{\sin(-\pi(\delta + \mathbf{i}y))} |e^{tf(\delta + \mathbf{i}y, w)}| \frac{\mathrm{d}y}{|w - \tau^{\delta + \mathbf{i}y} w|}. \end{aligned} \tag{3.3.14}$$

Note that Lemma 3.3.5 ((3.3.9) specifically) we see that the above maximum is bounded by $\mathrm{C}\exp(-th_q(\delta))$ where the constant C is same as in (3.3.9). Since $|\sigma^{u-n-2}| = |\sigma^{\delta-n-2}| \leq |\zeta_1^{\delta-n-2}|$

over the interval $[\zeta_1, \zeta_2]$ for $\delta \in (0, n \vee 1)$, we obtain

$$|D_{\zeta_1,\zeta_2}| \leq \mathrm{C}\exp(-h_q(\delta)) \int_{\zeta_1}^{\zeta_2} |\sigma^{u-n-2}| \mathrm{d}\sigma \leq \mathrm{C}\exp(-th_q(\delta))(\zeta_2 - \zeta_1)|\zeta_1^{\delta-n-2}|.$$

Thus, taking the limit as $\zeta_2 - \zeta_1 \to 0$ yields $|D_{\zeta_1,\zeta_2}| \to 0$ and completes the proof. □

Remark 3.3.7. We prove a higher order version of Proposition 3.3.6 later in Section 3.4 as Proposition 3.4.1 which includes the statement of the above Proposition when $L = 1$. However, we keep the above simple version for reader's convenience, which will serve as a guide in proving Proposition 3.4.1.

With the above results in place, we can now turn towards the main technical component of the proof of Proposition 3.2.4.

Proof of Proposition 3.2.4. Before proceeding with the proof, we fix some notations. Fix $s > 0$, and set $n = \lfloor s \rfloor + 1 \geq 1$ and $\alpha = s - \lfloor s \rfloor \in [0, 1)$ so that $s = n - 1 + \alpha$. Throughout the proof, we will denote C to be positive constant depending only on s, q – we will not mention this further. We will also use the big O notation. For two complex-valued functions $f_1(t)$ and $f_2(t)$ and $\beta \in \mathbb{R}$, the equations $f_1(t) = (1 + O(t^{\beta}))f_2(t)$ and $f_1(t) = f_2(t) + O(t^{\beta})$ have the following meaning: there exists a constant $\mathrm{C} > 0$ such that for all large enough t,

$$\left| \frac{f_1(t)}{f_2(t)} - 1 \right| \leq \mathrm{C} \cdot t^{\beta}, \text{ and } |f_1(t) - f_2(t)| \leq \mathrm{C} \cdot t^{\beta},$$

respectively. The constant $\mathrm{C} > 0$ value may change from line to line.

For clarity we divide the proof into seven steps. In Steps 1 and 2, we provide the upper and lower bounds for $|\mathcal{A}_s(t)|$ and $\mathrm{Re}[\mathcal{A}_s(t)]$ respectively and complete the proof of (3.2.15); in Steps 3–7, we verify the technical estimates assumed in the previous steps.

Step 1. Recall $\mathcal{A}_s(t)$ from (3.2.13). The goal of this step is to provide a different expression for $\mathcal{A}_s(t)$, which will be much more amenable to our analysis, as well as an upper bound for $|\mathcal{A}_s(t)|$. By Proposition 3.3.6, we have $\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\mathrm{tr}(K_{\zeta,t}) = \mathrm{tr}(K_{\zeta,t}^{(n)})$ and consequently using the expression in

(3.3.8) we have

$$\mathcal{A}_s(t) := (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \frac{\zeta^{-\alpha}}{(2\pi\mathbf{i})^2} \int_{\mathfrak{C}(\tau^{1-\frac{\delta}{2}})} \int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \Gamma(-u)\Gamma(1+u)(u)_n \zeta^{u-n} \frac{e^{tf(u,w)}\mathrm{d}u}{w-\tau^u w} \mathrm{d}w\mathrm{d}\zeta.$$

where $\delta \in (0,1)$ is chosen to be less than s. We now proceed to deform the u-contour and w-contour sequentially. As we explained in Remark 3.3.4, the integrand has no poles when $u = 1, 2, \ldots, n-1$. Hence u-contour can be deformed to $(s-\mathbf{i}\infty, s+\mathbf{i}\infty)$ as $s = n-1+\alpha \in (0,n)$.

Next, for the w-contour, we wish to deform it from $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$ to $\mathfrak{C}(\tau^{1-\frac{s}{2}})$. In order to do so, we need to ensure that we do not cross any poles. We observe that the potential sources of poles lie in the exponent $f(u,w) := \frac{(q-p)}{1+w\tau^{-1}} - \frac{(q-p)}{1+\tau^{u-1}w}$ (recalled from (3.3.1)) and in the denominator $w - \tau^u w$. Since for any $w \in \mathfrak{C}(\tau^{1-\frac{\delta'}{2}})$, where $\delta' \in (\delta, s)$, and $u \in (s-\mathbf{i}\infty, s+\mathbf{i}\infty)$, we have

$$|w-\tau^u w| \geq |w| - |\tau^u w| = \tau^{1-\frac{\delta'}{2}}(1-\tau^s) > 0, \quad |1+w\tau^{-1}| \geq |w\tau^{-1}| - 1 = \tau^{-\frac{\delta'}{2}} - 1 > 0,$$

$$\text{and } |1+\tau^{u-1}w| \geq 1 - |\tau^{u-1}w| = 1 - \tau^{s-\frac{\delta'}{2}} > 0.$$

Thus, we can deform the w-contour to $\mathfrak{C}(\tau^{1-\frac{s}{2}})$ as well without crossing any poles. With the change of variable $u = s + \mathbf{i}y$, $w = \tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta}$, and Euler's reflection formula we have

$$\mathcal{A}_s(t) = (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \frac{\zeta^{-1}}{4\pi^2} \int_{-\pi}^{\pi} \int_{\mathbb{R}} \frac{(s+\mathbf{i}y)_n \zeta^{\mathbf{i}y}}{\sin(-\pi(s+\mathbf{i}y))} e^{tf(s+\mathbf{i}y, \tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})} \frac{\mathrm{d}y}{1-\tau^{s+\mathbf{i}y}} \mathrm{d}\theta\mathrm{d}\zeta. \tag{3.3.15}$$

With this expression in hand, upper bound is immediate. By Lemma 3.3.5 ((3.3.9) specifically with $\rho \mapsto n-1$, $\delta \mapsto s$) pushing the absolute value inside the integrals we see that

$$|\mathcal{A}_s(t)| \leq \mathrm{C}\exp(-th_q(s)) \int_1^{e^{tB_q(\frac{s}{2})}} \frac{1}{\zeta}\mathrm{d}\zeta = \mathrm{C} \cdot tB_q(\tfrac{s}{2}) \exp(-th_q(s)) \tag{3.3.16}$$

for some constant $\mathrm{C} = \mathrm{C}(q,s) > 0$. Hence taking logarithm and dividing by t, we get

$$\limsup_{t\to\infty} |\mathcal{A}_s(t)| \leq -h_q(s) = -(q-p)\frac{1-\tau^{\frac{s}{2}}}{1+\tau^{\frac{s}{2}}}. \tag{3.3.17}$$

Step 2. In this step, we provide a lower bound for $\mathrm{Re}[\mathcal{A}_s(t)]$. Set $\varepsilon = t^{-2/5} > 0$. For each $k \in \mathbb{Z}$, set $v_k = -\frac{2\pi}{\log\tau}k$ and consider the interval $V_k := [v_k - \varepsilon^2, v_k + \varepsilon^2]$. Also set $A_\varepsilon := \{\theta \in [-\pi, \pi] : |e^{\mathbf{i}\theta} - 1| \le \varepsilon|\log\tau|\}$. We divide the triple integral in (3.3.15) into following parts

$$\mathcal{A}_s(t) = \sum_{k\in\mathbb{Z}} (\mathbf{I})_k + (\mathbf{II}) + (\mathbf{III}), \tag{3.3.18}$$

where

$$(\mathbf{I})_k := \int_1^{e^{tB_q(\frac{s}{2})}} \int_{A_\varepsilon} \int_{V_k} \frac{(-1)^n}{4\pi^2\zeta} \frac{(s+\mathbf{i}y)_n \zeta^{\mathbf{i}y}}{\sin(-\pi(s+\mathbf{i}y))} \frac{e^{tf(s+\mathbf{i}y,\tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})}\mathrm{d}y}{1-\tau^{s+\mathbf{i}y}} \mathrm{d}\theta\mathrm{d}\zeta, \tag{3.3.19}$$

$$(\mathbf{II}) := \int_1^{e^{tB_q(\frac{s}{2})}} \int_{A_\varepsilon} \int_{\mathbb{R}\setminus\cup_k V_k} \frac{(-1)^n}{4\pi^2\zeta} \frac{(s+\mathbf{i}y)_n \zeta^{\mathbf{i}y}}{\sin(-\pi(s+\mathbf{i}y))} \frac{e^{tf(s+\mathbf{i}y,\tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})}\mathrm{d}y}{1-\tau^{s+\mathbf{i}y}} \mathrm{d}\theta\mathrm{d}\zeta, \tag{3.3.20}$$

$$(\mathbf{III}) := \int_1^{e^{tB_q(\frac{s}{2})}} \int_{[-\pi,\pi]\cap A_\varepsilon^c} \int_{\mathbb{R}} \frac{(-1)^n}{4\pi^2\zeta} \frac{(s+\mathbf{i}y)_n \zeta^{\mathbf{i}y}}{\sin(-\pi(s+\mathbf{i}y))} \frac{e^{tf(s+\mathbf{i}y,\tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})}\mathrm{d}y}{1-\tau^{s+\mathbf{i}y}} \mathrm{d}\theta\mathrm{d}\zeta. \tag{3.3.21}$$

In subsequent steps we obtain the following estimates for each integral. We claim that we have

$$(\mathbf{I})_0 = (1 + O(t^{-\frac{1}{5}}))\frac{\mathrm{C}_0}{\sqrt{t}}\exp(-th_q(s)), \tag{3.3.22}$$

where $h_q(s)$ is defined in (3.1.3) and

$$\mathrm{C}_0 := \sqrt{\frac{(1+\tau^{\frac{s}{2}})^3}{4\pi(q-p)(\tau^{\frac{3s}{2}-2} - \tau^{2s-2})}} \frac{(-1)^n(s)_n}{\sin(-\pi s)(1-\tau^s)} > 0. \tag{3.3.23}$$

When s is an integer the above constant is defined in a limiting sense. Note that C_0 is indeed positive as $n = \lfloor s\rfloor + 1$. Furthermore, we claim that we have the following upper bounds for the other integrals:

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} |(\mathbf{I})_k| \le \mathrm{C}t^{-\frac{13}{10}}\exp(-th_q(s)). \tag{3.3.24}$$

where $v_k = -\frac{2\pi}{\log\tau}k$ and

$$|(\mathbf{II})|, |(\mathbf{III})| \le \mathrm{C}t \exp\left(-th_q(s)\right) \exp(-\tfrac{1}{\mathrm{C}}t^{\frac{1}{5}}). \tag{3.3.25}$$

Assuming the validity of (3.3.22), (3.3.24) and (3.3.25) we can complete the proof of lower bound for (3.2.15). Following the decomposition in (3.3.18) we see that for all large enough t,

$$\begin{aligned}\mathrm{Re}[\mathcal{A}_s(t)] &\ge \mathrm{Re}[(\mathbf{I})_0] - \sum_{k\in Z\setminus\{0\}} |(\mathbf{I})_k| - |(\mathbf{II})| - |(\mathbf{III})| \\ &\ge \tfrac{1}{\sqrt{t}} \exp(-th_q(s)) \left[\tfrac{1}{2}\mathrm{C}_0 - \mathrm{C}t^{-\frac{4}{5}} - \mathrm{C}t^{\frac{3}{2}} \exp(-\tfrac{1}{\mathrm{C}}t^{\frac{3}{5}})\right] \ge \tfrac{\mathrm{C}_0}{4\sqrt{t}} \exp(-th_q(s)).\end{aligned}$$

Taking logarithms and dividing by t we get that $\liminf_{t\to\infty} \mathrm{Re}[\mathcal{A}_s(t)] \ge -h_q(s)$. Combining with (3.3.17) we arrive at (3.2.15).

Step 3. From this step on, we dedicate the proof to justifying the various equations and claims that appeared in **Step 2**. First in this step, we prove (3.3.25). Recall $(\mathbf{II})$ and $(\mathbf{III})$ defined in (3.3.20) and (3.3.21). For each of them, we push the absolute value around each term of the integrand. We use (3.3.9) from Lemma 3.3.5 to get

$$|(\mathbf{II})| \le \mathrm{C}\exp\left(t \sup_{\substack{y\in\mathbb{R}\setminus\cup_k V_k \\ |e^{\mathbf{i}\theta}-1|\le\varepsilon|\log\tau|}} \mathrm{Re}[f(s+\mathbf{i}y, \tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})]\right) \int_1^{e^{tB_q(\frac{s}{2})}} \frac{\mathrm{d}\zeta}{\zeta}, \tag{3.3.26}$$

$$|(\mathbf{III})| \le \mathrm{C}\exp\left(t \sup_{\substack{y\in\mathbb{R} \\ |e^{\mathbf{i}\theta}-1|>\varepsilon|\log\tau|}} \mathrm{Re}[f(s+\mathbf{i}y, \tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})]\right) \int_1^{e^{tB_q(\frac{s}{2})}} \frac{\mathrm{d}\zeta}{\zeta}. \tag{3.3.27}$$

Note that in (3.3.26), we have $|\tau^{\mathbf{i}y}-1| \ge |\tau^{\mathbf{i}t^{-\frac{4}{5}}}-1| \ge \frac{1}{2}|\log\tau|t^{-\frac{4}{5}}$ for all large enough t. Meanwhile in (3.3.27), $|\tau^{1-\frac{s}{2}}(e^{\mathbf{i}\theta}-1)| \ge \tau^{1-\frac{s}{2}}\varepsilon|\log\tau| = \tau^{1-\frac{s}{2}}|\log\tau|t^{-\frac{2}{5}}$. In either case, appealing to (3.3.4) in Lemma 3.3.1 with $\rho \mapsto s$ gives us that

$$f(s, \tau^{1-\frac{s}{2}}) - \mathrm{Re}[f(s+\mathbf{i}y, \tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})] \ge \tfrac{1}{\mathrm{C}} \cdot t^{-\frac{4}{5}}.$$

Substituting $f(s,\tau^{1-\frac{s}{2}})$ with $-h_q(s)$ and evaluating the integrals in (3.3.26) and (3.3.27) gives us (3.3.25).

Step 4. In this step and subsequent steps we prove (3.3.22) and (3.3.24). Recall that $v_k = -\frac{2\pi}{\log\tau}k$ and $\varepsilon = t^{-\frac{2}{5}}$. We first focus on the $(\mathbf{I})_k$ integral defined in (3.3.30). Our goal in this and next step is to show

$$(\mathbf{I})_k = (1+O(t^{-\frac{1}{5}}))\frac{\mathrm{C}_0(k)}{2\pi\sqrt{t}}\int_1^{e^{tB_q(\frac{s}{2})}}\frac{\zeta^{\mathbf{i}v_k}}{\zeta}\int_{-\varepsilon^2}^{\varepsilon^2}\zeta^{\mathbf{i}y}\exp(-th_q(s+\mathbf{i}y))\mathrm{d}y\mathrm{d}\zeta. \tag{3.3.28}$$

where

$$\mathrm{C}_0(k) := \sqrt{\frac{(1+\tau^{\frac{s}{2}})^3}{4\pi(q-p)(\tau^{\frac{3s}{2}-2}-\tau^{2s-2})}}\frac{(-1)^n(s+\mathbf{i}v_k)_n}{\sin(-\pi(s+\mathbf{i}v_k))(1-\tau^s)} \tag{3.3.29}$$

Towards this end, note that in the argument for (3.3.16), we push the absolute value around each term of the integrand. Thus, the upper bound achieved in (3.3.16) guarantees that the triple integral in $(\mathbf{I})_k$ is absolutely convergent. Thereafter, Fubini's theorem allows us to switch the order of integration inside $(\mathbf{I})_k$. By a change-of-variables, we see that

$$(\mathbf{I})_k = (-1)^n\int_1^{e^{tB_q(\frac{s}{2})}}\frac{\zeta^{\mathbf{i}v_k-1}}{4\pi^2}\int_{-\varepsilon^2}^{\varepsilon^2}\frac{(s+\mathbf{i}y+\mathbf{i}v_k)_n\zeta^{\mathbf{i}y}}{\sin(-\pi(s+\mathbf{i}y+\mathbf{i}v_k))}\int_{A_\varepsilon}\frac{e^{tf(s+\mathbf{i}y,\tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})}\mathrm{d}\theta}{1-\tau^{s+\mathbf{i}y}}\mathrm{d}y\mathrm{d}\zeta,$$

where recall $A_\varepsilon = \{\theta\in[-\pi,\pi] : |e^{\mathbf{i}\theta}-1|\le\varepsilon|\log\tau|\}$. Note that in this case range of y lies in a small window of $[-t^{-\frac{4}{5}}, t^{-\frac{4}{5}}]$. As s is fixed, one can replace $(s+\mathbf{i}y+\mathbf{i}v_k)_n$, $\sin(-\pi(s+\mathbf{i}y+\mathbf{i}v_k))$, and $1-\tau^{s+\mathbf{i}y}$ by $(s+\mathbf{i}v_k)_n$, $\sin(-\pi(s+\mathbf{i}v_k))$, and $1-\tau^s$ with an expense of $O(t^{-\frac{4}{5}})$ term (which can be chosen independent of k). We thus obtain

$$(\mathbf{I})_k = \frac{(-1)^n(s+\mathbf{i}v_k)_n(1+O(t^{-\frac{4}{5}}))}{\sin(-\pi(s+\mathbf{i}v_k))(1-\tau^s)}\int_1^{e^{tB_q(\frac{s}{2})}}\frac{\zeta^{\mathbf{i}v_k}}{4\pi^2\zeta}\int_{-\varepsilon^2}^{\varepsilon^2}\zeta^{\mathbf{i}y}\int_{A_\varepsilon}e^{tf(s+\mathbf{i}y,\tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})}\mathrm{d}\theta\mathrm{d}y\mathrm{d}\zeta. \tag{3.3.30}$$

We now evaluate the θ-integral in the above expression. We claim that

$$\int_{A_\varepsilon} e^{tf(s+\mathbf{i}y,\tau^{1-\frac{s}{2}}e^{\mathbf{i}\theta})}\mathrm{d}\theta = (1+O(t^{-\frac{1}{5}}))\sqrt{\frac{\pi(1+\tau^{\frac{s}{2}})^3}{t(q-p)(\tau^{\frac{3s}{2}-2}-\tau^{2s-2})}}\exp(-th_q(s+\mathbf{i}y)) \tag{3.3.31}$$

Note that (3.3.28) follows from (3.3.31). Hence we focus on proving (3.3.31) in next step.

Step 5. In this step we prove (3.3.31). For simplicity we let $u = s+\mathbf{i}y$ temporarily. Taylor expanding the exponent appearing in l.h.s. of (3.3.31) around $\theta = -\frac{y}{2}\log\tau$ and using the fact $\partial_z f(u,z)|_{z=\tau^{1-\frac{u}{2}}} = 0$, we get

$$\begin{aligned}\text{l.h.s. of (3.3.31)} &= \int_{A_\varepsilon} e^{tf(u,\tau^{1-\frac{u}{2}}e^{\mathbf{i}(\theta+\frac{y}{2}\log\tau)})}\mathrm{d}\theta \\ &= \exp(tf(u,\tau^{1-\frac{u}{2}}))\int_{A_\varepsilon}\exp\left(-\frac{t}{2}\partial_z^2 f(u,\tau^{1-\frac{u}{2}})(\theta+\tfrac{y}{2}\log\tau)^2+O(t^{-\frac{1}{5}})\right)\mathrm{d}\theta.\end{aligned} \tag{3.3.32}$$

Note that we have replaced the higher order terms by $O(t^{-\frac{1}{5}})$ in the exponent above as θ, y are at most of the order $O(t^{-\frac{2}{5}})$. Furthermore, for all t large enough,

$$\begin{aligned}A_\varepsilon &= \{\theta\in[-\pi,\pi] : |e^{\mathbf{i}\theta}-1|\le\varepsilon|\log\tau|\} \\ &= \{\theta\in[-\pi,\pi] : |\sin\tfrac{\theta}{2}|\le\tfrac{1}{2}\varepsilon|\log\tau|\}\supset\{\theta\in[-\pi,\pi] : |\theta|\le\varepsilon|\log\tau|\}\end{aligned}$$

As $y\in[-\varepsilon^2,\varepsilon^2]$, we see that $A_\varepsilon\supset\{\theta\in[-\pi,\pi] : |\theta+\frac{y}{2}\log\tau|\le\frac{1}{2}\varepsilon|\log\tau|\}$ for all large enough t. Thus on A_ε^c we have $|\theta+\frac{y}{2}\log\tau|\ge\frac{1}{2}t^{-\frac{2}{5}}|\log\tau|$. Furthermore for small enough y, by (3.3.2), we have $\mathrm{Re}[\partial_z^2 f(u,\tau^{1-\frac{u}{2}})] > 0$. Hence the above integral can be approximated by Gaussian integral. In particular, we have

$$\text{r.h.s. of (3.3.32)} = (1+O(t^{-\frac{1}{5}}))\exp(tf(u,\tau^{1-\frac{u}{2}}))\sqrt{\frac{2\pi}{t\partial_z^2 f(u,\tau^{1-\frac{u}{2}})}} \tag{3.3.33}$$

Observe that as $u = s+\mathbf{i}y$ and y is at most $O(t^{-\frac{4}{5}})$, $\partial_z^2 f(u,\tau^{1-\frac{u}{2}})$ in r.h.s. of (3.3.33) can be replaced by $\partial_z^2 f(s,\tau^{1-\frac{s}{2}})$ by adjusting the order term. Recall the expression for $\partial_z^2 f(s,\tau^{1-\frac{s}{2}})$ from (3.3.2) and observe that from the definition of f and h_q from (3.3.1) and (3.1.3) we have $f(u,\tau^{1-\frac{u}{2}}) = h_q(s+\mathbf{i}y)$. We thus arrive at (3.3.31).

Step 6. With the expression of $(\mathbf{I})_k$ obtained in (3.3.28), in this step we prove (3.3.22) and (3.3.24). As y varies in the window of $y \in [-t^{-\frac{4}{5}}, t^{-\frac{4}{5}}]$, by Taylor expansion we may replace $th_q(s+\mathbf{i}y)$ appearing in the r.h.s. of (3.3.28) by $t(h_q(s)+\mathbf{i}yh_q'(s))$ at the expense of an $O(t^{-\frac{3}{5}})$ term. Upon making a change of variable $r = \log\zeta - th_q'(s)$ we thus have

$$\begin{aligned}(\mathbf{I})_k &= (1+O(t^{-\frac{1}{5}}))\frac{\mathrm{C}_0(k)}{2\pi\sqrt{t}}e^{-th_q(s)}\int_{-th_q'(s)}^{tB_q(\frac{s}{2})-th_q'(s)} e^{\mathbf{i}v_k(r+th_q'(s))}\int_{-\varepsilon^2}^{\varepsilon^2} e^{\mathbf{i}yr}\mathrm{d}y\mathrm{d}r \\ &= (1+O(t^{-\frac{1}{5}}))\frac{\mathrm{C}_0(k)}{2\pi\sqrt{t}}e^{-th_q(s)}\int_{-th_q'(s)}^{tB_q(\frac{s}{2})-th_q'(s)} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r}-e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r.\end{aligned} \tag{3.3.34}$$

We claim that for $k=0$, (which implies $v_k = 0$) we have

$$\int_{-th_q'(s)}^{tB_q(\frac{s}{2})-th_q'(s)} \frac{e^{\mathbf{i}\varepsilon^2 r}-e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r = 2\pi(1+O(t^{-\frac{1}{5}})) \tag{3.3.35}$$

For $k \neq 0$, we have

$$\left|\int_{-th_q'(s)}^{tB_q(\frac{s}{2})-th_q'(s)} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r}-e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r\right| \le \mathrm{C}t^{-\frac{4}{5}} \tag{3.3.36}$$

where $\mathrm{C} > 0$ can be chosen free of k. Assuming (3.3.35) and (3.3.36) we may now complete the proof of (3.3.22) and (3.3.24). Indeed, for $k=0$ upon observing that $\mathrm{C}_0 = \mathrm{C}_0(0)$ (recall (3.3.23) and (3.3.29)), in view of (3.3.34) and (3.3.35) we get (3.3.22). Whearas for $k \neq 0$, thanks to the estimate in (3.3.36), in view of (3.3.34), we have

$$\sum_{k\in\mathbb{Z}\backslash\{0\}} |(\mathbf{I})_k| \le \mathrm{C}t^{-\frac{13}{10}}\exp(-th_q(s))\sum_{k\in\mathbb{Z}\backslash\{0\}}|\mathrm{C}_0(k)|. \tag{3.3.37}$$

For $y \neq 0$, $|\frac{(s+\mathbf{i}y)_n}{\sin(-\pi(s+\mathbf{i}y))}| \le \mathrm{C}|y|^n e^{-|y|/\mathrm{C}}$ forces r.h.s. of (3.3.37) to be summable proving (3.3.24).

Step 7. In this step we prove (3.3.35) and (3.3.36). Recalling that $\varepsilon^2 = t^{-\frac{4}{5}}$, we see that

$$\int_{-th_q'(s)}^{tB_q(\frac{s}{2})-th_q'(s)} \frac{e^{\mathbf{i}\varepsilon^2 r} - e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r = \int_{-t^{1/5}h_q'(s)}^{t^{1/5}B_q(\frac{s}{2})-t^{1/5}h_q'(s)} \frac{2\sin r}{r}\mathrm{d}r. \tag{3.3.38}$$

Following the definition of h_q and B_q in Proposition 3.2.1 we observe that $-h_q'(s) = \frac{\tau^{\frac{s}{2}}\log\tau}{(1+\tau^{\frac{s}{2}})^2} < 0$ and

$$B_q(s) - h_q'(s) = \frac{1-\tau^s+\tau^{\frac{s}{2}}s\log\tau}{s(1+\tau^{\frac{s}{2}})} = -sB_q'(s) > 0,$$

where $B_q'(s) < 0$ follows from (3.2.1). Thus as B_q is strictly decreasing (Proposition 3.2.1 (a)) we have $B_q(\frac{s}{2}) > B_q(s) > h_q'(s)$. Thus the integral on r.h.s. of (3.3.38) can be approximated by $(1+O(t^{-1/5}))\int_{\mathbb{R}}\frac{2\sin r}{r}\mathrm{d}r = 2\pi(1+O(t^{-1/5}))$. This proves (3.3.35). We now focus on proving (3.3.36). Towards this end, we divide the integral appearing in (3.3.36) into three regions as follows

$$\begin{aligned}\text{l.h.s. of (3.3.36)} \le \left|\int_{-th_q'(s)}^{-1} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r}-e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r\right| + \left|\int_{-1}^{1} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r}-e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r\right| \\ + \left|\int_{1}^{tB_q(\frac{s}{2})-th_q'(s)} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r}-e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r\right|.\end{aligned} \tag{3.3.39}$$

Note that for the second term appearing in r.h.s. of (3.3.39) can be bounded by $4t^{-\frac{4}{5}}$ using

$$\left|\int_{-1}^{1} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{2\sin(\varepsilon^2 r)}{r}\mathrm{d}r\right| \le \int_{-1}^{1}\left|\frac{2\sin(\varepsilon^2 r)}{r}\right|\mathrm{d}r \le 4\varepsilon^2 = 4t^{-\frac{4}{5}}.$$

For the first term appearing in r.h.s. of (3.3.39), by making a change of variable $r \mapsto r\frac{v_k-\varepsilon^2}{v_k+\varepsilon^2}$ we observe the following identity:

$$\int_{-th_q'(s)}^{-1} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r = \int_{-th_q'(s)\frac{v_k+\varepsilon^2}{v_k-\varepsilon^2}}^{-\frac{v_k+\varepsilon^2}{v_k-\varepsilon^2}} e^{\mathbf{i}v_k(r+th_q'(s))}\frac{e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r.$$

This leads to

$$\int_{-th'_q(s)}^{-1} e^{\mathbf{i}v_k(r+th'_q(s))}\frac{e^{\mathbf{i}\varepsilon^2 r}-e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r = \int_{-th'_q(s)}^{-th'_q(s)\frac{v_k+\varepsilon^2}{v_k-\varepsilon^2}} e^{\mathbf{i}v_k(r+th'_q(s))}\frac{e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r + \int_{-\frac{v_k+\varepsilon^2}{v_k-\varepsilon^2}}^{-1} e^{\mathbf{i}v_k(r+th'_q(s))}\frac{e^{-\mathbf{i}\varepsilon^2 r}}{\mathbf{i}r}\mathrm{d}r.$$

In the first integral the length of the interval is $O(t^{1/5})$. However, the integrand itself is $O(t^{-1})$. For the second integral, the length of the interval is $O(t^{-4/5})$, and the integrand itself is $O(1)$. Note that this is only possible when $k \neq 0$ (forcing $v_k \neq 0$). And indeed all the O terms can be taken to be free of v_k (and hence of k). Combining this we get that the first term appearing in r.h.s of (3.3.39) can be bounded by $\mathrm{C}t^{-\frac{4}{5}}$. An exact analogous argument provides the same bound for the third term in r.h.s. of (3.3.39) as well. This proves (3.3.36) completing the proof. $\square$

3.4 Bounds for the Higher order terms

The goal of this section is to establish bounds for the higher-order term $\mathcal{B}_s(t)$ defined in (3.2.14). First, recall the Fredholm determinant formula from (3.1.10). Using the $\mathrm{tr}(K_{\zeta,t}^{\wedge L})$ notation from (3.1.9) we may rewrite $\mathcal{B}_s(t)$ as follows.

$$\mathcal{B}_s(t) = (-1)^n \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha}\frac{\mathrm{d}^n}{\mathrm{d}\zeta^n}\left[1+\sum_{L=2}^{\infty}\mathrm{tr}(K_{\zeta.t}^{\wedge L})\right]\mathrm{d}\zeta. \tag{3.4.1}$$

We claim that we could exchange the various integrals, derivatives and sums appearring in the r.h.s. of (3.4.1) and obtain $\mathcal{B}_s(t)$ through term-by-term differentiation, i.e.

$$\mathcal{B}_s(t) = (-1)^n \sum_{L=2}^{\infty}\int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha}\partial_\zeta^n(\mathrm{tr}(K_{\zeta,t}^{\wedge L}))\mathrm{d}\zeta. \tag{3.4.2}$$

Towards this end, we devote Section 3.4.1 to its justification. Following the technical lemmas in Section 3.4.1, we proceed to prove Proposition 3.2.5 in Section 3.4.2.

3.4.1 Interchanging sums, integrals and derivatives

Recall from (3.3.8) the definition of $K^{(n)}_{\zeta,t}$. As a starting point of our analysis, we introduce the following notations before providing the bounds on $|\partial^n_\zeta \mathrm{tr}(K^{\wedge L}_{\zeta,t})|$. For any $n, L \in \mathbb{Z}_{>0}$, define

$$\mathfrak{M}(L,n) := \{\vec{m} = (m_1, \dots, m_L) \in (\mathbb{Z}_{\geq 0})^L : m_1 + \dots + m_L = n\}, \tag{3.4.3}$$

and $\binom{n}{\vec{m}} := \frac{n!}{m_1! \cdots m_L!}$. Furthermore, for any $L \in \mathbb{Z}_{>0}$, $\zeta \in \mathbb{R}_{>0}$ and $\vec{m} \in \mathfrak{M}(L,n)$, let

$$I_\zeta(\vec{m}) := \int \dots \int \det(K^{(m_i)}_{\zeta,t}(w_i, w_j))^L_{i,j=1} \prod_{i=1}^L \mathrm{d}w_i \tag{3.4.4}$$

where w_i-contour lies on $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$. We also set $|\vec{m}|_{>0} := |\{i \mid i \in \mathbb{Z} \cap [1, L], m_i > 0\}|$, i.e. the number of positive m_i in $\vec{m}$.

To begin with, the next two lemma investigate the term-by-term n-th derivatives of $\mathrm{tr}(K^{\wedge L}_{\zeta,t})$ that appear on the r.h.s. of (3.4.2). The following should be regarded as a higher order version of Proposition 3.3.6.

Proposition 3.4.1. *Fix $n, L \in \mathbb{Z}_{>0}$ and let $\mathfrak{M}(L,n)$ be defined as in (3.4.3). Recall the function $B_q(x)$ from Proposition 3.2.1. For any $t > 0$, the function $\zeta \mapsto \mathrm{tr}(K^{\wedge L}_{\zeta,t})$ is infinitely differentiable at each $\zeta \in [1, e^{tB_q(\frac{\delta}{2})}]$, with*

$$\partial^n_\zeta \mathrm{tr}(K^{\wedge L}_{\zeta,t}) = \frac{1}{L!} \sum_{\vec{m} \in \mathfrak{M}(L,n)} \binom{n}{\vec{m}} I_\zeta(\vec{m}), \tag{3.4.5}$$

where the r.h.s of (3.4.5) converges absolutely uniformly. Furthermore, there exists a constant $\mathrm{C} = \mathrm{C}(n, \delta, q) > 0$ such that for all $\vec{m} \in \mathfrak{M}(L,n)$ we have

$$|I_\zeta(\vec{m})| \leq \mathrm{C}^L L^{\frac{L}{2}} \zeta^{L\delta - n} e^{-th_q(\delta)}, \quad |\partial^n_\zeta \mathrm{tr}(K^{\wedge L}_{\zeta,t})| \leq \frac{\mathrm{C}^L}{L!} L^n L^{\frac{L}{2}} \zeta^{L\delta - n} e^{-th_q(\delta)}. \tag{3.4.6}$$

Proof. The proof idea is same as that of Proposition 3.3.6, but it's more cumbersome notationally. For clarity we split the proof into four steps. In the first step, we introduce some necessary notations. In Steps 2-3, we prove (3.4.5) and in the final step, we prove (3.4.6).

Step 1. In this step we summarize the notation we will require in the proof of (3.4.5). We fix $L \in \mathbb{Z}_{>0}, \delta \in (0,1), t > 0$, and $\zeta_1, \zeta_2 > 0$ and recall $B_q(x)$ from Proposition 3.2.1.

We define $\vec{\xi}_k \in [1, e^{tB_q(\frac{\delta}{2})}]^L$ to be the vector whose first k entries are ζ_2 and the rest $L-k$ entries are ζ_1:

$$\vec{\xi}_k := (\xi_{k,1}, \xi_{k,2}, \ldots, \xi_{k,L}) := (\underbrace{\zeta_2\,,\ \zeta_2\,,\ \ldots\ ,\ \zeta_2}_{k \text{ times}}\,,\ \underbrace{\zeta_1\,,\ \zeta_1\,,\ \ldots\ ,\ \zeta_1}_{L-k \text{ times}}), \quad k = 0, 1, \ldots, L.$$

For any $\vec{m} = (m_1, m_2, \ldots, m_L) \in (\mathbb{Z}_{\geq 0})^L$ we define the following integral of mixed parameters

$$I^{(k)}_{\zeta_1,\zeta_2}(\vec{m}) := \int \ldots \int \det(K^{(m_i)}_{\xi_{k,i},t}(w_i, w_j))_{i,j=1}^L \prod_{i=1}^L \mathrm{d}w_i. \tag{3.4.7}$$

where w_i-contour lies on $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$. $I^{(k)}_{\zeta_1,\zeta_2}(\vec{m})$ serves as an interpolation between $I_{\zeta_1}(\vec{m})$ and $I_{\zeta_2}(\vec{m})$ defined in (3.4.4) as k increases from 0 to L where the parameters ζ are now allowed to be different for different rows in the determinant.

We next define $\vec{e}_k = (e_{k,1}, e_{k,2}, \ldots, e_{k,L})$ to be the unit vector with 1 in the k-th position and 0 elsewhere. With the above notations in place, for each $j, k \in \{1, 2, \ldots, L\}$ and $\vec{m} \in (\mathbb{Z}_{\geq 0})^L$ we set

$$\mathfrak{L}^{(1)}_{\zeta_1,\zeta_2}(\vec{m}; k) := \frac{1}{\zeta_2 - \zeta_1}\left[I^{(k)}_{\zeta_1,\zeta_2}(\vec{m}) - I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m}) - (\zeta_2 - \zeta_1) I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k)\right], \tag{3.4.8}$$

$$\mathfrak{L}^{(2)}_{\zeta_1,\zeta_2}(\vec{m}; j, k) := I^{(j)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k) - I^{(j-1)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k). \tag{3.4.9}$$

Note that we define (3.4.8) modelling after D_{ζ_1,ζ_2} in the proof of Proposition 3.3.6. Here, the only differences between the three determinants of the respective $I^{(k)}_{\zeta_1,\zeta_2}(\vec{m})$'s lie in the k-th row, i.e. $K^{(m_k)}_{\zeta_2,t}$ v.s. $K^{(m_k)}_{\zeta_1,t}$ v.s. $K^{(m_k+1)}_{\zeta_1,t}$. So we have isolated the differences and tried to reduce the question of differentiability to row-wise in (3.4.8). Meanwhile, (3.4.9) "measures" the distance between $I^{(k)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k)$ and $I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m} + \vec{e}_k)$ where they differ only in $\xi_{k,k} = \zeta_2$ or ζ_1 for $K^{(m_k)}_{\xi_{k,k},t}$ on the k-th row of the determinant.

We finally remark that all the w_i-contours in the integrals appearing throughout the proof are on $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$ – we will not mention this further. We would also drop (w_i, w_j) from $K^{(m_i)}_{\bullet,t}(w_i, w_j)$ when it is clear from the context.

Step 2. We show the infinite differentiability of $\mathrm{tr}(K^{\wedge L}_{\zeta,t})$ by proving (3.4.5) in this step. The proof proceeds via induction on n. When $n = 0$, observe that (3.4.5) recovers the formula of $\mathrm{tr}(K^{\wedge L}_{\zeta,t})$. This constitutes the base case. To prove the induction step, suppose (3.4.5) holds for $n = N$. Then for $n = N + 1$, we fix $\zeta_1, \zeta_2 > 0$. Without loss of generality, we assume $\zeta_1 + 1 > \zeta_2 > \zeta_1$ and consider

$$D_{\zeta_1,\zeta_2} := \frac{\partial^N_\zeta \mathrm{tr}(K^{\wedge L}_{\zeta_2,t}) - \partial^N_\zeta \mathrm{tr}(K^{\wedge L}_{\zeta_1,t})}{\zeta_2 - \zeta_1} - \frac{1}{L!} \sum_{\vec{m}\in\mathfrak{M}(L,N+1)} \binom{N+1}{\vec{m}} I_{\zeta_1}(\vec{m}). \tag{3.4.10}$$

To prove (3.4.5), it suffices to show $|D_{\zeta_1,\zeta_2}| \to 0$ as $\zeta_2 \to \zeta_1$. Towards this end, we first claim that for all $\vec{m} \in \mathfrak{M}(L, N)$ and for all $j, k \in \{1, 2, \dots, L\}$ we have

$$\left|\mathfrak{L}^{(1)}_{\zeta_1,\zeta_2}(\vec{m}; k)\right| \to 0, \text{ and } \left|\mathfrak{L}^{(2)}_{\zeta_1,\zeta_2}(\vec{m}; j, k)\right| \to 0, \text{ as } \zeta_2 \to \zeta_1, \tag{3.4.11}$$

where $\mathfrak{L}^{(1)}_{\zeta_1,\zeta_2}(\vec{m}; k)$ and $\mathfrak{L}^{(2)}_{\zeta_1,\zeta_2}(\vec{m}; j, k)$ are defined in (3.4.8) and (3.4.9) respectively. We postpone the proof of (3.4.11) to the next step. Assuming its validity, we now proceed to complete the induction step.

Towards this end, we first manipulate the expression appearing in r.h.s. of (3.4.10). A simple combinatorial fact shows

$$\sum_{\vec{m}\in\mathfrak{M}(L,N+1)} \binom{N+1}{\vec{m}} I_{\zeta_1}(\vec{m}) = \sum_{k=1}^{L} \sum_{\vec{m}\in\mathfrak{M}(L,N)} \binom{N}{\vec{m}} I_{\zeta_1}(\vec{m} + \vec{e}_k),$$

where $\vec{e}_k$ is defined in Step 1. Substituting this combinatorics back into the r.h.s. of (3.4.10) and using the induction step for $n = N$, allows us to rewrite D_{ζ_1,ζ_2} as follows:

$$\text{r.h.s. of (3.4.10)} = \frac{1}{L!}\sum_{\vec{m}\in\mathfrak{M}(L,N)}\binom{N}{\vec{m}}\left[\frac{I_{\zeta_2}(\vec{m})-I_{\zeta_1}(\vec{m})}{\zeta_2-\zeta_1}-\sum_{k=1}^{L} I_{\zeta_1}(\vec{m}+\vec{e}_k)\right]. \tag{3.4.12}$$

Recalling the definition of $I_\zeta(\vec{m})$ in (3.4.4) and that of $I^{(k)}_{\zeta_1,\zeta_2}(\vec{m})$ in (3.4.7), we see that $\sum_{k=1}^{L}[I^{(k)}_{\zeta_1,\zeta_2}(\vec{m})-I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m})]$ telescopes to $I_{\zeta_2}(\vec{m})-I_{\zeta_1}(\vec{m})$. Furthermore, if we recall $\mathfrak{L}^{(1)}_{\zeta_1,\zeta_2}(\vec{m};k)$ and $\mathfrak{L}^{(2)}_{\zeta_1,\zeta_2}(\vec{m};j,k)$ from (3.4.8) and (3.4.9) respectively, we observe that

$$I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m}+\vec{e}_k)-I_{\zeta_1}(\vec{m}+\vec{e}_k)=I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m}+\vec{e}_k)-I^{(0)}_{\zeta_1,\zeta_2}(\vec{m}+\vec{e}_k)=\sum_{j=1}^{k}\mathfrak{L}^{(2)}_{\zeta_1,\zeta_2}(\vec{m};j,k).$$

Combining these observations, we have

$$\begin{aligned}\text{r.h.s. of (3.4.12)} &= \frac{1}{L!}\sum_{\vec{m}\in\mathfrak{M}(L,N)}\binom{N}{\vec{m}}\sum_{k=1}^{L}\frac{\left[I^{(k)}_{\zeta_1,\zeta_2}(\vec{m})-I^{(k-1)}_{\zeta_1,\zeta_2}(\vec{m})-(\zeta_2-\zeta_1)I_{\zeta_1}(\vec{m}+\vec{e}_k)\right]}{\zeta_2-\zeta_1}\\ &= \frac{1}{L!}\sum_{\vec{m}\in\mathfrak{M}(L,N)}\binom{N}{\vec{m}}\sum_{k=1}^{L}\left[\mathfrak{L}^{(1)}_{\zeta_1,\zeta_2}(\vec{m};k)+\sum_{j=1}^{k-1}\mathfrak{L}^{(2)}_{\zeta_1,\zeta_2}(\vec{m};j,k)\right].\end{aligned} \tag{3.4.13}$$

Clearly r.h.s. of (3.4.13) goes to zero as $\zeta_2 \to \zeta_1$ whenever (3.4.11) is true. Thus by induction we have (3.4.5).

Step 3. In this step we prove (3.4.11). Recall $\mathfrak{L}^{(1)}_{\zeta_1,\zeta_2}(\vec{m};k)$ from (3.4.8). Following the definition of $I^{(k)}_{\zeta_1,\zeta_2}(\vec{m})$ from (3.4.7) we have

$$\begin{aligned}\left|\mathfrak{L}^{(1)}_{\zeta_1,\zeta_2}(\vec{m};k)\right| \le \int\cdots\int \frac{1}{\zeta_2-\zeta_1}\Big|&\det(K^{(m_i)}_{\xi_{k,i},t})_{i,j=1}^{L}-\det(K^{(m_i)}_{\xi_{k-1,i},t})_{i,j=1}^{L}\\ &-(\zeta_2-\zeta_1)\det(K^{(m_i+e_{k,i})}_{\xi_{k-1,i},t})_{i,j=1}^{L}\Big|\prod_{i=1}^{L}\mathrm{d}w_i.\end{aligned}$$

Recall that in the above expression, up to a constant, the three determinants differ only in the k-th row. Hence the above expression can be written as $\int\cdots\int|\det(A)|\prod_{i=1}^{L}\mathrm{d}w_i$, where the entries of

A are given as follows:

$$
\begin{aligned}
A_{i,j} &= K^{(m_i)}_{\zeta_2,t}(w_i, w_j), \quad i < k, \quad A_{i,j} = K^{(m_i)}_{\zeta_1,t}(w_i, w_j), \quad i > k, \\
A_{k,j} &= \frac{1}{\zeta_2 - \zeta_1}[K^{(m_k)}_{\zeta_2,t}(w_k, w_j) - K^{(m_k)}_{\zeta_1,t}(w_k, w_j) - (\zeta_2 - \zeta_1)K^{(m_k+1)}_{\zeta_1,t}(w_k, w_j)] \\
&= \frac{1}{2\pi\mathbf{i}} \int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \Gamma(-u)\Gamma(1+u)R_{\zeta_1,\zeta_2;m_k}(u)e^{tf(u,w_k)} \frac{\mathrm{d}u}{w_j - \tau^u w_k},
\end{aligned}
$$

where $R_{\zeta_1,\zeta_2;m_k}(u)$ is same as in (3.3.13). As m_i's are at most n, by Lemma 3.3.5 ((3.3.10) specifically), we can get a constant $\mathrm{C} > 0$ depending only on n, δ, and q, so that

$$
|A_{i,j}| \le \mathrm{C}(\zeta_1^{\delta-m_k} + \zeta_2^{\delta-m_k}) \exp(-th_q(\delta)) \le \mathrm{C}(1 + \zeta_2^{\delta}) \exp(-th_q(\delta))
$$

for all $i \neq k$. For $A_{k,j}$, we follow the same argument as in Proposition 3.3.6 (along the lines of (3.3.14)) to get

$$
\begin{aligned}
|A_{k,j}| \le \frac{\tau^{1-\frac{\delta}{2}}}{2\pi} \int_{\zeta_1}^{\zeta_2} \left|\sigma^{\delta+\mathbf{i}y-m_k-2}\right| \mathrm{d}\sigma \\
\cdot \max_{w_j,w_k \in \mathfrak{C}(\tau^{1-\frac{\delta}{2}})} \int_{\mathbb{R}} \left| \frac{(\delta+\mathbf{i}y)_{m_k+2}}{\sin(-\pi(\delta+\mathbf{i}y))} e^{tf(\delta+\mathbf{i}y,w_k)} \right| \frac{\mathrm{d}y}{|w_j - \tau^{\delta+\mathbf{i}y} w_k|}.
\end{aligned}
$$

Note that by Lemma 3.3.5 ((3.3.9) specifically) we see that the above maximum is bounded by $\mathrm{C}\exp(-th_q(\delta))$ where again as m_i's are at most n, the constant C can be chosen dependent only on n,δ, and q. Since $|\sigma^{u-n-2}| = |\sigma^{\delta-m_k-2}| \le |\zeta_1^{\delta-m_k-2}| \le |\zeta_1^{\delta-2}|$ over the interval $[\zeta_1, \zeta_2]$ for $\delta \in (0, 1)$, we obtain

$$
|A_{k,j}| \le \mathrm{C}\exp(-th_q(\delta)) \int_{\zeta_1}^{\zeta_2} |\sigma|^{\delta-m_k-2} \mathrm{d}\sigma \le \mathrm{C}\exp(-th_q(\delta))\zeta_1^{\delta-2}(\zeta_2 - \zeta_1).
$$

As all the above estimates on $|A_{i,j}|$ are uniform in w_i's, using Hadamard inequality we have

$$
\int \cdots \int |\det(A)| \prod_{i=1}^{L} \mathrm{d}w_i \le \mathrm{C}^L L^{\frac{L}{2}} \exp(-tLh_q(\delta))(1 + \zeta_2^{\delta})^{L-1}\zeta_1^{\delta-2}(\zeta_2 - \zeta_1)
$$

Taking $\zeta_2 \to \zeta_1$ above, we get the first part of (3.4.11). The proof of the second part of (3.4.11) follows similarly by observing that the corresponding determinants also differ only in one row. One can then deduce the second part of (3.4.11) using the uniform estimates of the kernel and difference of kernels given in (3.3.10) and (3.3.11) respectively. As the proof follows exactly in the lines of above arguments, we omit the technical details.

Step 4. In this step we prove (3.4.6).

Recall the definition of $I_\zeta(\vec{m})$ from (3.4.4). By Hadamard's inequality and Lemma 3.3.5 we have

$$\begin{aligned} |\det(K_{\zeta,t}^{(m_i)})_{i,j=1}^L| &\le L^{\frac{L}{2}} \prod_{i=1}^{L} \max_{w_i,w_j \in \mathfrak{C}(\tau^{1-\frac{\delta}{2}})} |K_{\zeta,t}^{(m_i)}(w_i,w_j)| \\ &\le L^{\frac{L}{2}} \prod_{i=1}^{L} \mathrm{C}\zeta^{\delta-m_i} \exp(-th_q(\delta)) = \mathrm{C}^L L^{\frac{L}{2}} \zeta^{L\delta-n} \exp(-th_q(\delta)), \end{aligned} \tag{3.4.14}$$

where the last equality follows as $\sum_{i=1}^L m_i = n$. Note that here also $\mathrm{C} > 0$ can be chosen to be dependent only on n, δ, and q as m_i's are at most n. Recall that w_i-contour in $I_\zeta(\vec{m})$ lies on $\mathfrak{C}(\tau^{1-\frac{\delta}{2}})$. Thus in view of (3.4.14) adjusting the constant C we obtain first inequality of (3.4.6).

For the second inequality, We observe the following recurrence relation:

$$|\mathfrak{M}(L,n)| = |\{\vec{m} = (m_1,\ldots,m_L) \in \mathbb{Z}_{\ge 0}^L, \sum_{i=1}^L m_i = n\}| \le L \cdot |\mathfrak{M}(L,n-1)|. \tag{3.4.15}$$

It follows immediately that $|\mathfrak{M}(L,n)| \le L^n$. Observe that for each $\vec{m} \in \mathfrak{M}(L,n)$, $\binom{n}{\vec{m}}$ is bounded from above by $n!$. Thus collectively with (3.4.5) we have

$$|\partial_\zeta^n \mathrm{tr}(K_{\zeta,t}^{\wedge L})| \le \frac{n! L^n}{L!} \max_{\vec{m} \in \mathfrak{M}(L,n)} |I_\zeta(\vec{m})|.$$

Applying the first inequality of (3.4.6) above leads to the second inequality of (3.4.6) completing the proof.

□

Lemma 3.4.2. *Fix $n \in \mathbb{Z}_{>0}$, $\zeta \in [1, e^{tB_q(\frac{s}{2})}]$, and $t > 0$. Then*

$$\partial_\zeta^n \left(\sum_{L=1}^{\infty} \mathrm{tr}(K_{\zeta,t}^{\wedge L}) \right) = \sum_{L=1}^{\infty} \partial_\zeta^n (\mathrm{tr}(K_{\zeta,t}^{\wedge L})).$$

Proof. On account of [131, Proposition 4.2]), it suffices to verify the following conditions:

1. $\sum_{L=1}^{\infty} \mathrm{tr}(K_{\zeta,t}^{\wedge L})$ converges absolutely pointwise for $\zeta \in [1, e^{tB_q(\frac{s}{2})}]$;

2. the absolute derivative series $\sum_{L=1}^{\infty} \partial_\zeta^n (\mathrm{tr}(K_{\zeta,t}^{\wedge L}))$ converges uniformly for $\zeta \in [1, e^{tB_q(\frac{s}{2})}]$.

By Proposition 3.4.1, we can pass the derivative inside the trace in (2). Both (1) and (2) follow from (3.4.6) in Proposition 3.4.1 as $\sum_{L=1}^{\infty} \frac{1}{L!} \mathrm{C}^L L^n L^{\frac{L}{2}} \zeta^{L\delta - n} \exp(-tLh_q(\delta)) < \infty$ for each $\zeta \in [1, e^{tB_q(\frac{s}{2})}]$. □

Now, with the results from Lemmas 3.4.1 and 3.4.2, we are poised to justify the interchanges of operations leading to (3.4.2).

Proposition 3.4.3. *For fixed $n, L \in \mathbb{Z}_{\geq 0}$, $\zeta \in [1, e^{tB_q(\frac{s}{2})}]$ and $t > 0$,*

$$\int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} \partial_\zeta^n \left[1 + \sum_{L=2}^{\infty} \mathrm{tr}(K_{\zeta,t}^{\wedge L}) \right] \mathrm{d}\zeta = \sum_{L=2}^{\infty} \sum_{\vec{m} \in \mathfrak{M}(L,n)} \binom{n}{\vec{m}} \frac{1}{L!} \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} I_\zeta(\vec{m}) \mathrm{d}\zeta. \tag{3.4.16}$$

Proof. Thanks to Lemma 3.4.2 we can switch the order of derivative and sum to get

$$\text{l.h.s. of (3.4.16)} = \int_1^{e^{tB_q(\frac{s}{2})}} \sum_{L=2}^{\infty} \zeta^{-\alpha} \partial_\zeta^n (\mathrm{tr}(K_{\zeta,t}^{\wedge L})) \mathrm{d}\zeta.$$

We next justify the interchange of the integral and the sum in above expression. Note that via the estimate in (3.4.6) we have

$$\int_1^{e^{tB_q(\frac{s}{2})}} \sum_{L=2}^{\infty} \zeta^{-\alpha} |\partial_\zeta^n (\mathrm{tr}(K_{\zeta,t}^{\wedge L}))| \mathrm{d}\zeta \leq \sum_{L=2}^{\infty} \frac{1}{L!} \mathrm{C}^L L^n L^{\frac{L}{2}} \exp(-th_q(\delta)) \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{L\delta - n - \alpha} \mathrm{d}\zeta < \infty.$$

Hence Fubini's theorem justifies the exchange of summation and integration. Finally we arrive at r.h.s. of (3.4.16) by using the higher order derivative identity (see (3.4.5)) from Proposition 3.4.1.

$\square$

3.4.2 Proof of Proposition 3.2.5

Finally, in this subsection we present the proof of Proposition 3.2.5 via obtaining an upper-bound for $|\mathcal{B}_s(t)|$, defined in (3.2.14).

Recall $I_\zeta(\vec{m})$ from (3.4.4). We first introduce the following technical lemma that upper bounds the absolute value of the integral $\int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} I_\zeta(\vec{m}) \mathrm{d}\zeta$ and will be an important ingredient in the proof of Proposition 3.2.5.

Lemma 3.4.4. *Fix $s > 0$ so that $\alpha := s - \lfloor s \rfloor > 0$. Set $n = \lfloor s \rfloor + 1$. Fix $L \in \mathbb{Z}_{>0}$ with $L \geq 2$ and $\vec{m} \in \mathfrak{M}(L,n)$, where $\mathfrak{M}(L,n)$ is defined in* (3.4.3)*. There exists a constant* $\mathrm{C} = \mathrm{C}(q,s) > 0$ *such that*

$$\int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} |I_\zeta(\vec{m})| \mathrm{d}\zeta \leq \mathrm{C}^L L^{\frac{L}{2}} \exp(-th_q(s) - \tfrac{1}{\mathrm{C}}t). \tag{3.4.17}$$

where $I_\zeta(\vec{m})$ is defined in (3.4.4) *and the functions B_q and h_q are defined in Proposition 3.2.1.*

Proof. As we obtain upper bounds for the LHS of (3.4.17) differently depending on the value of L, we split the proof into two steps as follows. Fix $L_0 = 2(n+1)$. In Step 1, we prove the inequality for when $2 \leq L \leq L_0$ and in Step 2, we consider the case when $L > L_0$. In both steps, we deform the w-contours in $I_\zeta(\vec{m})$ appropriately to achieve its upper bound.

Step 1. In this step, we prove (3.4.17) for when $2 \leq L \leq L_0$. Fix $\vec{m} = (m_1, \ldots, m_L) \in \mathfrak{M}(L,n)$, where $\mathfrak{M}(L,n)$ is defined in (3.4.3) and set

$$\rho_i := \begin{cases} m_i + \frac{\alpha}{L} - \frac{1}{|\vec{m}|_{>0}} & \text{if } m_i > 0 \\ \frac{\alpha}{L} & \text{if } m_i = 0. \end{cases} \tag{3.4.18}$$

where we recall that $|\vec{m}|_{>0} = |\{i \mid i \in \mathbb{Z}, m_i > 0\}|$.

Recall the definition of $I_\zeta(\vec{m})$ in (3.4.4). Note that each $K_{\zeta,t}^{(m_i)}(w_i, w_j)$ (see (3.3.8)) are themselves complex integral over $\delta + \mathbf{i}\mathbb{R}$. As $\alpha > 0$ and $L \leq L_0 = 2(n+1)$ we may take the δ appearing in the kernel in $K^{(m_i)}_{\zeta,t}$ less than all the ρ_i's. Note that this is only possible when $\alpha > 0$. This is why we assumed this in the hypothesis here and as well as in the statement of Proposition 3.2.5.

In what follows we show that the contours of $K^{(m_i)}_{\zeta,t}(w_i, w_j)$ followed by w_i-contours can be deformed appropriately without crossing any pole in $I_\zeta(\vec{m})$. Indeed for each $K^{(m_i)}_{\zeta,t}$ in $I_\zeta(\vec{m})$ we can write

$$K^{(m_i)}_{\zeta,t}(w_i, w_j) = \frac{1}{2\pi\mathbf{i}}\int_{\rho_i-\mathbf{i}\infty}^{\rho_i+\mathbf{i}\infty} \Gamma(-u_i)\Gamma(1+u_i)(u_i)_n\zeta^{u_i-n}e^{f(u_i,w_i)}\frac{\mathrm{d}u_i}{w_j - \tau^{u_i}w_i}.$$

As each $\rho_i \in (0, m_i \vee 1)$ (see (3.4.18)), by Remark 3.3.4, the above equality is true as we do not cross any poles in the integrand. Ensuing this change, we claim that we can deform the w_i-contour to $\mathfrak{C}(\tau^{1-\frac{\rho_i}{2}})$ one by one without crossing any pole in $I_\zeta(\vec{m})$. Similar to the argument given in the beginning of the proof of Proposition 3.2.4, we note that as we deform the w_i-contours potential sources of poles in $I_\zeta(\vec{m})$ lie in the exponent $f(u_i, w_i) := \frac{(q-p)}{1+w_i\tau^{-1}} - \frac{(q-p)}{1+\tau^{u_i-1}w_i}$ (recalled from (3.3.1)) and in the denominator $w_j - \tau^{u_i}w_i$.

Take $w_i \in \mathfrak{C}(\tau^{1-\frac{\delta_i}{2}})$, $\delta_i, \in [\delta, \rho_i]$, and $u_i \in \rho_i + \mathbf{i}\mathbb{R}$. Observe that

$$|w_j - \tau^{u_i}w_i| \geq |w_j| - |\tau^{u_i}w_i| \geq \tau^{1-\frac{\delta_j}{2}} - \tau^{1+\rho_i-\frac{\delta_i}{2}} > 0,$$

$$|1 + w_i\tau^{-1}| \geq |w_i\tau^{-1}| - 1 \geq \tau^{-\frac{\delta_i}{2}} - 1, \quad |1 + \tau^{u_i-1}w_i| \geq 1 - |\tau^{u_i-1}w_i| \geq 1 - \tau^{\rho_i-\frac{\delta_i}{2}}.$$

This ensures that each w_i-contour can be taken as $\mathfrak{C}(\tau^{1-\frac{\rho_i}{2}})$ without crossing any pole.

Permitting these contour deformations, we wish to apply Lemma 3.3.5, (3.3.9) specifically. Indeed we apply (3.3.9) with $\rho, \delta \mapsto \rho_i$, $w \mapsto w'$, $w' \mapsto w_j$. Note that we indeed have $|w_j| \neq \tau^{1+\frac{\rho_i}{2}}$ here. We thus obtain

$$|K^{(m_i)}_{\zeta,t}(w_i, w_j)| \leq \mathrm{C}\zeta^{\rho_i-m_i}\exp(-th_q(\rho_i)). \tag{3.4.19}$$

Here, C is supposed to be dependent on m_i, ρ_i, and q. Note that ρ_i are in turn dependent on m_i, s and L. Since L is at most $L_0 = 2(n+1)$, there are at most finitely many choices of m_i's which in turn produced finitely many choices of ρ_i's. As s is fixed, all of the ρ_i's are uniformly bounded away from 0. Hence we can choose the constant C to be dependent only s and q (recall that n is also dependent on s).

Observe that as $\vec{m} \in \mathfrak{M}(L,n)$ defined in (3.4.3), we have $\sum m_i = n$ and consequently $\sum \rho_i = n-1+\alpha = s$. In view of the estimate in (3.4.19) and the definition of $I_\zeta(\vec{m})$ from (3.4.4), by Hadamard's inequality, we obtain

$$|I_\zeta(\vec{m})| \le \mathrm{C}^L L^{\frac{L}{2}} \zeta^{s-n} \exp\left(-t\sum_{i=1}^{L} h_q(\rho_i)\right) = \mathrm{C}^L L^{\frac{L}{2}} \zeta^{-1+\alpha} \exp\left(-t\sum_{i=1}^{L} h_q(\rho_i)\right).$$

Thus

$$\int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha} |I_\zeta(\vec{m})| \mathrm{d}\zeta \le \mathrm{C}^L L^{\frac{L}{2}} \exp\left(-t\sum_{i=1}^{L} h_q(\rho_i)\right) \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-1} \mathrm{d}\zeta. \tag{3.4.20}$$

Observe that $\int_x^y \zeta^{-1} d\zeta = \log\frac{y}{x}$. We appeal to the subadditivity $h_q(x)+h_q(y) > h_q(x+y)$ in Proposition 3.2.1 to get that $\sum_{i=1}^L h_q(\rho_i) \ge h_q(s-\rho_1)+h_q(\rho_1)$. Note that here we used the fact that $L \ge 2$. This leads to

$$\text{r.h.s. of (3.4.20)} \le \mathrm{C}^L L^{\frac{L}{2}} tB_q(\tfrac{s}{2}) \exp(-th_q(s)) \exp(-t(h_q(s-\rho_1)+h_q(\rho_1)-h_q(s))) \tag{3.4.21}$$

Note that from (3.4.18), $\rho_i \ge \frac{\alpha}{L} \ge \frac{\alpha}{L_0}$, this forces $\frac{\alpha}{L_0} \le s-\rho_1, \rho_1 \le s-\frac{\alpha}{L_0}$. Appealing to the strict subadditivity in (3.2.2) gives us that $h_q(s-\rho_1)+h_q(\rho_1)-h_q(s)$ can be lower bounded by a constant $\frac{1}{\mathrm{C}} > 0$ depending only on s and q. Adjusting the constant C we can absorb $tB_q(\frac{s}{2})$ appearing in r.h.s. of (3.4.21), to get (3.4.17), completing our work for this step.

Step 2. In this step, we prove (3.4.17) for the rest of the cases when $L > L_0$. Fix $\vec{m} = (m_1,\ldots,m_L) \in \mathfrak{M}(L,n)$. Recall the definition of $I_\zeta(\vec{m})$ in (3.4.4). Note that each $K_{\zeta,t}^{(m_i)}(w_i,w_j)$ (see (3.3.8)) is a complex integral over $\delta + \mathbf{i}\mathbb{R}$. Here we set $\delta = \min(\frac{1}{2},\frac{s}{2})$. Thanks to (3.4.6) we have

$$|I_\zeta(\vec{m})| \le \mathrm{C}^L L^{\frac{L}{2}} \zeta^{L\delta-n} \exp(-tLh_q(\delta)),$$

where the constant C depends only on n, δ, and q and thus only on s and q. This leads to

$$\int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha}|I_\zeta(\vec{m})|\mathrm{d}\zeta \le \mathrm{C}^L L^{\frac{L}{2}} \exp(-tLh_q(\delta)) \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha-n+L\delta}\mathrm{d}\zeta. \tag{3.4.22}$$

Recall that $s = n-1+\alpha$. As $L \ge 2(n+1)$ and $\delta = \min(\frac{1}{2}, \frac{s}{2})$ we have $L\delta - n - \alpha > 0$ in this case. Thus, we can upper bound the integral in (3.4.22) to get

$$\text{r.h.s. of (3.4.22)} \le \mathrm{C}^L L^{\frac{L}{2}} \exp(-tLh_q(\delta)) \frac{\exp(tB_q(\frac{s}{2})(-s+L\delta))}{-s+L\delta}. \tag{3.4.23}$$

We incorporate $\frac{1}{-s+L\delta}$ into the constant C, Recall the definition of $B_q(x)$ from Proposition (3.2.1). We have $xB_q(x) = h_q(x)$. As $B_q(x)$ is strictly decreasing for $x > 0$, (Proposition 3.2.1 (a), (b)) we have

$$\begin{aligned}\text{r.h.s. of (3.4.23)} &\le \mathrm{C}^L L^{\frac{L}{2}} \exp(-2th_q(\tfrac{s}{2}) - tL\delta(B_q(\delta) - B_q(\tfrac{s}{2}))) \\ &\le \mathrm{C}^L L^{\frac{L}{2}} \exp(-2th_q(\tfrac{s}{2})) \le \mathrm{C}^L L^{\frac{L}{2}} \exp(-th_q(s) - \tfrac{1}{\mathrm{C}}t),\end{aligned}$$

where the last inequality above follows from (3.2.2) by observing that by subadditivity we can get a constant $\mathrm{C} = \mathrm{C}(q,s) > 0$ such that $2h_q(\frac{s}{2}) - h_q(s) \ge \frac{1}{\mathrm{C}}$. This completes the proof. □

With Lemma 3.4.4, we are now ready to prove Proposition 3.2.5.

Proof of Proposition 3.2.5. Recall the definition of $\mathcal{B}_s(t)$ as defined in (3.2.14). Appealing to (3.4.1) and Proposition (3.4.3) we get that

$$|\mathcal{B}_s(t)| = \sum_{L=2}^{\infty} \frac{1}{L!} \sum_{\vec{m}\in\mathfrak{M}(L,n)} \binom{n}{\vec{m}} \int_1^{e^{tB_q(\frac{s}{2})}} \zeta^{-\alpha}|I_\zeta(\vec{m})|\mathrm{d}\zeta \tag{3.4.24}$$

Note that $\binom{n}{\vec{m}}$ is bounded from above by $n!$, and by (3.4.15) we have $|\mathfrak{M}(L,n)| \le L^n$. Applying these inequalities along with the estimate in Lemma 3.4.4 we have that

$$\text{r.h.s. of (3.4.24)} \le \exp(-th_q(s) - \tfrac{1}{\mathrm{C}}t) \sum_{L=2}^{\infty} \frac{1}{L!} \mathrm{C}^L L^{\frac{L}{2}} L^n$$

for some constant $C = C(q, s) > 0$. By Stirling's formula, $\sum_{L=2}^{\infty} \frac{1}{L!} C^L L^{\frac{L}{2}} L^n$ converges and hence adjusting the constant C, we obtain (3.2.16) completing the proof of the proposition. $\square$

3.5 Comparison to TASEP

In this section, we compute explicit expression for the upper tail rate function for TASEP (ASEP with $q = 1$) with step initial data and show that it matches with general ASEP rate function Φ_+ defined in (3.1.4).

Indeed, the large deviation problem for TASEP is already solved in [211] and is formulated in terms of Exponential Last Passage Percolation (LPP) model (Theorem 1.6 in [211]).

In order to state the connection between TASEP and Exponential LPP, we briefly recall the Exponential LPP model. Let Π_N be the set of all upright paths π in $\mathbb{Z}^2_{>0}$ from $(1, 1)$ to (N, N). Let $w(i, j), (i, j) \in \mathbb{Z}^2_{>0}$ be independent exponential distributed random variables with parameter 1. The last passage value for (N, N) is defined to be

$$\mathcal{H}(N) := \max\Big\{ \sum_{(i,j)\in\pi} w(i, j); \pi \in \Pi_N \Big\}.$$

As with the ASEP, for TASEP, we also set $H_0^{q=1}(t)$ to be the number of particles to the right of origin at time t. It is well known (see [211] for example) that $H_0^{q=1}(t)$ is related to the last passage value $\mathcal{H}(N)$ in the following way

$$\mathbb{P}\left(-H_0^{q=1}(t) + \tfrac{t}{4} \geq \tfrac{t}{4}y\right) = \mathbb{P}(\mathcal{H}(M_t) \geq t), \quad \text{where } M_t = \lfloor \tfrac{t}{4}(1-y) \rfloor + 1. \tag{3.5.1}$$

Theorem 3.5.1. *For $y \in (0, 1)$ we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(-H_0^{q=1}(t) + \tfrac{t}{4} \geq \tfrac{t}{4}y\right) = -\Phi_+(y). \tag{3.5.2}$$

where Φ_+ is defined in (3.1.4).

The idea of the proof of Theorem 3.5.1 is to use large deviation principle for $\mathcal{H}(N)$ which appears in Theorem 1.6 in [211] followed by an application of the relation (3.5.1). The only impediment is that the Johansson result appears in a variational form.

Let us recall Theorem 1.6 in [211]. According to Eq (1.21) in [211] (with $\gamma = 1$), the upper tail of $\mathcal{H}(N)$ satisfy the following large deviation principle

$$\lim_{N\to\infty} \tfrac{1}{N} \log \mathbb{P}(\mathcal{H}(N) \geq Nz) = -J(z), \quad z \geq 4. \tag{3.5.3}$$

where the rate function J is given by

$$\begin{aligned} J(t) &:= \inf_{x\geq t}[G_V(x) - G_V(4)],\ t \geq 4,\ \text{where} \\ G_V(x) &:= -2\int_{\mathbb{R}} \log|x-r|\mathrm{d}\mu_V(r) + V(r),\ x \geq 4. \end{aligned} \tag{3.5.4}$$

Here $V(x) = x$ is defined on $[0,\infty)$, and the measure μ_V is the unique minimizer of $I_V(\mu)$ over $\mathcal{M}(\mathbb{R}_{\geq 0})$, the set of probability measures on $[0,\infty)$. $I_V(\cdot)$ is known as the *logarithmic entropy in presence of the external field V* and is given by

$$I_V(\mu) := -\iint_{\mathbb{R}^2} \log|x_1 - x_2|\mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2) + \int_{\mathbb{R}} V(x)\mathrm{d}\mu(x), \quad \mu \in \mathcal{M}(\mathbb{R}_{\geq 0}).$$

The logarithmic entropy $I_V(\mu)$ is well studied in both mathematical and physics literature and has several applications to random matrix theory and related models. We refer to [287] and [200] and the references there in for more details.

The form of the rate function defined in (3.5.4) is not exactly same as in [211]. However, one can show the rate function J defined in (3.5.4) is same as Eq (2.15) in [211] using the properties of minimizing measure (see Theorem 1.3 in [287] or Eq (1.6) in [157]). Such an expression for the rate function is derived using Coulomb gas theory. We refer to [211], [164], and [127] for treatment on the LDP problems of such nature.

Proof of Theorem 3.5.1. For clarity we split the proof into two steps.

Step 1. We claim that J defined in (3.5.4) has the following explicit expression.

$$J(t) = \sqrt{t^2 - 4t} - 2\log\frac{t - 2 + \sqrt{t^2 - 4t}}{2}, \quad t \geq 4. \tag{3.5.5}$$

We will prove (3.5.5) in Step 2. Here we assume its validity and conclude the proof of (3.5.2).

Towards this end, fix $y \in (0, 1)$ and K large enough such that $[y - \frac{1}{K}, y + \frac{1}{K}] \subset (0, 1)$. Recall the definition of M_t from (3.5.1). Note that for all large enough t, we have $\frac{4}{1-y+K^{-1}}M_t \leq t \leq \frac{4}{1-y-K^{-1}}M_t$. Thus

$$\mathbb{P}\big(\mathcal{H}(M_t) \geq \tfrac{4}{1-y+K^{-1}}M_t\big) \geq \mathbb{P}\left(-H_0^{q=1}(t) + \tfrac{t}{4} \geq \tfrac{t}{4}y\right) \geq \mathbb{P}\big(\mathcal{H}(M_t) \geq \tfrac{4}{1-y-K^{-1}}M_t\big).$$

Taking logarithms on each side, dividing by M_t and then taking $t \to \infty$ we get

$$\begin{aligned} -J\big(\tfrac{4}{1-y+K^{-1}}\big) &\geq \limsup_{t\to\infty} \frac{1}{M_t}\mathbb{P}\left(-H_0^{q=1}(t) + \tfrac{t}{4} \geq \tfrac{t}{4}y\right) \\ &\geq \liminf_{t\to\infty} \frac{1}{M_t}\mathbb{P}\left(-H_0^{q=1}(t) + \tfrac{t}{4} \geq \tfrac{t}{4}y\right) \geq -J\big(\tfrac{4}{1-y-K^{-1}}\big). \end{aligned} \tag{3.5.6}$$

where we used the upper tail large deviation principle for $\mathcal{H}(N)$ from (3.5.3). Observe that $\frac{M_t}{t} \to \frac{1-y}{4}$, and using (3.5.5) we see that

$$\frac{1-y}{4}J\big(\frac{4}{1-y}\big) = \frac{1-y}{4}\left(\frac{4\sqrt{y}}{1-y} - 2\log\frac{2(1+y) - 4\sqrt{y}}{2(1-y)}\right) = \Phi_+(y),$$

where Φ_+ is defined in (3.1.4). Thus taking $K \to \infty$ in (3.5.6) we arrive at (3.5.2).

Step 2. We now turn our attention to prove (3.5.5). It is well known that for $V(x) = x$, the minimizer μ_V is given by the *Marchenko-Pastur* measure (see Equation 3.3.2 and Proposition 5.3.7 in [200] with $\lambda = 1$):

$$\mathrm{d}\mu_V(x) = \frac{\sqrt{4x - x^2}}{2\pi x}\mathbf{1}_{x\in[0,4]}\mathrm{d}x.$$

Recall $G_V(x)$ defined in (3.5.4). Using the Cauchy Transform for μ_V (see the last unnumbered equation in Page 200 of [200]) we get that for $x > 4$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int \log|x-r|\mathrm{d}\mu_V(r) = \frac{1}{2} - \frac{\sqrt{x^2-4x}}{2x},$$

which implies $G_V(z) - G_V(4) = \int_4^z \frac{\sqrt{x^2-4x}}{x}\mathrm{d}x$. Thus $G_V(z) - G_V(4)$ is strictly increasing in y and whence by (3.5.4) we have

$$J(t) = \int_4^t \frac{\sqrt{x^2-4x}}{x}\mathrm{d}x.$$

To compute the above integral, we make the change of variable $x \mapsto \frac{(z+1)^2}{z}$ so that $\mathrm{d}x = (1-\frac{1}{z^2})\mathrm{d}z$ and $x^2 - 4x = \frac{(z^2-1)^2}{z^2}$. Set $a = \frac{t-2}{2} + \frac{\sqrt{t^2-4t}}{2}$ to get

$$\int_4^t \frac{\sqrt{x^2-4x}}{x}\mathrm{d}x = \int_1^a \frac{(z-1)^2}{z^2}\mathrm{d}z = \left[z - \frac{1}{z} - 2\log z\right]_1^a = a - \frac{1}{a} - 2\log a.$$

Plugging the value of a we get (3.5.5) completing the proof. □

Chapter 4: Law of iterated logarithms and fractal properties of the KPZ equation

4.1 Introduction

We study the Kardar-Parisi-Zhang (KPZ) equation, a stochastic PDE which is formally written

$$\partial_t \mathcal{H} = \tfrac{1}{2}\partial_{xx}\mathcal{H} + \tfrac{1}{2}(\partial_x \mathcal{H})^2 + \xi, \qquad \mathcal{H} := \mathcal{H}(t,x) \qquad (t,x) \in [0,\infty) \times \mathbb{R}. \tag{4.1.1}$$

Here $\xi = \xi(t,x)$ is the space time white noise. The KPZ equation was introduced in [218] for studying the fluctuation of growing interfaces and since then, it has found links to many systems including directed polymers, last passage percolation, interacting particle systems, and random matrices via its connections to the *KPZ universality class* (see [166, 278, 113, 281]).

The KPZ equation, as given in (4.1.1), is ill-posed as a stochastic PDE due to the presence of the nonlinear term $(\partial_x \mathcal{H})^2$. The physically relevant notion of solution for the KPZ equation is given by the *Cole-Hopf solution* which is defined as

$$\mathcal{H}(t,x) := \log \mathcal{Z}(t,x),$$

where $\mathcal{Z}(t,x)$ is the solution of the stochastic heat equation (SHE):

$$\partial_t \mathcal{Z} = \tfrac{1}{2}\partial_{xx}\mathcal{Z} + \xi\mathcal{Z}, \qquad \mathcal{Z} := \mathcal{Z}(t,x). \tag{4.1.2}$$

Throughout this paper, we work with the fundamental solution $\mathcal{Z}^{\mathbf{nw}}(t,x)$ of (4.1.2) and the associated Cole-Hopf solution $\mathcal{H}^{\mathbf{nw}}(t,x) := \log \mathcal{Z}^{\mathbf{nw}}(t,x)$ which corresponds to the SHE being started from the delta initial measure, i.e., $\mathcal{Z}^{\mathbf{nw}}(0,x) = \delta_{x=0}$. For any $t > 0$, $\mathcal{Z}^{\mathbf{nw}}(t,x)$ is strictly positive

[168] which makes the Cole-Hopf solution $\mathcal{H}^{\mathbf{nw}}(t,x)$ well-defined. The corresponding initial data of the KPZ equation is termed as the *narrow wedge* initial data.

The ubiquity of the SHE is discernible in many applications stretching from modeling the density of the particles diffusing through random environments [255, 222, 24, 108] to the partition function of the continuum directed random polymer model [4, 80, 68]. The solution theory for the SHE is standard [316, 278, 105]; based on Itô integral theory or martingale problems. The mathematical theory of the KPZ equation however has unleashed new challenges in recent years. Most notably, the study of the KPZ equation can now be classified into three broad directions, namely, to understand how the KPZ equation approximates the interface fluctuation of the random growth models, to build a robust solution theory of the KPZ equation and to unveil fine properties and asymptotics of the solution of the KPZ equation. The Cole-Hopf solution of the KPZ equation coincides with the limits of certain growth processes [**Lin20**, 49, 112, 111, 178, 107]. The KPZ equation being a testing ground for the nonlinear stochastic PDEs, stirs up intense recent innovations in the theory of singular PDEs including regularity structures [192], paracontrolled distributions [186, 185], energy solution [184] and renormalisation group [236] methods. In this paper, we seek to pursue the third direction, i.e., to unravel finer properties of the Cole-Hopf solution of the KPZ equation.

In this paper, we consider the following 1 : 2 : 3 scaled version of the KPZ height function:

$$\mathfrak{h}_t(\alpha,x) := \frac{\mathcal{H}^{\mathbf{nw}}(\alpha t, t^{2/3}x) + \frac{\alpha t}{24}}{t^{1/3}}. \tag{4.1.3}$$

where t specifies the time scale and α measures the time judged on that scale, x measures the space judged on $t^{2/3}$ scale. Although the presence of t and α bears a stain of redundancy, the notation introduced in (4.1.3) will be useful in stating and proving many of our results. For $\alpha = 1$, we will often use the shorthand $\mathfrak{h}_t(x) := \mathfrak{h}_t(1,x)$ and $\mathfrak{h}_t := \mathfrak{h}_t(0)$. We will call the stochastic process $\mathfrak{h}_t$ indexed by the time parameter t as the *KPZ temporal process*. In a seminal work, [6] showed that

$$\mathfrak{h}_t \xrightarrow{d} 2^{-1/3} TW_{\mathrm{GUE}}, \quad \text{as } t \to \infty.$$

Here, TW_{GUE} is the Tracy-Widom GUE distribution. The *KPZ scaling* of the fluctuation, space and time, i.e., the ratio of the corresponding scaling exponents being $1:2:3$ and TW_{GUE} as the limit of the fluctuations are the characteristics of the models in the KPZ universality class. Recently, [280, 315] have announced proofs of the convergence of the spatial process $\mathfrak{h}_t(x)$ (upto a parabola) to the universal limiting process of the KPZ universality class, namely the *KPZ fixed point* as t goes to ∞.

Our objects of study are the large *peaks and valleys* of the KPZ temporal process as the KPZ equation approaches the KPZ fixed point. Such study for any generic one-dimensional stochastic process with a macroscopic limiting profile usually starts up with two questions: *What are the scalings of the large peaks and valleys*? *Do they converge to any limit under such scaling*? For a Brownian motion $\mathfrak{B}_t$, these questions are answered via the (Brownian) *law of iterated logarithms* (LIL). Under the $\sqrt{t}$ scaling, the fluctuation of the Brownian motion $\mathfrak{B}_t$ has the Gaussian limit. At the onset of this macroscopic Gaussianity, the peaks and valleys of $\mathfrak{B}_t/\sqrt{t}$ under further scaling by $\sqrt{2\log\log t}$ stays in between -1 and 1. The extra scaling by an iterated logarithmic factor $\sqrt{2\log\log t}$ inflicts the name 'law of iterated logarithms'.

Our first main result which is stated as follows concerns with the law of iterated logarithms of the KPZ equation started from the narrow wedge initial data.

Theorem 4.1.1. *With probability* 1*, we have*

$$\limsup_{t\to\infty} \frac{2^{\frac{1}{3}}\mathfrak{h}_t}{(\log\log t)^{2/3}} = \left(\frac{3}{4}\right)^{\frac{2}{3}}, \quad \textit{and} \quad \liminf_{t\to\infty} \frac{2^{\frac{1}{3}}\mathfrak{h}_t}{(\log\log t)^{1/3}} = -(12)^{\frac{1}{3}}.$$

The above law of iterated logarithms reveals the scaling of the large peaks and valleys of $\mathfrak{h}_t$. As we may see, the scalings for limsup and liminf differ from each other. This naturally gives rise to the following two questions:

(1) *What are the origins of the scalings* $(\log\log t)^{2/3}$ *and* $(\log\log t)^{1/3}$*?*

The scaling of the large peaks and valleys for the KPZ height fluctuation are in fact orchestrated by the Tracy-Widom GUE distribution which is the scaling limit of $2^{\frac{1}{3}}\mathfrak{h}_t$ as t goes to ∞. This is in line with the LIL for the Brownian motion where the exponent $1/2$ of $(\log\log t)$ factor stems from the Gaussian tail decay of the limiting law. For the KPZ equation, the peaks and valleys have different scaling thanks to the distinct decay exponents of the upper and lower tail probabilities of the Tracy-Widom GUE. If X is a Tracy-Widom GUE random variable, then, the probability of X being higher than s (i.e., upper tail probability) decays as $\exp(-\frac{4}{3}(1+o(1))s^{3/2})$ and the probability of X being lower (i.e., lower tail probability) than $-s$ decays as $\exp(-\frac{1}{12}(1+o(1))s^3)$. So, the upper tail decay exponent is $3/2$ which induce the scaling $(\log\log t)^{2/3}$ for the peaks of the KPZ temporal process whereas the lower tail exponent being 3 is the source of the scaling $(\log\log t)^{1/3}$ of the valleys. Interestingly, as one may observe, the values of the limsup and liminf in Theorem 4.1.1 are seemingly connected to the constants $4/3$ and $1/12$ of the respective tail decays of the Tracy-Widom GUE distribution. This association is commensurate with the Brownian LIL and predicted in other works (discussed in Section 4.1.2).

(2) *How the LILs will vary with the initial data?*

Based on the LIL for the narrow wedge solution, one may insinuate that the scaling of the peaks and valleys of the KPZ temporal process under other initial condition will be governed by the tail exponents of the limiting random variables. It follows from Theorem 1.1 and 1.4 of [115] that for a wide class of initial data, the upper tail exponents of the limiting r.v. of the KPZ equation under KPZ scaling is $3/2$ and the lower tail exponent is at least 3. By drawing the analogy with the narrow wedge case, we conjecture that correct scaling of the peaks and valleys of the KPZ height fluctuation will be $(\log\log t)^{2/3}$ and $(\log\log t)^{1/3}$ respectively. In particular, we believe that such scaling of the peaks and valleys will hold for all classes of initial data which are considered in [115] including the bounded initial data and Brownian initial data. Proving these claims is beyond the scope of the present paper since some of the major tools that we use are not available for the KPZ height function under other initial data. However, we hope to explore this direction in future works.

Our next objective is to quantify how often the peaks and valleys of the KPZ temporal process exceed a given level. This entails to studying the *upper level sets* $\{t > t_0 : \mathfrak{h}_t \geq \gamma(\log\log t)^{2/3}\}$

and *lower level sets* $\{t > t_0 : \mathfrak{h}_t \leq -\gamma(\log\log t)^{1/3}\}$ for different values of γ where $\gamma > 0$ is a tuning parameter and t_0 is an arbitrary constant. In particular, we study the *macroscopic* fractal nature of the level sets. For brevity, we mainly focus on the study of the upper level sets in this paper.

Fractal nature of the level sets of the KPZ equation is intimately connected to the moment growth of the SHE which is captured through the *Lyapunov exponents*, i.e., the limit of $t^{-1}\mathbb{E}[(\mathcal{Z}^{\mathbf{nw}}(t,0))^k]$ as $t \to \infty$ for any integer k. The nonlinear nature of the Lyapunov exponents of the SHE (predicted by *Kardar's formula* [**Kardar87**]) suggests an abundance of the large peaks of the SHE. This is manifested through the existence of infinitely many scales for the peaks, a property often called as *multifractality*. In contrast, the peaks of a scaled Brownian motion $\mathfrak{B}_t/\sqrt{t}$ only show a single scale as time t increases to infinity. This latter property is named as *monofractality*. In the following, we give a mathematical definition of these two different natures of the (macroscopic) fractality.

Definition 4.1.2 (Mono- and Multifractality)**.** Let X be a stochastic process. Suppose there exists a non-random gauge function g such that $g(r)$ increases to ∞ as $r \to \infty$ and

$$\limsup_{r\to\infty} \frac{X(r)}{g(r)} = 1 \qquad \text{a.s.}$$

Fix a scalar $\gamma, t_0 > 0$. Define

$$\Xi_{X,g}(\gamma) := \Big\{t > t_0 : \frac{X(t)}{g(t)} > \gamma\Big\}.$$

We denote the (Barlow-Taylor) *macroscopic Hausdorff dimension* (see Definition 4.2.5) of any Borel set $\mathfrak{F}$ by $\mathrm{Dim}_{\mathbb{H}}(\mathfrak{F})$. The tall peaks of X is *multifractal* in gauge g when there exist infinitely many length scales $\gamma_1 > \gamma_2 > \ldots > 0$ such that, with probability one,

$$\mathrm{Dim}_{\mathbb{H}}\big(\Xi_{X,g}(\gamma_{i+1})\big) < \mathrm{Dim}_{\mathbb{H}}\big(\Xi_{X,g}(\gamma_i)\big).$$

Whereas the peaks of X with gauge function g is *monofractal* when for some $\gamma_0 > 0$,

$$\mathrm{Dim}_{\mathbb{H}}(\Xi_{X,g}(\gamma)) = \begin{cases} \text{Constant} & \gamma \leq \gamma_0, \\ 0 & \gamma > \gamma_0. \end{cases}$$

By the law of iterated logarithms, the gauge function of a scaled Brownian motion $\mathfrak{B}_t/\sqrt{t}$ is dictated as $(2\log\log t)^{1/2}$. It follows from the works of [223, 302, 257] that the Brownian motion with such choice of the gauge function is monofractal. However, the macroscopic nature of the peaks undergoes a transition under the exponential transformation of the time variable underpinning the Brownian motion. For instance, the Ornstein-Uhlenbeck process which is defined as $U(t) := \exp(-t/2)\mathfrak{B}_{e^t}$ for $t \in \mathbb{R}$ is multifractal in the gauge function $(2\log t)^{1/2}$.

Our second main result which is stated below shows that the KPZ temporal process is monofractal in the gauge function $(\frac{3}{4\sqrt{2}}\log\log t)^{2/3}$. Whereas under the exponential transformation of the time variable, the peaks of the KPZ temporal process exhibits multifractality.

Theorem 4.1.3. *Consider the rescaled height function $\mathfrak{h}_t$ of the KPZ equation and the exponential time-changed process $\mathfrak{G}(t) := \mathfrak{h}_{e^t}$. Then, we have the following: $\mathfrak{h}_t$ is monofractal with positive probability in gauge function $(\log\log t)^{2/3}$, i.e., for every $t_0, \gamma > 0$,*

$$\mathrm{Dim}_{\mathbb{H}}\Big\{t \geq e^e : \frac{\mathfrak{h}_t}{(\log\log t)^{2/3}} \geq \gamma\Big\} \overset{a.s.}{=} \begin{cases} 1 & \text{when } \gamma \leq \big(\frac{3}{4\sqrt{2}}\big)^{\frac{2}{3}}, \\ 0 & \text{when } \gamma > \big(\frac{3}{4\sqrt{2}}\big)^{\frac{2}{3}}. \end{cases} \tag{4.1.4}$$

In contrast, $\mathfrak{G}(t)$ is multifractal in gauge function $(3/4\sqrt{2})^{2/3}(\log t)^{2/3}$. In fact,

$$\mathrm{Dim}_{\mathbb{H}}\Big\{t \geq e^e : \frac{\mathfrak{G}(t)}{(3/4\sqrt{2})^{2/3}(\log t)^{2/3}} \geq \gamma\Big\} \overset{a.s.}{=} 1 - \gamma^{3/2}, \quad \text{for } \gamma \in [0, 1]. \tag{4.1.5}$$

Note that (4.1.4) shows that the peaks of $\mathfrak{h}_t$ are monofractal in the gauge function $(\log\log t)^{2/3}$.

On the other hand, the multifractality of the peaks of $\mathfrak{G}(t)$ is clear from (4.1.5) since

$$\Xi_{\mathfrak{G}(t),(3/4\sqrt{2})^{2/3}(\log t)^{2/3}}(\gamma_2) \overset{a.s}{=} 1-\gamma_2^{3/2} < 1-\gamma_1^{3/2} \overset{a.s}{=} \Xi_{\mathfrak{G}(t),(3/4\sqrt{2})^{2/3}(\log t)^{2/3}}(\gamma_1)$$

for $0 \le \gamma_1 < \gamma_2 \le 1$. This raises the following three interesting questions.

(1) *What is the minimal speed up needed for the time variable to see transition from monofractality to multifractality of the peaks of the KPZ equation?*

We are indebted to Davar Khoshnevisan for asking this question. By carefully studying the outreach of our tools, we expect to see the appearance of multifractality of the peaks under the transformation $t \mapsto \mathfrak{h}_{e^{(\log t)^a}}$ for any $a > 1$. Due to lack of detailed information on the correlation decay of the KPZ temporal process, we are unable to make precise prediction of the fractality under the transformation $t \mapsto \mathfrak{h}_{t^a}$ for any $a > 1$. We expect that the monofractality will still survive under such transformations. This expectation is based on the intuition that the mono- and multi-fractality are closely tied to the two point correlation function of the associated process. While monofractality is tied to the presence of strong correlation, the multifractality is rather prevalent when the correlation decays. In [117], the authors had investigated the correlation of the KPZ temporal process. They had shown that for any two time points $t_1 < t_2$ such that $t_2 - t_1 > t_1$, the correlation function between $\mathfrak{h}_{t_1}$ and $\mathfrak{h}_{t_2}$ decays as $(t_1/t_2)^{1/3}$, i.e., the correlation function is both upper and lower bounded by some constant multiples of $(t_1/t_2)^{1/3}$. This implies that when $t_2 - t_1$ grows like a higher order polynomial in t_1, the correlation between the KPZ height function at t_1 and t_2 decays to zero as t_1 increases. On the other hand, for any $t_1 < t_2 = ct_1$ for some constant $c > 1$, the correlation between $\mathfrak{h}_{t_1}$ and $\mathfrak{h}_{t_2}$ remains bounded away from 0 while the correlation between $\mathfrak{h}_{e^{(\log t_1)^a}}$ and $\mathfrak{h}_{e^{(\log t_2)^a}}$ rapidly goes to 0 as t_1 goes to ∞. This is one of the main reasons which lead us to believe that the transition from monofractality to multifractality happens under the transformation $t \mapsto e^{(\log t)^a}$ for some $a > 1$.

(2) *Is there a similar notion of macroscopic fractality for the valleys? What are the macroscopic fractal properties of the valleys of the KPZ height function?*

The fractal properties of the valleys can be studied using the lower level sets. For instance, if X is a stochastic process such that $\liminf_{r\to-\infty} X(r)/f(r) = -1$ almost surely for some gauge function f, then, the multifractality and/or monofractality of the valleys of X can be defined in the same way as in Definition 4.1.2 using the macroscopic Hausdorff dimension of the following lower level sets

$$\widehat{\Xi}_{X,f}(\gamma) := \Big\{t > t_0 : \frac{X(t)}{f(t)} < -\gamma\Big\}.$$

For studying the valleys of $\mathfrak{h}_t$, the natural choice of the gauge function is $(6\log\log t)^{1/3}$ as shown by Theorem 4.1.1. Using the tools of this paper, we expect that one can show monofractality of the valleys of $\mathfrak{h}_t$ in the gauge function $(6\log\log t)^{1/3}$. Furthermore, drawing the analogy with (4.1.5), we also expect the following equality holds

$$\mathrm{Dim}_{\mathbb{H}}\Big(\widehat{\Xi}_{\mathfrak{G}(t),(6\log t)^{1/3}}(\gamma)\Big) \overset{a.s}{=} 1-\gamma^3.$$

While the fractal properties of the valleys seem extremely exciting, for brevity, we restrict ourselves only to exploring the peaks of the KPZ temporal process in this paper.

(3) *What is expected about the peaks and valleys of the KPZ fixed point in the temporal direction?*

It is believed that $\mathfrak{h}_t(\alpha, x)$ weakly converges as a time-space process to the KPZ fixed point (started from the narrow wedge data) which has recently been constructed in [251] via its transition probability and simultaneously in [138] via the *Airy sheet*. Very recently, [280, 315] announced proofs of a special case of this conjecture, namely the weak convergence of the spatial process $x \mapsto (2\alpha^{-1})^{1/3}(\mathfrak{h}_t(\alpha, x) + \frac{x^2}{2})$ to the Airy$_2$ process (introduced in [275]) for any fixed $\alpha > 0$. In light of this conjecture, we expect that the law of iterated logarithms of the KPZ fixed point in the temporal direction bear the same scaling as in Theorem 4.1.1. Moreover, the macroscopic nature of the peaks and valleys of the KPZ equation as revealed in the above discussion is expected to be reflective of the case for the KPZ fixed point. Although, our proof techniques which will be touched on in Section 4.1.1 are very much likely to be applicable for the KPZ fixed point, we defer from proving results analogous to Theorem 4.1.1 and 4.1.3 for the KPZ fixed point.

Proving the law of iterated logarithms and the fractal properties of the KPZ equation requires information on the growth of $\mathfrak{h}_{t_1} - \mathfrak{h}_{t_2}$ for $t_1 > t_2 > 0$. When $t_1 - t_2$ is large, [117, Theorem 1.5] obtained upper and lower bounds on the tail probabilities of $\mathfrak{h}_{t_1} - \mathfrak{h}_{t_2}$. However, controlling the variations of the peaks in a smaller interval necessitates the study of the tail probabilities of the increments $\mathfrak{h}_{t_1} - \mathfrak{h}_{t_2}$ for $t_1 - t_2$ small. One of the main obstructions for studying the increments of $\mathfrak{h}_t$ in a small interval is the lack of uniform tail bounds of $\mathfrak{h}_t$ for all small $t > 0$. In the following two results, we seek to fill this gap. To state those results, we introduce the following notations:

$$\mathfrak{g}_t := \frac{\mathcal{H}^{\mathbf{nw}}(t,0) + \log\sqrt{2\pi t}}{(\pi t/4)^{1/4}}.$$

The first result proves a uniform bound on the upper tail probabilities of $\mathfrak{g}_t$ for all small $t > 0$.

Theorem 4.1.4. *Fix $\varepsilon > 0$. There exist $t_0 = t_0(\varepsilon) > 0$, $c = c(\varepsilon) > 0$, and $s_0 = s_0(\varepsilon) > 0$ such that for all $t \le t_0$ and $s \ge s_0$,*

$$\mathbf{P}(\mathfrak{g}_t \ge s) \le \exp\Big(- \tfrac{cs^2}{1+\sqrt{1+st^{1/4-4\varepsilon}}}\Big). \tag{4.1.6}$$

Remark 4.1.5. Note that the r.h.s. of (4.1.6) decays like Gaussian tails, i.e., $\exp(-cs^2)$ for some constant $c > 0$ as $t \downarrow 0$. This is embraced by the fact that $\mathfrak{g}_t$ weakly converges to a standard Gaussian distribution as t approaches 0 (shown in [6, Proposition 1.8]). On the other hand, for large t, the decay turns to $\exp(-cs^{3/2}t^{-1/8+2\varepsilon})$. The decay exponent $3/2$ accords with the finite time upper tail exponent (see [115, Theorem 1.10]) of the KPZ equation.

For the purpose of later use, we will only require the following loose bound which is free of the time variable and follows immediately from Theorem 4.1.4.

Corollary 4.1.6. *There exist $t_0 > 0$, $c > 0$, and $s_0 > 0$ such that for all $t \le t_0$ and $s \ge s_0$, we have* $\mathbf{P}(\mathfrak{g}_t \ge s) \le \exp(-cs^{3/2})$.

The next result shows an uniform bound on the lower tail probability of $\mathfrak{g}_t$ for all small $t > 0$.

Theorem 4.1.7. *There exist constants $t_0 \in (0,2]$, $s_0, c > 0$ such that for all $t \le t_0$, $s \ge s_0$,*

$$\mathbf{P}(\mathfrak{g}_t \le -s) \le e^{-cs^2}. \tag{4.1.7}$$

Remark 4.1.8. The decay exponent of the upper bound in (4.1.7) is consistent with the Gaussian limit of $\mathfrak{g}_t$ as t goes down to zero. It is worthwhile to note that Theorem 4.1.7 provides an upper bound to the lower tail probability which holds uniformly for all small $t > 0$. This should be contrasted with the work of [116, Theorem 1.1] which showed that the lower tail probability at finite time $t > 0$ decays as $\exp(-ct^{1/3}s^{5/2})$ for some constant $c > 0$. The interpolation between the exponents 2 and $5/2$ as one gradually increases time t from 0 to a finite value is not covered in Theorem 4.1.7.

Short time uniform tail bounds of Theorems 4.1.4 and 4.1.7 open directions to a plethora of new results. One of such directions is the study of modulus of continuity of the time-space process $\mathfrak{h}_t(\alpha, x)$. Our next and final main result proves a super-exponential tail bound of the modulus of continuity of $\mathfrak{h}_t(\alpha, x)$.

Theorem 4.1.9. *Fix $\varepsilon \in (0, \frac{1}{4})$ and any interval $[a,b] \subset \mathbb{R}_{\ge 1}$ and $[c,d] \subset \mathbb{R}$. Define* Norm : $([a,b]\times[c,d])^2 \to \mathbb{R}_{\ge 0}$

$$\mathrm{Norm}(\alpha_1, x_1; \alpha_2, x_2) = |x_1 - x_2|^{\frac{1}{2}}\Big(\log\frac{2|b-a|}{|x_1-x_2|}\Big)^{2/3} + |\alpha_1 - \alpha_2|^{\frac{1}{4}-\varepsilon}\Big(\log\frac{2|d-c|}{|\alpha_1-\alpha_2|}\Big)^{2/3} \tag{4.1.8}$$

and

$$C := \sup_{\alpha_1\neq\alpha_2, x_1\neq x_2} \frac{1}{\mathrm{Norm}(\alpha_1, x_1; \alpha_2, x_2)}\Big|\mathfrak{h}_t(\alpha_1, x_1) + \frac{x_1^2}{2\alpha_1} - \mathfrak{h}_t(\alpha_2, x_2) - \frac{x_2^2}{2\alpha_2}\Big|. \tag{4.1.9}$$

Then there exist $t_0(\varepsilon) > 0$, $s_0 = s_0(|b-a|, |c-d|, \varepsilon) > 0$ and $m = m(|b-a|, |c-d|, \varepsilon) > 0$ such that for all $s \ge s_0$ and $t \ge t_0$,

$$\mathbb{P}(C > s) \le e^{-ms^{3/2}}. \tag{4.1.10}$$

Remark 4.1.10. It was known ([48, Theorem 2.2]) that the fundamental solution of the SHE ($\mathcal{Z}^{\mathbf{nw}}(t,x)$) as a time-space process is almost surely Hölder continuous with the spatial and temporal Hölder exponents being less than $\frac{1}{2}$ and $\frac{1}{4}$ respectively. This indicates Hölder continuity of $\mathcal{H}^{\mathbf{nw}}(t,x)$ with same spatial and temporal Hölder exponents as that of $\mathcal{Z}^{\mathbf{nw}}(t,x)$. Theorem 4.1.9 corroborates this fact by giving tail bounds to the modulus of continuity.

4.1.1 Proof ideas

We start with discussing what makes our work hard to accomplish using other approaches. As a testing ground for non-linear SPDE's, the KPZ equation embraces a stack of new tools including regularity structures, paracontrolled distributions, energy solution method. Through its connection with the KPZ universality class, the KPZ equation advocates the usage of various techniques from integrable systems and random matrix theory. While these tools unveiled salient features of the KPZ equation in the past, many finer properties are still out of reach. One of the basic requirement for showing the law of iterated logarithms and the fractal nature of the KPZ height function level sets is to attain a delicate understanding of the modulus of continuity of the KPZ temporal process. This entails to knowing multi-point joint distribution of the KPZ equation. While the seminal paper [6] derived one point distribution of the narrow wedge solution of the KPZ equation, the exact formulas of more than one point does not seem to be on the horizon (see [150] for some recent progress in other positive temperature models). In [117, Theorem 1.5], the authors derived near-exponentially decaying bounds on the tail probabilities of the difference of the KPZ equation at two time points. Although these tail bounds were useful for finding the two time correlations of the KPZ equation, they failed to produce the modulus of continuity of the KPZ temporal process since those bounds are only valid when the two time points are far apart.

Our approach is mainly probabilistic while some of the key inputs bear an integrable origin. Two of such examples are the short time (upper) tail bounds of the KPZ equation (see Theorem 4.1.4) and the Gibbsian line ensemble. The short time upper tail will be derived using the integer moments of the SHE which has the recourse to some amenable contour integral formulas.

On the other hand, while the Gibbsian line ensemble owes its inception to some integrable system, it has so far been fostered by the probabilistic ideas. One of the other key tools which we will procure in the due course of this paper is the short time lower tail bound (see Theorem 4.1.7) which in contrast to the upper tail has its chassis made of core probabilistic ideas like Talagrand's concentration inequality.

Our first main tool is a multi-point composition law (see Proposition 4.2.11) which generalizes the two-point composition law of [117, Proposition 2.9]. In words, for any given set of time points $0 < t_1 < t_2 < \ldots < t_k$, this law constructs k independent random spatial profiles equivalent in law to the narrow wedge solution such that the KPZ temporal process at at t_i is obtained by exponential convolution of one of such independent profiles and $\mathfrak{h}_{t_{i-1}}(\cdot)$ for $i = 2, \ldots, k$.

Our second main tool is the Gibbsian line ensemble. More precisely, we use a special Gibbsian line ensemble called the KPZ line ensemble introduced by [109]. In short, KPZ line ensemble is a set of random curves whose lowest indexed curve has the same law as the narrow wedge solution of the KPZ equation. Furthermore, this set of random curves satisfies the $\mathbf{H}$-*Brownian Gibbs property* which ensures that the law of any fixed index curve in an interval only depends on the boundary value and can be described using the law of a Brownian bridge conditioned to have same boundary values, a connection elicited through a very explicit Radon-Nikodym derivative expression. As it was revealed in [109], the Brownian Gibbs property of the KPZ line ensemble imparts *stochastic monotonicity* on its lowest indexed curve, a property amenable to finding delicate tail bounds of the spatial profile of the KPZ equation. Furthermore, we also enrich the arsenal of the Gibbsian line ensemble by introducing and exploring a *short time KPZ line ensemble* (see Proposition 6.5.1) whose lowest indexed curve is the narrow wedge solution with *short-time KPZ scaling*, i.e., the scaling exponent of the fluctuation, space and time follows the ratio $1 : 2 : 4$. In order to distinguish, we would refer the KPZ line ensemble whose lowest indexed curve is narrow wedge solution with the KPZ scaling as the *long-time KPZ line ensemble*.

Our third main tool is the short time upper and lower tail bounds (Theorem 4.1.4 and 4.1.7) and the long time tail bounds of the KPZ equation from [116, 115] (summarized in Proposition 4.2.12-

4.2.14). The short time upper tail is derived using the contour integral formulas of the moments of the SHE whereas the short time lower tail (uniform in time) is obtained via controlling the tail estimates of the partition function of random polymer model whose continuum limit solves the SHE. We also improve the bounds available for the long time upper tail of the KPZ equation (see Proposition 4.8.5 of Section **??**), an important input for showing the fractal nature of the upper level sets in Theorem 4.1.3.

Now we proceed to discuss how we use those tools to prove our results. The one point tail estimates of the KPZ equation (from Theorem 4.1.4, 4.1.7 and Propositions 4.2.12-4.2.14) in conjugation with the tail bounds of the Brownian bridge fluctuations would allow us to derive delicate tail bounds of the spatial profile of the narrow wedge solution in finite intervals at the behest of the Brownian Gibbs property of the long and short time KPZ line ensembles. All these new tail estimates are detailed in Section 4.4. For any given $t_1 > t_2$, the two point composition law relates $\mathfrak{h}_{t_1}$ with the narrow wedge profile $\mathfrak{h}_{t_2}(1,\cdot)$ via an exponential convolution with another independent random spatial process which will be denoted as $\mathfrak{h}_{t_2\downarrow t_1}(\cdot)$ and has the same distribution as $\mathfrak{h}_{t_1}((t_2-t_1)/t_1,\cdot)$. Mating of this convolution principle with the tail bounds of the KPZ spatial process from Section 4.4 propagates the one point tail estimates to the tail bounds of the difference of the KPZ height functions at two time points. These ideas, inculcated in Propositions 4.5.1-4.5.4 of Section 4.5, will unfold to be a mainstay on which the proof of Theorem 4.1.9 rests.

Strategy for LIL: By the Borel-Cantelli lemmas, the law of iterated logarithms of Theorem 4.1.1 can be recast as showing that the infimum and supremum of the LIL adjusted temporal processes $\mathfrak{h}_t/(\log\log t)^{1/3}$ and $\mathfrak{h}_t/(\log\log t)^{2/3}$ respectively over the intervals $\mathcal{I}_n := [\exp(e^n), \exp(e^{n+1})]$ cannot stay further away from $-6^{1/3}$ and $(3/4\sqrt{2})^{2/3}$ infinitely often, i.e., we need to show that with high probability, the following holds

$$-(6(1+\epsilon))^{\frac{1}{3}} \leq \inf_{t\in\mathcal{I}_n} \frac{\mathfrak{h}_t}{(\log\log t)^{1/3}} \leq -(6(1-\epsilon))^{\frac{1}{3}}, \tag{4.1.11}$$

$$\Big(\frac{3}{4\sqrt{2}}(1-\epsilon)\Big)^{\frac{2}{3}} \leq \sup_{t\in\mathcal{I}_n} \frac{\mathfrak{h}_t}{(\log\log t)^{2/3}} \leq \Big(\frac{3}{4\sqrt{2}}(1+\epsilon)\Big)^{\frac{2}{3}}. \tag{4.1.12}$$

There are mainly two core ideas behind the proofs of (4.1.11) and (4.1.12). The proofs of the second inequality of (4.1.11) and the first inequality in (4.1.12) will be based on similar ideas whereas the proof of the first inequality of (4.1.11) and the second inequality of (4.1.12) will use a different strategy. We now divide the proof ideas of the inequalities in (4.1.11) and (4.1.12) into few steps. While these steps are not in chronological order, we mention the appropriate sections where these steps will be carried out.

Step 1: The first major idea which trickles down into the proof of the second inequality of (4.1.11) and the first inequality of (4.1.12) is to establish approximate independence of the increments of the KPZ height function over an increasing set of time intervals. Proposition 4.6.1 exactly proves this claim where we show that when the gaps between time points $t_1 < t_2 < \ldots < t_k$ are large, the multi-point distribution of the temporal process $\mathfrak{h}_t(0)$ at time points $t_1 < t_2 < \ldots < t_k$ is well approximated by joint distribution of a set of independent random variables. More precisely, we show that if $(t_{i+1}/t_i) - 1 \geq s > 0$ for $1 \leq i \leq k-1$, one can construct independent random variables $Y_1, Y_2, \ldots, Y_k$ on the same probability space such that $\mathbf{P}(|\mathfrak{h}_{t_i} - Y_i| \geq 1) \leq \exp(-cs^{1/2})$ for all $i = 1, 2, \ldots, k$. The construction of this new set of random variables is done via the multi-point composition law which demonstrates that for any given set of time points $0 < t_1 < \ldots < t_k$, $\mathfrak{h}_{t_{i+1}}(0)$ for $1 \leq i \leq k-1$ can be written in terms of the spatial profile of the KPZ equation at t_i and a spatial process $\mathfrak{h}_{t_{i+1}\downarrow t_i}(\cdot)$ such that $\{\mathfrak{h}_{t_{i+1}\downarrow t_i}(\cdot)\}_{1\leq i\leq k-1}$'s are independent processes in the same probability space as the white noise of the KPZ equation. The error of the approximations in Proposition 4.6.1 is obtained via Proposition 4.5.4 of Section 4.5 which provides tail bounds of the increments of the temporal process $\mathfrak{h}_t$ between two time points for varying choices of the gap size between the time points.

Step 2: This step finds the tail probabilities of the KPZ spatial process which are obtained by combining two-point composition law (in the time direction) of the KPZ equation with precise estimates on the tail probabilities of the maximal variation of the height profile of the KPZ equation along the spatial direction (see Section 4.4). While similar tail events are studied in [117], the results of Section 4.4 improves on many sides which includes bringing new ideas for improving

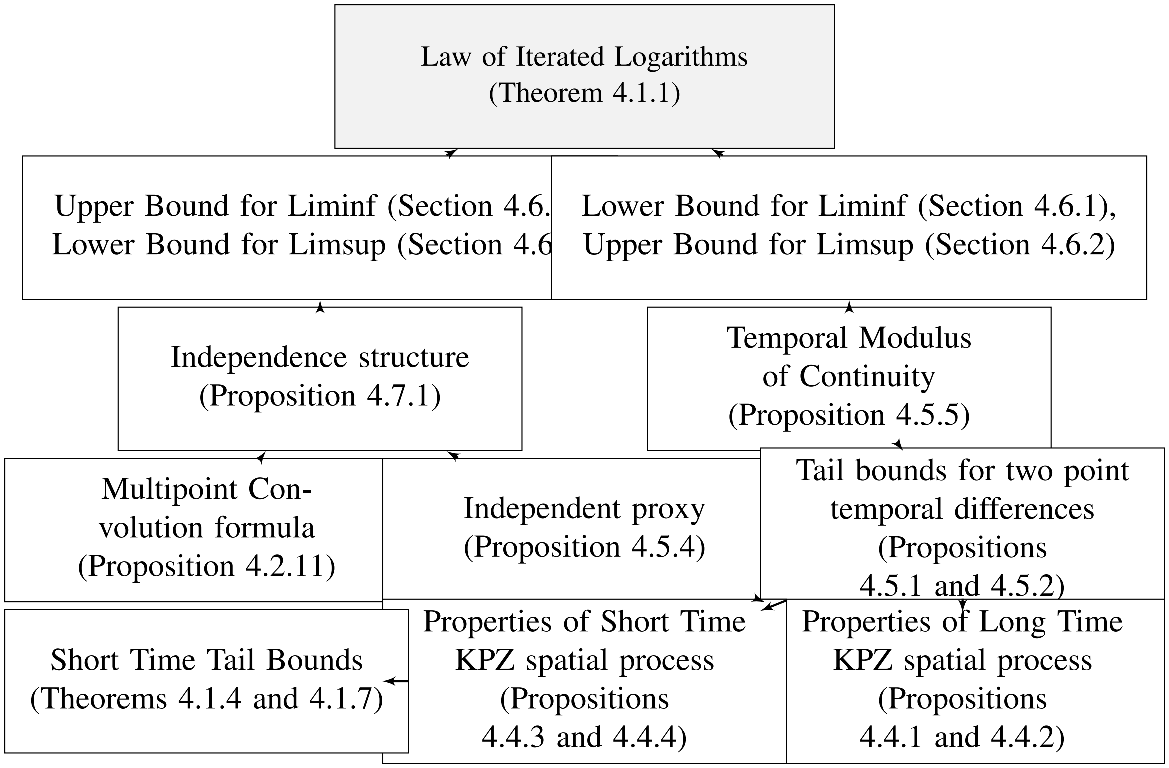

Figure 4.1: Flowchart of the proof of Theorem 4.1.1

the exponents of the tail probabilities of the maximal variation of the spatial profile of the KPZ height function on bounded intervals.

Step 3: Our other main idea behind the proofs of the law of iterated logarithms of the KPZ equation revolves around the tail probability bound for the spatio-temporal modulus of continuity of the KPZ equation of Theorem 4.1.9 which is proven in Section 4.5. The proof of Theorem 4.1.9 stands on the probability estimates derived from Propositions 4.5.1-4.5.2 which bound the tail probabilities of the KPZ height function between two time points. One of the important features of these tail estimates lies in the flexibility in varying the gap between two time points which were not known in the earlier works. The key ingredients for these estimates are twofold: (a) short time tail probabilities of the KPZ equation which are stated in Theorem 4.1.4 and Theorem 4.1.7 and, (b) short-time KPZ line ensemble which is defined in Lemma 6.5.1. Both of these tools are new and will be used in future for exploring many important properties of the KPZ equation.

In Figure 4.1, we summarize the above proof ideas of Theorem 4.1.1 using a flowchart.

Strategy for fractality: Much akin to the law of iterated logarithms, the proofs of mono- and multi-fractality of the KPZ equation heavily rely on the tail probabilities of the supremum and infimum of the KPZ temporal process in compact intervals. In the same spirit as in the proof of the law of iterated logarithms, the proofs of the fractality of the KPZ use two core ideas. The monofractality result which is stated in (4.1.4) of Theorem 4.1.3 uses the idea of independent proxies of Proposition 4.6.1 which we have discussed above. Furthermore, it leverages on precise estimates of the upper tail probability of the KPZ equation in long time. In Proposition 4.8.5 of Section **??**, we derive such estimate from the scratch which was unknown before and hefty to obtain otherwise. Combining the construction of independent proxy of Proposition 4.6.1 with Proposition 4.8.5, we show approximate independence between the upper level sets $\{\mathfrak{h}_t \geq (\log\log t)^{2/3}\gamma\}$ and $\{\mathfrak{h}_s \geq (\log\log s)^{2/3}\gamma\}$ when t and s are sufficiently far apart. We like to stress the fact that the mono-fractality result ((4.1.4) of Theorem 4.1.3) requires fast decoupling of the joint probabilities of events $\{\mathfrak{h}_t \geq (\log\log t)^{2/3}\gamma\}$ and $\{\mathfrak{h}_s \geq (\log\log s)^{2/3}\gamma\}$. While such decoupling results are obtained for the Brownian motion in [223, Lemma 3.5-3.6] without much ado, the situation for the KPZ equation is more complicated and hinges on getting fine estimates of the one-point upper tail probability. Based on similar techniques as in [115, Proposition 4.1], Proposition 4.8.7 of Section **??** provides such tail bounds which will be finally used in Proposition 4.7.1 for showcasing the decoupling in the KPZ upper tail probabilities. Unlike the monofractality, the key ingredient behind the proof of the multifractality result of (4.1.5) is the spatio-temporal modulus of continuity from Theorem 4.1.9. The summary of the proof ideas of Theorem 4.1.3 is shown in Figure 4.2 using a flowchart.

Our approach of studying the peaks and valleys of the KPZ equation has the potential to generalize for other models in the KPZ universality class. As it was mentioned earlier, our approach stands on the shoulders of three main components: multi-point composition law, Gibbsian line ensemble and one-point tail probabilities. For the zero temperature models like the last passage percolation model, Airy process and many more, the analogues of the multi-point composition law are easy to obtain and stated in terms of the *maximum convolution* instead of the exponential

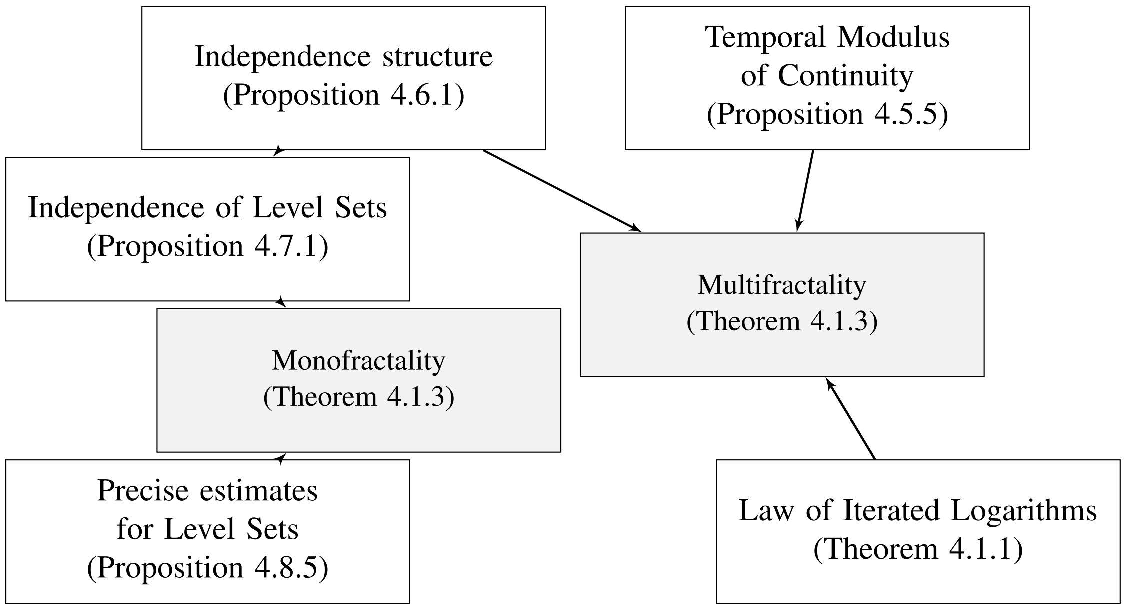

Figure 4.2: Flowchart of the proof of Theorem 4.1.3

convolution. Gibbsian line ensemble approach was first introduced by [109] for studying the Airy line ensemble and then, latter been applied in numerous zero temperature models. Furthermore, precise one-point tail estimates are available for many zero temperature models including the KPZ fixed point. Some of these technical appliances are also available for few positive temperature models such as the asymmetric simple exclusion process (ASEP), stochastic six vertex model, strictly weak lattice polymer model etc. With the aid of the above three proof components, the revelation of the landscape of the aforementioned models bears immense possibilities which we hope to explore in future.

4.1.2 Previous works

Studying *macroscopic landscapes* of stochastic processes is one of the most compelling research directions in probability theory. Starting from the middle of the previous century to the present time, Brownian motion serves as a fertile ground for doing alluring predictions on the landscape of the models in the Gaussian universality class and demonstrating those with lots of success. One of the main goals of this work is to showcase the KPZ equation as a representative of the models in the KPZ universality class when it comes to explaining the macroscopic landscape of its members under the KPZ scaling. Below, we review some of the previous works on the LIL and fractal properties of the models in the KPZ universality with the aim of comparing and contrasting those with our results.

Random matrix theory is intimately connected with the models of the KPZ universality class. In fact, the Tracy-Widom GUE distribution which became one of the characteristics of the fixed points of the universality class was borne out [**Tracy94**] as a by-product of a random matrix model. To be more precise, the limiting distribution of the largest eigenvalue λ_n^{GUE} of an $n \times n$ Gaussian unitary ensemble under centering by $\sqrt{2n}$ and scaling by $n^{-(1/6)}$ is known as the Tracy-Widom GUE distribution. One may also regard λ_n^{GUE} as the n-th element of the GUE minor process. From this point of view, it was an interesting open question to study the law of fractional logarithm of λ_n^{GUE} which was finally solved by [270]. The authors found the value of the limsup of $(\lambda_n^{\mathrm{GUE}} - \sqrt{2n})\sqrt{2}n^{1/6}$ under a normalization by $(\log n)^{2/3}$ when n goes to ∞. The authors had shown that the value of the limsup is almost surely equal to $(1/4)^{2/3}$. On the other hand, [270] had also shown that the liminf of $(\lambda_n^{\mathrm{GUE}} - \sqrt{2n})\sqrt{2}n^{1/6}$ under a normalization by $(\log n)^{1/3}$ is almost surely finite. They had conjectured that the liminf is almost surely equal to $-4^{1/3}$. To the best of our knowledge, the macroscopic Hausdorff dimensions of the level sets of λ_n^{GUE} are not known yet. Drawing the analogy with the KPZ equation, we conjecture that the peaks and valleys of λ_n^{GUE} are monofractal in the gauge functions $(\log n)^{2/3}$ and $(\log n)^{1/3}$ respectively.

Last passage percolation (LPP) is one of the widely studied models in the KPZ universality class. Due to the presence of endearing geometric properties, the study of the LPP model fueled lots of interests in the recent times. [244] had initiated the study on the laws of iterated logarithms in the case of integrable LPP models. In [244], the author had considered the LPP model in $\mathbb{Z}^2_{\geq 0}$ lattice where the weights of the lattice sites are independent exponential or, geometric random variables. It was shown in [244] that the limsup of point to point last passage percolation time from $(0,0)$ to (n,n) (centered by $4n$ and scaled by $(2^4 n)^{1/3}(\log\log n)^{2/3}$) is almost surely bounded between α_{sup} and $(3/4)^{2/3}$ for some $0 < \alpha_{\mathrm{sup}} \leq (3/4)^{2/3}$. In fact, [244] had conjectured that α_{sup} is equal to $(3/4)^{2/3}$. [244] had also investigated the liminf of the LPP model. It was shown that the last passage time between $(0,0)$ and (n,n) (centered by $4n$ and scaled by $(2^4 n)^{1/3}(\log\log n)^{1/3}$) is almost surely lower bounded by some constant. Recently, [36] have shown that the value of liminf is almost surely equal to a constant. However, not much is known about the exact value.

Fractal properties of the putative distributional limit of the models in the KPZ universality class, namely the KPZ fixed point has been investigated in few of the latest works. Recently, [138] gave a probabilistic construction of the KPZ fixed point as a distributional limit of the point-to-point Brownian last passage percolation model. The limiting space-time process which they named as the *directed landscape* led to a flurry of new discoveries. The study of the fractal geometry of the directed landscape has lately been initiated by [37, 45] who considered the problem of fractal dimension of some exceptional points along the spatial direction. In spite of the recent developments, the fractal nature of the space-time process of the directed landscape is still not fully understood. We hope that our results on fractality of the KPZ equation would shed some light for such study in future.

In the last decade, fractal properties of stochastic partial differential equations (SPDE) became an active area of research. The main focus of a vast majority of those works resided on the study of the large peaks of the SPDEs with multiplicative noise [**BC16**, 175, 85, 48, 201, 169, 104, 92, 90, 91]. The growth of the large peaks of the SPDEs is attested by the intermittency property which is the center of attention in the field of the research of complex multiscale system for last five-six decades. See introduction of [48] and [**CM94**, 222] for a detailed discussion. Recently, [223] investigated the fractal properties of the stochastic heat equation started from the constant initial data at the onset of intermittency and established the multifractal nature of the spatial process. Denote the solution of the SHE started from the constant initial data (i.e., $\mathcal{Z}^{\mathrm{flat}}(0,x) = 1$ for all $x \in \mathbb{R}$) by $\mathcal{Z}^{\mathrm{flat}}(t,x)$. Drawing on an earlier result of [94] which showed a fractional law of logarithm -

$$\limsup_{x\to\infty} \frac{\log \mathcal{Z}^{\mathrm{flat}}(t,x)}{t^{1/3}(\frac{3}{4\sqrt{2}}\log_+ x)^{2/3}} = 1 \quad \text{a.s.,} \tag{4.1.13}$$

Theorem 1.2 of [223] established the multifractal nature of the spatial process $\log \mathcal{Z}^{\text{flat}}(t, \cdot)$ for any fixed $t > 0$. The results of [223] is complemented by the study of the spatio-temporal fractal properties by [221] which showed that there are infinitely many different stretch scale (in the spatial direction) and time scale such that for any given stretch and time scale, the peaks of the spatio-temporal process of the stochastic heat equation attain non-trivial macroscopic Hausdorff dimensions. The idea of peaks of the stochastic heat equation forming complex multiscale system were also echoed in [324, 183, 182]. However, the macroscopic behavior of the KPZ temporal process as considered in this paper shows a different nature due to its slow decay of correlations in comparison to the KPZ equation along the spatial direction. For instance, our first result, Theorem 4.1.1 exhibits LIL for the KPZ temporal process as opposed to the fractional law of logarithm satisfied by the KPZ spatial process demonstrated in (4.1.13). In the same spirit, our second result (Theorem 4.1.3) which is reminiscent of a similar result in [223, Theorem 1.4] for 1-dimensional Brownian motion shows that the peaks of the KPZ temporal process exhibit a monofractal (see Definition 4.1.2) nature as time t goes to ∞. This is in contrast to the multifractal nature of the spatial process as shown in [223]. Nevertheless, Theorem 4.1.3 shows that the crossover to the multifractality of the KPZ temporal process happens under exponential transformation of the time variable. While the complete understanding of the spatio-temporal landscape of the KPZ equation is far-off to our present reach, we hope that our results will ignite further interests along this direction.

We end this section with a review on the tail probabilities of the KPZ equation, one of the key tools of this paper. Study of the KPZ tail probabilities had been taken up in many works [258, 122, 168] in the past. One of the recent major advances has been achieved in [116] which proved tight bounds to the lower tail probability of the KPZ equation started from the narrow wedge initial data. This sowed the seeds of a series of works [118, 310, 232, 79, 323, 78] which studied in detail the lower tail large deviation of the KPZ equation as time goes to ∞. The upper tail probabilities of the KPZ equation has been recently investigated by [115]. The same paper also initiated the study of the tail probabilities under general initial data. The upper tail large deviation was later found in [131] for narrow wedge initial data and in [180] for general initial data. In spite of these recent advances, not much were known about the evolution of the tail probabilities of the KPZ equation as time t goes to 0. In a very recent work, [249] showed the large deviation of the KPZ equation as t tends to 0. However, this does not shed much light on the uniform tail estimates of the KPZ height function starting from time equal to 0 to a finite value. Such uniform estimates which were reported in Theorems 4.1.4 and 4.1.7 will be instrumental in obtaining our other main results Theorems 4.1.1, 4.1.3 and 4.1.9.

Outline. Section 4.2 will introduce the basic frameworks of the KPZ line ensemble and the Barlow-Taylor macroscopic fractal theory. It will also introduce other useful tools including multipoint composition law, one-point tail probabilities of the KPZ equation, tail probabilities of the supremum and infimum of the KPZ spatial process. Section 4.3 will prove Theorems 4.1.4 and 4.1.7. This will be followed by Section 4.4 where we derive delicate tail bounds of the KPZ spatial process for finite and short time. Section 4.5 will study the temporal modulus of continuity of the KPZ equation and use it to prove Theorem 4.1.9. Based on the tools from Sections 4.2-4.5, the law of iterated logarithms of Theorem 4.1.1 will be proved in Section 4.6. The proof of the mono- and multifractality results of the KPZ equation from Theorem 4.1.3 will be given in Section 4.7. This last section will use an improved KPZ upper tail probability estimate which is proved in Proposition 4.8.5 of Appendix **??**.

4.2 Basic Framework and Tools

In this section, we will review three topics which are required for our subsequent analysis. One of the main topics of this section is the KPZ line ensemble and its Brownian Gibbs property. The KPZ line ensemble is a set of random curves whose lowest indexed curve is same in distribution with the narrow wedge solution of the KPZ equation. The **H**-Brownian Gibbs property of the KPZ line ensemble induces stochastic monotonicity of the spatial profile the KPZ equation, one of the major tools in our analysis. Lemma 6.5.2 of Section 4.2.1 will precisely state such monotonicity result. In a similar way as in [**CH16**], we will introduce a short-time version of the KPZ line ensemble which would play a key role in later sections to find the temporal modulus of continuity of the KPZ equation.

Our second main topic of this section is the Barlow-Taylor theory of macroscopic fractal properties of a stochastic process. In light of the expositions in [223, 23, 22], the notions of Barlow-Taylor *Hausdorff content* and *dimension* of any Borel set will be recalled. Some of the basic properties of the Barlow-Taylor Hausdorff dimension are presented in Proposition 4.2.6, 4.2.7 and 4.2.9 of Section 4.2.2.

Lastly, we recall some of the known facts about the KPZ equation including its *multipoint composition law* and the tail estimates of its one point distribution in Section 4.2.3.

4.2.1 KPZ line ensemble

Recall the general notion of line ensembles from Section 2 in [109].

Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \ldots)$ be an $\mathbb{N}\times\mathbb{R}$ indexed line ensemble. Fix $k_1 \le k_2$ with $k_1, k_2 \in \mathbb{N}$ and an interval $(a, b) \in \mathbb{R}$ and two vectors $\vec{x}, \vec{y} \in \mathbb{R}^{k_2-k_1+1}$. Given a continuous function $\mathbf{H} : \mathbb{R} \to [0, \infty)$ (Hamiltonian) and two measurable functions $f, g : (a, b) \to \mathbb{R}\cup\{\pm\infty\}$, the law $\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ on $\mathcal{L}_{k_1}, \ldots, \mathcal{L}_{k_2} : (a, b) \to \mathbb{R}$ has the following Radon-Nikodym derivative with respect to $\mathbb{P}_{\text{free}}^{k_1,k_2,(a,b),\vec{x},\vec{y}}$, the law of $k_2 - k_1 + 1$ many independent Brownian bridges taking values $\vec{x}$ at time a and $\vec{y}$ at time b:

$$\frac{d\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}}{d\mathbb{P}_{\text{free}}^{k_1,k_2,(a,b),\vec{x},\vec{y}}}(\mathcal{L}_{k_1}, \ldots, \mathcal{L}_{k_2}) = \frac{\exp\Big\{ - \sum_{i=k_1}^{k_2+1} \int \mathbf{H}\big(\mathcal{L}_i(x) - \mathcal{L}_{i-1}(x)\big)dx\Big\}}{Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}},$$

where $\mathcal{L}_{k_1-1} = f$, or ∞ if $k_1 = 1$; and $\mathcal{L}_{k_2+1} = g$. Here, $Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ is the normalizing constant which produces a probability measure.

We say $\mathcal{L}$ enjoys the $\mathbf{H}$-*Brownian Gibbs property* if, for all $K = \{k_1, \ldots, k_2\} \subset \mathbb{N}$ and $(a, b) \subset \mathbb{R}$, the following distributional equality holds:

$$\text{Law}\Big(\mathcal{L}_{K\times(a,b)} \text{ conditioned on } \mathcal{L}_{\mathbb{N}\times\mathbb{R}\setminus K\times(a,b)}\Big) = \mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g},$$

where $\vec{x} = (\mathcal{L}_{k_1}(a), \ldots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \ldots, \mathcal{L}_{k_2}(b))$, and where again $\mathcal{L}_{k_1-1} = f$, or ∞ if $k_1 = 1$; and $\mathcal{L}_{k_2+1} = g$.

Just as for the Markov property, there is a *strong* version of the $\mathbf{H}$-Brownian Gibbs property which is valid with respect to *stopping domains* which we now describe. A pair $(\mathfrak{a}, \mathfrak{b})$ of random variables is called a K-stopping domain if $\{\mathfrak{a} \leq a, \mathfrak{b} \geq b\} \in \mathfrak{F}_{\mathrm{ext}}(K \times (a, b))$, the σ-field generated by $\mathcal{L}_{(\mathbb{N}\times\mathbb{R})\setminus(K\times(a,b))}$. $\mathcal{L}$ satisfies the strong $\mathbf{H}$-Brownian Gibbs property if for all $K = \{k_1, \ldots, k_2\} \subset \mathbb{N}$ and K-stopping domain if $(\mathfrak{a}, \mathfrak{b})$, the conditional distribution of $\mathcal{L}_{K\times(\mathfrak{a},\mathfrak{b})}$ given $\mathfrak{F}_{\mathrm{ext}}(K \times (\mathfrak{a}, \mathfrak{b}))$ is $\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(l,r),\vec{x},\vec{y},f,g}$, where $\ell = \mathfrak{a}$, $r = \mathfrak{b}$, $\vec{x} = (\mathcal{L}_i(\mathfrak{a}))_{i\in K}$, $\vec{y} = (\mathcal{L}_i(\mathfrak{b}))_{i\in K}$, and where again $\mathcal{L}_{k_1-1} = f$, or ∞ if $k_1 = 1$; and $\mathcal{L}_{k_2+1} = g$.

The following lemma demonstrates a sufficient condition under which the strong H-Brownian Gibbs property holds.

Lemma 4.2.1 (Lemma 2.5 of [**CH16**])**.** *Any line ensemble which enjoys the* $\mathbf{H}$*-Brownian Gibbs property also enjoys the strong* $\mathbf{H}$*-Brownian Gibbs property.*

Line ensembles with the $\mathbf{H}$-Brownian Gibbs property benefit from certain stochastic monotonicities of the underlying measures. The following proposition shows that two line ensembles with the same index set can be coupled in such a way that if the boundary conditions of one ensemble dominates the other, then same is true for laws of the restricted curves.

Lemma 4.2.2 (Stochastic monotonicity: Lemmas 2.6 and 2.7 of [**CH16**])**.** *Fix finite intervals* $K \subset \mathbb{N}$ *and* $(a, b) \subset \mathbb{R}$*; and, for* $i \in \{1, 2\}$*, vectors* $\vec{x}_i = \big(x_i^{(k)} : k \in K\big)$ *and* $\vec{y}_i = \big(y_i^{(k)} : k \in K\big)$ *in* $\mathbb{R}^K$ *that satisfy* $x_2^{(k)} \leq x_1^{(k)}$ *and* $y_2^{(k)} \leq y_1^{(k)}$ *for* $k \in K$*; as well as measurable functions* $f_i : (a, b) \to \mathbb{R} \cup \{+\infty\}$ *and* $g_i : (a, b) \to \mathbb{R} \cup \{-\infty\}$ *such that* $f_2(s) \leq f_1(s)$ *and* $g_2(s) \leq g_1(s)$ *for* $s \in (a, b)$*. For* $i \in \{1, 2\}$*, let* $\mathbf{P}_i$ *denote the law* $\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x}_i,\vec{y}_i,f_i,g_i}$*, so that a* $\mathbf{P}_i$*-distributed random variable* $\mathcal{L}_i = \{\mathcal{L}_i^k(s)\}_{k\in K, s\in(a,b)}$ *is a* $K \times (a, b)$*-indexed line ensemble. If* $\mathbf{H} : [0, \infty) \to \mathbb{R}$ *is convex, then a common probability space may be constructed on which the two measures are supported such that, almost surely,* $\mathcal{L}_1^k(s) \geq \mathcal{L}_2^k(s)$ *for* $k \in K$ *and* $s \in (a, b)$*.*

In the present article, we will consider the following two kinds of $\mathbf{H}$:

$$\mathbf{H}_t^{\text{long}}(x) = t^{2/3}e^{t^{1/3}x} \quad \text{and,} \quad \mathbf{H}_t^{\text{short}}(x) = (\pi t/4)^{1/2}e^{(\pi t/4)^{1/4}x} \quad \text{for given } t > 0\,. \tag{4.2.1}$$

Clearly $\mathbf{H}_t^{\text{long}}(x)$ and $\mathbf{H}_t^{\text{short}}(x)$ in (4.2.1) are convex. Thus, Lemma 6.5.2 applies to any $\mathbf{H}_t^{\text{long}}$ or, $\mathbf{H}_t^{\text{short}}$-Brownian Gibbs line ensemble. The following proposition recalls the unscaled and scaled KPZ line ensemble constructed in [**CH16**] which satisfies $\mathbf{H}_t^{\text{long}}$-Brownian Gibbs property and introduces the short time KPZ line ensemble which exhibits $\mathbf{H}_t^{\text{short}}$-Brownian Gibbs property.

Lemma 4.2.3. *Let $t > 0$. There exists an $\mathbb{N}\times\mathbb{R}$-indexed line ensemble $\mathcal{H}_t = \{\mathcal{H}_t^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$ such that:*

1. *the lowest indexed curve $\mathcal{H}_t^{(1)}(x)$ is equal in distribution (as a process in x) to the Cole-Hopf solution $\mathcal{H}^{\mathbf{nw}}(t,x)$ of KPZ started from the narrow wedge initial data and the line ensemble $\mathcal{H}_t$ satisfies the $\mathbf{H}_1^{\text{long}}$-Brownian Gibbs property;*
2. *the scaled KPZ line ensemble $\{\mathfrak{h}_t^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$, defined by*

$$\mathfrak{h}_t^{(n)}(x) \;=\; t^{-1/3}\Big(\mathcal{H}_t^{(n)}\big(t^{2/3}x\big) + t/24\Big)\,,$$

   satisfies the $\mathbf{H}_t^{\text{long}}$-Brownian Gibbs property.

3. *and the scaled short time line ensemble $\{\mathfrak{g}_t^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$, defined by*

$$\mathfrak{g}_t^{(n)}(x) \;=\; (\pi t/4)^{-1/4}\Big(\mathcal{H}_t^{(n)}\big((\pi t/4)^{1/2}x\big) + \log\sqrt{2\pi t}\Big)\,, \tag{4.2.2}$$

   satisfies the $\mathbf{H}_t^{\text{short}}$-Brownian Gibbs property.

Proof. Part (1) is precisely part ((1) + (2)) of Theorem 2.15 of [**CH16**]. The proof of Part (2) and Part (3) follows from an easy change of variable lemma (Lemma 4.8.1). For brevity, we postpone the details of the proof of Lemma 4.8.1 to Section **??**. □

Remark 4.2.4. In part (3) of Theorem 2.15 [**CH16**] it is erroneously mentioned that the scaled KPZ line ensemble satisfies $\mathbf{H}_t$-Brownian Gibbs property with $\mathbf{H}_t(x) = e^{t^{1/3}x}$ (instead of $\mathbf{H}_t^{\mathrm{long}}$). This error was reported by Milind Hegde and has been communicated to the authors of [**CH16**]. They have acknowledged the error and are currently preparing an errata for the same. We use **H**-Brownian Gibbs property of line ensembles only in Section 4.4 and in some parts of the supplement file. More precisley, we use KPZ line ensemble to import some stochastic monotonicity properties of the KPZ equation which holds as long as the Hamiltonians are convex (see [**CH16**]). Indeed, the Hamiltonians remain convex even after modification. Therefore, changing Hamiltonians has nearly no effect in those places where **H**-Brownian Gibbs property been used. Furthermore, we never write the Hamiltonians explicitly in Section 4.4 and always refer to (4.2.1) for their definition. As a result, there are no changes in any equations or formulas.

The above result demonstrates that the lowest indexed curves $\mathfrak{h}_t^{(1)}$ and $\mathfrak{g}_t^{(1)}$ in the scaled long time and short time KPZ line ensemble have the laws of the centered and scaled narrow wedge solution $\mathfrak{h}_t(x) := \mathfrak{h}_t(1,x)$ and $\mathfrak{g}_t(x) := \mathfrak{g}_t(1,x)$ of the KPZ equation defined in (4.2.4). This property is crucial in extracting further probabilistic information of spatial processes $\mathfrak{h}_t(x)$ and $\mathfrak{g}_t(x)$ as done in Section 4.4. For the rest of the article, we will loosely say $\mathfrak{h}_t^{(n)}$ or $\mathfrak{g}_t^{(n)}$ satisfy the Brownian Gibbs property since the Hamiltonian will be clear from the context.

4.2.2 Barlow-Taylor's macroscopic fractal theory

Definition 4.2.5 (Hausdorff content and dimension)**.** For any Borel set $\mathcal{A} \subset \mathbb{R}$, the *n-th shell* of $\mathcal{A}$ is defined as $\mathcal{A} \cap \{(-e^{n+1}, -e^n] \cup [e^n, e^{n+1})\}$. Let us fix a number $c_0 > 0$, and the set $\mathcal{A} \subset \mathbb{R}$ and $\rho > 0$, define ρ-dimensional *Hausdorff content* of the n-th shell of $\mathcal{A}$ as

$$\nu_{n,\rho}(\mathcal{A}) := \inf \sum_{i=1}^{m} \Big(\frac{\mathrm{Length}(Q_i)}{e^n}\Big)^{\rho}$$

where the infimum is taken over all sets of intervals $Q_1, \ldots, Q_m$ of length greater than c_0 and covering n-th shell of $\mathcal{A}$. Define the ρ-dimensional Hausdorff content of the set $\mathcal{A}$ as a sum total of $\nu_{n,\rho}(\mathcal{A})$ as n varies over the set of all positive integers. Then, the Barlow-Taylor *macroscopic Hausdorff dimension* the set $\mathcal{A}$ is defined as the infimum over all $\rho > 0$ such that the ρ-dimensional Hausdorff content of $\mathcal{A}$ is finite, i.e.,

$$\mathrm{Dim}_{\mathbb{H}}(\mathcal{A}) := \inf\left\{\rho > 0 : \sum_{n=0}^{\infty} \nu_{n,\rho}(\mathcal{A}) < \infty\right\}.$$

From the definition, it follows that the macroscopic Hausdorff dimension of a bounded set is 0. Just as in the microscopic case, one has $\mathrm{Dim}_{\mathbb{H}}(E) \le \mathrm{Dim}_{\mathbb{H}}(F)$ when $E \subset F$. Furthermore, it has been observed in [223, Lemma 2.3] that the macroscopic Hausdorff dimension does not depend on c_0. These observations are summarized in the following proposition.

Proposition 4.2.6 ([23, 22, 223])**.** *Consider $E \subset \mathbb{R}$. Then,* $\mathrm{Dim}_{\mathbb{H}}(E)$ *does not depend on the value of c_0 of Definition 4.2.5 and* $\mathrm{Dim}_{\mathbb{H}}(E) \le \mathrm{Dim}_{\mathbb{H}}(F)$ *for $F \supset E$. Moreover,* $\mathrm{Dim}_{\mathbb{H}}(E) = 0$ *if E is bounded.*

Since the choice of $c_0 > 0$ does not matter, from now on we will work with $c_0 := 1$. We next mention a technical estimate on the Hausdorff content of any set. The following proposition, as stated in [223] is a macroscopic analogue of the classical Frostman lemma for microscopic Hausdorff dimension.

Proposition 4.2.7 (Lemma 2.5 of [223])**.** *Fix $n \in \mathbb{R}_{\ge 1}$, and suppose E is a subset of the shell $S_n := [-e^{n+1}, -e^n) \cup (e^n, e^{n+1}]$. Denote the Lebesgue measure of a Borel set $B \subset \mathbb{R}$ by* $\mathrm{Leb}(B)$*. Let μ be a finite Borel measure on $\mathbb{R}$ and define for $\rho > 0$,*

$$K_{n,\rho} := \sup\left\{\tfrac{\mu(Q)}{\mathrm{Leb}(Q)} : Q \textit{ is a Borel set in } S_n,\ \mathrm{Leb}(Q) \ge 1\right\}. \tag{4.2.3}$$

Then we have $\nu_{n,\rho}(E) \ge K_{n,\rho}^{-1} e^{-n\rho} \mu(E)$.

The above proposition will be used in Section 4.7 to show lower bound to the macroscopic Hausdorff dimension of the level sets of the KPZ equation. In the following, we recall the notion of *thickness* of a set (introduced in [223]), another important tool to bound the Hausdorff dimension.

Definition 4.2.8 (θ-Thickness)**.** Fix $\theta \in (0,1)$ and define

$$\Pi_n(\theta) := \bigcup_{\substack{0\le j\le e^{n(1-\theta)+1}-e^{n(1-\theta)} \\ j\in\mathbb{Z}}} \{e^n + je^{n\theta}\}.$$

We say $E \subset \mathbb{R}$ is θ-thick if there exist integer $M = M(\theta)$ such that $E \cap [x, x+e^{\theta n}] \neq \varnothing$ for all $x \in \Pi_n(\theta)$ and for all $n \ge M$.

The following result (Corollary 4.6 in [223]) provides a lower bound to the Hausdorff dimension of a given set in terms of its thickness.

Proposition 4.2.9. *If $E \subset \mathbb{R}$ is θ-thick for some $\theta \in (0,1)$, then* $\mathrm{Dim}_{\mathbb{H}}(E) \ge 1-\theta$.

4.2.3 KPZ equation results

We start with introducing the space-time scaling of the KPZ height function appropriate for the short time regime, i.e., the case when the time variable goes to 0.

$$\mathfrak{g}_t(\alpha,x) := \frac{\mathcal{H}^{\mathbf{nw}}(\alpha t, (\pi t/4)^{1/2}x) + \log\sqrt{2\pi\alpha t}}{(\pi t/4)^{1/4}}. \tag{4.2.4}$$

We will often use the shorthand notation $\mathfrak{g}_t(x) := \mathfrak{g}_t(1,x)$. In addition, we simply write $\mathfrak{g}_t := \mathfrak{g}_t(1,0)$ when $x=0$. The following lemma shows the spatial stationarity of the processes $\mathfrak{h}_t(\cdot)$ and $\mathfrak{g}_t(\cdot)$.

Lemma 4.2.10 (Stationarity)**.** *The one point distribution of $\mathfrak{h}_t(x) + \frac{x^2}{2}$ is independent of x and converges weakly to Tracy-Widom GUE distribution as $t \uparrow \infty$. On the other hand, the one point distribution of $\mathfrak{g}_t(x) + \frac{(\pi t/4)^{3/4}x^2}{2t}$ is independent of x and converges weakly to standard Gaussian distribution as $t \downarrow 0$.*

Proof. The first part was proved in Proposition 1.7 of [**CH16**]. By Proposition 1.4 of [6], we know $\mathcal{H}^{\mathbf{nw}}(t,z) + z^2/(2t)$ is stationary in z. As a result,

$$\mathfrak{g}_t(x) + \tfrac{(\pi t/4)^{3/4}x^2}{2t} = (\pi t/4)^{-1/4}\left[\mathcal{H}^{\mathbf{nw}}(t,(\pi t/4)^{1/2}x) + \tfrac{(\pi t/4)x^2}{2t} + \log\sqrt{2\pi t}\right]$$

is stationary in x. From Proposition 1.8 in [6], it follows that $\mathfrak{g}_t(0)$ converges weakly to standard Gaussian distribution as $t \downarrow 0$. $\square$

Our next result provides a multipoint composition law of the KPZ temporal process. In latter sections, this will be used to infer properties of multipoint distributions of $\mathfrak{h}_t$. Our proof of the multipoint composition law follow similar strategies as in [117, Proposition 2.9] which proves the two point composition law. For stating the law, we introduce the following notation. For $t > 0$, define a t-indexed composition map $I_t(f, g)$ between two functions $f(\cdot)$ and $g(\cdot)$ as

$$I_t(f,g) := t^{-1/3} \log \int_{-\infty}^{\infty} e^{t^{1/3}\left(f(t^{-2/3}y)+g(-t^{-2/3}y)\right)} dy\,. \tag{4.2.5}$$

Proposition 4.2.11. *For any fixed $t > 0$, $k \in \mathbb{N}$ and $1 < \alpha_1 < \alpha_2 < \ldots < \alpha_k$, there exist independent spatial processes $\mathfrak{h}_{\alpha_1 t\downarrow t}, \mathfrak{h}_{\alpha_2 t\downarrow \alpha_1 t}, \cdots, \mathfrak{h}_{\alpha_k t\downarrow \alpha_{k-1} t}$ supported on the same probability space as the KPZ equation solution such that:*

1. *$\mathfrak{h}_{\alpha_i t\downarrow\alpha_{i-1}t}(\cdot)$ is distributed according to the law of the process $\mathfrak{h}_{\alpha_{i-1}t}((\alpha_i - \alpha_{i-1})/\alpha_{i-1}, \cdot)$;*

2. *$\mathfrak{h}_{\alpha_i t\downarrow\alpha_{i-1}t}(\cdot)$ is independent of $\mathfrak{h}_{\alpha_{i-1}t}(\cdot)$; and*

3. *$\mathfrak{h}_{\alpha_{i-1}t}(\frac{\alpha_i}{\alpha_{i-1}}, 0) = I_{\alpha_{i-1}t}\big(\mathfrak{h}_{\alpha_{i-1}t}, \mathfrak{h}_{\alpha_i t\downarrow\alpha_{i-1}t}\big)$.*

Proof. For $s < t$ and $x, y \in \mathbb{R}$, let $\mathcal{Z}^{\mathbf{nw}}_{s,x}(t, y)$ be the solution at time t and position y of the SHE started at time s with Dirac delta initial data at position x. We will show that for any $0 < t_1 < \ldots < t_k$ and $y_1, \ldots, y_k \in \mathbb{R}$, there exists independent spatial processes $\mathcal{Z}_{y_2}(t_2 \downarrow t_1, \cdot), \ldots, \mathcal{Z}_{y_k}(t_k \downarrow t_{k-1}, \cdot)$ coupled on a probability space upon which the space-time white noise of the KPZ equation is defined such that

$$\mathcal{Z}^{\mathbf{nw}}(t_i, y_i) = \mathcal{Z}^{\mathbf{nw}}_{0,0}(t_i, y_i) = \int_{\mathbb{R}} \mathcal{Z}^{\mathbf{nw}}_{0,0}(t_{i-1}, x)\mathcal{Z}_{y_i}(t_i \downarrow t_i, x)dx\,, \tag{4.2.6}$$

and the law of $\mathcal{Z}_{y_i}(t_i \downarrow t_{i-1}, \cdot)$ is same as that of $\mathcal{Z}^{\mathbf{nw}}(t_i - t_{i-1}, y_i - \cdot)$ for $2 \le i \le k$. Expressing the convolution and interchange properties in terms of $\mathfrak{h}_t$ immediately yields the proposition.

We now return to show (4.2.6). The above convolution formula is known when $k = 2$ (see [117]). We extend the proof given in [117] for $k > 2$ using the chaos series for the SHE (see [105, 316, 4] for background). We write $\vec{s} = (s_1, \ldots, s_\ell) \in \mathbb{R}^\ell_{\geq 0}$, $\vec{x} = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$ and define the set of ordered times

$$\Delta_\ell(s,t) = \{\vec{s} : s \leq s_1 \leq s_2 \leq \ldots \leq s_\ell \leq t\}.$$

For any $0 \leq s < t$ and $x, y \in \mathbb{R}$, $\mathcal{Z}^{\mathbf{nw}}_{s,x}(t,y)$ is given as the following chaos series expansion (see Theorem 2.2 of [105]):

$$\mathcal{Z}^{\mathbf{nw}}_{s,x}(t,y) = \sum_{\ell=0}^{\infty} \int_{\Delta_\ell(s,t)} \int_{\mathbb{R}^\ell} P_{\ell;s,x;t,y}(\vec{s},\vec{x}) d\xi^{\otimes \ell}(\vec{s},\vec{x}). \tag{4.2.7}$$

The integration in (4.2.7) is a multiple Itô stochastic integral against the white noise ξ and the term $P_{\ell;s,x;t,y}(\vec{s},\vec{x})$ is the density function for a one-dimensional Brownian motion starting from (s,x) to go through the time-space points $(s_1,x_1), \ldots, (s_\ell,x_\ell)$ and ends up at (t,y). This transition density has the following product formula using the Gaussian heat kernel $p(s,y) := (2\pi s)^{-1/2} \exp(-y^2/2s)$ and the conventions $s_0 = s$, $s_{\ell+1} = t$, $x_0 = x$ and $x_{\ell+1} = y$:

$$P_{\ell;s,x;t,y}(\vec{s},\vec{x}) = \prod_{j=0}^{\ell} p(s_{i+1} - s_i, x_{i+1} - x_i).$$

For any $0 \leq s < t$, the heat kernel $p(\cdot,\cdot)$ satisfies the simple convolution identity

$$p(t,x) = \int p(s,y) p(t-s, x-y) dy. \tag{4.2.8}$$

Fix $2 \leq i \leq k$. By using the fact that the sum of indicator functions gives the value one, we may replace $\int_{\Delta_\ell(0,t_i)}$ in (4.2.7) by the quantity $\sum_{j=0}^{\ell} \int_{\Delta_\ell(0,t_i)} \mathbf{1}_{s_j \leq t_{i-1} < s_{j+1}}$. As a consequence, we get

$$\mathcal{Z}^{\mathbf{nw}}_{0,0}(t_i,y_i) = \sum_{\ell=0}^{\infty} \sum_{j=0}^{\ell} \int_{\Delta_\ell(s_j,t_i)} \int_{\mathbb{R}^k} \mathbf{1}_{s_j \leq t_{i-1} < s_{j+1}} P_{\ell;0,0;t_i,y_i}(\vec{s},\vec{x}) d\xi^{\otimes \ell}(\vec{s},\vec{x}). \tag{4.2.9}$$

For $1 \leq a \leq b \leq k$, $\vec{s}_{[a,b]}$ denotes $(s_a, \ldots, s_b)$ and likewise for $\vec{x}$. Using these notations and

(4.2.8), we may write

$$\mathbf{1}_{s_j \le t_{i-1} < s_{j+1}} P_{\ell;0,0;t_i,y_i}(\vec{s},\vec{x}) = \mathbf{1}_{s_j \le t_{i-1} < s_{j+1}} \int_{\mathbb{R}} P_{j;0,0;t_{i-1},z}(\vec{s}_{[1,j]},\vec{x}_{[1,j]}) \tag{4.2.10}$$

$$\times P_{\ell-j;t_{i-1},z;t_i,y_i}(\vec{s}_{[j+1,\ell]},\vec{x}_{[j+1,\ell]})dz. \tag{4.2.11}$$

We now insert the above display into (4.2.9). We also replace $\int_{\Delta_\ell(0,t_i)} \mathbf{1}_{s_j \le t_{i-1} < s_{j+1}}$ by the product of the integral $\int_{\Delta_j(0,t_{i-1})} \int_{\Delta_{\ell-j}(t_{i-1},t_i)}$ and relabel $\vec{s}_{[1,j]} = \vec{u}$, $\vec{s}_{[j+1,\ell]} = \vec{v}$, $\vec{x}_{[1,j]} = \vec{a}$, $\vec{x}_{[j+1,\ell]} = \vec{b}$, $P_{j;0,0;t_{i-1},z}(\vec{u},\vec{a}) = P^{0,0}_{j;t_{i-1},z}(\vec{u},\vec{a}) = P_{j;t_{i-1},z}(\vec{u},\vec{a})$ and $P_{\ell-j;t_{i-1},z;t_i,y_i}(\vec{v},\vec{b}) = P^{t_{i-1},z}_{\ell-j;t_i,y_i}(\vec{v},\vec{b})$. Using the fact that the white noise integration can be split since the times range over disjoint intervals, we find

$$\mathcal{Z}^{\mathbf{nw}}_{0,0}(t_i,y_i) = \sum_{\ell=0}^{\infty}\sum_{j=0}^{\ell} \int_{\Delta_j(0,t_{i-1})} \int_{\Delta_{\ell-j}(t_{i-1},t_i)} \int_{\mathbb{R}^i} \int_{\mathbb{R}^{\ell-j}} \int_{\mathbb{R}} P_{j;t_{i-1},z}(\vec{u},\vec{a}) \tag{4.2.12}$$

$$\times P^{t_{i-1},z}_{\ell-j;t_i,y_i}(\vec{v},\vec{b}) dz d\xi^{\otimes j}(\vec{u},\vec{a}) d\xi^{\otimes \ell-j}(\vec{v},\vec{b}). \tag{4.2.13}$$

By the change of variables $m = \ell - j$, the double sum $\sum_{\ell=0}^{\infty}\sum_{j=0}^{\ell}$ can be replaced by $\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}$. We bring the integral in z to the outside. Note that the reordering of integrals and sums is readily justified since all sums are convergent in L^2 (with respect to the probability space on which ξ is defined – see, for example, [105, Theorem 2.2] for details). As a result, we get

$$\mathcal{Z}^{\mathbf{nw}}_{0,0}(t_i,y_i) = \int_{\mathbb{R}} dz \Big(\sum_{j=0}^{\infty} \int_{\Delta_j(0,t_{i-1})} P_{j;t_{i-1},z}(\vec{u},\vec{a}) d\xi^{\otimes j}(\vec{u},\vec{a}) \Big)$$
$$\times \Big(\sum_{m=0}^{\infty} \int_{\Delta_{\ell-j}(t_{i-1},t_i)} P^{t_{i-1},z}_{\ell-j;t_i,y_i}(\vec{v},\vec{b}) d\xi^{\otimes \ell-j}(\vec{v},\vec{b}) \Big).$$

Comparing with (4.2.7), we may now recognize that

$$\mathcal{Z}^{\mathbf{nw}}_{0,0}(t_{i-1},z) = \sum_{j=0}^{\infty} \int_{\Delta_j(0,t_{i-1})} P_{j;t_{i-1},z}(\vec{u},\vec{a}) d\xi^{\otimes j}(\vec{u},\vec{a}),$$

for any $z \in \mathbb{R}$ whereas the stochastic process

$$\mathcal{Z}_{y_i}(t_i \downarrow t_{i-1}, z) := \sum_{m=0}^{\infty} \int_{\Delta_{\ell-j}(t_{i-1},t_i)} P^{t_{i-1},z}_{\ell-j;t_i,y_i}(\vec{v},\vec{b}) d\xi^{\otimes_{\ell-j}}(\vec{v},\vec{b})$$

is same in distribution with $\mathcal{Z}^{\mathbf{nw}}_{t_{i-1},z}(t_i, y_i)$. Furthermore, $\mathcal{Z}^{\mathbf{nw}}_{0,0}(t_{i-1},\cdot)$ and $\mathcal{Z}_{y_i}(t_i \downarrow t_{i-1},\cdot)$ are independent since they are defined with respect to disjoint portions of the space-time white noise. Due to the same reason, $\mathcal{Z}_{y_i}(t_i \downarrow t_{i-1},\cdot)$ and $\mathcal{Z}_{y_j}(t_{j-1} \downarrow t_{j-1},\cdot)$ are independent for any $1 \leq i < j \leq k$. Recall the *interchange* property of the SHE: namely that, for $s < t$ and $y \in \mathbb{R}$ fixed, $\mathcal{Z}^{\mathbf{nw}}_{s,x}(t,y)$ is equal in law as a process in x to $\mathcal{Z}^{\mathbf{nw}}_{s,y}(t,x)$ – the change between the two expressions is in the interchange of x and y. By the interchange property, the spatial process $\mathcal{Z}^{\mathbf{nw}}_{t_{i-1},\cdot}(t_i, y_i)$ has same law as $\mathcal{Z}^{\mathbf{nw}}_{0,0}(t_i - t_{i-1}, y_i - \cdot)$. This completes the proof of (4.2.6). □

In the following result, we collect the one point tail probabilities of the temporal process $\mathfrak{h}_t$ which are proved in [116, 115]. We state the results from [117] which has used same notations as ours. These results hold for any finite time $t > 0$. Since the short time scaling of the KPZ equation has the Gaussian limit, the same tail bounds as in the forthcoming result does not hold as $t \downarrow 0$. The short time tail bounds which are stated in Theorem 4.1.6 and 4.1.7 should be contrasted with the following proposition.

Proposition 4.2.12 (Proposition 2.11 and 2.12 from [117])**.** *For any $t_0 > 0, \varepsilon > 0$, there exist $s_0 = s_0(t_0) > 0$ and $c = c(t_0) > 0$ such that, for $t > t_0$, $s > s_0$ and $x \in \mathbb{R}$,*

$$\mathbb{P}\Big(\Big|\mathfrak{h}_t(x) + \tfrac{x^2}{2}\Big| \geq s\Big) \leq \exp\big(-cs^{3/2}\big). \tag{4.2.14}$$

As one may notice, the constant of the tail bound in the above proposition is left imprecise. For deriving tail bounds of Section 4.4 and 4.5, we do not need precise tail estimates. However, in Section 4.6 and 4.7, we require precise bounds only in the case when the time variable t is large. The following proposition quotes relevant tail bounds from [116, 115, 79, 323] for large t.

Proposition 4.2.13. *Fix $t_0 > 0$ large and $\varepsilon \in (0,1)$. Then, there exist $s_0 = s_0(t_0,\varepsilon) > 0$ and $c = c(t_0,\varepsilon) > 0$ such that, for $t > t_0$, $c(\log t)^{2/3} > s > s_0$ and $x \in \mathbb{R}$,*

$$\exp\big(-\tfrac{4\sqrt{2}}{3}(1+\varepsilon)s^{3/2}\big) \le \mathbb{P}\Big(\mathfrak{h}_t(x) + x^2/2 \ge s\Big) \le \exp\big(-\tfrac{4\sqrt{2}}{3}(1-\varepsilon)s^{3/2}\big), \tag{4.2.15}$$

$$\exp\big(-\tfrac{1}{6}(1+\varepsilon)s^{3}\big) \le \mathbb{P}\Big(\mathfrak{h}_t(x) + x^2/2 \le -s\Big) \le \exp\big(-\tfrac{1}{6}(1-\varepsilon)s^{3}\big). \tag{4.2.16}$$

Proof. Since $\mathfrak{h}_t(x) + \frac{x^2}{2}$ is stationary in x, it suffices to prove (4.2.15) and (4.2.16) for $x = 0$. From the specifications of the upper and lower bounds of the upper tail probabilities in Theorem 1.10 (part (a)) of [115], (4.2.15) follows immediately. It remains to show (4.2.16). Theorem 1.1 of [116] which is recently been strengthened in [79, 323] proves that for any given $\varepsilon, t_0 > 0$, there exists $s_0 = (t_0,\varepsilon) > 0$ such that for all $s \ge s_0$ and $t \ge t_0$,

$$\mathbb{P}\big(\mathfrak{h}_t(0) \le -s\big) \le e^{-\frac{4\sqrt{2}(1-2\varepsilon)}{15}t^{\frac{1}{3}}s^{\frac{5}{2}}} + e^{-Ks^3-\varepsilon s t^{\frac{1}{3}}} + e^{-\frac{(1-2\varepsilon)}{6}s^3} \tag{4.2.17}$$

and,

$$\mathbb{P}\big(\mathfrak{h}_t(0) \le -s\big) \ge e^{-\frac{4\sqrt{2}}{15}(1+\varepsilon)t^{\frac{1}{3}}s^{\frac{5}{2}}} + e^{-\frac{1}{6}(1+\varepsilon)s^3}. \tag{4.2.18}$$

The first inequality of (4.2.16) follows from (4.2.18). Note that $s^{5/2}t^{1/3} \gg s^3$ and $\varepsilon s t^{1/3} \gg s^3$ when we have $(\log t)^{2/3} \gg s$. By choosing s_0 and c large, we may bound the r.h.s. of (4.2.17) by $\exp(-(1-\varepsilon)s^3/6)$ for all $s \ge s_0$ satisfying $c(\log t)^{2/3} > s$. This proves (4.2.16). □

The next result, which is proved in [117], provides tail bounds on the supremum and infimum of the spatial process $\mathfrak{h}_t(\cdot)$ for any fixed time $t > 0$.

Proposition 4.2.14 (Proposition 4.1 and 4.2 from [117])**.** *For any $t_0 > 0$ and $\nu \in (0,1]$, there exist $s_0 = s_0(t_0,\nu) > 0$ and $c = c(t_0,\nu) > 0$ such that, for $t \ge t_0$ and $s > s_0$,*

$$\mathbb{P}(\mathsf{A}) \le \exp\big(-c_2 s^{3/2}\big) \quad \textit{where} \quad \mathsf{A} = \Big\{\sup_{x\in\mathbb{R}}\Big(\mathfrak{h}_t(x) + \tfrac{(1-\nu)x^2}{2}\Big) \ge s\Big\},$$

$$\mathbb{P}(\mathsf{B}) \leq \exp\big(-cs^{5/2}\big) \qquad \textit{where} \quad \mathsf{B} = \Big\{ \inf_{x\in\mathbb{R}} \Big(\mathfrak{h}_t(x) + \tfrac{(1-\nu)x^2}{2} \Big) \leq -s \Big\}.$$

4.3 Short Time Tail Bounds

The main goal of this section is to prove Theorem 4.1.4 and 4.1.7 which describe uniform bounds for the one point tail probabilities of the KPZ height function as time variable t goes to 0. The proof of Theorem 4.1.4 which is given in Section 4.3.1 will use the exact formulas of the integer moments of the SHE. These formulas are put forward by Kardar [**Kardar87**] using the techniques of replica Bethe ansatz. See [179] for a discussion on different approaches to prove those formulas rigorously. On the other hand, the proof of Theorem 4.1.7 which is contained in Section 4.3.2 will be based on core probabilistic aspect like Gaussian concentration.

4.3.1 Upper Tail

Our starting point which is the content of the following proposition is to provide upper bounds to the exponential moments of $\mathfrak{g}_t$. Using these moment estimates, the proof of Theorem 4.1.4 will be completed in the ensuing subsection.

Proposition 4.3.1. *Fix $\varepsilon > 0$. There exist $t_0 = t_0(\varepsilon) > 0$, $C = C(\varepsilon) > 0$, and $s_0 = s_0(\varepsilon) > 0$, such that for all $t \leq t_0$, $s \geq s_0$ and $k := \lfloor s(\pi t/4)^{-1/4} \rfloor$ we have*

$$\mathbf{E}\big[\exp\big(k(\pi t/4)^{1/4}\mathfrak{g}_{2t}\big)\big] \leq \exp\big(C(s^3 t^{1/4-4\varepsilon} + s^2)\big). \tag{4.3.1}$$

Proof. For any positive integer k, we recall the k-moment formula for $\mathcal{Z}^{\mathbf{nw}}(2t,0)$ (see [68, 179])

$$\mathbf{E}\left[\mathcal{Z}^{\mathbf{nw}}(2t,0)^k e^{\frac{kt}{12}}\right] = \sum_{\lambda\vdash k} \frac{k!}{\prod m_j!} \prod_{i=1}^{\ell(\lambda)} \frac{e^{\frac{t\lambda_i^3}{12}}}{2\pi} \int_{\mathbb{R}^{\ell(\lambda)}} \prod_{i=1}^{\ell(\lambda)} \frac{dz_i e^{-t^{\frac{1}{3}}\lambda_i z_i^2}}{t^{\frac{1}{3}}\lambda_i} \prod_{i<j}^{\ell(\lambda)} \frac{\frac{t^{\frac{2}{3}}(\lambda_i-\lambda_j)^2}{4} + (z_i-z_j)^2}{\frac{t^{\frac{2}{3}}(\lambda_i+\lambda_j)^2}{4} + (z_i-z_j)^2}.$$

Here the sum $\lambda \vdash k$ is over all positive integer partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ of k where $\lambda_1 \geq \lambda_2 \geq \ldots$. Furthermore, $m_i = m_i(\lambda) := |\{j \mid \lambda_j = i\}|$ and $\ell(\lambda) := |\{j \mid \lambda_j > 0\}|$. Note that each terms of the product inside the integral is less than 1. Bounding those terms by 1 and evaluating the left over Gaussian integral, we have

$$\mathbf{E}\left[\mathcal{Z}^{\mathbf{nw}}(2t,0)^k e^{\frac{kt}{12}}\right] \leq \sum_{\lambda \vdash k} \frac{k!}{\prod m_j!} \prod_{i=1}^{\ell(\lambda)} \frac{e^{\frac{t\lambda_i^3}{12}}}{2\pi} \prod_{i=1}^{\ell(\lambda)} \frac{\sqrt{\pi}}{t^{\frac{1}{2}}\lambda_i^{3/2}} \leq \sum_{\lambda \vdash k} \frac{k! e^{\frac{tk^3}{12}}}{(4\pi t)^{\frac{\ell(\lambda)}{2}} \prod m_j!}.$$

The last inequality in above equation follows by using $\lambda_i^{3/2} \geq 1$ and $\sum_i \lambda_i^3 \leq k^3$. Expressing the left hand side of the above display in terms of $\mathfrak{g}_{2t}$ we get

$$\mathbf{E}e^{k(\pi t/2)^{1/4}\mathfrak{g}_{2t}} = \mathbf{E}\left[(\mathcal{Z}^{\mathbf{nw}}(2t,0)\sqrt{4\pi t})^k\right] \leq e^{\frac{tk^3-tk}{12}} \sum_{\lambda \vdash k} (4\pi t)^{\frac{k-\ell(\lambda)}{2}} \frac{k!}{\prod m_j!}. \tag{4.3.2}$$

We choose t_0 and s_0 such that $2^{5/2}t_0^{\varepsilon}(\pi/2)^{1/4} \leq \frac{1}{2}$ and $s_0 \geq 2(\pi t_0/2)^{1/4}$. Then for all $t \leq t_0$ and $s \geq s_0$, we set $k = k(t) := \lfloor s(\pi t/2)^{-1/4} \rfloor$. By the condition on t_0, s_0 and k, we always have $k \geq 2$. We further have $k \leq s(\pi t/2)^{-1/4}$ which implies $t \leq \frac{2s^4}{\pi k^4}$. Bounding t with this inequality, combining it with the estimate $k! \leq k^{k-m_1}m_1!$ and using those in the r.h.s. of (4.3.2) yields

$$\mathbf{E}e^{k(\pi t/2)^{1/4}\mathfrak{g}_{2t}} \leq e^{\frac{s^4}{6\pi k}} \sum_{\lambda \vdash k} \left(2^{3/4}s\right)^{2k-2\ell(\lambda)} \frac{k^{k-m_1}k^{2\ell(\lambda)-2k}}{\prod\limits_{j\geq 2} m_j!}. \tag{4.3.3}$$

Throughout the rest, we provide bound for the r.h.s. of (4.3.3). We separate our analysis into three cases depending on the location of s.

Case-1. $s \leq t^{-1/4+\varepsilon}$. Observe that $k-\ell(\lambda) = \sum_{j\geq 2}(j-1)m_j$ and $2\ell(\lambda)-m_1-k = -\sum_{j\geq 3}(j-2)m_j$. We extend the range of $m_2, m_3, m_4, \ldots$ over all non-negative integers in (4.3.3). Taking first the sum w.r.t. m_2 shows

$$\text{r.h.s. of (4.3.3)} \leq e^{\frac{s^4}{6\pi k}} \sum_{j=3}^{\infty} \sum_{m_j=0}^{\infty} \left[2^{\frac{3}{2}}s^2\right]^{\sum\limits_{j\geq 3}(j-1)m_j} \frac{k^{-\sum\limits_{j\geq 3}(j-2)m_j}}{\prod\limits_{j\geq 3} m_j!} \sum_{m_2=0}^{\infty} \frac{(2^{\frac{3}{2}}s^2)^{m_2}}{m_2!}. \tag{4.3.4}$$

Note that the inner sum w.r.t. m_2 is equal to $\exp(2^{3/2}s^2)$. We may now write the r.h.s. of the above display as

$$e^{\frac{s^4}{6\pi k}} e^{2^{\frac{3}{2}} s^2} \prod_{j=3}^{\infty} \sum_{m_j=0}^{\infty} \left[2^{\frac{3}{2}} s^2\right]^{(j-1)m_j} \frac{k^{-(j-2)m_j}}{m_j!} = \exp\left(\tfrac{s^4}{6\pi k} + 2^{\frac{3}{2}} s^2 + \tfrac{2^3 s^4 k^{-1}}{1-2^{\frac{3}{2}} s^2 k^{-1}}\right),$$

where the equality is obtained by taking sum w.r.t. $m_3, m_4, \ldots$ separately and simplifying the product. With this equality, we get

$$\text{r.h.s. of (4.3.4)} \le \exp\left(\tfrac{2s^3(\pi t/2)^{1/4}}{6\pi} + 2^{\frac{3}{2}} s^2 + \tfrac{2^4 s^3 (\pi t/2)^{1/4}}{1-2^{5/2} s (\pi t/2)^{1/4}}\right) \le \exp\left(C s^3 t^{1/4} + C s^2\right),$$

where the last inequality is obtained by using the facts $k^{-1} \le 2s^{-1}(\pi t/2)^{1/4}$, $s \le t^{-1/4+\varepsilon}$ and $t \le t_0$ with $2^{5/2} t_0^{\varepsilon} (\pi/2)^{1/4} \le \frac{1}{2}$. This proves (4.3.1) for $s \le t^{-1/4+\varepsilon}$.

Case-2. $s \ge t^{-1/4-\varepsilon}$. We assume $t_0 \le \frac{1}{4\pi}$. Recall the definition of k. Since $k+1 \ge s/(\pi t/2)^{1/4}$, we may bound $s^{2k-2\ell(\lambda)}$ by $(\pi t/2)^{(k-\ell(\lambda))/2}(k+1)^{2k-2\ell(\lambda)}$. Combining this with the facts $k! \le k^k$ and $\prod m_j! \ge 1$, we get

$$\text{r.h.s. of (4.3.2)} \le e^{\frac{s^4}{6\pi k}} \sum_{\lambda \vdash k} (4\pi t)^{\frac{k-\ell(\lambda)}{2}} k^{k-m_1} \cdot k! \big(1 + \tfrac{1}{k}\big)^{2(k-\ell(\lambda))} \le e^{\frac{s^4}{6\pi k}} k^{2k},$$

where we bound $(1+1/k)^{2(k-\ell(\lambda))}$ by 1 and the number of partitions of k by k^k to get the last inequality. Since we are in the case $s \ge t^{-1/4-\varepsilon}$, we have $s^4 k^{-1} \le s^3 (\pi t/2)^{1/4}$ and $k \ln k \le c s t^{-1/4} \ln(s t^{-1/4}) \le c s^3 t^{1/4}$. Due to these inequalities, the r.h.s. of the above display is bounded by $\exp(c s^3 t^{1/4})$ for some constant $c > 0$. Combining this with (4.3.3) shows $\mathbf{E}[\exp(k(\pi t/2)^{1/4} \mathfrak{g}_{2t})] \le \exp(c s^3 t^{1/4})$.

Case-3. $t^{-1/4+\varepsilon} \le s \le t^{-1/4-\varepsilon}$. Define $\tilde{s} = t^{-1/4-\varepsilon}$ and $\tilde{k} := \lfloor \tilde{s} (\pi t/2)^{-1/4} \rfloor$. Note that $k \le \tilde{k}$ since $s \le t^{-1/4-\varepsilon}$. Using the Hölder's inequality, we know $\mathbf{E} \exp(k(\pi t/2)^{1/4} \mathfrak{g}_{2t})$ is bounded by $\big(\mathbf{E} \exp(\tilde{k}(\pi t/2)^{1/4} \mathfrak{g}_{2t})\big)^{k/\tilde{k}}$. By **Case-2**, we know $\mathbf{E} \exp(\tilde{k}(\pi t/2)^{1/4} \mathfrak{g}_{2t}) \le \exp(c \tilde{s}^3 t^{1/4})$ for all

$t \leq t_0 = \frac{1}{4\pi}$. Combining these observations shows

$$\mathbf{E}\exp(k(\tfrac{\pi t}{2})^{1/4}\mathfrak{g}_{2t}) \leq \exp(ck\tilde{s}^3 t^{1/4}/\tilde{k}) \leq \exp(ct^{-3/4-3\varepsilon}t^{1/4}st^{1/4+\varepsilon}) = \exp(cst^{-1/4-2\varepsilon}),$$

where the second inequality follows from the definition of $\tilde{k}$ and $\tilde{s}$. Since $s \geq t^{-1/4+\varepsilon}$, the last term of the above display is bounded by $\exp(cs^3 t^{1/4-4\varepsilon})$. This completes the proof for **Case-3**.

Combining all cases we get (4.3.1). This completes the proof. □

Proof of Theorem 4.1.4

We introduce the notations $f_{t,s} := \frac{1}{C+\sqrt{C^2+3Cst^{1/4-4\varepsilon}}}$, $\tilde{s} := sf_{t,s}$ and $\tilde{k} := \lfloor \tilde{s}(\pi t/2)^{-1/4} \rfloor$ where the constant C is same as in (4.3.1). By Markov's inequality,

$$\begin{aligned}\mathbf{P}(\mathfrak{g}_{2t} \geq s) = \mathbf{P}\left(e^{\tilde{k}(\pi t/2)^{1/4}\mathfrak{g}_{2t}} \geq e^{\tilde{k}s(\pi t/2)^{1/4}}\right) &\leq \exp(-\tilde{k}s(\tfrac{\pi t}{2})^{\frac{1}{4}})\mathbb{E}\big[\exp(\tilde{k}(\tfrac{\pi t}{2})^{\frac{1}{4}}\mathfrak{g}_{2t})\big] \\ &\leq \exp\left(Cs^3 f_{t,s}^3 t^{\frac{1}{4}-4\varepsilon} + Cs^2 f_{t,s}^2 - \tilde{k}s(\tfrac{\pi t}{2})^{\frac{1}{4}}\right), \quad (4.3.5)\end{aligned}$$

where the last inequality follows from Proposition 4.3.1. We choose s_0 large enough such that for all $s \geq s_0$ and $t \leq t_0$ we have $\tilde{k}s(\pi t/2)^{1/4} \geq \frac{11}{12}s^2 f_{t,s}$. From the definition of $f_{t,s}$, it follows $Cf_{t,s} \leq C\frac{1}{2C} = \frac{1}{2}$, and $Cst^{1/4-4\varepsilon}f_{t,s}^2 \leq Cst^{1/4-4\varepsilon}\frac{1}{3Cst^{1/4-4\varepsilon}} = \frac{1}{3}$. Plugging all these inequalities in the right side of (4.3.5) yields

$$\mathbf{P}(\mathfrak{g}_{2t} \geq s) \leq \exp\left(-\tfrac{s^2 f_{t,s}}{12}\right) \leq \exp\left(-\tfrac{C's^2}{1+\sqrt{1+st^{1/4-4\varepsilon}}}\right)$$

for all $t \leq t_0$, $s \geq s_0$ and some constant $C' > 0$. This completes the proof.

4.3.2 Lower Tail

Our proof of Theorem 4.1.7 will utilize ideas from [168]. In [168], the author provided an upper bound to the lower tail probability of $\mathcal{H}^{\mathbf{nw}}$. However, it was not clear whether the same bound holds for $\mathfrak{g}_t$, i.e., centering $\mathcal{H}^{\mathbf{nw}}$ with $\log\sqrt{2\pi t}$ and scaling by $(\pi t/4)^{1/4}$. Our analysis will demonstrate that it is indeed possible to derive similar tail bound for $\mathfrak{g}_t$. The main tool of our proof of Theorem 4.1.7 are some properties of the directed random polymer partition functions and its convergence to the solution of the SHE. Below, we introduce relevant notations.

Let $\Xi := \{\mathcal{E}(i,x) : i \in \mathbb{N}, x \in \mathbb{Z}\}$ be a collection of independent standard normal random variables. We call such collections as *lattice environment*. Let $\{S_i\}_{i\geq 0}$ be a simple symmetric random walk on $\mathbb{Z}$ starting at $S_0 = 0$ independent of Ξ. Denote the law of $\{S_i\}_{i\geq 0}$ by $\mathbb{P}_S$. At inverse temperature $\beta > 0$, the directed polymer partition function $Z_n^{(\Xi)}(\beta)$ is defined as

$$Z_n^{(\Xi)}(\beta) := \mathbf{E}_S\left[\exp\left\{\beta\sum_{i=1}^{n}\mathcal{E}(i,S_i)\right\}\mathbf{1}_{S_n=0}\right]$$

where the expectation $\mathbb{E}_S$ is taken w.r.t. $\mathbb{P}_S$. Define $\beta_n := (t/2n)^{1/4}$. From [5], we know as $n \to \infty$

$$\frac{1}{(\pi t/2)^{1/4}}\left[\log\sqrt{\frac{n\pi}{2}} + \log\frac{Z_n^{(\Xi)}(\beta_n)}{\mathbf{E}Z_n^{(\Xi)}(\beta_n)}\right] \xrightarrow{d} \mathfrak{g}_t, \quad \text{for each } t > 0. \tag{4.3.6}$$

To complete the proof of Theorem 4.1.7, we need the following two lemmas. Lemma 4.3.2 is originally from a part of the proof of Theorem 1.5 in [84].

Lemma 4.3.2 (Lemma 1 of [168]). *Let Ξ and Ξ' be two independent lattice environments. Let $S^{(1)}$ and $S^{(2)}$ be two independent simple symmetric random walks starting at origin. Denote the expectation w.r.t. the joint law of $S^{(1)}$ and $S^{(2)}$ by $\mathbf{E}_{S^{(1)},S^{(2)}}$. Then we have*

$$\log Z_n^{(\Xi)}(\beta) \geq \log Z_n^{(\Xi')}(\beta) - \beta d_n(\Xi,\Xi')\sqrt{\mathfrak{D}_{\Xi'}(S^{(1)},S^{(2)})},$$

where $d_n(\Xi,\Xi')^2 := \sum_{i=1}^{n}\sum_{|x|\leq i}(\mathcal{E}(i,x)-\mathcal{E}'(i,x))^2$ and

$$\mathfrak{D}_{\Xi'}(S^{(1)},S^{(2)}) := \frac{1}{Z_n^{(\Xi')}(\beta)^2}\mathbf{E}_{S^{(1)},S^{(2)}}\left[\sum_{i=1}^{n}\mathbf{1}_{S_i^{(1)}=S_i^{(2)}}e^{\beta\sum_{i=1}^{n}(\mathcal{E}'(i,S_i^{(1)})+\mathcal{E}'(i,S_i^{(2)}))}\mathbf{1}_{S_n^{(1)}=S_n^{(2)}=0}\right].$$

The next lemma is similar to Lemma 2 of [168]. To state the lemma, we introduce for any

$n \in \mathbb{N}$, $t > 0$ and $C > 0$

$$A_{n,t,C} := \left\{ \Xi' : Z_n^{(\Xi')}(\beta_n) \geq \sqrt{\tfrac{2}{n\pi}} \mathbf{E} Z_n^{(\Xi)}(\beta_n), \mathfrak{D}_{\Xi'}(S^{(1)}, S^{(2)}) \leq C\sqrt{n} \right\}.$$

Lemma 4.3.3. *For any given $\varepsilon > 0$, there exist constants $t_0 = t_0(\varepsilon) \in (0, 2]$ and $C = C(\varepsilon) > 0$ satisfying the following: for any $t \leq t_0$, there exists $n_t \in \mathbb{N}$ such that for all $n \geq n_t$, we have $\mathbf{P}(A_{n,t,C}) \geq \frac{1}{2} - \varepsilon$.*

Our proof of the above lemma uses some of the ideas from the proof of Lemma 2 of [168]. However, there is a major difference between these two results. Unlike Lemma 2 of [168], Lemma 4.3.3 provides a lower bound to $\mathbf{P}(A_{n,t,C})$ which does not depend on t. On the other hand, the lower bound of Lemma 2 of [168] is valid for all $n \geq 1$ which is not the case in Lemma 4.3.3. Since we are interested in the evolution of tail probabilities of $Z_n^{(\Xi)}((t/2n)^{1/4})$ as n grows large, the probability bound of $A_{n,t,C}$ for large n is more relevant to our analysis than a uniform bound for all $n \geq 1$. Furthermore, the independence of the lower bound of $\mathbf{P}(A_{n,t,C})$ from t enables us in Theorem 4.1.7 to derive bounds on the lower tail probability of $\mathfrak{g}_t$ uniform in t. Before proceeding to the proof of Lemma 4.3.3, we will show Theorem 4.1.7 by assuming Lemma 4.3.3.

Proof of Theorem 4.1.7

Fix $\varepsilon \in (0, \frac{1}{2})$. We choose $t_0 = t_0(\varepsilon) \in (0, 2]$ as defined in Lemma 4.3.3. Fix $t \leq t_0$. From Lemma 4.3.3 we pick $C > 0$ and $n_t \in \mathbb{N}$ such that for all $n \geq n_t$, $\mathbf{P}(A_{n,t,C}) \geq \frac{1}{4}$. Fix $n \geq n_t$. Consider any $\Xi' \in A_{n,t,C}$. By Lemma 4.3.2, we have

$$\begin{aligned}
\log Z_n^{(\Xi)}(\beta_n) &\geq \log Z_n^{(\Xi')}(\beta_n) - \beta_n d_n(\Xi, \Xi')\sqrt{\mathfrak{D}_{\Xi'}(S^{(1)}, S^{(2)})} \\
&\geq \log \sqrt{\tfrac{2}{n\pi}} + \log \mathbf{E} Z_n^{(\Xi)}(\beta_n) - \beta_n d_n(\Xi, \Xi')\sqrt{C n^{1/2}},
\end{aligned}$$

where the second inequality follows since $\Xi' \in A_{n,t,C}$. Rearranging the above inequality and using the fact that it holds for any $\Xi' \in A_{n,t,C}$ shows

$$\frac{1}{(\frac{\pi t}{2})^{\frac{1}{4}}}\left[\log\sqrt{\frac{n\pi}{2}}+\log\frac{Z_n^{(\Xi)}(\beta_n)}{\mathbf{E}Z_n^{(\Xi)}(\beta_n)}\right] \geq -\frac{\sqrt{C}}{\pi^{\frac{1}{4}}}\inf_{\Xi'\in A_{n,t,C}} d_n(\Xi,\Xi').$$

Thus, for all $s > 0$,

$$\mathbf{P}\left(\frac{1}{(\frac{\pi t}{2})^{\frac{1}{4}}}\left[\log\sqrt{\frac{n\pi}{2}}+\log\frac{Z_n^{(\Xi)}(\beta_n)}{\mathbf{E}Z_n^{(\Xi)}(\beta_n)}\right] \leq -s\right) \leq \mathbf{P}\left(d_n(\Xi, A_{n,t,C}) \geq \frac{s\pi^{\frac{1}{4}}}{\sqrt{C}}\right), \tag{4.3.7}$$

where $d_n(\Xi, A_{n,t,C}) := \inf_{\Xi'\in A_{n,t,C}} d_n(\Xi,\Xi')$. Since $\mathbf{P}(A_{n,t,C}) \geq \frac{1}{2}-\varepsilon$, applying Theorem 3 of [168] (Talagrand's inequality) shows $\mathbf{P}(d_n(\Xi, A_{n,t,C}) \geq u+\sqrt{4\log 2}) \leq e^{-u^2/2}$. Applying this probability bound into the r.h.s. of the above display yields

$$\text{r.h.s. of (4.3.7)} \leq \exp\left(-\frac{1}{2}\left\{s\pi^{-1/4}C^{-1/2}-\sqrt{4\log 2}\right\}^2\right) \leq e^{-cs^2} \tag{4.3.8}$$

for some positive constant $c > 0$ and for all $s \geq s_0$ where neither s_0 nor c does depend on n or t. Due to the weak convergence of (4.3.6), we have

$$\mathbf{P}\left(\frac{1}{(\pi t/2)^{1/4}}\left[\log\sqrt{\frac{n\pi}{2}}+\log\frac{Z_n^{(\Xi)}(\beta_n)}{\mathbf{E}Z_n^{(\Xi)}(\beta_n)}\right] \leq -s\right) \overset{n\to\infty}{\to} \mathbf{P}(\mathfrak{g}_t \leq -s).$$

Combining this convergence with (4.3.7) and (4.3.8) shows the desired conclusion.

Proof of Lemma 4.3.3

Recall $\beta_n = (t/2n)^{1/4}$. By Proposition 1.8 of [6], $\mathfrak{g}_t$ converges weakly to the standard Gaussian distribution implying $\lim_{t\to 0}\mathbf{P}(\mathfrak{g}_t \geq 0) = \frac{1}{2}$. We choose the largest $t_0 = t_0(\varepsilon) \in (0,2]$ such that $\mathbf{P}(\mathfrak{g}_t \geq 0) \geq \frac{1}{2} - \frac{\varepsilon}{2}$ for all $t \leq t_0$. For simplicity in notations, we set

$$\mathfrak{L}_n := \sum_{i=1}^{n} \mathbf{1}_{S_i^{(1)}=S_i^{(2)}} \cdot \mathbf{1}_{S_n^{(1)}=S_n^{(2)}=0} \cdot e^{\beta_n \sum_{i=1}^n(\mathcal{E}'(i,S_i^{(1)})+\mathcal{E}'(i,S_i^{(2)}))}, \quad L_n = \sum_{i=1}^{n} \mathbf{1}_{S_i^{(1)}=S_i^{(2)}}.$$

Recall that $\mathfrak{O}_{\Xi'}(S^{(1)}, S^{(2)})$ is equal to $\mathbb{E}_{S^{(1)},S^{(2)}}[\mathfrak{L}_n]/(Z_n^{(\Xi')}(\beta))^2$. By simple probability bounds, we get

$$\begin{aligned}\mathbf{P}(A_{n,t,C}) &\geq \mathbf{P}\left(Z_n^{(\Xi')}(\beta_n) \geq \sqrt{\tfrac{2}{n\pi}}\mathbf{E}Z_n^{(\Xi)}(\beta_n), \mathbf{E}_{S^{(1)}S^{(2)}}(\mathfrak{L}_n) \leq \tfrac{2C}{\sqrt{n\pi^2}}(\mathbf{E}Z_n^{(\Xi)}(\beta_n))^2\right)\\ &\geq \mathbf{P}\left(\tfrac{Z_n^{(\Xi')}(\beta_n)}{\mathbf{E}Z_n^{(\Xi)}(\beta_n)} \geq \sqrt{\tfrac{2}{n\pi}}\right) - \mathbf{P}\left(\tfrac{\mathbf{E}_{S^{(1)}S^{(2)}}(\mathfrak{L}_n)}{(\mathbf{E}Z_n^{(\Xi)}(\beta_n))^2} > \tfrac{2C}{\sqrt{n\pi^2}}\right).\end{aligned} \tag{4.3.9}$$

We claim that for any $t \leq t_0$, there exists $n_t \in \mathbb{N}$ such that for all $n \geq n_t$,

$$\mathbf{P}\left(\tfrac{Z_n^{(\Xi')}(\beta_n)}{\mathbf{E}Z_n^{(\Xi)}(\beta_n)} \geq \sqrt{\tfrac{2}{n\pi}}\right) \geq \tfrac{1}{2} - \tfrac{3\varepsilon}{4}, \quad \mathbf{P}\left(\tfrac{\mathbf{E}_{S^{(1)}S^{(2)}}(\mathfrak{L}_n)}{(\mathbf{E}Z_n^{(\Xi)}(\beta_n))^2} > \tfrac{2C}{\sqrt{n\pi^2}}\right) \leq \tfrac{\varepsilon}{4}. \tag{4.3.10}$$

Substituting the above inequalities into the r.h.s. of (4.3.9) completes the proof of Lemma 4.3.3. Thus, it suffices to show that the above inequalities hold for all large n. To see the first inequality of (4.3.10), we first note that $\mathbf{E}Z_n^{(\Xi)}(\beta_n) = \mathbf{E}Z_n^{(\Xi')}(\beta_n)$ and write

$$\mathbf{P}\left(\frac{Z_n^{(\Xi')}(\beta_n)}{\mathbf{E}Z_n^{(\Xi')}(\beta_n)} \geq \sqrt{\frac{2}{n\pi}}\right) = \mathbf{P}\left(\frac{1}{(\frac{\pi t}{2})^{\frac{1}{4}}}\left[\log\sqrt{\frac{n\pi}{2}} + \log\frac{Z_n^{(\Xi')}(\beta_n)}{\mathbf{E}Z_n^{(\Xi')}(\beta_n)}\right] \geq 0\right).$$

By the weak convergence in (4.3.6) and $\mathbb{P}(\mathfrak{g}_t \geq 0) \geq \frac{1}{2} - \frac{\varepsilon}{2}$, it follows that the right side of the above display is greater than $\frac{1}{2} - \frac{3\epsilon}{4}$ for all large n. This proves the first inequality of (4.3.10). For the second inequality, note that $\mathbf{E}Z_n^{(\Xi)}(\beta_n) = e^{n\beta_n^2/2}$. By Fubini, we have

$$\begin{aligned}\mathbf{E}_\Xi\mathbf{E}_{S^{(1)}S^{(2)}}[\mathfrak{L}_n] &= \mathbf{E}_{S^{(1)}S^{(2)}}\Big[\sum_{i=1}^n \mathbf{1}_{S_i^{(1)}=S_i^{(2)}} \cdot \mathbf{1}_{S_n^{(1)}=S_n^{(2)}=0} \cdot \prod_{j=1}^n \mathbf{E}_\Xi\Big(e^{\beta_n(\mathcal{E}(j,S_j^{(1)})+\mathcal{E}(j,S_j^{(2)}))}\Big)\Big]\\ &= e^{n\beta_n^2}\mathbf{E}_{S^{(1)}S^{(2)}}\Big[\sum_{i=1}^n \mathbf{1}_{S_i^{(1)}=S_i^{(2)}} \cdot \mathbf{1}_{S_n^{(1)}=S_n^{(2)}=0} \cdot \exp\big(\beta_n^2\sum_{i=1}^n \mathbf{1}_{S_i^{(1)}=S_i^{(2)}}\big)\Big]\\ &= (\mathbf{E}Z_n^{(\Xi)}(\beta)_n)^2\mathbf{E}_{S^{(1)}S^{(2)}}\big[L_n e^{\beta_n^2 L_n}\mathbf{1}_{S_n^{(1)}=S_n^{(2)}=0}\big].\end{aligned}$$

Applying Markov's inequality and using the above expression of $\mathbf{E}_\Xi\mathbf{E}_{S^{(1)}S^{(2)}}[\mathfrak{L}_n]$ shows

$$\mathbf{P}\left(\tfrac{\mathbf{E}_{S^{(1)}S^{(2)}}(\mathfrak{L}_n)}{(\mathbf{E}Z_n^{(\Xi)}(\beta_n))^2} > \tfrac{2C}{\sqrt{n\pi^2}}\right) \leq \tfrac{\sqrt{n\pi^2}}{2C}\mathbf{E}_{S^{(1)}S^{(2)}}\left[L_n e^{\beta_n^2 L_n}\mathbf{1}_{S_n^{(1)}=S_n^{(2)}=0}\right]$$

$$\leq \frac{\sqrt{n\pi^2}}{2C}\mathbf{P}(S_n^{(1)} = S_n^{(2)} = 0)\mathbf{E}_{S^{(1)}S^{(2)}}\left[L_n e^{\beta_n^2 L_n} \mid S_n^{(1)} = S_n^{(2)} = 0\right].$$

By Stirling's approximation, there exists constant $a > 0$ such that $\mathbf{P}(S_n^{(1)} = S_n^{(2)} = 0) = \frac{1}{2^{2n}}\binom{n}{n/2}^2 \leq \frac{a}{n}$ for all n. Since $\beta_n = (t/2n)^{1/4}$, we have $L_n e^{\beta_n^2 L_n} = L_n e^{(t/2n)^{1/2} L_n} \leq L_n e^{n^{-1/2} L_n}$ for all $t \leq t_0 \leq 2$. Furthermore, Lemma 3 in [168] proves

$$\sup_{n\geq 1} n^{-\frac{1}{2}}\mathbf{E}_{S^{(1)}S^{(2)}}\left[L_n e^{n^{-1/2}L_n} \mid S_n^{(1)} = S_n^{(2)} = 0\right] = K < \infty.$$

Thus for all $t \leq t_0$ we have a constant $K' > 0$ (free of t) so that $\mathbf{P}\left(\frac{\mathbf{E}_{S^{(1)}S^{(2)}}(\mathfrak{L}_n)}{(\mathbf{E}Z_n^{(\Xi)}(\beta_n))^2} > \frac{2C}{\sqrt{n\pi^2}}\right) \leq \frac{K'}{C}$. Taking C large shows the second inequality of (4.3.10) for all large n.

4.4 Tail Bounds of the KPZ Spatial Process

In this section, we prove delicate tail bounds on several events of the long and short time spatial processes $\mathfrak{h}_t(\cdot)$ and $\mathfrak{g}_t(\cdot)$ respectively. Four propositions will be proved in this section; two of them are about the supremum and the infimum of the spatial process $\mathfrak{h}_t$ and other two are devoted on similar results about $\mathfrak{g}_t$. One may notice similarities between Proposition 4.4.1, 4.4.2 and Theorem 1.3 of [117] since both bound the tail probabilities of the supremum and/or infimum of the KPZ height differences between spatial points. However, in comparison to [117, Theorem 1.3], the bounds in Proposition 4.4.1 and 4.4.2 improve on multiple aspects (e.g., decay exponents) which turn out to be extremely useful for proving the results of Section 4.5. The main ingredients of the proofs of this sections are: (1) tail bounds from Section 4.3 and (2) Brownian Gibbs property of the line ensemble discussed in Section 4.2. From this time forth, we will denote complement of any set B by $\neg\mathsf{B}$.

Proposition 4.4.1. *Fix $\kappa > 0$ and $\alpha \in [\frac{3}{2}, 2]$. There exist constant $c > 0$, $t_0 > 0$ such that for all $t \geq t_0$, $\beta \in (0, 1]$ and $s \geq s_0(t_0)$ we have*

$$\mathbf{P}\Big(\inf_{|y|\leq \beta^{2\kappa}s^{2-\alpha}}(\mathfrak{h}_t(y) - \mathfrak{h}_t(0)) \leq -\tfrac{7}{8}\beta^{\kappa}s\Big) \leq e^{-cs^{\alpha}}. \qquad (4.4.1)$$

Proof. Let us define

$$\mathsf{A} := \left\{ \inf_{y\in[0,\beta^{2\kappa}s^{2-\alpha}]} (\mathfrak{h}_t(y) - \mathfrak{h}_t(0)) \le -\tfrac{7}{8}\beta^{\kappa} s \right\}, \quad \mathsf{B} := \left\{ \mathfrak{h}_t(\beta^{\kappa}s^{1-\frac{\alpha}{3}}) - \mathfrak{h}_t(0) \le -\tfrac{3s^{2\alpha/3}}{4} \right\}.$$

Observe that $\mathbf{P}(\mathsf{A}) \le \mathbf{P}(\mathsf{A}\cap\neg\mathsf{B}) + \mathbf{P}(\mathsf{B})$. In what follows, we show that there exists $s_0 = s_0(t_0)$, $c > 0$ such that for all $s \ge s_0$ and $t > t_0$,

$$\mathbf{P}(\mathsf{B}) \le \exp(-cs^{\alpha}), \qquad \mathbf{P}(\mathsf{A}\cap\neg\mathsf{B}) \le \exp(-cs^{\alpha}). \tag{4.4.2}$$

(4.4.2) will bound $\mathbf{P}(\mathsf{A})$. By repeating the same argument for the interval $[-\beta^{2\kappa}s^{2-\alpha}, 0]$, one can show $\mathbb{P}\Big(\inf_{y\in[-\beta^{2\kappa}s^{2-\alpha},0]}(\mathfrak{h}_t(y) - \mathfrak{h}_t(0)) \le -\frac{7}{8}\beta^{\kappa}s\Big) \le e^{-cs^{\alpha}}$. This will complete the proof of this proposition. Throughout the rest, we prove (4.4.2). For the first one, note that B is contained in the union of $\{\mathfrak{h}_t(\beta^{\kappa}s^{1-\alpha/3}) \le -5s^{2\alpha/3}/8\}$ and $\{\mathfrak{h}_t(0) \ge s^{2\alpha/3}/8\}$. By the union bound,

$$\mathbf{P}(\mathsf{B}) \le \mathbf{P}\Big(\mathfrak{h}_t(\beta^{\kappa}s^{1-\frac{\alpha}{3}}) + \tfrac{\beta^{2\kappa}s^{2-\frac{2\alpha}{3}}}{2} \le -\tfrac{5s^{2\alpha/3}}{8} + \tfrac{\beta^{2\kappa}s^{2-\frac{2\alpha}{3}}}{2}\Big) + \mathbf{P}\Big(\mathfrak{h}_t(0) \ge \tfrac{s^{2\alpha/3}}{8}\Big). \tag{4.4.3}$$

Due to the stationarity, $\mathfrak{h}_t(\beta^{\kappa}s^{1-\frac{\alpha}{3}}) + \beta^{2\kappa}s^{2-\frac{2\alpha}{3}}/2$ is same in distribution with $\mathfrak{h}_t(0)$. Furthermore we have the inequality $-5s^{2\alpha/3}8+\beta^{2\kappa}s^{2-\frac{2\alpha}{3}}/2 \le -s^{2\alpha/3}/8$ because $\alpha \ge 3/2$ and $\beta \le 1$. Combining we get

$$\text{r.h.s. of (4.4.3)} \le \mathbf{P}\Big(\mathfrak{h}_t(0) \le -s^{2\alpha/3}/8\Big) + \mathbf{P}\Big(\mathfrak{h}_t(0) \ge s^{2\alpha/3}/8\Big).$$

Using Proposition 4.2.12 we bound $\mathbf{P}(\mathfrak{h}_t(0) \le -s^{2\alpha/3}/8)$ and $\mathbf{P}(\mathfrak{h}_t(0) \ge s^{2\alpha/3}/8)$ by $\exp(-cs^{\alpha})$ for some constant $c > 0$. Substituting these bound into the right side of the above display yields $\mathbf{P}(\mathsf{B}) \le 2\exp(-cs^{\alpha})$.

For the second inequality in (4.4.2) we use the Brownian Gibbs Property of the KPZ line ensemble. See Figure 8.2 and its caption for more details. Denote $\mathcal{I}_{s,\beta} := (0, \beta^{\kappa}s^{1-\alpha/3})$. Recall that $\mathfrak{h}_t$ is the lowest indexed curve $\mathfrak{h}_t^{(1)}$ of the KPZ line ensemble $\{\mathfrak{h}_t^{(n)}\}_{n\in\mathbb{N}}$. Let $\mathcal{F}_s := \mathcal{F}_{\mathbf{ext}}(\{1\}, \mathcal{I}_{s,\beta})$

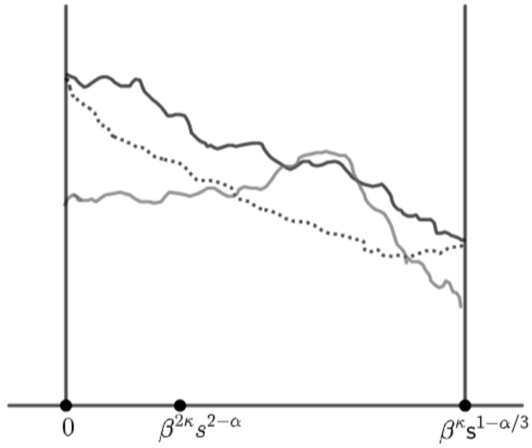

Figure 4.3: Illustration for the proof of second inequality in (4.4.2): The two solid black curves denote first two curves of a KPZ line ensemble $\{\mathfrak{h}_t^{(n)}\}_{n\geq 0}$ inside the interval $\mathcal{I}_{s,\beta}$. The dotted line denotes the top line $\widehat{\mathfrak{h}}_t^{(1)}$ of another KPZ line line ensemble coupled with $\{\mathfrak{h}_t^{(n)}\}_{n\geq 0}$ such that (i) $\widehat{\mathfrak{h}}_t^{(1)}$ has the same end point as $\mathfrak{h}_t^{(1)}$, (ii) always stays below of $\mathfrak{h}_t^{(1)}$ and $\widehat{\mathfrak{h}}_t^{(2)} = -\infty$. The law of $\widehat{h}_t^{(1)}$ is same a Brownian bridge between 0 and $\beta^\kappa s^{1-\alpha/3}$. Recall $\mathcal{A}$ denotes the event that $\mathfrak{h}_t^{(1)}(y)$ goes below of $\mathfrak{h}_t^{(1)}(0)$ by $-7\beta^\kappa s/8$ for some y in $(0, \beta^\kappa s^{1-\alpha/3})$. The event $\neg\mathsf{B}$ controls that $h_t^{(1)}(\beta^\kappa s^{1-\alpha/3})$ does not fall much below than $h_t^{(1)}(0)$. By the monotone coupling, we have $\mathbb{P}_s(\mathsf{A}\cap\neg\mathsf{B}) \leq \mathbb{P}_{\mathbf{free}}(\mathsf{A}\cap\neg\mathsf{B})$. Here, $\mathbb{P}_s$ is the law of $\mathfrak{h}_t^{(1)}$ conditioned on $\mathfrak{h}_t^{(2)}$ and everything outside of $\mathcal{I}_{s,\beta}$ whereas $\mathbb{P}_{\mathbf{free}}$ is the law of a Brownian bridge with same end point as $\mathfrak{h}_t^{(1)}$. Finally, $\mathbb{P}_{\mathbf{free}}(\mathsf{A}\cap\neg\mathsf{B})$ is estimated using the tail probability estimates of a Brownian bridge.

be the σ-algebra generated by $\{\mathfrak{h}_t^{(1)}(x) : x \in \mathbb{R}\backslash\mathcal{I}_{s,\beta}\}$ and $\{\mathfrak{h}_t^{(n)}(x) : x \in \mathbb{R}\}_{n\in\mathbb{N}_{\geq 2}}$. Note that $\neg\mathsf{B}$ is measurable w.r.t. $\mathcal{F}_s$. Thus, we may write

$$\mathbf{P}\big(\mathsf{A}\cap\neg\mathsf{B}\big) = \mathbf{E}\left[\mathbf{1}_{\neg\mathsf{B}}\mathbf{E}[\mathbf{1}_{\mathsf{A}}|\mathcal{F}_s]\right] = \mathbf{E}\left[\mathbf{1}_{\neg\mathsf{B}}\mathbf{P}_s(\mathsf{A})\right]. \tag{4.4.4}$$

where $\mathbf{P}_s := \mathbf{P}_{\mathbf{H}_t^{\text{long}}}^{1,1,(0,\beta^\kappa s^{1-\frac{\alpha}{3}}),\mathfrak{h}_t^{(1)}(0),\mathfrak{h}_t^{(1)}(\beta^\kappa s^{1-\frac{\alpha}{3}}),+\infty,\mathfrak{h}_t^{(2)}}$. By the monotone coupling (Lemma 6.5.2) $\mathbf{P}_s(\mathsf{A}) \leq \mathbf{P}_{\mathbf{free}}(\mathsf{A})$, where $\mathbf{P}_{\mathbf{free}} := \mathbf{P}_{\mathbf{H}_t^{\text{long}}}^{1,1,(0,\beta^\kappa s^{1-\frac{\alpha}{3}}),\mathfrak{h}_t^{(1)}(0),\mathfrak{h}_t^{(1)}(\beta^\kappa s^{1-\frac{\alpha}{3}}),+\infty,-\infty}$ is the law of a Brownian Bridge $\mathfrak{B}$ on $[0, \beta^\kappa s^{1-\frac{\alpha}{3}}]$ with $\mathfrak{B}(0) := \mathfrak{h}_t(0)$ and $\mathfrak{B}(\beta^\kappa s^{1-\frac{\alpha}{3}}) := \mathfrak{h}_t(\beta^\kappa s^{1-\frac{\alpha}{3}})$. Since $\beta \in (0,1]$ and $\alpha \geq 3/2$, we have $\beta^\kappa s^{1-\alpha/3} \geq \beta^{2\kappa}s^{2-\alpha}$. By the affine equivariance of the law of Brownian bridges

$$\big\{\mathfrak{B}(x) : x \in \mathcal{I}_{s,\beta}\big\} \stackrel{d}{=} \Big\{\widetilde{\mathfrak{B}}(x) + \tfrac{\mathfrak{h}_t(\beta^\kappa s^{1-\frac{\alpha}{3}})-\mathfrak{h}_t(0)}{\beta^\kappa s^{1-\frac{\alpha}{3}}}y : x \in \mathcal{I}_{s,\beta}\Big\},$$

where $\widetilde{\mathfrak{B}}$ is a Brownian Bridge on $[0, \beta^\kappa s^{1-\frac{\alpha}{3}}]$ starting and ending at 0. Combining these observations with (4.4.4) shows

$$\begin{aligned}
\mathbf{P}(\mathsf{A}\cap\neg\mathsf{B}) &\le \mathbb{E}[\mathbf{1}_{\neg\mathsf{B}}\mathbf{P}_s(\mathsf{A})]\\
&= \mathbf{E}\Big[\mathbf{1}_{\neg\mathsf{B}}\mathbf{P}\big(\inf_{y\in[0,\beta^{2\kappa}s^{2-\alpha}]}\big[\widetilde{\mathfrak{B}}(y)+\tfrac{\mathfrak{h}_t(\beta^\kappa s^{1-\frac{\alpha}{3}})-\mathfrak{h}_t(0)}{\beta^\kappa s^{1-\frac{\alpha}{3}}}y\big]\le -\tfrac{7}{8}\beta^\kappa s\big|\mathcal{F}_s\big)\Big]\\
&\le \mathbf{E}\Big[\mathbf{1}_{\neg\mathsf{B}}\mathbf{P}\big(\inf_{y\in[0,\beta^{2\kappa}s^{2-\alpha}]}\widetilde{\mathfrak{B}}(y)-\tfrac{3s^{2\alpha/3}\beta^{2\kappa}s^{2-\alpha}}{4\beta^\kappa s^{1-\frac{\alpha}{3}}}\le -\tfrac{7}{8}\beta^\kappa s\big|\mathcal{F}_s\big)\Big]\\
&\le \mathbf{P}\Big(\inf_{y\in[0,\beta^{2\kappa}s^{2-\alpha}]}\widetilde{\mathfrak{B}}(y)\le -\tfrac{\beta^\kappa s}{8}\Big) = \mathbf{P}\Big(\tfrac{1}{\beta^\kappa s^{1-\frac{\alpha}{2}}}\inf_{y\in[0,\beta^{2\kappa}s^{2-\alpha}]}\widetilde{\mathfrak{B}}(y)\le -\tfrac{s^{\alpha/2}}{8}\Big).
\end{aligned}$$

The inequality in the third line follows since

$$\inf_{y\in[0,\beta^{2\kappa}s^{2-\alpha}]}\left\{\widetilde{\mathfrak{B}}(y)+\frac{\mathfrak{h}_t(\beta^\kappa s^{1-\frac{\alpha}{3}})-\mathfrak{h}_t(0)}{\beta^\kappa s^{1-\frac{\alpha}{3}}}y\right\}\ge \inf_{y\in[0,\beta^{2\kappa}s^{2-\alpha}]}\widetilde{\mathfrak{B}}(y)-\frac{3s^{2\alpha/3}\beta^{2\kappa}s^{2-\alpha}}{4\beta^\kappa s^{1-\frac{\alpha}{3}}}$$

on the event $\neg\mathsf{B}$. The next inequality follows by neglecting the indicator. The last probability is clearly bounded by $\exp(-cs^\alpha)$ by tail estimates of Brownian motion. This proves (**II**) and hence, completes the proof of this proposition. □

Proposition 4.4.2. *Fix $\kappa > 0$. There exist constant $c > 0$, $t_0 > 0$ such that for all $t \ge t_0$ and $\beta \in (0,1]$ and $s \ge s_0(t_0)$ we have*

$$\mathbf{P}\Big(\sup_{|y|\le\frac{1}{16}\beta^{2\kappa}\sqrt{s}}(\mathfrak{h}_t(y)-\mathfrak{h}_t(0))\ge\beta^\kappa s\Big)\le e^{-cs^{3/2}}. \tag{4.4.5}$$

Proof. Set $p = \beta^{2\kappa}\sqrt{s}/16$. Let Sup_1 and Sup_2 be the supremum value of $\mathfrak{h}_t(y)-\mathfrak{h}_t(0)$ for $y\in[0,p]$ and $y\in[-p,0]$ respectively. In what follows, we only bound $\mathbf{P}(\mathsf{Sup}_1\ge\beta^\kappa s)$. One can bound $\mathbf{P}(\mathsf{Sup}_2\ge\beta^\kappa s)$ analogously. Let χ be the infimum of y in $[0,p]$ such that $\mathfrak{h}_t(y)-\mathfrak{h}_t(0)\ge\beta^\kappa s$. If there is no such y, define χ to be $+\infty$. Note that $\mathbf{P}(\mathsf{Sup}_1\ge\beta^\kappa s)=\mathbf{P}(\chi\le p)$. We can write this event as a disjoint union of A_1 and A_2 which are defined as

$$\mathsf{A}_1 := \{\chi\le p, \mathfrak{h}_t(\chi)-\mathfrak{h}_t(p)<\tfrac{\beta^\kappa s}{8}\},\quad \mathsf{A}_2 := \{\chi\le p, \mathfrak{h}_t(\chi)-\mathfrak{h}_t(p)\ge\tfrac{\beta^\kappa s}{8}\}.$$

In what follows, we show there exist $s_0 = s_0(t_0) > 0$ and constant $c > 0$ such that for all $s \ge s_0$ and $t \ge t_0$, we have

$$\mathbf{P}(\mathsf{A}_1) \le \exp(-cs^{3/2}), \qquad \mathbf{P}(\mathsf{A}_2) \le \tfrac{1}{2}\mathbf{P}(\chi \le p) + \exp(-cs^{3/2}). \tag{4.4.6}$$

As $\mathbf{P}(\chi \le p) = \mathbf{P}(\mathsf{A}_1) + \mathbf{P}(\mathsf{A}_2)$, combining the above two inequalities show $\mathbf{P}(\chi \le p) \le 4\exp(-cs^{\alpha})$. Thus, proving (4.4.5) boils down to showing (4.4.6). We first prove $\mathbf{P}(\mathsf{A}_1) \le \exp(-cs^{3/2})$. By the continuity of the spatial process $\mathfrak{h}_t(\cdot)$, we have $\mathfrak{h}_t(\chi) = \mathfrak{h}_t(0) + \beta^{\kappa}s$ on the event $\{\chi \le p\}$. Thus

$$\mathbf{P}(\mathsf{A}_1) \le \mathbf{P}(\mathfrak{h}_t(0) - \mathfrak{h}_t(p) \le -7\beta^{\kappa}s/8) \le \mathbf{P}(\inf_{y\in[0,p]} (\mathfrak{h}_t(y) - \mathfrak{h}_t(p)) \le -7\beta^{\kappa}s/8).$$

The r.h.s. of the above inequality is bounded by $\exp(-cs^{3/2})$ due to Proposition 4.4.1 and the stationarity of spatial process $\mathfrak{h}_t(x) + \frac{x^2}{2}$. This proves the first inequality of (4.4.6). Now we turn to show the second inequality of (4.4.6). Consider the following event

$$\mathsf{B} := \Big\{\mathfrak{h}_t(0) \in [-s/4, s/4], \mathfrak{h}_t(\beta^{\kappa}\sqrt{s}) \in [-3s/4, s/4]\Big\}.$$

Observe that $\mathbf{P}(\mathsf{A}_2) \le \mathbf{P}(\mathsf{A}_2 \cap \mathsf{B}) + \mathbf{P}(\neg\mathsf{B})$. By Proposition 4.2.12, we get $\mathbf{P}(\neg\mathsf{B}) \le \exp(-cs^{3/2})$ for some constant $c > 0$ and all large s and t. It suffices to show

$$\mathbf{P}(\mathsf{A}_2 \cap \mathsf{B}) \le 2^{-1}\mathbf{P}(\chi \le p). \tag{4.4.7}$$

Towards this end, we use the strong Brownian Gibbs property of the KPZ line ensemble. Let $\mathcal{F}_s = \mathcal{F}_{\mathbf{ext}}(\{1\}, (\chi, \beta^{\kappa}\sqrt{s}))$ be the σ-algebra generated by $\{\mathfrak{h}_t^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$ outside $\{\mathfrak{h}_t^{(1)}(x)\}_{x\in(\chi,\beta^{\kappa}\sqrt{s})}$. By the tower property of the conditional expectation, we have

$$\mathbf{P}(\mathsf{A}_2 \cap \mathsf{B}) = \mathbf{E}\left[\mathbf{1}_{\{\chi\le p\}\cap\mathsf{B}}\mathbf{E}(\mathbf{1}_{\mathsf{D}}|\mathcal{F}_s)\right] = \mathbf{E}\left[\mathbf{1}_{\{\chi\le p\}\cap\mathsf{B}}\mathbf{P}_s(\mathsf{D})\right], \tag{4.4.8}$$

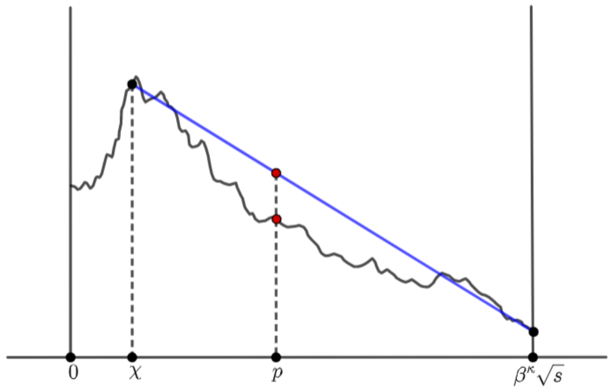

Figure 4.4: Illustration for the proof of (4.4.7): The solid curves denotes $\mathfrak{h}_t^{(1)}$ and χ is the nearest point from 0 in $[0, p]$ where $\mathfrak{h}_t^{(1)}(y)$ exceeds $\mathfrak{h}_t^{(1)}(0) + \beta^k s$. If no such point exists, then, $\chi = +\infty$. The blue line connects $(\chi, \mathfrak{h}_t^{(1)}(\chi))$ with $(\beta^k\sqrt{s}, \mathfrak{h}_t^{(1)}(\beta^k\sqrt{s}))$. The event D indicates a rapid downfall of $\mathfrak{h}_t^{(1)}$ from χ to p and the event B ensures that the line joining the endpoints of $\mathfrak{h}_t$ from 0 to $\beta^{2k}\sqrt{s}$ does not become too steep. Since $\mathfrak{h}_t(\chi) = \mathfrak{h}_t(0) + \beta^\kappa s$, B also controls the slope of the line joining the endpoints of $\mathfrak{h}_t$ from χ to $\beta^{2k}\sqrt{s}$. The rapid downfall of the black solid curve induced by D enforces the blue line staying above the black curve. Here $\mathbb{P}_s$ is the law of the black solid curve and $\mathbb{P}_{\mathbf{free}}$ is the law of free Brownian bridge, i.e., the law of the top line of a KPZ line ensemble with $\widehat{\mathfrak{h}}_t^{(2)} = -\infty$ and $\widehat{\mathfrak{h}}_t^{(1)}$ has the same end point as $\mathfrak{h}_t^{(1)}$. By the monotone coupling, $\mathbb{P}_s(\mathsf{D}) \le \mathbb{P}_{\mathbf{free}}(\mathsf{D})$ for the event D. The probability of a Brownian bridge staying below the interpolating line of its end point is less than $\frac{1}{2}$. This shows $\mathbb{P}_s(\mathsf{D})$ is bounded above by $\frac{1}{2}$.

where $\mathsf{D} := \{\mathfrak{h}_t(\chi) - \mathfrak{h}_t(p) \ge \frac{1}{8}\beta^\kappa s\}$ and $\mathbf{P}_s := \mathbf{P}_{\mathbf{H}_t^{\mathrm{long}}}^{1,1,(\chi,\beta^\kappa\sqrt{s}),\mathfrak{h}_t^{(1)}(\chi),\mathfrak{h}_t^{(1)}(\beta^\kappa\sqrt{s}),+\infty,\mathfrak{h}_t^{(2)}}$. We show that $\mathbf{P}_s(\mathsf{D}) \le \frac{1}{2}$ on the event $\{\chi \le p\} \cap \mathsf{B}$. By Lemma 6.5.2, $\mathbf{P}_s(\mathsf{D}) \le \mathbf{P}_{\mathbf{free}}(\mathsf{D})$, where $\mathbf{P}_{\mathbf{free}} := \mathbf{P}_{\mathbf{H}_t^{\mathrm{long}}}^{1,1,(\chi,\beta^\kappa\sqrt{s}),\mathfrak{h}_t^{(1)}(\chi),\mathfrak{h}_t^{(1)}(\beta^\kappa\sqrt{s}),+\infty,-\infty}$ is the law of a Brownian Bridge $\mathfrak{B}(\cdot)$ on $[\chi, \beta^\kappa\sqrt{s}]$ with $\mathfrak{B}(\chi) := \mathfrak{h}_t(\chi)$ and $\mathfrak{B}(\beta^\kappa\sqrt{s}) := \mathfrak{h}_t(\beta^\kappa\sqrt{s})$. Let us define

$$\mathfrak{B}_{\mathrm{Int}}(y) := \tfrac{\beta^\kappa\sqrt{s}-y}{\beta^\kappa\sqrt{s}-\chi}\mathfrak{h}_t(\chi) + \tfrac{y-\chi}{\beta^\kappa\sqrt{s}-\chi}\mathfrak{h}_t(\beta^\kappa\sqrt{s}) \quad \text{for } y \in [\chi, \beta^\kappa\sqrt{s}].$$

On the event $\{\chi \le p\} \cap \mathsf{B}$, we have $\mathfrak{h}_t(\chi) = \mathfrak{h}_t(0) + \beta^\kappa s$ by the continuity of the spatial process $\mathfrak{h}_t$ and hence, $\mathfrak{h}_t(\beta^\kappa\sqrt{s}) - \mathfrak{h}_t(\chi)$ is bounded below by $\frac{-3s}{4} - \mathfrak{h}_t(0) - \beta^\kappa s$ which is further lower bounded by $-2s$. This shows $\mathfrak{B}_{\mathrm{Int}}(p) \ge \mathfrak{h}_t(\chi) - \frac{\beta^\kappa s}{8}$. However, we know that $\mathfrak{B}(p) \le \mathfrak{h}_t(\chi) - \frac{\beta^\kappa s}{8}$ on the event D. This shows $\mathfrak{B}(p) \le \mathfrak{B}_{\mathrm{Int}}(p)$ on the event $\{\chi \le p\} \cap \mathsf{B}$. However, since $\mathfrak{B}$ is a Brownian bridge and $\mathfrak{B}_{\mathrm{Int}}$ is the linear interpolation of the end points of $\mathfrak{B}$, $\mathbf{P}(\mathfrak{B}(p) \le \mathfrak{B}_{\mathrm{Int}}(p)) = \frac{1}{2}$. This implies $\mathbf{P}_s(\mathsf{D}) \le \frac{1}{2}$ on the event $\{\chi \le p\} \cap \mathsf{B}$. Substituting this bound into the r.h.s. of (4.4.8) shows (4.4.7). This completes the proof. $\square$

Proposition 4.4.3. *Fix $a \in \mathbb{R}$ and $\delta > 0$. There exist $t_0 \in (0,1)$ and an absolute constant $c > 0$ such that for all $t \le t_0$, $s \ge s_0(t_0)$ satisfying $(|a| + \delta)^2 - a^2 \le \frac{s}{2^8}$, we have*

$$\mathbf{P}\Big(\sup_{x\in[a,a+\delta]} \Big(\mathfrak{g}_t((4^3t/\pi^3)^{1/8}x) + x^2/2\Big) \ge s\Big) \le e^{-cs^{3/2}}. \tag{4.4.9}$$

$$\mathbf{P}\Big(\inf_{x\in[a,a+\delta]} \Big(\mathfrak{g}_t((4^3t/\pi^3)^{1/8}x) + x^2/2\Big) \le -s\Big) \le e^{-cs^2} + e^{-cs^2t^{-1/8}\delta^{-1}}. \tag{4.4.10}$$

Proof. We introduce the shorthand $\tilde{\mathfrak{g}}_t(x) := \mathfrak{g}_t((4^3t/\pi^3)^{1/8}x)$ which will be used throughout this proof. We divide the proof of this proposition in two stages. We prove (4.4.9) and (4.4.10) in *Stage-1* and *Stage-2* respectively.

Stage-1: Proof of (4.4.9). Define

$$\mathsf{C}_{[a,a+\delta]} := \Big\{\sup_{x\in[a,a+\delta]} (\tilde{\mathfrak{g}}_t(x) + x^2/2) \ge s\Big\}, \quad \mathsf{C}'_{[a,a+\delta]} := \Big\{\sup_{x\in[a,a+\delta]} (\tilde{\mathfrak{g}}_t(x) - \tilde{\mathfrak{g}}_t(a)) \ge \frac{s}{4}\Big\},$$
$$\mathsf{D}_w := \Big\{-\tfrac{s}{32} \le \tilde{\mathfrak{g}}_t(w) + \tfrac{w^2}{2} \le \tfrac{s}{32}\Big\},$$

where $w \in \mathbb{R}$. We seek to show that $\mathbf{P}(\mathsf{C}_{[a,a+\delta]}) \le \exp(-cs^{3/2})$ for all large s and small t. The stationarity in x of the process $\tilde{\mathfrak{g}}_t(x) + \frac{x^2}{2}$ (Lemma 4.2.10) with Corollary 4.1.6 and Theorem 4.1.7 yields $\mathbf{P}(\neg\mathsf{D}_w) \le \exp(-cs^{3/2})$ for all $w \in \mathbb{R}$. This will be used throughout the proof. On the event $\mathsf{C}_{[a,a+\delta]} \cap \mathsf{D}_a$, there exists $x \in [a, a+\delta]$ such that

$$\tilde{\mathfrak{g}}_t(x) \ge s - x^2/2 \ge s - (a+\delta)^2/2 \ge \tfrac{31s}{32} + \tilde{\mathfrak{g}}_t(a) + a^2/2 - (a+\delta)^2/2 \ge \tfrac{s}{4} + \tilde{\mathfrak{g}}_t(a),$$

where the second inequality follows since $x \le a+\delta$, the third inequality follows since $\tilde{\mathfrak{g}}_t(a) + \frac{a^2}{2} \le s/32$ on D_a and the last inequality holds since $(a+\delta)^2 - a^2 \le s/2^8$. The above inequalities shows $\mathsf{C}_{[a,a+\delta]} \cap \mathsf{D}_a \subset \mathsf{C}'_{[a,a+\delta]}$ which implies $\mathbf{P}(\mathsf{C}_{[a,a+\delta]}) \le \mathbf{P}(\neg\mathsf{D}_a) + \mathbf{P}(\mathsf{C}'_{[a,a+\delta]})$. Recall that

$\mathbf{P}(\neg \mathsf{D}_a) \leq \exp(-cs^{3/2})$. To complete the proof, it suffices to show that $\mathbf{P}(\mathsf{C}'_{[a,a+\delta]}) \leq \exp(-cs^{3/2})$ for large s and small t. This we do as follows.

Let σ be the infimum of $y \in [a, a+\delta]$ such that $\tilde{\mathfrak{g}}_t(y) - \tilde{\mathfrak{g}}_t(a) \geq \frac{s}{4}$, with the convention that $\sigma = \infty$ if no such point exists. Define $\mathsf{B} := \{\tilde{\mathfrak{g}}_t(a+\delta) - \tilde{\mathfrak{g}}_t(\sigma) \leq -\frac{s}{8}\}$ and write

$$\mathbf{P}(\mathsf{C}'_{[a,a+\delta]}) = \mathbf{P}(\sigma \leq a+\delta) = \mathbf{P}(\{\sigma \leq a+\delta\} \cap \mathsf{B}) + \mathbf{P}(\{\sigma \leq a+\delta\} \cap \neg\mathsf{B}).$$

On the event $\{\sigma \leq a+\delta\}$, we have $\tilde{\mathfrak{g}}_t(\sigma) = \tilde{\mathfrak{g}}_t(a) + \frac{s}{4}$. This implies $\tilde{\mathfrak{g}}_t(a+\delta) - \tilde{\mathfrak{g}}_t(a) = \tilde{\mathfrak{g}}_t(a+\delta) - \tilde{\mathfrak{g}}_t(\sigma) + s/4 \geq -s/8$ on $\{\sigma \leq a+\delta\} \cap \neg\mathsf{B}$ and hence,

$$\begin{aligned}\mathbf{P}(\{\sigma \leq a+\delta\} \cap \neg\mathsf{B}) &\leq \mathbf{P}\big(\tilde{\mathfrak{g}}_t(a+\delta) + (a+\delta)^2/2 - \tilde{\mathfrak{g}}_t(a) - a^2/2 \geq -\tfrac{s}{8}\big) \\ &\leq \mathbf{P}\big(\tilde{\mathfrak{g}}_t(a+\delta) + (a+\delta)^2/2 > \tfrac{s}{16}\big) + P\big(\tilde{\mathfrak{g}}_t(a) + a^2/2 \leq -\tfrac{s}{16}\big) \leq \exp(-cs^{3/2}),\end{aligned} \tag{4.4.11}$$

where the second inequality follows from the union bound and the last inequality follows by combining the stationarity of $\tilde{\mathfrak{g}}_t(x) + \frac{x^2}{2}$ with Corollary 4.1.6 and Theorem 4.1.7.

Now we proceed to bound $\mathbf{P}(\{\sigma \leq a+\delta\} \cap \mathsf{B})$. By the union bound, we have

$$\mathbf{P}(\{\sigma \leq a+\delta\} \cap \mathsf{B}) \leq \mathbf{P}(\{\sigma \leq a+\delta\} \cap \mathsf{B} \cap \mathsf{D}_a \cap \mathsf{D}_{a+4\delta}) + \mathbf{P}(\neg\mathsf{D}_a) + \mathbf{P}(\neg\mathsf{D}_{a+4\delta}). \tag{4.4.12}$$

We know $\mathbf{P}(\neg\mathsf{D}_a) + \mathbf{P}(\neg\mathsf{D}_{a+4\delta})$ is bounded above by $\exp(-cs^{3/2})$ for some constant $c > 0$. In what follows, we show that

$$\mathbf{P}(\{\sigma \leq a+\delta\} \cap \mathsf{B} \cap \mathsf{D}_a \cap \mathsf{D}_{a+4\delta}) \leq \tfrac{1}{2}\mathbf{P}(\sigma \leq a+\delta). \tag{4.4.13}$$

Combining this inequality with (4.4.12) and (4.4.11) shows that $\mathbf{P}(\mathsf{C}'_{[a,a+\delta]}) \leq 2^{-1}\mathbf{P}(\mathsf{C}'_{[a,a+\delta]}) + \exp(-cs^{3/2})$ for all large s and small t. By simplifying this inequality, we get the desired result. It remains to show (4.4.13) whose proof is similar to that of (4.4.7). To avoid repetition, we sketch the key ideas without details. The main tool that we use is the Brownian Gibbs property of the short time KPZ line ensemble $\{\mathfrak{g}_t^{(n)}\}_{n\in\mathbb{N}}$ (Recall Definition (4.2.2)). By the tower property, we write the left hand side of (4.4.13) as $\mathbf{E}[\mathbf{1}_{\{\sigma\le a+\delta\}\cap\mathcal{D}_a\cap\mathsf{D}_{a+4\delta}}\mathbf{P}_s(\mathsf{B})]$ where

$$\mathbf{P}_s := \mathbf{P}_{\mathbf{H}_t^{\text{short}}}^{1,1,(4^3t/\pi^3)^{1/8}(\sigma,a+4\delta),\tilde{\mathfrak{g}}_t^{(1)}(\sigma),\tilde{\mathfrak{g}}_t^{(1)}(a+4\delta),+\infty,\mathfrak{g}_t^{(2)}}.$$

By monotone coupling, $\mathbf{P}_s(\mathsf{B}) \le \mathbf{P}_{\mathbf{free}}(\mathsf{B})$ where $\mathbf{P}_{\mathbf{free}}$ is the law of a free Brownian bridge between $(4^3t/\pi^3)^{1/8}\sigma$ and $(4^3t/\pi^3)^{1/8}(a+4\delta)$ with the value of the end points being $\tilde{\mathfrak{g}}_t(\sigma)$ and $\tilde{\mathfrak{g}}_t(a+4\delta)$. On the event $\{\sigma \le a+\delta\}\cap\mathsf{D}_a\cap\mathsf{D}_{a+4\delta}\cap\mathsf{B}$, the value of the Brownian bridge at $(4^3t/\pi^3)^{1/8}(a+\delta)$ has to be lower than the value of the line joining two end points of the Brownian bridge. The probability of this is bounded by $1/2$ which shows $\mathbf{P}_{\mathbf{free}}(\mathsf{B}) \le 1/2$ on $\{\sigma \le a+\delta\}\cap\mathsf{D}_a\cap\mathsf{D}_{a+4\delta}$. Hence, we get $\mathbf{E}[\mathbf{1}_{\{\sigma\le a+\delta\}\cap\mathsf{D}_a\cap\mathsf{D}_{a+4\delta}}\mathbf{P}_s(\mathsf{B})]$ is less than $\mathbf{P}(\sigma \le a+\delta)/2$. This shows (4.4.13) and hence, completes the proof of (4.4.9).

Stage-2: Proof of (4.4.10). Let us define the following two events:

$$\mathsf{B}_{[a,a+\delta]} = \{a^2/2 + \inf_{x\in[a,a+\delta]}\tilde{\mathfrak{g}}_t(x) \le -s\}, \qquad \mathsf{E}_w := \{\tilde{\mathfrak{g}}_t(w) + w^2/2 \ge -s/4\}$$

for $w\in\mathbb{R}$. Note that $\mathbf{P}\big(\inf_{x\in[a,a+\delta]}\big(\tilde{\mathfrak{g}}_t(x)+\frac{x^2}{2}\big) \le -s\big)$ is bounded by $\mathbf{P}(\mathsf{B}_{[a,a+\delta]})$. Furthermore,

$$\mathbf{P}(\mathsf{B}_{[a,a+\delta]}) \le \mathbf{P}(\neg\mathsf{E}_a) + \mathbf{P}(\neg\mathsf{E}_{a+\delta}) + \mathbf{P}(\mathsf{B}_{[a,a+\delta]}\cap\mathsf{E}_a\cap\mathsf{E}_{a+\delta}).$$

Due to the spatial stationarity of the process $\tilde{\mathfrak{g}}_t(x)+x^2/2$ (Lemma 4.2.10) and Theorem 4.1.7, we have $\mathbf{P}(\neg\mathsf{E}_{a+\delta}) = \mathbf{P}(\neg\mathsf{E}_0) \le \exp(-cs^2)$ for all large s and small t. To complete the proof of (4.4.10), it suffices to show

$$\mathbf{P}(\mathsf{B}_{[a,a+\delta]}\cap\mathsf{E}_a\cap\mathsf{E}_{a+\delta}) \le \exp(-cs^2t^{-1/8}\delta^{-1}). \tag{4.4.14}$$

To show the above inequality, we use the Brownian-Gibbs property of the short time KPZ line ensemble. Recall from (4.2.4) and (4.2.2) that $\big\{\tilde{\mathfrak{g}}_t((4^3t/\pi^3)^{-1/8}w)\big\}_{w\in\mathbb{R}}$ is same in distribution with $\mathfrak{g}_t^{(1)}(\cdot)$ where $\mathfrak{g}_t^{(1)}$ is the lowest indexed curve of the short-time KPZ line ensemble defined in (3) of Lemma 6.5.1. Let us set $a' := (4^3t/\pi^3)^{1/8}a$ and $\delta' := (4^3t/\pi^3)^{1/8}\delta$ for convenience. Let $\mathcal{F}_s := \mathcal{F}_{\mathbf{ext}}(\{1\}, (a', a'+\delta'))$ be the σ-algebra generated by $\{\tilde{\mathfrak{g}}_t^{(n)}(x)\}_{n\in\mathbb{N}_{\geq 2}, x\in\mathbb{R}}$ outside $\{\tilde{\mathfrak{g}}_t^{(1)}(x)\}_{x\in\mathbb{R}\setminus(a',a'+\delta')}$. Consider the following two measures

$$\mathbf{P}_s := \mathbf{P}_{\mathbf{H}_t^{\text{short}}}^{1,1,(a',a'+\delta'),\tilde{\mathfrak{g}}_t(a),\tilde{\mathfrak{g}}_t(a+\delta),\infty,\mathfrak{g}_t^{(2)}}, \quad \mathbf{P}_{\mathbf{free}} := \mathbf{P}_{\mathbf{H}_t^{\text{short}}}^{1,1,(a',a'+\delta'),\tilde{\mathfrak{g}}_t(a),\tilde{\mathfrak{g}}_t(a+\delta),\infty,-\infty},$$

where $\mathbf{P}_{\mathbf{free}}$ denotes the law of a Brownian bridge on $[a', a'+\delta']$ with the boundary values $\tilde{\mathfrak{g}}_t(a)$ and $\tilde{\mathfrak{g}}_t(a+\delta)$ respectively. By the strong Brownian Gibbs property for the short-time KPZ line ensemble,

$$\mathbf{P}(\mathsf{B}_{[a,a+\delta]} \cap \mathsf{E}_a \cap \mathsf{E}_{a+\delta}) = \mathbf{E}\left[\mathbf{1}_{\mathsf{E}_a}\mathbf{1}_{\mathsf{E}_{a+\delta}}\mathbf{E}(\mathsf{B}_{[a,a+\delta]}|\mathcal{F}_s)\right] = \mathbf{E}\left[\mathbf{1}_{\mathsf{E}_a}\mathbf{1}_{\mathsf{E}_{a+\delta}}\mathbf{P}_s(\mathsf{B}_{[a,a+\delta]})\right].$$

Due to the monotone coupling, we know $\mathbf{P}_s(\mathsf{B}_{[a,a+\delta]}) \leq \mathbf{P}_{\mathbf{free}}(\mathsf{B}_{[a,a+\delta]})$. Let $\mathfrak{B}$ be a Brownian bridge on $[0, \delta']$ with $\mathfrak{B}(0) = \mathfrak{B}(\delta') = 0$. Then, the law of $\mathfrak{B}(x) + \tilde{\mathfrak{g}}_t(a)\frac{\delta'-x}{\delta'} + \tilde{\mathfrak{g}}_t(a+\delta)\frac{x}{\delta'}$ is same as $\mathbf{P}_{\mathbf{free}}$. So, we have

$$\mathbf{P}(\mathsf{E}_a \cap \mathsf{E}_{a+\delta} \cap \mathsf{B}_{[a,a+\delta]}) \leq \mathbf{E}\Big[\mathbf{1}_{\mathsf{E}_a}\mathbf{1}_{\mathsf{E}_{a+\delta}}\mathbf{P}\Big(\tfrac{a^2}{2} + \inf_{x\in[0,\delta']}\big[\mathfrak{B}(x) + \tilde{\mathfrak{g}}_t(a)\tfrac{\delta'-x}{\delta'} + \tilde{\mathfrak{g}}_t(a+\delta)\tfrac{x}{\delta'}\big] \leq -s\Big)\Big]$$

$$\leq \mathbf{E}\Big[\mathbf{1}_{\mathsf{E}_a}\mathbf{1}_{\mathsf{E}_{a+\delta}}\mathbf{P}\Big(\tfrac{a^2}{2} + \inf_{x\in[0,\delta']}\big[\mathfrak{B}(x) - \big(\tfrac{s}{4} + \tfrac{a^2}{2}\big)\tfrac{\delta'-x}{\delta'} - \big(\tfrac{s}{4} + \tfrac{(a+\delta)^2}{2}\big)\tfrac{x}{\delta'}\big] \leq -s\Big)\Big] \tag{4.4.15}$$

$$\leq \mathbf{P}\Big(\tfrac{a^2-(a+\delta)^2}{2} - \tfrac{s}{4} + \inf_{x\in[0,\delta']}\big[\mathfrak{B}(x) + \tfrac{[(a+\delta)^2-a^2](\delta'-x)}{2\delta'}\big] \leq -s\Big). \tag{4.4.16}$$

The inequality in (4.4.15) follows by noting that $\tilde{\mathfrak{g}}_t(a) + a^2/2$ and $\tilde{\mathfrak{g}}_t(a+\delta) + (a+\delta)^2/2$ are at least $-s/4$ on the event on $(\mathsf{E}_a \cap \mathsf{E}_{a+\delta})$. The last inequality in (7.3.22) follows by dropping the indicators $\mathbf{1}_{\mathsf{E}_a}$ and $\mathbf{1}_{\mathsf{E}_{a+\delta}}$ from inside the expectation. Recall that $(|a| + |\delta|)^2 - a^2 \leq s/2^8$. Using this inequality to bound in the last line of the above display yields

$$\text{r.h.s. of (7.3.22)} \le \mathbf{P}\Big(\inf_{x\in[0,\delta']} \big[\mathfrak{B}(x) + \tfrac{[(a+\delta)^2-a^2](\delta'-x)}{2\delta'}\big] \le -\tfrac{3s}{4} + \tfrac{s}{2^9}\Big). \tag{4.4.17}$$

Note that $|(a+\delta)^2 - a^2| \le 2|a|\delta + \delta^2 = (|a|+\delta)^2 - a^2 \le s/2^8$ by the hypothesis. Adjusting the bound on the drift term in (4.4.17), we get r.h.s. of (7.3.22) $\le \mathbb{P}\Big(\inf_{x\in[0,\delta']} \mathfrak{B}(x) \le -\frac{3s}{4} + \frac{s}{2^8}\Big)$. Upper bounding $-\frac{3s}{4} + \frac{s}{2^9}$ by $-\frac{s}{4}$, we get

$$\text{r.h.s. of (4.4.17)} \le \mathbb{P}\Big(\inf_{x\in[0,\delta']} \mathfrak{B}(x) \le -\tfrac{s}{4}\Big) = \mathbf{P}\Big(\inf_{x\in[0,1]} \widetilde{\mathfrak{B}}(x) \le -\tfrac{s}{4\sqrt{\delta'}}\Big) \le \exp(-cs^2/\delta')$$

Here, $\widetilde{\mathfrak{B}}$ is a Brownian bridge on $[0,1]$ with $\widetilde{\mathfrak{B}}(0) = \widetilde{\mathfrak{B}}(1) = 0$. The equality in the above display follows from the scale invariance property of the Brownian bridge. The last inequality is obtained by bounding the tail probability of the infimum of a Brownian bridge using reflection principle. Noting that $\delta' \le 2t^{1/8}\delta$, we get (4.4.14) from (4.4.17) and hence obtain (4.4.9), completing the proof. □

Our next and final proposition of this section bounds the tail probabilities of the supremum and infimum of the spatial process $x \mapsto \mathfrak{g}_t(x) + (\pi t/4)^{3/4}x^2/(2t)$. Proof of this proposition is similar to that of Proposition 4.1 and 4.2 of [117] and thus deferred to Section **??**.

Proposition 4.4.4. *Let $\nu > 0$. There exist $t_0 = t_0(\nu) \in (0,1)$, $c = c(\nu) > 0$ and $s = s(\nu) > 0$ such that for all $t \le t_0$ and $s \ge s_0$, we have*

$$\mathbf{P}\Big(\sup_{x\in\mathbb{R}} \big(\mathfrak{g}_t(x) + \tfrac{(\pi t/4)^{3/4}(1-\nu)x^2}{2t}\big) \ge s\Big) \le \exp(-cs^{3/2}),$$
$$\mathbf{P}\Big(\inf_{x\in\mathbb{R}} \big(\mathfrak{g}_t(x) + \tfrac{(\pi t/4)^{3/4}(1+\nu)x^2}{2t}\big) \le -s\Big) \le \exp(-cs^2).$$

4.5 Spatio-Temporal Modulus of Continuity

The main goal of this section is to study the temporal modulus of continuity of the KPZ equation and use it for proving Theorem 4.1.9. The proof of Theorem 4.1.9 requires detailed study of the tail probabilities for difference of the KPZ height function at two distinct time points. This will be explored in Proposition 4.5.1 and 4.5.2. In particular, Proposition 4.5.1 will study the tail estimates when two time points are close to each other and Proposition 4.5.2 will focus on the case when the time points are far apart. With these result in hand, we show the Hölder continuity of the sample path of $\mathfrak{h}_t$ in Proposition 4.5.5. Below, we first state those propositions; prove Theorem 4.1.9; and then, complete proving those proposition in three ensuing subsections.

Proposition 4.5.1. *Fix $\varepsilon \in (0, \frac{1}{4})$. There exist $t_0 = t_0(\varepsilon) \geq 1, c = c(\varepsilon) > 0$, and $s_0 = s_0(\varepsilon) > 0$ such that for all $t \geq t_0$, $s \geq s_0$ and $\beta \leq (0, 1]$ satisfying $\beta t \leq \frac{1}{t_0}$, we have*

$$\mathbf{P}(\mathfrak{h}_t(1+\beta, 0) - \mathfrak{h}_t(1, 0) \geq \beta^{1/4-\varepsilon} s) \leq \exp(-cs^{3/2}), \tag{4.5.1}$$

$$\mathbf{P}(\mathfrak{h}_t(1+\beta, 0) - \mathfrak{h}_t(1, 0) \leq -\beta^{1/4-\varepsilon} s) \leq \exp(-cs^{2}). \tag{4.5.2}$$

Proposition 4.5.2. *Fix $t_0 > 0$. There exist $c = c(t_0) > 0$, and $s_0 = s_0(t_0) > 0$ such that for all $t \geq t_0$ satisfying $\beta t \geq t_0$, $\beta \in (0, 1]$ and $s \geq s_0$,*

$$\mathbf{P}(\mathfrak{h}_t(1+\beta, 0) - \mathfrak{h}_t(1, 0) \geq \beta^{1/4} s) \leq \exp(-cs^{3/2}), \tag{4.5.3}$$

$$\mathbf{P}(\mathfrak{h}_t(1+\beta, 0) - \mathfrak{h}_t(1, 0) \leq -\beta^{1/4} s) \leq \exp(-cs^{2}). \tag{4.5.4}$$

Remark 4.5.3. Note that Proposition 4.5.1 and 4.5.2 together bounds the upper and lower tail probabilities of the difference of the KPZ height function at any two time points irrespective of their distance. This is in sharp contrast with Theorem 1.5 of [117] which was able to prove some tail bounds of the KPZ height difference only under the assumption that the two associated time points are far apart. While Proposition 4.5.2 may appear to share the same spirit as [117, Theorem 1.5] since they both work under the assumption of the time points being distant from each other, however, the tail bounds of Proposition 4.5.2 improve on the decay exponents in comparison with those in [117]. That being said, we expect that same tail bounds as in (4.5.3) and (4.5.4) hold even when the exponent of β is $\frac{1}{3}$ instead of $\frac{1}{4}$. Nevertheless, the present tail bounds of Proposition 4.5.1

and 4.5.2 are sufficient for proving main results of this paper.

Proposition 4.5.1 and 4.5.2 will be proved in Section 4.5.1 and 4.5.2 respectively. The following proposition is in the same vein as Proposition 4.5.2.

Proposition 4.5.4. *Fix $t_0 > 0$. For any given $\beta > 0$, recall the spatial process $\mathfrak{h}_{(1+\beta)t\downarrow t}(\cdot)$ from Proposition 4.2.11. There exist $c = c(t_0) > 0$, and $s_0 = s_0(t_0) > 0$ such that for all $t \geq t_0$, $s \geq s_0, \beta \geq 1$ with $t \geq t_0$ we have*

$$\mathbf{P}(\mathfrak{h}_t(1+\beta,0) - \mathfrak{h}_{(t+\beta t)\downarrow t}(0) \geq s) \leq \exp(-cs^{3/2})$$
$$\mathbf{P}(\mathfrak{h}_t(1+\beta,0) - \mathfrak{h}_{(t+\beta t)\downarrow t}(0) \leq -s) \leq \exp(-cs^{2}).$$

The proofs of Proposition 4.5.2 and Proposition 4.5.4, both use the representation $h_t(1+\beta,0) = I_t(\mathfrak{h}_t, \mathfrak{h}_{(1+\beta)t\downarrow t})$ (Recall I_t from (4.2.5)). In fact, the proof of Proposition 4.5.4 is ditto to that of Proposition 4.5.2 upto switching the role of $\mathfrak{h}_t$ and $\mathfrak{h}_{(1+\beta)t\downarrow t}$. For avoiding repetitions, we will only prove Proposition 4.5.2 here and postpone the proof of Proposition 4.5.4 to Section **??**.

Although Theorem 4.1.9 captures the tail bounds for spatio-temporal modulus of continuity, we record a stronger tail bounds of the modulus of continuity of the KPZ temporal process. This result which will proved in Section 4.5.3 is useful for several estimates in Section 4.6 and 4.7.

Proposition 4.5.5 (Temporal moulus of continuity)**.** *Fix $\varepsilon \in (0, \frac{1}{4})$. There exist $t_0 = t_0(\varepsilon), s_0 = s_0(\varepsilon) > 0$ and $c = c(\varepsilon) > 0$, such that for all $a, t \geq 0$ with $at \geq t_0$ and $s \geq s_0$,*

$$\mathbf{P}\left(\sup_{\tau\in[0,a]} \frac{\mathfrak{h}_t(a+\tau,0) - \mathfrak{h}_t(a,0)}{(\tau/a)^{\frac{1}{4}-\varepsilon}\log^{2/3}\frac{2a}{\tau}} \geq a^{1/3}s\right) \leq e^{-cs^{3/2}}, \tag{4.5.5}$$

$$\mathbf{P}\left(\inf_{\tau\in[0,a]} \frac{\mathfrak{h}_t(a+\tau,0) - \mathfrak{h}_t(a,0)}{(\tau/a)^{\frac{1}{4}-\varepsilon}\log^{1/2}\frac{2a}{\tau}} \leq -a^{1/3}s\right) \leq e^{-cs^{2}}. \tag{4.5.6}$$

Proof of Theorem 4.1.9. Fix $\varepsilon > 0$. Take $t_0(\varepsilon)$ from Proposition 4.5.1. Fix any $t \geq t_0$. Fix $[c,d] \subset [1,\infty)$. We claim that there exists $s_0(c,d,\varepsilon) > 0$ and $m(c,d,\varepsilon) > 0$ such that for all

$\alpha_1 \neq \alpha_2 \in [c,d]$ and $x \in \mathbb{R}$, we have

$$\mathbf{P}\left(|\mathfrak{h}_t(\alpha_1,x) + \tfrac{x^2}{2\alpha_1} - \mathfrak{h}_t(\alpha_2,x) + \tfrac{x^2}{2\alpha_2}| \geq (\alpha_1-\alpha_2)^{\frac14-\varepsilon}s\right) \leq \exp(-ms^{3/2}). \tag{4.5.7}$$

By stationarity we may assume $x = 0$. Assume $\alpha_1 > \alpha_2$. If $\alpha_1 - \alpha_2 < 1$, by the scaling property: $\mathfrak{h}_t(\alpha_1,0) - \mathfrak{h}_t(\alpha_2,0) = \alpha_2^{1/3}[\mathfrak{h}_t(\alpha_1/\alpha_2,0) - \mathfrak{h}_t(1,0)]$, and using Proposition 4.5.1 and 4.5.2 together, we have (4.5.7). For $\alpha_1 - \alpha_2 > 1$, by union bound and Proposition 4.2.12 we have

$$\begin{aligned}\mathbf{P}\left(|\mathfrak{h}_t(\alpha_1,0) - \mathfrak{h}_t(\alpha_2,0)| \geq (\alpha_1-\alpha_2)^{\frac14-\varepsilon}s\right) &\leq \mathbf{P}\left(|\mathfrak{h}_t(\alpha_1,0)| \geq s/2\right) + \mathbf{P}\left(|\mathfrak{h}_t(\alpha_2,0)| \geq s/2\right) \\ &\leq \exp(-ms^{3/2}).\end{aligned}$$

Next fix $[a,b] \subset \mathbb{R}$. By Theorem 1.3 from [117], there exists $s_0(c,d,\varepsilon) > 0$ and $m(c,d,\varepsilon) > 0$ such that for all $\alpha \in [c,d]$ and $|x_1 - x_2| \leq 1$ with $x_1, x_2 \in [a,b] \subset \mathbb{R}$ we have

$$\mathbf{P}\left(|\mathfrak{h}_t(\alpha,x_1) + \tfrac{x_1^2}{2\alpha} - \mathfrak{h}_t(\alpha,x_2) + \tfrac{x_2^2}{2\alpha}| \geq (x_1-x_2)^{\frac12}s\right) \leq \exp(-ms^{3/2}). \tag{4.5.8}$$

Utilizing the tail bounds of two point differences from (4.5.7) and (4.5.8), one may get the modulus of continuity result of Theorem 4.1.9 via Lemma 2.8 in [138]. This completes the proof. □

4.5.1 Proof of Proposition 4.5.1

We will prove (4.5.2) and (4.5.1) in *Stage-1* and *Stage-2* respectively. We start with introducing relevant notations which will be used throughout the proof. Fix $t_0 > 0$, $\varepsilon \in (0, \frac14)$ and set $\kappa = \frac14 - \varepsilon$. By the composition law

$$\mathfrak{h}_t(1+\beta,0) - \mathfrak{h}_t(1,0) = t^{-\frac13}\log\int_{\mathbb{R}} e^{t^{\frac13}\left(\mathfrak{h}_t(1,t^{-\frac23}y) + \mathfrak{h}_{(t+\beta t)\downarrow t}(-t^{-\frac23}y) - \mathfrak{h}_t(1,0)\right)}dy, \tag{4.5.9}$$

where $\mathfrak{h}_{(t+\beta t)\downarrow t}(\cdot)$ is independent of $\mathfrak{h}_t(1,\cdot)$ and is distributed as $\mathfrak{h}_t(\beta,\cdot)$. We define $\widetilde{\mathfrak{h}}_t(\beta,\cdot):\mathbb{R}\to\mathbb{R}$ and $\widetilde{\mathfrak{g}}_{\beta t}(\cdot):\mathbb{R}\to\mathbb{R}$ by $\widetilde{\mathfrak{h}}_t(\beta,\cdot):=\mathfrak{h}_{(t+\beta t)\downarrow t}(\cdot)$ and

$$\widetilde{\mathfrak{h}}_t(\beta,t^{-\frac{2}{3}}y)=t^{-\frac{1}{3}}\left(\frac{\pi\beta t}{4}\right)^{\frac{1}{4}}\left(\widetilde{\mathfrak{g}}_{\beta t}(z)+z^2/2\right)+\frac{\frac{\beta t}{24}-\log\sqrt{2\pi\beta t}}{t^{\frac{1}{3}}}-\frac{y^2}{2\beta t^{\frac{4}{3}}},$$

where $z=(\pi\beta^5t^5/4)^{-1/8}y$. Note that $\widetilde{\mathfrak{g}}_{\beta t}(x)$ is distributed as $\mathfrak{g}_{\beta t}((4^3t/\pi^3)^{1/8}x)$ and independent of $\mathfrak{h}_t(1,\cdot)$. Writing the r.h.s. of (4.5.9) in terms of of $\widetilde{\mathfrak{g}}_{\beta t}$ yields

$$\begin{aligned}\mathfrak{h}_t(1+\beta,0)-\mathfrak{h}_t(1,0)&=\frac{\beta t^{2/3}}{24}+t^{-1/3}\log\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi\beta t}}\exp\Big\{-\frac{y^2}{2\beta t}+t^{\frac{1}{3}}\left(\mathfrak{h}_t(t^{-\frac{2}{3}}y)-\mathfrak{h}_t(0)\right)\\&\qquad+\left(\frac{\pi\beta t}{4}\right)^{\frac{1}{4}}\left[\widetilde{\mathfrak{g}}_{\beta t}\left(\frac{-y}{(\pi\beta^5t^5/4)^{1/8}}\right)+\frac{y^2}{2(\pi\beta^5t^5/4)^{1/4}}\right]\Big\}\,dy\\&=:\frac{\beta t^{2/3}}{24}+t^{-1/3}\log\int_{\mathbb{R}}X_t(\beta,y)dy,\end{aligned}\tag{4.5.10}$$

where the space-time stochastic process $X_t(\beta,y):\mathbb{R}_{>0}\times\mathbb{R}\to\mathbb{R}_{\geq 0}$ is defined by the above relation. We seek for an upper bound and a lower bound for the r.h.s. of (4.5.10) which will prove (4.5.1) and (4.5.2) respectively.

Stage-1. Define $\mathfrak{Int}(\beta,t):=[-t^{2/3}\beta^{2\kappa},t^{2/3}\beta^{2\kappa}]$. From (4.5.10), $\mathfrak{h}_t(1+\beta,0)-\mathfrak{h}_t(1,0)$ is bounded below by $t^{-1/3}\log\int_{\mathfrak{Int}(\beta,t)}X_t(\beta,y)dy$. This implies

$$\mathbb{P}\big(\mathfrak{h}_t(1+\beta,0)-\mathfrak{h}_t(1,0)\leq-\beta^{\frac{1}{4}-\varepsilon}s\big)\leq\mathbb{P}\Big(\log\int_{\mathfrak{Int}(\beta,t)}X_t(\beta,y)dy\leq-\beta^{\frac{1}{4}-\varepsilon}t^{\frac{1}{3}}s\Big).\tag{4.5.11}$$

By the definition of $X_t(\cdot,\cdot)$ we have

$$\begin{aligned}t^{-1/3}\log\int_{\mathfrak{Int}(\beta,t)}X_t(\beta,y)dy&\geq t^{-1/3}\log\int_{\mathfrak{Int}(\beta,t)}\frac{e^{-y^2/(2\beta t)}}{\sqrt{2\pi\beta t}}dy+\inf_{|y|\leq\beta^{2\kappa}}(\mathfrak{h}_t(y)-\mathfrak{h}_t(0))\\&\quad+t^{-\frac{1}{3}}\left(\frac{\pi\beta t}{4}\right)^{\frac{1}{4}}\inf_{|y|\leq(\pi\beta^5t^5/4)^{-1/8}t^{2/3}\beta^{2\kappa}}\left(\widetilde{\mathfrak{g}}_{\beta t}(y)+y^2/2\right).\end{aligned}\tag{4.5.12}$$

The first term on the r.h.s. is deterministic. Using the Gaussian integral bound, we can write

$$t^{-\frac{1}{3}}\log\int_{\mathfrak{Int}(\beta,t)}\frac{e^{-y^2/(2\beta t)}}{\sqrt{2\pi\beta t}}dy \geq t^{-\frac{1}{3}}\log(1-e^{-t^{1/3}\beta^{4\kappa-1}/2}) \geq -2t^{-\frac{1}{3}}e^{-t^{\frac{1}{3}}\beta^{4\kappa-1}/2}, \tag{4.5.13}$$

where the last inequality follows since $\log(1-x) \geq -x$ for any $x \in (0,1)$. Note that $4\kappa - 1 < 0$. For any given $s_0(\varepsilon)$, choosing $t_0(\varepsilon)$ large, we may bound $t^{-1/3}e^{-t^{1/3}\beta^{4\kappa-1}/2}$ by $\beta^\kappa s_0/8$ for all $t \geq t_0$, and $\beta \leq t_0^{-2}$. This shows there exists $t_0(\varepsilon)$ large such that the r.h.s. of (4.5.13) is bounded below by $-\beta^\kappa s/4$ for all $t \geq t_0$, $\beta t \leq t_0^{-1}$ and $s \geq s_0$. By the inequality (4.5.12), (4.5.13) and the union bound, the right side of (4.5.11) is bounded by $\mathbb{P}(\mathsf{A}_1) + \mathbb{P}(\mathsf{A}_2)$ for all $t \geq t_0$, $\beta t \leq t_0^{-1}$ and $s \geq s_0$ where

$$\mathsf{A}_1 := \{\inf_{|y|\leq\beta^{2\kappa}} \mathfrak{h}_t(y) - \mathfrak{h}_t(0) \leq -\beta^\kappa s/8\},$$
$$\mathsf{A}_2 := \{\inf_{|y|\leq(\pi\beta^5t^5/4)^{-1/8}t^{2/3}\beta^{2\kappa}} (\widetilde{\mathfrak{g}}_{\beta t}(y) + y^2/2) \leq -\beta^{\kappa-\frac{1}{4}}t^{1/12}s/8\}.$$

By setting $\alpha = 2$ in Lemma 4.4.1, we get $\mathbf{P}(\mathsf{A}_1) \leq \exp(-cs^2)$ from (4.4.1). In order to bound $\mathbf{P}(\mathsf{A}_2)$, we use Lemma 4.4.3. Mapping $a \mapsto -(\pi\beta^5t^5/4)^{-1/8}t^{2/3}\beta^{2\kappa}$, $\delta \mapsto 2(\pi\beta^5t^5/4)^{-1/8}t^{2/3}\beta^{2\kappa}$ and $s \mapsto -\frac{1}{8}\beta^{\kappa\frac{1}{4}}t^{1/12}s$ and choosing $s_0(\varepsilon)$ large, we note $(|a|+\delta)^2 - a^2 \leq s/2^8$ for all $s \geq s_0$. With those choice of a, δ, s in hand, the condition of Lemma 4.4.3 is satisfied and hence, (4.4.10) yields

$$\begin{aligned}\mathbf{P}(\mathsf{A}_2) &\leq \exp(-cs^2t^{1/6}\beta^{2\kappa-\frac{1}{2}}) + \exp(-cs^2t^{1/6}\beta^{2\kappa-\frac{1}{2}}(\beta t)^{-1/8}t^{-1/24}\beta^{\kappa-\frac{3}{8}}) \\ &\leq \exp(-cs^2t^{1/6}\beta^{2\kappa-\frac{1}{2}}) + \exp(-cs^2\beta^{3\kappa-1}) \leq \exp(-cs^2).\end{aligned}$$

Combining the upper bounds on $\mathbb{P}(\mathsf{A}_1)$ and $\mathbb{P}(\mathsf{A}_2)$ and using those to bound the right side of (4.5.11) completes the proof of (4.5.2).

Stage-2: Here we prove (4.5.1). According to (4.5.10), $\mathfrak{h}_t(1+\beta,0) - \mathfrak{h}_t(1,0)$ equals $\beta t^{2/3}/24 + t^{-1/3}\log\int X_t(\beta,y)dy$. For all $t \geq t_0$ and $\beta > 0$ satisfying $\beta t \leq t_0^{-1}$, $\beta t^{2/3}$ is less than $\beta^{1/3}t_0^{-2/3}$. We can choose $s_0(\varepsilon) > 0$ large such that $\beta t^{2/3}/24 \leq \beta^{1/4-\varepsilon}s/2$ for all $s \geq s_0$, $t \geq t_0$ and β satisfying

$\beta t \leq t_0^{-1}$. Thus, for all $s \geq s_0$, we have

$$\mathbb{P}(\mathfrak{h}_t(1+\beta,0) - \mathfrak{h}_t(1,0) \geq \beta^{1/4-\varepsilon} s) \leq \mathbb{P}\big(t^{-1/3}\log \int X_t(\beta,y)dy \geq \beta^{1/4-\varepsilon}s/2\big). \quad (4.5.14)$$

Our objective is to the upper bound the r.h.s. of the above inequality. To this end, let us denote $\mathfrak{Int}_s(\beta,t) := [-\frac{1}{64}t^{2/3}\beta^{2\kappa}\sqrt{s}, \frac{1}{64}t^{2/3}\beta^{2\kappa}\sqrt{s}]$. By the union bound, we may write

$$\text{r.h.s. of (4.5.14)} \leq \underbrace{\mathbf{P}\Big(\int_{\mathfrak{Int}_s(\beta,t)} X_t(\beta,y)dy \geq e^{\frac{s}{2}t^{\frac{1}{3}}\beta^{\frac{1}{4}-\varepsilon}}\Big)}_{=:(\mathbf{I})} + \underbrace{\mathbf{P}\Big(\int_{\mathbb{R}\setminus\mathfrak{Int}_s(\beta,t)} X_t(\beta,y)dy \geq e^{\frac{s}{2}t^{\frac{1}{3}}\beta^{\frac{1}{4}-\varepsilon}}\Big)}_{=:(\mathbf{II})}.$$

We will show that (**I**) and (**II**) are bounded above by $\exp(-cs^{3/2})$ for some constant $c > 0$ in *Step I* and *Step II* respectively. Substituting these bounds into the right side of the above inequality completes the proof of (4.5.1).

Step I: Using similar ideas as in (4.5.12), we have

$$\begin{aligned} t^{-1/3}\log\int_{\mathfrak{Int}_s(\beta,t)} X_t(\beta,y)dy &\leq t^{-1/3}\log\int_{\mathfrak{Int}_s(\beta,t)} \frac{e^{-y^2/(2\beta t)}}{\sqrt{2\pi\beta t}}dy + \sup_{|y|\leq \frac{1}{64}\beta^{2\kappa}\sqrt{s}} (\mathfrak{h}_t(y) - \mathfrak{h}_t(0)) \\ &\quad + t^{-1/3}\left(\tfrac{\pi\beta t}{4}\right)^{\frac{1}{4}} \sup_{|y|\leq \frac{1}{64}(\pi\beta^5 t^5/4)^{-1/8}t^{2/3}\beta^{2\kappa}\sqrt{s}} \Big(\widetilde{\mathfrak{g}}_{\beta t}(y) + y^2/2\Big). \end{aligned}$$

Since $(2\pi\beta t)^{-1/2}\int_{\mathfrak{Int}_s(\beta,t)} e^{-y^2/2\beta t}dy < 1$, from the above inequality and the union bound, it follows that (**I**) $\leq \mathbf{P}(\mathsf{A}_3) + \mathbf{P}(\mathsf{A}_4)$ where

$$\begin{aligned} \mathsf{A}_3 &:= \{\sup_{|y|\leq \frac{1}{64}\beta^{2\kappa}\sqrt{s}} (\mathfrak{h}_t(y) - \mathfrak{h}_t(0)) \geq \beta^{\kappa}s/8\}, \\ \mathsf{A}_4 &:= \{\sup_{|y|\leq \frac{1}{64}(\pi\beta^5 t^5/4)^{-1/8}t^{2/3}\beta^{2\kappa}\sqrt{s}} \Big(\widetilde{\mathfrak{g}}_{\beta t}(y) + y^2/2\Big) \geq \beta^{\kappa-\frac{1}{4}}t^{1/12}s/8\}. \end{aligned}$$

Indeed, from Lemma 4.4.2, we know $\mathbf{P}(\mathsf{A}_3) \leq \exp(-cs^{3/2})$. In what follows, we claim and prove that $\mathbf{P}(\mathsf{A}_4) \leq \exp(-cs^{3/2})$ for all large s and some constant $c > 0$. Let us denote $\mathfrak{M} :=$

$\frac{1}{64}(4/\pi)^{1/8}\beta^{2\kappa-\frac{5}{8}}t^{1/24}\sqrt{s}$ and $\delta := \frac{1}{2^{14}}\beta^{\frac{3}{8}-\kappa}t^{1/24}\sqrt{s}$. Define $N := \lceil \mathfrak{M}/\delta \rceil$. For any $a \in \mathbb{R}$, define

$$\mathsf{B}_{[a,a+\delta]} = \{\sup_{y\in[a,a+\delta]} \left(\widetilde{\mathfrak{g}}_{\beta t}(y) + y^2/2\right) \geq \beta^{\kappa-\frac{1}{4}}t^{1/12}s/8\}.$$

Notice that $\mathsf{A}_4 \subset \cup_{i=-N-1}^{N} \mathsf{B}_{[i\delta,(i+1)\delta]}$. Hence, by the union bound

$$\mathbf{P}(\mathsf{A}_4) \leq \sum_{i=-N-1}^{N} \mathbf{P}(\mathsf{B}_{[i\delta,(i+1)\delta]}). \tag{4.5.15}$$

In what follows, we seek to bound $\mathbf{P}(\mathsf{B}_{[a,a+\delta]})$ for $a \in \{-(N+1)\delta, -N\delta, \ldots, N\delta\}$. To this end, we wish to apply Lemma 4.4.3. It is readily checked that we have $|(|a|+|\delta|)^2 - a^2| \leq \frac{\beta^{\kappa-\frac{1}{4}}t^{1/12}s}{2^{11}}$ for $a \in \{-(N+1)\delta, -N\delta, \ldots, N\delta\}$. Thus with the substitutions $t \mapsto \beta t$, $s \mapsto \beta^{\kappa-\frac{1}{4}}t^{1/12}s$, and $\delta \mapsto \frac{1}{2^{14}}\beta^{\frac{3}{8}-\kappa}t^{1/24}\sqrt{s}$ in Lemma 4.4.3 we have

$$\begin{aligned}\mathbf{P}(\mathsf{B}_{[a,a+\delta]}) &\leq \exp(-cs^{3/2}t^{1/8}\beta^{\frac{3\kappa}{2}-\frac{3}{8}}) + \exp(-cs^2t^{1/6}\beta^{2\kappa-\frac{1}{2}}(\beta t)^{-1/8}t^{-1/24}\beta^{\kappa-\frac{3}{8}}s^{-1/2}) \\ &\leq \exp(-cs^{3/2}t^{1/8}\beta^{\frac{3\kappa}{2}-\frac{3}{8}}) + \exp(-cs^{3/2}\beta^{3\kappa-1}).\end{aligned}$$

Substituting this upper bound into the r.h.s. of (4.5.15) and using the fact that $2(N+1) \leq 4N \leq 2^{11}\beta^{3\kappa-1}$, we get

$$\mathbf{P}(\mathsf{A}_4) \leq 2^{11}\beta^{3\kappa-1}\left[\exp(-cs^{3/2}t^{1/8}\beta^{\frac{3\kappa}{2}-\frac{3}{8}}) + \exp(-cs^{3/2}\beta^{3\kappa-1})\right] \leq \exp(-cs^{3/2}).$$

This completes the proof of the claim. Combining the bounds on $\mathbf{P}(\mathcal{A}_3)$ and $\mathbf{P}(\mathcal{A}_4)$ shows $(\mathbf{I}) \leq \exp(-cs^{3/2})$ for all large s.

Step II: Define $\tilde{y} := y/(\pi\beta^5t^5/4)^{1/8}$. Recall the definition of $X_t(\beta,y)$ from (4.5.10). Adjusting the parabolic term inside the exponent, we may rewrite

$$X_t(\beta,y) = \frac{1}{\sqrt{2\pi\beta t}}\exp\{-y^2/(4\beta t) + t^{\frac{1}{3}}\left(\mathfrak{h}_t(yt^{-\frac{2}{3}}) - \mathfrak{h}_t(0)\right) + (\tfrac{\pi\beta t}{4})^{\frac{1}{4}}[\widetilde{\mathfrak{g}}_{\beta t}(\tilde{y}) + \tilde{y}^2/4]\}$$

$$\leq \exp\{t^{\frac{1}{3}}\sup_{z\in\mathbb{R}}(\mathfrak{h}_t(z)-\mathfrak{h}_t(0))+(\tfrac{\pi\beta t}{4})^{\frac{1}{4}}\sup_{z\in\mathbb{R}}[\widetilde{\mathfrak{g}}_{\beta t}(z)+z^2/4]\}\frac{1}{\sqrt{2\pi\beta t}}e^{-y^2/(4\beta t)},$$

where the last inequality follows by fixing the quadratic term in y and taking supremum of the rest of the terms over $y\in\mathbb{R}$. Integrating both sides of the last inequality over $\mathbb{R}\backslash\mathfrak{Int}_s(\beta,t)$ and taking log on both sides yields shows $\frac{1}{t^{\frac{1}{3}}}\log\int_{\mathbb{R}\backslash\mathfrak{Int}_s(\beta,t)}X_t(\beta,y)dy$ is bounded by

$$-\frac{s\beta^{4\kappa-1}}{2^{15}}+\sup_{z\in\mathbb{R}}\mathfrak{h}_t(z)-\mathfrak{h}_t(0)+\frac{(\frac{\pi\beta t}{4})^{\frac{1}{4}}}{t^{\frac{1}{3}}}\sup_{z\in\mathbb{R}}\big(\widetilde{\mathfrak{g}}_{\beta t}(z)+\tfrac{z^2}{4}\big), \tag{4.5.16}$$

where $-\frac{1}{2^{15}}s\beta^{4\kappa-1}$ is an upper bound to the logarithm of the Gaussian integral term. To bound (**II**) using the above inequality, we introduce the following events:

$$\mathsf{A}_5:=\{\sup_{y\in\mathbb{R}}\mathfrak{h}_t(y)\geq\tfrac{s}{2^{17}}\},\quad \mathsf{A}_6:=\big\{\mathfrak{h}_t(0)\leq-\tfrac{s}{2^{17}}\big\},\quad \mathsf{A}_7:=\{\sup_{z\in\mathbb{R}}\Big(\widetilde{\mathfrak{g}}_{\beta t}(z)+\tfrac{z^2}{4}\Big)\geq\tfrac{s}{8}\}.$$

Note that on $\neg\mathsf{A}_5\cap\neg\mathsf{A}_6\cap\neg\mathsf{A}_7$, we get

$$(4.5.16)\leq-2^{-15}s\beta^{4\kappa-1}+2^{-16}s+\beta^{1/4}t^{-1/12}s/8\leq\beta^{1/4}s/4-2^{-16}s\beta^{4\kappa-1},$$

for any $\beta<1$. Owing to this and the union bound, we have (**II**) $\leq\mathbf{P}(\mathsf{A}_5)+\mathbf{P}(\mathsf{A}_6)+\mathbf{P}(\mathsf{A}_7)$. From Proposition 4.2.12 and 4.2.14 with $\nu=1$, we get $\mathbf{P}(\mathsf{A}_5),\mathbf{P}(\mathsf{A}_6)\leq\exp(-cs^{3/2})$. Lemma 4.4.4 shows $\mathbf{P}(\mathsf{A}_7)\leq\exp(-cs^{3/2})$. Combining these bounds with the above inequality proves (**II**) $\leq\exp(-cs^{3/2})$ for all s large and β small. This completes the proof of (4.5.1).

4.5.2 Proof of Proposition 4.5.2

Recall the composition law

$$\mathfrak{h}_t(1+\beta,0)=\frac{1}{t^{1/3}}\log\int_{\mathbb{R}}e^{t^{1/3}\big(\mathfrak{h}_t(1,t^{-2/3}y)+\beta^{1/3}\widehat{\mathfrak{h}}_{(t+\beta t)\downarrow t}(-(\beta t)^{-2/3}y)\big)}dy, \tag{4.5.17}$$

where $\widehat{\mathfrak{h}}_{\beta t}(x) := \frac{\mathfrak{h}_{(t+\beta t)\downarrow t}(\beta^{2/3}x)}{\beta^{1/3}}$. We prove (4.5.3) and (4.5.4) in *Stage-1* and *Stage-2* respectively.

Stage-1: Proof of (4.5.3): We use the following notation $\mathfrak{h}_t^{\nabla}(y) := \mathfrak{h}_t(y) - \mathfrak{h}_t(0)$ throughout this proof. Subtract $\mathfrak{h}_t(1,0)$ from both sides of (4.8.10). Furthermore, subtracting and adding the parabola $\frac{y^2}{4\beta t}$ inside the exponent of the integrand on the r.h.s. of (4.8.10) shows

$$\begin{aligned}\mathfrak{h}_t(1+\beta,0) - \mathfrak{h}_t(1,0) &= \frac{1}{t^{\frac{1}{3}}}\log\int_{\mathbb{R}} e^{-\frac{y^2}{4\beta t}+t^{\frac{1}{3}}\left(\mathfrak{h}_t^{\nabla}(t^{-\frac{2}{3}}y)+\beta^{\frac{1}{3}}\widehat{\mathfrak{h}}_{\beta t}(-\beta^{-2/3}t^{-2/3}y)+\frac{y^2}{4\beta t^{4/3}}\right)}dy \\ &\leq \beta^{1/3}\sup_{y\in\mathbb{R}}\left(\widehat{\mathfrak{h}}_{\beta t}(y)+y^2/4\right) + \frac{1}{t^{1/3}}\log\int_{\mathbb{R}} e^{-\frac{y^2}{4\beta t}+t^{1/3}\mathfrak{h}_t^{\nabla}(t^{-2/3}y)}dy. \qquad (4.5.18)\end{aligned}$$

Let us consider the following events.

$$\mathsf{A}_1 := \{\sup_{x\in\sqrt{\beta s}/32}\mathfrak{h}_t^{\nabla}(x) \geq \beta^{1/4}s/4\}, \quad \mathsf{A}_2 := \{\sup_{x\in\mathbb{R}}\left(\widehat{\mathfrak{h}}_{\beta t}(y)+y^2/4\right) \geq s/4\}$$

$$\mathsf{A}_3 := \{\sup_{|x|\in\mathbb{R}}\mathfrak{h}_t(x) \geq 2^{-14}s\}, \quad \mathsf{A}_4 := \{\mathfrak{h}_t(0) \leq -2^{-14}s\}.$$

To complete the proof of (4.5.3), we need the following lemma.

Lemma 4.5.6. $\{\mathfrak{h}_t(1+\beta,0) - \mathfrak{h}_t(1,0) \geq \beta^{1/4}s\} \subset (\mathsf{A}_1\cup\mathsf{A}_2\cup\mathsf{A}_3\cup\mathsf{A}_4)$.

Before proceeding to prove Lemma 4.5.6, we show how this will imply (4.5.3). From the above lemma and the union bound, we get $\mathbf{P}(\mathfrak{h}_t(1+\beta,0)-\mathfrak{h}_t(1,0) \geq \beta^{1/4}s) \leq \sum_{i=1}^4\mathbf{P}(\mathsf{A}_i)$. By Lemma 4.4.2 with $\kappa = \frac{1}{4}$ we get that $\mathbf{P}(\mathsf{A}_1) \leq \exp(-cs^{3/2})$. By Proposition 4.2.14 with $\nu = \frac{1}{2}$ and $\nu = 0$ we get $\mathbf{P}(\mathsf{A}_2) \leq \exp(-cs^{3/2})$ and $\mathbf{P}(\mathsf{A}_3) \leq \exp(-cs^{3/2})$ respectively. The one point tail estimate in Proposition 4.2.12 yields $\mathbf{P}(\mathsf{A}_4) \leq \exp(-cs^{3/2})$. Combining all these bounds completes the proof of (4.5.3) modulo Lemma 4.5.6.

Proof of Lemma 4.5.6: Define $\widehat{\mathfrak{Int}}_s(\beta,t) := \frac{1}{32}t^{2/3}\sqrt{\beta s}$. Observe the following two inequalities

$$\int_{\widehat{\mathfrak{Int}}_s(\beta,t)} e^{-\frac{y^2}{4\beta t}+t^{1/3}\mathfrak{h}_t^{\nabla}(t^{-2/3}y)}dy \leq \sup_{|x|\leq\sqrt{\beta s}/32}\mathfrak{h}_t^{\nabla}(x) + t^{-1/3}\log\sqrt{4\pi\beta t}, \qquad (4.5.19)$$

$$\int_{\mathbb{R}\setminus\widehat{\mathfrak{Int}}_s(\beta,t)}\exp\Big(-\frac{y^2}{4\beta t}+t^{1/3}\mathfrak{h}_t^{\nabla}(t^{-2/3}y)\Big)dy \leq \sup_{x\in\mathbb{R}}\mathfrak{h}_t^{\nabla}(x) + t^{-1/3}\log\int_{\mathbb{R}\setminus\widehat{\mathfrak{Int}}_s(\beta,t)} e^{-\frac{y^2}{4\beta t}}dy$$

$$\leq \sup_{x\in\mathbb{R}} \mathfrak{h}_t^{\nabla}(x) + t^{-1/3}\log\sqrt{4\pi\beta t} - s/2^{13}, \quad (4.5.20)$$

where the last inequality follows from the bounds on the Gaussian tail integral. On $\neg \mathsf{A}_1$ and $(\neg\mathsf{A}_3 \cap \neg\mathsf{A}_4)$, we have

$$\text{r.h.s. of (4.5.19)} \leq \tfrac{1}{4}\beta^{\frac{1}{4}}s + t^{-\frac{1}{3}}\log\sqrt{4\pi\beta t}, \quad \text{r.h.s. of (4.5.20)} \leq t^{-\frac{1}{3}}\log\sqrt{4\pi\beta t} + \tfrac{1}{4}\beta^{1/4}s$$

respectively. Thus on $\neg(\mathsf{A}_1 \cup \mathsf{A}_2 \cup \mathsf{A}_3 \cup \mathsf{A}_4)$ we get

$$\begin{aligned}\text{r.h.s. of (4.8.11)} &\leq \tfrac{1}{4}\beta^{1/3}s + t^{-1/3}\log 2 + t^{-1/3}\log\sqrt{4\pi\beta t} + \frac{1}{4}\beta^{1/4}s \\ &\leq \tfrac{1}{2}\beta^{1/4}s + (2\pi\beta)^{1/3}(16\pi\beta t)^{-1/3}\log(16\pi\beta t) < \beta^{1/4}s.\end{aligned}$$

The last inequality is true for all large enough s since $\sup_{r>0} r^{-1/3}\log r$ is bounded. This shows $\neg(\mathsf{A}_1 \cup \mathsf{A}_2 \cup \mathsf{A}_3 \cup \mathsf{A}_4)$ is contained in $\{\mathfrak{h}_t(1+\beta,0) \leq \mathfrak{h}_t(1,0) + \beta^{1/4}s\}$ and hence, completes the proof of the lemma.

Proof of (4.5.2): Restricting the integral in (4.8.10) over the region $\{|y| \leq t^{2/3}\beta^{1/2}\}$ yields

$$\begin{aligned}\mathfrak{h}_t(1+\beta,0) &\geq \frac{1}{t^{1/3}}\log\int_{|y|\leq t^{2/3}\beta^{1/2}} \exp\left(t^{1/3}\big(\mathfrak{h}_t(t^{-2/3}y) + \beta^{1/3}\widehat{\mathfrak{h}}_{\beta t}(-\beta^{-2/3}t^{-2/3}y)\big)\right)dy \\ &\geq \beta^{1/3}\inf_{y\in\mathbb{R}}\left(\widehat{\mathfrak{h}}_{\beta t}(y) + \tfrac{y^2}{4}\right) + \inf_{|y|\leq\beta^{1/2}}\mathfrak{h}_t(y) + t^{-\frac{1}{3}}\log\int_{|y|\leq t^{2/3}\beta^{1/2}}\exp\left(-\tfrac{y^2}{4\beta t}\right)dy. \quad (4.5.21)\end{aligned}$$

From the Gaussian tail bound, we have

$$t^{-1/3}\log\int_{|y|\leq t^{2/3}\beta^{1/2}}\exp\left(-\tfrac{y^2}{4\beta t}\right)dy \geq t^{-1/3}\log\sqrt{4\pi\beta t} - 2t^{-1/3}\exp\left(-\tfrac{t^{1/3}}{4}\right). \quad (4.5.22)$$

We now claim and prove that there exists $s_0 = s_0(t_0) > 0$ such that $\{\mathfrak{h}_t(1+\beta,0) - \mathfrak{h}_t(1,0) \leq -\beta^{1/4}s\} \subset \mathsf{A}_5 \cup \mathsf{A}_6$ for all $s \geq s_0$ and $\beta > 0$ satisfying $\beta t \geq t_0$ where

$$\mathsf{A}_5 := \{\inf_{|y|\leq\beta^{1/2}}\mathfrak{h}_t(y) \leq \mathfrak{h}_t(0) - \beta^{\frac{1}{4}}s\}, \quad \mathsf{A}_6 := \{\inf_{y\in\mathbb{R}}\big(\widehat{\mathfrak{h}}_{\beta t}(y) + y^2/4\big) \leq -s/4\}.$$

To see this, using (4.5.21) and (4.5.22), we have

$$\text{r.h.s. of (4.5.21)} \geq -\tfrac{1}{4}\beta^{\frac{1}{3}}s + \mathfrak{h}_t(0) - \beta^{\frac{1}{4}}s + t^{-\frac{1}{3}}\log\sqrt{4\pi\beta t} - 2t^{-1/3}\exp\left(-\tfrac{t^{1/3}}{4}\right)$$

on $\neg(\mathsf{A}_5 \cup \mathsf{A}_6)$. Note that $\log\sqrt{4\pi\beta t}/t^{1/3}$ is bounded below by $\log(4\pi t_0)/2t_0^{1/3}$ for all $t, \beta > 0$ satisfying $t \geq t_0$ and $\beta t \geq t_0$. Furthermore, $\exp(-t^{1/3})/t^{1/3}$ converge to 0 as t increases to ∞. This shows there exists $s_0 = s_0(t_0) > 0$ such that for all $t \geq t_0$, $s \geq s_0$ and β satisfying $\beta t \geq t_0$, the r.h.s. of the above display is greater than $\mathfrak{h}_t(1,0) - \beta^{1/4}s$. This shows $\neg(\mathsf{A}_5 \cup \mathsf{A}_6) \subset \{\mathfrak{h}_t(1+\beta, 0) > \mathfrak{h}_t(1,0) - \beta^{1/4}s\}$ and hence, the claim. From the above claim, we have $\mathbf{P}(\mathfrak{h}_t(1+\beta,0) - \mathfrak{h}_t(1,0) \leq -\beta^{1/4}s) \leq \mathbf{P}(\mathsf{A}_5) + \mathbf{P}(\mathsf{A}_6)$. Using Lemma 4.4.1, we see that $\mathbf{P}(\mathsf{A}_5) \leq e^{-cs^2}$ and Proposition 4.2.14 implies $\mathbf{P}(\mathsf{A}_6) \leq e^{-cs^{5/2}}$. Thus, $\mathbf{P}(\mathfrak{h}_t(1+\beta,0) - \mathfrak{h}_t(0) \leq -\beta^{1/4}s) \leq \mathbf{P}(\mathsf{A}_5) + \mathbf{P}(\mathsf{A}_6) \leq e^{-cs^2}$. This completes the proof.

4.5.3 Proof of Proposition 4.5.5

Fix $\varepsilon \in (0, \frac{1}{4})$. From the scaling of $\mathfrak{h}_t$, it follows that $\mathfrak{h}_t(\alpha, 0) = \alpha^{1/3}\mathfrak{h}_{\alpha t}(1,0)$ for any $\alpha, t > 0$. Hence, it suffices to prove the result for $a = 1$. In the following, we first set up few notations and recall relevant result that we use in this proof. Consider the following events

$$\mathsf{B}_1 := \left\{\sup_{\tau\in[0,1]} \frac{\mathfrak{h}_t(1+\tau,0)-\mathfrak{h}_t(1,0)}{\tau^{\frac{1}{4}-\varepsilon}\log^{2/3}\frac{2}{\tau}} \geq s\right\}, \quad \mathsf{B}_2 := \left\{\inf_{\tau\in[0,1]} \frac{\mathfrak{h}_t(1+\tau,0)-\mathfrak{h}_t(1,0)}{\tau^{\frac{1}{4}-\varepsilon}\log^{1/2}\frac{2}{\tau}} \leq -s\right\}.$$

Set $\kappa_1 = \frac{1}{4} - \varepsilon$ and $\kappa_2 = \frac{1}{12} + \varepsilon$. For any $\alpha_1 > \alpha_2 \geq 1$, define

$$\mathfrak{h}^{\nabla}_{t,\alpha_1,\alpha_2} := \mathfrak{h}_t(\alpha_1, 0) - \mathfrak{h}_t(\alpha_2, 0) = \alpha_2^{1/3}(\mathfrak{h}_{t\alpha_2}(\tfrac{\alpha_1}{\alpha_2}, 0) - \mathfrak{h}_{t\alpha_2}(1,0)),$$

and set $\beta = \frac{\alpha_1}{\alpha_2} - 1$. Combining Proposition 4.5.1 and Proposition 4.5.2, we get $t_0 = t_0(\varepsilon) > 0$, $s_0 = s_0(\varepsilon) > 0$ and $c = c(\varepsilon) > 0$ such that for all $s \geq s_0$ and $2\alpha_2 \geq \alpha_1 > \alpha_2 \geq 1$,

$$\mathbf{P}\left(\frac{\mathfrak{h}^{\nabla}_{t,\alpha_1,\alpha_2}}{(\alpha_1-\alpha_2)^{\kappa_1}} \geq \alpha_2^{\kappa_2}s\right) \leq e^{-cs^{3/2}}, \quad \mathbf{P}\left(\frac{\mathfrak{h}^{\nabla}_{t,\alpha_1,\alpha_2}}{(\alpha_1-\alpha_2)^{\kappa_1}} \leq -\alpha_2^{\kappa_2}s\right) \leq e^{-cs^2}. \tag{4.5.23}$$

Now we proceed to complete the proof. We first construct a dyadic mesh of points of the interval $[1,2]$ and prove the tail bounds of the modulus of continuity over that mesh. Finally, the tail bounds of the modulus of continuity will be extended for all points of $[1,2]$. To begin with, we consider the dyadic partitions $\{\bigcup_{k=1}^{2^n} \mathcal{J}_k^{(n)}\}_{n\in\mathbb{N}}$ of the interval $[1,2]$: $\mathcal{J}_k^{(n)} := [\alpha_{k-1}^{(n)}, \alpha_k^{(n)}]$, with $\alpha_k^{(n)} := 1 + \frac{k}{2^n}$, for $k = 0, 1, \ldots, 2^n$. We now define

$$\mathsf{A}_{\mathrm{up}}(s) := \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n}\Big\{\mathfrak{h}^{\nabla}_{t,\alpha_k^{(n)},\alpha_{k-1}^{(n)}} \geq (\alpha_{k-1}^{(n)})^{\kappa_2}(\alpha_k^{(n)} - \alpha_{k-1}^{(n)})^{\kappa_1}(n\log 2)^{\frac{2}{3}}s\Big\},$$
$$\mathsf{A}_{\mathrm{low}}(s) := \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{2^n}\Big\{\mathfrak{h}^{\nabla}_{t,\alpha_k^{(n)},\alpha_{k-1}^{(n)}} \leq -(\alpha_{k-1}^{(n)})^{\kappa_2}(\alpha_k^{(n)} - \alpha_{k-1}^{(n)})^{\kappa_1}(n\log 2)^{\frac{1}{2}}s\Big\}.$$

By the union bound, we write

$$\mathbb{P}(\mathsf{A}_{\mathrm{up}}(s)) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{2^n}\mathbb{P}\Big(\mathfrak{h}^{\nabla}_{t,\alpha_k^{(n)},\alpha_{k-1}^{(n)}} \geq (\alpha_{k-1}^{(n)})^{\kappa_2}(\alpha_k^{(n)} - \alpha_{k-1}^{(n)})^{\kappa_1}(n\log 2)^{\frac{2}{3}}s\Big). \tag{4.5.24}$$

Applying (4.5.23) in the r.h.s. of (4.5.24), we get

$$\mathbb{P}(\mathsf{A}_{\mathrm{up}}(s)) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{2^n}\exp\big(-cn\log 2s^{\frac{3}{2}}\big) \leq \sum_{n=1}^{\infty}\exp\big(-n\log 2(cs^{\frac{3}{2}}-1)\big) \leq \exp\big(-\tfrac{c}{2}s^{\frac{3}{2}}\big).$$

Fix $\tau \in [\frac{1}{2^{k+1}}, \frac{1}{2^k})$. By continuity of the process $\mathfrak{h}_t(\cdot, 0)$, we have the following on $\neg\mathsf{A}_{\mathrm{up}}(s)$

$$\begin{aligned}
\mathfrak{h}_t(1+\tau,0) - \mathfrak{h}_t(1,0) &= \sum_{n=1}^{\infty}\Big[\mathfrak{h}_t\Big(\tfrac{1}{2^n}\lfloor 2^n(1+\tau)\rfloor, 0\Big) - \mathfrak{h}_t\Big(\tfrac{1}{2^{n-1}}\lfloor 2^{n-1}(1+\tau)\rfloor, 0\Big)\Big] \\
&\leq \sum_{n=1}^{\infty}\Big(\tfrac{\lfloor 2^{n-1}(1+\tau)\rfloor}{2^{n-1}}\Big)^{\kappa_2}\Big(\tfrac{\lfloor 2^n(1+\tau)\rfloor - 2\lfloor 2^{n-1}(1+\tau)\rfloor}{2^n}\Big)^{\kappa_1}(n\log 2)^{\frac{2}{3}}s \\
&\leq 2\sum_{n=k+1}^{\infty}\Big(\tfrac{\lfloor 2^n(1+\tau)\rfloor - 2\lfloor 2^{n-1}(1+\tau)\rfloor}{2^n}\Big)^{\kappa_1}(n\log 2)^{\frac{2}{3}}s \leq c'\tfrac{(k+1)^{2/3}s}{2^{\kappa_1(k+1)}}.
\end{aligned}$$

Clearly for the given range of τ, last term is bounded by $c''s\tau^{\kappa_1}\log^{2/3}\frac{2}{\tau}$. Thus $\mathsf{B}_1 \subset \mathsf{A}_{\mathrm{up}}(s/c'')$ which proves (4.5.5). Similarly we get $\mathsf{B}_2 \subset \mathsf{A}_{\mathrm{low}}(s/\tilde{c})$ for some constant $\tilde{c} > 0$ and using similar summation trick as in (4.5.24), we have $\mathbf{P}(\mathsf{A}_{\mathrm{low}}(s)) \leq e^{-cs^2}$. This proves (4.5.6) and hence, completes the proof of the desired results.

4.6 Law Of Iterated Logarithms

The main goal of this section is to prove Theorem 4.1.1. We will prove the liminf and the limsup result in Section 4.6.1 and 4.6.2 respectively. One of the key ideas of our proof is to approximate multi-point distributions of the KPZ temporal process $\mathfrak{h}_t$ with a set of independent random variables using the multipoint composition law (Proposition 4.2.11). The following proposition captures this idea.

Proposition 4.6.1. *For any* $0 = t_0 < t_1 < t_2 < t_3 < \ldots < t_m$*, with* $\mathfrak{s} := \min_i |\exp(t_{i+1} - t_i) - 1|$*, there exist independent random variables* $Y_1, Y_2, \ldots, Y_m$ *and some constants* $s_0 = s_0(t_0) > 0, c = c(t_0) > 0$ *such that for all* $x\mathfrak{s}^{1/3} \geq s_0$ *and* $1 \leq i \leq m$*,*

$$Y_i \stackrel{d}{=} (1 - e^{-(t_i - t_{i-1})})^{1/3} \mathfrak{h}_{e^{t_i} - e^{t_{i-1}}} \;\; \textit{and} \;\; \mathbf{P}\left(|\mathfrak{h}_{e^{t_i}} - Y_i| \geq x\right) \leq \exp(-cx^{3/2}\mathfrak{s}^{1/2}). \tag{4.6.1}$$

Proof. Denote $\tilde{t}_i := e^{t_i}$ and $\tilde{\beta}_i := (\tilde{t}_i - \tilde{t}_{i-1})/\tilde{t}_{i-1}$. For any $1 \leq i \leq m$, define $Y_i := (1+\tilde{\beta}_i)^{-1/3}\mathfrak{h}_{\tilde{t}_i \downarrow \tilde{t}_{i-1}}$. Recall from Proposition 4.2.11 that $\{\mathfrak{h}_{\tilde{t}_i \downarrow \tilde{t}_{i-1}}\}_{i=1}^m$ are set of independent random variables and $\mathfrak{h}_{\tilde{t}_i \downarrow \tilde{t}_{i-1}}$ is same in distribution with $(1 - (\tilde{t}_{i-1}/\tilde{t}_i))^{-1/3}\mathfrak{h}_{\tilde{t}_i - \tilde{t}_{i-1}}$. From this, it follows that $Y_1, \ldots, Y_m$ are independent and $Y_i \stackrel{d}{=} (1 - \exp(-(t_i - t_{i-1})))^{1/3}\mathfrak{h}_{e^{t_i} - e^{t_{i-1}}}$. Furthermore, applying Proposition 4.5.4 with setting $t := \tilde{t}_{i-1}$, $\beta := \tilde{\beta}_i$ and $s := x\tilde{\beta}_i^{1/3}$, there exists $s_0 > 0$ such that for all x satisfying $x\mathfrak{s}^{1/3} \geq s_0$,

$$\mathbf{P}\Big(\big|\mathfrak{h}_{\tilde{t}_{i-1}}(1 + \tilde{\beta}_i, 0) - \mathfrak{h}_{\tilde{t}_i \downarrow \tilde{t}_{i-1}}(1, 0)\big| \geq x\tilde{\beta}_i^{1/3}\Big) \leq \exp(-cx^{3/2}\tilde{\beta}_i^{1/2})$$

for some absolute constant $c > 0$ which does not depend on $t_1, \ldots, t_m$. Note that $\mathfrak{h}_{\tilde{t}_{i-1}}(1 + \tilde{\beta}_i, 0)$ is equal to $(1 + \tilde{\beta}_i)^{1/3}\mathfrak{h}_{\tilde{t}_i}$. Furthermore, $\mathfrak{h}_{\tilde{t}_i \downarrow \tilde{t}_{i-1}}(1, 0) \stackrel{d}{=} (1 + \tilde{\beta}_i)^{1/3}Y_i$. As a result, we obtain

$$\mathbf{P}\big(|\mathfrak{h}_{\tilde{t}_i} - Y_i| \geq x\tilde{\beta}_i^{1/3}(1 + \tilde{\beta}_i)^{-1/3}\big) \leq \exp(-cx^{3/2}\tilde{\beta}_i^{1/2}) \leq \exp(-cx^{3/2}\mathfrak{s}^{1/2}),$$

where the last inequality follows since $\tilde{\beta}_i \geq \min_i(e^{t_i - t_{i-1}} - 1) = \mathfrak{s}$. (4.6.1) follows from the above inequality. $\square$

4.6.1 Proof of Liminf

In this section, we will prove that the liminf of $\mathfrak{h}_t/(\log\log t)^{1/3}$ is almost surely equal to $-6^{1/3}$. For any given $\epsilon > 0$, we show that the following hold

$$\underbrace{-\big(6(1+\epsilon)\big)^{1/3} \leq \liminf_{t\to\infty} \frac{\mathfrak{h}_t}{(\log\log t)^{1/3}},}_{\mathfrak{LimInf}_l} \qquad \underbrace{\liminf_{t\to\infty} \frac{\mathfrak{h}_t}{(\log\log t)^{1/3}} \leq -\big(6(1-\epsilon)\big)^{1/3}}_{\mathfrak{LimInf}_u}$$

with probability 1 in Section 4.6.1 and 4.6.1 respectively. By letting $\epsilon \to 0$ in the above two inequalities, it follows that $\liminf \mathfrak{h}_t/(\log\log t)^{1/3}$ is equal to $-6^{1/3}$.

Proof of $\mathfrak{LimInf}_u$

For any $n \in \mathbb{N}$, define $\mathcal{I}_n := [\exp(e^n), \exp(e^{n+1})]]$. Fix any $\epsilon \in (0,1)$ and set $\gamma := (6(1-\epsilon))^{1/3}$. We will show that

$$\sum_{n=1}^{\infty} \mathbf{P}\Big(\inf_{t\in\mathcal{I}_n} \frac{\mathfrak{h}_t}{(\log\log t)^{1/3}} \geq -\gamma\Big) < \infty. \tag{4.6.2}$$

Clearly then by the Borel-Cantelli lemma, we have $\mathfrak{LimInf}_u$ with probability 1. Choose $\eta > 0$ small such that $(\frac{1}{6}+\eta)(\gamma+2\eta)^3 < 1$. We define $\zeta := (\frac{1}{6}+\eta)(\gamma+2\eta)^3$. Fix $\theta \in (\zeta, 1)$ and choose $\delta \in (0, \theta-\zeta)$. For any $n \geq 1$, we consider the following sub-intervals of $\mathcal{I}_n$,

$$\mathcal{I}_n^{(j)} := [e^{e^n + (j-1)e^{n\theta}}, e^{e^n + je^{\theta n}}], \quad 1 \leq j \leq \mathcal{M}_\theta := \lfloor e^{n-\theta n+1} - e^{n-n\theta} \rfloor. \tag{4.6.3}$$

By the union bound, we have

$$\mathbf{P}\big(\inf_{t\in\mathcal{I}_n} \frac{\mathfrak{h}_t}{(\log\log t)^{\frac{1}{3}}} \geq -\gamma\big) \leq \sum_{j=1}^{\mathcal{M}_\theta} \mathbf{P}\big(\inf_{t\in\mathcal{I}_n^{(j)}} \frac{\mathfrak{h}_t}{(\log\log t)^{\frac{1}{3}}} \geq -\gamma\big) \leq \sum_{j=1}^{\mathcal{M}_\theta} \mathbf{P}(\mathsf{A}_n^{(j)}), \tag{4.6.4}$$

where $\mathsf{A}_n^{(j)} := \left\{ \inf_{t\in\mathcal{I}_n^{(j)}} \mathfrak{h}_t \geq -(n+1)^{1/3}\gamma \right\}$. The last inequality in above follows since $\max_{t\in\mathcal{I}_n^{(j)}} \log\log t \leq (n+1)$. Now we bound each term of the above sum. We now claim that there exists constants $c_1, c_2 > 0$ such that

$$\mathbf{P}(\mathsf{A}_n^{(j)}) \leq \exp(-ce^{n(\theta-\delta)}e^{-n\zeta}) + \exp\left(n(\theta-\delta) - c_2(\exp(e^{n\delta})-1)^{1/2}\right) \tag{4.6.5}$$

for all $1 \leq j \leq n$ and all large n. We first assume (4.6.5) and complete the proof of (4.6.2). Using (4.6.5), we may estimate the r.h.s. of (4.6.4) as

$$\text{r.h.s. of (4.6.4)} \leq e^{n-n\theta+1}\Big(e^{-ce^{n(\theta-\delta)}e^{-n\zeta}} + e^{n(\theta-\delta)-c_2(\exp(e^{n\delta})-1)^{1/2}}\Big). \tag{4.6.6}$$

Here, the factor $e^{n-n\theta+1}$ is an upper bound to the number of summands in (4.6.4). Recalling that $\theta > \zeta+\delta$, we observe that the r.h.s. of the above display can be bounded by $\exp(-c_1e^{n\omega})$ for some constant $c_1 > 0$ and $\omega \in (0,1)$ for all large n. This shows the sum in (4.6.2) is finite and hence, completes the proof of $\mathfrak{LimInf}_u$ modulo (4.6.5) which we show as follows.

Fix $j \in \{1,\ldots,\mathcal{M}_\theta\}$ and some constant $a > 1$. We choose a sequence $t_1 < t_2 < \cdots < t_{L_n} \in [e^n + (j-1)e^{n\theta}, e^n + je^{n\theta}]$ such that $\min|t_{i+1}-t_i| \geq e^{n\delta}$ and $a^{-1}(e^{n(\theta-\delta)}) \leq L_n \leq a(e^{n(\theta-\delta)})$. Applying Proposition 4.6.1, we get $Y_1, Y_2, \ldots, Y_{L_n}$ such that (4.6.1) (with $\mathfrak{s} \geq e^{n\delta}$) will be satisfied for the above choice of $t_1, t_2, \ldots, t_{L_n}$. As a result, we get

$$\begin{aligned}
\mathbf{P}(\mathsf{A}_n^{(j)}) &\leq \mathbf{P}\big(\min_{1\leq i\leq L_n} \mathfrak{h}_{e^{t_i}} \geq -(n+1)^{1/3}\gamma\big) \\
&\leq \mathbf{P}\big(\min_{1\leq i\leq L_n} Y_i \geq -(n+1)^{1/3}\gamma - 1\big) + \sum_{i=1}^{L_n} \mathbf{P}(\mathfrak{h}_{e^{t_i}} - Y_i \geq 1) \\
&\leq \prod_{i=1}^{L_n} \mathbf{P}\big(Y_i \geq -(n+1)^{1/3}\gamma - 1\big) + a\exp\left(n(\theta-\delta) - c(\exp(e^{n\delta})-1)^{1/2}\right),
\end{aligned} \tag{4.6.7}$$

where in the last line we use the independence of Y_i to write $\mathbf{P}\big(\min_{1\leq i\leq L_n} Y_i \geq -(n+1)^{1/3}\gamma - 1\big)$ as a product over $\mathbf{P}\big(Y_i \geq -(n+1)^{1/3}\gamma - 1\big)$ and use the inequality in (4.6.1) to bound $\mathbf{P}(\mathfrak{h}_{e^{t_i}} - Y_i \geq 1)$.

Using the distributional identity of (4.6.1) , we get

$$\begin{aligned}\prod_{i=1}^{L_n}\mathbf{P}\left(Y_i \geq -(n+1)^{1/3}\gamma - 1\right) &= \prod_{i=1}^{L_n}\mathbf{P}\left((1-e^{-(t_i-t_{i-1})})^{1/3}\mathfrak{h}_{e^{t_i}-e^{t_{i-1}}} \geq -(n+1)^{1/3}\gamma - 1\right)\\ &\leq \prod_{i=1}^{L_n}\mathbf{P}\big(\mathfrak{h}_{e^{t_i}-e^{t_{i-1}}} \geq -n^{1/3}(\gamma+\eta)\big)\\ &\leq \big[1-\exp(-(\tfrac{1}{6}+\eta)n(\gamma+2\eta)^3)\big]^{L_n}\\ &\leq \exp\big(-L_n\exp(-(\tfrac{1}{6}+\eta)n(\gamma+2\eta)^3)\big) \leq e^{-e^{n(\theta-\delta-\zeta)}/a},\end{aligned}$$

where the first inequality follows by noting that $(1-e^{-(t_i-t_{i-1})})^{-1/3}((n+1)^{1/3}\gamma+1) \leq n^{1/3}(\gamma+\eta)$ for all large n and the second inequality follows due to (4.2.16) of Proposition 4.2.13. The last inequality follows since $L_n \geq e^{n(\theta-\delta)}/a$ and $\zeta = (\frac{1}{6}+\eta)(\gamma+2\eta)^3$. Substituting the inequality of the above display into the r.h.s. of (4.6.7) yields the inequality (4.6.5). This completes the proof.

Proof of $\mathfrak{LimInf}_l$

Fix $t_0 > 0$. Define $\psi : \mathbb{R}_{>1} \to \mathbb{R}_{>0}$ as $\psi(\alpha) = \alpha^{1/3}(\log\log\alpha)^{1/3}$. Let $\alpha_n := 2^n$ and $k_n := \lfloor(\log\log\alpha_n)^4\rfloor$ for any $n \in \mathbb{N}$. Let us denote $\mathcal{I}_n := [\alpha_n, \alpha_{n+1}]$ and its k_n many equal length subintervals as $\mathcal{I}_n^{(j)} := [(1+\frac{j-1}{k_n})\alpha_n, (1+\frac{j}{k_n})\alpha_n]$ for $1 \leq j \leq k_n$. Fix $\epsilon > 0$. Set $s = (6(1+\epsilon))^{\frac{1}{3}}$. We will show that

$$\sum_{n=1}^{\infty}\mathbf{P}\Big(\inf_{\alpha\in\mathcal{I}_n}\tfrac{\mathfrak{h}_{t_0}(\alpha,0)}{\psi(\alpha)} \leq -s\Big) < \infty. \tag{4.6.8}$$

Applying (4.6.8) and Borel-Cantelli lemma, we can conclude that $\liminf_{\alpha\to\infty}\frac{\mathfrak{h}_{t_0}(\alpha,0)}{\psi(\alpha)} = \liminf_{t\to\infty}\frac{\mathfrak{h}_t}{(\log\log t)^{1/3}} \geq -s$ with probability 1 where the equality is obtained by substituting $t = \alpha t_0$. Letting $\epsilon \to 0$ on the r.h.s. of the above inequality yields $\mathfrak{LimInf}_l$. It boils down to showing (4.6.8) which we do as follows. We claim that there exist constant $c_1 > 0$ and $c_2 > 1$ such that

$$\mathbf{P}\Big(\inf_{\alpha\in\mathcal{I}_n}\tfrac{\mathfrak{h}_{t_0}(\alpha,0)}{\psi(\alpha)} \leq -s\Big) \leq (\log\log\alpha_n)^4\Big(e^{-c_1(\log\log\alpha_n)^{7/6}} + e^{-c_2\log\log\alpha_n}\Big) \tag{4.6.9}$$

for all large n. Recall that $\alpha_n = 2^n$. Substituting this into the r.h.s. of the above inequality, we see that (4.6.8) holds modulo the last inequality. We now proceed to prove this last inequality. Let N be the smallest positive integer such that $\alpha_N \geq e^e$. For any $n \geq N$, using the union bound we have

$$\mathbf{P}\Big(\inf_{\alpha\in\mathcal{I}_n} \tfrac{\mathfrak{h}_{t_0}(\alpha,0)}{\psi(\alpha)} \leq -s\Big) \leq \sum_{j=1}^{k_n} \mathbf{P}\Big(\inf_{\alpha\in\mathcal{I}_n^{(j)}} \tfrac{\mathfrak{h}_{t_0}(\alpha,0)}{\psi(\alpha)} \leq -s\Big). \tag{4.6.10}$$

In what follows, we will show

$$\mathbf{P}\Big(\inf_{\alpha\in\mathcal{I}_n^{(j)}} \tfrac{\mathfrak{h}_{t_0}(\alpha,0)}{\psi(\alpha)} \leq -s\Big) \leq e^{-c_1(\log\log\alpha_n)^{7/6}} + e^{-c_2\log\log\alpha_n} \tag{4.6.11}$$

for all $1 \leq j \leq n$, n large and some constant $c_1 > 0$ and $c_2 > 1$. Substituting the above inequality into right side of (4.6.10) and recalling that $k_n \leq (\log\log\alpha_n)^4$ show (4.6.9). We now focus on proving (4.6.11). Fix any $j \in \{1,\ldots,k_n\}$. Denote the left and right end point of $\mathcal{I}_n^{(j)}$ by a_n and b_n. We choose $\eta \in (0,1)$ such that $(1-\eta)^4(1+\epsilon) > 1$. Using the fact that $\psi(\alpha)$ is an increasing function of α, we get

$$\begin{aligned}
\mathbf{P}\Big(\inf_{\alpha\in\mathcal{I}_n^{(j)}} \tfrac{\mathfrak{h}_{t_0}(\alpha,0)}{\psi(\alpha)} \leq -v\Big) &\leq \mathbf{P}\Big(\inf_{\alpha\in\mathcal{I}_n^{(j)}} \mathfrak{h}_{t_0}(\alpha,0) \leq -s\psi(a_n)\Big) \\
&\leq \mathbf{P}\Big(\inf_{\alpha\in\mathcal{I}_n^{(j)}} \mathfrak{h}_{t_0}(\alpha,0) - \mathfrak{h}_{t_0}(a_n,0) \leq -\eta s\psi(a_n)\Big) \\
&\quad + \mathbf{P}\Big(\mathfrak{h}_{t_0}(a_n,0) \leq -(1-\eta)s\psi(a_n)\Big),
\end{aligned} \tag{4.6.12}$$

where the last inequality follows by the union bound. We now apply (4.5.6) of Proposition 4.5.5 and (4.2.16) of Proposition 4.2.13 to bound the first and the second term of the right side of the last inequality. To apply (4.5.6), we set $\varepsilon = \frac{1}{8}$. We may now write

$$\begin{aligned}
&\mathbf{P}\Big(\inf_{\alpha\in\mathcal{I}_n^{(j)}} \mathfrak{h}_{t_0}(\alpha,0) - \mathfrak{h}_{t_0}(a_n,0) \leq -\eta s\psi(a_n)\Big) \\
&\quad \leq \mathbf{P}\Big(\inf_{\tau\in[0,k_n^{-1}]} \tfrac{\mathfrak{h}_{t_0}(a_n+\tau,0)-\mathfrak{h}_{t_0}(a_n,0)}{(\tau/a_n)^{1/8}(\log|\tau/a_n|)^{1/2}} \leq -\tfrac{\eta s\psi(a_n)}{k_n^{-1/8}(\log|k_n|)^{1/2}}\Big)
\end{aligned}$$

$$\leq \exp\big(-c(\eta s)^2 k_n^{1/4}(\log|k_n|)^{-1}(\log\log\alpha_n)^{2/3}\big), \tag{4.6.13}$$

where the second inequality follows by applying (4.5.6). Since $k_n = \lfloor(\log\log\alpha_n)^4\rfloor$, we get the following bound

$$k_n^{1/4}(\log|k_n|)^{-1}(\log\log\alpha_n)^{2/3} \geq (\log\log\alpha_n)^{\frac{1}{2}+\frac{2}{3}} = (\log\log\alpha_n)^{7/6}$$

for all large n. By substituting inequality into the r.h.s. of (4.6.13), we may bound the first term in the r.h.s. of (4.6.12) by $\exp(-c_1(\log\log\alpha_n)^{7/6})$ for all large integer n where c_1 is a positive which does not depend on n. On the other hand, (4.2.16) of Proposition 4.2.13 implies

$$\mathbf{P}\Big(\mathfrak{h}_{t_0}(a_n,0) \leq -(1-\eta)s\psi(a_n)\Big) \leq e^{-(1-\eta)^4(1+\epsilon)(\log\log a_n)} \leq e^{-c_2\log\log\alpha_n} \tag{4.6.14}$$

for all large n where c_2 is a constant greater than 1. The second inequality of the above display follows since $a_n \geq \alpha_n$ and $(1-\eta)^4(1+\epsilon) > 1$. Combining the bounds in (4.6.13) and (4.6.14) and substituting those into (4.6.12) shows (4.6.11). This completes the proof of $\mathfrak{Liminf}_l$.

4.6.2 Proof of Limsup

The main goal of this section is to prove the limsup result of the law of iterated logarithms for which we need to show that for any $\epsilon \in (0,1)$,

$$\underbrace{\limsup_{t\to\infty}\frac{\mathfrak{h}_t}{(\log\log t)^{2/3}} \geq \left(\frac{3(1-\epsilon)}{4\sqrt{2}}\right)^{2/3}}_{\mathfrak{LimSup}_l}, \quad \underbrace{\limsup_{t\to\infty}\frac{\mathfrak{h}_t}{(\log\log t)^{2/3}} \leq \left(\frac{3(1+\epsilon)}{4\sqrt{2}}\right)^{2/3}}_{\mathfrak{LimSup}_u}$$

with probability 1. In what follows, we first show $\mathfrak{LimSup}_u$. As we discuss in the next section, $\mathfrak{LimSup}_u$ will imply that the macroscopic Hausdorff dimension of the level sets $\{t \geq e^e : \mathfrak{h}_t/(\log\log t)^{2/3} \geq (3(1+\epsilon)/4\sqrt{2})^{2/3}\}$ are equal to 0 with probability 1 for any $\epsilon > 0$ proving partially (4.1.4).

Proof of $\mathfrak{LimSup}_u$

Fix $\epsilon, \theta \in (0,1)$ and $t_0 > 0$. Define $\phi : [e^e, \infty) \to \mathbb{R}$ by $\phi(x) = x^{1/3}(\log\log x)^{2/3}$. We note that $\phi(x)$ is increasing in x. Define $s := (3(1+\epsilon)/4\sqrt{2})^{2/3}$. Fix $\delta \in (0,1)$. We will make the choice δ precise in due course of the proof. For any $n \in \mathbb{N}$, we define $\alpha_n := (1+\delta)^i$ and denote $\mathcal{I}_n := [\alpha_n, \alpha_{n+1}]$. We claim and prove that

$$\sum_{n\geq N} \mathbf{P}\Big(\sup_{\alpha\in\mathcal{I}_n} \tfrac{\mathfrak{h}_t(\alpha,0)}{\phi(\alpha)} \geq s\Big) < \infty \tag{4.6.15}$$

for all $\epsilon > 0$. Then by Borel-Cantelli lemma, we get $\limsup_{\alpha\to\infty} \frac{\mathfrak{h}_t(\alpha,0)}{\phi(\alpha)} \leq s$ holds with probability 1 for all large t. $\mathfrak{LimSup}_u$ now follows by noting

$$\begin{aligned}
\limsup_{t\to\infty} \tfrac{\mathfrak{h}_t}{(\log\log t)^{2/3}} &= \limsup_{\alpha\to\infty} \tfrac{\mathfrak{h}_{\alpha t_0}}{(\log\log \alpha t_0)^{2/3}} \\
&= \limsup_{\alpha\to\infty} \Big[\tfrac{\mathfrak{h}_{t_0}(\alpha,0)}{\alpha^{1/3}(\log\log\alpha)^{2/3}} \cdot \big(\tfrac{\log\log \alpha t_0}{\log\log\alpha}\big)^{2/3}\Big] = \limsup_{\alpha\to\infty} \tfrac{\mathfrak{h}_{t_0}(\alpha,0)}{\phi(\alpha)}.
\end{aligned}$$

To prove (4.6.15) we show the following: there exists and $c > 1$ such that

$$\mathbf{P}\Big(\sup_{\alpha\in\mathcal{I}_n} \tfrac{\mathfrak{h}_t(\alpha,0)}{\phi(\alpha)} \geq s\Big) \leq \exp(-c\log\log\alpha_n) \tag{4.6.16}$$

for all large n and t. Let N be the smallest positive integer such that $\alpha_N \geq e^e$. For $n \geq N$ and $\eta \in (0,1)$, we have

$$\begin{aligned}
\mathbf{P}\Big(\sup_{\alpha\in\mathcal{I}_n} \frac{\mathfrak{h}_{t_0}(\alpha,0)}{\phi(\alpha)} \geq s\Big) &\leq \mathbf{P}\Big(\sup_{\alpha\in\mathcal{I}_n} \mathfrak{h}_{t_0}(\alpha,0) \geq s\phi(\alpha_i)\Big) \\
&\leq \mathbf{P}\Big(\sup_{\alpha\in\mathcal{I}_n} \mathfrak{h}_{t_0}(x,0) - \mathfrak{h}_{t_0}(\alpha_i,0) \geq \eta s\phi(\alpha_i)\Big) + \mathbf{P}\Big(\tfrac{\mathfrak{h}_{t_0}(\alpha_i,0)}{\phi(\alpha_i)} \geq (1-\eta)s\Big),
\end{aligned} \tag{4.6.17}$$

where the first inequality follows since ϕ is an increasing function of α and the second inequality follows by the union bound. We proceed to bound the two terms in the r.h.s. of the last display. For the first term, we seek to apply (4.5.5) of Proposition 4.5.5. We set $\varepsilon = \frac{1}{8}$ in Proposition 4.5.5, and define $r := \sup_{\tau\in(0,\delta]} \tau^{\frac{1}{8}}\big(\log(1/\tau)\big)^{\frac{2}{3}}$. It straightforward to see that r decreases to 0 as δ goes to 0. We may now write

$$\mathbf{P}\Big(\sup_{\alpha\in\mathcal{I}_n} \mathfrak{h}_{t_0}(\alpha,0)-\mathfrak{h}_{t_0}(\alpha_n,0)\geq \eta s\phi(\alpha_i)\Big)\leq \mathbf{P}\Big(\sup_{\tau\in(0,\delta]}\tfrac{\mathfrak{h}_{t_0}((1+\tau)\alpha_n,0)-\mathfrak{h}_{t_0}(\alpha_n,0)}{\tau^{1/8}\log^{2/3}(1/\tau)}\geq \eta\tfrac{s}{r}\phi(\alpha_n)\Big)$$
$$\leq \exp\big(-c\big(s\eta r^{-1}(\log\log\alpha_n)^{2/3}\big)^{3/2}\big) \qquad (4.6.18)$$

where the last inequality follows from Proposition 4.5.5. For any fixed η, we choose $\delta > 0$ small such that $c(\eta r^{-1})^{3/2} > 1$. For such choice of δ, the r.h.s. of the last line of the above display will be bounded above by $\exp(-c_1\log\log\alpha_n)$ for some constant $c_1 > 1$. This bounds the first term in the r.h.s. of (4.6.17). We now proceed to bound the second term. Note that $\mathfrak{h}_{t_0}(\alpha_n,0)/\alpha_n^{1/3}$ is same as $\mathfrak{h}_{\alpha_n t_0}(1,0)=\mathfrak{h}_{\alpha_n t_0}$. Using the second inequality of (4.2.15) in Proposition 4.2.13, for all large t and n

$$\mathbf{P}\Big(\tfrac{\mathfrak{h}_{t_0}(\alpha_n,0)}{\phi(\alpha_n)}\geq(1-\eta)s\Big)\leq \exp\Big(-\tfrac{4\sqrt{2}}{3}(1-\gamma)^{5/2}s^{3/2}\log\log\alpha_n\Big).$$

Recall that η is chosen in a way such that $(1-\eta)^{5/2}(1+\epsilon) > 0$. Since $\frac{4\sqrt{2}}{3}(1-\gamma)^{5/2}s^{3/2} = (1-\eta)^{5/2}(1+\epsilon)$, the r.h.s. of the above display is bounded by $\exp(-c_2\log\log\alpha_n)$ for some constant $c_2 > 0$. Combining this upper bound with the bounds in (4.6.18) an substituting those into the r.h.s. of (4.6.17) yields (4.6.16). This completes the proof.

Proof of $\mathfrak{LimSup}_l$

We prove $\mathfrak{LimSup}_l$ using similar argument as in the proof of $\mathfrak{LimInf}_u$ (see Section 4.6.1). Recall the definitions of the interval $\mathcal{I}_n$ from Section 4.6.1. Set $\gamma := (3(1-\epsilon)/4\sqrt{2})^{2/3}$. Due to Borel-Cantelli lemma, it suffices to show

$$\sum_{n=1}^{\infty}\mathbf{P}\Big(\sup_{t\in\mathcal{I}_n}\tfrac{\mathfrak{h}_t}{(\log\log t)^{1/3}}\leq\gamma\Big)<\infty. \qquad (4.6.19)$$

Choose $\eta > 0$ such that $\zeta := (\frac{4\sqrt{2}}{3} + \eta)(\gamma + 2\eta)^{3/2} < 1$. Fix $\theta \in (\zeta, 1)$ and $\delta \in (0, \theta - \zeta)$. With this θ, recall the definition of the subintervals $\{\mathcal{I}_n^{(j)}\}_{j=1}^{\mathcal{M}_\theta}$ from (4.6.3). Set $\tilde{\mathsf{A}}_n^{(j)} := \{\sup_{t \in \mathcal{I}_n^{(j)}} \mathfrak{h}_t \leq (n+1)^{2/3}\gamma\}$. By union bound $\mathbf{P}\Big(\sup_{t\in\mathcal{I}_n} \frac{\mathfrak{h}_t}{(\log\log t)^{1/3}} \leq \gamma\Big) \leq \sum_{j=1}^{\mathcal{M}_\theta} \mathbf{P}(\tilde{\mathsf{A}}_n^{(j)})$. In a similar way as in (4.6.5), we claim that there exists $c_1, c_2 > 0$ such that

$$\mathbf{P}(\tilde{\mathsf{A}}_n^{(j)}) \leq \exp(-c_1 e^{n(\theta-\delta)} e^{-n\zeta}) + \exp\left(n(\theta - \delta) - c_2(\exp(e^{n\delta}) - 1)^{1/2}\right) \tag{4.6.20}$$

for all $1 \leq j \leq n$ and all large n. Using this upper bound on $\mathbf{P}(\tilde{\mathsf{A}}_n^{(j)})$, we may bound each term in the sum (4.6.19) exactly in the same way as in (4.6.6). Since $\theta > \zeta + \delta$ by our choice, we may also bound each term of the sum in (4.6.19) by $\exp(-e^{n\omega})$ for some $\omega \in (0, 1)$. This shows the finiteness of the sum in (4.6.19). To complete the proof, it boils down to showing (4.6.20) which we do as follows.

Fix $j \in \{1, \ldots, \mathcal{M}_\theta\}$ and some constant $a > 1$. We choose a sequence $t_1 < t_2 < \cdots < t_{L_n} \in [e^n + (j-1)e^{n\theta}, e^n + je^{n\theta}]$ such that $\min|t_{i+1} - t_i| \geq e^{n\delta}$ and $a^{-1}(e^{n(\theta-\delta)}) \leq L_n \leq a(e^{n(\theta-\delta)})$. Proposition 4.6.1 shows the existence of independent r.v. $Y_1, Y_2, \ldots, Y_{L_n}$ satisfying (4.6.1) (with $\mathfrak{s} \geq e^{n\delta}$) for the above choice of $t_1, t_2, \ldots, t_{L_n}$. Using similar ideas as in (4.6.7), we can write

$$\begin{aligned}
\mathbf{P}(\tilde{\mathsf{A}}_n^{(j)}) &\leq \mathbf{P}\big(\sup_{1\leq i\leq L_n} \mathfrak{h}_{e^{t_i}} \leq (n+1)^{2/3}\gamma\big) \\
&\leq \mathbf{P}\big(\sup_{1\leq i\leq L_n} Y_i \leq (n+1)^{2/3}\gamma + 1\big) + \sum_{i=1}^{L_n} \mathbf{P}(Y_i - \mathfrak{h}_{e^{t_i}} \geq 1) \\
&\leq \prod_{i=1}^{L_n} \mathbf{P}\big(Y_i \leq (n+1)^{2/3}\gamma + 1\big) + a\exp\left(n(\theta-\delta) - c(\exp(e^{n\delta}) - 1)^{1/2}\right)
\end{aligned}$$

Now we apply the distributional identity of (4.6.1) to write

$$\begin{aligned}
\prod_{i=1}^{L_n} \mathbf{P}\left(Y_i \leq (n+1)^{2/3}\gamma + 1\right) &= \prod_{i=1}^{L_n} \mathbf{P}\left((1 - e^{-(t_i - t_{i-1})})^{1/3} \mathfrak{h}_{e^{t_i} - e^{t_{i-1}}} \leq (n+1)^{2/3}\gamma + 1\right) \\
&\leq \prod_{i=1}^{L_n} \mathbf{P}\big(\mathfrak{h}_{e^{t_i} - e^{t_{i-1}}} \leq n^{2/3}(\gamma + \eta)\big)
\end{aligned}$$

$$\leq \left[1 - \exp(-(\eta + \tfrac{4\sqrt{2}}{3})n(\gamma + 2\eta)^{3/2})\right]^{L_n}$$
$$\leq \exp\big(- L_n \exp(-(\eta + \tfrac{4\sqrt{2}}{3})n(\gamma + 2\eta)^{3/2})\big) \leq e^{-e^{n(\theta-\delta-\zeta)}/a},$$

where the first inequality follows by noting that $(1 - e^{-(t_i - t_{i-1})})^{-1/3}((n+1)^{1/3}\gamma + 1) \leq n^{1/3}(\gamma + \eta)$ for all large n and the second inequality follows due to (4.2.15) of Proposition 4.2.13. The last inequality follows since $L_n \geq e^{n(\theta-\delta)}/a$ and $\zeta = (\eta + \frac{4\sqrt{2}}{3})(\gamma + 2\eta)^{3/2}$. Substituting the inequality in the above display into the r.h.s. of (4.6.20) completes the proof of (4.6.20).

4.7 Mono- and Multifractality of the KPZ equation

The aim of this section is to prove Theorem 4.1.3. The monofractality result of the KPZ equation which is stated in (4.1.4) will be proved in Section 4.7.1 where the multifractality result of (4.1.5) will be proved in Section 4.7.2.

4.7.1 Monofractality: Proof of 4.1.4

By the inequality $\mathfrak{LimSup}_u$, we know that the limsup of $\mathfrak{h}_t/(\log\log t)^{2/3}$ as t goes to ∞ is strictly less than γ with probability 1 for any $\gamma > (3/4\sqrt{2})^{2/3}$. This shows $\{t \geq e^e : \mathfrak{h}_t/(\log\log t)^{2/3} \geq \gamma\}$ is almost surely bounded. From Proposition 4.2.6, it follows that the Barlow-Taylor Hausdorff dimension of a bounded set is zero. This shows $\mathrm{Dim}_{\mathbb{H}}(\{t \geq e^e : \mathfrak{h}_t/(\log\log t)^{2/3} \geq \gamma\}) \overset{a.s.}{=} 0$ for any $\gamma > (3/4\sqrt{2})^{2/3}$. We now proceed to prove $\mathrm{Dim}_{\mathbb{H}}(\{t \geq e^e : \mathfrak{h}_t/(\log\log t)^{2/3} \geq \gamma\}) = 1$ with probability 1 for any $\gamma \leq (3/4\sqrt{2})^{2/3}$. For this, it suffices to show that

$$\mathrm{Dim}_{\mathbb{H}}(\mathcal{P}_{\mathfrak{h}}) \overset{a.s.}{=} 1, \quad \text{where } \mathcal{P}_{\mathfrak{h}} := \left\{t \geq e^e : \tfrac{\mathfrak{h}_t}{(\log\log t)^{2/3}} \geq \tfrac{3}{4\sqrt{2}}\right\}. \tag{4.7.1}$$

Throughout the rest of this section, we show (4.7.1). Denote $\gamma_0 := (3/4\sqrt{2})^{2/3}$ and set $\mathsf{A}_s := \left\{\frac{\mathfrak{h}_s}{(\log\log s)^{2/3}} \geq \gamma_0\right\}$ for any $s > 0$. For showing (4.7.1), we need the following two propositions. These two propositions will shed light on the nature of dependence between A_t and A_s when t and s are far from each other and 1-dimensional Hausdorff content (see Definition 4.2.5) of the the set

$\mathcal{P}_{\mathfrak{h}}$. We first complete the proof of (4.7.1) using these two propositions and then, those will be proved in two ensuing subsections.

We are now ready to state Proposition 4.7.1 which will demonstrate that A_t and A_s are approximately independent when t and s are sufficiently far apart.

Proposition 4.7.1. *There exist $T_0 > 0$, such that for all $t > s \geq T_0$ with*

$$t \geq s(\log\log t)^3(\log\log s + \log\log t)^2, \tag{4.7.2}$$

we have $\mathbf{P}(\mathsf{A}_s \cap \mathsf{A}_t) = (1 + o(1))\mathbf{P}(\mathsf{A}_s)\mathbf{P}(\mathsf{A}_t)$, where $o(1)$ goes to zero as $s, t \to \infty$.

The next proposition will investigate 1-dimensional Hausdorff contents of the set $\mathcal{P}_{\mathfrak{h}}$.

Proposition 4.7.2. *Denote $\mathcal{V}_n := [-e^n, e^n]$ and $\mathcal{S}_0 := \mathcal{V}_0, \mathcal{S}_{n+1} := \mathcal{V}_{n+1} \setminus \mathcal{V}_n$ for $n \in \mathbb{N}$. For any Borel set G, define $\mu(G) := \mathrm{Leb}\left(\mathcal{P}_{\mathfrak{h}} \cap G\right)$. We have*

$$\sum_{n=4}^{\infty} e^{-n}\mu(\mathcal{S}_n) \overset{a.s.}{=} \infty. \tag{4.7.3}$$

Assuming Proposition 4.7.1 and 4.7.2, we proceed to complete the proof of (4.7.1).

Proof of (4.7.1). Recall the definition of ρ-dimensional Hausdorff content $\nu_{n,\rho}$ from Definition 4.2.5. By Proposition 4.2.7, there exists some constant $K_{1,n} > 0$ (defined in (4.2.3)) such that $\nu_{n,1}(\mathcal{P}_{\mathfrak{h}}) \geq K_{1,n}^{-1}e^{-n}\mu(\mathcal{P}_{\mathfrak{h}})$. Since $\mu(Q) \leq \mathrm{Leb}(Q)$ for any finite Borel set Q, $K_{1,n}$ is less than 1. This implies $\nu_{n,1}(\mathcal{P}_{\mathfrak{h}}) \geq e^{-n}\mu(\mathcal{P}_{\mathfrak{h}})$. Combining this inequality with (4.7.3) of Proposition 4.7.2 yields $\sum_{n=4}^{\infty} \nu_{n,1}(\mathcal{P}_{\mathfrak{h}}) \geq \sum_{n=4}^{\infty} e^{-n}\mu(\mathcal{S}_n) \overset{a.s.}{=} \infty$. From Definition 4.2.5 it now follows that $\mathrm{Dim}_{\mathbb{H}}(\mathcal{P}_{\mathfrak{h}}) = 1$ occurs with probability 1. This completes the proof. $\square$

Proof of Proposition 4.7.1. By Proposition 2.7 (FKG inequality) in [117] we know $\mathbf{P}(\mathsf{A}_t \cap \mathsf{A}_s) \geq \mathbf{P}(\mathsf{A}_t)\mathbf{P}(\mathsf{A}_s)$. It suffices to show that $\mathbf{P}(\mathsf{A}_t \cap \mathsf{A}_s) \leq (1+o(1))\mathbf{P}(\mathsf{A}_t)\mathbf{P}(\mathsf{A}_s)$ as $s, t \to \infty$. For showing this, we use Proposition 4.6.1. Fix $t > s > T_0$ such that t, s satisfy the inequality (4.7.2). Note that $(\log\log t)^{-1/2}(\frac{t}{s} - 1)^{1/3} \to \infty$ as $s, t \to \infty$. By Proposition 4.6.1, there exists a r.v. Y independent of $\mathfrak{h}_s$ and constant $c > 0$ such that

$$Y \overset{d}{=} \left(1 - \tfrac{s}{t}\right)^{\frac{1}{3}} \mathfrak{h}_{t-s}, \quad \mathbf{P}\big(|\mathfrak{h}_t - Y| \geq (\log\log t)^{-\frac{1}{2}}\big) \leq e^{-c(\frac{t}{s}-1)^{\frac{1}{2}}(\log\log t)^{-\frac{3}{4}}}. \tag{4.7.4}$$

Using the above display and the union bound of the probability, we write

$$\begin{aligned}
\mathbf{P}(\mathsf{A}_s \cap \mathsf{A}_t) &\leq \mathbf{P}\big(\{\mathsf{A}_s \cap \mathsf{A}_t\} \cap \{|\mathfrak{h}_t - Y| \leq (\log\log t)^{-1/2}\}\big) + \mathbf{P}\big(|\mathfrak{h}_t - Y| \geq (\log\log t)^{-1/2}\big) \\
&\leq \mathbf{P}\Big(\{\mathfrak{h}_s \geq \gamma_0(\log\log s)^{2/3}\} \cap \{Y \geq \gamma_0(\log\log t)^{2/3} - (\log\log t)^{-1/2}\}\Big) \\
&\qquad + \exp\big(-c(t/s-1)^{1/2}(\log\log t)^{-3/4}\big) \\
&\leq \mathbf{P}\Big(\mathfrak{h}_s \geq \gamma_0(\log\log s)^{2/3}\big)\mathbf{P}\big(Y \geq \gamma_0(\log\log t)^{2/3} - (\log\log t)^{-1/2}\Big) \\
&\qquad + \exp\big(-c(\log\log t)^{3/4}\log\big(\log s\log t\big)\big) \\
&\leq \mathbf{P}\Big(\mathfrak{h}_s \geq \gamma_0(\log\log s)^{2/3}\Big)\,\mathbf{P}\Big(\mathfrak{h}_{t-s} \geq \gamma_0(\log\log t)^{2/3} - (\log\log t)^{-1/2}\Big) \\
&\qquad + o(1)(\log\log t\log t\log\log s\log s)^{-1} \\
&\leq \frac{(16\pi\gamma_0^{3/2})^{-2}(1+o(1))}{\log\log s\log s\log\log t\log t} = (1+o(1))\mathbf{P}(\mathsf{A}_t)\mathbf{P}(\mathsf{A}_s),
\end{aligned}$$

where the inequality in the second line follows by observing that

$$\mathsf{A}_t \cap \{|\mathfrak{h}_t - Y| \leq (\log\log t)^{-1/2}\} \subset \big\{Y \geq \gamma_0(\log\log t)^{2/3} - (\log\log t)^{-1/2}\big\}$$

and using the probability bound in (4.7.4). The next inequality follows by the independence of $\mathfrak{h}_s$ and Y and through the following observation

$$\exp\big(-c(t/s-1)^{1/2}(\log\log t)^{-3/4}\big) \leq \exp\big(-c(\log\log t)^{3/4}\log\big(\log s\log t\big)\big)$$

which is a consequence of the fact that t, s satisfy the condition (4.7.2). The inequality in the sixth line follows by noting $Y \stackrel{d}{=} (1 - s/t)^{1/3}\mathfrak{h}_{t-s}$ and observing

$$\exp\big(- c(\log\log t)^{3/4} \log\big(\log s \log t\big)\big) = o(1) \cdot (\log\log t)^{-1}(\log t)^{-1}(\log\log s)^{-1}(\log s)^{-1}.$$

The last equality follows by using Proposition 4.8.5 of the Section **??**. This completes the proof of Proposition 4.7.1. □

Proof of Proposition 4.7.2. Fix $\varepsilon > 0$. Let $N_0 = N_0(\varepsilon) > T_0$ be such that for any $t, s \geq e^{N_0}$ satisfying (4.7.2), we have

$$(1-\varepsilon) \leq \tfrac{\mathbf{P}(\mathsf{A}_s)\log s \log\log s}{(16\pi\gamma_0^{3/2})^{-1}} \leq (1+\varepsilon), \qquad (1-\varepsilon) \leq \tfrac{\mathbf{P}(\mathsf{A}_t)\log t \log\log t}{(16\pi\gamma_0^{3/2})^{-1}} \leq (1+\varepsilon),$$
$$(1-\varepsilon) \leq \tfrac{\mathbf{P}(\mathsf{A}_t \cap \mathsf{A}_s)}{\mathbf{P}(\mathsf{A}_t)\mathbf{P}(\mathsf{A}_s)} \leq (1+\varepsilon).$$

Note that the first two inequalities hold due to Proposition 4.8.5 of the Section **??** and the last inequality holds due to Proposition 4.7.1. Next we define a subsequence $\{N_k\}$ recursively as follows: $N_1 := \max\{N_0, e^{e^{10}}\}$, $N_{k+1} = N_k + 10\log\log N_k$, for $k \in \mathbb{N}$. Consider the following random variables

$$S_M := \sum_{k=1}^{M} e^{-N_k}\mu(\mathcal{S}_{N_k}), \quad M \in \mathbb{N}, \qquad S_\infty := \sum_{k=1}^{\infty} e^{-N_k}\mu(\mathcal{S}_{N_k}).$$

Define $\kappa := (1 - e^{-1})$. For $M \in \mathbb{N}$, we will show that

$$\tfrac{\mathbf{E}[S_M]}{1-\varepsilon} \geq (1+o(1))\tfrac{\kappa \log\log\log N_M}{16\pi\gamma_0^{3/2}}, \quad \tfrac{\mathbf{E}[S_{N,M}^2]-(1+\varepsilon)(\mathbf{E}[S_{N,M}])^2}{(1+\varepsilon)(1+o(1))} \leq \tfrac{\kappa^2\log\log\log N_M}{80\pi\gamma_0^{3/2}}, \tag{4.7.5}$$

where the term $o(1)$ goes to 0 as M goes to ∞. By assuming the above inequality, we first complete the proof of Proposition 4.7.2. We seek to show $\mathbf{P}(S_\infty = \infty) \geq 1$. Note that $\mathbb{P}(S_\infty = \infty) = \mathbb{P}(\lim_{M\to\infty} S_M = \infty)$. We may now write

$$\mathbf{P}\big(\lim_{M\to\infty} S_M = \infty\big) \geq \liminf_{M\to\infty} \mathbf{P}\big(S_M \geq \tfrac{1}{\sqrt{\log\log\log N_M}}\mathbf{E}S_{N,M}\big)$$

$$\geq \liminf_{M\to\infty} \frac{(1-(\log\log\log N_M)^{-1/2})^2(\mathbf{E}[S_{N,M}])^2}{\mathbf{E}[S_{N,M}^2]}$$
$$\geq \liminf_{M\to\infty} \frac{(1-(\log\log\log N_M)^{-1/2})^2\cdot\frac{(1-\varepsilon)^2}{1+\varepsilon}}{(1+o(1))4\pi\gamma_0^{3/2}(\log\log\log N_M)^{-1}+1} = \frac{(1-\varepsilon)^2}{1+\varepsilon}. \tag{4.7.6}$$

The first inequality in the above display follows since $(\log\log\log N_M)^{-1/2}\mathbf{E}[S_M]$ goes to ∞ thanks to the first inequality of (4.7.5). We obtained the second inequality by applying Paley-Zygmund inequality [268] for the random variable S_M with setting $\delta := (\log\log\log N_M)^{-1/2}$. The third inequality follows by noticing from (4.7.5) that

$$\tfrac{\mathbf{E}[S_M^2]}{(\mathbf{E}[S_M])^2} \leq \tfrac{1+\varepsilon}{(1-\varepsilon)^2}\Big((1+o(1))4\pi\gamma_0^{3/2}(\log\log\log N_M)^{-1}+1\Big).$$

From (4.7.6), Proposition 4.7.2 follows by letting ε to 0 and observing that $S_\infty \leq \sum_{n=4}^{\infty} e^{-n}\mu(\mathcal{S}_n)$.

Throughout the rest, we prove (4.7.5). Note that

$$\mathbf{E}[S_M] = \sum_{k=1}^{M} e^{-N_k}\int_{e^{N_k-1}}^{e^{N_k}} \mathbf{P}(\mathsf{A}_s)\mathrm{d}s \geq \sum_{k=1}^{M} e^{-N_k}\int_{e^{N_k-1}}^{e^{N_k}} \frac{(1-\varepsilon)(16\pi\gamma_0^{3/2})^{-1}}{\log\log s\log s}\mathrm{d}s \tag{4.7.7}$$
$$\geq \sum_{k=1}^{M} \frac{\kappa(16\pi\gamma_0^{3/2})^{-1}}{N_k\log N_k}. \tag{4.7.8}$$

where the first inequality follows since $\mathbf{P}(\mathsf{A}_s) \geq (1-\varepsilon)(16\pi\gamma_0^{3/2}\log s\log\log s)^{-1}$ for all $s \geq e^{N_0}$ and the second inequality follows since $\log s \leq N_k$ for all $s \in [e^{N_k-1}, e^{N_k}]$. To lower bound the r.h.s. of (4.7.8), we note

$$\sum_{n=N_{k-1}}^{N_k-1} \frac{1}{n\log n\log\log n} \leq \sum_{n=N_{k-1}}^{N_k-1} \frac{(\log\log N_{k-1})^{-1}}{N_{k-1}\log N_{k-1}} \leq \frac{10\log\log N_{k-1}}{N_{k-1}\log N_{k-1}\log\log N_{k-1}},$$

where the first inequality is straightforward and the second inequality follows since $|N_k - N_{k-1}| = 10(1+o(1))\log\log N_{k-1}$. It is worth noting that the r.h.s. of the last inequality is equal to

$10/(N_{k-1}\log N_{k-1})$. Using the above display, we may write

$$\sum_{k=1}^{M}[N_k\log N_k]^{-1}\geq \tfrac{1}{10}\sum_{n=N_0}^{N_k}[n\log n\log\log n]^{-1}=\tfrac{1}{10}(1+o(1))\log\log\log N_M, \tag{4.7.9}$$

where $o(1)$ goes to 0 as M goes to ∞. This implies the first inequality of (4.7.5). Now we proceed to prove the second inequality of (4.7.5). We introduce the notation $\mathfrak{Int}(n,m):=e^{-n-m}\int_{e^{n-1}}^{e^n}\int_{e^{m-1}}^{e^m}\mathbf{P}(\mathsf{A}_t\cap\mathsf{A}_s)\mathrm{d}t\mathrm{d}s$. Observe that

$$\mathbf{E}[S_M^2]=\sum_{k=1}^{M}\sum_{\ell=1}^{M}\mathfrak{Int}(N_k,N_\ell)=\underbrace{\sum_{k=1}^{M}\mathfrak{Int}(N_k,N_k)}_{(\mathbf{I})}+\underbrace{\sum_{k\neq\ell}^{M}\mathfrak{Int}(N_k,N_\ell)}_{(\mathbf{II})}.$$

We first bound $(\mathbf{I})$. Using the inequality $\mathbf{P}(\mathsf{A}_s\cap\mathsf{A}_t)\leq\mathbf{P}(\mathsf{A}_t)\leq(1+\varepsilon)(16\pi\gamma_0^{3/2}\log\log t\log t)^{-1}$ for any $s,t\in[e^{N_k-1},e^{N_k}]$, we see

$$\int_{e^{N_k-1}}^{e^{N_k}}\int_{e^{N_k-1}}^{e^{N_k}}\mathbf{P}(\mathsf{A}_s\cap\mathsf{A}_t)\mathrm{d}s\mathrm{d}t\leq\frac{(1+\varepsilon)(e^{N_k}-e^{N_k-1})^2}{16\pi\gamma_0^{3/2}(N_k-1)\log(N_k-1)}.$$

Multiplying both sides by e^{-2N_k} and summing over k as k varies in $[1,M]\cap\mathbb{Z}_{\geq1}$ yields

$$\begin{aligned}(\mathbf{I})\leq\sum_{k=1}^{M}\frac{\kappa^2(1+\varepsilon)}{16\pi\gamma_0^{3/2}(N_k-1)\log(N_k-1)}&\leq\frac{\kappa^2(1+\varepsilon)}{80\pi\gamma_0^{3/2}}\sum_{n=N_0}^{N_M}\frac{C}{(n-1)\log(n-1)\log\log(n-1)}\\&=\kappa^2(1+o(1))\frac{(1+\varepsilon)\log\log\log N_M}{80\pi\gamma_0^{3/2}}.\end{aligned}\tag{4.7.10}$$

The equality in the last line follows since $\sum_{n=N_0}^{N_M}((n-1)\log(n-1)\log\log(n-1))^{-1}=(1+o(1))\log\log\log(n)$. It remains to explain the second inequality of the above display. To see this, notice that for any $k\in\mathbb{N}$,

$$\frac{(\log(N_k-1))^{-1}}{(N_k-1)}\leq\frac{2\log\log(N_{k-1})}{(N_k-1)\log(N_k-1)\log\log(N_k-1)}\leq 2\sum_{n=N_{k-1}}^{N_k}\frac{(\log\log(n-1))^{-1}}{10(n-1)\log(n-1)}.$$

The first inequality follows since $2\log\log N_{k-1} \geq \log\log(N_k - 1)$ whereas the second inequality is obtained by noting that $|N_k - N_{k-1}| \leq 10\log\log N_{k-1}$. Now we bound (**II**). Fix any $t \in [e^{N_k - 1}, e^{N_k}]$ and $s \in [e^{N_\ell - 1}, e^{N_\ell}]$ for $k > \ell \in \mathbb{N}$. Using this information, we write

$$t/s \geq e^{N_k - N_\ell - 1} \geq e^{10\log\log(N_{k-1})-1} \geq e^{5\log\log(N_k)+\log 4} = 4(\log N_k)^5 \geq 4(\log\log t)^5,$$

where the second inequality follows since $N_k - N_\ell \geq N_k - N_{k-1} \geq 10\log\log N_{k-1}$ and the third inequality is obtained by noting that $10\log\log(N_{k-1}) \geq 5\log\log N_k + 1 + \log 4$. From the above display, it follows that t and s satisfy (4.7.2). Due to Proposition 4.7.1, we have $\mathbf{P}(\mathsf{A}_t \cap \mathsf{A}_s) \leq (1+\varepsilon)\mathbf{P}(\mathsf{A}_t)\mathbf{P}(\mathsf{A}_s)$. This implies

$$\begin{aligned}(\mathbf{II}) &= 2\sum_{\ell=1}^{M}\sum_{k=\ell+1}^{M} e^{-N_k - N_\ell}\int_{e^{N_k-1}}^{e^{N_k}}\int_{e^{N_\ell-1}}^{e^{N_\ell}} \mathbf{P}(\mathsf{A}_s \cap \mathsf{A}_t)\mathrm{d}s\mathrm{d}t \\ &\leq 2(1+\varepsilon)\sum_{\ell=1}^{M}\sum_{k=\ell+1}^{M} e^{-N_k - N_\ell}\int_{e^{N_k-1}}^{e^{N_k}}\int_{e^{N_\ell-1}}^{e^{N_\ell}} \mathbf{P}(\mathsf{A}_s)\mathbf{P}(\mathsf{A}_t)\mathrm{d}s\mathrm{d}t \leq (1+\varepsilon)\big(\mathbf{E}[S_M]\big)^2. \qquad (4.7.11)\end{aligned}$$

Combining (4.7.10) and (4.7.11) yields (4.7.5). This completes the proof. □

4.7.2 Multifractality: Proof of 4.1.5

Recall the definition of the exponential time changed process $\mathfrak{G}(t)$. Define

$$\Lambda_\gamma := \{t \geq e \mid \mathfrak{G}(t) \geq \gamma\left(\tfrac{3}{4\sqrt{2}}\log t\right)^{2/3}\}, \quad \gamma \in \mathbb{R}. \qquad (4.7.12)$$

Due to Theorem 4.1.1, we know $\limsup_{t\to\infty} \frac{\mathfrak{G}(t)}{(3\log t/4\sqrt{2})^{2/3}} \overset{a.s.}{=} 1$, which shows that Λ_γ is almost surely bounded for $\gamma > 1$. This proves $\mathrm{Dim}_{\mathbb{H}}(\Lambda_\gamma) = 0$ with probability 1 when $\gamma > 1$. In the rest of the section, we focus on showing (4.1.5) for $\gamma \in (0,1]$. We divide the proof into two stages. The first stage will show the upper bound $\mathrm{Dim}_{\mathbb{H}}(\Lambda_\gamma) \leq 1 - \gamma^{3/2}$ and the lower bound $\mathrm{Dim}_{\mathbb{H}}(\Lambda_\gamma) \geq 1 - \gamma^{3/2}$ will be shown in the second stage.

Stage 1: Proof of $\mathrm{Dim}_{\mathbb{H}}(\Lambda_\gamma) \leq 1 - \gamma^{3/2}$**.** Recall the definition of ρ-dimensional Hausdorff content

$\nu_{n,\rho}$ from Definition 4.2.5. The main step of the proof is to show that

$$\sum_{n=1}^{\infty} \mathbb{E}\big[\nu_{n,\rho}\big(\Lambda_\gamma\big)\big] < \infty, \quad \forall \rho > 1-(1-\epsilon)\gamma^{3/2}, \epsilon \in (0,1). \tag{4.7.13}$$

This immediately implies that $\sum_{n=1}^{\infty} \nu_{n,1-(1-\epsilon)\gamma^{3/2}}(\Lambda_\gamma) < \infty$ almost surely for all $\epsilon \in (0,1)$ and hence, $\mathrm{Dim}_{\mathbb{H}}(\Lambda_\gamma) \leq 1-(1-\epsilon)\gamma^{3/2}$. From this upper bound, the result will follow by taking ϵ to 0. Below, we state a lemma showing a technical estimate which will be required to bound $\mathbb{E}\big[\nu_{n,1-(1-\epsilon)\gamma^{3/2}}\big(\Lambda_\gamma\big)\big]$ for any $n \in \mathbb{N}$. After that, we will proceed to complete the proof of the upper bound which will be followed by the proof of the lemma.

Lemma 4.7.3. *Fix $\epsilon \in (0,1)$. We have*

$$\mathbf{P}\left(\Lambda_\gamma \cap [m, m+1] \neq \varnothing\right) \leq 2m^{-(1-\epsilon)^{3/2}\gamma^{3/2}+o(1)} \log m, \tag{4.7.14}$$

where $o(1)$ term goes to zero as $m \to \infty$.

Final steps of the upper bound proof. Fix $\epsilon > 0$ and take any $\rho > 1-(1-\epsilon)^{3/2}\gamma^{3/2}$. For any $n \geq 1$, define $\Xi_n := [-e^{n+1}, -e^n) \cup (e^n, e^{n+1}]$. From the definition of $\nu_{n,\rho}$, it follows that

$$\nu_{n,\rho}(\Lambda_\gamma) \leq \sum_{m \in \mathbb{Z}_{>0}} e^{-n\rho} \mathbf{1}_{\Lambda_\gamma \cap [m,m+1] \neq \varnothing} \cdot \mathbf{1}_{[m,m+1] \subset \Xi_n}.$$

Taking expectation on both sides, we get

$$\begin{aligned}
\mathbf{E}\left[\nu_{n,\rho}(\Lambda_\gamma)\right] &\leq e^{-n\rho} \sum_{m \in \mathbb{Z}_{>0}} \mathbf{1}_{[m,m+1] \in \Xi_n} \cdot \mathbf{P}\left(\Lambda_\gamma \cap [m, m+1] \neq \varnothing\right) \\
&\leq e^{-n\rho} \sum_{m \in \mathbb{Z}_{>0}} 2m^{-(1-\epsilon)^{3/2}\gamma^{3/2}+o(1)} \log m \cdot \mathbf{1}_{[m,m+1] \in \Xi_n} \\
&\leq e^{-n\rho} \cdot 2e^{n+1} \cdot 2ne^{-(1-\epsilon)^{3/2}\gamma^{3/2} n} = 4ne^{n(1-\rho-(1-\epsilon)^{3/2}\gamma^{3/2})+1}.
\end{aligned} \tag{4.7.15}$$

The second inequality follows from Lemma 5.5.18. We get the third inequality by observing that the number of non-zero terms in the sum is bounded by $2e^{n+1}$ and each non-zero term is bounded above by $2ne^{-(1-\epsilon)^{3/2}\gamma^{3/2}n}$. The upper bound of $\mathbf{E}[\nu_{n,\rho}(\Lambda_\gamma)]$ in (4.7.15) is summable over n whenever $\rho > 1-(1-\epsilon)\gamma^{3/2}$. This shows (4.7.13). Alluding to the discussion after (4.7.13), we get the proof of $\mathrm{Dim}_{\mathbb{H}}(\Lambda_\gamma) \le 1-\gamma^{3/2}$. □

Proof of Lemma 5.5.18. Define $B_m := \lceil \log m \rceil$. We divide the interval $[e^m, e^{m+1}]$ into B_m many intervals $\{\mathcal{I}_j^m\}_{j=1}^m$ where $\mathcal{I}_j^m := [x_{j-1}^{(m)}, x_j^{(m)}]$ and, $x_j^{(m)} := e^m(1+\frac{(e-1)j}{B_m})$, for $j = 1,\ldots,B_m$. We may now write

$$\mathbf{P}(\Lambda_\gamma \cap [m,m+1] \neq \varnothing) \le \mathbf{P}\Big(\sup_{t\in[m,m+1]} \mathfrak{H}(t) \ge \gamma\big(\tfrac{3}{4\sqrt{2}}\log m\big)^{\frac{2}{3}}\Big)$$
$$= \mathbf{P}\Big(\sup_{t\in[e^m,e^{m+1}]} \mathfrak{h}_t \ge \gamma\big(\tfrac{3}{4\sqrt{2}}\log m\big)^{\frac{2}{3}}\Big) \le \sum_{j=1}^{B_m} \mathbf{P}\Big(\sup_{t\in\mathcal{I}_j^m} \mathfrak{h}_t \ge \gamma\big(\tfrac{3}{4\sqrt{2}}\log m\big)^{\frac{2}{3}}\Big), \tag{4.7.16}$$

where the last inequality follows by the union bound. In what follows, we show that

$$\mathbf{P}\Big(\sup_{t\in\mathcal{I}_j^m} \mathfrak{h}_t \ge \gamma\big(\tfrac{3}{4\sqrt{2}}\log m\big)^{2/3}\Big) \le 2m^{-(1-\epsilon)^{3/2}\gamma^{3/2}+o(1)}, \tag{4.7.17}$$

where $o(1)$ term converges to 0 as m goes to ∞ uniformly for all $j = 1,\ldots,B_m$. From (4.7.17), (4.7.14) of Lemma 5.5.18 will follow by noting that there are at most $\log m$ terms in the sum (4.7.16). Fix $j \in \{1,\ldots,B_m\}$. For convenience, we use shorthand x_j and x_{j-1} to denote $x_j^{(m)}$ and $x_{j-1}^{(m)}$ respectively. Consider the events

$$\mathsf{A}_{j,m} := \Big\{\sup_{t\in\mathcal{I}_j^m} \big(\mathfrak{h}_t - \big(\tfrac{x_{j-1}}{t}\big)^{\frac{1}{3}}\mathfrak{h}_{x_{j-1}}\big) \ge \epsilon\gamma\big(\tfrac{3\log m}{4\sqrt{2}}\big)^{\frac{2}{3}}\Big\}, \mathsf{B}_{j,m} := \Big\{\mathfrak{h}_{x_{j-1}} \ge (1-\epsilon)\gamma\big(\tfrac{3\log m}{4\sqrt{2}}\big)^{\frac{2}{3}}\Big\}.$$

Note that

$$\sup_{t\in\mathcal{I}_j^m} \mathfrak{h}_t \le \sup_{t\in\mathcal{I}_j^m}\Big(\mathfrak{h}_t - \big(\tfrac{x_{j-1}}{t}\big)^{1/3}\mathfrak{h}_{x_{j-1}}\Big) + \sup_{t\in\mathcal{I}_j^m}\Big(\tfrac{x_{j-1}}{t}\Big)^{1/3}\mathfrak{h}_{x_{j-1}}$$
$$= \sup_{t\in\mathcal{I}_j^m}\Big(\mathfrak{h}_t - \big(\tfrac{x_{j-1}}{t}\big)^{1/3}\mathfrak{h}_{x_{j-1,m}}\Big) + \max\Big\{\big(\tfrac{x_{j-1}}{x_j}\big)^{1/3}\mathfrak{h}_{x_{j-1}}, \mathfrak{h}_{x_{j-1}}\Big\}.$$

Due to the above inequality, we have $\left\{\sup_{t\in\mathcal{I}_j^m}\mathfrak{h}_t \geq \gamma\big(\frac{3}{4\sqrt{2}}\log m\big)^{2/3}\right\}\subset \mathsf{A}_{j,m}\cup\mathsf{B}_{j,m}$. By the union bound, we get

$$\mathbf{P}\Big(\sup_{t\in\mathcal{I}_j^m}\mathfrak{h}_t \geq \gamma\big(\tfrac{3}{4\sqrt{2}}\log m\big)^{2/3}\Big)\leq \mathbf{P}(\mathsf{A}_{j,m})+\mathbf{P}(\mathsf{B}_{j,m}). \tag{4.7.18}$$

In what follows, we claim and prove that

$$m^{\gamma^{3/2}}\mathbf{P}(\mathsf{A}_{j,m}) = o(1), \quad \text{and} \quad \mathbf{P}(\mathsf{B}_{j,m}) = m^{-(1-\epsilon)^{3/2}\gamma^{3/2}+o(1)}, \tag{4.7.19}$$

where $o(1)$ terms converge to 0 as $m\to\infty$ uniformly for all $j\in\{1,\ldots,B_m\}$. Substituting the bounds of (4.7.19) into the r.h.s. of (4.7.18) shows (4.7.18). To complete the proof of this lemma, it suffices to to show (4.7.19).

By noting that $\log x_{j-1,m}\in[m,m+1]$, we use (4.2.15) of Proposition 4.2.13 to get

$$\mathbf{P}(\mathsf{B}_{j,m})\leq \exp\Big(-(1+o(1))\gamma^{3/2}(1-\epsilon)^{3/2}\log m\Big) = m^{-(1-\epsilon)^{3/2}\gamma^{3/2}+o(1)},$$

where the $o(1)$ term goes to zero as $m\to\infty$ uniformly for all j. This proves the bound on $\mathbf{P}(\mathsf{B}_{j,m})$ in (4.7.19). Now we proceed to prove the bound on $\mathbf{P}(\mathsf{A}_{j,m})$. To this end, recall that $h_t(\alpha,0)=\alpha^{1/3}h_{\alpha t}$ for any $\alpha,t>0$. Using this, we may write

$$\begin{aligned}\mathbf{P}(\mathsf{A}_{j,m}) &= \mathbf{P}\Big(\sup_{t\in\mathcal{I}_m^j}\big(\tfrac{x_{j-1}}{t}\big)^{1/3}\big(\mathfrak{h}_{x_{j-1}}(\tfrac{t}{x_{j-1}},0)-\mathfrak{h}_{x_{j-1}}(1,0)\big)\geq \epsilon\gamma\big(\tfrac{3}{4\sqrt{2}}\log m\big)^{2/3}\Big)\\ &\leq \mathbf{P}\Big(\sup_{\tau\in[0,\frac{e-1}{B_m}]}\big(\mathfrak{h}_{x_{j-1}}(1+\tau,0)-\mathfrak{h}_{x_{j-1}}(1,0)\big)\geq \epsilon\gamma\big(\tfrac{x_j}{x_{j-1}}\big)^{1/3}\big(\tfrac{3}{4\sqrt{2}}\log m\big)^{2/3}\Big),\end{aligned} \tag{4.7.20}$$

where the second inequality follows since $(t^{-1}x_{j-1})^{1/3}$ is bounded below by $(x_j^{-1}x_{j-1})^{1/3}$ for any

$t \in \mathcal{I}_j^m$. Setting $r := \sup_{\tau\in(0,(e-1)/B_m]} \tau^{1/8}\log^{2/3}(1/\tau) < \infty$, we get

$$\text{r.h.s. of (4.7.20)} \leq \mathbf{P}\Big(\sup_{\tau\in[0,\frac{e-1}{B_m}]} \frac{\mathfrak{h}_{x_{j-1}}(1+\tau,0)-\mathfrak{h}_{x_{j-1}}(1,0)}{\tau^{1/8}\log^{2/3}(1/\tau)} \geq \frac{\epsilon\gamma}{r}\big(\tfrac{x_j}{x_{j-1}}\big)^{1/3}\big(\tfrac{3}{4\sqrt{2}}\log m\big)^{2/3}\Big). \tag{4.7.21}$$

Applying Proposition 4.5.5 with $\varepsilon = \frac{1}{8}$, $\delta = \frac{e-1}{B_m}$, $a = 1$, we get

$$\text{r.h.s. of (4.7.21)} \leq \exp\Big(-c\big(\frac{\epsilon\gamma}{r}\big)^{\frac{3}{2}}\big(\tfrac{x_j}{x_{j-1}}\big)^{\frac{1}{2}}\big(\tfrac{3}{4\sqrt{2}}\log m\big)\Big) \leq \exp(-C(\log m)^{1+\frac{3}{32}}) = o(m^{-\gamma^{3/2}})$$

for all large m. Here, C is a constant which will only depend ϵ. The second inequality follows since $r^{-\frac{3}{2}} \geq c_1(\log m)^{\frac{3}{32}}$ for some $c_1 > 0$ and $(x_j/x_{j-1}) \geq 1$. This proves the first bound in (4.7.19) and hence, completes the proof of the lemma. $\square$

Stage 2: Proof of $\mathrm{Dim}_{\mathbb{H}}(\Lambda_\gamma) \geq 1-\gamma^{3/2}$**.** To prove the lower bound, we use similar techniques used as in [223, (4.14) of Theorem 4.7]. Recall the definition of 'thickness' of a set from Definition 4.2.8. We seek to use to use Proposition 4.2.9. Let us fix $\theta \in (\gamma^{3/2}, 1)$. Recall Λ_γ from (4.7.12). We will show that Λ_γ is θ-thick with probability 1. This will prove the almost sure lower bound $\mathrm{Dim}_{\mathbb{H}}(\Lambda_\gamma) \geq 1-\gamma^{3/2}$ via Proposition 4.2.9. Let us define

$$\mathsf{D}_n := \left\{\Lambda_\gamma \cap [x, x+e^{\theta n}] = \varnothing, \text{ for some } x \in \Pi_n(\theta)\right\}.$$

The θ-thickness of Λ_γ will follow through the Borel-Cantelli lemma if the following holds

$$\sum_{n=1}^{\infty} \mathbf{P}(\mathsf{D}_n) < \infty. \tag{4.7.22}$$

Showing the above display will be the main focus of the rest of the proof.

Recall the definition of the interval $\mathcal{I}_n$ and its $\mathcal{M}_\theta$ many sub-intervals $\{\mathcal{I}_n^{(j)}\}_{j=1}^{\mathcal{M}_\theta}$ from Section 4.6.1. Let us denote the end points of the sub-intervals $\{\mathcal{I}_n^{(j)}\}_{j=1}^{\mathcal{M}_\theta}$ as $x_n^{(1)}, \ldots, x_n^{(\mathcal{M}_\theta)}$ such that

$\mathcal{I}_n^{(j)} = [\exp(x_n^{(j-1)}), \exp(x_n^{(j)})]$. Let us define

$$\mathsf{B}_n^{(j)} := \Big\{ \sup_{t\in[x_n^{(j-1)},x_n^{(j)}]} \mathfrak{G}(t) \le \gamma\big(\tfrac{3}{4\sqrt{2}}\big)^{\frac{2}{3}}(n+1)^{\frac{2}{3}} \Big\}.$$

From the definition of $\mathsf{B}_n^{(j)}$, it follows that $\mathsf{D}_n \subset \cup_{j=1}^{\mathcal{M}_\theta} \mathsf{B}_n^{(j)}$. By the union bound, we get $\mathbf{P}(\mathsf{D}_n) \le \sum_j \mathbf{P}(\mathsf{B}_n^{(j)})$. We will now show (4.7.22) by proving a bound (uniform on j and n) on $\mathbf{P}(\mathsf{B}_n^{(j)})$.

Choose $\eta > 0$ such that $\zeta := (\frac{4\sqrt{2}}{3} + \eta)(\gamma(\frac{3}{4\sqrt{2}})^{\frac{2}{3}} + 2\eta)^{3/2} < \theta$ and pick $\delta \in (0, \theta - \zeta)$. We now claim and prove that there exists $c_1, c_2 > 0$ such that

$$\mathbf{P}(\mathsf{B}_n^{(j)}) \le \exp(-ce^{n(\theta-\delta)}e^{-n\zeta}) + \exp\big(n(\theta-\delta) - c_2(\exp(e^{n\delta}) - 1)^{1/2}\big) \tag{4.7.23}$$

for all $1 \le j \le n$ and all large n. Using the above inequality, we may bound $\mathbf{P}(\mathsf{D}_n)$ by $\exp(n - n\theta + 1)$ times the r.h.s. of (4.7.23). Since $\theta > \zeta + \delta$, we can bound $\mathbf{P}(\mathsf{D}_n)$ by $\exp(-e^{n\omega})$ for some $\omega \in (0,1)$ and for all large n. This shows (4.7.22) and hence, completes the proof modulo (4.7.23) which is finally remained to be shown. By the identification $\mathfrak{G}(t) = h_{e^t}$ and $\mathcal{I}_n^{(j)} = [\exp(x_n^{(j-1)}), \exp(x_n^{(j)})]$, it is straightforward to see that

$$\mathsf{B}_n^{(j)} = \Big\{ \sup_{t\in\mathcal{I}_n^{(j)}} \mathfrak{h}_t \le \gamma\big(\tfrac{3}{4\sqrt{2}}\big)^{\frac{2}{3}}(n+1)^{\frac{2}{3}} \Big\}.$$

Due to this identity, (4.7.23) now follows from the proof of (4.6.20), completing the proof. □

4.8 Auxiliary results

Lemma 4.8.1. *Fix $\beta > 0$ and a constant $C = C(\beta) \in \mathbb{R}$. If a line ensemble $\mathcal{L}$ satisfies the* $\mathbf{H}$*-Brownian Gibbs property, then the line ensemble $\mathcal{D}$ defined by*

$$\mathcal{D}_i(x) := \tfrac{1}{\sqrt{\beta}}\mathcal{L}_i(\beta x) + C \tag{4.8.1}$$

satisfies the $\mathbf{H}_\beta$-Brownian Gibbs property where $\mathbf{H}_\beta(x) := \beta e^{\sqrt{\beta}x}$.

Proof. Fix $a, b \in \mathbb{R}$ with $a < b$. Fix $k_1 \le k_2$ with $k_1, k_2 \in \mathbb{N}$. Set $K = \{k_1, \ldots, k_2\}$. The conditional law of $\mathcal{L}_{K\times(a\beta,b\beta)}$ conditioned on $\mathcal{L}_{\mathbb{N}\times\mathbb{R}\setminus K\times(a\beta,b\beta)}$ is given by

$$\mathbf{P}_{\mathbf{H}}^{k_1,k_2,(a\beta,b\beta),\vec{x},\vec{y},f,g}$$

where $\mathcal{L}_{k_1-1} = f$, or ∞ if $k_1 = 1$ and $\mathcal{L}_{k_2+1} = g$, and $\vec{x} = (\mathcal{L}_{k_1}(a\beta), \ldots, \mathcal{L}_{k_2}(a\beta)) = (\sqrt{\beta}(\mathcal{D}_{k_1}(a)-C), \ldots, \sqrt{\beta}(\mathcal{D}_{k_2}(a)-C))$ and $\vec{y} = (\mathcal{L}_{k_1}(b\beta), \ldots, \mathcal{L}_{k_2}(b\beta)) = (\sqrt{\beta}(\mathcal{D}_{k_1}(b)-C), \ldots, \sqrt{\beta}(\mathcal{D}_{k_2}(b)-C))$. Note that under the scaling (4.8.1), the underlying law of free Brownian bridges are still free Brownian bridges but with endpoints $\vec{x}' = (\mathcal{D}_{k_1}(a), \ldots, \mathcal{D}_{k_2}(a))$ and $\vec{y}' = (\mathcal{D}_{k_1}(b), \ldots, \mathcal{D}_{k_2}(b))$. On the other hand, the Radon Nikodym derivative is proportional to

$$\begin{aligned}
&\exp\left(-\sum_{i=k_1+1}^{k_2-1}\int_{a\beta}^{b\beta}\mathbf{H}(\mathcal{L}_i(x)-\mathcal{L}_{i-1}(x))\,\mathrm{d}x\right)\\
&=\exp\left(-\sum_{i=k_1+1}^{k_2-1}\int_{a\beta}^{b\beta}\mathbf{H}(\sqrt{\beta}[\mathcal{D}_i(x/\beta)-\mathcal{D}_{i-1}(x/\beta)])\,\mathrm{d}x\right)\\
&=\exp\left(-\sum_{i=k_1+1}^{k_2-1}\int_{a}^{b}\beta\mathbf{H}(\sqrt{\beta}[\mathcal{D}_i(x)-\mathcal{D}_{i-1}(x)])\,\mathrm{d}x\right)\\
&=\exp\left(-\sum_{i=k_1+1}^{k_2-1}\int_{a}^{b}\mathbf{H}_\beta(\mathcal{D}_i(x)-\mathcal{D}_{i-1}(x))\,\mathrm{d}x\right).
\end{aligned}$$

This completes the proof. □

Proposition 4.8.2. *Let $\nu > 0$. There exist $t_0 = t_0(\nu) \in (0,1)$, $c = c(\nu) > 0$ and $s = s(\nu) > 0$ such that for all $t \le t_0$ and $s \ge s_0$, we have*

$$\mathbf{P}\Big(\inf_{x\in\mathbb{R}}\big(\mathfrak{g}_t(x) + \tfrac{(\pi t/4)^{3/4}(1+\nu)x^2}{2t}\big) \le -s\Big) \le \exp(-cs^2), \tag{4.8.2}$$

$$\mathbf{P}\Big(\sup_{x\in\mathbb{R}}\big(\mathfrak{g}_t(x) + \tfrac{(\pi t/4)^{3/4}(1-\nu)x^2}{2t}\big) \ge s\Big) \le \exp(-cs^{3/2}). \tag{4.8.3}$$

Proof. **Proof of** (4.8.2)**:** Set $\widetilde{\mathfrak{g}}_t(x) = \mathfrak{g}_t((4^3t/\pi^3)^{1/8}x)$. Fix $\nu > 0$. For any $A \subset \mathbb{R}$ and $m \in \mathbb{Z}$, define

$$\mathcal{G}_A = \left\{ \inf_{x\in A} \left(\widetilde{\mathfrak{g}}_t(x) + \tfrac{(1+\nu)x^2}{2} \right) \le -s \right\}, \quad \mathcal{D}_m = \left\{ \widetilde{\mathfrak{g}}_t(m) + \tfrac{m^2}{2} \ge -\tfrac{s}{4} - \tfrac{\nu m^2}{4} \right\}.$$

We seek to bound $\mathbf{P}(\mathcal{G}_{\mathbb{R}})$. Using the union bound, we have

$$\mathbf{P}(\mathcal{G}_{\mathbb{R}}) \le \mathbf{P}(\mathcal{G}_{(-\infty,-16\nu^{-1}]}) + \mathbf{P}(\mathcal{G}_{[-16\nu^{-1},16\nu^{-1}]}) + \mathbf{P}(\mathcal{G}_{[16\nu^{-1}.\infty)}). \tag{4.8.4}$$

Let $N = \lfloor 16\nu^{-1} \rfloor$. Note that we are allowed to choose the threshold of s, namely s_0 dependent on ν. So, we can choose s large enough so that $s \ge 15N$ is satisfied. Choosing $\delta = 1$ and $t_0 \le 1$ in Proposition 4.4.3, via the union bound we get

$$\mathbf{P}\left(\mathcal{G}_{[-16\nu^{-1},16\nu^{-1}]}\right) = \mathbf{P}\left(\bigcup_{i=-N-1}^{N} \mathcal{G}_{[i,i+1]} \right) \le N\exp(-cs^2) \le \exp(-c's^2), \tag{4.8.5}$$

where c' depends on N. We now turn to bound $\mathbf{P}(\mathcal{G}_{[16\nu^{-1},\infty)})$. Due to the spatial stationarity of $\tilde{\mathfrak{g}}_t(x) + \frac{x^2}{2}$, similar argument can be used to bound $\mathbf{P}(\mathcal{G}_{(-\infty,-16\nu^{-1}]})$. We start by writing

$$\mathbf{P}(\mathcal{G}_{[16\nu^{-1},\infty)}) \le \sum_{m=N}^{\infty} \mathbf{P}(\neg\mathcal{D}_m) + \sum_{m=N}^{\infty} \mathbf{P}(\mathcal{D}_m \cap \mathcal{D}_{m+1} \cap \mathcal{G}_{[m,m+1]}). \tag{4.8.6}$$

Note that for all s large enough, we have for all m, $\mathbf{P}(\neg\mathcal{D}_m) \le \exp\left(-c\left(\frac{s}{4} + \frac{\nu m^2}{4}\right)^2\right) \le \exp(-c'(s^2 + |m|^4))$. Here the constant c' depends on ν. Hence the first sum in r.h.s. of (4.8.6) is clearly bounded $\exp(-cs^2)$. For the second sum, we will invoke the Brownian Gibbs property. The rest of the calculations is in similar spirit with proof of Proposition 4.4.3. Define the event

$$\mathcal{G}'_{[m,m+1]} = \left\{ \tfrac{(1+\nu)(m+1)^2}{2} + \inf_{x\in[m,m+1]} \widetilde{\mathfrak{g}}_t(x) \le -s \right\} \supset \mathcal{G}_{[m,m+1]}.$$

From (4.2.4) and (4.2.2), it follows that $\left\{\widetilde{\mathfrak{g}}_t((4^3t/\pi^3)^{-1/8}w)\right\}_{w\in\mathbb{R}}$ is same in distribution as $\mathfrak{g}_t^{(1)}(\cdot)$ where $\mathfrak{g}_t^{(1)}$ is the top curve of the short-time line ensemble defined in (3) of Lemma 6.5.1. Let us set $a = (4^3t/\pi^3)^{1/8}m$ and $b = (4^3t/\pi^3)^{1/8}(m+1)$ for convenience. Let $\mathcal{F}_s = \mathcal{F}_{\mathbf{ext}}(\{1\},(a,b))$

be the σ-algebra generated by $\{\mathfrak{g}_t^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$ outside $\{\mathfrak{g}_t^{(1)}(x)\}_{x\in(a,b)}$. By the strong Brownian Gibbs property for the short-time line ensemble we have

$$\mathbf{P}(\mathcal{D}_m \cap \mathcal{D}_{m+1} \cap \mathcal{G}'_{[m,m+1]}) = \mathbf{E}\left[\mathbf{1}_{\mathcal{D}_m}\mathbf{1}_{\mathcal{D}_{m+1}}\mathbf{E}(\mathcal{G}'_{[m,m+1]}|\mathcal{F}_s)\right] = \mathbf{E}\left[\mathbf{1}_{\mathcal{D}_m}\mathbf{1}_{\mathcal{D}_{m+1}}\mathbf{P}_s(\mathcal{G}'_{[m,m+1]})\right].$$

where $\mathbf{P}_s := \mathbf{P}_{\mathbf{G}_t}^{1,1,(a,b),\mathfrak{g}_t(m),\mathfrak{g}_t(m+1),\infty,\mathfrak{g}_t^{(2)}}$. By monotone coupling Lemma $\mathbf{P}_s(\mathcal{G}'_{[m,m+1]}) \le \mathbf{P}_{\mathbf{free}}(\mathcal{G}'_{[m,m+1]})$, where $\mathbf{P}_{\mathbf{free}} := \mathbf{P}_{\mathbf{G}_t}^{1,1,(a,b),\mathfrak{g}_t(m),\mathfrak{g}_t(m+1),\infty,-\infty}$ is the law of a Brownian bridge on $[a,b]$ with entry value $\mathfrak{g}_t(m)$ and exit value $\mathfrak{g}_t(m+1)$. Thus if $\mathfrak{B}$ be a Brownian bridge on $[0,b-a]$ independent of the σ-algebra $\mathcal{F}_s$, we get that $\mathbf{P}(\mathcal{D}_m \cap \mathcal{D}_{m+1} \cap \mathcal{G}'_{[m,m+1]})$ is atmost

$$\mathbf{E}\left[\mathbf{1}_{\mathcal{D}_m}\mathbf{1}_{\mathcal{D}_{m+1}}\mathbf{P}\left(\tfrac{(1+\nu)(m+1)^2}{2} + \inf_{x\in[0,b-a]}\left(\mathfrak{B}(x) + \tfrac{\mathfrak{g}_t(m)(b-a-x)+\mathfrak{g}_t(m+1)x}{b-a}\right) \le -s \,\middle|\, \mathcal{F}_s\right)\right].$$

On $(\mathcal{D}_m \cap \mathcal{D}_{m+1})$, $\mathfrak{g}_t(m)$ is atleast $-\frac{s}{4} - \frac{(1+\nu/2)m^2}{2}$ and $\mathfrak{g}_t(m+1)$ is atleast $-\frac{s}{4} - \frac{(1+\nu/2)(m+1)^2}{2}$. We use these inequalities and then neglect the indicator events above. Then using the fact that $\mathfrak{B}(\cdot)$ is independent of $\mathcal{F}_s$ we get that

$$\begin{aligned}
&\mathbf{P}(\mathcal{D}_m \cap \mathcal{D}_{m+1} \cap \mathcal{G}'_{[m,m+1]}) \\
&\le \mathbf{P}\left(\inf_{x\in[0,b-a]}\left[\mathfrak{B}(x) - \tfrac{(1+\nu/2)(2m+1)x}{2(b-a)}\right] \le -\tfrac{3s}{4} + \tfrac{\nu}{4}(m+1)^2 - \tfrac{(1+\nu/2)(2m+1)}{2}\right) \\
&\le \mathbf{P}\left(\inf_{x\in[0,b-a]}\mathfrak{B}(x) \le -\tfrac{3s}{4} + \tfrac{\nu}{4}(m+1)^2 - \tfrac{(1+\nu/2)(2m+1)}{2}\right).
\end{aligned} \tag{4.8.7}$$

We note that $\frac{1}{8}\nu(m+1)^2 \ge m+1 \ge \frac{(1-\nu/2)(2m+1)}{2}$. Furthermore for $t \le \frac{\pi^3}{4^3}$, $b-a \le 1$. Hence

$$\begin{aligned}
\text{r.h.s. of (4.8.7)} \le \mathbf{P}\left(\sup_{x\in[0,b-a]}\mathfrak{B}(x) \ge \tfrac{3s}{4} + \tfrac{\nu}{8}m^2\right) &\le \exp\left[-c(\tfrac{3s}{4} + \tfrac{\nu m^2}{8})^2\right] \\
&\le \exp\left[-c'(s^2+m^4)\right],
\end{aligned}$$

where $c' > 0$ is dependent on ν. Clearly this implies the second sum in r.h.s. of (4.8.6) is bounded by $\exp(-cs^2)$. Overall we get $\mathbf{P}(\mathcal{G}_{[16\nu^{-1},\infty)}) \le \exp(-cs^2)$. Similar analysis on the negative side yields $\mathbf{P}(\mathcal{G}_{(-\infty,-16\nu^{-1}]}) \leq \exp(-cs^2)$. Plugging the bounds in (4.8.4) gives the desired result.

Proof of (4.8.3)**.** For the supremum process, the proof is similar to Proposition 4.2 in [117]. For the aid of the reader, we point out the key changes that one needs to do in their proof.

Recall that $\widetilde{\mathfrak{g}}_t(x) = \mathfrak{g}_t((4^3t/\pi^3)^{1/8}x)$. We replace $\mathfrak{h}_t$ in the proof of Proposition 4.2 in [117] with $\widetilde{\mathfrak{g}}_t(\cdot)$. In what follows we justify briefly how almost every step in their analysis holds true even after this replacement.

Note that just like $\mathfrak{h}_t(\cdot)$, $\widetilde{\mathfrak{g}}(\cdot)$ is also stationary when the parabola $\frac{x^2}{2}$ is added to it. Furthermore, $\widetilde{\mathfrak{g}}_t(0)$ also has similar one point lower tail and upper tail estimates (holds for all t small enough). The only difference here is decay in short time lower tail is not as fast as that of long time upper tail. However, it is not hard to check that having $\mathbf{P}(\mathfrak{g}_t(0) \leq -s) \leq \exp(-cs^{3/2})$ suffices. This enables us to reduce the proof to proving the $\widetilde{\mathfrak{g}}_t$ analogue of Eq. (26) in [117].

Next we justify the second part of their proof where the Brownian Gibbs property of $\mathfrak{h}_t$ is applied. Here one needs to be careful as it is the process $\mathfrak{g}_t(\cdot)$ (instead of $\widetilde{\mathfrak{g}}_t(\cdot)$) that satisfies the Brownian Gibbs property with $\mathbf{H}_t^{\text{short}}$ as Hamiltonian. However, as we will explain in a moment, the arguments present in [117] after Eq. (26) still holds in our case. Indeed, $\mathbf{H}_t^{\text{short}}$ being convex, monotone coupling still holds. Hence one arrives at an analogue of Eq. (27) in [117] for $\widetilde{\mathfrak{g}}_t$ with the measures $\mathbf{P}_{\mathbf{H}_t}$ suitably redefined. Next to arrive at Eq. (28), we still demand a given point on the Brownian bridge to stay above the line formed by linearly interpolating its endpoints. Because of the change from $\widetilde{\mathfrak{g}}_t$ to $\mathfrak{g}_t$, the Brownian bridge is not of the same length as considered in [117]. But the above probability is still $\frac{1}{2}$ leading to our analogue of Eq. (28) in [117].

The rest of the algebraic calculations is applicable to our case as well which leads to the last math display of their proof (with $\mathfrak{h}_t^{(1)} = \mathfrak{h}_t$ replaced by $\widetilde{\mathfrak{g}}_t$). Finally one invokes the stationarity of $\widetilde{\mathfrak{g}}_t(x) + \frac{x^2}{2}$ proving the $\widetilde{\mathfrak{g}}_t$ analogue of Eq. (26) in [117]. This completes the proof. □

Proposition 4.8.3. *Fix $t_0 > 0$. For any given $\beta > 0$, recall the spatial process $\mathfrak{h}_{(1+\beta)t\downarrow t}(\cdot)$ from Proposition 4.2.11. There exist $c = c(t_0) > 0$, and $s_0 = s_0(t_0) > 0$ such that for all $t \geq t_0$,*

$s \geq s_0, \beta \geq 1$ *with* $t \geq t_0$ *we have*

$$\mathbf{P}(\mathfrak{h}_t(1+\beta,0) - \mathfrak{h}_{(t+\beta t)\downarrow t}(0) \geq s) \leq \exp(-cs^{3/2}) \tag{4.8.8}$$

$$\mathbf{P}(\mathfrak{h}_t(1+\beta,0) - \mathfrak{h}_{(t+\beta t)\downarrow t}(0) \leq -s) \leq \exp(-cs^{2}). \tag{4.8.9}$$

Proof. Recall the composition law from

$$\mathfrak{h}_t(1+\beta,0) = \frac{1}{t^{1/3}} \log \int_{\mathbb{R}} e^{t^{1/3}\left(\mathfrak{h}_t(1,t^{-2/3}y)+\beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(-(\beta t)^{-2/3}y)\right)} dy, \tag{4.8.10}$$

where $\widetilde{\mathfrak{h}}_{\beta t}(x) := \frac{\mathfrak{h}_{(t+\beta t)\downarrow t}(\beta^{2/3}x)}{\beta^{1/3}}$. We prove (4.8.8) and (4.8.9) in *Stage-1* and *Stage-2* respectively.

Stage-1: Proof of (4.8.8): We consider the following events:

$$\mathsf{A}_1 := \left\{ \sup_{|x| \leq \frac{1}{32}\beta^{-2/3}\sqrt{s}} (\widetilde{\mathfrak{h}}_{\beta t}(x) - \widetilde{\mathfrak{h}}_{\beta t}(0)) \geq \tfrac{1}{4}\beta^{-1/3}s \right\}, \quad \mathsf{A}_2 := \left\{ \sup_{x\in\mathbb{R}} \left(\mathfrak{h}_t(y) + \tfrac{y^2}{4}\right) \geq \tfrac{s}{4} \right\}$$
$$\mathsf{A}_3 := \left\{ \sup_{x\in\mathbb{R}} \widetilde{\mathfrak{h}}_{\beta t}(x) \geq \tfrac{s}{2^{14}} \right\}, \quad \mathsf{A}_4 := \left\{ \widetilde{\mathfrak{h}}_{\beta t}(0) \leq -\tfrac{s}{2^{14}} \right\},$$

Lemma 4.8.4. $\{\mathfrak{h}_t(1+\beta,0) - \beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(0) \geq s\} \subset (\mathsf{A}_1 \cup \mathsf{A}_2 \cup \mathsf{A}_3 \cup \mathsf{A}_4)$ *for large enough s.*

Before proceeding to prove Lemma 4.8.4, we show how this will imply (4.8.8). From the above lemma and the union bound, we get $\mathbf{P}(\mathfrak{h}_t(1+\beta,0) - \beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(0) \geq s) \leq \sum_{i=1}^4 \mathbf{P}(\mathsf{A}_i)$. By Proposition 4.4.2 with $\beta \mapsto \beta^{-1}$ and $\kappa = \frac{2}{3}$ we get that $\mathbf{P}(\mathsf{A}_1) \leq \exp(-cs^{3/2})$. By Proposition 4.2.14, we get $\mathbf{P}(\mathsf{A}_2) \leq \exp(-cs^{3/2})$ and $\mathbf{P}(\mathsf{A}_3) \leq \exp(-cs^{3/2})$. The one point tail estimate in Proposition 4.2.12 yields $\mathbf{P}(\mathsf{A}_4) \leq \exp(-cs^{3/2})$. This proves (4.8.8).

Proof of Lemma 4.8.4. Assume $\neg(\mathsf{A}_1 \cup \mathsf{A}_2 \cup \mathsf{A}_3 \cup \mathsf{A}_4)$ holds. Note that $\neg(\mathsf{A}_1 \cup \mathsf{A}_2 \cup \mathsf{A}_3 \cup \mathsf{A}_4) = (\neg\mathsf{A}_1) \cap (\neg\mathsf{A}_2) \cap (\neg\mathsf{A}_3) \cap (\neg\mathsf{A}_4)$. Subtracting and adding the parabola $\frac{y^2}{4t}$ inside the exponent of the term on r.h.s. of (4.8.10) shows

$$\mathfrak{h}_t(1+\beta,0) = \frac{1}{t^{\frac{1}{3}}} \log \int_{\mathbb{R}} e^{-\frac{y^2}{4t}+t^{\frac{1}{3}}\left(\mathfrak{h}_t(t^{-\frac{2}{3}}y)+\beta^{\frac{1}{3}}\widetilde{\mathfrak{h}}_{\beta t}(-(\beta t)^{-2/3}y)+\frac{y^2}{4t^{4/3}}\right)} dy$$

$$
\begin{aligned}
&\le \sup_{y\in\mathbb{R}}\left(\mathfrak{h}_t(y)+\tfrac{y^2}{4}\right)+\frac{1}{t^{1/3}}\log\int_{\mathbb{R}} e^{-\frac{y^2}{4t}+t^{1/3}\beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(-(\beta t)^{-2/3}y)}dy \\
&\le \tfrac{s}{4}+\frac{1}{t^{1/3}}\log\int_{\mathbb{R}} e^{-\frac{y^2}{4t}+t^{1/3}\beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(-(\beta t)^{-2/3}y)}dy, \qquad (4.8.11)
\end{aligned}
$$

where we have bound $\sup_{y\in\mathbb{R}}\left(\mathfrak{h}_t(y)+\frac{y^2}{4}\right)$ by $\frac{s}{4}$ since we have assumed $\neg\mathsf{A}_2$ holds. Now we divide the above integral into two parts in the following way: $\int_{|y|\le\frac{1}{32}t^{2/3}\sqrt{s}}+\int_{|y|\ge\frac{1}{32}t^{2/3}\sqrt{s}}$. For the first integral we have

$$
\begin{aligned}
&\frac{1}{t^{\frac13}}\log\int_{|y|\le\frac{1}{32}t^{2/3}\sqrt{s}}\exp\left(-\tfrac{y^2}{4t}+t^{1/3}\beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(-\beta^{-2/3}t^{-2/3}y)\right)dy \\
&\le \beta^{1/3}\sup_{|x|\le\frac{1}{32}\beta^{-2/3}\sqrt{s}}\widetilde{\mathfrak{h}}_{\beta t}(x)+t^{-1/3}\log\int_{|y|\le\frac{1}{32}t^{2/3}\sqrt{\beta s}}e^{-\frac{y^2}{4t}}dy \\
&\le \beta^{1/3}\sup_{|x|\le\frac{1}{32}\beta^{-2/3}\sqrt{s}}\widetilde{\mathfrak{h}}_{\beta t}(x)+\tfrac{1}{t^{\frac13}}\log\sqrt{4\pi t}\le\beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(0)+\tfrac{s}{4}+\tfrac{1}{t^{\frac13}}\log\sqrt{4\pi t}. \qquad (4.8.12)
\end{aligned}
$$

Here the last integral is bounded by extending the range of integration to $\mathbb{R}$. The final inequality follows from the fact that $\neg\mathsf{A}_1$ holds. For the other integral observe that

$$
\frac{1}{t^{\frac13}}\log\int_{|y|\ge\frac{1}{32}t^{2/3}\sqrt{s}}\exp\left(-\tfrac{y^2}{4t}+(\beta t)^{1/3}\widetilde{\mathfrak{h}}_{\beta t}((\beta t)^{-2/3}y)\right)dy \qquad (4.8.13)
$$

$$
\le \sup_{x\in\mathbb{R}}\beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(x)+t^{-1/3}\log\int_{|y|\ge\frac{1}{32}t^{2/3}\sqrt{s}}e^{-\frac{y^2}{4t}}dy \qquad (4.8.14)
$$

Using the fact that $\int_{|y|\ge p}e^{-y^2/4t}\mathrm{d}y\le 4\sqrt{\pi t}\cdot e^{-p^2/4t}$ as long as $p\ge\sqrt{2t}$, we get that

$$
\int_{|y|\ge\frac{1}{32}t^{2/3}\sqrt{s}}e^{-\frac{y^2}{4t}}dy\le 4\sqrt{\pi t}\cdot\exp\left(-\tfrac{(t^{2/3}\sqrt{s})^2}{(32)^2\cdot 4t}\right)=4\sqrt{\pi t}\cdot\exp\left(-\tfrac{st^{1/3}}{2^{12}}\right).
$$

Plugging this bound back we get

$$
(4.8.14)\le\beta^{\frac13}\sup_{x\in\mathbb{R}}\widetilde{\mathfrak{h}}_{\beta t}(x)+t^{-\frac13}\log 4\sqrt{\pi t}-\frac{s}{2^{12}}\le\beta^{\frac13}\widetilde{\mathfrak{h}}_{\beta t}(0)+\tfrac{s}{4}+\tfrac{1}{t^{\frac13}}4\log\sqrt{\pi t}. \qquad (4.8.15)
$$

where the last inequality follows from the fact that $\neg\mathsf{A}_3\cap\neg\mathsf{A}_4$ holds. Thus for large enough s we get

$$
\begin{aligned}
\text{r.h.s. of (4.8.11)} &\le \tfrac{1}{4}s + t^{-1/3}\log 2 + \beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(0) + t^{-1/3}\log 4\sqrt{\pi t} + \tfrac{1}{4}s \\
&\le \tfrac{1}{2}s + \tfrac{1}{2}t^{-1/3}\log(64\pi t) + \beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(0) < \beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(0) + \beta^{1/3}s,
\end{aligned}
$$

which entails $\mathfrak{h}_t(1+\beta,0) < \beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(0) + \beta^{1/3}s$ on $\neg(\mathsf{A}_1 \cup \mathsf{A}_2 \cup \mathsf{A}_3 \cup \mathsf{A}_4)$ completing the proof. $\square$

Stage-2: Proof of (4.8.9): In this stage we seek to find a lower bound for $\mathfrak{h}_t(1+\beta,0)$. Towards this end, we recall (4.8.10) and lower bound the expression by integrating over the smaller region $\{|y| \le t^{2/3}\}$. We thus have

$$
\begin{aligned}
\mathfrak{h}_t(1+\beta,0) &\ge \frac{1}{t^{\frac{1}{3}}}\log \int_{|y|\le t^{2/3}} e^{-\frac{3y^2}{4t} + t^{\frac{1}{3}}\left(\mathfrak{h}_t(t^{-\frac{2}{3}}y) + \beta^{\frac{1}{3}}\widetilde{\mathfrak{h}}_{\beta t}(-(\beta t)^{-2/3}y) + \frac{3y^2}{4t^{4/3}}\right)} dy \\
&\ge \inf_{y\in\mathbb{R}}\left(\mathfrak{h}_t(y) + \tfrac{3y^2}{4}\right) + \beta^{1/3}\inf_{|y|\le\beta^{-2/3}}\widetilde{\mathfrak{h}}_{\beta t}(y) + \frac{1}{t^{1/3}}\log\int_{|y|\le t^{2/3}} e^{-\frac{3y^2}{4t}}dy \\
&\ge \inf_{y\in\mathbb{R}}\left(\mathfrak{h}_t(y) + \tfrac{3y^2}{4}\right) + \beta^{1/3}\inf_{|y|\le\beta^{-2/3}}\widetilde{\mathfrak{h}}_{\beta t}(y) + \tfrac{1}{t^{\frac{1}{3}}}\log\sqrt{\tfrac{4\pi t}{3}} - \tfrac{2}{t^{\frac{1}{3}}}e^{-3t^{\frac{1}{3}}/4}, \qquad (4.8.16)
\end{aligned}
$$

where the last inequality follows from Gaussian tail bounds. Next we consider the following events:

$$
\mathsf{A}_5 := \{\inf_{|y|\le\beta^{-2/3}}\widetilde{\mathfrak{h}}_{\beta t}(y) \le \widetilde{\mathfrak{h}}_{\beta t}(0) - \tfrac{1}{4}\beta^{-1/3}s\}, \quad \mathsf{A}_6 := \left\{\inf_{y\in\mathbb{R}}\left(\mathfrak{h}_t(y) + \tfrac{y^2}{4}\right) \le -\tfrac{s}{4}\right\}.
$$

From Proposition 4.4.1, we see that $\mathbf{P}(\mathsf{A}_5) \le e^{-cs^2}$ and Proposition 4.2.14 implies $\mathbf{P}(\mathsf{A}_6) \le e^{-cs^{5/2}}$. On $\neg(\mathsf{A}_5 \cup \mathsf{A}_6)$ we observe that

$$
\text{r.h.s. of (4.8.16)} \ge -\frac{s}{4} + \beta^{\frac{1}{3}}\widetilde{\mathfrak{h}}_{\beta t}(0) - \tfrac{s}{4} + \tfrac{1}{t^{\frac{1}{3}}}\log\sqrt{\tfrac{4\pi t}{3}} - \tfrac{2}{t^{\frac{1}{3}}}e^{-3t^{\frac{1}{3}}/4} > \beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(0) - s
$$

for all large enough $s \ge s_0(t_0)$. Thus, $\mathbf{P}(\mathfrak{h}_t(1+\beta,0) - \beta^{1/3}\widetilde{\mathfrak{h}}_{\beta t}(0) \le -s) \le \mathbf{P}(\mathsf{A}_5) + \mathbf{P}(\mathsf{A}_6) \le e^{-cs^2}$. This completes the proof. $\square$

Proposition 4.8.5. *Set $\gamma_0 = (3/4\sqrt{2})^{2/3}$. Define $b_t := (\log\log t)^{-7/6}$. Then, for any fixed constant $K \in \mathbb{R}$,*

$$\mathbf{P}\Big(\frac{\mathfrak{h}_t}{1+Kb_t} \geq \gamma_0(\log\log t)^{2/3}\Big) = \frac{(16\pi)^{-1}(1+o(1))}{\gamma_0^{3/2}\log t\log\log t}.$$

where $o(1)$ term converges to 0 as t goes to ∞.

Our proof of Proposition 4.8.5 is closely in line with the proof of Proposition 4.1 of [115]. It will use a Laplace transform formula for $\mathcal{Z}^{\mathbf{nw}}(T,0)$ proved in [65]. It connects $\mathcal{Z}^{\mathbf{nw}}(T,0)$ with the Airy point process $\mathbf{a}_1 > \mathbf{a}_2 > \ldots$, a well studied determinantal point process in random matrix theory (see, e.g., [7, Section 4.2]).

Throughout the rest, we use the following shorthand notations.

$$\mathcal{I}_s(x) := \frac{1}{1+\exp(t^{\frac{1}{3}}(x-s))}, \qquad \mathcal{J}_s(x) := \log\big(1+\exp(t^{\frac{1}{3}}(x-s))\big).$$

Proposition 4.8.6 (Theorem 1 of [65])**.** *For all $s \in \mathbb{R}$,*

$$\mathbb{E}_{\mathrm{KPZ}}\Big[\exp\Big(-\exp\big(t^{\frac{1}{3}}(\mathfrak{h}_t(0)-s)\big)\Big)\Big] = \mathbb{E}_{\mathrm{Airy}}\left[\prod_{k=1}^{\infty}\mathcal{I}_s(\mathbf{a}_k)\right]. \tag{4.8.17}$$

The following proposition proves an upper and a lower bound on the r.h.s. of (4.8.17). We use these bounds to complete the proof of Proposition 4.8.5. We defer the proof of Proposition 4.8.7 to Section 4.8.

Proposition 4.8.7. *Fix any constants $K_1, K_2, K_3 > 0$ with $K_1 < K_2$. Recall b_t from Proposition 4.8.5. There exists $t_0 = t_0(K_1,K_2,K_3) > 0$ and two sequences $\{\mathfrak{p}_t\}_{t\geq t_0}$, $\{\mathfrak{q}_t\}_{t\geq t_0}$ such that for all $t \geq t_0$, $K_1(\log\log t)^{2/3} \leq s \leq K_2(\log\log t)^{2/3}$ and $K \in [-K_3, K_3]$,*

$$1 - \mathbb{E}\Big[\prod_{k=1}^{\infty}\mathcal{I}_{(1+Kb_t)s}(\mathbf{a}_k)\Big] \leq (1+\mathfrak{p}_t)\frac{1}{16\pi s^{3/2}}e^{-\frac{4}{3}s^{3/2}}, \tag{4.8.18}$$

$$1 - \mathbb{E}\Big[\prod_{k=1}^{\infty}\mathcal{I}_{(1+Kb_t)s}(\mathbf{a}_k)\Big] \geq (1+\mathfrak{q}_t)\frac{1}{16\pi s^{3/2}}e^{-\frac{4}{3}s^{3/2}} \tag{4.8.19}$$

and $\mathfrak{p}_t \to 0$, $\mathfrak{q}_t \to 0$ *as* $t \to \infty$.

Proof of Proposition 4.8.5. Define

$$s := \gamma_0(1 + Kb_t)(\log\log t)^{2/3}, \quad \overline{s} := \gamma_0(1 + (K+1)b_t)(\log\log t)^{2/3}$$

and $\theta(s) := \exp\big(-\exp\big(t^{\frac{1}{3}}(\mathfrak{h}_t - s)\big)\big)$. By (4.8.17), we know $\mathbb{E}_{\mathrm{KPZ}}[\theta(s)] = \mathbb{E}_{\mathrm{Airy}}[\prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)]$. Note that

$$\theta(s) \leq \mathbf{1}(\mathfrak{h}_t(0) \leq \overline{s}) + \mathbf{1}(\mathfrak{h}_t(0) > \overline{s}) \exp(-\exp(b_t s t^{1/3}))$$

which after rearranging, taking expectations and applying (4.8.17) will lead to

$$\mathbb{P}(\mathfrak{h}_t(0) > \overline{s}) \leq \Big(1 - \exp(-\exp(b_t s t^{\frac{1}{3}}))\Big)^{-1} \Big(1 - \mathbb{E}_{\mathrm{Airy}}\big[\prod_{k=1}^{\infty} \mathcal{I}_s(\mathbf{a}_k)\big]\Big).$$

We may write $1 - \exp(-\exp(b_t s t^{\frac{1}{3}})) = 1 + o(t)$. Combining this with (4.8.18) yields

$$\mathbb{P}(\mathfrak{h}_t(0) \geq \overline{s}) \leq (1 + o(1)) \frac{1}{16\pi s^{3/2}} e^{-\frac{4}{3}s^{3/2}}$$

for all large t.

We turn now to prove the lower bound. By Markov's inequality, we get

$$\mathbb{P}(\mathfrak{h}_t(0) \leq s) = \mathbb{P}\Big(\theta(\overline{s}) \geq \exp\big(-e^{-b_t s t^{1/3}}\big)\Big) \leq \exp\big(e^{-b_t s t^{1/3}}\big) \cdot \mathbb{E}[\theta(\overline{s})]$$

which after rearranging yields $1 - \exp\big(-e^{-b_t s t^{1/3}}\big)\mathbb{P}(\mathfrak{h}_t(0) \leq s) \geq 1 - \mathbb{E}\left[\theta(\overline{s})\right]$. Finally, applying (4.8.19) to the right hand side of the above display shows the lower bound. $\square$

Proof of Proposition 4.8.7

Proof of (4.8.18). Define $\overline{s} := (1+Kb_t)s$. Define $\mathsf{A}^{(-)} := \{\mathbf{a}_1 \le (1-\tilde{K}b_t)\overline{s}\}$ for some $\tilde{K} \in [0, K_3]$ and note the following lower bound

$$\mathbb{E}_{\text{Airy}}\Big[\prod_{k=1}^{\infty} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\Big] \ge \mathbb{E}_{\text{Airy}}\Big[\prod_{k=1}^{\infty} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\mathbf{1}(\mathsf{A}^{(-)})\Big]. \tag{4.8.20}$$

We show a lower bound to the right hand side of the above display. We set $k_0 := \lfloor \frac{2}{3\pi}\overline{s}^{\frac{9}{4}+2b_t} \rfloor$. By the inequality $\mathcal{J}_{\overline{s}}(\mathbf{a}_k) \le \exp(-\tilde{K}t^{\frac{1}{3}}\overline{s}b_t)$ which follows on the event $\mathsf{A}^{(-)}$, we observe that

$$\prod_{k=1}^{k_0} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\mathbf{1}_{\mathsf{A}^{(-)}} = \exp\Big(-\sum_{k=1}^{k_0}\mathcal{J}_{\overline{s}}(\mathbf{a}_k)\Big)\mathbf{1}_{\mathsf{A}^{(-)}} \ge \exp\Big(-\tfrac{2}{3\pi}\overline{s}^{\frac{9}{4}+2b_t}e^{-\tilde{K}\overline{s}b_t t^{\frac{1}{3}}}\Big). \tag{4.8.21}$$

We now focus on bounding $\prod_{k>k_0}\mathcal{I}_{\overline{s}}(\mathbf{a}_k)$ from below on the event $\mathsf{A}^{(-)}$. By the result of [116, Proposition 4.5], for any $\epsilon, \delta \in (0,1)$ the probability space corresponding to the Airy point process can be augmented so that there exists a random variable C^{Ai}_{ϵ} satisfying

$$(1+\epsilon)\lambda_k - C^{\text{Ai}}_{\epsilon} \le \mathbf{a}_k \le (1-\epsilon)\lambda_k + C^{\text{Ai}}_{\epsilon} \quad \text{for all } k \ge 1 \quad \text{and} \quad \mathbb{P}(C^{\text{Ai}}_{\epsilon} \ge s) \le e^{-s^{1-\delta}}$$

for all $s \ge s_0$ where $s_0 = s_0(\epsilon, \delta)$ is a constant. Here, λ_k is the k-th zero of the Airy function (see [116, Proposition 4.6]) and we fix some $\delta \in (0, \epsilon)$. Define $\phi(s) := s^{\frac{3+8\epsilon/3}{2(1-\delta)^2}}$ and observe that

$$\prod_{k>k_0}\mathcal{I}_{\overline{s}}(\mathbf{a}_k) \ge \prod_{k>k_0}\mathcal{I}_{\overline{s}}(\mathbf{a}_k)\mathbf{1}(C^{\text{Ai}}_{\epsilon} \le \phi(\overline{s})) \ge \exp\Big(-\sum_{k>k_0}\mathcal{J}_{\overline{s}}\big((1-\epsilon)\lambda_k + \phi(\overline{s})\big)\Big). \tag{4.8.22}$$

Using tail probability of C^{Ai}_{ϵ}, we have $\mathbb{P}(C^{\text{Ai}}_{\epsilon} \le \phi(\overline{s})) \ge 1 - \exp\big(-s^{\frac{3}{2}+\frac{4}{3}\epsilon}\big)$. We now claim that for $k \ge k_0$

$$\mathcal{J}_{\overline{s}}\Big((1-\epsilon)\lambda_k + \phi(\overline{s})\Big) \le e^{t^{1/3}\left(-\overline{s}-(1-\epsilon)(3\pi k/2)^{2/3}+\phi(\overline{s})\right)} \le e^{t^{1/3}\left(-\overline{s}-(1-\epsilon)(k-k_0)^{2/3}\right)}. \tag{4.8.23}$$

To see this note that for all $k \geq k_0$,

$$\lambda_k \leq -\Big(\frac{3\pi k}{2}\Big)^{\frac{3}{2}} \quad \text{and,} \quad (1-\epsilon)(\frac{3\pi k}{2})^{\frac{2}{3}} - \phi(\overline{s}) \geq (1-\epsilon)\Big(\frac{3\pi}{2}(k-k_0)\Big)^{\frac{1}{3}}.$$

The first and second inequalities are consequences of [116, Proposition 4.6] and [116, Lemma 5.6] respectively. Summing both sides of (4.8.23) over $k > k_0$ in (4.8.23), approximating the sum by the corresponding integral, and evaluating shows

$$\sum_{k>k_0} \mathcal{J}_{\overline{s}}((1-\epsilon)\lambda_k + \phi(\overline{s})) \leq Ct^{-\frac{1}{3}} \exp(-\overline{s}t^{\frac{1}{3}}) \tag{4.8.24}$$

for some constant $C > 0$. Now, we substitute (4.8.24) into the r.h.s. of (4.8.22) to write

$$\prod_{k>k_0} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\mathbf{1}(C_{\epsilon}^{\mathrm{Ai}} \leq \phi(\overline{s})) \geq \exp\Big(-\mathrm{C}t^{-\frac{1}{3}}\exp(-\overline{s}t^{\frac{1}{3}})\Big).$$

Applying (4.8.21) in combination with the above inequality shows

$$\text{l.h.s. of (4.8.20)} \geq \exp\Big(-\tfrac{2}{3\pi}\overline{s}^{\frac{9}{4}+2b_t}e^{-\tilde{K}b_t\overline{s}t^{\frac{1}{3}}} - \tfrac{C}{t^{\frac{1}{3}}}e^{-\overline{s}t^{\frac{1}{3}}}\Big)\mathbb{P}\big(\{C_{\epsilon}^{\mathrm{Ai}} \leq \phi(\overline{s})\} \cap \mathsf{A}^{(-)}\big). \tag{4.8.25}$$

First we note that

$$\exp\Big(-\frac{2}{3\pi}s^{\frac{9}{4}+2b_t}e^{-\tilde{K}b_t\overline{s}t^{\frac{1}{3}}} - Ct^{-\frac{1}{3}}e^{-\overline{s}t^{\frac{1}{3}}}\Big) = 1 + o(1)$$

as $t \to \infty$. Using the tail bound of $C_{\epsilon}^{\mathrm{Ai}} \leq \phi(\overline{s})$, we may now write

$$\begin{aligned}\mathbb{P}\big(\{C_{\epsilon}^{\mathrm{Ai}} \leq \phi(\overline{s})\} \cap \mathsf{A}^{(-)}\big) &\geq 1 - \mathbb{P}(C_{\epsilon}^{\mathrm{Ai}} \geq \phi(\overline{s})) - \mathbb{P}(\neg\mathsf{A}^{(-)}) \\ &\geq 1 - e^{-\overline{s}^{\frac{3}{2}+\frac{4}{3}\epsilon}} - \frac{(1+o(1))}{16\pi s^{\frac{3}{2}}}e^{-\frac{4}{3}s^{\frac{3}{2}}}\end{aligned} \tag{4.8.26}$$

for all large t. The second inequality above also uses

$$\mathbb{P}(\neg\mathsf{A}^{(-)}) = \mathbb{P}(\mathbf{a}_1 \geq (1-\tilde{K}b_t)s) \leq (1+o(1))\frac{1}{16\pi s^{3/2}}\exp(-\tfrac{4}{3}s^{\frac{3}{2}})$$

which holds when t is sufficiently large (see [75, Theorem 1]). Substituting (4.8.26) into the right hand side of (4.8.25) yields (4.8.18). $\square$

Proof of (4.8.19). Now we show an upper bound on $\mathbb{E}\big[\prod_{k=1}^{\infty} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\big]$, where recall that $\overline{s} = (1 + Kb_t)s$. Define $\mathsf{A}^{(+)} := \big\{\mathbf{a}_1 \le (1 + \tilde{K}b_t)\overline{s}\big\}$ for some $\tilde{K} \in [0, K_3]$. We split $\mathbb{E}\big[\prod_{k=1}^{\infty} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\big]$ into two different parts shown as follows

$$\mathbb{E}\Big[\prod_{k=1}^{\infty} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\Big] \le \mathbb{E}\Big[\prod_{k=1}^{\infty} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\mathbf{1}(\mathsf{A}^{(+)})\Big] + \mathbb{P}(\neg\mathsf{A}^{(+)}) \cdot \exp(-\tilde{K}b_t\overline{s}t^{\frac{1}{3}}). \tag{4.8.27}$$

Let us denote $\chi^{\mathrm{Ai}}(s) := \#\{\mathbf{a}_i \ge s\}$. Fix $\epsilon \in (0, 1)$, $c \in (0, \frac{2}{3\pi})$ and define

$$\mathsf{B} := \Big\{\chi^{\mathrm{Ai}}(-\epsilon\overline{s}) - \mathbb{E}\big[\chi^{\mathrm{Ai}}(-\epsilon\overline{s})\big] \ge -c(\epsilon\overline{s})^{\frac{3}{2}}\Big\}.$$

We split the first term on the r.h.s. of (4.8.27) as follows

$$\mathbb{E}\Big[\prod_{k=1}^{\infty} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\mathbf{1}(\mathsf{A}^{(+)})\Big] \le \mathbb{E}\Big[\prod_{k=1}^{\infty} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\mathbf{1}\big(\mathsf{B}\cap\mathsf{A}^{(+)}\big)\Big] + \mathbb{E}\Big[\mathbf{1}\big((\neg\mathsf{B})\cap\mathsf{A}^{(+)}\big)\Big].$$

We now bound each term on the right hand side of the above display. Note that

$$\prod_{k=1}^{\infty} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\mathbf{1}(\mathsf{B}) \le \exp\Big(-\Big(\frac{2}{3\pi} - c\Big)(\epsilon\overline{s})^{\frac{3}{2}}e^{-(1+\epsilon)\overline{s}t^{\frac{1}{3}}}\Big)$$

holds on the event **B**. As a consequence, we get

$$\mathbb{E}\Big[\prod_{k=1}^{\infty} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\mathbf{1}\big(\mathsf{B}\cap\mathsf{A}^{(+)}\big)\Big] \le \exp\Big(-\Big(\frac{2}{3\pi} - c\Big)(\epsilon\overline{s})^{\frac{3}{2}}e^{-(1+\epsilon)\overline{s}t^{\frac{1}{3}}}\Big) \cdot \mathbb{P}(\mathsf{A}^{(+)}). \tag{4.8.28}$$

We may bound the r.h.s. of (4.8.28) by $(1 - \exp(-\zeta\overline{s}t^{1/3}))\mathbb{P}(\mathsf{A}^{(+)})$ for some $\zeta > 0$ as t gets large. On the other hand, we note that there exists $t_\delta > 0$ such that $\mathbb{P}(\neg\mathsf{B}) \le e^{-c(\epsilon\overline{s})^{3-\delta}}$ for all $t > t_\delta$.

Substituting these bounds into the r.h.s. of (4.8.27) shows

$$1 - \mathbb{E}\Big[\prod_{k=1}^{\infty} \mathcal{I}_{\overline{s}}(\mathbf{a}_k)\Big] \geq \mathbf{P}(\neg\mathsf{A}^{(+)}) + \mathbb{P}(\mathsf{A}^{(+)})(e^{-\zeta\overline{s}t^{\frac{1}{3}}} - e^{-\tilde{K}b_t\overline{s}t^{1/3}}) - e^{-c(\zeta\overline{s})^{3-\delta}}. \tag{4.8.29}$$

Thanks to [75, Theorem 1] we know

$$\mathbb{P}(\neg\mathsf{A}^{(+)}) \geq (1+o(1))\frac{1}{16\pi s^{3/2}}\exp\big(-\tfrac{4}{3}s^{\frac{3}{2}}\big).$$

Recall that $K_1(\log\log t)^{2/3} \leq s \leq K_2(\log\log t)^{2/3}$. In this range we have

$$\exp(-\zeta\overline{s}t^{1/3}) + \exp(-\widetilde{K}b_t\overline{s}t^{1/3}) + \exp(-c(\epsilon\overline{s})^{3-\delta}) = o(1)\cdot\exp(-4s^{3/2}/3).$$

Thus as t grows large, the r.h.s. of (4.8.29) is lower bounded by

$$(1+o(1))(16\pi s^{3/2})^{-1}\exp(-4s^{3/2}/3).$$

This completes the proof of (4.8.19) and hence also of Proposition 4.8.7. □

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Chapter 5: Localization of the continuum directed random polymer

5.1 Introduction

The continuum directed random polymer (CDRP) is a continuum version of the discrete directed polymer measures modeled by a path interacting with a space-time white noise that first appeared in [4]. It arises as a scaling limit of the 1+1 dimensional directed polymers in the "intermediate disorder regime" and can be defined through the Kardar-Parisi-Zhang (KPZ) equation with narrow wedge initial data (see Section 6.1.2). A folklore favorite region conjecture on directed polymers states that under strong disorder, the midpoint (or any other point) distribution of a point-to-point directed polymer is asymptotically localized in a region of stochastically bounded diameter ([88], [44], Section 5.1.1).

In light of this conjecture, we initiate such study of the CDRP's long-time localization behaviors. Our main result, stated in Section 6.1.2, asserts that any point at a fixed proportional location on the point-to-point CDRP relative to its length converges to an explicit density function when centered around its almost surely unique random mode. A similar result for the endpoint of point-to-line CDRP is also obtained, confirming the favorite region conjecture for the CDRP. In this process, through the connections between the CDRP and the KPZ equation with narrow wedge initial data, we have shown properties such as ergodicity and Bessel behaviors around the maximum for the latter. These and other results are summarized in Section 5.1.2 and explained in fuller detail in Section 6.1.2.

As an effort to understand the broader localization phenomena, our main theorems (Theorems 5.1.4, 5.1.5) confirm the favorite region conjecture for the *first* non-stationary integrable model and are the *first* to obtain pointwise localization along the *entire path* (Theorem 5.1.4). The first rigorous localization result for directed polymers in random environment appeared in [84] and proved the existence of “favorite sites" in the Gaussian environment, which was later extended to general environments in [102]. This notion of localization is known as the *strong localization* and is weaker than the favorite region conjecture (See Section 5.1.1 for discussions on different notions of localizations). The only other case where the favorite region conjecture is proved, is the mid and endpoint localizations of the point-to-point and point-to-line one-dimensional stationary log-gamma polymers in [101]. The proofs of the results and the specificity of the locales (mid/endpoints) in [101] relied on the stationary boundary condition of the model, which reduced the endpoint distribution to exponents of simple random walks [294]. For CDRP, the absence of a similar stationarity necessitates a new approach towards the favorite region conjecture, which extends to every point on the polymer's path. Conversely, as we do not rely on integrability other than the Gibbs property, our proof for the CDRP has the potential to generalize to other integrable models. Other works that have considered localization along the whole path include the pathwise localization of the parabolic Anderson polymer model in [100] and that of the discrete polymer in Gaussian environments [43] [40]. Lastly, accompanying our localization results, we establish the convergence of the scaled favorite points to the almost sure unique maximizer of the Airy_2 process minus a parabola and the geodesics of the directed landscape respectively (Theorem 5.1.8).

Finally, from the perspective of the KPZ universality class, our paper is an innovative application of several fundamental new techniques and results that have recently emerged in the community. These include the Brownian Gibbs resampling property [**CH16**], the weak convergence from the KPZ line ensemble to the Airy line ensemble [280], the tail estimates of the KPZ equation with narrow wedge initial data [115, 116, 117] as well as probabilistic properties of the Airy line ensemble from [140]. In particular, although the Gibbs property has been utilized before in works such as [140, 81, 117, 119], we overcome a unique challenge of quantifying the Gibbs property precisely on a *symmetric* random interval around the joint local maximizer of two independent copies of the KPZ equation with narrow wedge initial data. This issue is resolved after we prescribe the joint law of the KPZ equations around the desired interval. A more detailed description of our main technical innovations is available in Section 5.1.4.

Presently, we begin with the background of our model and related key concepts.

5.1.1 Introducing the CDRP through discrete directed lattice polymers

Directed polymers in random environments were first introduced in statistical physics literature by Huse and Henley [202] to study the phase boundary of the Ising model with random impurities. Later, it was mathematically reformulated as a random walk in a random environment by Imbrie and Spencer [206] and Bolthausen [61]. Since then immense progress has been made in understanding this model (see [99] for a general introduction and [181, 44] for partial surveys).

In the $(d + 1)$- dimensional discrete polymer case, the random environment is specified by a collection of zero-mean i.i.d. random variables $\{\omega = \omega(i, j) \mid (i, j) \in \mathbb{Z}_+ \times \mathbb{Z}^d\}$. Given the environment, the energy of the n-step nearest neighbour random walk $(S_i)_{i=0}^n$ starting and ending at the origin (one can take the endpoint to be any suitable $\mathbf{x} \in \mathbb{R}^d$ as well) is given by $H_n^\omega(S) := \sum_{i=1}^n \omega(i, S_i)$. The **point-to-point** polymer measure on the set of all such paths is then defined as

$$\mathbf{P}_{n,\beta}^\omega(S) = \frac{1}{Z_{n,\beta}^\omega} e^{\beta H_n^\omega(S)} \mathbf{P}(S), \tag{5.1.1}$$

where $\mathbf{P}(S)$ is the uniform measure on set of all n-step nearest neighbour paths starting and ending at origin, β is the inverse temperature, and $Z_{n,\beta}^\omega$ is the partition function. Meanwhile, one can also consider the **point-to-line** polymer measures where the endpoint is 'free' and the reference measure $\mathbf{P}$ is given by n-step simple symmetric random walks. In the polymer measure, there is a competition between the *entropy* of paths and the *disorder strength* of the environment. Under this competition, two distinct regimes appear depending on the inverse temperature β [103]:

- *Weak Disorder*: When β is small or equivalently in high temperature regime, intuitively the disorder strength diminishes. The walk is dominated by the entropy and exhibits diffusive behaviors. This type of entropy domination is termed as *weak disorder*.

- *Strong Disorder*: If β is large and positive or equivalently the temperature is low but remains positive, the polymer measure concentrates on singular paths with high energies and the diffusive behavior is no longer guaranteed. This type of disorder strength domination is known as the *strong disorder*.

The precise definitions of weak and strong disorder regimes are available in [103]. Furthermore, there exists a critical inverse temperature $\beta_c(d)$ that depends on the dimension d such that weak disorder holds for $0 \leq \beta < \beta_c$ and strong disorder for $\beta > \beta_c$. When $d = 1$ or $d = 2$, $\beta_c = 0$, i.e. all positive β fall into the strong disorder regime.

The rest of the article focuses on $d = 1$. While for $\beta = 0$, the path fluctuations are of the order $\sqrt{n}$ via Brownian considerations, the situation is much more complex in the strong disorder regime. The following two phenomena are conjectured:

- *Superdiffusivity*: Under strong disorder, the polymer measure is believed to be in the KPZ universality class and paths have typical fluctuations of the order $n^{2/3}$. This widely conjectured phenomenon is physics literature is known as superdiffusion (see [202], [203], [218], [235]) and has been rigorously proven in specific situations (see [245],[274], [212],[84], [253]). But much remains unknown especially for $d \geq 2$.

- *Localization and the favorite region conjecture*: The polymer exhibits certain localization phenomena under strong disorder. The favorite region conjecture speculates that any point on the path of a point-to-point directed polymer is asymptotically localized in a region of stochastically bounded diameter (see [88], [42] for partial survey.)

We remark that there exist many different notions of localizations. In addition to the favorite region one discussed above and the strong localization in [84] mentioned earlier, the atomic localization [313] and the geometric localization [44] were studied in [44] for simple random walks and later extended to general reference walks in [42]. Both of [44] and [42] provided sufficient criteria for the existence of the 'favorite region' of order one for the endpoint in arbitrary dimension. Yet in spite of the sufficiency, it is unknown how to check them for standard directed polymers. We refer the readers to Bates' thesis [41] for a more detailed survey on this topic.

Meanwhile, even though the critical inverse temperature $\beta_c(1) = 0$ for $d = 1$, one might scale the inverse temperature with the length of the polymer critically to capture the transition between weak and strong disorder. In this spirit, the seminal work of [5] considered *an intermediate disordered regime* where $\beta = \beta_n = n^{-1/4}$ and n is the length of the polymer. [5] showed that the partition function Z^{ω}_{n,β_n} has a universal scaling limit given by the solution of the Stochastic Heat Equation (SHE) for ω with finite exponential moments. Furthermore, under the diffusive scaling, the polymer path itself converges to a universal object called the Continuous Directed Random Polymer (denoted as CDRP hereafter) which appeared first in [4] and depended on a continuum external environment given by the space-time white noise.

More precisely, given a white noise ξ on $[0, t] \times \mathbb{R}$, CDRP is a path measure on the space of $C([0, t])$ (continuous functions on $[0, t]$) for each realization of ξ. Conditioned on the environment, the CDRP is a continuous Markov process with the same quadratic variation as the Brownian motion but is singular w.r.t. the Brownian motion ([5]). Due to this singularity w.r.t. the Wiener measure, expressing the CDRP path measure in a Gibbsian form similar to (5.1.1) is challenging. Instead, one can construct a consistent family of finite dimensional distributions using the partition functions which uniquely specify the path measure (see [4] or Section 6.1.2).

As the CDRP sits between weak and strong disorder regimes, it exhibits weak disorder type behaviors in the short-time regime ($t \downarrow 0$) and strong ones in the long-time regime ($t \uparrow \infty$). Indeed, the log partition function of CDRP is Gaussian in the short time limit (see [6]), which provides evidence for weak disorder. Upon varying the endpoint of the CDRP measure, the log partition function becomes a random function of the endpoint and converges to the parabolic Airy_2 process under the $1 : 2 : 3$ KPZ scaling (see [280, 315]) with the characteristic $2/3$ exponent. This alludes to the superdiffusivity in the strong disorder regime. That said, the theory of universality class alone does not shed insight on the possible localization phenomena of the CDRP measures.

5.1.2 Summary of Results

The purpose of the present article is to study the localization phenomena for the long-time CDRP measure. The following summarizes our results, which we elaborate on individually in Section 6.1.2. Our first two results affirm the favorite region conjecture which so far has only been proven for the mid and endpoints of the log-gamma polymer model in [101].

- For a point-to-point CDRP of length t, the quenched density of its pt-point with fixed $p \in (0,1)$ when centered around its almost sure unique mode (which is the maximizer of the probability density function) $\mathcal{M}_{p,t}$, converges weakly to a density proportional to $e^{-\mathcal{R}_2(x)}$. Here, $\mathcal{R}_2$ is a two-sided 3D-Bessel process with diffusion coefficient 2 defined in (5.5.2)(Theorem 5.1.4).

- For a point-to-line CDRP of length t, the quenched density of its endpoint when centered around its almost sure unique mode $\mathcal{M}_{*,t}$ converges weakly to a density proportional to $e^{-\mathcal{R}_1(x)}$. $\mathcal{R}_1$ is a two-sided 3D-Bessel process with diffusion coefficient 1 (Theorem 5.1.5).

- The random mode $\mathcal{M}_{*,t}$ of length-t point-to-line CDRP's endpoint upon $2^{-1/3}t^{2/3}$ scaling converges in law to the unique maximum of the Airy$_2$ process minus a parabola; the random mode $\mathcal{M}_{p,t}$ of the pt point of point-to-point CDRP of length t upon $t^{2/3}$ scaling converges to $\Gamma(p\sqrt{2})$, the Directed landscape's geodesic from $(0,0)$ to $(0, p\sqrt{2})$ (Theorem 5.1.8).

Next, the well-known KPZ equation with narrow wedge initial data forms the log-partition function of the CDRP. Our main results below shed light on some of its local information:

- *Ergodicity*: The spatial increment of the KPZ equation with narrow wedge initial data as $t \to \infty$ converges weakly to a standard two-sided Brownian motion (Theorem 5.1.11).

- The sum of two independent copies of the KPZ equation with narrow wedge initial data when re-centered around its maximum converges to a two-sided 3D-Bessel process with diffusion coefficient 2 (Theorem 5.1.10).

These results provide a comprehensive characterization of the localization picture for the CDRP model. We present the formal statements of the results in the next subsection.

5.1.3 The model and the main results

In order to define the CDRP model we use the stochastic heat equation (SHE) with multiplicative noise as our building blocks. Namely, we consider a four-parameter random field $\mathcal{Z}(x,s;y,t)$ defined on

$$\mathbb{R}^4_{\uparrow} := \{(x,s;y,t) \in \mathbb{R}^4 : s < t\}.$$

For each $(x,s) \in \mathbb{R}\times\mathbb{R}$, the field $(y,t) \mapsto \mathcal{Z}(x,s;y,t)$ is the solution of the SHE starting from location x at time s, i.e., the unique solution of

$$\partial_t \mathcal{Z} = \tfrac{1}{2}\partial_{xx}\mathcal{Z} + \mathcal{Z}\cdot\xi, \qquad (y,t) \in \mathbb{R}\times(s,\infty),$$

with Dirac delta initial data $\lim_{t\downarrow s}\mathcal{Z}(x,s;y,t) = \delta(x-y)$. Here $\xi = \xi(x,t)$ is the space-time white noise. The SHE itself enjoys a well-developed solution theory based on Itô integral and chaos expansion [48, 316] also [113, 278]. Via the Feynmann-Kac formula ([203, 99]) the four-parameter random field can be written in terms of chaos expansion as

$$\mathcal{Z}(x,s;y,t) = \sum_{k=0}^{\infty}\int_{\Delta_{k,s,t}}\int_{\mathbb{R}^k}\prod_{\ell=1}^{k+1} p(y_\ell - y_{\ell-1}, s_\ell - s_{\ell-1})\xi(y_\ell, s_\ell)d\vec{y}\,d\vec{s}, \tag{5.1.2}$$

with $\Delta_{k,s,t} := \{(s_\ell)_{\ell=1}^k : s < s_1 < \cdots < s_k < t\}$, $s_0 = s, y_0 = x, s_{k+1} = t$, and $y_{k+1} = y$. Here $p(x,t) := (2\pi t)^{-1/2}\exp(-x^2/(2t))$ denotes the standard heat kernel. The field $\mathcal{Z}$ satisfies several other properties including the Chapman-Kolmogorov equations [4, Theorem 3.1]. For all $0 \le s < r < t$, and $x, y \in \mathbb{R}$ we have

$$\mathcal{Z}(x,s;y,t) = \int_{\mathbb{R}} \mathcal{Z}(x,s;z,r)\mathcal{Z}(z,r;y,t)dz. \tag{5.1.3}$$

For all $(x, s; y, t) \in \mathbb{R}^4_{\uparrow}$, we also set

$$\mathcal{Z}(x, s; *, t) := \int_{\mathbb{R}} \mathcal{Z}(x, s; y, t) dy. \tag{5.1.4}$$

Definition 5.1.1 (Point-to-point CDRP)**.** Conditioned on the white noise ξ, let $\mathbf{P}^{\xi}$ be a measure $C([s, t])$ whose finite dimensional distribution is given by

$$\mathbf{P}^{\xi}(X(t_1) \in dx_1, \ldots, X(t_k) \in dx_k) = \frac{1}{\mathcal{Z}(x, s; y, t)} \prod_{j=0}^{k} \mathcal{Z}(x_j, t_j, ; x_{j+1}, t_{j+1}) dx_1 \ldots dx_k. \tag{5.1.5}$$

for $s = t_0 \le t_1 < \cdots < t_k \le t_{k+1} = t$, with $x_0 = x$ and $x_{k+1} = y$.

The measure $\mathbf{P}^{\xi}$ also depends on x and y but we suppress it from our notations. We will also use the notation $\mathsf{CDRP}(x, s; y, t)$ and write $X \sim \mathsf{CDRP}(x, s; y, t)$ when $X(\cdot)$ is a random continuous function on $[s, t]$ with $X(s) = x$ and $X(t) = y$ and its finite dimensional distributions given by (5.1.5) conditioned on ξ.

Definition 5.1.2 (Point-to-line CDRP)**.** Conditioned on the white noise ξ, we also let $\mathbf{P}^{\xi}_{*}$ be a measure $C([s, t])$ whose finite dimensional distributions are given by

$$\mathbf{P}^{\xi}_{*}(X(t_1) \in dx_1, \ldots, X(t_k) \in dx_k) = \frac{1}{\mathcal{Z}(x, s; *, t)} \prod_{j=0}^{k} \mathcal{Z}(x_j, t_j, ; x_{j+1}, t_{j+1}) dx_1 \ldots dx_k. \tag{5.1.6}$$

for $s = t_0 \le t_1 < \cdots < t_k \le t_{k+1} = t$, with $x_0 = x$ and $x_{k+1} = *$.

Remark 5.1.3. Note that the Chapman-Kolmogorov equations (5.1.3) and (5.1.4) ensure that the finite dimensional distributions in (5.1.5) and (5.1.6) are consistent, and that $\mathbf{P}^{\xi}$ and $\mathbf{P}^{\xi}_{*}$ are probability measures. The measure $\mathbf{P}^{\xi}_{*}$ also depends on x but we again suppress it from our notations. We similarly use $\mathsf{CDRP}(x, y; *, t)$ to refer to $\mathbf{P}^{\xi}_{*}$.

Theorem 5.1.4 (Pointwise localization for point-to-point CDRP)**.** *Fix any $p \in (0, 1)$. Let $X \sim$ $\mathsf{CDRP}(0, 0; 0, t)$ and let $f_{p,t}(\cdot)$ denotes the density of $X(pt)$ which depends on the white noise ξ. Then, for all $t > 0$ the random density $f_{p,t}$ has almost surely a unique mode $\mathcal{M}_{p,t}$. Furthermore,* *as $t \to \infty$, we have the following convergence in law*

$$f_{p,t}(x + \mathcal{M}_{p,t}) \xrightarrow{d} r_2(x) := \frac{e^{-\mathcal{R}_2(x)}}{\int_{\mathbb{R}} e^{-\mathcal{R}_2(y)} dy}, \tag{5.1.7}$$

in the uniform-on-compact topology. Here $\mathcal{R}_2(\cdot)$ is a two-sided 3D-Bessel process with diffusion coefficient 2 *defined in Definition 5.5.2.*

Theorem 5.1.5 (Endpoint localization for point-to-line CDRP)**.** *Let $X \sim \mathsf{CDRP}(0,0;*,t)$ and let $f_t(\cdot)$ denotes the density of $X(t)$ which depends on the white noise ξ. Then for $t > 0$, the random density f_t has almost surely a unique mode $\mathcal{M}_{*,t}$. Furthermore, as $t \to \infty$, we have the following convergence in law*

$$f_{*,t}(x + \mathcal{M}_{*,t}) \xrightarrow{d} r_1(x) := \frac{e^{-\mathcal{R}_1(x)}}{\int_{\mathbb{R}} e^{-\mathcal{R}_1(y)} dy}, \tag{5.1.8}$$

in the uniform-on-compact topology. Here $\mathcal{R}_1(\cdot)$ is a two-sided 3D-Bessel process with diffusion coefficient 1 *defined in Definition 5.5.2.*

Remark 5.1.6. In Proposition 5.7.1 we show that for a two-sided 3D-Bessel process $\mathcal{R}_\sigma$ with diffusion coefficient $\sigma > 0$, $\int_{\mathbb{R}} e^{-\mathcal{R}_\sigma(y)} dy$ is finite almost surely. Thus $r_1(\cdot)$ and $r_2(\cdot)$ defined in (5.1.8) and (5.1.7) respectively are valid random densities. Theorems 5.1.4 and 5.1.5 derive explicit limiting probability densities for the quenched distributions of the endpoints of the point-to-line polymers and the pt-point of point-to-point polymers when centered around their respective modes, providing a complete description of the localization phenomena in the CDRP model. More concretely, it shows that the corresponding points are concentrated in a microscopic region of order one around their "favorite points" (see Corollary 5.7.3).

We next study the random modes $\mathcal{M}_{*,t}$ and $\mathcal{M}_{p,t}$. The "favorite point" $\mathcal{M}_{p,t}$ is of the order $t^{2/3}$ and converges in distribution upon scaling. The limit is given in terms of the directed landscape constructed in [138, 251] which arises as an universal full scaling limit of several zero temperature models [141]. Below we briefly introduce this limiting model in order to state our next result.

The directed landscape $\mathcal{L}$ is a random continuous function $\mathbb{R}^4_{\uparrow} \to \mathbb{R}$ that satisfies the metric composition law

$$\mathcal{L}(x,s;y,t) = \max_{z\in\mathbb{R}} \left[\mathcal{L}(x,s;z,r) + \mathcal{L}(z,r;y,t)\right], \tag{5.1.9}$$

with the property that $\mathcal{L}(\cdot,t_i;\cdot,t_i+s_i^3)$ are independent for any set of disjoint intervals $(t_i,t_i+s_i^3)$, and as a function in x,y, $\mathcal{L}(x,t;y,t+s^3) \stackrel{d}{=} s\cdot\mathcal{S}(x/s^2,y/s^2)$, where $\mathcal{S}(\cdot,\cdot)$ is a parabolic Airy Sheet. We omit definitions of the parabolic Airy Sheet (see Definition 1.2 in [138]) except that $\mathcal{S}(0,\cdot) \stackrel{d}{=} \mathcal{A}(\cdot)$ where $\mathcal{A}$ is the parabolic Airy$_2$ process and $\mathcal{A}(x)+x^2$ is the (stationary) Airy$_2$ process constructed in [275]

Definition 5.1.7 (Geodesics of the directed landscape)**.** For $(x,s;y,t)\in\mathbb{R}^4_{\uparrow}$, a geodesic from (x,s) to (y,t) of the directed landscape is a random continuous function $\Gamma:[s,t]\to\mathbb{R}$ such that $\Gamma(s)=x$ and $\Gamma(t)=y$ and for any $s\le r_1<r_2<r_3\le t$ we have

$$\mathcal{L}(\Gamma(r_1),r_1;\Gamma(r_3),r_3) = \mathcal{L}(\Gamma(r_1),r_1;\Gamma(r_2),r_2) + \mathcal{L}(\Gamma(r_2),r_2;\Gamma(r_3),r_3).$$

Thus geodesics precisely contain the points where the equality holds in (5.1.9). Given any $(x,s;y,t)\in\mathbb{R}^4_{\uparrow}$, by Theorem 12.1 in [138], it is known that almost surely there is a unique geodesic Γ from (x,s) to (y,t).

We are now ready to state our favorite point scaling result.

Theorem 5.1.8 (Favorite Point Limit)**.** *Fix any $p\in(0,1)$. Consider $\mathcal{M}_{p,t}$ and $\mathcal{M}_{*,t}$ defined almost surely in Theorems 5.1.4 and 5.1.5 respectively. As $t\to\infty$ we have*

$$2^{-1/3}t^{-2/3}\mathcal{M}_{*,t} \xrightarrow{d} \mathcal{M}, \qquad t^{-2/3}\mathcal{M}_{p,t} \xrightarrow{d} \Gamma(p\sqrt{2})$$

where $\mathcal{M}$ is the almost sure unique maximizer of the Airy$_2$ *process minus a parabola, and $\Gamma:[0,\sqrt{2}]\to\mathbb{R}$ is the almost sure unique geodesic of the directed landscape from $(0,0)$ to $(0,\sqrt{2})$.*

Remark 5.1.9. Theorem 5.1.8 shows that the random mode fluctuates in the order of $t^{2/3}$. This corroborates the fact that CDRP undergoes superdiffusion as $t \to \infty$. We remark that the $\mathcal{M}_{*,t}$ convergence was anticipated in [256] modulo a conjecture about convergence of scaled KPZ equation to the parabolic Airy$_2$ process. This conjecture was later proved in [315, 280].

The proof of Theorem 5.1.4 relies on establishing fine properties of the partition function $\mathcal{Z}(x,t) := \mathcal{Z}(0,0;x,t)$, or more precisely, properties of the log-partition function $\log \mathcal{Z}(x,t)$. For delta initial data, $\mathcal{Z}(x,t) > 0$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$ almost surely [168]. Thus the logarithm of the partition function $\mathcal{H}(x,t) := \log \mathcal{Z}(x,t)$ is well-defined. It formally solves the KPZ equation:

$$\partial_t \mathcal{H} = \tfrac{1}{2}\partial_{xx}\mathcal{H} + \tfrac{1}{2}(\partial_x \mathcal{H})^2 + \xi, \qquad \mathcal{H} = \mathcal{H}(x,t), \quad (x,t) \in \mathbb{R} \times [0,\infty). \tag{5.1.10}$$

The KPZ equation was introduced in [218] to study the random growing interfaces and since then it has been extensively studied in both the mathematics and the physics communities. We refer to [166, 278, 113, 281, 87, 124] for partial surveys.

As a stochastic PDE, (5.1.10) is ill-posed due to the presence of the nonlinear term $\frac{1}{2}(\partial_x \mathcal{H})^2$. The above notion of solutions from the logarithm of the solution of SHE is referred to as the Cole-Hopf solution. The corresponding initial data is called the narrow wedge initial data for the KPZ equation. Other notions of solutions, such as regularity structures [192, 191], paracontrolled distributions [186, 188], and energy solutions [184, 187], have been shown to coincide with the Cole-Hopf solution within the class of initial datas the theory applies.

To prove Theorem 5.1.4, one needs to understand how multiple copies of the KPZ equation behave around the maximum of their sum. We present below our first main result that studies the limiting behavior of sum of two independent copies of KPZ equation re-centered around the maximizer of the sum, which we often refer to as the joint maximizer in the subsequent text as $t \to \infty$.

Theorem 5.1.10 (Bessel behavior around the maximizer)**.** *Fix $k = 1$ or $k = 2$. Consider k in-*

dependent copies of the KPZ equation $\{\mathcal{H}_i(x,t)\}_{i=1}^k$ *started from the narrow wedge initial data. For each* $t > 0$*, almost surely, the process* $x \mapsto \sum_{i=1}^k \mathcal{H}_i(x,t)$ *has a unique maximizer, say* $\mathcal{P}_{k,t}$*. Furthermore, as* $t \to \infty$*, we have the following convergence in law*

$$R_k(x,t) := \sum_{i=1}^{k} \left[\mathcal{H}_i(\mathcal{P}_{k,t},t) - \mathcal{H}_i(x+\mathcal{P}_{k,t},t)\right] \xrightarrow{d} \mathcal{R}_k(x) \tag{5.1.11}$$

in the uniform-on-compact topology. Here $\mathcal{R}_k(x)$ *is a two-sided Bessel process with diffusion coefficient* k*.*

The next result captures the behaviors of the increments of $\mathcal{H}(\cdot,t)$ and complements Theorem 5.1.10. It's a by-product of our analysis and doesn't appear in the proof of Theorem 5.1.4.

Theorem 5.1.11 (Ergodicity of the KPZ equation)**.** *Consider the KPZ equation* $\mathcal{H}(x,t)$ *started from the narrow wedge initial data. As* $t \to \infty$*, we have the following convergence in law*

$$\mathcal{H}(x,t) - \mathcal{H}(0,t) \xrightarrow{d} B(x)$$

in the uniform-on-compact topology. Here $B(x)$ *is a two-sided standard Brownian motion.*

Remark 5.1.12. For a Brownian motion on a compact interval, the law of the process when re-centered around its maximum is absolutely continuous w.r.t. Bessel process. In light of Theorem 5.1.11, one expects the Bessel process as a limit in Theorem 5.1.10. The diffusion coefficient is k because there are k independent copies of the KPZ equation.

Remark 5.1.13. We prove (5.1.11) for $k = 1$ and $k = 2$ only, where $k = 1$ case relates to Theorem 5.1.5 and the $k = 2$ case relates to Theorem 5.1.4. Our proof strategy for Theorem 5.1.10 can be adapted for general $k \geq 3$ and Remark 5.4.13 explains the missing pieces for the proof of (5.1.11) for general k. While Theorem 5.1.10 for general k is an interesting result, due to brevity and the lack of applications to our localization problem, we restrict to only $k = 1, 2$.

A useful property in establishing the ergodicity of a given Markov process is the strong Feller property. For instance, [193] studied the strong Feller property for singular SPDEs to establish ergodicity for a multicomponent KPZ equation. However, [193] techniques and results are limited to only periodic boundary conditions, i.e. on torus domain, and are thus inaccessible for the KPZ equation with narrow-wedge initial data.

In addition to the strong Feller property, we can also probe the KPZ equation's ergodicity through the lens of the KPZ universality class. Often viewed as the fundamental positive temperature model of the latter, the KPZ equation shares the same $1 : 2 : 3$ scaling exponents and universal long-time behaviors expected or proven for other members of the class. A widely-held belief about the KPZ universality class is that under the $1 : 2 : 3$ scaling and in the large scale limit, all models in the class converge to an universal scaling limit, the KPZ fixed point [138, 251]. This very conjecture has been recently proved for the KPZ equation in [280, 315]. Here we recall a special case of the statement in [280] useful to us later. Consider the $1 : 2 : 3$ scaling of the KPZ equation (the scaled KPZ equation)

$$\mathfrak{h}_t(x) := t^{-1/3}\left(\mathcal{H}(t^{2/3}x, t) + \tfrac{t}{24}\right).$$

Then $2^{1/3}\mathfrak{h}_t(2^{1/3}x)$ converges to the parabolic Airy$_2$ process as $t \to \infty$. Note that the parabolic Airy$_2$ process is the marginal of the parabolic Airy Sheet, which is a canonical object in the construction of the KPZ fixed point and the related directed landscape (see [138, 280]).

On the KPZ fixed point level, ergodicity and behaviors around the maximum are better understood. Under the zero temperature setting, numerous results and techniques address the ergodicity question for the KPZ fixed point. For instance, due to the $1 : 2 : 3$ scaling invariance, ergodicity of the fixed point is equivalent to the local Brownian behavior ([251, Theorem 4.14 and 4.15]) or can be deduced in [273] using coupling techniques applicable only in zero temperature settings.

Meanwhile, [139] showed that local Brownianity and local Bessel behaviors around the maximizer hold for any process which is absolutely continuous w.r.t. Brownian motions on every compact set. The scaled KPZ equation possesses such property [**CH16**] and its ergodicity question can be transformed into local Brownian behaviors of the scaled KPZ equation. Note that we have

$$\mathcal{H}(x,t) - \mathcal{H}(0,t) = t^{-1/3}\left(\mathfrak{h}_t(t^{-2/3}x) - \mathfrak{h}_t(0)\right).$$

However, the law of $\mathfrak{h}_t$ changes with respect to time and the diffusive scaling precisely depends on t. Therefore it is unclear how to extend the soft techniques in [139, Lemma 4.3] for the KPZ equation to address the limiting local Brownian behaviors in above setting.

Another recent line of inquiries regarding the behavior around the maxima is the investigation of the fractal nature of exceptional times for the KPZ fixed point with multiple maximizers [119, 136]. In [119], the authors computed the Hausdorff dimension of the set of times for the KPZ fixed point with at least two maximizers and was extended to the case of exactly k maximizers in [136]. The latter work relied on a striking property of the KPZ fixed point where it becomes stationary in t after recentering at the maximum with Bessel initial conditions. This property considerably simplified their analysis. Other initial data were then accessed through a transfer principle from [288]. Unfortunately, analogous properties for the KPZ equation are not available.

5.1.4 Proof Ideas

In this section we sketch the key ideas behind the proofs of our main results. For brevity, we present a heuristic argument for the proofs of Theorem 5.1.4 and the related Theorem 5.1.10 with the $k = 2$ case only. The proofs for the point-to-line case (Theorem 5.1.5) and the related $k = 1$ case of Theorem 5.1.10 and ergodicity (Theorem 5.1.11) follow from similar ideas. Meanwhile, the methods related to the uniqueness and convergence of random modes (Theorem 5.1.8) are of a different flavor. We present them directly in Section 5.3 as the arguments are more straightforward.

Recall from Theorem 5.1.4 that $f_{p,t}$ denotes the quenched density of $X(pt)$ for $X \sim \mathsf{CDRP}(0,0;0,t)$. To simplify our discussion below, we let $p = \frac{1}{2}$ and replace t by $2t$. (5.1.5) gives us

$$f_{\frac{1}{2},2t}(x) = \frac{\mathcal{Z}(0,0;x,t)\mathcal{Z}(x,t;0,2t)}{\mathcal{Z}(0,0;0,2t)}.$$

Recall the chaos expansion for $\mathcal{Z}(x,s;y,t)$ from (5.1.2). Note that $\mathcal{Z}(0,0;x,t)$ and $\mathcal{Z}(x,t;0,2t)$ are independent for using different sections of the noise ξ. A change of variable and symmetry yields that $\mathcal{Z}(x,t;0,2t)$ is same in distribution as $\mathcal{Z}(0,0;x,t)$ as a process in x. Thus as a process in x, $\mathcal{Z}(0,0;x,t)\mathcal{Z}(x,t;0,2t) \stackrel{d}{=} e^{\mathcal{H}_1(x,t)+\mathcal{H}_2(x,t)}$ were $\mathcal{H}_1(x,t)$ and $\mathcal{H}_2(x,t)$ are independent copies of the KPZ equation with narrow wedge initial data. This puts Theorem 5.1.4 in the framework of Theorem 5.1.10. Viewing the density around its unique random mode $\mathcal{M}_{\frac{1}{2},2t}$ (that is the maximizer), we may thus write $f_{\frac{1}{2},2t}(x+\mathcal{M}_{\frac{1}{2},2t})$ as

$$\frac{e^{-R_2(x,t)}}{\int\limits_{\mathbb{R}} e^{-R_2(y,t)}dy},$$

where $R_2(x,t)$ is defined in (5.1.11). For simplicity, let us use the notation $\mathcal{P} = \mathcal{M}_{\frac{1}{2},2t}$.

The rest of the argument hinges on the following two results:

(i) *Bessel convergence*: $R_2(x,t)$ converges weakly to 3D-Bessel process with diffusion coefficient 2 in the uniform-on-compact topology (Theorem 5.1.10).

(ii) *Controlling the tails*: $\int_{[-K,K]^c} e^{-R_2(y,t)}dy$ can be made arbitrarily small for all large t by taking large K (Proposition 5.7.2).

Theorem 5.1.4 then follows from the above two items by standard analysis. We now explain the ideas behind items (i) and (ii) and our principal tool is the Gibbsian line ensemble, which is an object of integrable origin often used in probabilistic settings. More precisely, we use the *KPZ line ensemble* (recalled in Proposition 6.5.1), i.e. a set of random continuous functions whose lowest indexed curve is same in distribution as the narrow wedge solution of the KPZ equation. The law of the lowest indexed curve enjoys a Gibbs property called the **H**-Brownian Gibbs property. This property states that the law of the lowest indexed curve conditioned on an interval depends only on the curve indexed one below and the starting and ending points. Furthermore, this conditional law is absolutely continuous w.r.t. a Brownian bridge of the same starting and ending points with an explicit expression of the Radon-Nikodym derivative.

We now recast (i) in the language of Gibbsian line ensemble. Note that $R_2(x,t)$ is a sum of two independent KPZ equations viewed from the maximum of the sume ((5.1.11)). Accessing its distribution requires a precise description of the conditional joint law of the top curves of two independent copies of the KPZ line ensemble on random intervals around the joint maximizer. Thus (i) reduces to the following results, which we elaborate on individually:

(a) Two Brownian bridges when viewed around the maximum of their sum can be given by two pairs of non-intersecting Brownian bridges to either side of the maximum (Proposition 5.4.10).

(b) For a suitable $K(t) \uparrow \infty$, the Radon-Nikodym derivatives associated with the KPZ line ensembles (see (5.2.3) for the precise expression of Radon-Nikodym derivative) on the random interval $[\mathcal{P} - K(t), \mathcal{P} + K(t)]$ containing the maximizer goes to 1.

Combining the above two ideas, we can conclude the joint law of

$$(D_1(x,t), D_2(x,t)) := (\mathcal{H}_1(\mathcal{P},t) - \mathcal{H}_1(\mathcal{P}+x,t), \mathcal{H}_2(\mathcal{P}+x,t) - \mathcal{H}_2(\mathcal{P},t)) \tag{5.1.12}$$

on $x \in [-K(t), K(t)]$ is close to two-sided pair of non-intersecting Brownian bridges with the same starting point and appropriate endpoints. Upon taking $t \to \infty$, one obtains a two-sided Dyson Brownian motion $(\mathcal{D}_1, \mathcal{D}_2)$ defined in Definition 5.5.1 as a distributional limit. Proposition 5.6.1 is the precise rendering of this fact. Finally a 3D-Bessel process emerges as the difference of two parts of the Dyson Brownian motion: $\mathcal{D}_1(\cdot) - \mathcal{D}_2(\cdot)$ (see Lemma 5.5.3).

Before expanding upon items (a) and (b), let us explain the reasons behind our approach. Since our desired random interval includes the maximizer of two independenet copies or the joint maximizer, it is not a stopping domain and is inaccessible by classical properties such as the strong Gibbs property for KPZ line ensemble. Note that a similar context of the KPZ fixed point appeared in [119], where the authors used Gibbs property on random intervals defined to the right of the maximizer in their proof. However, [119] relied on a path decomposition of Markov processes at certain spatial times from [254], which states that conditioned on the maximizer, the process to the *right* of the maximizer is Markovian. However in our case, the intervals around the maximum is *two-sided*. Thus the abstract setup of [254] is not suited for our case. Thus, the precise description of the law given in item (a) is indispensable to our argument.

Next, one needs an exact comparison of the Brownian law and KPZ law to transition between the two. Traditional tools such as stochastic monotonicity for the KPZ line ensembles help obtain one-sided bounds for monotone events. Especially for tail estimates of the KPZ equation, it reduces the problem to the setting of Brownian bridges, which can be treated classically. However, this approach only produces a one-sided bound, which is insufficient for the precise convergence we need. Hence we treat the Radon-Nikodym derivative directly to exactly compare the two laws.

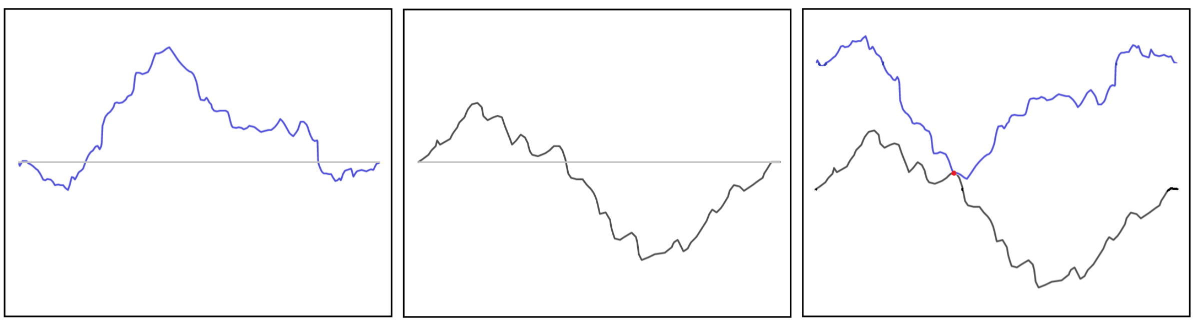

Figure 5.1: First idea for the proof: The first two figures depict two independent Brownian bridges 'blue' and 'black' on $[0,1]$ starting and ending at zero. We flip the blue one and shift it appropriately so that when it is superimposed with the black one, the blue curve always stays above the black one and touches the black curve at exactly one point. The superimposed figure is shown in third figure. The red point denotes the 'touching' point or equivalently the joint maximizer. Conditioned on the max data, the trajectories on the left and right of the red points are given by two pairs of non-intersecting Brownian bridges with appropriate end points.

To describe the result in item (a), consider two independent Brownian bridges $\overline{B}_1$ and $\overline{B}_2$ on $[0,1]$ both starting and ending at zero. See Figure 5.1. Let $M =: \operatorname{argmax}(\overline{B}_1(x) + \overline{B}_2(x))$. We study the conditional law of $(\overline{B}_1, \overline{B}_2)$ given the max data: $(M, \overline{B}_1(M), \overline{B}_2(M))$. The key fact from Proposition 5.4.10 is that conditioned on the max data

$$(\overline{B}_1(M) - \overline{B}_1(M-x), \overline{B}_2(M-x) - \overline{B}_2(M))_{x\in[0,M]}, \quad (\overline{B}_1(M) - \overline{B}_1(x), \overline{B}_2(x) - \overline{B}_2(M))_{x\in[M,1]}$$

are independent and each is a non-intersecting Brownian bridge with appropriate end points (see

Definition 5.4.4). The proof proceeds to show such a decomposition at the level of discrete random walks before taking diffusive limits to get the same for Brownian motions and finally for Brownian bridges. The details are presented in Section 5.4.

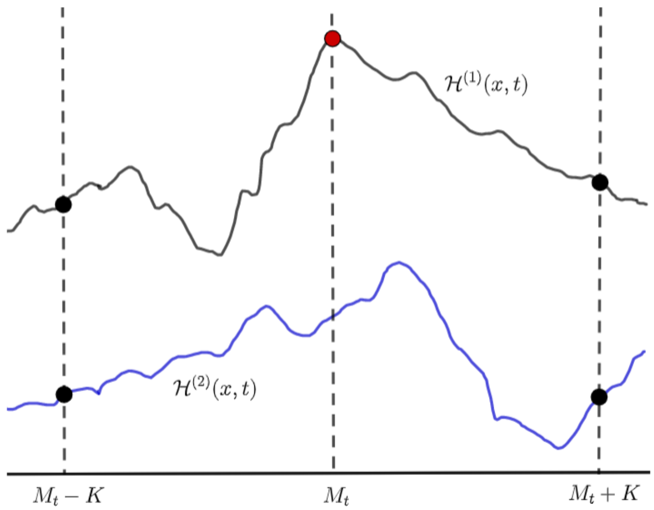

Figure 5.2: Second idea for the proof: For all “good" boundary data and max data, with high probability, there is an uniform separation of order $t^{1/3}$ between the first two curves on the random interval $[M_t - K, M_t + K]$.

To illustrate the idea behind item (b), let us consider an easier yet pertinent scenario. Let $\mathcal{H}^{(1)}(x,t)$ and $\mathcal{H}^{(2)}(x,t))$ be the first two curves of the KPZ line ensemble. Let $M_t = \operatorname{argmax} \mathcal{H}^{(1)}(x,t)$. We consider the interval $I_t := [M_t - K, M_t + K]$. See Figure 5.2. We show that

1. The maximum is not too high: $\mathcal{H}^{(1)}(M_t,t) - \mathcal{H}^{(1)}(M_t \pm K,t) = O(1)$,

2. The gap at the end points is sufficiently large: $\mathcal{H}^{(1)}(M_t \pm K,t) - \mathcal{H}^{(2)}(M_t \pm K,t) = O(t^{1/3})$.

3. The fluctuations of the second curve on I_t are $O(1)$.

Under the above favorable class of boundary data: $\mathcal{H}^{(1)}(M_t \pm K,t), \mathcal{H}^{(2)}(\cdot,t)$ and the max data: $(M_t, \mathcal{H}^{(1)}(M_t,t))$, we show that the conditional fluctuations of the first curve are $O(1)$. This forces a uniform separation between the first two curves throughout the random interval I_t. Consequently the Radon-Nikodym derivative in (5.2.3) converges to 1 as $t \to \infty$.

We rely on tail estimates for the KPZ equation as well as some properties of the Airy line ensemble which are the distributional limits of the scaled KPZ line ensemble defined in (5.2.6) to conclude such a statement rigorously. Section 5.2 contains a review of the necessary tools. Note that the rigorous argument for the Radon-Nikodym derivative in the proof of Theorem 5.1.4 (Proposition 5.6.1) is more involved. Indeed, one needs to consider another copy of line ensemble and argue that similar uniform separation holds for both when viewed around the joint maximum $\mathcal{P}$. We also take $K = K(t) \uparrow \infty$ and the separation length is consequently different.

We have argued so far that $(D_1(x,t), D_2(x,t))$ defined in (5.1.12) jointly converges to a two-sided Dyson Brownian motion. This convergence holds in the uniform-on-compact topology. However, this does not address the question about behavior of the tail integral in (ii)

$$\int_{[-K,K]^c} e^{D_2(y,t)-D_1(y,t)}\mathrm{d}y.$$

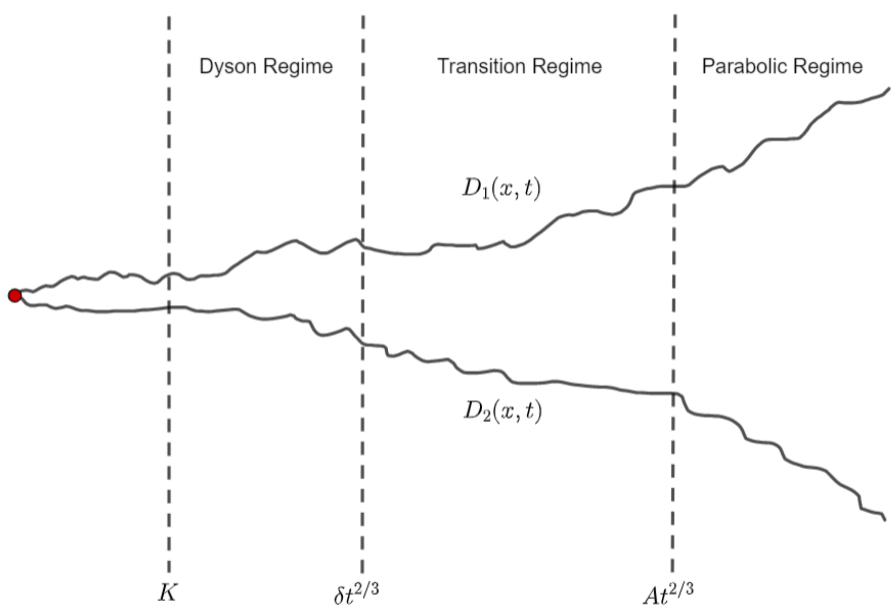

Figure 5.3: Third idea for the proof: The three regimes

To control the tail, we divide the tail integral into three parts based on the range of integration (See Figure 5.3):

- *Dyson regime:* The law of $(\mathcal{D}_1(x,t), \mathcal{D}_2(x,t))$ on the interval $[0, \delta t^{2/3}]$ is comparable to that of the Dyson Brownian motions for small δ and for large t. For Dyson Brownian motions, w.h.p. $\mathcal{D}_1(x) - \mathcal{D}_2(x) \geq \varepsilon|x|^{1/4}$ for all large enough $|x|$. This translates to $(D_1(x,t), D_2(x,t))$ and provides a decay estimate over this interval.

- *Parabolic Regime:* The maximizer $\mathcal{P}$ lies in a window of order $t^{2/3}$ region w.h.p.. On the other hand, the KPZ equation upon centering has a parabolic decay: $\mathcal{H}(x,t) + \frac{t}{24} \approx -\frac{x^2}{2t} + O(t^{1/3})$. Thus taking A large enough ensures w.h.p. $D_1(x,t) \approx \frac{x^2}{4t}$ and $D_2(x,t) \approx -\frac{x^2}{4t}$ on the interval $[At^{2/3}, \infty)$. These estimates give a rapid decay of our integral in this regime.

- *Transition Regime:* Between the two regimes, we use soft arguments of non-intersecting brownian bridges to ensure that $D_1(x,t) - D_2(x,t) \geq \rho t^{1/3}$ w.h.p. uniformly on $[\delta t^{2/3}, At^{2/3}]$.

Proposition 5.5.6 and Proposition 5.7.2 are the precise manifestations of the above idea. Proposition 5.5.6 provides decay estimates in the Dyson and transition regimes for Brownian objects. Proposition 5.7.2 translates the estimates in Proposition 5.5.6 to D_1, D_2 for the "shallow tail regime" (see Figure 5.10). The parabolic regime or the "deep tail" in Section 5.7 is addressed in Proposition 5.7.2.

Outline

The remainder of the paper is organized as follows. Section 5.2 reviews some of the existing results related to the KPZ line ensemble and its zero temperature counterpart, the Airy line ensemble. We then prove the existence and uniqueness of random modes in Theorem 5.1.8 in Section 5.3. Section 5.4 is dedicated to the behaviors of the Brownian bridges around their joint maximum. Two important objects are defined in this Section: the Bessel bridges and the non-intersecting Brownian bridges. Several properties of these two objects are subsequently proved in Section 5.5. The proofs of Theorems 5.1.10 and 5.1.11 comprise section 5.6. Finally in Section 5.7, we complete the proofs of Theorems 5.1.4 and 5.1.5. Appendix 5.8 contains a convergence result about non-intersecting random walks used in Section 5.4.

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5.2 Basic framework and tools

Remark on Notations

Throughout this paper we use $\mathrm{C} = \mathrm{C}(\alpha, \beta, \gamma, \ldots) > 0$ to denote a generic deterministic positive finite constant that may change from line to line, but dependent on the designated variables $\alpha, \beta, \gamma, \ldots$. We will often write C_α in case we want to stress the dependence of the constant to the variable α. We will use serif fonts such as $\mathsf{A}, \mathsf{B}, \ldots$ to denote events as well as $\mathsf{CDRP}, \mathsf{DBM} \ldots$ to denote laws. The distinction will be clear from the context. The complement of an event A will be denoted as $\neg\mathsf{A}$.

In this section, we present the necessary background on the directed landscape and Gibbsian line ensembles including the Airy line ensemble and the KPZ line ensemble as well as known results on these objects that are crucial in our proofs.

5.2.1 The directed landscape and the Airy line ensemble

We recall the definition of the directed landscape and several related objects from [138, 140]. The directed landscape is the central object in the KPZ universality class constructed as a scaling limit of the Brownian Last Passage percolation (BLPP). We recall the setup of the BLPP below to define the directed landscape.

Definition 5.2.1 (Directed landscape)**.** Consider an infinite collection $B := (B_k(\cdot))_{k\in\mathbb{Z}}$ of independent two-sided Brownian motions with diffusion coefficient 2. For $x \le y$ and $n \le m$, the last passage value from (x, m) to (y, n) is defined by

$$B[(x,m) \to (y,n)] = \sup_{\pi} \sum_{k=n}^{m} [B_k(\pi_k) - B_k(\pi_{k-1})],$$

where the supremum is over all $\pi \in \Pi_{m,n}(x,y) := \{\pi_m \le \cdots \le \pi_n \le \pi_{n-1} \mid \pi_m = x, \pi_{n-1} = y\}$. Now for any $(x,s;y,t) \in \mathbb{R}^4_{\uparrow}$, we denote $(x,s)_n := (s + 2xn^{-1/3}, -\lfloor sn \rfloor)$ and $(y,t)_n := (t + 2yn^{-1/3}, -\lfloor tn \rfloor)$ and define

$$\mathcal{L}_n(x,s;y,t) := n^{1/6} B_n[(x,s)_n \to (y,t)_n] - 2(t-s)n^{2/3} - 2(y-x)n^{1/3}.$$

The directed landscape $\mathcal{L}$ is the distributional limit of $\mathcal{L}_n$ as $n \to \infty$ with respect to the uniform convergence on compact subsets of $\mathbb{R}^4_{\uparrow}$. By [138], the limit exists and is unique.

The marginal $\mathcal{A}_1(x) := \mathcal{L}(0,0;x,1)$ is known as the parabolic Airy$_2$ process. In [275] the Airy$_2$ process $\mathcal{A}_1(x) + x^2$ was constructed as the scaling limit of the polynuclear growth model. At the same time, $\mathcal{A}_1(x)$ can also be viewed as the top curve of the Airy line ensemble, which we define formally below in the approach of [109].

Definition 5.2.2 (Brownian Gibbs Property)**.** Recall the general notion of line ensembles from Section 2 in [109]. Fix $k_1 \le k_2$ with $k_1, k_2 \in \mathbb{N}$ and an interval $(a,b) \in \mathbb{R}$ and two vectors $\vec{x}, \vec{y} \in \mathbb{R}^{k_2-k_1+1}$. Given two measurable functions $f, g : (a,b) \to \mathbb{R} \cup \{\pm\infty\}$, let $\mathbb{P}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}_{\text{nonint}}$ be the law of $k_2 - k_1 + 1$ many independent Brownian bridges (with diffusion coefficient 2) $\{B_i : [a,b] \to \mathbb{R}\}_{i=k_1}^{k_2}$ with $B_i(a) = x_i$ and $B_i(b) = y_i$ conditioned on the event that

$$f(x) > B_{k_1}(x) > B_{k_1+1}(x) > \cdots > B_{k_2}(x) > g(x), \quad \text{for all } x \in [a,b].$$

Then the $\mathbb{N} \times \mathbb{R}$ indexed line ensemble $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \ldots)$ is said to enjoy the *Brownian Gibbs property* if, for all $K = \{k_1, \ldots, k_2\} \subset \mathbb{N}$ and $(a, b) \subset \mathbb{R}$, the following distributional equality holds:

$$\mathrm{Law}\Big(\mathcal{L}_{K\times(a,b)} \text{ conditioned on } \mathcal{L}_{\mathbb{N}\times\mathbb{R}\backslash K\times(a,b)}\Big) = \mathbb{P}_{\text{nonint}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g},$$

where $\vec{x} = (\mathcal{L}_{k_1}(a), \ldots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \ldots, \mathcal{L}_{k_2}(b))$, $\mathcal{L}_{k_1-1} = f$ (or ∞ if $k_1 = 1$) and $\mathcal{L}_{k_2+1} = g$.

Definition 5.2.3 (Airy line ensemble)**.** The Airy line ensemble $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots)$ is the unique $\mathbb{N} \times \mathbb{R}$-indexed line ensemble satisfying Brownian Gibbs property whose top curve $\mathcal{A}_1(\cdot)$ is the parabolic Airy$_2$ process. The existence and uniqueness of $\mathcal{A}$ follow from [109] and [151] respectively.

The Airy line ensemble is in fact a strictly ordered line ensemble in the sense that almost surely,

$$\mathcal{A}_k(x) > \mathcal{A}_{k+1}(x) \text{ for all } k \in \mathbb{N}, x \in \mathbb{R}. \tag{5.2.1}$$

(5.2.1) follows from the Brownian Gibbs property and the fact that for each $x \in \mathbb{R}$, $(\mathcal{A}_k(x)+x^2)_{k\geq 1}$ is equal in distribution to the Airy point process. The latter is strictly ordered. In [140], the authors studied several probabilistic properties of the Airy line ensembles such as tail estimates and modulus of continuity. Below we state an extension of one of such results used later in our proof.

Proposition 5.2.4. *Fix $k \geq 1$. There exists a universal constant $\mathrm{C}_k > 0$ such that for all $m > 0$ and $R \geq 1$ we have*

$$\mathbf{P}\left(\sup_{\substack{x\neq y\in[-R,R]\\ |x-y|\leq 1}} \frac{|\mathcal{A}_k(x) + x^2 - \mathcal{A}_k(y) - y^2|}{\sqrt{|x-y|}\log^{\frac{1}{2}}\frac{2}{|x-y|}} \geq m\right) \leq \mathrm{C}_k \cdot R\exp\left(-\tfrac{1}{\mathrm{C}_k}m^2\right). \tag{5.2.2}$$

Proof. Fix $k \geq 1$. By [140, Lemma 6.1] there exists a constant C_k such that for all $x, y \in \mathbb{R}$ with

$|x - y| \leq 1$, we have

$$\mathbf{P}\left(|\mathcal{A}_k(x) + x^2 - \mathcal{A}_k(y) - y^2| \geq m\sqrt{x-y}\right) \leq \mathrm{C}_k \exp\left(-\tfrac{1}{\mathrm{C}_k} m^2\right).$$

Thus applying Lemma 3.3 in [140] (with $d = 1, T = [-R, R], r_1 = 1, \alpha_1 = \frac{1}{2}, \beta_1 = 2$) and adjusting the value of C_k yields (5.2.2). □

5.2.2 KPZ line ensemble

Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \ldots)$ be an $\mathbb{N} \times \mathbb{R}$-indexed line ensemble. Fix $k_1 \leq k_2$ with $k_1, k_2 \in \mathbb{N}$ and an interval $(a, b) \in \mathbb{R}$ and two vectors $\vec{x}, \vec{y} \in \mathbb{R}^{k_2 - k_1 + 1}$. Given a continuous function $\mathbf{H} : \mathbb{R} \to [0, \infty)$ (Hamiltonian) and two measurable functions $f, g : (a, b) \to \mathbb{R} \cup \{\pm\infty\}$, the law $\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ on $\mathcal{L}_{k_1}, \ldots, \mathcal{L}_{k_2} : (a, b) \to \mathbb{R}$ has the following Radon-Nikodym derivative with respect to $\mathbb{P}_{\text{free}}^{k_1,k_2,(a,b),\vec{x},\vec{y}}$, the law of $k_2 - k_1 + 1$ many independent Brownian bridges (with diffusion coefficient 1) taking values $\vec{x}$ at time a and $\vec{y}$ at time b:

$$\frac{d\mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}}{d\mathbb{P}_{\text{free}}^{k_1,k_2,(a,b),\vec{x},\vec{y}}}(\mathcal{L}_{k_1}, \ldots, \mathcal{L}_{k_2}) = \frac{\exp\left\{-\sum_{i=k_1}^{k_2+1} \int \mathbf{H}\big(\mathcal{L}_i(x) - \mathcal{L}_{i-1}(x)\big) dx\right\}}{Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}}, \tag{5.2.3}$$

where $\mathcal{L}_{k_1-1} = f$, or ∞ if $k_1 = 1$; and $\mathcal{L}_{k_2+1} = g$. Here, $Z_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ is the normalizing constant which produces a probability measure. We say $\mathcal{L}$ enjoys the **H**-*Brownian Gibbs property* if, for all $K = \{k_1, \ldots, k_2\} \subset \mathbb{N}$ and $(a, b) \subset \mathbb{R}$, the following distributional equality holds:

$$\mathrm{Law}\Big(\mathcal{L}_{K \times (a,b)} \text{ conditioned on } \mathcal{L}_{\mathbb{N} \times \mathbb{R} \setminus K \times (a,b)}\Big) = \mathbb{P}_{\mathbf{H}}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g},$$

where $\vec{x} = (\mathcal{L}_{k_1}(a), \ldots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \ldots, \mathcal{L}_{k_2}(b))$, and where again $\mathcal{L}_{k_1-1} = f$, or ∞ if $k_1 = 1$; and $\mathcal{L}_{k_2+1} = g$.

In the following text, we consider a specific class of $\mathbf{H}$ such that

$$\mathbf{H}_t(x) = t^{2/3} e^{t^{1/3} x}. \tag{5.2.4}$$

The next proposition then recalls the unscaled and scaled KPZ line ensemble constructed in [**CH16**] with $\mathbf{H}_t$-Brownian Gibbs property.

Proposition 5.2.5 (Theorem 2.15 in [**CH16**])**.** *Let $t \geq 1$. There exists an $\mathbb{N} \times \mathbb{R}$-indexed line ensemble $\mathcal{H}_t = \{\mathcal{H}_t^{(n)}(x)\}_{n \in \mathbb{N}, x \in \mathbb{R}}$ such that:*

(a) the lowest indexed curve $\mathcal{H}_t^{(1)}(x)$ is equal in distribution (as a process in x) to the Cole-Hopf solution $\mathcal{H}(x,t)$ of the KPZ equation started from the narrow wedge initial data and the line ensemble $\mathcal{H}_t$ satisfies the $\mathbf{H}_1$-Brownian Gibbs property;

(b) the scaled KPZ line ensemble $\{\mathfrak{h}_t^{(n)}(x)\}_{n \in \mathbb{N}, x \in \mathbb{R}}$, defined by

$$\mathfrak{h}_t^{(n)}(x) := t^{-1/3}\Big(\mathcal{H}_t^{(n)}\big(t^{2/3}x\big) + t/24\Big) \tag{5.2.5}$$

satisfies the $\mathbf{H}_t$-Brownian Gibbs property. Furthermore, for any interval $(a,b) \subset \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $t \geq 1$,

$$\mathbf{P}\left(Z_{\mathbf{H}_t}^{1,1,(a,b),\mathfrak{h}_t^{(1)}(a),\mathfrak{h}_t^{(1)}(b),\infty,\mathfrak{h}_t^{(2)}} < \delta\right) \leq \varepsilon,$$

where $Z_{\mathbf{H}_t}^{1,1,(a,b),\mathfrak{h}_t^{(1)}(a),\mathfrak{h}_t^{(1)}(b),\infty,\mathfrak{h}_t^{(2)}}$ is the normalizing constant defined in (5.2.3).

Remark 5.2.6. In part (3) of Theorem 2.15 [**CH16**] it is erroneously mentioned that the scaled KPZ line ensemble satisfies $\mathbf{H}_t$-Brownian Gibbs property with $\mathbf{H}_t(x) = e^{t^{1/3}x}$ (instead of $\mathbf{H}_t(x) = t^{2/3}e^{t^{1/3}x}$ from (5.2.4)). This error was reported by Milind Hegde and has been acknowledged by the authors of [**CH16**], who are currently preparing an errata for the same.

More recently, it has also been shown in [149] that the KPZ line ensemble as defined in Proposition 6.5.1 is unique as well. We will make extensive use of this scaled KPZ line ensemble $\mathfrak{h}_t^{(n)}(x)$

in our proofs in later sections. For $n = 1$, we also adopt the shorthand notation:

$$\mathfrak{h}_t(x) := \mathfrak{h}_t^{(1)}(x) = t^{-1/3}\left(\mathcal{H}(t^{2/3}x, t) + \tfrac{t}{24}\right). \tag{5.2.6}$$

Note that for t large, the Radon-Nikodym derivative in (5.2.3) attaches heavy penalty if the curves are not ordered. Thus, intuitively at $t \to \infty$, one expects to get completely ordered curves, where the $\mathbf{H}_t$-Brownian Gibbs property will be replaced by the usual Brownian Gibbs property (see Definition 5.2.2) for non-intersecting Brownian bridges. Thus it's natural to expect the scaled KPZ line ensemble to converge to the Airy line ensemble. Along with the recent progress on the tightness of KPZ line ensemble [321] and characterization of Airy line ensemble [151], this remarkable result has been recently proved in [280].

Proposition 5.2.7 (Theorem 2.2 (4) in [280])**.** *Consider the KPZ line ensemble and the Airy line ensemble defined in Proposition 6.5.1 and Definition 5.2.3 respectively. For any $k \geq 1$, we have*

$$(2^{1/3}\mathfrak{h}_t^{(i)}(2^{1/3}x))_{i=1}^k \xrightarrow{d} (\mathcal{A}_i(x))_{i=1}^k,$$

in the uniform-on-compact topology.

The $2^{1/3}$ factor in Proposition 5.2.7 corrects the different diffusion coefficient used when we define the Brownian Gibbs property and $\mathbf{H}_t$ Brownian Gibbs property. We end this section by recalling several known results and tail estimates for the scaled KPZ equation with narrow wedge initial data.

Proposition 5.2.8. *Recall $\mathfrak{h}_t(x)$ from* (5.2.6)*. The following results hold:*

(a) For each $t > 0$, $\mathfrak{h}_t(x) + x^2/2$ is stationary in x.

(b) Fix $t_0 > 0$. There exists a constant $\mathrm{C} = \mathrm{C}(t_0) > 0$ such that for all $t \geq t_0$ and $m > 0$ we have

$$\mathbf{P}(|\mathfrak{h}_t(0)| \geq m) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^{3/2}\right).$$

(c) *Fix $t_0 > 0$ and $\beta > 0$. There exists a constant $\mathrm{C} = \mathrm{C}(\beta, t_0) > 0$ such that for all $t \geq t_0$ and $m > 0$ we have*

$$\mathbf{P}\left(\sup_{x\in\mathbb{R}} \left(\mathfrak{h}_t(x) + \tfrac{x^2}{2}(1-\beta)\right) \geq m\right) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}} m^{3/2}\right).$$

The results in Proposition 5.2.8 is a culmination of results from several papers. Part (a) follows from [6, Corollary 1.3 and Proposition 1.4]. The one-point tail estimates for KPZ equation are obtained in [115, 116]. One can derive part (b) from those results or can combine the statements of Proposition 2.11 and 2.12 in [117] to get the same. Part (c) is Proposition 4.2 from [117].

5.3 Uniqueness and convergence of random modes

In this section we prove the uniqueness of random modes that appears in Theorems 5.1.4 and 5.1.5 and prove Theorem 5.1.8 which claims the convergences of random modes to appropriate limits. The following lemma settles the uniqueness question.

Lemma 5.3.1. *Fix $p \in (0,1)$ and $t > 0$. Recall $f_{p,t}$ and $f_{*,t}$ from Theorem 5.1.4 and 5.1.5. Then $f_{*,t}$ has almost surely a unique mode $\mathcal{M}_{*,t}$ and $f_{p,t}$ has almost surely a unique mode $\mathcal{M}_{p,t}$. Furthermore for any $t_0 > 0$, there exist a constant $\mathrm{C}(p, t_0) > 0$ such that for all $t > t_0$ we have*

$$\mathbf{P}(t^{-2/3}|\mathcal{M}_{p,t}| > m) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right), \quad \textit{and} \quad \mathbf{P}(t^{-2/3}|\mathcal{M}_{*,t}| > m) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right). \quad (5.3.1)$$

Proof. We first prove the point-to-point case. Fix $p \in (0,1)$ and set $q = 1-p$. Take $t > 0$. Throughout the proof $\mathrm{C} > 0$ will depend on p, we won't mention this further.

Note that (5.1.5) implies that the density $f_{p,t}(x)$ is proportional to $\mathcal{Z}(0,0;x,pt)\mathcal{Z}(x,pt;0,t)$ and that $\mathcal{Z}(0,0;x,pt)$ and $\mathcal{Z}(x,pt;0,t)$ are independent. By time reversal property of SHE we have $\mathcal{Z}(x,pt;0,t) \stackrel{d}{=} \mathcal{H}(x,qt)$ as functions in x. Using the $1:2:3$ scaling from (5.2.6) we may write

$$f_{p,t}(x) \stackrel{d}{=} \frac{1}{\widetilde{Z}_{p,t}} \exp\left(t^{1/3}p^{1/3}\mathfrak{h}_{pt,\uparrow}(p^{-2/3}t^{-2/3}x) + t^{1/3}q^{1/3}\mathfrak{h}_{qt,\downarrow}(q^{-2/3}t^{-2/3}x)\right) \tag{5.3.2}$$

where $\mathfrak{h}_{t,\uparrow}(x)$ and $\mathfrak{h}_{t,\downarrow}(x)$ are independent copies of the scaled KPZ line ensemble $\mathfrak{h}_t(x)$ defined in (5.2.6) and $\widetilde{Z}_{p,t}$ is the normalizing constant. Thus it suffices to study the maximizer of

$$\mathcal{S}_{p,t}(x) := p^{1/3}\mathfrak{h}_{pt,\uparrow}(p^{-2/3}x) + q^{1/3}\mathfrak{h}_{qt,\downarrow}(q^{-2/3}x). \tag{5.3.3}$$

Note that maximizer of $f_{p,t}$ can be retrieved from that of $\mathcal{S}_{p,t}$ by a $t^{-2/3}$ scaling.

We first claim that for all $m > 0$ we have

$$\mathbf{P}(\mathsf{A}_1) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right), \quad \text{where } \mathsf{A}_1 := \left\{\mathfrak{h}_{pt,\uparrow}(p^{-2/3}x) > \mathfrak{h}_{pt,\uparrow}(0) \text{ for some } |x| > m\right\} \tag{5.3.4}$$

$$\mathbf{P}(\mathsf{A}_2) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right), \quad \text{where } \mathsf{A}_2 := \left\{\mathfrak{h}_{qt,\downarrow}(q^{-2/3}x) > \mathfrak{h}_{qt,\downarrow}(0) \text{ for some } |x| > m\right\}. \tag{5.3.5}$$

Let us prove (5.3.4). Define

$$\mathsf{D}_1 := \left\{\sup_{x\in\mathbb{R}}\left(\mathfrak{h}_{pt,\uparrow}(p^{-2/3}x) + \frac{x^2}{4p^{4/3}}\right) \le \frac{m^2}{8p^{4/3}}\right\}, \quad \mathsf{D}_2 := \left\{|\mathfrak{h}_{pt,\uparrow}(0)| \le \frac{m^2}{16p^{4/3}}\right\}.$$

Note that on D_2, $\mathfrak{h}_{pt,\uparrow}(0) \in [-\frac{m^2}{16p^{4/3}}, \frac{m^2}{16p^{4/3}}]$, whereas on D_1, for all $|x| > m$ we have

$$\mathfrak{h}_{pt,\uparrow}(p^{-2/3}x) < \tfrac{m^2}{8p^{4/3}} - \tfrac{m^2}{4p^{4/3}} = -\tfrac{m^2}{8p^{4/3}}.$$

Thus $\mathsf{A}_1 \subset \neg\mathsf{D}_1 \cup \neg\mathsf{D}_2$ where A_1 is defined in (5.3.4). On the other hand, by Proposition 5.2.8(c) with $\beta = \frac{1}{2}$ and Proposition 5.2.8 (b) we have

$$\mathbf{P}(\mathsf{D}_1) > 1 - \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right), \quad \mathbf{P}(\mathsf{D}_2) > 1 - \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right).$$

Hence by union bound we get $\mathbf{P}(\mathsf{A}_1) \le \mathbf{P}(\neg\mathsf{D}_1) + \mathbf{P}(\neg\mathsf{D}_2) \le \mathrm{C}\exp(-\frac{1}{\mathrm{C}}m^3)$. This proves (5.3.4). Proof of (5.3.5) is analogous.

Now via the Brownian Gibbs property $\mathfrak{h}_t$ is absolute continuous w.r.t. Brownian motion on every compact interval. Hence for each $t > 0$, $\mathcal{S}_{p,t}(x)$ defined in (5.3.3) has a unique maximum on any compact interval almost surely. But due to the bounds in (5.3.4) and (5.3.5), we see that

$$\mathbf{P}\left(\mathcal{S}_{p,t}(x) > \mathcal{S}_{p,t}(0) \text{ for some } |x| > m\right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^3\right). \tag{5.3.6}$$

Thus $\mathcal{S}_{p,t}(\cdot)$ has a unique maximizer almost surely. By the definitions of $f_{p,t}(x)$ and $\mathcal{S}_{p,t}(x)$ from (5.3.2) and (5.3.3), this implies $f_{p,t}(x)$ also has a unique maximizer $\mathcal{M}_{p,t}$ and we have that

$$\mathcal{M}_{p,t} \stackrel{d}{=} t^{2/3} \operatorname*{argmax}_{x\in\mathbb{R}} \mathcal{S}_{p,t}(x). \tag{5.3.7}$$

In view of (5.3.6), the above relation (5.3.7) leads to the first inequality in (5.3.1).

For the point-to-line case, note that via (5.1.6) and (5.2.6), $f_{*,t}(x)$ is proportional to $\exp(t^{1/3}\mathfrak{h}_t(t^{-2/3}x))$. The proofs of uniqueness of the maximizer and the second bound in (5.3.1) then follow by analogous arguments. This completes the proof. $\square$

In the course of proving the above lemma, we have also proved an important result that connects the random modes to the maximizers of the KPZ equations. We isolate this result as a separate lemma.

Lemma 5.3.2. *Consider three independent copies $\mathcal{H}, \mathcal{H}_\uparrow, \mathcal{H}_\downarrow$ of the KPZ equation started from the narrow wedge initial data. The random mode $\mathcal{M}_{p,t}$ of $f_{p,t}$ (defined in statement of Theorem 5.1.4) is same in distribution as the maximizer of*

$$\mathcal{H}_\uparrow(x, pt) + \mathcal{H}_\downarrow(x, qt).$$

Similarly one has that the random mode $\mathcal{M}_{*,t}$ of $f_{*,t}$ (defined in statement of Theorem 5.1.5) is same in distribution as the maximizer of $\mathcal{H}(x,t)$.

Proof of Theorem 5.1.8. Due to the identity in (5.3.7) we see that $t^{-2/3}\mathcal{M}_{p,t}$ is same in distribution as

$$\operatorname*{argmax}_{x\in\mathbb{R}} S_{p,t}(x)$$

where $\mathcal{S}_{p,t}(x)$ is defined in (5.3.3). By Proposition 5.2.7 we see that as $t \to \infty$

$$\mathcal{S}_{p,t}(x) \xrightarrow{d} 2^{-1/3}\left(p^{1/3}\mathcal{A}_{1,\uparrow}(2^{-1/3}p^{-2/3}x) + q^{1/3}\mathcal{A}_{1,\downarrow}(2^{-1/3}q^{-2/3}x)\right)$$

in the uniform-on-compact topology where $\mathcal{A}_{1,\uparrow}, \mathcal{A}_{1,\downarrow}$ are independent parabolic Airy_2 processes. Note that the expression in the r.h.s. of the above equation is the same as

$$\mathcal{A}(x) := 2^{-1/2}\left(\mathcal{A}_{\uparrow}^{(p\sqrt{2})}(x) + \mathcal{A}_{\downarrow}^{(q\sqrt{2})}(x)\right) \tag{5.3.8}$$

where $\mathcal{A}_{\uparrow}^{(p\sqrt{2})}(x), \mathcal{A}_{\downarrow}^{(q\sqrt{2})}(x)$ are independent Airy sheets of index $p\sqrt{2}$ and $q\sqrt{2}$ respectively. By Lemma 9.5 in [138] we know that $\mathcal{A}(x)$ has almost surely a unique maximizer on every compact set. Thus,

$$\operatorname*{argmax}_{x\in[-K,K]} \mathcal{S}_{p,t}(x) \xrightarrow{d} \operatorname*{argmax}_{x\in[-K,K]} \mathcal{A}(x). \tag{5.3.9}$$

Finally the decay bounds for the maximizer of $\mathcal{S}_{p,t}(x)$ from Lemma 5.3.1 and the decay bounds for the maximizer of $\mathcal{A}$ from [279] allow us to extend the weak convergence to the case of maximizers on the full line. However, by the definition of the geodesic of the directed landscape from Definition 6.1.6, we see that $\Gamma(p\sqrt{2}) \stackrel{d}{=} \operatorname{argmax}_{x\in\mathbb{R}} \mathcal{A}(x)$. This concludes the proof for the point-to-point case. For the point-to-line case, following Lemma 5.3.2 and recalling again the scaled KPZ line ensemble from (5.2.6), we have

$$2^{-1/3}t^{-2/3}\mathcal{M}_{*,t} = \operatorname*{argmax}_{x\in\mathbb{R}} \mathcal{H}(2^{1/3}t^{2/3}x, t) = \operatorname*{argmax}_{x\in\mathbb{R}} \left(t^{1/3}\mathfrak{h}_t(2^{1/3}x) - \tfrac{t}{24}\right) = \operatorname*{argmax}_{x\in\mathbb{R}} 2^{1/3}\mathfrak{h}_t(2^{1/3}x).$$

From Proposition 5.2.7 we know $2^{1/3}\mathfrak{h}_t(2^{1/3}x)$ converges in distribution to $\mathcal{A}_1(x)$ in the uniform-on-compact topology. Given the decay estimates for $\mathcal{M}_{*,t}$ from (5.3.1) and decay bounds for the maximizer of $\mathcal{A}_1$ from [138], we thus get that $\operatorname{argmax}_{x\in\mathbb{R}} 2^{1/3}\mathfrak{h}_t(2^{1/3}x)$ converges in distribution to $\mathcal{M}$, the unique maximizer of the parabolic Airy_2 process. This completes the proof. □

5.4 Decomposition of Brownian bridges around joint maximum

The goal of this section is to prove certain decomposition properties of Brownian bridges around the joint maximum outlined in Proposition 5.4.8 and Proposition 5.4.10. To achieve this goal, we first discuss several Brownian objects and their related properties in Section 5.4.1 which will be foundational for the rest of the paper. Then we prove Proposition 5.4.8 and 5.4.10 in the ensuing subsection. We refer to Figure 5.4 for the structure and various Brownian laws convergences in this and the next sections. The notation $p_t(y) := (2\pi t)^{-1/2}e^{-y^2/(2t)}$ for the standard heat kernel will appear throughout the rest of the paper.

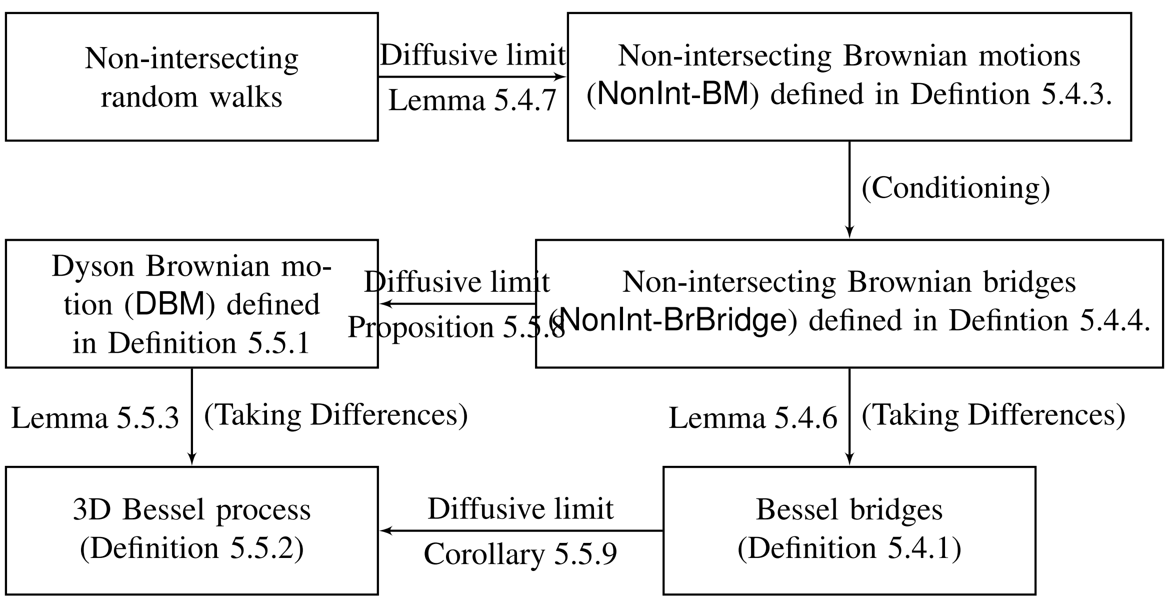

Figure 5.4: Relationship between different laws used in Sections 5.4 and 5.5.

5.4.1 Brownian objects

In this section we recall several objects related to Brownian motion, including the Brownian meanders, Bessel bridges, non-intersecting Brownian motions and non-intersecting Brownian bridges.

Definition 5.4.1 (Brownian meanders and Bessel bridges)**.** Given a standard Brownian motion $B(\cdot)$ on $[0,1]$, a standard Brownian meander $\mathfrak{B}_{\mathrm{me}} : [0,1] \to \mathbb{R}$ is a process defined by

$$\mathfrak{B}_{\mathrm{me}}(x) = (1-\theta)^{-\frac{1}{2}}|B(\theta + (1-\theta)x)|, \quad x \in [0,1],$$

where $\theta = \sup\{x \in [0,1] \mid B(x) = 0\}$. In general, we say a process $\mathfrak{B}_{\mathrm{me}} : [a,b] \to \mathbb{R}$ is a Brownian meander on $[a,b]$ if

$$\mathfrak{B}'_{\mathrm{me}}(x) := (b-a)^{-\frac{1}{2}}\mathfrak{B}_{\mathrm{me}}(a + x(b-a)), \quad x \in [0,1]$$

is a standard Brownian meander. A Bessel bridge $\mathcal{R}_{\mathrm{bb}}$ on $[a,b]$ ending at $y > 0$ is a Brownian meander $\mathfrak{B}_{\mathrm{me}}$ on $[a,b]$ subject to the condition (in the sense of Doob) $\mathfrak{B}_{\mathrm{me}}(b) = y$.

A Bessel bridge can also be realized as conditioning a 3D Bessel process to end at some point and hence the name. As we will not make use of this fact, we do not prove this in the paper.

Lemma 5.4.2 (Transition densities for Bessel Bridge)**.** *Let V be a Bessel bridge on $[0,1]$ ending at a. Then for $0 < t < 1$,*

$$\mathbf{P}(V(t) \in dx) = \frac{x}{at}\frac{p_t(x)}{p_1(a)}[p_{1-t}(x-a) - p_{1-t}(x+a)]dx, \quad x \in [0,\infty).$$

For $0 < s < t < 1$ and $x > 0$,

$$\mathbf{P}(V(t) \in dy \mid V(s) = x) = \frac{[p_{t-s}(x-y) - p_{t-s}(x+y)][p_{1-t}(y-a) - p_{1-t}(y+a)]}{[p_{1-s}(x-a) - p_{1-s}(x+a)]}dy, \quad y \in [0,\infty).$$

Proof. We recall the joint density formula for Brownian meander W on $[0,1]$ from [204]. For $0 = t_0 < t_1 < t_2 < \cdots < t_k \le 1$:

$$\mathbf{P}(W(t_1) \in dx_1, \ldots, W(t_k) \in dx_k) = \frac{x_1}{t} p_{t_1}(x_1) \Psi(\tfrac{x_k}{\sqrt{1-t_k}}) \prod_{j=1}^{k-1} g(x_j, x_{j+1}; t_{j+1} - t_j) \prod_{j=1}^{k} dx_j$$

where

$$g(x_j, x_{j+1}; t_{j+1} - t_j) := [p_{t_{j+1}-t_j}(x_j - x_{j+1}) - p_{t_{j+1}-t_j}(x_j + x_{j+1})],$$
$$\Psi(x) := (2/\pi)^{\frac{1}{2}} \int_0^x e^{-u^2/2} du.$$

The joint density is supported on $[0,\infty)^k$. We now use Doob-h transform to get the joint density for Bessel bridge on $[0,1]$ ending at a. For $0 = t_0 < t_1 < t_2 < \cdots < t_k < 1$:

$$\mathbf{P}(V(t_1) \in dx_1, \ldots, V(t_k) \in dx_k) = \frac{x_1}{at_1} \frac{p_{t_1}(x_1)}{p_1(a)} \prod_{j=1}^{k} g(x_j, x_{j+1}; t_{j+1} - t_j) \prod_{j=1}^{k} dx_j$$

where $x_{k+1} = a$ and $t_{k+1} = 1$. Formulas in Lemma 5.4.2 is obtained easily from the above joint density formula. $\square$

Definition 5.4.3 (Non-intersecting Brownian motions)**.** We say a random continuous function $W(t) = (W_1(t), W_2(t)) : [0,1] \to \mathbb{R}^2$ is a pair of non-intersecting Brownian motion ($\mathsf{NonInt\text{-}BM}$ in short) if its distribution is given by the following formulas:

(a) We have for any $y_1, y_2 \in \mathbb{R}$

$$\mathbf{P}(W_1(1) \in \mathrm{d}y_1, W_2(1) \in \mathrm{d}y_2) = \frac{\mathbf{1}\{y_1 > y_2\}(y_1 - y_2)p_1(y_1)p_1(y_2)}{\int_{r_1 > r_2}(r_1 - r_2)p_1(r_1)p_1(r_2)\mathrm{d}r_1\mathrm{d}r_2} \mathrm{d}y_1\mathrm{d}y_2. \tag{5.4.1}$$

(b) For $0 < t < 1$, we have

$$\begin{aligned}
&\mathbf{P}(W_1(t) \in \mathrm{d}y_1, W_2(t) \in \mathrm{d}y_2) \\
&= \frac{\mathbf{1}\{y_1 > y_2\}(y_1 - y_2)p_t(y_1)p_t(y_2) \int_{r_1 > r_2} \det(p_{1-t}(y_i - r_j))_{i,j=1}^2 \mathrm{d}r_1 \mathrm{d}r_2}{t \int_{r_1 > r_2} (r_1 - r_2) p_1(r_1) p_1(r_2) \mathrm{d}r_1 \mathrm{d}r_2} \mathrm{d}y_1 \mathrm{d}y_2.
\end{aligned} \tag{5.4.2}$$

(c) For $0 < s < t \leq 1$ and $x_1 > x_2$, we have

$$\begin{aligned}
&\mathbf{P}(W_1(t) \in \mathrm{d}y_1, W_2(t) \in \mathrm{d}y_2 | W_1(s) = x_1, W_2(s) = x_2) \\
&= \mathbf{1}\{y_1 > y_2\} \frac{\det(p_{t-s}(y_i - x_j))_{i,j=1}^2 \int_{r_1 > r_2} \det(p_{1-t}(y_i - r_j))_{i,j=1}^2 \mathrm{d}r_1 \mathrm{d}r_2}{\int_{r_1 > r_2} \det(p_{1-s}(x_i - r_j))_{i,j=1}^2 \mathrm{d}r_1 \mathrm{d}r_2} \mathrm{d}y_1 \mathrm{d}y_2.
\end{aligned} \tag{5.4.3}$$

We call $W^{[0,M]}$ a NonInt-BM on $[0, M]$ if $(M^{-1/2} W_1^{[0,M]}(Mx), M^{-1/2} W_2^{[0,M]}(Mx))$ is a NonInt-BM on $[0, 1]$.

Definition 5.4.4 (Non intersection Brownian bridges)**.** A 2-level non-intersecting Brownian bridge (NonInt-BrBridge in short) $V = (V_1, V_2)$ on $[0, 1]$ ending at (z_1, z_2) with $(z_1 \neq z_2)$ is a NonInt-BM on $[0, 1]$ defined in Definition 5.4.3 subject to the condition (in the sense of Doob) $V(1) = z_1, V(1) = z_2$. It is straight forward to check we have the following formulas for the distribution of V:

(a) For $0 < t < 1$, we have

$$\mathbf{P}(V_1(t) \in \mathrm{d}y_1, V_2(t) \in \mathrm{d}y_2) = \frac{\mathbf{1}\{y_1 > y_2\}(y_1 - y_2)p_t(y_1)p_t(y_2)}{t(z_1 - z_2)p_1(z_1)p_1(z_2)} \det(p_{1-t}(y_i - z_j))_{i,j=1}^2 \mathrm{d}y_1 \mathrm{d}y_2.$$

(b) For $0 < s < t \leq 1$ and $x_1 > x_2$, we have

$$\begin{aligned}
&\mathbf{P}(V_1(t) \in \mathrm{d}y_1, V_2(t) \in \mathrm{d}y_2 | V_1(s) = x_1, V_2(s) = x_2) \\
&= \frac{\det(p_{t-s}(y_i - x_j))_{i,j=1}^2 \det(p_{1-t}(y_i - z_j))_{i,j=1}^2}{\det(p_{1-s}(x_i - z_j))_{i,j=1}^2} \mathrm{d}y_1 \mathrm{d}y_2.
\end{aligned}$$

Just like NonInt-BM, we call $V^{[0,M]}$ a NonInt-BrBridge on $[0, M]$ if $(\frac{1}{\sqrt{M}}V_1^{[0,M]}(Mx), \frac{1}{\sqrt{M}}V_2^{[0,M]}(Mx))$ is a NonInt-BrBridge on $[0, 1]$.

Remark 5.4.5. It is possible to specify the distributions for a n-level non-intersecting Brownian bridge. However, the notations tend to get cumbersome due to the possibility of some paths sharing the same end points. We refer to Definition 8.1 in [155] for a flavor of such formulas. We remark that in this paper we will focus exclusively on the $n = 2$ case with distinct endpoints.

The following Lemma connects NonInt-BrBridge with Bessel bridges.

Lemma 5.4.6 (NonInt-BrBridge to Bessel bridges)**.** *Let $V = (V_1, V_2)$ be a* NonInt-BrBridge *on $[0, 1]$ ending at (z_1, z_2) with $z_1 > z_2$. Then, as functions in x, we have $V_1(x) - V_2(x) \stackrel{d}{=} \sqrt{2}\mathcal{R}_{\text{bb}}(x)$ where $\mathcal{R}_{\text{bb}} : [0, 1] \to \mathbb{R}$ is a Bessel bridge (see Definition 5.4.1) ending at $(z_1 - z_2)/\sqrt{2}$.*

The proof of Lemma 5.4.6 is based on the following technical lemma that discusses how NonInt-BM comes up as a limit of non-intersecting random walks.

Lemma 5.4.7. *Let X_j^i be i.i.d. $\mathrm{N}(0, 1)$ random variables. Let $S_0^{(i)} = 0$ and $S_k^{(i)} = \sum_{j=1}^k X_j^i$. Consider $Y_n(t) = (Y_{n,1}(t), Y_{n,2}(t)) := (\frac{S_{nt}^{(1)}}{\sqrt{n}}, \frac{S_{nt}^{(2)}}{\sqrt{n}})$ an $\mathbb{R}^2$ valued process on $[0, 1]$ where the in-between points are defined by linear interpolation. Then conditioned on the non-intersecting event $\Lambda_n := \cap_{j=1}^n \{S_j^{(1)} > S_j^{(2)}\}$, $Y_n \xrightarrow{d} W$, where $W(t) = (W_1(t), W_2(t))$ is distributed as* NonInt-BM *defined in Definition 5.4.3.*

We defer the proof of this lemma to the Appendix as it roughly follows standard calculations based on the Karlin-McGregor formula [219].

Proof of Lemma 5.4.6. Let X_j^i to be i.i.d. $\mathrm{N}(0, 1)$ random variables. Let $S_0^{(i)} = 0$ and $S_k^{(i)} = \sum_{j=1}^k X_j^i$. Set $Y_n(t) = (\frac{S_{nt}^{(1)}}{\sqrt{n}}, \frac{S_{nt}^{(2)}}{\sqrt{n}})$ an $\mathbb{R}^2$ valued process on $[0, 1]$ where the in-between points are defined by linear interpolation. By Lemma 5.4.7, conditioned on the non-intersecting event $\Lambda_n := \cap_{j=1}^n \{S_j^{(1)} > S_j^{(2)}\}$, Y_n converges to $W = (W_1, W_2)$, a NonInt-BM on $[0, 1]$ defined in Definition 5.4.3. On the other hand, classical results from [204] tell us, $(S_{nt}^{(1)} - S_{nt}^{(2)})/\sqrt{n}$ conditioned on Λ_n converges weakly to $\sqrt{2}\mathfrak{B}_{\mathrm{me}}(t)$, where $\mathfrak{B}_{\mathrm{me}}(\cdot)$ is a Brownian meander defined in Definition 5.4.1. The $\sqrt{2}$ factor comes because $S_k^{(1)} - S_k^{(2)}$ is random walk with variance 2. Thus

$$W_1(\cdot) - W_2(\cdot) \stackrel{d}{=} \sqrt{2}\mathfrak{B}_{\mathrm{me}}(\cdot).$$

From [204], $\mathfrak{B}_{\mathrm{me}}$ is known to be Markov process. Hence the law of $W_1 - W_2$ depends on $(W_1(1), W_2(1))$ only through $W_1(1) - W_2(1)$. In particular conditioning on $(W_1(1) = z_1, W_2(1) = z_2)$, for any $z_1 > z_2$, makes W to be a NonInt-BrBridge on $[0, 1]$ ending at (z_1, z_2) and the conditional law of $\frac{1}{\sqrt{2}}(W_1 - W_2)$ is then a Bessel bridge ending at $\frac{1}{\sqrt{2}}(z_1 - z_2)$. This completes the proof. $\square$

5.4.2 Decomposition Results

In this section we prove two path decomposition results around the maximum for a single Brownian bridge and for a sum of two Brownian bridges. The first one is for a single Brownian bridge.

Proposition 5.4.8 (Bessel bridge decomposition)**.** *Let $\overline{B} : [a, b] \to \mathbb{R}$ be a Brownian bridge with $\overline{B}(a) = x$ and $\overline{B}(b) = y$. Let M be the almost sure unique maximizer of $\overline{B}$. Consider $W_\ell : [a, M] \to \mathbb{R}$ defined as $W_\ell(x) = \overline{B}(M) - \overline{B}(M + a - x)$, and $W_{\mathrm{r}} : [M, b] \to \mathbb{R}$ defined as $W_{\mathrm{r}}(x) = \overline{B}(M) - \overline{B}(x)$. Then, conditioned on $(M, \overline{B}(M))$,*

(a) $W_\ell(\cdot)$ and $W_{\mathrm{r}}(\cdot)$ are independent.

(b) $W_\ell(\cdot)$ is a Bessel bridge on $[a, M]$ starting at zero and ending at $\overline{B}(M) - x$.

(c) $W_{\mathrm{r}}(\cdot)$ is a Bessel bridge on $[M, b]$ starting at zero and ending at $\overline{B}(M) - y$.

Recall that the Bessel bridges are defined in Definition 5.4.1.

Remark 5.4.9. There is a technical issue in considering the regular conditional distribution of W_ℓ, W_{r} separately as the objects are defined on intervals of random length. Instead we should always view W_ℓ, W_{r} appended together as one random function defined on the deterministic interval $[a, b]$, as done in our proofs in Sections 5.6 and 5.7. However here in Proposition 5.4.8 (as well as in Proposition 5.4.10), we state their distributions separately for simplicity.

Proof. We will prove the result for $a = 0$, $b = 1$ and $x = 0$; the general case then follows from Brownian scaling and translation property of bridges. We recall the classical result of Brownian motion decomposition around maximum from [145]. Consider a map $\Upsilon : C([0,1]) \to C([0,1]) \times C([0,1])$ given by

$$(\Upsilon f)_-(t) := M^{-\frac{1}{2}}[f(M) - f(M(1-t))], \quad t \in [0,1],$$
$$(\Upsilon f)_+(t) := (1-M)^{-\frac{1}{2}}[f(M) - f(M+(1-M)t)], \quad t \in [0,1],$$

where $M = M(f) := \inf\{t \in [0,1] \mid f(s) \le f(t), 0 \le s \le 1\}$ is the left-most maximizer of f. We set $(\Upsilon f)_- \equiv (\Upsilon f)_+ \equiv 0$ if $M = 0$ or $M = 1$. We also define

$$(\Upsilon^M f)_-(t) := M^{1/2}(\Upsilon f)_-(\tfrac{t}{M}), \quad t \in [0,M],$$
$$(\Upsilon^M f)_+(t) := (1-M)^{\frac{1}{2}}(\Upsilon f)_+(\tfrac{t-M}{1-M}), \quad t \in [M,1].$$

Given a standard Brownian motion B on $[0,1]$, by Theorem 1 in [145], $\Upsilon(B)$ is independent of $M = M(B)$ and $\Upsilon(B)_-$ and $\Upsilon(B)_+$ are independent Brownian meanders on $[0,1]$. By the Brownian scaling and the fact that $\Upsilon(B)$ is independent of $M(B)$, conditioned on $M(B)$, we see that $(\Upsilon^M B)_-$ and $(\Upsilon^M B)_+$ are independent Brownian meanders on $[0,M]$ and $[M,1]$ respectively. Observe that $(\Upsilon^M f)_-(M) = f(M)$ and $(\Upsilon^M f)_+(1) = f(M) - f(1)$ for any $f \in C([0,1])$. Thus conditioning on $(B(M) = v, B(1) = y)$ is equivalent to conditioning on $((\Upsilon^M B)_-(M) = v, (\Upsilon^M B)_+(1) = v-y)$. By definition the conditional law of Brownian meanders upon conditioning their end points are Bessel bridges. Thus conditioning on $(M = m, B(M) = v, B(1) = y)$, we see that $(\Upsilon^M B)_-$ and $(\Upsilon^M B)_+$ are independent Bessel bridges on $[0,M]$ and $[M,1]$ ending at v and $v-y$ respectively. But the law of a Brownian motion conditioned on $(M = m, B(M) = v, B(1) = y)$ is same as the law of a Brownian bridge $\overline{B}$ on $[0,1]$ starting at 0 and ending at y, conditioned on $(M(\overline{B}) = m, \overline{B}(M) = v)$.

Identifying $(\Upsilon^M \overline{B})_-$ and $(\Upsilon^M \overline{B})_+$ with W_ℓ and W_{r} gives us the desired result. $\square$

The next proposition show that for two Brownian bridges the decomposition around the joint maximum is given by non-intersecting Brownian bridges.

Proposition 5.4.10 (Non-intersecting Brownian bridges decomposition)**.** *Let $\overline{B}_1, \overline{B}_2 : [a,b] \to \mathbb{R}$ be independent Brownian bridges such that $\overline{B}_i(a) = x_i, \overline{B}_i(b) = y_i$. Let M be the almost sure unique maximizer of $(\overline{B}_1(x) + \overline{B}_2(x))$ on $[0,1]$. Define $\overline{V}_\ell(x) : [0, M-a] \to \mathbb{R}^2$ and $\overline{V}_r : [0, b-M] \to \mathbb{R}^2$ as follows:*

$$\overline{V}_\ell(x) := (\overline{B}_1(M) - \overline{B}_1(M-x), -\overline{B}_2(M) + \overline{B}_2(M-x))$$
$$\overline{V}_r(x) := (\overline{B}_1(M) - \overline{B}_1(M+x), -\overline{B}_2(M) + \overline{B}_2(M+x))$$

Then, conditioned on $(M, \overline{B}_1(M), \overline{B}_2(M))$,

(a) $\overline{V}_\ell(\cdot)$ and $\overline{V}_r(\cdot)$ are independent.

(b) $\overline{V}_\ell(\cdot)$ is a $\mathsf{NonInt\text{-}BrBridge}$ *on $[0, M-a]$ ending at $(\overline{B}_1(M) - x_1, x_2 - \overline{B}_2(M))$.*

(c) $\overline{V}_r(\cdot)$ is a $\mathsf{NonInt\text{-}BrBridge}$ *on $[0, b-M]$ ending at $(\overline{B}_1(M) - y_1, y_2 - \overline{B}_2(M))$.*

Recall that $\mathsf{NonInt\text{-}BrBridge}$*s are defined in Definition 5.4.4.*

As in the proof of Proposition 5.4.8, to prove Proposition 5.4.10 we rely on a decomposition result for Brownian motions instead. To state the result we introduce the Ω map which encodes the trajectories of around the joint maximum of the sum of two functions.

Definition 5.4.11. For any $f = (f_1, f_2) \in C([0,1] \to \mathbb{R}^2)$, we define $\Omega f \in C([-1,1] \to \mathbb{R}^2)$ as follows:

$$(\Omega f)_1(t) = \begin{cases} [f_1(M) - f_1(M(1+t))]M^{-1/2} & -1 \le t \le 0 \\ [f_1(M) - f_1(M + (1-M)t)](1-M)^{-1/2} & 0 \le t \le 1 \end{cases}$$

$$(\Omega f)_2(t) = \begin{cases} -[f_2(M) - f_2(M(1+t))]M^{-1/2} & -1 \le t \le 0 \\ -[f_2(M) - f_2(M+(1-M)t)](1-M)^{-1/2} & 0 \le t \le 1 \end{cases}$$

where $M = \inf\{t \in [0,1] : f_1(s) + f_2(s) \le f_1(t) + f_2(t), \forall s \in [0,1]\}$ is the left most maximizer. We set $(\Omega f) \equiv (0,0)$ if $M = 0$ or 1 on $[0,1]$. With this we define two functions in $C([0,1] \to \mathbb{R}^2)$ as follows

$$(\Omega f)_+(x) := ((\Omega f)_1(x), (\Omega f)_2(x)), \quad x \in [0,1]$$
$$(\Omega f)_-(x) := ((\Omega f)_1(-x), (\Omega f)_2(-x)), \quad x \in [0,1].$$

We are now ready to state the corresponding result for Brownian motions.

Lemma 5.4.12. *Suppose $B = (B_1, B_2)$ are independent Brownian motions on $[0,1]$ with $B_i(0) = x_i$. Let*

$$M = \operatorname*{argmax}_{t\in[0,1]}(B_1(t) + B_2(t)).$$

Then $(\Omega B)_+, (\Omega B)_-$ are independent and distributed as non-intersecting Brownian motions on $[0,1]$ (see Definition 5.4.3). Furthermore, $(\Omega B)_+, (\Omega B)_-$ are independent of M.

We first complete the proof of Proposition 5.4.10 assuming the above Lemma.

Proof of Proposition 5.4.10. Without loss of generality, we set $a = 0$ and $b = 1$. Let $B_1, B_2 : [0,1] \to \mathbb{R}$ be two independent Brownian bridges with $B_i(0) = x_i$ and denote $M = \operatorname{argmax}_{x\in[0,1]} B_1(x) + B_2(x)$. Consider

$$V_\ell(x) := (B_1(M) - B_1(M-x), -B_2(M) + B_2(M-x))_{x\in[0,M]}$$
$$V_r(x) := (B_1(M) - B_1(M+x), -B_2(M) + B_2(M+x))_{x\in[0,1-M]}$$

By Lemma 5.4.12, conditioned on M and after Brownian re-scaling, we have where V_r, V_ℓ are conditionally independent and $V_r \sim W^{[0,1-M]}$ and $V_\ell \sim W^{[0,M]}$ where $W^{[0,\rho]}$ denote a NonInt-BM on $[0,\rho]$ defined in Definition 5.4.4. To convert the above construction to Brownian bridges, we observe that the map

$$(B_1(M), B_2(M), B_1(1), B_2(1)) \leftrightarrow (V_r(1-M), V_\ell(M))$$

is bijective. Indeed, we have that

$$\begin{pmatrix} B_1(M)=b_1,\ B_2(M)=b_2 \\ B_1(1)=y_1,\ B_2(1)=y_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} V_r(1-M)=(b_1-y_1,-b_2+y_2) \\ V_\ell(M)=(b_1-x_1,-b_2+x_2) \end{pmatrix}.$$

Thus conditioned on $(M=m, B_i(M)=b_i, B_i(1)=y_i)$, $V_r(\cdot)$ is now a NonInt-BrBridge Brownian bridge on $[0,1-m]$ ending at $(b_1-y_1,-b_2+y_2)$ and $V_\ell(\cdot)$ is a NonInt-BrBridge on $[0,m]$ ending at $(b_1-x_1,-b_2+x_2)$ where both are conditionally independent of each other. But the law of a Brownian motions conditioned on $(M=m, B_i(M)=b_i, B_i(1)=y_i)$ is same as the law of a Brownian bridges $\overline{B}$ on $[0,1]$ starting at (x_1,x_2) and ending at (y_1,y_2), conditioned on $(M=m, \overline{B}_i(M)=b_i)$. Thus this leads to the desired decomposition for Brownian bridges. $\square$

Let us now prove Lemma 5.4.12. The proof of Lemma 5.4.12 follows similar ideas from [145] and [204]. To prove such a decomposition holds, we first show it at the level of random walks. Then we take diffusive limit to get the same decomposition for Brownian motions.

Proof of Lemma 5.4.12. Let $X_j^{(i)} \overset{i.i.d.}{\sim} \mathrm{N}(0,1),\ i=1,2,\ j\ge 1$ and set $S_k^{(i)} = \sum_{j=1}^n X_j^{(i)}$. Define

$$M_n := \operatorname*{argmax}_{k=1}^{n}\{S_k^{(1)}+S_k^{(2)}\},$$

and let $A_j^{(i)}$ be subsets of $\mathbb{R}$. Define

$$\mathcal{I} := \mathbf{P}\Bigg(\bigcap_{\substack{j=0\\ i=1,2}}^{k-1}\{S_k^{(i)}-S_j^{(i)} \in (-1)^{i+1}A_{k-j}^{(i)}\} \cap \bigcap_{\substack{j=k+1\\ i=1,2}}^{n}\{S_k^{(i)}-S_j^{(i)} \in (-1)^{i+1}A_j^{(i)}\} \cap \{M_n=k\}\Bigg). \quad (5.4.4)$$

Noting that the event $\{M_n = k\}$ is the same as

$$\bigcap_{\substack{j=0\\i=1,2}}^{k-1} \{S_k^{(1)} + S_k^{(2)} > S_j^{(1)} + S_j^{(2)}\} \bigcap_{\substack{j=k+1\\i=1,2}}^{n} \{S_k^{(1)} + S_k^{(2)} > S_j^{(1)} + S_j^{(2)}\},$$

we have

$$\begin{aligned}\text{r.h.s of (5.4.4)} = \mathbf{P}\Bigg(&\bigcap_{\substack{j=0\\i=1,2}}^{k-1} \{S_k^{(i)} - S_j^{(i)} \in (-1)^{i+1} A_{k-j}^{(i)}, S_k^{(1)} + S_k^{(2)} > S_j^{(1)} + S_j^{(2)}\} \\ &\cap \bigcap_{\substack{j=k+1\\i=1,2}}^{n} \{S_k^{(i)} - S_j^{(i)} \in (-1)^{i+1} A_j^{(i)}, S_k^{(1)} + S_k^{(2)} > S_j^{(1)} + S_j^{(2)}\}\Bigg).\end{aligned}$$

We also observe that the pairs $(S_k^{(1)} - S_j^{(1)}, S_k^{(2)} - S_j^{(2)})_{j=0}^{k-1}$ and $(S_k^{(1)} - S_j^{(1)}, S_k^{(2)} - S_j^{(2)})_{j=k+1}^{n}$ are independent of each other and as X_j^i is symmetric

$$\begin{aligned}(S_k^{(1)} - S_j^{(1)}, S_k^{(2)} - S_j^{(2)})_{j=0}^{k-1} &\overset{(d)}{=} (S_{k-j}^{(1)}, -S_{k-j}^{(2)})_{j=0}^{k-1}\\ (S_k^{(1)} - S_j^{(1)}, S_k^{(2)} - S_j^{(2)})_{j=k+1}^{n} &\overset{(d)}{=} (S_{j-k}^{(1)}, -S_{j-k}^{(2)})_{j=k+1}^{n}.\end{aligned}$$

Thus,

$$\mathcal{I} = \mathbf{P}\Bigg(\bigcap_{\substack{j=0\\i=1,2}}^{k-1} \{S_j^{(i)} \in A_j^{(i)}, S_j^{(1)} > S_j^{(2)}\}\Bigg) \cdot \mathbf{P}\Bigg(\bigcap_{\substack{j=1\\i=1,2}}^{n-k} \{S_j^{(i)} \in A_j^{(i)}, S_j^{(1)} > S_j^{(2)}\}\Bigg). \tag{5.4.5}$$

Based on (5.4.5), we obtain that

$$\begin{aligned}\frac{\mathcal{I}}{\mathbf{P}(M_n = k)} = \mathbf{P}\Bigg(&\bigcap_{\substack{j=0\\i=1,2}}^{k-1} \{S_j^{(i)} \in A_j^{(i)}\} | \bigcap_{j=1}^{k} \{S_j^{(1)} > S_j^{(2)}\}) \\ &\cdot \mathbf{P}\Bigg(\bigcap_{\substack{j=1\\i=1,2}}^{n-k} \{S_j^{(i)} \in A_j^{(i)}\} | \bigcap_{j=1}^{n-k} \{S_j^{(1)} > S_j^{(2)}\}\Bigg)\end{aligned} \tag{5.4.6}$$

where we utilize the fact $\mathbf{P}(M_n = k) = \mathbf{P}(\cap_{j=1}^{k} S_j^{(1)} > S_j^{(2)})\mathbf{P}(\cap_{j=1}^{n-k} S_j^{(1)} > S_j^{(2)})$. The above splitting essentially shows that conditioned on the maximizer, the left and right portion of the maximizer are independent non-intersecting random walks.

We now consider $Z_n(t) = (\frac{S_{nt}^{(1)}}{\sqrt{n}}, \frac{S_{nt}^{(2)}}{\sqrt{n}})$ on $[0,1]$ where it is linearly interpolated in between. By Donsker's invariance principle, $Z_n \Rightarrow B = (B_1, B_2)$ independent Brownian motions on $[0,1]$. Recall Ω from Definition 5.4.11. Clearly $\mathbf{P}(B \in \text{Discontinuity of } \Omega) = 0$, so

$$(\Omega Z_n)_+ \Rightarrow (\Omega B)_+ \text{ and } (\Omega Z_n)_- \Rightarrow (\Omega B)_-.$$

On the other hand, following (5.4.6) we see that conditioned on $M_n = k$, $(\Omega Z_n)_+ \stackrel{(d)}{=} Y_{n-k}$ and $(\Omega Z_n)_- \stackrel{(d)}{=} Y_k$ are independent where $Y_n(\cdot)$ is the linearly interpolated non-intersecting random walk defined in Proposition 5.4.7. As $k, n \to \infty$, $Y_k(\cdot), Y_{n-k}(\cdot) \xrightarrow{d} W$ where W is the non-intersecting Brownian motion on $[0,1]$ defined in Definition 5.4.3. At the same time, $\frac{M_n}{n} \Rightarrow M$, which has density $\propto \frac{1}{\sqrt{t(1-t)}}$ on $[0,1]$. Thus, $(\Omega B)_+, (\Omega B)_-, M$ are independent and $(\Omega B)_+ \stackrel{(d)}{=} (\Omega B)_- \stackrel{(d)}{=} W$. □

Remark 5.4.13. We expect similar decomposition results to hold for 3 or more Brownian motions or bridges around the maximizer of their sums. More precisely, if M is the maximizer of $B_1(x) + B_2(x) + B_3(x)$, where B_i are independent Brownian motion on $[0,1]$, we expect the law of

$$(B_1(M) - B_1(M+x), B_2(M) - B_2(M+x), B_3(M) - B_3(M+x))$$

to be again Brownian motions but their sum conditioned to be positive (its singular conditioning; so requires some care to define properly). Indeed, such a statement can be proven rigorously at the level of random walks. Then a possible approach is to take diffusive limit of random walks under conditioning and prove existence of weak limits. Due to lack of results for such conditioning event, proving such a statement require quite some technical work. Since it is extraneous for our purpose, we do not pursue this direction here.

5.5 Bessel bridges and non-intersecting Brownian bridges

In this section, we study diffusive limits and separation properties of Bessel bridges and non-intersecting Brownian bridges. The central object that appears in this section is the Dyson Brownian motion [**dyson1962brownian**] which are intuitively several Brownian bridges conditioned on non-intersection. In Section 5.5.1, we recall Dyson Brownian motion and study different properties of it. In Section 5.5.2 we prove a technical estimate that indicates the two parts of non-intersecting Brownian bridges have uniform separation and derive the diffusive limits of non-intersecting Brownian bridges. The precise renderings of these results are given in Proposition 5.5.6 and Proposition 5.5.8.

5.5.1 Diffusive limits of Bessel bridges and NonInt-BrBridge

We first recall the definition of Dyson Brownian motion. Although they are Brownian motions conditioned on non-intersection, since the conditioning event is singular, such an interpretation needs to be justified properly. There are several ways to rigorously define the Dyson Brownian motion, either through the eigenvalues of Hermitian matrices with Brownian motions as entries or as a solution of system of stochastic PDEs. In this paper, we recall the definition via specifying the entrance law and transition densities (see [265] and [317, Section 3] for example).

Definition 5.5.1 (Dyson Brownian motion)**.** A 2-level Dyson Brownian motion $\mathcal{D}(\cdot) = (\mathcal{D}_1(\cdot), \mathcal{D}_2(\cdot))$ is an $\mathbb{R}^2$ valued process on $[0, \infty)$ with $\mathcal{D}_1(0) = \mathcal{D}_2(0) = 0$ and with the entrance law

$$\mathbf{P}\left(\mathcal{D}_1(t) \in \mathrm{d}y_1, \mathcal{D}_2(t) \in \mathrm{d}y_2\right) = \mathbf{1}\{y_1 > y_2\}\frac{(y_1 - y_2)^2}{t} p_t(y_1)p_t(y_2)\mathrm{d}y_1\mathrm{d}y_2, \quad t > 0. \tag{5.5.1}$$

For $0 < s < t < \infty$ and $x_1 > x_2$, its transition densities are given by

$$\begin{aligned}&\mathbf{P}\left(\mathcal{D}_1(t) \in \mathrm{d}y_1, \mathcal{D}_2(t) \in \mathrm{d}y_2 \mid \mathcal{D}_1(s) = x_1, \mathcal{D}_2(s) = x_2\right) \\ &\qquad = \mathbf{1}\{y_1 > y_2\}\frac{y_1 - y_2}{x_1 - x_2} \det(p_{t-s}(x_i - y_j))_{i,j=1}^2 \mathrm{d}y_1\mathrm{d}y_2.\end{aligned} \tag{5.5.2}$$

The above formulas can be extended to n-level Dyson Brownian motions with (see [317, Section 3]) but for the rest of the paper we only require the $n = 2$ case. So, we will refer to the 2-level object defined above loosely as Dyson Brownian motion or DBM in short.

We next define the Bessel processes via specifying the entrance law and transition densities which are also well known in literature (see [284, Chapter VI.3]).

Definition 5.5.2 (Bessel Process)**.** A 3D Bessel process $\mathcal{R}_1$ with diffusion coefficient 1 is an $\mathbb{R}$-valued process on $[0,\infty)$ with $\mathcal{R}_1(0) = 0$ and with the entrance law

$$\mathbf{P}(\mathcal{R}_1(t) \in dy) = \tfrac{2y^2}{t} p_t(y)\mathrm{d}y, \quad x \in [0,\infty), \quad t > 0.$$

For $0 < s < t < \infty$ and $x > 0$, its transition densities are given by

$$\mathbf{P}(\mathcal{R}_1(t) \in dy \mid \mathcal{R}_1(s) = x) = \frac{y}{x}[p_{t-s}(x-y) - p_{t-s}(x+y)]dy, \quad y \in [0,\infty).$$

More generally, $\mathcal{R}_\sigma(\cdot)$ is a 3D Bessel process with diffusion coefficient $\sigma > 0$ if $\sigma^{-1/2}\mathcal{R}_\sigma(\cdot)$ is a 3D Bessel process with diffusion coefficient 1.

In this paper we will only deal with 3-dimensional Bessel processes. Thus we will just loosely refer to the above processes as Bessel processes.

DBM is directly linked with Bessel processes. Indeed the difference of the two paths of DBM is known (see [158] for example) to be a 3D Bessel process with diffusion coefficient 2. This fact can be proven easily via SPDE or the Hermitian matrices interpretation of DBM. Since we use this result repeatedly in later sections we record it as a lemma below.

Lemma 5.5.3 (Dyson to Bessel)**.** *Let $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2)$ be a* DBM*. Then, as a function in x, we have $\mathcal{D}_1(x) + \mathcal{D}_2(x) \stackrel{d}{=} \sqrt{2}B(x)$ and $\mathcal{D}_1(x) - \mathcal{D}_2(x) \stackrel{d}{=} \mathcal{R}_2(x)$ where $B(x)$ is a Brownian motion and $\mathcal{R}_2 : [0,\infty) \to \mathbb{R}$ is a Bessel process (see Definition 5.5.2) with diffusion coefficient* 2*.*

We end this subsection by providing two lemmas that compare the densities of NonInt-BrBridge and DBM.

Lemma 5.5.4. *Suppose the pair of random variables (U_1, U_2) has joint probability density function:*

$$\mathbf{P}(U_1 \in \mathrm{d}y_1, U_2 \in \mathrm{d}y_2) = \frac{(y_1 - y_2)^2}{t} p_t(y_1) p_t(y_2), \ y_1 > y_2. \tag{5.5.3}$$

Conditioning on (U_1, U_2), we consider a NonInt-BrBridge (V_1, V_2) *on $[0,t]$ ending at (U_1, U_2), see Definition 5.4.4. Then unconditionally, (V_1, V_2) is equal in distribution as* DBM $(\mathcal{D}_1, \mathcal{D}_2)$ *on* $[0,t]$. *(see Definition 5.5.1).*

Lemma 5.5.5. *Fix $\delta, M > 0$. Consider a* NonInt-BrBridge (V_1, V_2) *on $[0,1]$ ending at (a_1, a_2) (see Definition 5.4.4), where $a_1 > a_2$. Then, there exists a constant $\mathrm{C}_{M,\delta} > 0$ such that for all $t \in (0,\delta)$, $y_1 > y_2$ and $-M \le a_2 < a_1 \le M$,*

$$\frac{\mathbf{P}(V_1(t) \in \mathrm{d}y_1, V_2(t) \in \mathrm{d}y_2)}{\mathbf{P}(\mathcal{D}_1(t) \in \mathrm{d}y_1, \mathcal{D}_2(t) \in \mathrm{d}y_2)} \le \mathrm{C}_{M,\delta}, \tag{5.5.4}$$

where $(\mathcal{D}_1, \mathcal{D}_2)$ is a DBM *defined in Definition 5.5.1.*

Proof of Lemma 5.5.4. To show that (V_1, V_2) is equal in distribution to $(\mathcal{D}_1, \mathcal{D}_2)$ on $[0,t]$, it suffices to show that (V_1, V_2) has the same finite dimensional distribution as $(\mathcal{D}_1, \mathcal{D}_2)$ on $[0,t]$. Fix any $k \in \mathbb{N}$, and $0 < s_1 < \ldots < s_k < t$ and $y_1 > y_2$. Using Brownian scaling and the formulas from Definition 5.4.4 we have

$$\begin{aligned}
&\mathbf{P}\bigg(\bigcap_{i=1}^{k}\{V_1(s_i) \in \mathrm{d}x_{i,1}, V_2(s_i) \in \mathrm{d}x_{i,2}\} | U_1 = y_1, U_2 = y_2\bigg) \\
&\qquad = \frac{(x_{1,1} - x_{1,2})}{s_1} p_{s_1}(x_{1,1}) p_{s_1}(x_{1,2}) \prod_{m=1}^{k-1} \det(p_{s_{m+1}-s_m}(x_{m+1,i} - x_{m,j}))_{i,j=1}^2 \\
&\qquad\qquad \cdot \frac{\det(p_{t-s_k}(x_{k,i} - y_j))_{i,j=1}^2}{\frac{1}{t}(y_1 - y_2) p_t(y_1) p_t(y_2)} \prod_{i=1}^{k} \mathrm{d}x_{i,1} \mathrm{d}x_{i,2},
\end{aligned}$$

where the above density is supported on $\{x_{i,1} > x_{i,2} \mid i = 1, 2, \ldots, k\}$. For convenience, in the rest of the calculations, we drop $\prod_{i=1}^{k} \mathrm{d}x_{i,1} \mathrm{d}x_{i,2}$ from the above formula. In view of the marginal density of (U_1, U_2) given by (5.5.3), we thus have that

$$
\begin{aligned}
&\mathbf{P}\Big(\bigcap_{i=1}^{k}\{V_1(s_i) \in \mathrm{d}x_{i,1}, V_2(s_i) \in \mathrm{d}x_{i,2}\}\Big) \\
&= \int_{y_1>y_2} \mathbf{P}\Big(\bigcap_{i=1}^{k}\{V_1(s_i) \in \mathrm{d}x_{i,1}, V_2(s_i) \in \mathrm{d}x_{i,2}\} | U_1 = y_1, U_2 = y_2\Big) \frac{(y_1 - y_2)^2}{t} p_t(y_1) p_t(y_2) \mathrm{d}y_1 \mathrm{d}y_2 \\
&= \frac{(x_{1,1} - x_{1,2})}{s_1} p_{s_1}(x_{1,1}) p_{s_1}(x_{1,2}) \prod_{m=1}^{k-1} \det(p_{s_{m+1}-s_m}(x_{m+1,i} - x_{m,j}))_{i,j=1}^2 \\
&\qquad\qquad\qquad \cdot \int_{y_1>y_2} (y_1 - y_2) \det(p_{t-s_k}(x_{k,i} - y_j))_{i,j=1}^2 \mathrm{d}y_1 \mathrm{d}y_2.
\end{aligned}
$$

But given the transition densities for DBM from (5.5.2). we know that

$$
\int_{y_1>y_2} (y_1 - y_2) \det(p_{t-s_k}(x_{k,i} - y_j))_{i,j=1}^2 \mathrm{d}y_1 \mathrm{d}y_2 = x_{k,1} - x_{k,2}.
$$

Plugging this back we get

$$
\begin{aligned}
&\mathbf{P}\Big(\bigcap_{i=1}^{k}\{V_1(s_i) \in \mathrm{d}x_{i,1}, V_2(s_i) \in \mathrm{d}x_{i,2}\}\Big) \\
&= \frac{(x_{1,1} - x_{1,2})^2}{s_1} p_{s_1}(x_{1,1}) p_{s_1}(x_{1,2}) \prod_{m=1}^{k-1} \frac{x_{m+1,1} - x_{m+1,2}}{x_{m,1} - x_{m,2}} \det(p_{s_{m+1}-s_m}(x_{m+1,i} - x_{m,j}))_{i,j=1}^2.
\end{aligned}
$$

Using the entrance law and transition densities formulas for DBM from Definition 5.5.1, we see that the above formula matches with the finite dimensional density formulas for DBM. This completes the proof. □

Proof of Lemma 5.5.5. Fix any arbitrary $y_1 > y_2$ and $t \in (0, \delta)$ Recall the density formulas for NonInt-BrBridge and DBM from Definitions 5.4.4 and 5.5.1. We have

$$
\begin{aligned}
\text{l.h.s of (5.5.4)} &= \frac{\det(p_{1-t}(y_i - a_j))_{i,j=1}^2}{(y_1 - y_2)(a_1 - a_2) p_1(a_1) p_1(a_2)} && (5.5.5) \\
&= \frac{p_{1-t}(y_1 - a_2) p_{1-t}(y_2 - a_1)}{(y_1 - y_2)(a_1 - a_2) p_1(a_1) p_1(a_2)} \left[e^{\frac{(y_1-y_2)(a_1-a_2)}{1-t}} - 1 \right]. && (5.5.6)
\end{aligned}
$$

If $(y_1 - y_2)(a_1 - a_2) \geq 1 - t$, then

$$\text{r.h.s. of (5.5.5)} \leq \frac{\det(p_{1-t}(y_i - a_j))_{i,j=1}^2}{(1-t)p_1(a_1)p_1(a_2)} \leq \tfrac{1}{(1-t)^2} e^{\frac{a_1^2+a_2^2}{2}} \leq \tfrac{1}{(1-\delta)^2} e^{M^2}.$$

If $(y_1 - y_2)(a_1 - a_2) \leq 1 - t$, we utilize the elementary inequality that $\gamma(e^{\frac{1}{\gamma}} - 1) \leq e - 1$, for all $\gamma \geq 1$. Indeed, taking $\gamma = \frac{1-t}{(y_1-y_2)(a_1-a_2)} \geq 1$ in this case we have

$$\text{r.h.s. of (5.5.6)} \leq \frac{p_{1-t}(y_1 - a_2)p_{1-t}(y_2 - a_1)}{(1-t)p_1(a_1)p_1(a_2)}(e-1) \leq \tfrac{2}{(1-t)^2} e^{\frac{a_1^2+a_2^2}{2}} \leq \tfrac{2}{(1-\delta)^2} e^{M^2}.$$

Combining both cases yields the desired result. $\square$

5.5.2 Uniform separation and diffusive limits

The main goal of this subsection is to prove Proposition 5.5.6 and Proposition 5.5.8. Proposition 5.5.6 highlights a uniform separation between the two parts of the NonInt-BrBridge defined in Definition 5.4.4, while Proposition 5.5.8 shows DBMs are obtained as diffusive limits of NonInt-BrBridges.

Proposition 5.5.6. *Fix $M > 0$. Let $(V_1^{(n)}, V_2^{(n)})$ be a sequence of* NonInt-BrBridge*s (see Definition 5.4.4) on $[0,1]$ beginning at 0 and ending at $(a_1^{(n)}, a_2^{(n)})$. Suppose that $a_1^{(n)} - a_2^{(n)} > \frac{1}{M}$ and $|a_i^{(n)}| \leq M$ for all n and $i = 1,2$. Then for all $\rho > 0$, we have*

$$\limsup_{\theta\to\infty}\limsup_{n\to\infty} \mathbf{P}\left(\int_\theta^n \exp\left(-\sqrt{n}\big[V_1^{(n)}(\tfrac{y}{n}) - V_2^{(n)}(\tfrac{y}{n})\big]\right) \mathrm{d}y \geq \rho\right) = 0. \tag{5.5.7}$$

Recall that by Lemma 5.4.6, the difference of the two parts of NonInt-BrBridge is given by a Bessel bridge (upto a constant). Hence we can recast the above result in terms of separations between Bessel bridges from the x-axis as well.

Corollary 5.5.7. *Fix $M > 0$. Let $\mathcal{R}_{\mathrm{bb}}^{(n)}$ be a sequence of Bessel bridges (see Definition 5.4.1) on $[0,1]$ beginning at 0 and ending at $a^{(n)}$. Suppose that $M > a_1^{(n)} > \frac{1}{M}$ for all n. Then for all $\rho > 0$,* *we have*

$$\limsup_{\theta\to\infty}\limsup_{n\to\infty}\mathbf{P}\left(\int_\theta^n \exp\left(-\sqrt{n}\mathcal{R}_{\mathrm{bb}}^{(n)}(\tfrac{y}{n})\right)\mathrm{d}y \geq \rho\right) = 0.$$

Proof of Proposition 5.5.6. We fix $\delta \in (0, \frac{1}{4})$. To prove the inequality in (5.5.7), we divide the integral from θ to n into two parts: $(\theta, n\delta)$ and $[n\delta, n)$ for some $\delta \in (0,1)$ and n large and prove each one separately. For the interval $(n\delta, n)$ interval, we use the fact that the non-intersecting Brownian bridges $V_1^{(n)}(y), V_2^{(n)}(y)$ are separated by a uniform distance when away from 0. For the smaller interval $(\theta, n\delta)$ close to 0, we define a $\mathsf{Gap}_{n,\theta,\delta}$ event that occurs with high probability and utilize Lemmas 5.5.4 and 5.5.5 to transform the computations of NonInt-BrBridge into those of the DBM to simplify the proof.

We now fill out the details of the above road-map. First, as $(V_1^{(n)}, V_2^{(n)})$ are non-intersecting Brownian bridges on $[0,1]$ starting from 0 and ending at two points which are within $[-M, M]$ and are separated by at least $\frac{1}{M}$, for every $\lambda, \delta > 0$, there exists $\alpha(M, \delta, \lambda) > 0$ small enough such that

$$\mathbf{P}\left(V_1^{(n)}(y) - V_2^{(n)}(y) \geq \alpha, \forall y \in [\delta, 1]\right) \geq 1 - \lambda. \tag{5.5.8}$$

(5.5.8) implies that with probability at least $1 - \lambda$,

$$\int_{n\delta}^n \exp\left(-\sqrt{n}\left[V_1^{(n)}(\tfrac{y}{n}) - V_2^{(n)}(\tfrac{y}{n})\right]\right)\mathrm{d}y \leq (n - n\delta)e^{-\sqrt{n}\alpha} \tag{5.5.9}$$

which converges to 0 as $n \to \infty$. Next we define the event

$$\mathsf{Gap}_{n,\theta,\delta} := \left\{\sqrt{n}\left[V_1^{(n)}(\tfrac{y}{n}) - V_2^{(n)}(\tfrac{y}{n})\right] \geq y^{\frac{1}{4}}, \forall y \in [\theta, n\delta]\right\}.$$

We claim that $\neg\mathsf{Gap}_{n,\theta,\delta}$ event is negligible in the sense that

$$\limsup_{\theta\to\infty}\limsup_{n\to\infty}\mathbf{P}(\neg\mathsf{Gap}_{n,\theta,\delta}) = 0. \tag{5.5.10}$$

Note that on $\mathsf{Gap}_{n,\theta,\delta}$ event, we have

$$\int_\theta^{n\delta} \exp\left(-\sqrt{n}\big[V_1^{(n)}(\tfrac{y}{n}) - V_2^{(n)}(\tfrac{y}{n})\big]\right) \mathrm{d}y \le \int_\theta^{n\delta} \exp(-y^{1/4})\mathrm{d}y \tag{5.5.11}$$

which goes to zero as $n \to \infty$, followed by $\theta \to \infty$. In view of the probability estimates from (5.5.8) and (5.5.9), combining (5.5.10) and (5.5.11) yields

$$\limsup_{\theta\to\infty}\limsup_{n\to\infty}\mathbf{P}\left(\int_\theta^{n} \exp\left(-\sqrt{n}\big[V_1^{(n)}(\tfrac{y}{n}) - V_2^{(n)}(\tfrac{y}{n})\big]\right) \mathrm{d}y \ge \rho\right) \le \lambda. \tag{5.5.12}$$

Since λ is arbitrary, (5.5.12) completes the proof. Hence it suffices to show (5.5.10). Towards this end, by the properties of the conditional expectation, if we condition on the values of $V_1^{(n)}(2\delta), V_2^{(n)}(2\delta)$, we have that

$$\begin{aligned}\mathbf{P}(\neg\mathsf{Gap}_{n,\theta,\delta}) &= \mathbf{E}\left[\mathbf{P}\left(\neg\mathsf{Gap}_{n,\theta,\delta} | V_1^{(n)}(2\delta), V_2^{(n)}(2\delta)\right)\right] \\ &= \int_{y_1>y_2} \mathbf{P}_{y_1,y_2}(\neg\mathsf{Gap}_{n,\theta,\delta})\mathbf{P}(V_1^{(n)}(2\delta) \in \mathrm{d}y_1, V_2^{(n)}(2\delta) \in \mathrm{d}y_2) \end{aligned} \tag{5.5.13}$$

where $\mathbf{P}_{y_1,y_2}$ is the conditional law of $\mathsf{NonInt\text{-}BrBridge}$ conditioned on $(V_1^{(n)}(2\delta) = y_1, V_2^{(n)}(2\delta) = y_2)$. Note that $\mathsf{Gap}_{n,\theta,\delta}$ event depends only on the $[0, \delta]$ path of the $\mathsf{NonInt\text{-}BrBridge}$. Thus by Markovian property of the $\mathsf{NonInt\text{-}BrBridge}$, $\mathbf{P}_{y_1,y_2}(\mathsf{Gap}_{n,\theta,\delta})$ can be computed by assuming the $\mathsf{NonInt\text{-}BrBridge}$ is on $[0, 2\delta]$ and ends at (y_1, y_2).

On the other hand, $\mathbf{P}(V_1^n(2\delta) \in \mathrm{d}y_1, V_2^n(2\delta) \in \mathrm{d}y_2)$ is the probability density function of the marginal density of $\mathsf{NonInt\text{-}BrBridge}$ on $[0, 1]$. Via Lemma 5.5.5, this is comparable to the density of $(\mathcal{D}_1(2\delta), \mathcal{D}_2(2\delta))$, where $\mathcal{D}$ follows DBM law defined in Definition 5.5.1. Thus by (5.5.4) the r.h.s of (5.5.13) is bounded from above by

$$\text{r.h.s of (5.5.13)} \le \mathrm{C}_{M,2\delta} \int \mathbf{P}_{y_1,y_2}(\neg \mathsf{Gap}_{n,\theta,\delta}) \mathbf{P}(\mathcal{D}_1(2\delta) \in \mathrm{d}y_1, \mathcal{D}_2(2\delta) \in \mathrm{d}y_2) \mathrm{d}y_1 \mathrm{d}y_2$$
$$= \mathrm{C}_{M,2\delta} \cdot \mathbf{P}_{\text{Dyson}}(\neg \mathsf{Gap}_{n,\theta,\delta}). \tag{5.5.14}$$

Here the notation $\mathbf{P}_{\text{Dyson}}$ means the law of the paths (V_1, V_2) is assumed to follow DBM law. With this notation, the last equality of (5.5.14) follows from Lemma 5.5.4. From the density formulas of DBM from Definition 5.5.1, it is clear that DBM is invariant under diffusive scaling, i.e.

$$\sqrt{n}(\mathcal{D}_1(\tfrac{\cdot}{n}), \mathcal{D}_2(\tfrac{\cdot}{n})) \stackrel{d}{=} (\mathcal{D}_1(\cdot), \mathcal{D}_2(\cdot)) \tag{5.5.15}$$

and by Lemma 5.5.3, $\mathcal{D}_1(\cdot) - \mathcal{D}_2(\cdot) = \mathcal{R}_2(\cdot)$, a 3D Bessel process with diffusion coefficient 2. Thus, we obtain that for any $n \in \mathbb{N}$,

$$\mathbf{P}_{\text{Dyson}}(\neg \mathsf{Gap}_{n,\theta,\delta}) \le \mathbf{P}(\mathcal{R}_2(y) \le y^{1/4}, \text{ for some } y \in [\theta, \infty)). \tag{5.5.16}$$

Meanwhile, Motoo's theorem [257] tells us that

$$\limsup_{\theta \to \infty} \mathbf{P}(\mathcal{R}_2(y) \le y^{1/4}, \text{ for some } y \in [\theta, \infty)) = 0. \tag{5.5.17}$$

Hence (5.5.14), (5.5.16) and (5.5.17) imply (5.5.10). This completes the proof. □

We now state our results related to the diffusive limits of $\mathsf{NonInt\text{-}BrBridge}$ (defined in Definition 5.4.4) and Bessel bridges (defined in Definition 5.4.1) with varying endpoints.

Proposition 5.5.8. *Fix $M > 0$. Let $V^{(n)} = (V_1^{(n)}, V_2^{(n)}) : [0, a_n] \to \mathbb{R}$ be a sequence of* $\mathsf{NonInt\text{-}BrBridge}$*s (defined in Definition 5.4.4) with $V_i^{(n)}(0) = 0$ and $V_i^{(n)}(a_n) = z_i^{(n)}$. Suppose that for all $n \ge 1$ and $i = 1, 2$, $M > a_n > \frac{1}{M}$ and $|z_i^{(n)}| < \frac{1}{M}$. Then as $n \to \infty$ we have:*

$$\sqrt{n}\big(V_1^{(n)}(\tfrac{t}{n}), V_2^{(n)}(\tfrac{t}{n})\big) \xrightarrow{d} (\mathcal{D}_1(t), \mathcal{D}_2(t))$$

in the uniform-on-compact topology. Here $\mathcal{D}$ is a DBM *defined in Definition 5.5.1.*

In view of Lemma 5.4.6 and Lemma 5.5.3, Proposition 5.5.8 also leads to the following corollary.

Corollary 5.5.9. *Fix $M > 0$. Let $\mathcal{R}_{\mathrm{bb}}^{(n)} : [0, a_n] \to \mathbb{R}$ be a sequence of Bessel bridges (defined in Definition 5.4.1) with $\mathcal{R}_{\mathrm{bb}}^{(n)}(0) = 0$ and $\mathcal{R}_{\mathrm{bb}}^{(n)}(a_n) = z^{(n)}$. Suppose for all $n \geq 1$, $M > a_n > \frac{1}{M}$ and $|z^{(n)}| < \frac{1}{M}$. Then as $n \to \infty$ we have:*

$$\sqrt{n}\mathcal{R}_{\mathrm{bb}}^{(n)}(\tfrac{t}{n}) \xrightarrow{d} \mathcal{R}_1(t)$$

in the uniform-on-compact topology. Here $\mathcal{R}_1$ is a Bessel process with diffusion coefficient 1*, defined in Definition 5.5.2.*

Proof of Proposition 5.5.8. For convenience, we drop the superscript (n) from V_1, V_2 and z_i's. We proceed by showing convergence of one-point densities and transition densities of $\sqrt{n}(V_1(\frac{t}{n}), V_2(\frac{t}{n}))$ to that of DBM and then verifying the tightness of the sequence. Fix any $t > 0$. For each fixed $y_1 > y_2$, it is not hard to check that we have as $n \to \infty$

$$\frac{a_n\sqrt{n}\det(p_{a_n - \frac{t}{n}}(\frac{y_i}{\sqrt{n}} - z_j))_{i,j=1}^2}{(z_1 - z_2)p_{a_n}(z_1)p_{a_n}(z_2)} \to y_1 - y_2. \tag{5.5.18}$$

uniformly over $a_n \in [\frac{1}{M}, M]$ and $z_1, z_2 \in [-M, M]$.

Utilizing the one-point densities and transition densities formulas for NonInt-BrBridge of length 1 in Definition 5.4.4, we may perform a Brownian rescaling to get analogous formulas for V_1, V_2 which are NonInt-BrBridge of length a_n. Then by a change of variable, the density of $(\sqrt{n}V_1(\frac{t}{n}), \sqrt{n}V_2(\frac{t}{n}))$ is given by

$$\frac{a_n(y_1 - y_2)p_t(y_1)p_t(y_2)}{t(z_1 - z_2)p_{a_n}(z_1)p_{a_n}(z_2)}\sqrt{n}\det(p_{a_n - \frac{t}{n}}(\tfrac{y_i}{\sqrt{n}} - z_j))_{i,j=1}^2.$$

Using (5.5.18) we see that for each fixed $y_1 > y_2$, the above expression goes to $\frac{(y_1-y_2)^2}{t}p_t(y_1)p_t(y_2)$

which matches with (5.5.1).

Similarly for the transition probability, letting $0 < s < t < a_n$, $y_1 > y_2$ and $x_1 > x_2$, we have

$$\begin{aligned}&\mathbf{P}\left(\sqrt{n}V_1(\tfrac{t}{n}) \in \mathrm{d}y_1, \sqrt{n}V_2(\tfrac{t}{n}) \in \mathrm{d}y_2 \mid \sqrt{n}V_1(\tfrac{s}{n}) \in \mathrm{d}x_1, \sqrt{n}V_2(\tfrac{s}{n}) \in \mathrm{d}x_2\right)\\ &= \det(p_{t-s}(y_i - x_j))_{i,j=1}^2 \frac{\det(p_{a_n-\frac{t}{n}}(\frac{y_i}{\sqrt{n}} - z_j))_{i,j=1}^2}{\det(p_{a_n-\frac{s}{n}}(\frac{x_i}{\sqrt{n}} - z_j))_{i,j=1}^2}\mathrm{d}y_1\mathrm{d}y_2. \end{aligned} \tag{5.5.19}$$

Applying (5.5.18) we see that as $n \to \infty$

$$\text{r.h.s of (5.5.19)} \to \det(p_{t-s}(x_i - y_j))_{i,j=1}^2 \cdot \frac{y_1 - y_2}{x_1 - x_2}.$$

which matches with (5.5.2). This verifies the finite dimensional convergence by Scheffe's theorem. For tightness we will show that there exists a constant $\mathrm{C}_{K,M} > 0$ such that for all $0 < s < t < K$,

$$\sum_{i=1}^{2} \mathbf{E}\left[\left(\sqrt{n}V_i(\tfrac{t}{n}) - \sqrt{n}V_i(\tfrac{s}{n})\right)^4\right] \le \mathrm{C}_{K,M}(t-s)^2. \tag{5.5.20}$$

We compute the above expectation by comparing with DBM as was done in the proof of Proposition 5.5.6. Using definition of the conditional expectation we have

$$\begin{aligned}&\mathbf{E}\left[\left(\sqrt{n}V_i(\tfrac{t}{n}) - \sqrt{n}V_i(\tfrac{s}{n})\right)^4\right]\\ &= \int_{y_1>y_2} \mathbf{E}\left[\left(\sqrt{n}V_i(\tfrac{t}{n}) - \sqrt{n}V_i(\tfrac{s}{n})\right)^4 \mid V_1(\tfrac{K}{n}) = y_1, V_1(\tfrac{K}{n}) = y_2\right] \mathbf{P}(V_1(\tfrac{K}{n}) \in \mathrm{d}y_1, V_2(\tfrac{K}{n}) \in \mathrm{d}y_2)\\ &\le \mathrm{C}_{K,M} \int_{y_1>y_2} \mathbf{E}\left[\left(\sqrt{n}V_i(\tfrac{t}{n}) - \sqrt{n}V_i(\tfrac{s}{n})\right)^4 \mid V_1(\tfrac{K}{n}) = y_1, V_1(\tfrac{K}{n}) = y_2\right] \mathbf{P}(\mathcal{D}_1(\tfrac{K}{n}) \in \mathrm{d}y_1, \mathcal{D}_2(\tfrac{K}{n}) \in \mathrm{d}y_2)\end{aligned}$$

where the last inequality follows from Lemma 5.5.5 by taking n large enough. Here $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2)$ follows DBM law. Due to Lemma 5.5.4 and (5.5.15) the last integral above is precisely $\mathbf{E}[(\mathcal{D}_i(t) - \mathcal{D}_i(s))^4]$. Hence it suffices to show

$$\mathbf{E}[(\mathcal{D}_i(t) - \mathcal{D}_i(s))^4] \le \mathrm{C}(t-s)^2. \tag{5.5.21}$$

By Lemma 5.5.3, we see $\sqrt{2}B(x) := \mathcal{D}_1(x) + \mathcal{D}_2(x)$ and $\sqrt{2}\mathcal{R}(x) := \mathcal{D}_1(x) - \mathcal{D}_2(x)$ are a standard Brownian motion and a 3D Bessel process with diffusion coefficient 1 respectively. We have

$$\mathbf{E}[(\mathcal{D}_i(t) - \mathcal{D}_i(s))^4] \le \mathrm{C}\left[\mathbf{E}[(\mathcal{R}(t) - \mathcal{R}(s))^4] + \mathbf{E}[(B(t) - B(s))^4]\right].$$

We have $\mathbf{E}[(B(t)-B(s))^4] = 3(t-s)^2$, whereas for $\mathcal{R}(\cdot)$, we use Pitman's theorem [284, Theorem VI.3.5], to get that $\mathcal{R}(t) \stackrel{d}{=} 2M(t) - B(t)$, where B is a Brownian motion and $M(t) = \sup_{u \le t} B(u)$. Thus,

$$\begin{aligned}\mathbf{E}[(\mathcal{R}(t) - \mathcal{R}(s))^4] &\le \mathrm{C}\left[\mathbf{E}[(M(t) - M(s))^4] + \mathbf{E}[(B(t) - B(s))^4]\right] \\ &\le \mathrm{C}\left[\mathbf{E}\big[\big(\sup_{s \le u \le t} B(u) - B(s)\big)^4\big] + \mathbf{E}[(B(t) - B(s))^4]\right].\end{aligned}$$

Clearly both the expressions above are at most $\mathrm{C}(t-s)^2$. This implies (5.5.21) completing the proof. □

5.6 Ergodicity and Bessel behavior of the KPZ equation

The goal of this section is to prove Theorems 5.1.10 and 5.1.11. As the proof of the latter is shorter and illustrates some of the ideas behind the proof of the former, we first tackle Theorem 5.1.11 in Section 5.6.1. After that in Section 5.6.2, we state a general version of the $k = 2$ case of Theorem 5.1.10, namely Proposition 5.6.1. This proposition will then be utilized in the proof of Theorem 5.1.4. Finally in Section 5.6.3, we show how to obtain Theorem 5.1.10 from Proposition 5.6.1.

5.6.1 Proof of Theorem 5.1.11

For clarity we divide the proof into several steps.

Step 1. In this step, we introduce necessary notations used in the proof and explain the heuristic idea behind the proof.

Fix any $a > 0$. Consider any Borel set A of $C([-a, a])$ which is also a continuity set of a two-sided Brownian motion $B(x)$ restricted to $[-a, a]$. By Portmanteau theorem, it suffices to show

$$\mathbf{P}((\mathcal{H}(\cdot, t) - \mathcal{H}(0, t) \in A) \to \mathbf{P}(B(\cdot) \in A). \tag{5.6.1}$$

For simplicity let us write $\mathbf{P}_t(A) := \mathbf{P}((\mathcal{H}(\cdot, t) - \mathcal{H}(0, t) \in A)$. Using (5.2.6) we have $\mathcal{H}(x, t) - \mathcal{H}(0, t) = t^{1/3}(\mathfrak{h}_t(t^{-2/3}x) - \mathfrak{h}_t(0))$. Recall that $\mathfrak{h}_t(\cdot) = \mathfrak{h}_t^{(1)}(\cdot)$ can be viewed as the top curve of the KPZ line ensemble $\{\mathfrak{h}_t^{(n)}(\cdot)\}_{n\in\mathbb{N}}$ which satisfies the $\mathbf{H}_t$-Brownian Gibbs property with $\mathbf{H}_t$ given by (5.2.4).

Note that at the level of the scaled KPZ line ensembles we are interested in understanding the law of $\mathfrak{h}_t^{(1)}(\cdot)$ restricted to a very small interval: $x \in [-t^{-2/3}a, t^{-2/3}a]$. At such a small scale, we expect the Radon-Nikodym derivative appearing in (5.2.3) to be very close to 1. Hence the law of top curve should be close to a Brownian bridge with appropriate end points. To get rid of the endpoints we employ the following strategy, which is also illustrated in Figure 5.5 and its caption.

- We start with a slightly larger but still vanishing interval $I_t := (-t^{-\alpha}, t^{-\alpha})$ with $\alpha = \frac{1}{6}$ say. We show that conditioned on the end points $\mathfrak{h}_t^{(1)}(-t^{-\alpha}), \mathfrak{h}_t^{(1)}(t^{-\alpha})$ of the first curve and the second curve $\mathfrak{h}_t^{(2)}$, the law of $\mathfrak{h}_t^{(1)}$ is close to that of a Brownian bridge on I_t starting and ending at $\mathfrak{h}_t^{(1)}(-t^{-\alpha})$ and $\mathfrak{h}_t^{(1)}(t^{-\alpha})$ respectively.

- Once we probe further into an even narrower window of $[-t^{2/3}a, t^{2/3}a]$, the Brownian bridge no longer feels the effect of the endpoints and one gets a Brownian motion in the limit.

Step 2. In this step and next step, we give a technical roadmap of the heuristics presented in **Step 1**. Set $\alpha = \frac{1}{6}$ and consider the small interval $I_t = (t^{-\alpha}, t^{-\alpha})$. Let $\mathcal{F}$ be the σ-field generated by

$$\mathcal{F} := \sigma\left(\{\mathfrak{h}_t^{(1)}(x)\}_{x\in I_t^c}, \{\mathfrak{h}_t^{(n)}(\cdot)\}_{n\geq 2}\right). \tag{5.6.2}$$

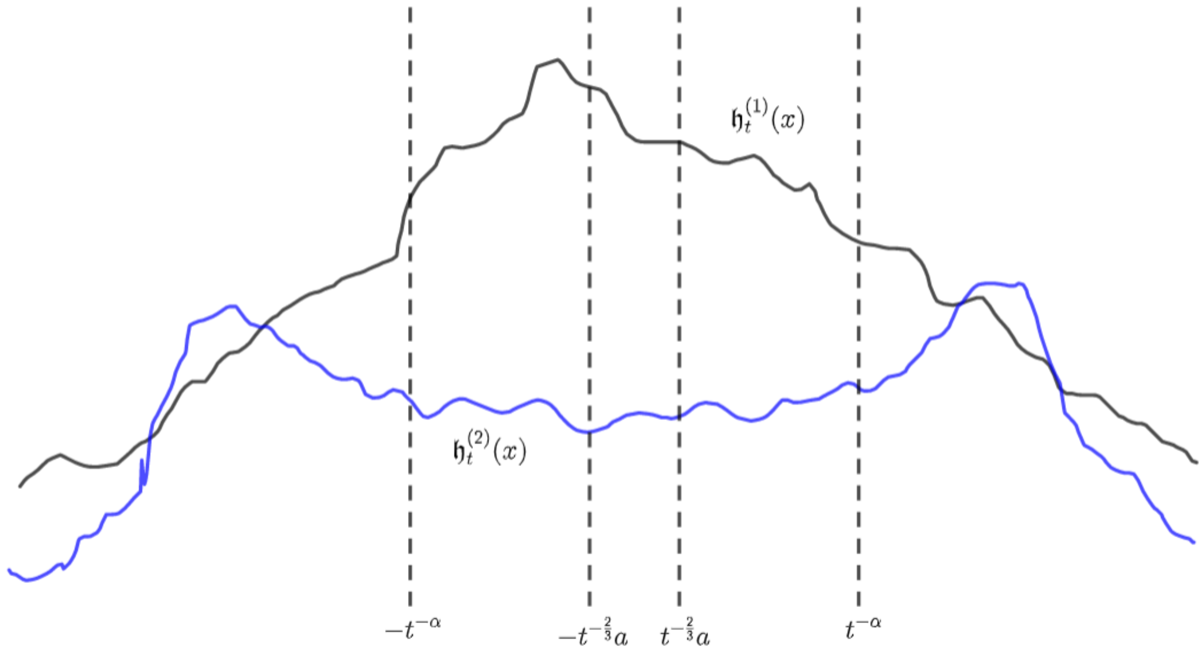

Figure 5.5: Illustration of the proof of Theorem 5.1.11. In a window of $[t^{-\alpha}, t^{\alpha}]$, the curves $\mathfrak{h}_t^{(1)}(x), \mathfrak{h}_t^{(2)}(x)$ attains an uniform gap with high probability. This allows us to show law of $\mathfrak{h}_t^{(1)}$ on that small patch is close to a Brownian bridge. Upon zooming in a the tiny interval $[-t^{2/3}a, t^{2/3}a]$ we get a two-sided Brownian bridge as explained in **Step 1** of the proof.

Fix any arbitrary $\delta > 0$ and consider the following three events:

$$\mathsf{Gap}_t(\delta) := \left\{\mathfrak{h}_t^{(2)}(-t^{-\alpha}) \le \min\{\mathfrak{h}_t^{(1)}(t^{-\alpha}), \mathfrak{h}_t^{(1)}(-t^{-\alpha})\} - \delta\right\}, \tag{5.6.3}$$

$$\mathsf{Rise}_t(\delta) := \left\{\sup_{x\in I_t} \mathfrak{h}_t^{(2)}(x) \le \tfrac{1}{4}\delta + \mathfrak{h}_t^{(2)}(-t^{-\alpha})\right\}, \tag{5.6.4}$$

$$\mathsf{Tight}_t(\delta) := \left\{-\delta^{-1} \le \mathfrak{h}_t^{(1)}(t^{-\alpha}), \mathfrak{h}_t^{(1)}(-t^{-\alpha}) \le \delta^{-1}\right\}. \tag{5.6.5}$$

Note that all the above events are measurable w.r.t. $\mathcal{F}$. A visual interpretation of the above events are given later in Figure 5.6. Since the underlying curves are continuous almost surely, while specifying events over I_t, such as the $\mathsf{Rise}_t(\delta)$ event defined in (5.6.4), one may replace I_t by its closure $\overline{I}_t = [-t^{-\alpha}, t^{-\alpha}]$ as well; the events will remain equal almost surely. We will often make use of this fact, and will not make a clear distinction between I_t and $\overline{I}_t$.

We set

$$\mathsf{Fav}_t(\delta) := \mathsf{Gap}_t(\delta) \cap \mathsf{Rise}_t(\delta) \cap \mathsf{Tight}_t(\delta). \tag{5.6.6}$$

The $\mathsf{Fav}_t(\delta)$ event is a favorable event in the sense that given any $\varepsilon > 0$, there exists $\delta_0 \in (0,1)$ such that for all $\delta \in (0, \delta_0)$

$$\liminf_{t\to\infty} \mathsf{Fav}_t(\delta) \geq 1 - \varepsilon. \tag{5.6.7}$$

We will prove (5.6.7) in **Step 4**. For the moment, we assume this and continue with our calculations. We now proceed to find tight upper and lower bounds for $\mathbf{P}_t(A) = \mathbf{P}((\mathcal{H}(\cdot,t)-\mathcal{H}(0,t) \in A)$. Recall the σ-field $\mathcal{F}$ from (5.6.2). Note that using the tower property of the conditional expectation we have

$$\mathbf{P}_t(A) = \mathbf{E}\left[\mathbf{P}_t(A \mid \mathcal{F})\right] \geq \mathbf{E}\left[\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_t(A \mid \mathcal{F})\right]. \tag{5.6.8}$$

$$\mathbf{P}_t(A) = \mathbf{E}\left[\mathbf{P}_t(A \mid \mathcal{F})\right] \leq \mathbf{E}\left[\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_t(A \mid \mathcal{F})\right] + \mathbf{P}\left(\neg\mathsf{Fav}_t(\delta)\right). \tag{5.6.9}$$

Applying the $\mathbf{H}_t$-Brownian Gibbs property for the interval I_t we have

$$\mathbf{P}_t(A \mid \mathcal{F}) = \mathbf{P}_{\mathbf{H}_t}^{1,1,I_t,\mathfrak{h}_t(-t^{-\alpha}),\mathfrak{h}_t(t^{-\alpha}),\mathfrak{h}_t^{(2)}}(A) = \frac{\mathbf{E}_{\text{free},t}\left[W\mathbf{1}_A\right]}{\mathbf{E}_{\text{free},t}\left[W\right]}, \tag{5.6.10}$$

where

$$W := \exp\left(-t^{2/3}\int_{t^{-\alpha}}^{t^{-\alpha}} e^{t^{1/3}(\mathfrak{h}_t^{(2)}(x)-\mathfrak{h}_t^{(1)}(x))}\mathrm{d}x\right) \tag{5.6.11}$$

and $\mathbf{P}_{\text{free},t} := \mathbf{P}_{\text{free}}^{1,1,I_t,\mathfrak{h}_t(-t^{-\alpha}),\mathfrak{h}_t(t^{-\alpha})}$ and $\mathbf{E}_{\text{free},t} := \mathbf{E}_{\text{free}}^{1,1,I_t,\mathfrak{h}_t(-t^{-\alpha}),\mathfrak{h}_t(t^{-\alpha})}$ are the probability and the expectation operator respectively for a Brownian bridge $B_1(\cdot)$ on I_t starting at $\mathfrak{h}_t(-t^{-\alpha})$ and ending at $\mathfrak{h}_t(t^{-\alpha})$. Note that the second equality in (5.6.10) follows from (5.2.3). We now seek to find upper and lower bounds for r.h.s. of (5.6.10). For W, we have the trivial upper bound: $W \le 1$. For the lower bound, we claim that there exists $t_0(\delta) > 0$, such that for all $t \ge t_0$, we have

$$\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_{\text{free},t}(W \ge 1-\delta) \ge \mathbf{1}\{\mathsf{Fav}_t(\delta)\}(1-\delta). \tag{5.6.12}$$

Note that (5.6.12) suggests that the W is close to 1 with high probability. This is the technical expression of the first conceptual step that we highlighted in **Step 1**. In the similar spirit for the second conceptual step, we claim that there exists $t_0(\delta) > 0$, such that for all $t \ge t_0$, we have

$$\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\left|\mathbf{P}_{\text{free},t}(A) - \gamma(A)\right| \le \mathbf{1}\{\mathsf{Fav}_t(\delta)\}\cdot\delta, \tag{5.6.13}$$

where $\gamma(A) := \mathbf{P}(B(\cdot) \in A) \in [0,1]$. We remark that the l.h.s. of (5.6.12) and (5.6.13) are random variables measurable w.r.t. $\mathcal{F}$. The inequalities above hold pointwise. We will prove (5.6.12) and (5.6.13) in **Step 5** and **Step 6** respectively. We next complete the proof of the Theorem 5.1.11 assuming the above claims.

Step 3. In this step we assume (5.6.7), (5.6.12), and (5.6.13) and complete the proof of (5.6.1). Fix any $\varepsilon \in (0,1)$. Get a $\delta_0 \in (0,1)$, so that (5.6.7) is true for all $\delta \in (0,\delta_0)$. Fix any such $\delta \in (0,\delta_0)$. Get $t_0(\delta)$ large enough so that both (5.6.12) and (5.6.13) hold for all $t \ge t_0$. Fix any such $t \ge t_0$.

As $W \le 1$, we note that on the event $\mathsf{Fav}_t(\delta)$,

$$\begin{aligned}
\frac{\mathbf{E}_{\text{free},t}\left[W\mathbf{1}_A\right]}{\mathbf{E}_{\text{free},t}\left[W\right]} &\ge \mathbf{E}_{\text{free},t}\left[W\mathbf{1}_A\right] \\
&\ge (1-\delta)\mathbf{P}_{\text{free},t}\left(A \cap \{W \ge 1-\delta\}\right) \\
&\ge (1-\delta)\mathbf{P}_{\text{free},t}(A) - (1-\delta)\mathbf{P}_{\text{free},t}(W < 1-\delta) \\
&\ge (1-\delta)\mathbf{P}_{\text{free},t}(A) - (1-\delta)\delta,
\end{aligned}$$

where in the last line we used (5.6.12). Plugging this bound back in (5.6.8) we get

$$\begin{aligned}
\mathbf{P}_t(A) &\geq (1-\delta)\mathbf{E}\left[\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_{\text{free},t}(A)\right] - (1-\delta)\delta \\
&\geq (1-\delta)\mathbf{E}\left[\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\gamma(A) - \delta\right] - (1-\delta)\delta \\
&= \gamma(A)(1-\delta)\mathbf{P}(\mathsf{Fav}_t(\delta)) - 2\delta(1-\delta).
\end{aligned}$$

where the inequality in the penultimate line follows from (5.6.13). Taking $\liminf$ both sides as $t \to \infty$, in view of (5.6.7) we see that

$$\liminf_{t\to\infty} \mathbf{P}_t(A) \geq (1-\delta)(1-\varepsilon)\gamma(A) - 2\delta(1-\delta).$$

Taking $\liminf_{\delta\downarrow 0}$ and using the fact that ε is arbitrary, we get that $\liminf_{t\to\infty} \mathbf{P}_t(A) \geq \gamma(A)$. Similarly for the upper bound, on the event $\mathsf{Fav}_t(\delta)$ we have

$$\frac{\mathbf{E}_{\text{free},t}\left[W\mathbf{1}_A\right]}{\mathbf{E}_{\text{free},t}\left[W\right]} \leq \frac{\mathbf{P}_{\text{free},t}(A)}{(1-\delta)\mathbf{P}_{\text{free},t}(W \geq 1-\delta)} \leq \frac{1}{(1-\delta)^2}\mathbf{P}_{\text{free},t}(A),$$

where we again use (5.6.12) for the last inequality. Inserting the above bound in (5.6.9) we get

$$\begin{aligned}
\mathbf{P}_t(A) &\leq \frac{1}{(1-\delta)^2}\mathbf{E}\left[\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_{\text{free},t}(A)\right] + \mathbf{P}\left(\neg\mathsf{Fav}_t(\delta)\right) \\
&\leq \frac{\delta}{(1-\delta)^2} + \frac{1}{(1-\delta)^2}\mathbf{E}\left[\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\gamma(A)\right] + \mathbf{P}\left(\neg\mathsf{Fav}_t(\delta)\right) \\
&\leq \frac{\delta}{(1-\delta)^2} + \frac{1}{(1-\delta)^2}\gamma(A) + \mathbf{P}\left(\neg\mathsf{Fav}_t(\delta)\right).
\end{aligned}$$

The inequality in the penultimate line above follows from (5.6.13). Taking $\limsup$ both sides as $t \to \infty$, in view of (5.6.7) we see that

$$\limsup_{t\to\infty} \mathbf{P}_t(A) \leq \frac{\delta}{(1-\delta)^2} + \frac{1}{(1-\delta)^2}\gamma(A) + \varepsilon.$$

As before taking $\limsup_{\delta\downarrow 0}$ and using the fact that ε is arbitrary, we get that $\limsup_{t\to\infty} \mathbf{P}_t(A) \leq$

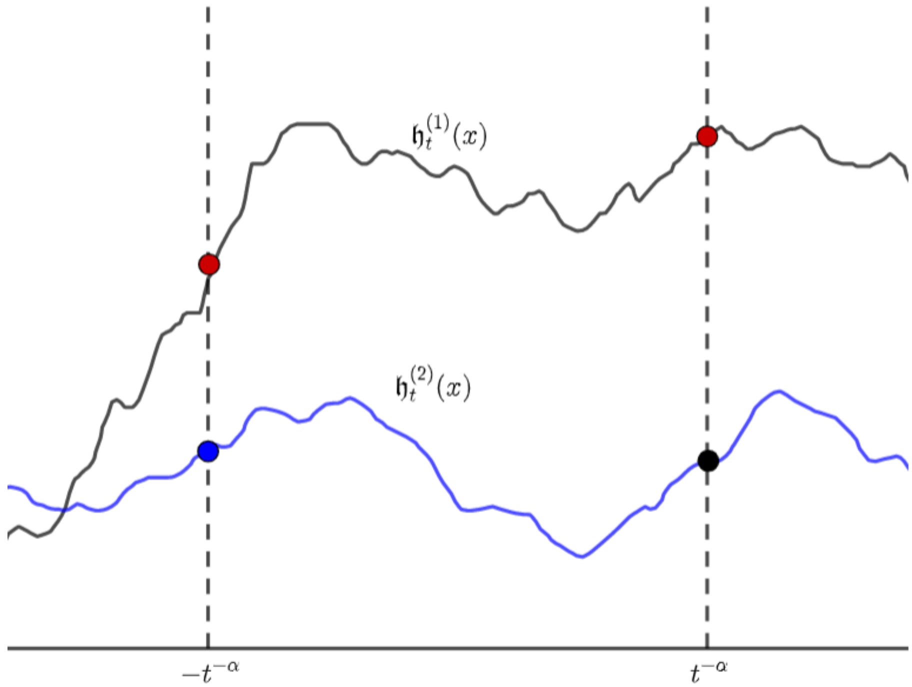

Figure 5.6: In the above figure $\mathsf{Gap}_t(\delta)$ defined in (5.6.3) denotes the event that the value of the blue point is smaller than the value of each of the red points at least by δ, The $\mathsf{Rise}_t(\delta)$ event defined in (5.6.4) requires *no* point on the whole blue curve (restricted to $I_t = (-t^{-\alpha}, t^{-\alpha})$) exceed the value of the blue point by a factor $\frac{1}{4}\delta$ (i.e., there is no significant rise). The $\mathsf{Tight}_t(\delta)$ defined in (5.6.5) event ensures the value of the red points are within $[-\delta^{-1}, \delta^{-1}]$. The $\mathsf{Fluc}_t^{(i)}(\delta)$ event defined in (5.6.15) signifies every value of every point on the i-th curve (restricted to I_t) is within $\frac{1}{4}\delta$ distance away from its value on the left boundary: $\mathfrak{h}_t^{(1)}(-t^{-\alpha})$. Finally, $\mathsf{Sink}_t(\delta)$ event defined in (5.6.20) denotes the event that no point on the black curve (restricted to I_t) drops below the value of the red points by a factor larger than $\frac{1}{4}\delta$, (i.e., there is no significant sink).

$\gamma(A)$. With the matching upper bound for $\liminf$ derived above, we thus arrive at (5.6.1), completing the proof of Theorem 5.1.11.

Step 4. In this step we prove (5.6.7). Fix any $\delta > 0$. Recall the distributional convergence of KPZ line ensemble to Airy line ensemble from Proposition 5.2.7. By the Skorokhod representation theorem, we may assume that our probability space are equipped with $\mathcal{A}_1(x)$ $\mathcal{A}_2(x)$, such that as $t \to \infty$, almost surely we have

$$\max_{i=1,2} \sup_{x\in[-1,1]} |2^{1/3}\mathfrak{h}_t^{(i)}(2^{1/3}x) - \mathcal{A}_i(x)| \to 0. \tag{5.6.14}$$

For $i = 1, 2$, consider the event

$$\mathsf{Fluc}_t^{(i)}(\delta) := \left\{ \sup_{x \in I_t} |\mathfrak{h}_t^{(i)}(x) - \mathfrak{h}_t^{(i)}(-t^{-\alpha})| \le \tfrac{1}{4}\delta \right\}. \tag{5.6.15}$$

See Figure 5.6 and its caption for an interpretation of this event. We claim that for every $\delta > 0$,

$$\liminf_{t\to\infty} \mathbf{P}\left(\mathsf{Fluc}_t^{(i)}(\delta)\right) = 1. \tag{5.6.16}$$

Let us complete the proof of (5.6.7) assuming (5.6.16). Fix any $\varepsilon > 0$. Note that $\{|\mathfrak{h}_t^{(1)}(-t^{-\alpha}) - \mathfrak{h}_t^{(1)}(t^{-\alpha})| \le \frac{1}{4}\delta\} \supset \mathsf{Fluc}_t^{(1)}(\delta)$. Recall $\mathsf{Gap}_t(\delta)$ from (5.6.3). Observe that

$$\begin{aligned} \neg\mathsf{Gap}_t(\delta) \cap \left\{ |\mathfrak{h}_t^{(1)}(-t^{-\alpha}) - \mathfrak{h}_t^{(1)}(t^{-\alpha})| \le \tfrac{1}{4}\delta \right\} &\subset \left\{ \mathfrak{h}_t^{(2)}(-t^{\alpha}) - \mathfrak{h}_t^{(1)}(-t^{-\alpha}) \ge -\tfrac{5}{4}\delta \right\} \\ &\subset \left\{ \inf_{x\in[-1,0]} [\mathfrak{h}_t^{(2)}(x) - \mathfrak{h}_t^{(1)}(x)] \ge -\tfrac{5}{4}\delta \right\}. \end{aligned}$$

Using these two preceding set relations, by union bound we have

$$\begin{aligned} \mathbf{P}\left(\neg\mathsf{Gap}_t(\delta)\right) &\le \mathbf{P}\left(|\mathfrak{h}_t^{(1)}(-t^{-\alpha}) - \mathfrak{h}_t^{(1)}(t^{-\alpha})| \ge \tfrac{1}{4}\delta \right) + \mathbf{P}\left(\neg\mathsf{Gap}_t(\delta) \cap |\mathfrak{h}_t^{(1)}(-t^{-\alpha}) - \mathfrak{h}_t^{(1)}(t^{-\alpha})| \le \tfrac{1}{4}\delta \right) \\ &\le \mathbf{P}\left(\neg\mathsf{Rise}_t^{(1)}(\delta)\right) + \mathbf{P}\left(\inf_{x\in[-1,0]} [\mathfrak{h}_t^{(2)}(x) - \mathfrak{h}_t^{(1)}(x)] \ge -\tfrac{5}{4}\delta \right). \end{aligned}$$

As $t \to \infty$, the first term goes to zero due (5.6.16) and by Proposition 5.2.7, the second term goes to

$$\mathbf{P}\left(\inf_{x\in[-1,0]} [\mathcal{A}_2(2^{-1/3}x) - \mathcal{A}_1(2^{-1/3}x)] \ge -\tfrac{5}{4\cdot 2^{1/3}}\delta \right).$$

But by (5.2.1) we know Airy line ensembles are strictly ordered. Thus the above probability can be made arbitrarily small be choose δ small enough. In particular, there exists a $\delta_1 \in (0, 1)$ such that for all $\delta \in (0, \delta_1)$ the above probability is always less than $\frac{\varepsilon}{2}$. This forces

$$\liminf_{t\to\infty} \mathbf{P}\left(\mathsf{Gap}_t(\delta)\right) \ge 1 - \tfrac{\varepsilon}{2}. \tag{5.6.17}$$

Recall $\mathsf{Rise}_t(\delta)$ from (5.6.4). Clearly $\mathsf{Rise}_t(\delta) \subset \mathsf{Fluc}_t^{(2)}(\delta)$. Thus for every $\delta > 0$,

$$\liminf_{t\to\infty} \mathbf{P}(\mathsf{Rise}_t(\delta)) = 1. \tag{5.6.18}$$

Finally using Proposition 5.2.8 (a) and (b) we see that $\mathfrak{h}_t^{(1)}(t^{-\alpha}), \mathfrak{h}_t^{(1)}(t^{-\alpha})$ are tight. Thus there exists $\delta_2 \in (0,1)$ such that for all $\delta \in (0, \delta_2)$, we have

$$\liminf_{t\to\infty} \mathbf{P}\left(\mathsf{Tight}_t(\delta)\right) \ge 1 - \tfrac{\varepsilon}{2}. \tag{5.6.19}$$

Combining (5.6.17), (5.6.18), (5.6.19), and recalling the definition of $\mathsf{Fav}_t(\delta)$ from (5.6.6), by union bound we get (5.6.7) for all $\delta \in (0, \min\{\delta_1, \delta_2\})$.

Let us now prove (5.6.16). Recall $\mathsf{Fluc}_t^{(i)}(\delta)$ from (5.6.15). Define the event:

$$\mathsf{Conv}_t(\delta) := \left\{ \sup_{x\in[-1,1]} |\mathfrak{h}_t^{(i)}(x) - 2^{-1/3}\mathcal{A}_i(2^{-1/3}x)| \le \tfrac{1}{16}\delta \right\}.$$

Observe that

$$\left\{\neg\mathsf{Fluc}_t^{(i)}(\delta), \mathsf{Conv}_t(\delta)\right\} \subset \left\{ \sup_{|x|\le 2^{-1/3}t^{-\alpha}} \left[\mathcal{A}_i(x) - \mathcal{A}_i(-2^{-1/3}t^{-\alpha})\right] \ge \tfrac{2^{1/3}}{8}\delta \right\}.$$

Thus by union bound

$$\begin{aligned}\mathbf{P}\left(\neg\mathsf{Fluc}_t^{(i)}(\delta)\right) &\le \mathbf{P}\left(\neg\mathsf{Conv}_t(\delta)\right) + \mathbf{P}\left(\neg\mathsf{Fluc}_t^{(i)}(\delta), \mathsf{Conv}_t(\delta)\right) \\ &\le \mathbf{P}\left(\neg\mathsf{Conv}_t(\delta)\right) + \mathbf{P}\left(\sup_{|x|\le 2^{-1/3}t^{-\alpha}} \left[\mathcal{A}_i(x) - \mathcal{A}_i(-2^{-1/3}t^{-\alpha})\right] \ge \tfrac{2^{1/3}}{8}\delta\right).\end{aligned}$$

By (5.6.14), the first term above goes to zero as $t \to \infty$, whereas the second term goes to zero as $t \to \infty$, via modulus of continuity of Airy line ensembles from Proposition 5.2.4. Note that in Proposition 5.2.4 the modulus of continuity is stated for $\mathcal{A}_i(x) + x^2$. However, in the above scenario since we deal with a vanishing interval $[-2^{-1/3}t^{-\alpha}, 2^{-1/3}t^{-\alpha}]$, the parabolic term does not play any role. This establishes (5.6.16).

Step 5. In this step we prove (5.6.12). Let us consider the event

$$\mathsf{Sink}_t(\delta) := \left\{ \inf_{x\in I_t} \mathfrak{h}_t^{(1)}(x) \geq -\tfrac{1}{4}\delta + \min\{\mathfrak{h}_t(-t^{-\alpha}), \mathfrak{h}_t(t^{-\alpha})\} \right\}. \tag{5.6.20}$$

See Figure 5.6 and its caption for an interpretation of this event. Recall $\mathsf{Gap}_t(\delta)$ and $\mathsf{Rise}_t(\delta)$ from (5.6.3) and (5.6.4). Observe that on the event $\mathsf{Gap}_t(\delta) \cap \mathsf{Rise}_t(\delta)$, we have $\sup_{x\in I_t} \mathfrak{h}_t^{(2)}(x) \leq \min\{\mathfrak{h}_t(-t^{-\alpha}), \mathfrak{h}_t(t^{-\alpha})\} - \frac{3}{4}\delta$. Thus on $\mathsf{Gap}_t(\delta) \cap \mathsf{Rise}_t(\delta) \cap \mathsf{Sink}_t(\delta)$, we have

$$\inf_{x\in I_t} \left[\mathfrak{h}_t^{(1)}(x) - \mathfrak{h}_t^{(2)}(x) \right] \geq \tfrac{1}{2}\delta.$$

Recall W from (5.6.11). On the event $\{\inf_{x\in I_t} \left[\mathfrak{h}_t^{(1)}(x) - \mathfrak{h}_t^{(2)}(x) \right] \geq \frac{1}{2}\delta\}$ we have the pointwise inequality

$$W > \exp(-2t^{2/3-\alpha} e^{-\frac{1}{2}t^{1/3}\delta}) \geq 1 - \delta,$$

where we choose a $t_1(\delta) > 0$ so that the last inequality is true for all $t \geq t_1$. Thus for all $t \geq t_1$,

$$\mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_{\mathrm{free},t}(W \geq 1 - \delta) \geq \mathbf{1}\{\mathsf{Fav}_t(\delta)\}\mathbf{P}_{\mathrm{free},t}(\mathsf{Sink}_t(\delta)). \tag{5.6.21}$$

Recall that $\mathbf{P}_{\mathrm{free},t}$ denotes the law of a Brownian bridge $B_1(\cdot)$ on I_t starting at $B_1(-t^{-\alpha}) = \mathfrak{h}_t(-t^{-\alpha})$ and ending at $B_1(t^{-\alpha}) = \mathfrak{h}_t(t^{-\alpha})$. Let us consider another Brownian bridge $\widetilde{B}_1(\cdot)$ on I_t starting and ending at $\min\{\mathfrak{h}_t(-t^{-\alpha}), \mathfrak{h}_t(t^{-\alpha})\}$. By standard estimates for Brownian bridge (see Lemma 2.11 in [**CH16**] for example)

$$\mathbf{P}\left(\inf_{x\in I_t} \widetilde{B}_1(x) \geq -\tfrac{1}{4}\delta + \min\{\mathfrak{h}_t(-t^{-\alpha}), \mathfrak{h}_t(t^{-\alpha})\} \right) = 1 - \exp\left(-\tfrac{\delta^2}{8|I_t|}\right) = 1 - \exp\left(-\tfrac{\delta^2}{16}t^{\alpha}\right).$$

Note that $B_1(\cdot)$ is stochastically larger than $\widetilde{B}_1(\cdot)$. Since the above event is increasing, we thus have $\mathbf{P}_{\mathrm{free},t}$ $(\mathsf{Sink}_t(\delta))$ is at least $1-\exp\left(-\frac{\delta^2}{16}t^{\alpha}\right)$. We now choose $t_2(\delta) > 0$, such that $1-\exp\left(-\frac{\delta^2}{16}t^{\alpha}\right) \geq$

$1-\delta$. Taking $t_0 = \max\{t_1, t_2\}$, we thus get (5.6.12) from (5.6.21).

Step 6. In this step we prove (5.6.13). As before consider the Brownian bridge $B_1(\cdot)$ on I_t starting at $B_1(-t^{-\alpha}) = \mathfrak{h}_t(-t^{-\alpha})$ and ending at $B_1(t^{-\alpha}) = \mathfrak{h}_t(t^{-\alpha})$. We may write B_1 as

$$B_1(x) = \mathfrak{h}_t^{(1)}(-t^{-\alpha}) + \frac{x+t^{-\alpha}}{2t^{-\alpha}}(\mathfrak{h}_t^{(1)}(t^{-\alpha}) - \mathfrak{h}_t^{(1)}(-t^{-\alpha})) + \overline{B}(x).$$

where $\overline{B}$ is a Brownian bridge on I_t starting and ending at zero. Thus,

$$t^{1/3}(B_1(t^{-2/3}x) - B_1(0)) = t^{1/3}\left[\overline{B}(t^{-2/3}x) - \overline{B}(0)\right] + \tfrac{1}{2}t^{\alpha-1/3}x(\mathfrak{h}_t^{(1)}(t^{-\alpha}) - \mathfrak{h}_t^{(1)}(-t^{-\alpha})). \tag{5.6.22}$$

Recall that $\alpha = \frac{1}{6}$. By Brownian scaling, $B_*(x) := t^{1/3}\overline{B}(t^{-2/3}x)$ is a Brownian bridge on the large interval $[-\sqrt{t}, \sqrt{t}]$ starting and ending at zero. By computing the covariances, it is easy to check that as $t \to \infty$, $B_*(x) - B_*(0)$ converges weakly to a two-sided Brownian motion $B(\cdot)$ on $[-a, a]$. This gives us the weak limit for the first term on the r.h.s. of (5.6.22). For the second term, recall the event $\mathsf{Tight}_t(\delta)$ from (5.6.5). As $|x| \le a$, on $\mathsf{Tight}_t(\delta)$, we have

$$\tfrac{1}{2}t^{\alpha-1/3}x(\mathfrak{h}_t^{(1)}(t^{-\alpha}) - \mathfrak{h}_t^{(1)}(-t^{-\alpha})) \le t^{-1/6}a\delta^{-1}.$$

This gives an uniform bound (uniform over the event $\mathsf{Fav}_t(\delta)$) on the second term in (5.6.22). Thus as long as the boundary data is in $\mathsf{Tight}_t(\delta)$, $\mathbf{P}_{\text{free},t}(A) \to \gamma(A)$ where $\gamma(A) = \mathbf{P}(B(\cdot) \in A)$. This proves (5.6.13).

5.6.2 Dyson Behavior around joint maximum

In this subsection we state and prove Proposition 5.6.1.

Proposition 5.6.1 (Dyson behavior around joint maximum)**.** *Fix $p \in (0,1)$. Set $q = 1-p$. Consider 2 independent copies of the KPZ equation $\mathcal{H}_\uparrow(x,t)$, and $\mathcal{H}_\downarrow(x,t)$, both started from the narrow wedge initial data. Let $\mathcal{M}_{p,t}$ be the almost sure unique maximizer of the process*

$x \mapsto (\mathcal{H}_\uparrow(x,pt) + \mathcal{H}_\downarrow(x,qt))$ which exists via Lemma 5.3.1. Set

$$\begin{aligned} D_1(x,t) &:= \mathcal{H}_\uparrow(\mathcal{M}_{p,t},pt) - \mathcal{H}_\uparrow(x+\mathcal{M}_{p,t},pt), \\ D_2(x,t) &:= \mathcal{H}_\downarrow(x+\mathcal{M}_{p,t},qt) - \mathcal{H}_\downarrow(\mathcal{M}_{p,t},qt). \end{aligned} \tag{5.6.23}$$

As $t \to \infty$, we have the following convergence in law

$$(D_1(x,t), D_2(x,t)) \xrightarrow{d} (\mathcal{D}_1(x), \mathcal{D}_2(x)) \tag{5.6.24}$$

in the uniform-on-compact topology. Here $\mathcal{D} = (\mathcal{D}_1, \mathcal{D}_2) : \mathbb{R} \to \mathbb{R}^2$ is a two-sided DBM*, that is, $\mathcal{D}_+(\cdot) := \mathcal{D}(\cdot) \mid_{[0,\infty)}$ and $\mathcal{D}_-(\cdot) := \mathcal{D}(-\cdot) \mid_{(-\infty,0]}$ are independent copies of* DBM *defined in Definition 5.5.1.*

For clarity, the proof is completed over several subsections (Sections 5.6.2-5.6.2) below and we refer to Figure 5.7 for the structure of the proof.

KPZ line ensemble framework

In this subsection, we convert Proposition 5.6.1 into the language of scaled KPZ line ensemble defined in Proposition 6.5.1. We view $\mathcal{H}_\uparrow(x,t) = \mathcal{H}_{t,\uparrow}^{(1)}(x), \mathcal{H}_\downarrow(x,t) = \mathcal{H}_{t,\downarrow}^{(1)}(x)$ as the top curves of two (unscaled) KPZ line ensembles: $\{\mathcal{H}_{t,\uparrow}^{(n)}(x), \mathcal{H}_{t,\downarrow}^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$. Following (5.2.5) we define their scaled versions:

$$\mathfrak{h}_{t,\uparrow}^{(n)}(x) := t^{-1/3}\left(\mathcal{H}_{t,\uparrow}^{(n)}(t^{2/3}x) + \tfrac{t}{24}\right), \qquad \mathfrak{h}_{t,\downarrow}^{(n)}(x) := t^{-1/3}\left(\mathcal{H}_{t,\downarrow}^{(n)}(t^{2/3}x) + \tfrac{t}{24}\right).$$

Along with the full maximizer $\mathcal{M}_{p,t}$, we will also consider local maximizer defined by

$$\mathcal{M}_{p,t}^M := \underset{x\in[-Mt^{2/3},Mt^{2/3}]}{\operatorname{argmax}} (\mathcal{H}_{pt,\uparrow}^{(1)}(x) + \mathcal{H}_{qt,\downarrow}^{(1)}(x)), \qquad M \in [0,\infty]. \tag{5.6.25}$$

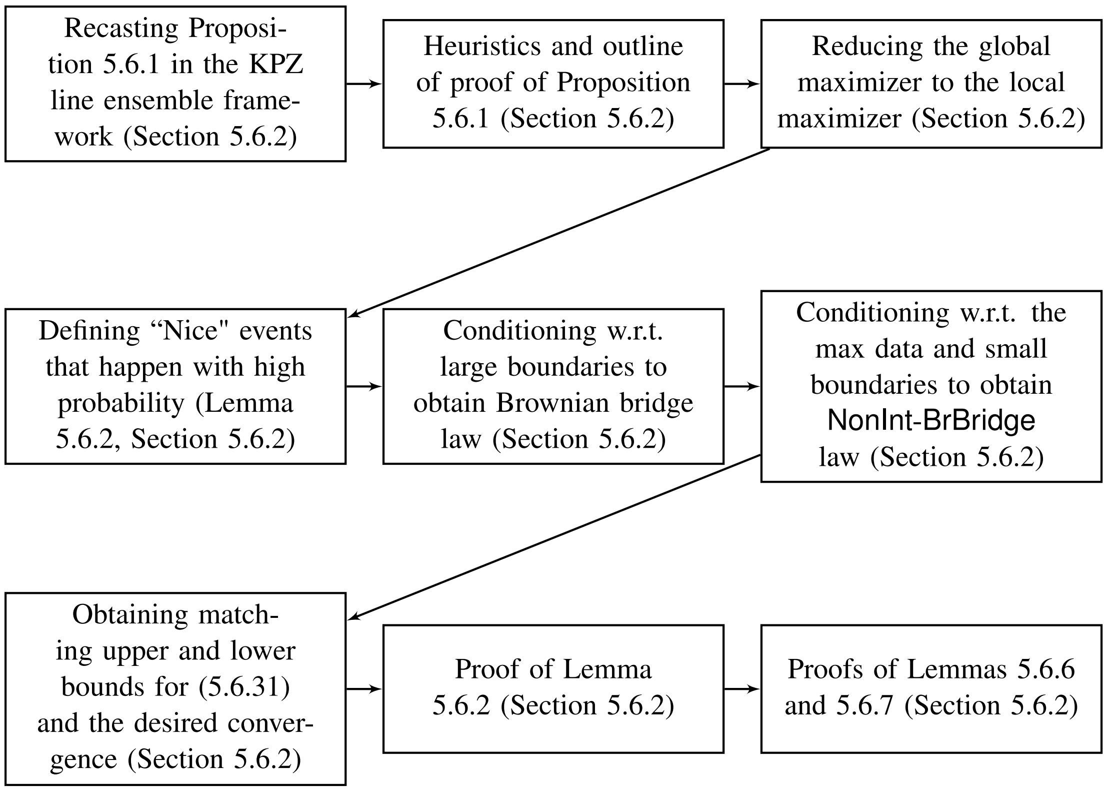

Figure 5.7: Structure of Section 5.6.2.

For each $M > 0$, $\mathcal{M}_{p,t}^M$ is unique almost surely by $\mathbf{H}_t$-Brownian Gibbs property. We now set

$$\begin{aligned} Y_{M,t,\uparrow}^{(n)}(x) &:= p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(n)}\big((pt)^{-2/3}\mathcal{M}_{p,t}^M\big) - p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(n)}\big(p^{-2/3}x\big), \\ Y_{M,t,\downarrow}^{(n)}(x) &:= q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(n)}\big(q^{-2/3}x\big) - q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(n)}\big((qt)^{-2/3}\mathcal{M}_{p,t}^M\big). \end{aligned} \tag{5.6.26}$$

Taking into account of (5.6.23) and all the above new notations, it can now be checked that for each $t > 0$,

$$D_1(x,t) \stackrel{d}{=} t^{1/3}Y_{\infty,t,\uparrow}^{(1)}\big(t^{-2/3}(\mathcal{M}_{p,t}^\infty + x)\big), \quad D_2(x,t) \stackrel{d}{=} t^{1/3}Y_{\infty,t,\downarrow}^{(1)}\big(t^{-2/3}(\mathcal{M}_{p,t}^\infty + x)\big), \tag{5.6.27}$$

both as functions in x. Thus it is equivalent to verify Proposition 5.6.1 for the above $Y_{\infty,t,\uparrow}^{(1)}, Y_{\infty,t,\downarrow}^{(1)}$ expressions. In our proof we will mostly deal with local maximizer version, and so for convenience we define:

$$D_{M,t,\uparrow}(x) := t^{1/3} Y^{(1)}_{M,t,\uparrow}\big(t^{-2/3}(\mathcal{M}^M_{p,t} + x)\big), \quad D_{M,t,\downarrow}(x) = t^{1/3} Y^{(1)}_{M,t,\downarrow}\big(t^{-2/3}(\mathcal{M}^M_{p,t} + x)\big). \qquad (5.6.28)$$

where $Y^{(1)}_{M,t,\uparrow}, Y^{(1)}_{M,t,\downarrow}$ are defined in (5.6.26). We will introduce several other notations and parameters later in the proof. For the moment, the minimal set of notations introduced here facilitate our discussion of ideas and outline of the proof of Proposition 5.6.1 in the next subsection.

Ideas and Outline of Proof of Proposition 5.6.1

Before embarking on a rather lengthy proof, in this subsection we explain the core ideas behind the proof and provide an outline for the remaining subsections.

First we contrast the proof idea with that of Theorem 5.1.11. Indeed, similar to the proof of Theorem 5.1.11, from (5.6.27) we see that at the level $Y^{(1)}_{\infty,t,\uparrow}, Y^{(1)}_{\infty,t,\downarrow}$ we are interested in understanding their laws restricted to a very small symmetric interval of order $O(t^{-2/3})$ around the point $t^{-2/3}\mathcal{M}^\infty_{p,t}$. However, the key difference from the conceptual argument presented at the beginning of the proof if Theorem 5.1.11 is that the centered point $t^{-2/3}\mathcal{M}^\infty_{p,t}$ is random and it does not go to zero. Rather by Theorem 5.1.8 it converges in distribution to a nontrivial random quantity (namely $\Gamma(p\sqrt{2})$). Hence one must take additional care of this random point. This makes the argument significantly more challenging compared to that of Theorem 5.1.11.

We now give a road-map of our proof. At this point, readers are also invited to look into Figure 5.8 alongside the explanation offered in its caption.

- As noted in Lemma 5.3.1, the random centering $t^{-2/3}\mathcal{M}^\infty_{p,t}$ has decaying properties and can be approximated by $t^{-2/3}\mathcal{M}^M_{p,t}$ by taking large enough M. Hence on a heuristic level it suffices to work with the local maximizers instead. In Subsection 5.6.2, this heuristics will be justified rigorously. We will show there how to pass from $Y^{(1)}_{\infty,t,\uparrow}, Y^{(1)}_{\infty,t,\downarrow}$ defined in (5.6.27) to their finite centering analogs: $Y^{(1)}_{M,t,\uparrow}, Y^{(1)}_{M,t,\downarrow}$. The rest of the proof then boils down to

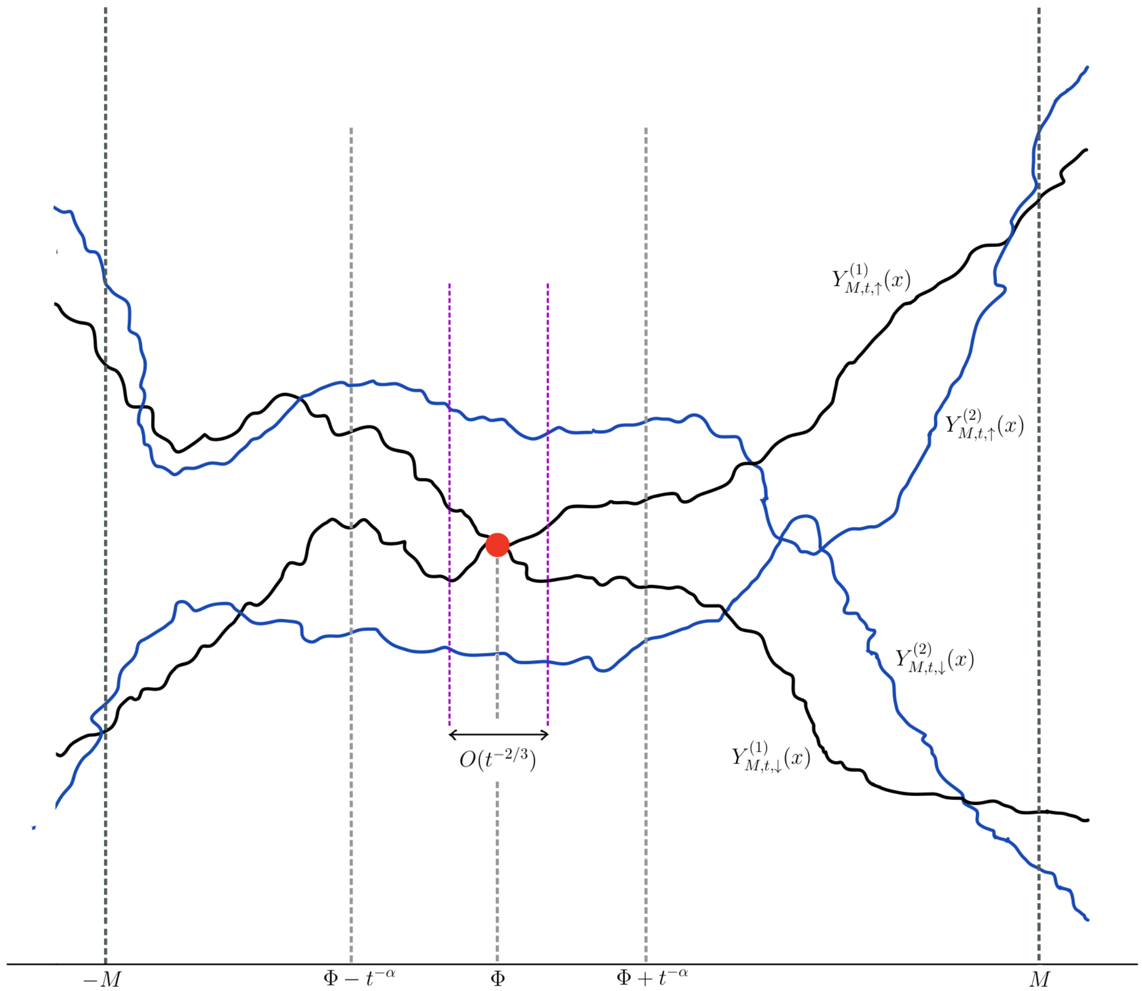

Figure 5.8: An overview of the proof for Proposition 5.6.1. The top and bottom black curves are $Y^{(1)}_{M,t,\uparrow}$ and $Y^{(1)}_{M,t,\downarrow}$ respectively. Note that the way they are defined in (5.6.26), $Y^{(1)}_{M,t,\uparrow}(x) \geq Y^{(1)}_{M,t,\downarrow}(x)$ with equality at $x = \Phi = t^{-2/3}\mathcal{M}^{M}_{p,t}$ labelled as the red dot in the above figure. The blue curves are $Y^{(1)}_{M,t,\uparrow}, Y^{(2)}_{M,t,\downarrow}$. There is no such ordering within blue curves. They may intersect among themselves as well as with the black curves. With $\alpha = \frac{1}{6}$, we consider the interval $K_t = (\Phi - t^{-\alpha}, \Phi + t^{-\alpha})$. In this vanishing interval around Φ, the curves will be ordered with high probability. In fact, with high probability, there will be a uniform separation. For instance, for small enough δ, we will have $Y^{(2)}_{M,t,\uparrow}(x) - Y^{(1)}_{M,t,\uparrow}(x) \geq \frac{1}{4}\delta$, and $Y^{(1)}_{M,t,\downarrow}(x) - Y^{(2)}_{M,t,\downarrow}(x) \geq \frac{1}{4}\delta$, for all $x \in K_t$ wth high probability. This will allow us to conclude black curves are behave approximately like two-sided NonInt-BrBridges on that narrow window. Then upon going into a even smaller window of $O(t^{-2/3})$, the two-sided NonInt-BrBridges turn into a two-sided DBM.

analyzing the laws of the latter.

- We now fix a $M > 0$ for the rest of the proof. Our analysis will now operate with $\mathcal{M}^{M}_{p,t}$. For simplicity, let us also use the notation

$$\Phi := t^{-2/3}\mathcal{M}_{p,t}^{M} \tag{5.6.29}$$

for the rest of the proof. We now perform several conditioning on the laws of the curves. Recall that by Proposition 6.5.1, $\{\mathfrak{h}_{pt,\uparrow}^{(n)}(\cdot)\}_{n\in\mathbb{N}}$ $\{\mathfrak{h}_{qt,\downarrow}^{(n)}(\cdot)\}_{n\in\mathbb{N}}$ satisfy the $\mathbf{H}_{pt}$-Brownian Gibbs property and the $\mathbf{H}_{qt}$-Brownian Gibbs property respectively with $\mathbf{H}_t$ given by (5.2.4). Conditioned on the end points of $\mathfrak{h}_{pt,\uparrow}^{(1)}(\pm Mp^{-2/3})$ and $\mathfrak{h}_{qt,\downarrow}^{(1)}(\pm Mq^{-2/3})$ and the second curves $\mathfrak{h}_{pt,\uparrow}^{(2)}(\cdot)$ and $\mathfrak{h}_{qt,\downarrow}^{(2)}(\cdot)$, the laws of $\mathfrak{h}_{pt,\uparrow}^{(1)}(\cdot)$, and $\mathfrak{h}_{pt,\uparrow}^{(1)}(\cdot)$ are absolutely continuous w.r.t. Brownian bridges with appropriate end points. This conditioning is done in Subsection 5.6.2.

- We then condition further on *Max data* : $\mathcal{M}_{p,t}^{M}, \mathfrak{h}_{pt,\uparrow}^{(1)}((pt)^{-2/3}\mathcal{M}_{p,t}^{M}), \mathfrak{h}_{qt,\downarrow}^{(1)}((qt)^{-2/3}\mathcal{M}_{p,t}^{M})$. Under this conditioning, via the decomposition result in Proposition 5.4.10, the underlying Brownian bridges mentioned in the previous point, when viewed from the joint maximizer, becomes two-sided NonInt-BrBridges defined in Definition 5.4.4. This viewpoint from the joint maximizer is given by $Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)}$. See Figure 5.8 for more details.

- We emphasize the fact that the deduction of NonInt-BrBridges done above is only for the underlying Brownian law. One still needs to analyze the Radon-Nikodym (RN) derivative. As we are interested in the laws of $Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)}$ on an interval of order $t^{-2/3}$ around Φ, we analyze the RN derivative only on a small interval around Φ. To be precise, we consider a slightly larger yet vanishing interval of length $2t^{-\alpha}$ for $\alpha = \frac{1}{6}$ around the random point Φ. We show that the RN derivative on this small random patch is close to 1. Thus upon further conditioning on the boundary data of this random small interval, the trajectories of $Y_{M,t,\uparrow}^{(1)}$ and $Y_{M,t,\downarrow}^{(1)}$ defined in (5.6.26) around Φ turns out to be close to two-sided NonInt-BrBridge with appropriate (random) endpoints.

- Finally, we zoom further into a tiny interval of order $O(t^{-2/3})$ symmetric around the random point Φ. Utilizing Lemma 5.5.3, we convert the two-sided NonInt-BrBridges to two-sided

DBMs.

We now provide an outline of the rest of the subsections. In Subsection 5.6.2 we reduce our proof from understanding laws around global maximizers to that of local maximizers. As explained in the above road-map, the proof follows by performing several successive conditioning. On a technical level, this requires defining several high probability events on which we can carry out our conditional analysis. These events are all defined in Subsection 5.6.2 and are claimed to happen with high probability in Lemma 5.6.2. We then execute the first layer of conditioning in Subsection 5.6.2. The two other layers of conditioning are done in Subsection 5.6.2. Lemma 5.6.6 and Lemma 5.6.7 are the precise technical expressions for the heuristic claims in the last two bullet points of the road-map. Assuming them, we complete the final steps of the proof in Subsection 5.6.2. Proof of Lemma 5.6.2 is then presented in Subsection 5.6.2. Finally, in Subsection 5.6.2, we prove the remaining lemmas: Lemma 5.6.6 and 5.6.7.

Global to Local maximizer

We now fill out the technical details of the road-map presented in the previous subsection. Fix any $a > 0$. Consider any Borel set A of $C([-a, a] \to \mathbb{R}^2)$ which is a continuity set of a two-sided DBM $\mathcal{D}(\cdot)$ restricted to $[-a, a]$. By Portmanteau theorem, it is enough to show that

$$\mathbf{P}((D_1(\cdot, t), D_2(\cdot, t)) \in A) \to \mathbf{P}(\mathcal{D}(\cdot) \in A), \tag{5.6.30}$$

where D_1, D_2 are defined in (5.6.23). In this subsection, we describe how it suffices to check (5.6.30) with $\mathcal{M}_{p,t}^{M}$. Recall $D_{M,t,\uparrow}(\cdot), D_{M,t,\downarrow}(\cdot)$ from (5.6.28). We claim that for all $M > 0$:

$$\lim_{t\to\infty} \mathbf{P}((D_{M,t,\uparrow}(\cdot), D_{M,t,\downarrow}(\cdot)) \in A) \to \mathbf{P}(\mathcal{D}(\cdot) \in A). \tag{5.6.31}$$

Note that when $\mathcal{M}_{p,t}^{\infty} = \mathcal{M}_{p,t}^{M}$, $(D_{M,t,\uparrow}(\cdot), D_{M,t,\downarrow}(\cdot))$ is exactly equal to

$$t^{1/3} Y_{\infty,t,\uparrow}^{(1)}\big(t^{-2/3}(\mathcal{M}_{p,t}^{\infty} + \cdot)\big) \ , \ t^{1/3} Y_{\infty,t,\downarrow}^{(1)}\big(t^{-2/3}(\mathcal{M}_{p,t}^{\infty} + \cdot)\big)$$

which via (5.6.27) is same in distribution as $D_1(\cdot,t), D_2(\cdot,t)$. Thus,

$$\left|\mathbf{P}((D_1(\cdot,t), D_2(\cdot,t)) \in A) - \mathbf{P}((D_{M,t,\uparrow}(\cdot), D_{M,t,\downarrow}(\cdot)) \in A)\right| \le 2\mathbf{P}(\mathcal{M}_{p,t} \neq \mathcal{M}_{p,t}^M).$$

Now given any $\varepsilon > 0$, by Lemma 5.3.1, we can take $M = M(\varepsilon) > 0$ large enough so that $2\mathbf{P}(\mathcal{M}_{p,t} \neq \mathcal{M}_{p,t}^M) \le \varepsilon$. Then upon taking $t \to \infty$ in the above equation, in view of (5.6.31), we see that

$$\limsup_{t\to\infty} ||\mathbf{P}((D_1(\cdot,t), D_2(\cdot,t)) \in A) - \mathbf{P}((\mathcal{D}(\cdot) \in A)| \le \varepsilon.$$

As ε is arbitrary, this proves (5.6.30). The rest of the proof is now devoted in proving (5.6.31).

Nice events

In this subsection, we focus on defining several events that are collectively 'nice' in the sense that they happen with high probability. We fix an $M > 0$ for the rest of the proof and work with the local maximizer $\mathcal{M}_{p,t}^M$ defined in (5.6.25). We will also make use of the notation Φ defined in (5.6.29) heavily in this and subsequent subsections. We now proceed to define a few events based on the location and value of the maximizer and values at the endpoints of an appropriate interval. Fix any arbitrary $\delta > 0$. Let us consider the event:

$$\mathsf{ArMx}(\delta) := \{\Phi \in [-M+\delta, M-\delta]\}. \tag{5.6.32}$$

The $\mathsf{ArMx}(\delta)$ controls the location of the local maximizer Φ. Set $\alpha = \frac{1}{6}$. We define tightness event that corresponds to the boundary of the interval of length $2t^{-\alpha}$ around Φ :

$$\mathsf{Bd}_{\uparrow}(\delta) := \mathsf{Bd}_{+,\uparrow}(\delta) \cap \mathsf{Bd}_{-,\uparrow}(\delta), \quad \mathsf{Bd}_{\downarrow}(\delta) := \mathsf{Bd}_{+,\downarrow}(\delta) \cap \mathsf{Bd}_{-,\downarrow}(\delta), \tag{5.6.33}$$

where

$$\mathsf{Bd}_{\pm,\uparrow}(\delta) := \left\{\left|\mathfrak{h}_{pt,\uparrow}^{(1)}\big(p^{-2/3}(\Phi \pm t^{-\alpha})\big) - \mathfrak{h}_{pt,\uparrow}^{(1)}\big(\Phi p^{-2/3}\big)\right| \le \tfrac{1}{\delta} t^{-\alpha/2}\right\} \tag{5.6.34}$$

$$\mathsf{Bd}_{\pm,\downarrow}(\delta) := \left\{\left|\mathfrak{h}_{qt,\downarrow}^{(1)}\big(q^{-2/3}(\Phi \pm t^{-\alpha})\big) - \mathfrak{h}_{qt,\downarrow}^{(1)}\big(\Phi q^{-2/3}\big)\right| \le \tfrac{1}{\delta} t^{-\alpha/2}\right\},$$

Finally we consider the gap events that provide a gap between the first curve and the second curve for each of the line ensemble:

$$\mathsf{Gap}_{M,\uparrow}(\delta) := \left\{p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}\big(\Phi p^{-2/3}\big) \ge p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}\big(\Phi p^{-2/3}\big) + \delta\right\}, \tag{5.6.35}$$

$$\mathsf{Gap}_{M,\downarrow}(\delta) := \left\{q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(1)}\big(\Phi q^{-2/3}\big) \ge q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(2)}\big(\Phi q^{-2/3}\big) + \delta\right\}. \tag{5.6.36}$$

We next define the 'rise' events which roughly says the second curves $\mathfrak{h}_{pt,\uparrow}^{(1)}$ and $\mathfrak{h}_{qt,\downarrow}^{(2)}$ of the line ensembles does not rise too much on a small interval of length $2t^{-\alpha}$ around $\Phi p^{-2/3}$ and $\Phi q^{-2/3}$ respectively.

$$\mathsf{Rise}_{M,\uparrow}(\delta) := \left\{\sup_{x\in[-t^{-\alpha},t^{-\alpha}]} p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}\big(\Phi p^{-2/3} + x\big) \le p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}\big(\Phi p^{-2/3}\big) + \tfrac{\delta}{4}\right\}, \tag{5.6.37}$$

$$\mathsf{Rise}_{M,\downarrow}(\delta) := \left\{\sup_{x\in[-t^{-\alpha},t^{-\alpha}]} q^{1/3}\mathfrak{h}_{qt,\downarrow}^{(2)}\big(\Phi q^{-2/3} + x\big) \le q^{1/3}\mathfrak{h}_{pt,\downarrow}^{(2)}\big(\Phi q^{-2/3}\big) + \tfrac{\delta}{4}\right\}. \tag{5.6.38}$$

Bd, Gap, Rise type events and their significance are discussed later in Subsection 5.6.2 in greater details. See also Figure 5.9 and its caption for explanation of some of these events. We put all the above events into one final event:

$$\mathsf{Nice}_M(\delta) := \left\{\mathsf{ArMx}(\delta) \cap \bigcap_{x\in\{\uparrow,\downarrow\}} \mathsf{Bd}_x(\delta) \cap \mathsf{Gap}_{M,x}(\delta) \cap \mathsf{Rise}_{M,x}(\delta)\right\}. \tag{5.6.39}$$

All the above events are dependent on t. But we have suppressed this dependence from the notations. The $\mathsf{Nice}_M(\delta)$ turns out to be a favorable event. We isolate this fact as a lemma below.

Lemma 5.6.2. *For any $M > 0$, under the above setup we have*

$$\liminf_{\delta\downarrow 0}\liminf_{t\to\infty} \mathbf{P}_t\left(\mathsf{Nice}_M(\delta)\right) = 1. \tag{5.6.40}$$

We postpone the proof of this technical lemma to Section 5.6.2 and for the moment we continue with the current proof of Proposition 5.6.1 assuming its validity.

Conditioning with respect to large boundaries

As alluded in Subsection 5.6.2, the proof involves conditioning on different σ-fields successively. We now specify all the different σ-fields that we will use throughout the proof. Set $\alpha = \frac{1}{6}$. We consider the random interval

$$K_t := (\Phi - t^{-\alpha}, \Phi + t^{-\alpha}). \tag{5.6.41}$$

Let us define:

$$\mathcal{F}_1 := \sigma\left(\left\{\mathfrak{h}_{pt,\uparrow}^{(1)}(p^{-2/3}x), \mathfrak{h}_{qt,\downarrow}^{(1)}(q^{-2/3}x)\right\}_{x\in(-M,M)^c}, \left\{\mathfrak{h}_{pt,\uparrow}^{(2)}(x), \mathfrak{h}_{qt,\downarrow}^{(2)}(x)\right\}_{x\in\mathbb{R}}\right) \tag{5.6.42}$$

$$\mathcal{F}_2 := \sigma\left(\Phi, \mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3}), \mathfrak{h}_{qt,\downarrow}^{(1)}(\Phi q^{-2/3})\right), \tag{5.6.43}$$

$$\mathcal{F}_3 := \sigma\left(\left\{\mathfrak{h}_{pt,\uparrow}^{(1)}(p^{-2/3}x), \mathfrak{h}_{qt,\downarrow}^{(1)}(q^{-2/3}x)\right\}_{x\in K_t^c}\right). \tag{5.6.44}$$

In this step we perform conditioning w.r.t. $\mathcal{F}_1$ for the expression on the l.h.s. of (5.6.31). We denote $\mathbf{P}_t(A) := \mathbf{P}\big((D_{M,t,\uparrow}(\cdot), D_{M,t,\downarrow}(\cdot)) \in A\big)$. Taking the $\mathsf{Nice}_M(\delta)$ event defined in (5.6.39) under consideration, upon conditioning with $\mathcal{F}_1$ we have the following upper and lower bounds:

$$\mathbf{P}_t(A) \ge \mathbf{P}_t(\mathsf{Nice}_M(\delta), A) = \mathbf{E}_t\left[\mathbf{P}_t(\mathsf{Nice}_M(\delta), A \mid \mathcal{F}_1)\right], \tag{5.6.45}$$

$$\mathbf{P}_t(A) \le \mathbf{P}_t(\mathsf{Nice}_M(\delta), A) + \mathbf{P}_t(\neg\mathsf{Nice}_M(\delta)) = \mathbf{E}_t\left[\mathbf{P}_t(\mathsf{Nice}_M(\delta), A \mid \mathcal{F}_1)\right] + \mathbf{P}_t(\neg\mathsf{Nice}_M(\delta)). \tag{5.6.46}$$

Note that the underlying measure consists of the mutually independent $\mathfrak{h}_{pt,\uparrow}^{(1)}(\cdot)$ and $\mathfrak{h}_{qt,\downarrow}^{(1)}(\cdot)$ which by Proposition 6.5.1 satisfy $\mathbf{H}_{pt}$ and $\mathbf{H}_{qt}$ Brownian Gibbs property respectively. Applying the respectively Brownian Gibbs properties and following (5.2.3) we have

$$\mathbf{P}_t(\mathsf{Nice}_M(\delta), A \mid \mathcal{F}_1) = \frac{\mathbf{E}_{\mathrm{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta),A} W_{\uparrow} W_{\downarrow}]}{\mathbf{E}_{\mathrm{free},t}[W_{\uparrow} W_{\downarrow}]}. \tag{5.6.47}$$

Here

$$W_{\uparrow} := \exp\left(-t^{2/3} \int_{-M}^{M} \exp\left(t^{1/3}\left[p^{1/3}\mathfrak{h}^{(2)}_{pt,\uparrow}(p^{-2/3}x) - p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x)\right]\right) \mathrm{d}x\right) \tag{5.6.48}$$

and

$$W_{\downarrow} := \exp\left(-t^{2/3} \int_{-M}^{M} \exp\left(t^{1/3}\left[q^{1/3}\mathfrak{h}^{(2)}_{qt,\downarrow}(q^{-2/3}x) - q^{1/3}\mathfrak{h}^{(1)}_{qt,\downarrow}(q^{-2/3}x)\right]\right) \mathrm{d}x\right). \tag{5.6.49}$$

In (5.6.47), $\mathbf{P}_{\mathrm{free},t}$ and $\mathbf{E}_{\mathrm{free},t}$ are the probability and the expectation operator respectively corresponding to the joint 'free' law for $(p^{1/3}\mathfrak{h}_{pt,\uparrow}(p^{-2/3}x), q^{1/3}\mathfrak{h}_{qt,\downarrow}(q^{-2/3}x))_{x\in[-M,M]}$ which by Brownian scaling is given by a pair of independent Brownian bridges $(B_1(\cdot), B_2(\cdot))$ on $[-M, M]$ with starting points $(p^{1/3}\mathfrak{h}_{pt,\uparrow}(-Mp^{-2/3}), q^{1/3}\mathfrak{h}_{qt,\downarrow}(-Mq^{-2/3}))$ and endpoints $(q^{1/3}\mathfrak{h}_{pt,\uparrow}(Mp^{-2/3}), q^{1/3}\mathfrak{h}_{qt,\downarrow}(Mq^{-2/3}))$.

Conditioning with respect to maximum data and small boundaries

In this subsection we perform conditioning on the numerator of r.h.s. of (5.6.47) w.r.t. $\mathcal{F}_2$ and $\mathcal{F}_3$ defined in (5.6.43) and (5.6.44). Recall that by Proposition 5.4.10, upon conditioning Brownian bridges on $\mathcal{F}_2$, the conditional laws around the joint local maximizer Φ over $[-M, M]$ is now given by two $\mathsf{NonInt\text{-}BrBridge}$s (defined in Definition 5.4.4) with appropriate lengths and endpoints. Indeed, based on Proposition 5.4.10, given $\mathcal{F}_1, \mathcal{F}_2$, we may construct the conditional laws for the two functions on $[-M, M]$:

Definition 5.6.3 (Nlarge Law)**.** Consider two independent $\mathsf{NonInt\text{-}BrBridge}$ V_ℓ^{large} and V_r^{large} with following description:

1. V_ℓ^{large} is a $\mathsf{NonInt\text{-}BrBridge}$ on $[0, \Phi + M]$ ending at

$$\left(p^{1/3}\left[\mathfrak{h}^{(1)}_{pt,\uparrow}(\Phi p^{-2/3}) - \mathfrak{h}^{(1)}_{pt,\uparrow}(-Mp^{-2/3})\right], q^{1/3}\left[\mathfrak{h}^{(1)}_{qt,\downarrow}(-Mq^{-2/3}) - \mathfrak{h}^{(1)}_{qt,\downarrow}(\Phi q^{-2/3})\right]\right),$$

2. V_r^{large} is a $\mathsf{NonInt\text{-}BrBridge}$ on $[0, M - \Phi]$ ending at

$$\left(p^{1/3}\left[\mathfrak{h}^{(1)}_{pt,\uparrow}(\Phi p^{-2/3}) - \mathfrak{h}^{(1)}_{pt,\uparrow}(Mp^{-2/3})\right], q^{1/3}\left[\mathfrak{h}^{(1)}_{qt,\downarrow}(Mq^{-2/3}) - \mathfrak{h}^{(1)}_{qt,\downarrow}(\Phi q^{-2/3})\right]\right).$$

We then define $B^{\mathsf{large}} : [-M, M] \to \mathbb{R}^2$ as follows:

$$B^{\mathsf{large}}(x) = \begin{cases} V_\ell(\Phi - x) & x \in [-M, \Phi] \\ V_r(x - \Phi) & x \in [\Phi, M] \end{cases}.$$

We denote the expectation and probability operator under above law for B^{large} (which depends on $\mathcal{F}_1, \mathcal{F}_2$) as $\mathbf{E}_{\mathsf{Nlarge}|2,1}$ and $\mathbf{P}_{\mathsf{Nlarge}|2,1}$.

Thus we may write

$$\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta),A} W_\uparrow W_\downarrow] = \mathbf{E}_{\text{free},t}[\mathbf{E}_{\mathsf{Nlarge}|2,1}[\mathbf{1}_{\mathsf{Nice}_M(\delta),A} W_\uparrow W_\downarrow]]. \tag{5.6.50}$$

Since $\mathsf{NonInt\text{-}BrBridge}$s are Markovian, we may condition further upon $\mathcal{F}_3$ to get $\mathsf{NonInt\text{-}BrBridge}$s again but on a smaller interval. To precisely define the law, we now give the following definitions:

Definition 5.6.4 (Nsmall law)**.** Consider two independent $\mathsf{NonInt\text{-}BrBridge}$ V_ℓ^{small} and V_r^{small} with the following descriptions:

1. V_ℓ^{small} is a $\mathsf{NonInt\text{-}BrBridge}$ on $[0, t^{-\alpha}]$ ending at

$$\left(p^{1/3}\left[\mathfrak{h}^{(1)}_{pt,\uparrow}(\Phi p^{-2/3}) - \mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}(\Phi - t^{-\alpha}))\right], q^{1/3}\left[\mathfrak{h}^{(1)}_{qt,\downarrow}(q^{-2/3}(\Phi - t^{-\alpha})) - \mathfrak{h}^{(1)}_{qt,\downarrow}(\Phi q^{-2/3})\right]\right),$$

2. V_r^{small} is a $\mathsf{NonInt\text{-}BrBridge}$ on $[0, t^{-\alpha}]$ ending at

$$\left(p^{1/3}\left[\mathfrak{h}^{(1)}_{pt,\uparrow}(\Phi p^{-2/3}) - \mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}(\Phi + t^{-\alpha}))\right], q^{1/3}\left[\mathfrak{h}^{(1)}_{qt,\downarrow}(q^{-2/3}(\Phi + t^{-\alpha})) - \mathfrak{h}^{(1)}_{qt,\downarrow}(\Phi q^{-2/3})\right]\right).$$

We then define $B^{\mathsf{small}} : [\Phi + t^{-\alpha}, \Phi - t^{-\alpha}] \to \mathbb{R}^2$ as follows:

$$B^{\mathsf{small}}(x) = \begin{cases} V_\ell(\Phi - x) & x \in [\Phi - t^{-\alpha}, \Phi] \\ V_r(x - \Phi) & x \in [\Phi, \Phi + t^{-\alpha}] \end{cases}.$$

We denote the the expectation and probability operators under the above law for B^{small} (which depends on $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$) as $\mathbf{E}_{\mathsf{Nsmall}|3,2,1}$ and $\mathbf{P}_{\mathsf{Nsmall}|3,2,1}$ respectively.

We thus have

$$\text{r.h.s. of (5.6.50)} = \mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}\mathbf{E}_{\mathsf{Nsmall}|3,2,1}[\mathbf{1}_A W_\uparrow W_\downarrow]]. \tag{5.6.51}$$

The $\mathbf{1}_{\mathsf{Nice}_M(\delta)}$ comes of the interior expectation above as $\mathsf{Nice}_M(\delta)$ is measurable w.r.t. $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ (see its definition in (5.6.39)).

Next note that due to the definition of $W_\uparrow, W_\downarrow$ from (5.6.48) and (5.6.49), we may extract certain parts of it which are measurable w.r.t. $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Indeed, we can write $W_\uparrow = W_{\uparrow,1}W_{\uparrow,2}$ and $W_\downarrow = W_{\downarrow,1}W_{\downarrow,2}$ where

$$W_{\uparrow,1} := \exp\left(-t^{2/3}\int_{K_t} \exp\left(t^{1/3}\left[p^{1/3}\mathfrak{h}^{(2)}_{pt,\uparrow}(p^{-2/3}x) - p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x)\right]\right)\mathrm{d}x\right) \tag{5.6.52}$$

$$W_{\uparrow,2} := \exp\left(-t^{2/3}\int_{[-M,M]\cap K_t^c} \exp\left(t^{1/3}\left[p^{1/3}\mathfrak{h}^{(2)}_{pt,\uparrow}(p^{-2/3}x) - p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x)\right]\right)\mathrm{d}x\right),$$

and

$$W_{\downarrow,1} := \exp\left(-t^{2/3}\int_{K_t} \exp\left(t^{1/3}\left[q^{1/3}\mathfrak{h}^{(2)}_{qt,\downarrow}(q^{-2/3}x) - q^{1/3}\mathfrak{h}^{(1)}_{qt,\downarrow}(q^{-2/3}x)\right]\right)\mathrm{d}x\right). \tag{5.6.53}$$

$$W_{\downarrow,2} := \exp\left(-t^{2/3}\int_{[-M,M]\cap K_t^c} \exp\left(t^{1/3}\left[q^{1/3}\mathfrak{h}^{(2)}_{qt,\downarrow}(q^{-2/3}x) - q^{1/3}\mathfrak{h}^{(1)}_{qt,\downarrow}(q^{-2/3}x)\right]\right) \mathrm{d}x\right),$$

where recall K_t from (5.6.41). The key observation is that $W_{\uparrow,2}, W_{\downarrow,2}$ are measurable w.r.t. $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. Thus we have

$$\text{r.h.s. of (5.6.51)} = \mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)} W_{\uparrow,2} W_{\downarrow,2} \cdot \mathbf{E}_{\mathsf{Nsmall}|3,2,1}[\mathbf{1}_A W_{\uparrow,1} W_{\downarrow,1}]]. \tag{5.6.54}$$

Remark 5.6.5. It is crucial to note that in (5.6.51) the event $\mathsf{Nice}_M(\delta)$ includes the event $\mathsf{ArMx}(\delta)$ defined in (5.6.32). Indeed, the $\mathsf{ArMx}(\delta)$ event is measurable w.r.t. $\mathcal{F}_1 \cup \mathcal{F}_2$ and ensures that $[\Phi - t^{-\alpha}, \Phi + t^{-\alpha}] \subset [-M, M]$ for all large enough t, which is essential for going from Nlarge law to Nsmall law. Thus such a decomposition is not possible for $\mathbf{E}_{\text{free},t}[W_\uparrow W_\downarrow]$ which appears in the denominator of r.h.s. of (5.6.47). Nonetheless, we may still provide a lower bound for $\mathbf{E}_{\text{free},t}[W_\uparrow W_\downarrow]$ as follows:

$$\mathbf{E}_{\text{free},t}[W_\uparrow W_\downarrow] \geq \mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)} W_\uparrow W_\downarrow] = \mathbf{E}_{\text{free},t}[W_{\uparrow,2} W_{\downarrow,2} \mathbf{1}_{\mathsf{Nice}_M(\delta)} \cdot \mathbf{E}_{\mathsf{Nsmall}|3,2,1}[W_{\uparrow,1} W_{\downarrow,1}]]. \tag{5.6.55}$$

With the deductions in (5.6.54) and (5.6.55), we now come to the task of analyzing $W_{\uparrow,1} W_{\downarrow,1}$ under Nsmall law. The following lemma ensures that on $\mathsf{Nice}_M(\delta)$, $W_{\uparrow,1} W_{\downarrow,1}$ is close to 1 under Nsmall law.

Lemma 5.6.6. *There exist $t_0(\delta) > 0$ such that for all $t \geq t_0$ we have*

$$\mathbf{1}_{\mathsf{Nice}_M(\delta)} \mathbf{P}_{\mathsf{Nsmall}|3,2,1}(W_{\uparrow,1} W_{\downarrow,1} > 1 - \delta) \geq \mathbf{1}_{\mathsf{Nice}_M(\delta)}(1 - \delta). \tag{5.6.56}$$

This allow us to ignore $W_{\uparrow,1} W_{\downarrow,1}$, in $\mathbf{E}_{\mathsf{Nsmall}|3,2,1}[\mathbf{1}_A W_{\uparrow,1} W_{\downarrow,1}]$. Hence it suffices to study $\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A)$. The following lemma then compares this conditional probability with that of DBM.

Lemma 5.6.7. *There exist $t_0(\delta) > 0$ such that for all $t \geq t_0$ we have*

$$\mathbf{1}_{\mathsf{Nice}_M(\delta)}|\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A) - \tau(A)| \leq \mathbf{1}_{\mathsf{Nice}_M(\delta)} \cdot \delta, \tag{5.6.57}$$

where $\tau(A) := \mathbf{P}(\mathcal{D}(\cdot) \in A)$, $\mathcal{D}$ being a two-sided DBM *defined in the statement of Proposition 5.6.1.*

We prove these two lemmas in Section 5.6.2. For now, we proceed with the current proof of (5.6.31) in the next section.

Matching Lower and Upper Bounds

In this subsection, we complete the proof of (5.6.31) by providing matching lower and upper bounds in the two steps below. We assume throughout this subsection that t is large enough, so that (5.6.56) and (5.6.57) holds.

Step 1: Lower Bound. We start with (5.6.45). Following the expression in (5.6.47), and our deductions in (5.6.50), (5.6.51), (5.6.54) we see that

$$\begin{aligned}
\mathbf{P}_t(A) &\geq \mathbf{E}_t\left[\mathbf{P}_t(\mathsf{Nice}_M(\delta), A \mid \mathcal{F}_1)\right] \\
&= \mathbf{E}\left[\frac{\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)} W_{\uparrow,2} W_{\downarrow,2} \cdot \mathbf{E}_{\mathsf{Nsmall}|3,2,1}[\mathbf{1}_A W_{\uparrow,1} W_{\downarrow,1}]]}{\mathbf{E}_{\text{free},t}[W_\uparrow W_\downarrow]}\right] && (5.6.58)\\
&\geq (1-\delta)\mathbf{E}_t\left[\frac{\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)} W_{\uparrow,2} W_{\downarrow,2} \cdot \mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A, W_{\uparrow,1} W_{\downarrow,1} > 1-\delta)]}{\mathbf{E}_{\text{free},t}[W_\uparrow W_\downarrow]}\right] && (5.6.59)
\end{aligned}$$

where in the last inequality we used the fact $W_{\uparrow,1} W_{\downarrow,1} \leq 1$. Now applying Lemma 5.6.6 and Lemma 5.6.7 successively we get

$$\begin{aligned}
&\mathbf{1}_{\mathsf{Nice}_M(\delta)} \mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A, W_{\uparrow,1} W_{\downarrow,1} > 1-\delta) \\
&\geq \mathbf{1}_{\mathsf{Nice}_M(\delta)}[\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A) - \mathbf{P}_{\mathsf{Nsmall}|3,2,1}(W_{\uparrow,1} W_{\downarrow,1} \leq 1-\delta)] \\
&\geq \mathbf{1}_{\mathsf{Nice}_M(\delta)}[\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A) - \delta] \\
&\geq \mathbf{1}_{\mathsf{Nice}_M(\delta)}[\tau(A) - 2\delta]
\end{aligned}$$

where recall $\tau(A) = \mathbf{P}(\mathcal{D}(\cdot) \in A)$. As $W_{\uparrow,1}W_{\downarrow,1} \le 1$ and probabilities are nonnegative, following the above inequalities we have

$$\mathbf{1}_{\mathsf{Nice}_M(\delta)}\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A, W_{\uparrow,1}W_{\downarrow,1} > 1-\delta) \ge \max\{0, \tau(A) - 2\delta\}\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow,1}W_{\downarrow,1}.$$

Substituting the above bound back to (5.6.59) and using the fact that $W_{\uparrow,2}W_{\downarrow,2}W_{\uparrow,1}W_{\downarrow,1} = W_{\uparrow}W_{\downarrow}$, we get

$$\begin{aligned}\mathbf{P}_t(A) &\ge (1-\delta)\max\{0, \tau(A) - 2\delta\}\mathbf{E}_t\left[\frac{\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow}W_{\downarrow}]}{\mathbf{E}_{\text{free},t}[W_{\uparrow}W_{\downarrow}]}\right] \\ &= (1-\delta)\max\{0, \tau(A) - 2\delta\}\mathbf{P}_t(\mathsf{Nice}_M(\delta)).\end{aligned}$$

In view of Lemma 5.6.2, taking $\liminf_{t\to\infty}$ followed by $\liminf_{\delta\downarrow 0}$ we get that $\liminf_{t\to\infty}\mathbf{P}_t(A) \ge \tau(A)$. This proves the lower bound.

Step 2: Upper Bound. We start with (5.6.46). Using the equality in (5.6.58) we get

$$\begin{aligned}\mathbf{P}_t(A) &\le \mathbf{E}\left[\frac{\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow,2}W_{\downarrow,2}\cdot\mathbf{E}_{\mathsf{Nsmall}|3,2,1}[\mathbf{1}_A W_{\uparrow,1}W_{\downarrow,1}]]}{\mathbf{E}_{\text{free},t}[W_{\uparrow}W_{\downarrow}]}\right] + \mathbf{P}_t(\neg\mathsf{Nice}_M(\delta)) \\ &\le \mathbf{E}\left[\frac{\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow,2}W_{\downarrow,2}\cdot\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(A)]}{\mathbf{E}_{\text{free},t}[W_{\uparrow}W_{\downarrow}]}\right] + \mathbf{P}_t(\neg\mathsf{Nice}_M(\delta)) \\ &\le (\tau(A)+\delta)\mathbf{E}\left[\frac{\mathbf{E}_{\text{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow,2}W_{\downarrow,2}]}{\mathbf{E}_{\text{free},t}[W_{\uparrow}W_{\downarrow}]}\right] + \mathbf{P}_t(\neg\mathsf{Nice}_M(\delta)). \end{aligned} \tag{5.6.60}$$

Let us briefly justify the inequalities presented above. Going from first line to second line we used the fact $W_{\uparrow,1}W_{\downarrow,1} \le 1$. The last inequality follows from Lemma 5.6.7 where recall that $\tau(A) = \mathbf{P}(\mathcal{D}(\cdot) \in A)$. Now note that by Lemma 5.6.6, on $\mathsf{Nice}_M(\delta)$,

$$\begin{aligned}\mathbf{E}_{\mathsf{Nsmall}|3,2,1}[W_{\uparrow,1}W_{\downarrow,1}] &\ge \mathbf{E}_{\mathsf{Nsmall}|3,2,1}[\mathbf{1}_{W_{\uparrow,1}W_{\downarrow,1}\ge 1-\delta}\cdot W_{\uparrow,1}W_{\downarrow,1}] \\ &\ge (1-\delta)\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(W_{\uparrow,1}W_{\downarrow,1} \ge 1-\delta) \ge (1-\delta)^2.\end{aligned}$$

Using the expression from (5.6.55) we thus have

$$\begin{aligned}\mathbf{E}_{\mathrm{free},t}[W_{\uparrow}W_{\downarrow}] &\geq \mathbf{E}_{\mathrm{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow,2}W_{\downarrow,2}\cdot\mathbf{E}_{\mathsf{Nsmall}|3,2,1}[W_{\uparrow,1}W_{\downarrow,1}]] \\ &\geq (1-\delta)^2\mathbf{E}_{\mathrm{free},t}[\mathbf{1}_{\mathsf{Nice}_M(\delta)}W_{\uparrow,2}W_{\downarrow,2}].\end{aligned}$$

Going back to (5.6.60), this forces

$$\text{r.h.s. of (5.6.60)} \leq \frac{\tau(A)+\delta}{(1-\delta)^2}+\mathbf{P}_t(\neg\mathsf{Nice}_M(\delta)).$$

In view of Lemma 5.6.2, taking $\limsup_{t\to\infty}$, followed by $\limsup_{\delta\downarrow 0}$ in above inequality we get that $\limsup_{t\to\infty}\mathbf{P}_t(A)\leq\tau(A)$. Along with the matching lower bound obtained in **Step 1** above, this establishes (5.6.31).

Proof of Lemma 5.6.2

Recall from (5.6.39) that $\mathsf{Nice}_M(\delta)$ event is an intersection of several kinds of events. To show (5.6.40), it suffices to prove the same for each of the events. That is, given an event E which is part of $\mathsf{Nice}_M(\delta)$ we will show

$$\limsup_{\delta\to\infty}\limsup_{t\to\infty}\mathbf{P}(\mathsf{E})=1. \tag{5.6.61}$$

Below we analyze each such possible choices for E separately.

$\mathsf{ArMx}(\delta)$ **event.** Recall $\mathsf{ArMx}(\delta)$ event from (5.6.32). As noted in (5.3.9),

$$\mathcal{M}_{p,t}^{M}\xrightarrow{d}\operatorname*{argmax}_{x\in[-M,M]}\mathcal{A}(x),$$

where $\mathcal{A}$ is defined in (5.3.8). Since $\mathcal{A}$ restricted to $[-M,M]$ is absolutely continuous with Brownian motion with appropriate diffusion coefficients, $\operatorname{argmax}_{x\in[-M,M]}\mathcal{A}(x)\in(-M,M)$ almost surely. In other words, maximum is not attained on the boundaries almost surely. But then

$$\begin{aligned}\liminf_{\delta\downarrow 0}\liminf_{t\to\infty}\mathbf{P}(\mathsf{ArMx}(\delta)) &= \liminf_{\delta\downarrow 0}\mathbf{P}(\operatorname*{argmax}_{x\in[-M,M]}\mathcal{A}(x)\in[-M+\delta,M-\delta]) \\ &= \mathbf{P}(\operatorname*{argmax}_{x\in[-M,M]}\mathcal{A}(x)\in(-M,M)) = 1.\end{aligned}$$

This proves (5.6.61) with $\mathsf{E}\mapsto\mathsf{ArMx}(\delta)$.

<u>$\mathsf{Bd}_{\uparrow}(\delta), \mathsf{Bd}_{\downarrow}(\delta)$ **events.**</u> We first define

$$\begin{aligned}\mathsf{Tight}_{\pm,\uparrow}(\lambda) &:= \left\{p^{1/3}\left|\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3})-\mathfrak{h}_{pt,\uparrow}^{(1)}(\pm Mp^{-2/3})\right|\le\tfrac{1}{\lambda}\right\}, \\ \mathsf{Tight}_{\pm,\downarrow}(\lambda) &:= \left\{q^{1/3}\left|\mathfrak{h}_{qt,\downarrow}^{(1)}(\Phi q^{-2/3})-\mathfrak{h}_{qt,\downarrow}^{(1)}(\pm Mq^{-2/3})\right|\le\tfrac{1}{\lambda}\right\},\end{aligned}$$

and set

$$\mathsf{Sp}(\lambda) := \mathsf{ArMx}(\lambda)\cap\mathsf{Tight}_{+,\uparrow}(\lambda)\cap\mathsf{Tight}_{-,\uparrow}(\lambda)\cap\mathsf{Tight}_{+,\downarrow}(\lambda)\cap\mathsf{Tight}_{-,\downarrow}(\lambda) \tag{5.6.62}$$

where $\mathsf{ArMx}(\lambda)$ is defined in (5.6.32). We claim that

$$\limsup_{\lambda\downarrow 0}\limsup_{t\to\infty}\mathbf{P}(\neg\mathsf{Sp}(\lambda))) = 0. \tag{5.6.63}$$

Let us assume (5.6.63) for the time being and consider the main task of analyzing the probability of the events $\mathsf{Bd}_{\uparrow}(\delta), \mathsf{Bd}_{\downarrow}(\delta)$ defined in (5.6.33). We have $\mathsf{Bd}_{\uparrow}(\delta) = \mathsf{Bd}_{+\uparrow}(\delta)\cap\mathsf{Bd}_{-,\uparrow}(\delta)$ where $\mathsf{Bd}_{\pm,\uparrow}(\delta)$ is defined in (5.6.34). Let us focus on $\mathsf{Bd}_{+,\uparrow}(\delta)$. Recall the σ-fields $\mathcal{F}_1, \mathcal{F}_2$ from (5.6.42) and (5.6.43). As described in Subsection 5.6.2, upon conditioning on $\mathcal{F}_1\cup\mathcal{F}_2$, the conditional law on $[-M,M]$ are given by Nlarge defined in Definition 5.6.3, which are made up of $\mathsf{NonInt\text{-}BrBridges}$ $V_{\ell}^{\mathsf{large}}, V_r^{\mathsf{large}}$ defined in Definition 5.6.3.

Note that applying Markov inequality conditionally we have

$$\mathbf{1}_{\mathsf{Sp}(\lambda)}\mathbf{P}\left(\mathsf{Bd}_{+,\uparrow}(\delta)\mid\mathcal{F}_1,\mathcal{F}_2\right)$$

$$= \mathbf{1}_{\mathsf{Sp}(\lambda)} \cdot \mathbf{P}\left(|\mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}(\Phi + t^{-\alpha})) - \mathfrak{h}^{(1)}_{pt,\uparrow}(\Phi p^{-2/3})| > \tfrac{1}{\delta} t^{-\alpha/2} \mid \mathcal{F}_1, \mathcal{F}_2\right)$$
$$\leq \mathbf{1}_{\mathsf{Sp}(\lambda)} \cdot \delta^2 t^{2\alpha} \cdot \mathbf{E}_{\mathsf{Nlarge}|2,1}\left[[V^{\mathsf{large}}_{r,1}(p^{-2/3}t^{-\alpha})]^4\right]$$

However, on $\mathbf{1}_{\mathsf{Sp}(\lambda)}$, the $\mathsf{NonInt\text{-}BrBridge}$ has length bounded away from zero and the endpoints are tight. Applying (5.5.20) with $K \mapsto 2, t \mapsto 1, s \mapsto 0, n \mapsto p^{2/3}t^{\alpha}, M \mapsto 1/\lambda$, for all large enough t we get $\mathbf{E}_{\mathsf{Nlarge}|2,1}\left[[V^{\mathsf{large}}_{r,1}(p^{-2/3}t^{-\alpha})]^4\right] \leq \mathrm{C}_{p,\lambda} t^{-2\alpha}$. Thus,

$$\limsup_{t\to\infty} \mathbf{P}\left(\neg\mathsf{Bd}_{+,\uparrow}(\delta)\right) \leq \limsup_{t\to\infty} \mathbf{P}(\neg\mathsf{Sp}(\lambda)) + \delta^2 \mathrm{C}_{p,\lambda}.$$

Taking $\delta \downarrow 0$, followed by $\lambda \downarrow 0$, in view of (5.6.63) we get $\limsup_{\delta\downarrow 0}\limsup_{t\to\infty} \mathbf{P}(\neg\mathsf{Bd}_{+,\uparrow}(\delta)) = 0$. Similarly one can conclude $\limsup_{\delta\downarrow 0}\limsup_{t\to\infty} \mathbf{P}(\neg\mathsf{Bd}_{-,\uparrow}(\delta)) = 0$ Thus, this two together yields $\liminf_{\delta\downarrow 0}\liminf_{t\to\infty} \mathbf{P}(\mathsf{Bd}_{\uparrow}(\delta)) = 1$. By exactly the same approach one can derive that $\mathbf{P}(\mathsf{Bd}_{\downarrow}(\delta))$ goes to 1 under the same iterated limit. Thus it remains to show (5.6.63).

Let us recall from (5.6.62) that $\mathsf{Sp}(\lambda)$ event is composed of four tightness events and one event about the argmax. We first claim that $\limsup_{\lambda\downarrow 0}\limsup_{t\to\infty} \mathbf{P}(\mathsf{Tight}_{x,y}(\lambda)) = 1$ for each $x \in \{+,-\}$ and $y \in \{\uparrow,\downarrow\}$. The earlier analysis of $\mathsf{ArMx}(\lambda)$ event in (5.6.62) then enforces (5.6.63). Since all the tightness events are similar, it suffices to prove any one of them say $\mathsf{Tight}_{+,\uparrow}$. By Proposition 6.5.1 we have the distributional convergence of $2^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(2^{1/3}x)$ to $\mathcal{A}_1(x)$ in the uniform-on-compact topology, where $\mathcal{A}_1(\cdot)$ is the parabolic Airy$_2$ process. As $\Phi \in [-M, M]$, we thus have

$$\limsup_{t\to\infty} \mathbf{P}(\mathsf{Tight}_{+,\uparrow}(\lambda)) \leq \limsup_{t\to\infty} \mathbf{P}\left(p^{1/3} \sup_{x\in[-M,M]} \left|\mathfrak{h}^{(1)}_{pt,\uparrow}(xp^{-2/3}) - \mathfrak{h}^{(1)}_{pt,\uparrow}(Mp^{-2/3})\right| \leq \tfrac{1}{\lambda}\right)$$
$$= \mathbf{P}\left(p^{1/3} \sup_{|x|\leq 2^{-1/3}M} \left|\mathcal{A}_1(xp^{-2/3}) - \mathcal{A}_1(2^{-1/3}Mp^{-2/3})\right| \leq \tfrac{2^{1/3}}{\lambda}\right).$$

For fixed p, M, by tightness of parabolic Airy$_2$ process on a compact interval, the last expression goes to one as $\lambda \downarrow 0$, which is precisely what we wanted to show.

$\underline{\mathsf{Gap}_{M,\uparrow}(\delta), \mathsf{Gap}_{M,\downarrow}(\delta)\ \textbf{events.}}$ Recall the definitions of $\mathsf{Gap}_{M,\uparrow}(\delta)$ and $\mathsf{Gap}_{M,\downarrow}(\delta)$ from(5.6.35) and (5.6.36).We begin with the proof of $\mathsf{Gap}_{M,\uparrow}(\delta)$. Let

$$\mathsf{Diff}_{M,\uparrow}(\delta) := \left\{ \inf_{|x|\le M} p^{1/3} \left(\mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x) - \mathfrak{h}^{(2)}_{pt,\uparrow}(p^{-2/3}x) \right) \ge \delta \right\}.$$

Note that $\Phi \in [-M, M]$. Thus $\mathsf{Gap}_{M,\uparrow}(\delta) \supset \mathsf{Diff}_{M,\uparrow}(\delta)$. Thus to show (5.6.61) with $\mathsf{E} \mapsto \mathsf{Gap}_{M,\uparrow}(\delta)$ it suffices to prove

$$\liminf_{\delta\downarrow 0} \liminf_{t\to\infty} \mathbf{P}(\mathsf{Diff}_{M,\uparrow}(\delta)) = 1, \tag{5.6.64}$$

We recall from Proposition 5.2.7 the distributional convergence of the KPZ line ensemble to the Airy line ensemble in the uniform-on-compact topology. By Skorokhod representation theorem, we may assume that our probability space is equipped with $\mathcal{A}_1(\cdot)$ and $\mathcal{A}_2(\cdot)$ such that almost surely as $t \to \infty$

$$\max_{i=1,2} \sup_{|x|\le Mp^{-2/3}} |2^{1/3}\mathfrak{h}^{(i)}_{t,\uparrow}(2^{1/3}x) - \mathcal{A}_i(x)| \to 0. \tag{5.6.65}$$

We thus have

$$\liminf_{t\to\infty} \mathbf{P}(\mathsf{Diff}_{M,\uparrow}(\delta)) = \mathbf{P}\left(\inf_{|x|\le M2^{-1/3}p^{-2/3}} p^{1/3} \left(\mathcal{A}_1(x) - \mathcal{A}_2(x)\right) \ge 2^{1/3}\delta \right). \tag{5.6.66}$$

As the Airy line ensemble is absolutely continuous w.r.t. non-intersecting Brownian motions, it is strictly ordered with touching probability zero (see (5.2.1)). Hence r.h.s. of (5.6.66) goes to zero as $\delta \downarrow 0$. This proves (5.6.64). The proof is similar for $\mathsf{Gap}_{M,\downarrow}(\delta)$.

$\underline{\mathsf{Rise}_{M,\uparrow}(\delta), \mathsf{Rise}_{M,\uparrow}(\delta)\ \textbf{events.}}$ Recall $\mathsf{Rise}_{M,\uparrow}(\delta), \mathsf{Rise}_{M,\uparrow}(\delta)$ events from (5.6.37) and (5.6.38). Due to their similarities, we only analyze the $\mathsf{Rise}_{M,\uparrow}(\delta)$ event. As with the previous case, we assume that our probability space is equipped with $\mathcal{A}_1(\cdot)$ and $\mathcal{A}_2(\cdot)$ (first two lines of the Airy line ensemble) such that almost surely as $t \to \infty$ (5.6.65) holds. Applying union bound we have

$$\begin{aligned}
\mathbf{P}\left(\neg \mathsf{Rise}_M(\delta)\right) &\le \mathbf{P}\left(\sup_{|x|\le Mp^{-2/3}} p^{1/3}|2^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}(2^{1/3}x)-\mathcal{A}_2(x)| \ge \tfrac{\delta}{16}\right) \\
&\qquad + \mathbf{P}\left(\neg \mathsf{Rise}_M(\delta), \sup_{|x|\le Mp^{-2/3}} p^{1/3}|2^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}(2^{1/3}x)-\mathcal{A}_2(x)| \le \tfrac{\delta}{16}\right) \\
&\le \mathbf{P}\left(\sup_{|x|\le Mp^{-2/3}} p^{1/3}|2^{1/3}\mathfrak{h}_{pt,\uparrow}^{(2)}(2^{1/3}x)-\mathcal{A}_2(x)| \ge \tfrac{\delta}{16}\right) \\
&\qquad + \mathbf{P}\Bigg(\sup_{\substack{x,y\in[-M,M]\\ |x-y|\le t^{-\alpha}}} p^{1/3}|\mathcal{A}_2(x)-\mathcal{A}_2(y)| \ge \tfrac{\delta}{8}\Bigg).
\end{aligned}$$

In the r.h.s. of above equation, the first term goes to zero as $t \to \infty$ by (5.6.65). The second term on the other hand goes to zero as $t \to \infty$ by modulus of continuity estimates for Airy line ensemble from Proposition 5.2.4. This shows, $\lim_{t\to\infty} \mathbf{P}(\mathsf{Rise}_{M,\uparrow}(\delta)) = 1$. Similarly one has $\lim_{t\to\infty} \mathbf{P}(\mathsf{Rise}_{M,\downarrow}(\delta)) = 1$ as well. This proves (5.6.61) for $\mathsf{E} \mapsto \mathsf{Rise}_{M,\uparrow}(\delta), \mathsf{Rise}_{M,\downarrow}(\delta)$.

We have thus shown (5.6.61) for all the events listed in (5.6.39). This establishes (5.6.40) concluding the proof of Lemma 5.6.2.

Proof of Lemma 5.6.6 and 5.6.7

In this subsection we prove Lemma 5.6.6 and 5.6.7.

Proof of Lemma 5.6.6. Recall $W_{\uparrow,1}$ and $W_{\downarrow,1}$ from (5.6.52) and (5.6.53) respectively. We claim that for all large enough t, on $\mathsf{Nice}_M(\delta)$ we have

$$\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(W_{\uparrow,1} > \sqrt{1-\delta}) \ge 1-\tfrac{1}{2}\delta, \quad \mathbf{P}_{\mathsf{Nsmall}|3,2,1}(W_{\downarrow,1} > \sqrt{1-\delta}) \ge 1-\tfrac{1}{2}\delta \tag{5.6.67}$$

simultaneously. (5.6.56) then follows via union bound. Hence we focus on proving (5.6.67). In the proof below we only focus on first part of (5.6.67) and the second one follows analogously. We

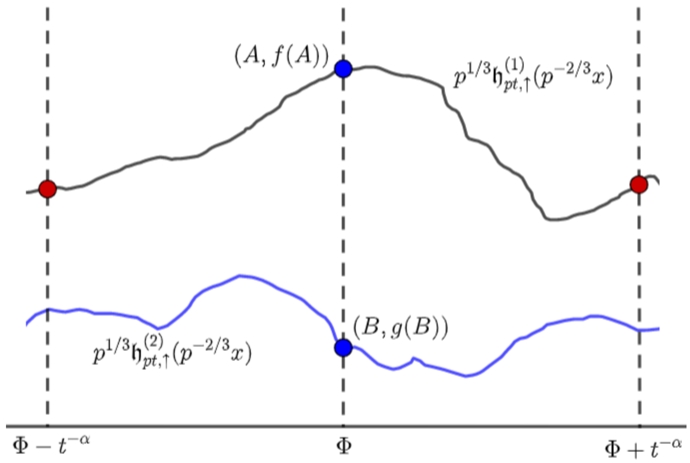

Figure 5.9: In the above figure we have plotted the curves $f(x) := p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x)$ (black) and $g(x) := p^{1/3}\mathfrak{h}^{(2)}_{pt,\uparrow}(p^{-2/3}x)$ (blue) restricted to the interval $K_t := (\Phi - t^{-\alpha}, \Phi + t^{-\alpha})$. For convenience, we have marked two blue points along with their values as $(A, f(A))$, $(B, g(B))$. $\mathsf{Gap}_{M,\uparrow}(\delta)$ defined in (5.6.35) denote the event that the blue points are separated by δ, i.e, $f(A) - g(B) \geq \delta$. The $\mathsf{Rise}_{M,\uparrow}(\delta)$ defined in (5.6.37) ensures *no* point on the blue curve (restricted to K_t) has value larger than $g(B) + \frac{1}{4}\delta$ (that is no significant rise). The $\mathsf{Bd}_{\uparrow}(\delta)$ event defined in (5.6.33) indicates the red points on the black curve are within $[f(A) - \frac{1}{\delta}t^{-\alpha/2}, f(A) + \frac{1}{\delta}t^{-\alpha/2}]$. The $\mathsf{Sink}_{\uparrow}(\delta)$ event defined in (5.6.68) ensures that *all* points on the black curve (restricted to K_t) have values larger than $f(A) - \frac{1}{4}\delta$ (that is no significant sink). Clearly then on $\mathsf{Sink}_{\uparrow}(\delta) \cap \mathsf{Rise}_{M,\uparrow}(\delta) \cap \mathsf{Gap}_{M,\uparrow}(\delta)$ for all $x \in K_t$, we have $f(x) - g(x) \geq f(A) - \frac{1}{4}\delta - g(B) - \frac{1}{4}\delta \geq \frac{1}{2}\delta$.

now define the 'sink' event:

$$\mathsf{Sink}_{\uparrow}(\delta) := \left\{ \inf_{x \in [-t^{-\alpha}, t^{-\alpha}]} p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(\Phi p^{-2/3} + x) \geq p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(\Phi p^{-2/3}) - \tfrac{\delta}{4} \right\}. \tag{5.6.68}$$

Recall $\mathsf{Rise}_{M,\uparrow}(\delta)$ and $\mathsf{Gap}_{M,\uparrow}(\delta)$ from (5.6.37) and (5.6.35). Note that on $\mathsf{Sink}_{\uparrow}(\delta) \cap \mathsf{Rise}_{M,\uparrow}(\delta) \cap \mathsf{Gap}_{M,\uparrow}(\delta)$ we have uniform separation between $\mathfrak{h}^{(1)}_{pt,\uparrow}$ and $\mathfrak{h}^{(2)}_{pt,\downarrow}$ on the interval $p^{-2/3}K_t$, that is

$$\inf_{x \in [\Phi - t^{-\alpha}, \Phi + t^{-\alpha}]} \left[p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x) - p^{1/3}\mathfrak{h}^{(2)}_{pt,\uparrow}(p^{-2/3}x) \right] \geq \tfrac{\delta}{2}. \tag{5.6.69}$$

See Figure 5.9 alongside its caption for further explanation of the above fact. But then (5.6.69) forces $W_{\uparrow,1} \geq \exp(-t^{2/3} 2t^{-\alpha} e^{-\frac{1}{4}t^{1/3}\delta})$ which can be made strictly larger than $\sqrt{1-\delta}$ for all large enough t. Thus,

$$\mathbf{1}_{\mathsf{Nice}_M(\delta)}\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(W_{\uparrow,1} > \sqrt{1-\delta}) \ge \mathbf{1}_{\mathsf{Nice}_M(\delta)}\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(\mathsf{Sink}_{\uparrow}(\delta)). \tag{5.6.70}$$

Now we divide the sink event into two parts: $\mathsf{Sink}_{\uparrow}(\delta) = \mathsf{Sink}_{+\uparrow}(\delta) \cap \mathsf{Sink}_{-,\uparrow}(\delta)$ where

$$\mathsf{Sink}_{\pm,\uparrow}(\delta) := \left\{\inf_{x\in[0,t^{-\alpha}]} p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3} \pm x) \ge p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3}) - \tfrac{\delta}{4}\right\},$$

In view of (5.6.70), to prove first part of (5.6.67), it suffices to show for all large enough t, on $\mathsf{Nice}_M(\delta)$ we have

$$\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(\mathsf{Sink}_{+,\uparrow}(\delta)) \ge 1 - \tfrac{\delta}{4}, \quad \mathbf{P}_{\mathsf{Nsmall}|3,2,1}(\mathsf{Sink}_{-,\uparrow}(\delta)) \ge 1 - \tfrac{\delta}{4}. \tag{5.6.71}$$

We only prove first part of (5.6.71) below. Towards this end, recall $Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)}$ from (5.6.26). Observe that

$$Y_{M,t,\uparrow}^{(1)}(\Phi + x) = p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3}) - p^{1/3}\mathfrak{h}_{pt,\uparrow}^{(1)}(\Phi p^{-2/3} + x).$$

Recall Nsmall law from Definition 5.6.4. Our discussion in Subsection 5.6.2 implies that under $\mathbf{P}_{\mathsf{Nsmall}|3.2,1}$,

$$(Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)})(\Phi + \cdot)|_{[0,t^{-\alpha}]} \stackrel{d}{=} V_r^{\mathsf{small}}(\cdot), \quad (Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)})(\Phi + \cdot)|_{[-t^{-\alpha},0]} \stackrel{d}{=} V_\ell^{\mathsf{small}}(-\cdot),$$

where recall that V_ℓ^{small} and V_r^{small} are conditionally independent $\mathsf{NonInt\text{-}BrBridge}$ on $[0, t^{-\alpha}]$ with appropriate end points, defined in Definition 5.6.4. In particular we have,

$$\mathbf{P}_{\mathsf{Nsmall}|3,2,1}(\mathsf{Sink}_{+,\uparrow}(\delta)) = \mathbf{P}_{\mathsf{Nsmall}|3,2,1}\left(\sup_{x\in[0,t^{-\alpha}]} V_{r,1}^{\mathsf{small}}(x) \le \tfrac{1}{4}\delta\right) \tag{5.6.72}$$

where $V_r^{\mathsf{small}} = (V_{r,1}^{\mathsf{small}}, V_{r,2}^{\mathsf{small}})$. Recall $\mathsf{Nice}_M(\delta)$ event from (5.6.39). It contains $\mathsf{Bd}_{\uparrow}(\delta)$ event defined in (5.6.33). On this event, $-\frac{1}{\delta} \le V_{r,1}^{\mathsf{Small}}(t^{-\alpha}), V_{r,2}^{\mathsf{Small}}(t^{-\alpha}) \le \frac{1}{\delta}t^{-\alpha/2}$. We consider another NonInt-BrBridge $U = (U_1, U_2)$ on $[0, t^{-\alpha}]$ with non-random endpoints $U_1(t^{-\alpha}) = U_2(t^{-\alpha}) = \frac{1}{\delta}t^{-\alpha/2}$. On $\mathsf{Bd}_{\uparrow}(\delta)$ event, by monotonicity of non-intersecting Brownian bridges (Lemma 2.6 in [109]), one may couple $U = (U_1, U_2)$ and V_r^{small} so that U_i always lies above $V_{r,i}^{\mathsf{small}}$ for $i = 1, 2$. Thus on $\mathsf{Bd}_{\uparrow}(\delta)$ event,

$$\mathbf{P}_{\mathsf{Nsmall}|3,2,1}\left(\sup_{x\in[0,t^{-\alpha}]} V_{r,1}^{\mathsf{small}}(x) \le \lambda t^{-\alpha/2}\right) \ge \mathbf{P}\left(\sup_{x\in[0,1]} t^{\alpha/2}U_1(xt^{-\alpha}) \le \lambda\right) \ge 1 - \tfrac{\delta}{4},$$

where the last inequality is true by taking λ large enough. This choice of λ is possible as by Brownian scaling, $t^{\alpha/2}U_1(xt^{-\alpha}), t^{\alpha/2}U_2(xt^{-\alpha})$ is NonInt-BrBridge on $[0, 1]$ ending at $(\frac{1}{\delta}, \frac{1}{\delta})$. Taking t large enough one can ensure $\lambda t^{-\alpha/2} \le \frac{\delta}{4}$. Using the equality in (5.6.72) we thus establish the first part of (5.6.71). The second part is analogous. This proves the first part of (5.6.67). The second part of (5.6.67) follows similarly. This completes the proof of Lemma 5.6.6.

Proof of Lemma 5.6.7. The idea behind this proof is Proposition 5.5.8, which states that a NonInt-BrBridge after Brownian rescaling converges in distribution to a DBM. The following fills out the details. Recall that

$$\mathbf{P}_{\mathsf{Nsmall}|3.2,1}(A) = \mathbf{P}_{\mathsf{Nsmall}|3.2,1}(D_{M,t,\uparrow}, D_{M,t,\downarrow}(\cdot) \in A).$$

Recall from (5.6.28) that $D_{M,t,\uparrow}, D_{M,t,\downarrow}$ is a diffusive scaling of $Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)}$ when centering at Φ, where $Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)}$ are defined in (5.6.26). Recall Nsmall law from Definition 5.6.4. Our discussion in Subsection 5.6.2 implies that under $\mathbf{P}_{\mathsf{Nsmall}|3.2,1}$,

$$(Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)})(\Phi + \cdot)|_{[0,t^{-\alpha}]} \stackrel{d}{=} V_r^{\mathsf{small}}(\cdot), \quad (Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)})(\Phi + \cdot)|_{[-t^{-\alpha},0]} \stackrel{d}{=} V_\ell^{\mathsf{small}}(-\cdot),$$

where V_ℓ^{small} and V_r^{small} are conditionally independent NonInt-BrBridge on $[0, t^{-\alpha}]$ with appropriate end points defined in Definition 5.6.4. Using Brownian scaling, we consider

$$V_\ell^0(x) := t^{\alpha/2} V_\ell^{\mathsf{small}}(xt^{-\alpha}), \quad V_r^0(x) := t^{\alpha/2} V_r^{\mathsf{small}}(xt^{-\alpha}),$$

which are now $\mathsf{NonInt\text{-}BrBridge}$ on $[0,1]$. Note that on $\mathsf{Bd}_\uparrow(\delta), \mathsf{Bd}_\downarrow(\delta)$ (defined in (5.6.33)), we see that endpoints of V_ℓ^0, V_r^0 are in $[-\frac{1}{\delta}, \frac{1}{\delta}]$. Thus as $\alpha = \frac{1}{6}$, performing another diffusive scaling by Proposition 5.5.8 we see that as $t \to \infty$

$$t^{1/4} V_\ell^0(xt^{-1/2}) \ , \ t^{1/4} V_r(xt^{-1/2})$$

converges to two independent copies of DBMs (defined in Definition 5.5.1) in the uniform-on-compact topology. Hence we get two-sided DBM convergence for the pair $(D_{M,t,\uparrow}, D_{M,t,\downarrow})$ under $\mathbf{P}_{\mathsf{Nsmall}|3.2,1}$ as long as $\mathbf{1}\{\mathsf{Nice}_M(\delta)\}$ holds. This proves (5.6.57).

5.6.3 Proof of Theorem 5.1.10

We take $p \mapsto \frac{1}{2}$ and $t \mapsto 2t$ in Proposition 5.6.1. Then by Lemma 5.3.2, $\mathcal{P}_{2,t}$ defined in the statement of Theorem 5.1.10 is same as $\mathcal{M}_{\frac{1}{2},2t}$ considered in Proposition 5.6.1. Its uniqueness is already justified in Lemma 5.3.1. Furthermore,

$$R_2(x,t) \stackrel{d}{=} D_1(x,t) - D_2(x,t),$$

as functions in x, where $R_2(x,t)$ is defined in (5.1.11) and D_1, D_2 are defined in (5.6.24). By Proposition 5.6.1 and Lemma 5.5.3 we get that $D_1(x,t) - D_2(x,t) \xrightarrow{d} \mathcal{R}_2(x)$ in the uniform-on-compact topology. This proves Theorem 5.1.10 for $k = 2$ case.

For $k = 1$ case, by Lemma 5.3.2, $\mathcal{P}_{1,t}$ is same as $\mathcal{M}_{*,t}$ which is unique almost surely by Lemma

5.3.1. This guarantees $\mathcal{P}_{1,t}$ is unique almost surely as well. Thus we are left to show

$$\mathcal{H}(\mathcal{P}_{1,t}, t) - \mathcal{H}(x + \mathcal{P}_{1,t}, t) \xrightarrow{d} \mathcal{R}_1(x). \tag{5.6.73}$$

where $\mathcal{R}_1(x)$ is a two-sided Bessel process with diffusion coefficient 1 defined in Definition 5.5.2. The proof of (5.6.73) is exactly similar to that of Proposition 5.6.1 with few minor alterations listed below.

1. Just as in Subsection 5.6.2, one may put the problem in (5.6.73) under the framework of KPZ line ensemble. Compared to Subsection 5.6.2, in this case, clearly there will be just one set of line ensemble.

2. Given the decay estimates for $\mathcal{M}_{*,t}$ from Lemma 5.3.1, it boils down to show Bessel behavior around local maximizers. The rigorous justification follows from a soft argument analogous to what is done in Subsection 5.6.2.

3. In the spirit of Subsection 5.6.2, one can define a similar $\mathsf{Nice}'_M(\delta)$ event but now for a single line ensemble. $\mathsf{Nice}'_M(\delta)$ will contain similar events, such as:

   - control on the location of local maximizer (analog of $\mathsf{ArMx}(\delta)$ event (5.6.32)),
   - control on the gap between first curve and second curve at the maximizer (analog of $\mathsf{Gap}_{M,\uparrow}(\delta)$ event (5.6.35)),
   - fluctuations of the first curve on a small interval say I around maximizer (analog of $\mathsf{Rise}_{M,\uparrow}(\delta)$ event (5.6.37),
   - and control on the value of the endpoints of I (analog of $\mathsf{Bd}_{\uparrow}(\delta)$ event (5.6.33)).

   On $\mathsf{Nice}'_M(\delta)$ event, the conditional analysis can be performed in the same manner.

4. Next, as in proof of Proposition 5.6.1, we proceed by three layers of conditioning. For first layer, we use the $\mathbf{H}_t$ Brownian Gibbs property of the single line ensemble under consideration. Next, conditioning on the location and values of the maximizer, we similarly apply the

same Bessel bridge decomposition result from Proposition 5.4.8 to convert the conditional law to that of the Bessel bridges over a large interval (see Subsection 5.6.2). Finally, analogous to Subsection 5.6.2, the third layer of conditioning reduces large Bessel bridges to smaller ones following the Markovian property of Bessel bridges, see Lemma 5.4.2.

5. Since a Bessel bridge say on $[0, 1]$ is a Brownian bridge conditioned to stay positive on $[0, 1]$, it has the Brownian scaling property and it admits monotonicity w.r.t. endpoints. These are two crucial tools that went into the Proof of Lemma 5.6.6 in Subsection 5.6.2. Thus the Bessel analogue of Lemma 5.6.6 can be derived using the scaling property and monotonicity stated above in the exact same way. Finally, the Bessel analogue of Lemma 5.6.7 can be obtained from Corollary 5.5.9. Indeed Corollary 5.5.9 ensures that small Bessel bridges converges to Bessel process under appropriate diffusive limits on the $\mathsf{Nice}'_M(\delta)$ event.

Executing all the above steps in an exact same manner as proof of Proposition 5.6.1, (5.6.73) is established. This completes the proof of Theorem 5.1.10.

5.7 Proof of localization theorems

In this section we prove our main results: Theorem 5.1.4 and Theorem 5.1.5. In Section 5.7.1 we study certain tail properties (Lemma 5.7.1 and Proposition 5.7.2) of the quantities that we are interested in and prove Theorem 5.1.4. Proof of Proposition 5.7.2 is then completed in Section 5.7.2 along with proof of Theorem 5.1.5.

5.7.1 Tail Properties and proof of Theorem 5.1.4

We first settle the question of finiteness of the Bessel integral appearing in the statements of Theorems 5.1.4 and 5.1.5 in the following Lemma.

Lemma 5.7.1. *Let $R_\sigma(\cdot)$ be a Bessel process with diffusion coefficient $\sigma > 0$, defined in Definition*

5.5.2. Then

$$\mathbf{P}\left(\int_{\mathbb{R}} e^{-R_\sigma(x)}\mathrm{d}x \in (0,\infty)\right) = 1.$$

Proof. Since $R_\sigma(\cdot)$ has continuous paths, $\sup_{x\in[0,1]} R_\sigma(x)$ is finite almost surely. Thus almost surely we have

$$\int_{\mathbb{R}} e^{-R_\sigma(x)}\mathrm{d}x \geq \int_0^1 e^{-R_\sigma(x)}\mathrm{d}x > 0.$$

On the other hand, by the classical result from [257] it is known that

$$\mathbf{P}(R_\sigma(x) < x^{1/4} \text{ infinitely often}) = 0.$$

Thus, there exists Ω such that $\mathbf{P}(\Omega) = 1$ and for all $\omega \in \Omega$, there exists $x_0(\omega) \in (0,\infty)$ such that

$$R_\sigma(x)(\omega) \geq x^{1/4} \text{ for all } x \geq x_0(\omega).$$

Hence for this ω,

$$\int_0^\infty e^{-R_\sigma(x)(\omega)}\mathrm{d}x = \int_0^{x_0(\omega)} e^{-R_\sigma(x)(\omega)}\mathrm{d}x + \int_{x_0(\omega)}^\infty e^{-R_\sigma(x)(\omega)}\mathrm{d}x < x_0(\omega) + \int_0^\infty e^{-x^{1/4}}\mathrm{d}x < \infty.$$

This establishes that $\int_{\mathbb{R}} e^{-R_\sigma(x)}\mathrm{d}x$ is finite almost surely. $\square$

Our next result studies the tail of the integral of the pre-limiting process.

Proposition 5.7.2. *Fix $p \in (0,1)$. Set $q = 1-p$. Consider 2 independent copies of the KPZ equation $\mathcal{H}_\uparrow(x,t)$, and $\mathcal{H}_\downarrow(x,t)$, both started from the narrow wedge initial data. Let $\mathcal{M}_{p,t}$ be the almost sure unique maximizer of the process $x \mapsto (\mathcal{H}_\uparrow(x,pt) + \mathcal{H}_\downarrow(x,qt))$ which exists via Lemma 5.3.1. Set*

$$\begin{aligned} D_1(x,t) &:= \mathcal{H}_\uparrow(\mathcal{M}_{p,t},pt) - \mathcal{H}_\uparrow(x+\mathcal{M}_{p,t},pt), \\ D_2(x,t) &:= \mathcal{H}_\downarrow(x+\mathcal{M}_{p,t},qt) - \mathcal{H}_\downarrow(\mathcal{M}_{p,t},qt). \end{aligned} \tag{5.7.1}$$

For all $\rho > 0$ we have

$$\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{P}\left(\int_{[-K,K]^c} e^{D_2(x,t)-D_1(x,t)}\,\mathrm{d}x \geq \rho\right) = 0. \tag{5.7.2}$$

As a corollary, we derive that for any $p \in (0,1)$ the pt-point density of point-to-point CDRP of length t indeed concentrates in a microscopic region of size $O(1)$ around the favorite point.

Corollary 5.7.3. *Recall the definition of* CDRP *and the notation* $\mathbf{P}^\xi$ *from Definition 6.1.1. Fix* $p \in (0,1)$. *Suppose* $X \sim \mathsf{CDRP}(0,0;0,t)$. *Consider* $\mathcal{M}_{p,t}$ *the almost sure unique mode of* $f_{p,t}$, *the quenched density of* $X(pt)$. *We have*

$$\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{P}^\xi\left(|X(pt)-\mathcal{M}_{p,t}| \geq K\right) = 0, \text{ in probability}.$$

One also has the analogous version of Proposition 5.7.2 involving one single copy of the KPZ equation viewed around its maximum. This leads to a similar corollary about tightness of the quenched endpoint distribution for point-to-line CDRP (see Definition 6.1.2) when re-centered around its mode. The details are skipped for brevity.

The proof of Proposition 5.7.2 is heavily technical and relies on the tools as well as notations from Proposition 5.6.1. For clarity, we first prove Corollary 5.7.3 and Theorem 5.1.4 assuming the validity of Proposition 5.7.2. The proof of Proposition 5.7.2 is then presented in Section 5.7.2.

Proof of Corollary 5.7.3. We have $\mathcal{Z}(0,0;x,pt) \stackrel{d}{=} e^{\mathcal{H}_\uparrow(x,pt)}$ and by time reversal property $\mathcal{Z}(x,pt;0,t) \stackrel{d}{=} e^{\mathcal{H}_\downarrow(x,qt)}$ as functions in x, where $\mathcal{H}_\uparrow, \mathcal{H}_\downarrow$ are independent copies of KPZ equation started from narrow wedge initial data. The uniqueness of the mode $\mathcal{M}_{p,t}$ for $f_{p,t}$ is already settled in Lemma 5.3.1. Thus, the quenched density of $X(pt) - \mathcal{M}_{p,t}$ is given by

$$f_{p,t}(x+\mathcal{M}_{p,t}) = \frac{\exp(D_2(x,t)-D_1(x,t))}{\int_{\mathbb{R}}\exp(D_2(y,t)-D_1(y,t))\mathrm{d}y}, \tag{5.7.3}$$

where $D_i(x,t), i = 1,2$ are defined in (5.6.23). Thus,

$$\mathbf{P}^{\xi}\left(|X(pt) - \mathcal{M}_{p,t}| \geq K\right) = \frac{\int\limits_{[-K,K]^c} e^{D_2(x,t)-D_1(x,t)}\,\mathrm{d}x}{\int\limits_{\mathbb{R}} e^{D_2(x,t)-D_1(x,t)}\,\mathrm{d}x} \leq \frac{\int\limits_{[-K,K]^c} e^{D_2(x,t)-D_1(x,t)}\,\mathrm{d}x}{\int\limits_{[-K,K]} e^{D_2(x,t)-D_1(x,t)}\,\mathrm{d}x}. \tag{5.7.4}$$

Notice that by by (5.7.2) the numerator of r.h.s. of (5.7.4) goes to zero in probability under the iterated limit $\limsup_{t\to\infty}$ followed by $\limsup_{K\to\infty}$. Whereas due to Proposition 5.6.1, under the iterated limit, the denominator converges in distribution to $\int_{\mathbb{R}} e^{-R_2(x)}\mathrm{d}x$ which is strictly positive by Lemma 5.7.1. Thus overall the r.h.s. of (5.7.4) goes to zero in probability under the iterated limit. This completes the proof. $\square$

Proof of Theorem 5.1.4. Fix any $p \in (0,1)$. Set $q = 1-p$. Recall from (5.7.3) that

$$f_{p,t}(x + \mathcal{M}_{p,t}) = \frac{\exp(D_2(x,t) - D_1(x,t))}{\int\limits_{\mathbb{R}} \exp(D_2(y,t) - D_1(y,t))\mathrm{d}y} \tag{5.7.5}$$

where $D_i(x,t), i = 1,2$ are defined in (5.6.23). Note that by Proposition 5.6.1, a continuous mapping theorem immediately implies that for any $K < \infty$

$$\frac{\exp(D_2(x,t) - D_1(x,t))}{\int_{-K}^{K} \exp(D_2(y,t) - D_1(y,t))\mathrm{d}y} \xrightarrow{d} \frac{e^{-\mathcal{R}_2(x)}}{\int_{-K}^{K} e^{-\mathcal{R}_2(y)}\mathrm{d}y} \tag{5.7.6}$$

in the uniform-on-compact topology. Here $\mathcal{R}_2$ is a 3D Bessel process with diffusion coefficient 2. For simplicity, we denote

$$\mathfrak{g}_t(x) := \exp(D_2(x,t) - D_1(x,t)) \text{ and } \mathfrak{g}(x) = \exp(-\mathcal{R}_2(x)).$$

We can then rewrite (5.7.5) as product of four factors:

$$f_{p,t}(x + \mathcal{M}_{p,t}) = \frac{\mathfrak{g}_t(x)}{\int_{\mathbb{R}} \mathfrak{g}_t(y)\mathrm{d}y} = \frac{\int_{-K}^{K} \mathfrak{g}_t(y)\mathrm{d}y}{\int_{\mathbb{R}} \mathfrak{g}_t(y)\mathrm{d}y} \cdot \frac{\int_{\mathbb{R}} \mathfrak{g}(y)\mathrm{d}y}{\int_{-K}^{K} \mathfrak{g}(y)\mathrm{d}y} \cdot \frac{\int_{-K}^{K} \mathfrak{g}(y)\mathrm{d}y}{\int_{\mathbb{R}} \mathfrak{g}(y)\mathrm{d}y} \cdot \frac{\mathfrak{g}_t(x)}{\int_{-K}^{K} \mathfrak{g}_t(y)\mathrm{d}y}.$$

Corollary 5.7.3 ensures

$$\frac{\int_{-K}^{K} \mathfrak{g}_t(y)\mathrm{d}y}{\int_{\mathbb{R}} \mathfrak{g}_t(y)\mathrm{d}y} = \mathbf{P}^{\xi}(|X(pt) - \mathcal{M}_{p,t}| \le K) \xrightarrow{p} 1$$

as $t \to \infty$ followed by $K \to \infty$. Lemma 5.7.1 with $\sigma = 2$ yields that $\int_{[-K,K]^c} \mathfrak{g}(y)\mathrm{d}y = \int_{[-K,K]^c} e^{-\mathcal{R}_2(y)}\mathrm{d}y \xrightarrow{p} 0$ as $K \to \infty$. Thus as $K \to \infty$

$$\frac{\int_{\mathbb{R}} \mathfrak{g}(y)\mathrm{d}y}{\int_{-K}^{K} \mathfrak{g}(y)\mathrm{d}y} \xrightarrow{p} 1.$$

Meanwhile, (5.7.6) yields that as $t \to \infty$,

$$\frac{\int_{-K}^{K} \mathfrak{g}(y)\mathrm{d}y}{\int_{\mathbb{R}} \mathfrak{g}(y)\mathrm{d}y} \cdot \frac{\mathfrak{g}_t(x)}{\int_{-K}^{K} \mathfrak{g}_t(y)\mathrm{d}y} \xrightarrow{d} \frac{\int_{-K}^{K} \mathfrak{g}(y)\mathrm{d}y}{\int_{\mathbb{R}} \mathfrak{g}(y)\mathrm{d}y} \cdot \frac{\mathfrak{g}(x)}{\int_{-K}^{K} \mathfrak{g}(y)\mathrm{d}y} = \frac{\mathfrak{g}(x)}{\int_{\mathbb{R}} \mathfrak{g}(y)\mathrm{d}y}.$$

in the uniform-on-compact topology. Thus, overall we get that $f_{p,t}(x + \mathcal{M}_{p,t}) \xrightarrow{d} \frac{\mathfrak{g}(x)}{\int_{\mathbb{R}} \mathfrak{g}(y)\mathrm{d}y}$, in the uniform-on-compact topology. This establishes (5.1.7), completing the proof of Theorem 5.1.4. $\square$

5.7.2 Proof of Proposition 5.7.2 and Theorem 5.1.5

Coming to the proof of Proposition 5.7.2, we note that the setup of Proposition 5.7.2 is same as that of Proposition 5.6.1. Hence all the discussions pertaining to Proposition 5.6.1 are applicable here. In particular, to prove Proposition 5.7.2, we will be using few notations and certain results from the proof of Proposition 5.6.1.

Proof of Proposition 5.7.2. Fix any $M > 0$. The proof of (5.6.24) proceeds by dividing the integral into two parts depending on the range:

$$U_1 := [-t^{2/3}M - \mathcal{M}_{p,t}, t^{2/3}M - \mathcal{M}_{p,t}]^c, \qquad \text{(Deep Tail)}$$

$$U_2 := [K, K]^c \cap [-t^{2/3}M - \mathcal{M}_{p,t}, t^{2/3}M - \mathcal{M}_{p,t}], \qquad \text{(Shallow Tail)}$$

and controlling each of them individually. See Figure 5.10 for details. In the following two steps, we control these two kind of tails respectively.

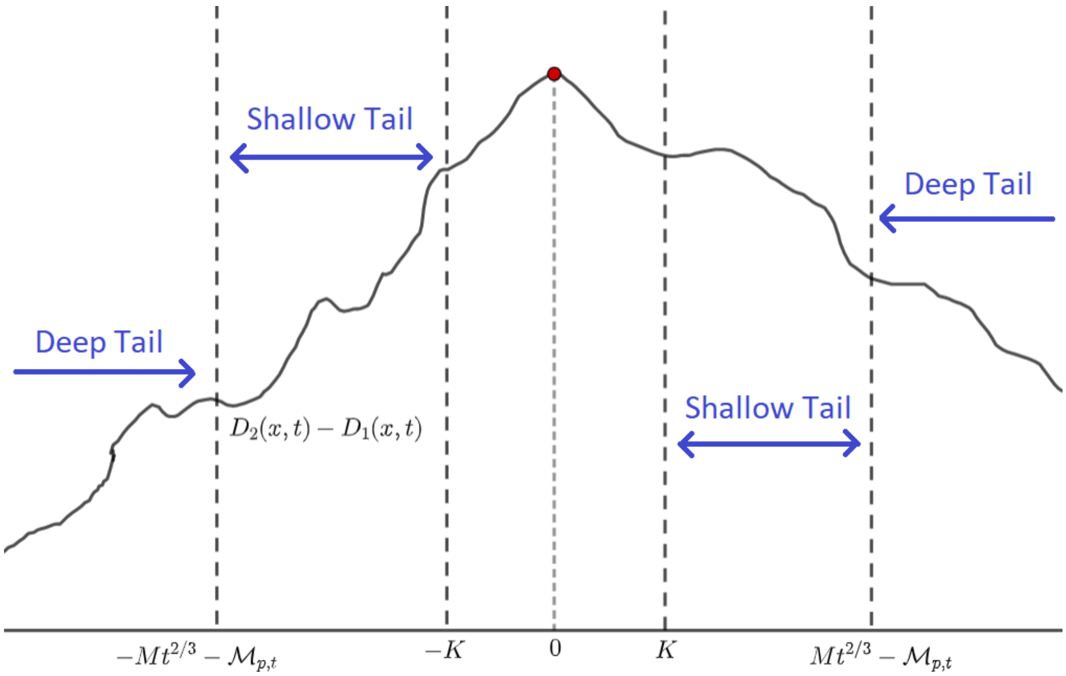

Figure 5.10: Illustration for the proof of Proposition 5.7.2. In Deep Tail region we use parabolic decay of KPZ line ensemble, and in Shallow Tail we use non-intersecting Brownian bridge separation estimates from Proposition 5.5.6.

Step 1. In this step, we control the Deep Tail region: U_1. The goal of this step is to show

$$\limsup_{t\to\infty} \mathbf{P}\left(\int_{U_1} e^{D_2(x,t)-D_1(x,t)}\, \mathrm{d}x \geq \tfrac{\rho}{2}\right) \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}M^3), \tag{5.7.7}$$

for some constant $\mathrm{C} = \mathrm{C}(p) > 0$. We now recall the framework of KPZ line ensemble discussed in Subsection 5.6.2. We define

$$\mathcal{S}_{p,t}(x) := p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x) + q^{1/3}\mathfrak{h}^{(1)}_{qt,\downarrow}(q^{-2/3}x) \tag{5.7.8}$$

where $\mathfrak{h}_{t,\uparrow}, \mathfrak{h}_{t,\downarrow}$ are scaled KPZ line ensembles corresponding to $\mathcal{H}_\uparrow, \mathcal{H}_\downarrow$, see (5.2.6). Observe that

$$D_2(x,t) - D_1(x,t) \stackrel{d}{=} t^{1/3}\left[\mathcal{S}_{p,t}(t^{-2/3}(x+\mathcal{M}_{p,t})) - \sup_{z\in\mathbb{R}} \mathcal{S}_{p,t}(z)\right],$$

where D_1, D_2 are defined in (5.7.1). Thus we have

$$\int_{U_1} \exp(D_2(x,t) - D_1(x,t))\mathrm{d}x \stackrel{d}{=} \int_{|x|\geq M} \exp\left(t^{1/3}\left[\mathcal{S}_{p,t}(x) - \sup_{z\in\mathbb{R}} \mathcal{S}_{p,t}(z)\right]\right)\mathrm{d}x$$

where U_1 is defined in (Deep Tail). Towards this end, we define two events

$$\mathsf{A} := \left\{\sup_{z\in\mathbb{R}} \mathcal{S}_{p,t}(z) \leq -\tfrac{M^2}{4}\right\}, \quad \mathsf{B} := \left\{\sup_{x\in\mathbb{R}} \left(\mathcal{S}_{p,t}(x) + x^2\right) > \tfrac{M^2}{4}\right\},$$

Note that on $\neg A \cap \neg B$, for all $|x| \geq M$, we have

$$\mathcal{S}_{p,t}(x) - \sup_{z\in\mathbb{R}} \mathcal{S}_{p,t}(z) \leq \tfrac{M^2}{4} + \tfrac{M^2}{4} - x^2 \leq \tfrac{M^2}{2} - \tfrac{3M^2}{4} - \tfrac{x^2}{4} \leq -\tfrac{M^2}{4} - \tfrac{x^2}{4}.$$

This forces

$$\int_{|x|\geq M} \exp\left(t^{1/3}\left[\mathcal{S}_{p,t}(x) - \sup_{z\in\mathbb{R}} \mathcal{S}_{p,t}(z)\right]\right)\mathrm{d}x \leq \int_{[-M,M]^c} \exp\left(-t^{1/3}(\tfrac{M^2}{2} + \tfrac{y^2}{4})\right)\mathrm{d}y,$$

which goes to zero as $t \to \infty$. Hence l.h.s. of (5.7.7) $\leq \mathbf{P}(\neg\mathsf{A}) + \mathbf{P}(\neg\mathsf{B})$. Hence it suffices to show

$$\mathbf{P}(\neg\mathsf{A}) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}M^3\right), \quad \mathbf{P}(\neg\mathsf{B}) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}M^3\right). \tag{5.7.9}$$

To prove the first part of (5.7.9), note that

$$\begin{aligned}\mathbf{P}(\neg A) &\leq \mathbf{P}\left(\mathcal{S}_{p,t}(0) \leq -\tfrac{M^2}{4}\right) \\ &\leq \mathbf{P}\left(p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(0) \leq -\tfrac{M^2}{8}\right) + \mathbf{P}\left(q^{1/3}\mathfrak{h}^{(1)}_{qt,\downarrow}(0) \leq -\tfrac{M^2}{8}\right) \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}M^3).\end{aligned}$$

where the last inequality follows by Proposition 5.2.8 (b), for some constant $\mathrm{C} = \mathrm{C}(p) > 0$. This proves the first part of (5.7.9). For the second part of (5.7.9), following the definition of $\mathcal{S}_{p,t}(x)$

from (5.7.8), and using the elementary inequality $\frac{1}{4p}+\frac{1}{4q}\ge 1$ by a union bound we have

$$\begin{aligned}\mathbf{P}\left(\sup_{x\in\mathbb{R}}\left(\mathcal{S}_{p,t}(x)+x^2\right)>\tfrac{M^2}{4}\right)\le \mathbf{P}\left(\sup_{x\in\mathbb{R}}\left(p^{1/3}\mathfrak{h}^{(1)}_{pt,\uparrow}(p^{-2/3}x)+\tfrac{x^2}{4p}\right)>\tfrac{M^2}{8}\right)\\ +\mathbf{P}\left(\sup_{x\in\mathbb{R}}\left(q^{1/3}\mathfrak{h}^{(1)}_{qt,\uparrow}(q^{-2/3}x)+\tfrac{x^2}{4q}\right)>\tfrac{M^2}{8}\right).\end{aligned} \tag{5.7.10}$$

Applying Proposition (5.2.8) (c) with $\beta=\frac{1}{2}$, we get that each of the terms on r.h.s. of (5.7.10) are at most $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^3)$ where $\mathrm{C}=\mathrm{C}(p)>0$. This establishes the second part of (5.7.9) completing the proof of (5.7.7).

Step 2. In this step, we control the Shallow Tail region: U_2. We first lay out the heuristic idea behind the Shallow Tail region controls. We recall the nice event $\mathsf{Sp}(\lambda)$ from (5.6.62) which occurs with high probability. Assuming $\mathsf{Sp}(\lambda)$ holds, we apply the the $\mathbf{H}_t$ Brownian Gibbs property of the KPZ line ensembles, and analyze the desired integral

$$\int_{U_2} e^{D_2(x,t)-D_1(x,t)}\mathrm{d}x$$

under the 'free' Brownian bridge law. Further conditioning on the information of the maximizer converts the free law into the law of the $\mathsf{NonInt\text{-}BrBridge}$ (defined in Definition 5.4.4). On $\mathsf{Sp}(\lambda)$, we may apply Proposition 5.5.6 to obtain the desired estimates for the 'free' law. One then obtain the desired estimates for KPZ law using the lower bound for the normalizing constant from Proposition 6.5.1 (b).

We now expand upon the technical details. In what follows we will only work with the right tail:

$$U_{+,2}:=[-t^{2/3}M-\mathcal{M}_{p,t},t^{2/3}M-\mathcal{M}_{p,t}]\cap[K,\infty)=[K,t^{2/3}M-\mathcal{M}_{p,t}]$$

and the argument for the left part of the shallow tail is analogous. Note that we also implicitly assumed $t^{2/3}M-\mathcal{M}_{p,t}\ge K$ above. Otherwise there is nothing to prove. As before we utilize the the notations defined in Subsection 5.6.2. Recall the local maximizer $\mathcal{M}_{p,t}^{M}$ defined in (5.6.25). Recall $Y_{M,t,\uparrow}^{(1)}, Y_{M,t,\downarrow}^{(1)}$ from (5.6.26). Set

$$\begin{aligned} \Gamma_{t,M,K} &:= \int_K^{Mt^{2/3}-\mathcal{M}_{p,t}} e^{-t^{1/3}\left[Y_{M,t,\uparrow}^{(1)}(t^{-2/3}(\mathcal{M}_{p,t}^{M}+x))-Y_{M,t,\downarrow}^{(1)}(t^{-2/3}(\mathcal{M}_{p,t}^{M}+x))\right]}\mathrm{d}x \\ &= \int_K^{Mt^{2/3}-\mathcal{M}_{p,t}} \exp(-D_{M,t,\uparrow}(x)+D_{M,t,\downarrow}(x))\mathrm{d}x, \end{aligned} \tag{5.7.11}$$

where the last equality follows from the definition of $D_{M,t,\uparrow}, D_{M,t,\downarrow}$ from (5.6.28). Recall that the only difference between D_1, D_2 (defined in (5.6.27)) and $D_{M,t,\uparrow}, D_{M,t,\downarrow}$ is that former is defined using the global maximizer $\mathcal{M}_{p,t}$ and the latter by local maximizer $\mathcal{M}_{p,t}^{M}$. However, Lemma 5.3.1 implies that with probability at least $1-\mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^3)$, we have $\mathcal{M}_{p,t} = \mathcal{M}_{p,t}^{M}$. Next, fix $\lambda > 0$. Consider $\mathsf{Sp}(\lambda)$ event defined in (5.6.62). We thus have

$$\mathbf{P}\left(\int_{U_{+,2}} e^{D_2(x,t)-D_1(x,t)}\mathrm{d}x \ge \frac{\rho}{4}\right) \le \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}M^3) + \mathbf{P}(\neg\mathsf{Sp}(\lambda)) + \mathbf{P}\left(\Gamma_{t,M,K} \ge \tfrac{\rho}{4}, \mathsf{Sp}(\lambda)\right). \tag{5.7.12}$$

We recall the σ-fields $\mathcal{F}_1, \mathcal{F}_2$ defined in (5.6.42) and (5.6.43). We first condition on $\mathcal{F}_1$. As noted in Subsection 5.6.2, since $\mathfrak{h}_{pt,\uparrow}^{(1)}$ and $\mathfrak{h}_{qt,\downarrow}^{(1)}$ are independent, applying $\mathbf{H}_{pt}$ and $\mathbf{H}_{qt}$ Brownian Gibbs property from Proposition 6.5.1 for $\mathfrak{h}_{pt,\uparrow}^{(1)}, \mathfrak{h}_{qt,\downarrow}^{(1)}$ respectively we have

$$\mathbf{P}\left(\Gamma_{t,M,K} \ge \tfrac{\rho}{2}, \mathsf{Sp}(\lambda)\right) = \mathbf{E}\left[\frac{\mathbf{E}_{\mathrm{free},t}\left[\mathbf{1}_{\Gamma_{t,M,K}\ge\frac{\rho}{4},\mathsf{Sp}(\lambda)} W_\uparrow W_\downarrow\right]}{\mathbf{E}_{\mathrm{free},t}\left[W_\uparrow W_\downarrow\right]}\right], \tag{5.7.13}$$

where $W_\uparrow$, $W_\downarrow$ are defined in (5.6.48) and (5.6.49). Here $\mathbf{P}_{\mathrm{free},t}$ and $\mathbf{E}_{\mathrm{free},t}$ are the probability and the expectation operator respectively corresponding to the joint 'free' law for $(p^{1/3}\mathfrak{h}_{pt,\uparrow}(p^{-2/3}x)$, and $q^{1/3}\mathfrak{h}_{qt,\downarrow}(q^{-2/3}x))_{x\in[-M,M]}$ which by Brownian scaling is given by a pair of independent Brownian bridges $(B_1(\cdot), B_2(\cdot))$ on $[-M,M]$ with starting points $(p^{1/3}\mathfrak{h}_{pt,\uparrow}(-Mp^{-2/3}), q^{1/3}\mathfrak{h}_{qt,\downarrow}(-Mq^{-2/3}))$ and endpoints $(q^{1/3}\mathfrak{h}_{pt,\uparrow}(Mp^{-2/3}), q^{1/3}\mathfrak{h}_{qt,\downarrow}(Mq^{-2/3}))$.

In addition, from the last part of Proposition 6.5.1 we know that for any given $\lambda > 0$, there exists $\delta(M,p,\lambda)>0$ such that

$$\mathbf{P}(\mathbf{E}_{\mathrm{free},t}[W_{\uparrow}W_{\downarrow}]>\delta)\geq 1-\lambda. \tag{5.7.14}$$

Since the weight $W_{\uparrow}W_{\downarrow}\in[0,1]$, (5.7.13) and (5.7.14) give us

$$\text{r.h.s. of (5.7.12)} \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}M^3)+\mathbf{P}(\neg\mathsf{Sp}(\lambda))+\lambda+\frac{1}{\delta}\mathbf{E}\left[\mathbf{P}_{\mathrm{free},t}\left(\Gamma_{t,M,K}\geq\tfrac{\rho}{4},\mathsf{Sp}(\lambda)\right)\right]. \tag{5.7.15}$$

Next we condition on $\mathcal{F}_2$ defined in (5.6.43). By Proposition 5.4.10, upon conditioning the free measure of two Brownian bridges when viewed around the maximizer are given by two $\mathsf{NonInt\text{-}BrBridge}$ (defined in Definition 5.4.4). The precise law is given by Nlarge law defined in Definition 5.6.3. Note that $\mathsf{Sp}(\lambda)$ is measurable w.r.t. $\mathcal{F}_1\cup\mathcal{F}_2$. By Reverse Fatou's Lemma and the tower property of conditional expectations, we obtain that

$$\begin{aligned}
&\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{E}\left[\mathbf{P}_{\mathrm{free},t}\left(\Gamma_{t,M,K}\geq\tfrac{\rho}{4},\mathsf{Sp}(\lambda)\right)\right]\\
&\leq\mathbf{E}\left[\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{1}_{\mathsf{Sp}(\lambda)}\mathbf{P}_{\mathsf{Nlarge}|2,1}\left(\Gamma_{t,M,K}\geq\tfrac{\rho}{4}\right)\right].
\end{aligned} \tag{5.7.16}$$

Following the Definition 5.6.3 and (5.7.11) we see that under Nlarge law,

$$\Gamma_{t,M,K}\stackrel{d}{=}\int_K^{Mt^{2/3}-\mathcal{M}_{p,t}}e^{-t^{1/3}\left[V_{r,1}^{\mathsf{large}}(t^{-2/3}x)-V_{r,2}^{\mathsf{large}}(t^{-2/3}x)\right]}\mathrm{d}x. \tag{5.7.17}$$

where $V_r^{\mathsf{large}}=(V_{r,1}^{\mathsf{large}},V_{r,2}^{\mathsf{large}})$ is a $\mathsf{NonInt\text{-}BrBridge}$ defined in Definition 5.6.3. Now notice that by the definition in (5.6.62), on the $\mathsf{Sp}(\lambda)$ event, the length of the Brownian bridges considered are bounded from below and above and the end points are tight. Following the equality in distribution in (5.7.17), the technical result of Proposition 5.5.6 precisely tells us that the term inside the expectation of r.h.s. of (5.7.16) is zero. Thus, going back to (5.7.15) we get that

$$\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{P}\left(\int_{U_{+,2}} e^{D_2(x,t)-D_1(x,t)}\mathrm{d}x \geq \frac{\rho}{4}\right) \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}M^3) + \limsup_{t\to\infty}\mathbf{P}(\neg\mathsf{Sp}(\lambda)) + \lambda.$$

Taking $\limsup_{\lambda\downarrow 0}$, in view of (5.6.63), we get that last two terms in r.h.s. of the above equation are zero. Similarly one can show the same bound for the integral under $U_{-,2} := [-t^{2/3}M - \mathcal{M}_{p,t}, t^{2/3}M - \mathcal{M}_{p,t}] \cap (-\infty, -K]$. Together with (5.7.7), we thus have

$$\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{P}\left(\int_{[-K,K]^c} e^{D_2(x,t)-D_1(x,t)}\mathrm{d}x \geq \rho\right) \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}M^3).$$

Taking $M \to \infty$ we get (5.7.2) completing the proof. □

Proof of Theorem 5.1.5. Recall from (5.1.6) that

$$f_{*,t}(x) = \frac{\mathcal{Z}(0,0;x,t)}{\mathcal{Z}(0,0;*,t)} = \frac{e^{\mathcal{H}(x,t)}}{\int_{\mathbb{R}} e^{\mathcal{H}(y,t)}\mathrm{d}y}.$$

The uniqueness of the mode $\mathcal{M}_{*,t}$ for $f_{*,t}$ is already proved in Lemma 5.3.1. Thus, we have

$$f_{*,t}(x+\mathcal{M}_{*,t}) = \frac{\exp\left(\mathcal{H}(\mathcal{M}_{*,t}+x,t) - \mathcal{H}(\mathcal{M}_{*,t},t)\right)}{\int\limits_{\mathbb{R}} \exp\left(\mathcal{H}(\mathcal{M}_{*,t}+y,t) - \mathcal{H}(\mathcal{M}_{*,t},t)\right)\mathrm{d}y}.$$

Just like in Proposition 5.7.2, we claim that

$$\limsup_{K\to\infty}\limsup_{t\to\infty}\mathbf{P}\left(\int_{[-K,K]^c} e^{\mathcal{H}(\mathcal{M}_{*,t}+y,t)-\mathcal{H}(\mathcal{M}_{*,t},t)}\mathrm{d}y \geq \rho\right) = 0. \tag{5.7.18}$$

The proof of (5.7.18) is exactly same as that of (5.7.2), where we divide the integral in (5.7.18) into a deep tail and a shallow tail and bound them individually. To avoid repetition, we just add few pointers for the readers. Indeed the two key steps of proof of Proposition 5.7.2 that bound the deep and shallow tails can be carried out for the (5.7.18) case. The deep tail regime follows an exact similar strategy as Step 1 of the proof of Proposition 5.7.2 and utilizes the same parabolic decay of the KPZ equation from Proposition 5.2.8. The analogous shallow tail regime also follows in a similar manner by using the uniform separation estimate for Bessel bridges from Corollary 5.5.7.

Now note that by Theorem 5.1.10 with $k = 1$, we have

$$\mathcal{H}(\mathcal{M}_{*,t} + x, t) - \mathcal{H}(\mathcal{M}_{*,t}, t) \xrightarrow{d} \mathcal{R}_1(x), \tag{5.7.19}$$

in the uniform-on-compact topology. Here $\mathcal{R}_1$ is a 3D-Bessel process with diffusion coefficient 1. With the tail decay estimate in (5.7.18) and the same for the Bessel process from Proposition 5.7.1, in view of (5.7.19) one can show $f_{*,t}(x + \mathcal{M}_{*,t}) \to \frac{e^{-\mathcal{R}_1(x)}}{\int_{\mathbb{R}} e^{-\mathcal{R}_1(y)} \mathrm{d}y}$ in the uniform-on-compact topology by following the analogous argument from the proof of Theorem 5.1.4. This completes the proof. □

5.8 Non-intersecting random walks

In this section we prove Lemma 5.4.7 that investigates the convergence of non-intersecting random walks to non-intersecting brownian motions. We remark that similar types of Theorems are already known in the literature such as [161], where the authors considered random walks to start at different locations. Since our walks starts at the same point, additional care is required.

We now recall Lemma 5.4.7 for readers' convenience.

Lemma 5.8.1. *Let X_j^i be i.i.d. $\mathrm{N}(0,1)$ random variables. Let $S_0^{(i)} = 0$ and $S_k^{(i)} = \sum_{j=1}^k X_j^i$. Consider $Y_n(t) = (Y_{n,1}(t), Y_{n,2}(t)) := (\frac{S_{nt}^{(1)}}{\sqrt{n}}, \frac{S_{nt}^{(2)}}{\sqrt{n}})$ an $\mathbb{R}^2$ valued process on $[0,1]$ where the in-between points are defined by linear interpolation. Then conditioned on the non-intersecting event $\Lambda_n := \cap_{j=1}^n \{S_j^{(1)} > S_j^{(2)}\}$, $Y_n \xrightarrow{d} W$, where $W(t) = (W_1(t), W_2(t))$ is distributed as* NonInt-BM *defined in Definition 5.4.3.*

Proof of Lemma 5.8.1. To show weak convergence, it suffices to show finite dimensional convergence and tightness. Based on the availability of exact joint densities for non-intersecting random walks from Karlin-McGregor formula [219], the verification of weak convergence is straightforward. So, we only highlight major steps of the computations below.

Step 1. One point convergence at $t = 1$. Note that

$$\mathbf{P}\left(|\sqrt{n}Y_{n,i}(t) - S^{(i)}_{\lfloor nt \rfloor}| > \sqrt{n}\varepsilon \mid \Lambda_n\right) \leq \frac{1}{\mathbf{P}(\Lambda_n)}\mathbf{P}\left(|X_{\lfloor nt \rfloor+1}| > \sqrt{n}\varepsilon\right) \leq \frac{C}{\varepsilon^2\sqrt{n}}$$

The last inequality above follows by Markov inequality and the classical result that $\mathbf{P}(\Lambda_n) \geq \frac{C}{\sqrt{n}}$ in Spitzer [297]. Thus it suffices to show finite dimensional convergence for the cadlag process:

$$(Z^{(1)}_{nt}, Z^{(2)}_{nt}) := \frac{1}{\sqrt{n}}(S^{(1)}_{\lfloor nt \rfloor}, S^{(2)}_{\lfloor nt \rfloor}). \tag{5.8.1}$$

We assume that n large enough so that $\frac{n-1}{M\sqrt{n}} \geq 1$ for some $M > 0$ to be chosen later. When $t = 1$, applying the Karlin-McGregor formula, we obtain that

$$\mathbf{P}(Z_n(1) \in \mathrm{d}y_1, Z_n(1) \in \mathrm{d}y_2|\Lambda_n) = \tau_n \cdot f_{n,1}(y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2$$

where

$$f_{n,1}(y_1, y_2) := \int_{a_1 > a_2} p_1(a_1)p_1(a_2)\det(p_{n-1}(a_i - y_j\sqrt{n}))^2_{i,j=1}\mathrm{d}a_1\mathrm{d}a_2,$$

and

$$\tau_n^{-1} := \int_{r_1 > r_2}\int_{a_1 > a_2} p_1(a_1)p_1(a_2)\det(p_{n-1}(a_i - r_j\sqrt{n}))^2_{i,j=1}\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}r_1\mathrm{d}r_2. \tag{5.8.2}$$

Note that here the Karlin-McGregor formula, after we have conditioned on the first step of the random walks with $X^1_1 = a_1 > X^2_1 = a_2$.

We will now show that $\frac{(n-1)^2}{\sqrt{n}}\tau_n^{-1}$ and $\frac{(n-1)^2}{\sqrt{n}}f_{n,1}(y_1, y_2)$ converges to a nontrivial limit. Observe that

$$\frac{(n-1)^2}{\sqrt{n}}\det(p_{n-1}(a_i-y_j\sqrt{n}))_{i,j=1}^2 = (n-1)p_{n-1}(a_1-y_2\sqrt{n})p_{n-1}(a_2-y_1\sqrt{n}) \\ \cdot \frac{n-1}{\sqrt{n}}[e^{\frac{\sqrt{n}(a_1-a_2)(y_1-y_2)}{n-1}}-1]. \tag{5.8.3}$$

Thus, as $n \to \infty$, we have

$$\frac{(n-1)^2}{\sqrt{n}}\det(p_{n-1}(a_i-y_j\sqrt{n}))_{i,j=1}^2 \to p_1(y_1)p_1(y_2)(a_1-a_2)(y_1-y_2). \tag{5.8.4}$$

Next we proceed to find a uniform bound for the expression in (5.8.3). Not that for $x, r \geq 1$, one has the elementary inequality $x^r \geq x^r - 1 \geq r(x-1)$. Now taking $r = \frac{n-1}{M\sqrt{n}}$ and $x = \exp(\frac{\sqrt{n}}{n-1}(a_1-a_2)(y_1-y_2)$ we get

$$\begin{aligned} \text{r.h.s. of (5.8.3)} &\leq \frac{1}{2\pi}\exp\left(-\frac{(a_1-y_2\sqrt{n})^2}{2n-2}-\frac{(a_2-y_1\sqrt{n})^2}{2n-2}+\frac{1}{M}(a_1-a_2)(y_1-y_2)\right) \\ &\leq \frac{1}{2\pi}\exp\left(-\frac{y_2^2}{4}-\frac{y_1^2}{4}+\frac{1}{M}(a_1-a_2)(y_1-y_2)+\frac{1}{M}(|a_1y_2|+|a_2y_1|)\right) \\ &\leq \frac{1}{2\pi}\exp\left(-\frac{y_2^2}{4}-\frac{y_1^2}{4}+\frac{2(a_1^2+y_1^2+a_2^2+y_2^2)}{M})\right), \end{aligned} \tag{5.8.5}$$

where the last inequality follows by several application of the elementary inequality $|xy| \leq \frac{1}{2}(x^2 + y^2)$. One can choose M large enough so that the uniform bound in (5.8.5) is integrable w.r.t. the measure $p_1(a_1)p_1(a_2)\mathrm{d}a_1\mathrm{d}a_2$. With the pointwise limit from (5.8.4), by dominated convergence theorem we have

$$\begin{aligned} \frac{(n-1)^2}{\sqrt{n}}f_{n,1}(y_1,y_2) = \frac{(n-1)^2}{\sqrt{n}}\int_{a_1>a_2} p_1(a_1)p_1(a_2)\det(p_{n-1}(a_i-y_j\sqrt{n}))_{i,j=1}^2\mathrm{d}a_1\mathrm{d}a_2 \\ \to p_1(y_1)p_1(y_2)(y_1-y_2)\int_{a_1>a_2}(a_1-a_2)p_1(a_1)p_1(a_2)\mathrm{d}a_1\mathrm{d}a_2. \end{aligned}$$

Similarly one can compute the pointwise limit for the integrand in τ_n^{-1} (defined in (5.8.2)) and the uniform bound in (5.8.5) works for the denominator as well. We thus have

$$\frac{(n-1)^2}{\sqrt{n}}\tau_n^{-1} \to \int_{a_1>a_2}\int_{r_1>r_2} p_1(r_1)p_1(r_2)(r_1-r_2)(a_1-a_2)p_1(a_1)p_1(a_2)\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}r_1\mathrm{d}r_2. \quad (5.8.6)$$

Plugging these limits back in (5.8.1), we arrive at (5.4.1) (the one point density formula for NonInt-BM) as the limit for (5.8.1).

Step 2. One point convergence at $0 < t < 1$**.** When $0 < t < 1$, with the Karlin-Mcgregor formula, we similarly obtain

$$\mathbf{P}(Z_{nt}^{(1)} \in \mathrm{d}y_1, Z_{nt}^{(2)} \in \mathrm{d}y_2 \mid \Lambda_n) = \tau_n \cdot f_{n,t}(y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2 \quad (5.8.7)$$

where τ_n is defined in (5.8.2) and

$$\begin{aligned} f_{n,t}(y_1,y_2) = \int_{r_1>r_2}\int_{a_1>a_2} p_1(a_1)p_1(a_2)\Big[&\det(p_{\lfloor nt\rfloor-1}(a_i-y_j\sqrt{n}))_{i,j=1}^2 \\ &n\cdot\det(p_{n-\lfloor nt\rfloor}(\sqrt{n}y_i-\sqrt{n}r_j))_{i,j=1}^2\Big]\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}r_1\mathrm{d}r_2. \end{aligned} \quad (5.8.8)$$

One can check that as $n \to \infty$, we have

$$\begin{aligned} n^{3/2}\det(p_{\lfloor nt\rfloor-1}(a_i-y_j\sqrt{n}))_{i,j=1}^2 &\to \tfrac{1}{t}p_t(y_1)p_t(y_2)(a_1-a_2)(y_1-y_2), \\ n\cdot\det(p_{n-\lfloor nt\rfloor}(\sqrt{n}y_i-\sqrt{n}r_j))_{i,j=1}^2 &\to \det(p_{1-t}(y_i-r_j))_{i,j=1}^2. \end{aligned}$$

One can provide uniformly integrable bound for the integrand in $f_{n,t}(y_1,y_2)$ in a similar fashion. Thus by dominated convergence theorem,

$$\begin{aligned} n^{3/2}f_{n,t}(y_1,y_2) \to \tfrac{1}{t}p_t(y_1)p_t(y_2)(y_1-y_2)&\int_{a_1>a_2} p_1(a_1)p_1(a_2)(a_1-a_2)\mathrm{d}a_1\mathrm{d}a_2 \\ &\int_{r_1>r_2}\det(p_{1-t}(y_i-r_j))_{i,j=1}^2\mathrm{d}r_1\mathrm{d}r_2. \end{aligned}$$

Using (5.8.6) we get that $\tau_n \cdot f_{n,t}(y_1,y_2)$ converges to (5.4.2), the one point density formula for

NonInt-BM.

Step 3. Transition density convergence. For the transition densities, let $0 < t_1 < t_2 < 1$, and fix $x_1 > x_2$. Another application of Karlin-McGregor formula tells us

$$\begin{aligned}
&\mathbf{P}(Z_{nt_2}^{(1)} \in \mathrm{d}y_1, Z_{nt_2}^{(2)} \in \mathrm{d}y_2 \mid Z_{nt_1}^{(1)} = x_1, Z_{nt_1}^{(2)} = x_2) \\
&= n \det(p_{\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor}(\sqrt{n}y_i - \sqrt{n}x_j))_{i,j=1}^2 \\
&\qquad \cdot \frac{\int\limits_{r_1>r_2} \det(p_{n-\lfloor nt_2 \rfloor}(\sqrt{n}y_i - \sqrt{n}r_j))_{i,j=1}^2 \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}y_1 \mathrm{d}y_2}{\int\limits_{r_1>r_2} \det(p_{n-\lfloor nt_1 \rfloor}(\sqrt{n}x_i - \sqrt{n}r_j))_{i,j=1}^2 \mathrm{d}r_1 \mathrm{d}r_2}.
\end{aligned} \tag{5.8.9}$$

One can check as $n \to \infty$

$$\text{r.h.s of (5.8.9)} \to \frac{\det(p_{t_2-t_1}(y_i - x_j))_{i,j=1}^2 \int_{r_1>r_2} \det(p_{1-t_2}(y_i - r_j))_{i,j=1}^2 \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}y_1 \mathrm{d}y_2}{\int_{r_1>r_2} \det(p_{1-t_1}(x_i - r_j))_{i,j=1}^2 \mathrm{d}r_1 \mathrm{d}r_2}$$

which is same as transition densities for NonInt-BM as shown in (5.4.3). This proves finite dimensional convergence.

Step 4. Tightness. To show tightness, by Kolmogorov tightness criterion, it suffices to show there exist $K > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\mathbf{E}\left[|Y_{n,i}(t) - Y_{n,i}(s)|^K \mid \Lambda_n\right] \leq \mathrm{C}_{K,n_0} \cdot (t-s)^2 \tag{5.8.10}$$

holds for all $0 \leq s < t \leq 1$.

Recall that $\mathbf{P}(\Lambda_n) \geq \frac{\mathrm{C}}{\sqrt{n}}$. For $t - s \leq \frac{1}{n}$ with $K \geq 5$ we have

$$\begin{aligned}
\mathbf{E}\left[|Y_{n,i}(t) - Y_{n,i}(s)|^K \mid \Lambda_n\right] &\leq \mathrm{C} \cdot \sqrt{n}\mathbf{E}\left[|Y_{n,i}(t) - Y_{n,i}(s)|^K\right] \\
&\leq \mathrm{C} \cdot \sqrt{n}\frac{(nt-ns)^K}{n^{K/2}}\mathbf{E}[|X_1^1|^K] \leq \mathrm{C}n^{\frac{1-K}{2}}(nt-ns)^2 \leq \mathrm{C}_K(t-s)^2.
\end{aligned}$$

Thus we may assume $t - s \geq 1/n$. Then it is enough to show (5.8.10) for $Z_{nt}^{(i)}$ (defined in (5.8.1)) instead. Note that if $t - s \in [n^{-1}, n^{-1/4}]$, we may take K large enough so $\frac{1}{4}(K-4) \geq 1$. Then we have

$$\mathbf{E}\left[|Z_{nt}^{(i)} - Z_{ns}^{(i)}|^K \mid \Lambda_n\right] \leq \mathrm{C} \cdot \sqrt{n}\mathbf{E}\left[|Z_{nt}^{(i)} - Z_{ns}^{(i)}|^K\right]$$
$$\leq \mathrm{C} \cdot \sqrt{n}(t-s)^{K/2} \leq \mathrm{C} \cdot n^{1/2-(K-4)/8}(t-s)^2$$

where in the last line we used the fact $(t-s)^{(K-4)/2} \leq n^{-(K-4)/8}$. As $\frac{1}{4}(K-4) \geq 1$, we have $\mathbf{E}\left[|Z_{nt}^{(i)} - Z_{ns}^{(i)}|^K \mid \Lambda_n\right] \leq \mathrm{C}(t-s)^2$ in this case. So, we are left with the case $t - s \geq n^{-1/4}$.

Let us assume $t = 0$, $s \geq n^{-\frac{1}{4}}$. As $ns \geq n^{3/4} \to \infty$, we will no longer make the distinction between ns and $\lfloor ns \rfloor$ in our computations. We use the pdf formula from (7.4.16) and (5.8.8) to get

$$\mathbf{E}[|Z_{ns}^{(i)}|^5] \leq \tau_n \int_{y_1>y_2} |y_i|^5 \int_{r_1>r_2} \int_{a_1>a_2} p_1(a_1)p_1(a_2)\det(p_{ns-1}(a_i - y_j\sqrt{n}))_{i,j=1}^2 \quad (5.8.11)$$
$$n \cdot \det(p_{n-ns}(\sqrt{n}y_i - \sqrt{n}r_j))_{i,j=1}^2\Big] \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}y_1\mathrm{d}y_2.$$

For the last determinant we may use

$$n \cdot \det(p_{n-ns}(\sqrt{n}y_i - \sqrt{n}r_j))_{i,j=1}^2 \mathrm{d}r_1\mathrm{d}r_2$$
$$\leq n \cdot p_{n-ns}(\sqrt{n}y_1 - \sqrt{n}r_1)p_{n-ns}(\sqrt{n}y_2 - \sqrt{n}r_2)\mathrm{d}r_1\mathrm{d}r_2$$

which integrates to 1 irrespective of the value of y_1, y_2. Thus

$$\text{r.h.s. of (5.8.11)} \leq \tau_n \int_{y_1>y_2} |y_i|^5 \int_{a_1,a_2} p_1(a_1)p_1(a_2)\det(p_{ns-1}(a_i - y_j\sqrt{n}))_{i,j=1}^2 \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}y_1\mathrm{d}y_2. \quad (5.8.12)$$

Making the change of variable $y_i = \sqrt{s}z_i$ and setting $m = ns$, we have

$$\text{r.h.s. of (5.8.12)} \leq \tau_n \cdot s^{\frac{5}{2}+1}\mathcal{I}_m,$$

where

$$\mathcal{I}_m := \int_{z_1>z_2} |z_i|^5 \int_{a_1>a_2} p_1(a_1)p_1(a_2)\det(p_{m-1}(a_i - z_j\sqrt{m}))_{i,j=1}^2 \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}z_1\mathrm{d}z_2.$$

We claim that $\frac{(m-1)^2}{\sqrt{m}}\mathcal{I}_m \le \mathrm{C}$ for some universal constant $\mathrm{C} > 0$. Clearly this integral is finite for each m. And by exact same approach in **Step 1**, one can show as $m \to \infty$,

$$\frac{(m-1)^2}{\sqrt{m}}\mathcal{I}_m := \int_{z_1>z_2} |z_i|^5 \int_{a_1>a_2} p_1(z_1)p_1(z_2)p_1(a_1)p_1(a_2)(a_1-a_2)(z_1-z_2)\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}z_1\mathrm{d}z_2.$$

Thus, $\frac{(m-1)^2}{\sqrt{m}}\mathcal{I} \le \mathrm{C}$ for all $m \ge 1$. Thus following (5.8.11), (5.8.12), in view of the above estimate we get

$$\mathbf{E}[|Z_{ns}^{(i)}|^5] \le \mathrm{C}\tau_n \frac{\sqrt{m}}{(m-1)^2} s^{\frac{5}{2}+1}.$$

However, by **Step 1**, $n^{3/2}\tau_n^{-1}$ converges to a finite positive constant. As $m = ns$, we thus get that the above term is at most $\mathrm{C}\cdot s^2$. The case $t \ne 0$ can be checked similarly using the formulas from (7.4.16) and (5.8.8) as well as transition densities formula (5.8.9). This completes the proof. □

Chapter 6: Short- and long-time path tightness of the continuum directed random polymer

6.1 Introduction

6.1.1 Background and motivation

Directed polymers in random environment can be considered as random walks interacting with a random external environment. First introduced and studied in [202], [206] and [61], they have since become a fertile ground for research in orthogonal polynomials, random matrices, stochastic PDEs, and integrable systems (see [99, 181, 44] and the references therein). In the $(1+1)$-dimensional discrete polymer case, the random environment is specified by a collection of zero-mean i.i.d. random variables $\{\omega = \omega(i,j) \mid (i,j) \in \mathbb{Z}_+ \times \mathbb{Z}\}$. Given the environment, the energy of the n-step nearest neighbour random walk $(S_i)_{i=0}^n$ starting at the origin is given by $H_n^\omega(S) := \sum_{i=1}^n \omega(i, S_i)$. The **point-to-line** polymer measure on the set of all such paths is then defined as

$$\mathbf{P}_{n,\beta}^{\omega}(S) = \frac{1}{Z_{n,\beta}^{\omega}} e^{\beta H_n^\omega(S)} \mathbf{P}(S),$$

where $\mathbf{P}(S)$ is the simple random walk measure, β is the inverse temperature, and $Z_{n,\beta}^\omega$ is the partition function.

A competition exists between the *entropy* of paths and the *energy* of the environment in this polymer measure. Spurred by this competition, two distinct regimes appear depending on the inverse temperature β. When $\beta = 0$ the polymer measure is the simple random walk; hence it is entropy-dominated and exhibits diffusive behavior. We refer to this scenario as *weak disorder*. For $\beta > 0$, the polymer measure concentrates on paths with high energies and the diffusive behavior ceases to be guaranteed. This type of energy domination is known as *strong disorder*. For the definitions and results on the precise separation between the two regimes as well as results on higher dimensions, we refer the readers to [103, 237, 97].

While the polymer behavior is characterized by diffusivity in weak disorder, the fluctuations of polymers in strong disorder are conjecturally characterized by two scaling exponents ζ and χ ([294], [5]):

$$\text{Fluctuation of the endpoint of the path: } |S_n| \sim n^{\zeta}, \tag{6.1.1}$$
$$\text{Fluctuation of the log partition function: } [\log Z^{\omega}_{n,\beta} - \rho(\beta)n] \sim n^{\chi}.$$

It is believed that directed polymers fall under the "Kardar-Parisi-Zhang (KPZ) universality class" (see [202, 203, 218, 235, 113]) with fluctuation exponent $\chi = \frac{1}{3}$ and transversal exponent $\zeta = \frac{2}{3}$. This instance of the transversal exponent appearing larger than the diffusive scaling exponent $\frac{1}{2}$ is called *superdiffusivity*. Crucially, the conjectured values for χ and ζ satisfy the "KPZ relation":

$$\chi = 2\zeta - 1. \tag{6.1.2}$$

At the moment, rigorous results on either exponent or the KPZ relation have been scarce. For directed polymers, $\zeta = 2/3$ has only been obtained for log-gamma polymers in [294, 29] and for certain semi-discrete polymers called O'Connell-Yor polymer [**lan**]. Upper and lower bounds on ζ have been established in [274, 253] under additional weight assumptions. For zero-temperature models, $\zeta = \frac{2}{3}$ has been established in [212, 72, 195, 138, 35]. Outside the temperature models, the KPZ relation in (6.1.2) has also been shown in other random growth models such as first passage percolation in [89] and [9] under the assumption that the exponents exist in a certain sense. In strong disorder, the polymer also exhibits certain localization phenomena (see [103, 44, 132] for partial surveys). In particular, the favorite region conjecture speculates that the endpoint of the polymer is asymptotically localized in a region of stochastically bounded diameter (see [100, 44, 42, 20, 132] for related results).

Given the conceptual pictures on the two extreme regimes, in the present paper, we consider polymer fluctuations in the *intermediate disorder regime*. Introduced in [5], the intermediate disorder regime corresponds to scaling the inverse temperature $\beta = \beta_n = n^{-1/4}$ with the length of the polymer n, which captures the transitions between the weak and strong disorders and retains features of both. Within this regime, [4] showed that the partition function for point-to-point directed polymers has a universal scaling limit given by the solution of the Stochastic Heat Equation (SHE) for environment with finite exponential moments. In addition, the polymer path itself converges to a universal object called the Continuum Directed Random Polymer (denoted as CDRP hereafter) under the diffusive scaling.

We consider point-to-point CDRP of length t. The main contribution of this paper can be summarized as follows.

(a) We show that as $t \downarrow 0$, the polymer paths behave diffusively and its annealed law converges in to the law of a Brownian bridge (Theorem 6.1.4).

(b) On the other hand, as $t \uparrow \infty$, the polymers have $t^{2/3}$ pathwise fluctuations. The latter result confirms superdiffusivity and the conjectural $2/3$ exponent for the CDRP (Theorem 6.1.7 (a)). Moreover, the strength of our result exceeds the conjecture in (6.1.1), which only claims endpoint tightness. Instead, in Theorem 6.1.7 (a), we prove that the annealed law of paths of point-to-point CDRP of length t are tight (as $t \uparrow \infty$) upon $t^{2/3}$ scaling. This marks the first result of path tightness among all positive-temperature models.

(c) We also show pointwise weak convergence of the polymer paths under the $t^{2/3}$ scaling to points on the geodesic of the directed landscape (Theorem 6.1.7 (b)). This ensures the $2/3$ scaling exponent is indeed tight. Modulo a conjecture on convergence of the KPZ sheet to the Airy Sheet (Conjecture 6.1.9), we obtain pathwise convergence of the rescaled CDRP to the geodesic of the directed landscape (Theorem 6.1.10).

These results provide a comprehensive picture of fluctuations of CDRP paths under short- and long-time scaling. Our short-time and long-time tightness results also extend to point-to-line

CDRP (Theorem 6.1.8). The formal statement of the main results are given in Section 6.1.2.

6.1.2 The model and the main results

We use the stochastic heat equation (SHE) with multiplicative noise to define the CDRP model. To start with, consider a four-parameter random field $\mathcal{Z}(x, s; y, t)$ defined on

$$\mathbb{R}^4_{\uparrow} := \{(x, s; y, t) \in \mathbb{R}^4 : s < t\}.$$

For each $(x, s) \in \mathbb{R} \times \mathbb{R}$, the function $(y, t) \mapsto \mathcal{Z}(x, s; y, t)$ is the solution of the SHE starting from location x at time s, i.e., the unique solution of

$$\partial_t \mathcal{Z} = \tfrac{1}{2}\partial_{xx}\mathcal{Z} + \mathcal{Z} \cdot \xi, \qquad (y, t) \in \mathbb{R} \times (s, \infty),$$

with Dirac delta initial data $\lim_{t \downarrow s} \mathcal{Z}(x, s; y, t) = \delta(x - y)$. Here $\xi = \xi(x, t)$ is the space-time white noise. The SHE itself enjoys a well-developed solution theory based on Itô integral and chaos expansion [48, 316] also [113, 278]. Via the Feynmann-Kac formula ([203, 99]) the four-parameter random field can be written in terms of chaos expansion as

$$\mathcal{Z}(x, s; y, t) = p(y - x, t - s) + \sum_{k=1}^{\infty} \int_{\Delta_{k,s,t}} \int_{\mathbb{R}^k} \prod_{\ell=1}^{k+1} p(y_\ell - y_{\ell-1}, s_\ell - s_{\ell-1}) \xi(y_\ell, s_\ell) d\vec{y}\, d\vec{s}, \quad (6.1.3)$$

with $\Delta_{k,s,t} := \{(s_\ell)_{\ell=1}^k : s < s_1 < \cdots < s_k < t\}$, $s_0 = s$, $y_0 = x$, $s_{k+1} = t$, and $y_{k+1} = y$. Here

$$p(x, t) := (2\pi t)^{-1/2} \exp(-x^2/(2t))$$

denotes the standard heat kernel. The field $\mathcal{Z}$ satisfies several other properties including the Chapman-Kolmogorov equations [4, Theorem 3.1]. For all $0 \leq s < r < t$, and $x, y \in \mathbb{R}$ we have

$$\mathcal{Z}(x, s; y, t) = \int_{\mathbb{R}} \mathcal{Z}(x, s; z, r)\mathcal{Z}(z, r; y, t)dz. \tag{6.1.4}$$

Definition 6.1.1 (Point-to-point CDRP)**.** Conditioned on the white noise ξ, let $\mathbf{P}^{\xi}$ be a measure on $C([s, t])$ whose finite-dimensional distribution is given by

$$\mathbf{P}^{\xi}(X(t_1) \in dx_1, \ldots, X(t_k) \in dx_k) = \frac{1}{\mathcal{Z}(x, s; y, t)} \prod_{j=0}^{k} \mathcal{Z}(x_j, t_j, ; x_{j+1}, t_{j+1})dx_1 \cdots dx_k. \tag{6.1.5}$$

for $s = t_0 \leq t_1 < \cdots < t_k \leq t_{k+1} = t$, with $x_0 = x$ and $x_{k+1} = y$. (6.1.4) ensure $\mathbf{P}^{\xi}$ is a valid probability measure. Note that $\mathbf{P}^{\xi}$ also depends on x and y but we suppress it from our notations. We will use the notation $\mathsf{CDRP}(x, s; y, t)$ and write $X \sim \mathsf{CDRP}(x, s; y, t)$ when $X(\cdot)$ is a random continuous function on $[s, t]$ with $X(s) = x$ and $X(t) = y$ and its finite-dimensional distributions given by (6.1.5) conditioned on ξ. We will also use the notation $\mathbf{P}^{\xi}, \mathbf{E}^{\xi}$ to denote the law and expectation conditioned on the noise ξ, and $\mathbf{P}, \mathbf{E}$ for the annealed law and expectation respectively.

Definition 6.1.2 (Point-to-line CDRP)**.** Conditioned on the white noise ξ, we let $\mathbf{P}^{\xi}_{*}$ be a measure $C([s, t])$ whose finite-dimensional distributions are given by

$$\mathbf{P}^{\xi}_{*}(X(t_1) \in dx_1, \ldots, X(t_k) \in dx_k) = \frac{1}{\mathcal{Z}(x, s; *, t)} \prod_{j=0}^{k} \mathcal{Z}(x_j, t_j, ; x_{j+1}, t_{j+1})dx_1 \cdots dx_k. \tag{6.1.6}$$

for $s = t_0 \leq t_1 < \cdots < t_k \leq t_{k+1} = t$, with $x_0 = x$ and $x_{k+1} = *$. Here $\mathcal{Z}(x, s; *, t) := \int_{\mathbb{R}} \mathcal{Z}(x, s; y, t)dy$. Note that the Chapman-Kolmogorov equations (6.1.4) ensure $\mathbf{P}^{\xi}_{*}$ is a probability measure. The measure $\mathbf{P}^{\xi}_{*}$ also depends on x but we again suppress it from our notations. We similarly use $\mathsf{CDRP}(x, y; *, t)$ to refer to random variables with $\mathbf{P}^{\xi}_{*}$ law.

Remark 6.1.3. In both [4] and [99], the authors considered a five-parameter random field $\mathcal{Z}_{\beta}(x, s; y, t)$

with inverse temperature β, which is the simultaneous solution of the stochastic heat equation

$$\partial_t \mathcal{Z}_\beta = \tfrac{1}{2}\partial_{xx}\mathcal{Z}_\beta + \beta \mathcal{Z}_\beta \xi, \qquad \lim_{t\downarrow s} \mathcal{Z}_\beta(x,s;y,t) = \delta_x(y).$$

and defined corresponding CDRP measures. Observe that when $\beta = 0$, the stochastic heat equation becomes the heat equation and the corresponding CDRP measures reduce to Brownian measures. Furthermore, for any $\beta > 0$, by the scaling property of the random field $\mathcal{Z}_\beta$, i.e. (iii) of Theorem 3.1 in [4], we have

$$\mathcal{Z}_\beta(x,s;y,t) \stackrel{d}{=} \beta^{-2}\mathcal{Z}_1(\beta^2 x, \beta^4 s; \beta^2 y, \beta^4 t),$$

Thus in this paper, we focus on exclusively on $\beta = 1$.

We now state our first main result which discusses the annealed convergence of the CDRP in the short-time regime to Brownian bridge law.

Theorem 6.1.4 (Annealed short-time convergence)**.** *Fix $\varepsilon > 0$. Let $X \sim \mathsf{CDRP}(0,0;0,\varepsilon)$. Consider the random function $Y^{(\varepsilon)} : [0,1] \to \mathbb{R}$ defined by $Y_t^{(\varepsilon)} := \frac{1}{\sqrt{\varepsilon}}X(\varepsilon t)$. Let $\mathbf{P}^\varepsilon$ denote the annealed law of $Y^{(\varepsilon)}$ on the space of continuous functions on $C([0,1])$. As $\varepsilon \downarrow 0$, $\mathbf{P}^\varepsilon$ converges weakly to $\mathbf{P}_B$, where $\mathbf{P}_B$ is the measure on $C([0,1])$ generated by a Brownian bridge on $[0,1]$ starting and ending at 0.*

Remark 6.1.5. The proof of Theorem 6.1.4 appears in Section 6.4.1. With minor modification in the proof, the above theorem can be extended to include endpoints of the form $x\sqrt{\varepsilon}$. The resulting distributional limit is then a Brownian bridge on $[0,1]$ starting at 0 and ending at x. We also remark that we expect Theorem 6.1.4 to hold true even in the quenched case. However, some of our arguments, in particular the tightness, do not generalize to the quenched case. We hope to explore this direction in future works.

Our next result concerns the tightness and annealed convergence of the CDRP in the long-time regime and gives a rigorous justification of the 2/3 scaling exponent discussed in Section 8.2. The limit is given in terms of the directed landscape constructed in [138, 251] which arises as a universal full scaling limit of several zero-temperature models [141]. Below we briefly introduce this limiting model before stating our result.

The directed landscape $\mathcal{L}$ is a random continuous function $\mathbb{R}^4_\uparrow \to \mathbb{R}$ that satisfies the metric composition law

$$\mathcal{L}(x, s; y, t) = \max_{z\in\mathbb{R}} \left[\mathcal{L}(x, s; z, r) + \mathcal{L}(z, r; y, t)\right], \tag{6.1.7}$$

with the property that $\mathcal{L}(\cdot, t_i; \cdot, t_i+s_i^3)$ are independent for any set of disjoint intervals $(t_i, t_i+s_i^3)$. As a function in x, y, $\mathcal{L}(x, t; y, t+s^3) \stackrel{d}{=} s \cdot \mathcal{S}(x/s^2, y/s^2)$, where $\mathcal{S}(\cdot, \cdot)$ is a parabolic Airy Sheet. We omit definitions of the parabolic Airy Sheet (see Definition 1.2 in [138]) except that $\mathcal{S}(0, \cdot) \stackrel{d}{=} \mathcal{A}(\cdot)$ where $\mathcal{A}$ is the parabolic Airy$_2$ process and $\mathcal{A}(x)+x^2$ is the (stationary) Airy$_2$ process constructed in [275].

Definition 6.1.6 (Geodesics of the directed landscape)**.** For $(x, s; y, t) \in \mathbb{R}^4_\uparrow$, a geodesic from (x, s) to (y, t) of the directed landscape is a random continuous function $\Gamma : [s, t] \to \mathbb{R}$ such that $\Gamma(s) = x$ and $\Gamma(t) = y$ and for any $s \le r_1 < r_2 < r_3 \le t$ we have

$$\mathcal{L}\left(\Gamma(r_1), r_1; \Gamma(r_3), r_3\right) = \mathcal{L}\left(\Gamma(r_1), r_1; \Gamma(r_2), r_2\right) + \mathcal{L}\left(\Gamma(r_2), r_2; \Gamma(r_3), r_3\right).$$

Thus the geodesic precisely contain the points where the equality holds in (6.1.7). Given any $(x, s; y, t) \in \mathbb{R}^4_\uparrow$, by Theorem 12.1 in [138], it is known that almost surely there is a unique geodesic Γ from (x, s) to (y, t).

Theorem 6.1.7 (Long-time CDRP path tightness)**.** *Fix $\varepsilon > (0, 1]$. $V \sim \mathsf{CDRP}(0, 0; 0, \varepsilon^{-1})$. Define a random continuous function $L^{(\varepsilon)} : [0, 1] \to \mathbb{R}$ as $L_t^{(\varepsilon)} := \varepsilon^{2/3} V(\varepsilon^{-1}t)$. We have the following:*

(a) Let $\mathbf{P}^\varepsilon$ denote the annealed law of $L^{(\varepsilon)}$, which is viewed as a random variable in the space of continuous functions on $[0, 1]$ equipped with uniform topology and Borel σ-algebra. The sequence $\mathbf{P}^\varepsilon$ is tight w.r.t. ε.

(b) *For each $t \in (0,1)$, $L_t^{(\varepsilon)}$ converges weakly to $\Gamma(t\sqrt{2})$, where $\Gamma(\cdot)$ is the geodesic of directed landscape from $(0,0)$ to $(0,\sqrt{2})$.*

The above path tightness result under $2/3$ scaling is first such result among all positive-temperature models. Part (b) of the above theorem shows that this $2/3$ scaling is indeed correct: upon this scaling, the CDRP paths have pointwise non-trivial weak limit.

In the same spirit, we have the following short- and long-time tightness result for point-to-line CDRP.

Theorem 6.1.8 (Point-to-line CDRP path tightness)**.** *Fix $\varepsilon \in (0,1]$. Suppose $X \sim \mathsf{CDRP}(0,0;*,\varepsilon)$ and $V \sim \mathsf{CDRP}(0,0;*,\varepsilon^{-1})$. Define two random continuous functions $Y_*^{(\varepsilon)}, L_*^{(\varepsilon)} : [0,1] \to \mathbb{R}$ as $Y_*^{(\varepsilon)}(t) := \varepsilon^{-1/2}X(\varepsilon t)$ and $L_*^{(\varepsilon)}(t) := \varepsilon^{2/3}V(\varepsilon^{-1}t)$. We have the following:*

(a) *If we let $\mathbf{P}^{\varepsilon}_{*,\mathcal{S}}$ denote the annealed law of $Y_*^{(\varepsilon)}(\cdot)$, then as $\varepsilon \downarrow 0$, $\mathbf{P}^{\varepsilon}_{*,\mathcal{S}}$ converges weakly to $\mathbf{P}_{B_*}$, where $\mathbf{P}_{B_*}$ is the measure on $C([0,1])$ generated by a standard Brownian motion.*

(b) *If we let $\mathbf{P}^{\varepsilon}_{*,\mathcal{L}}$ denote the annealed law of $L_*^{(\varepsilon)}(\cdot)$, then the sequence $\mathbf{P}^{\varepsilon}_{*,\mathcal{L}}$ is tight w.r.t. ε.*

(c) *$L_*^{(\varepsilon)}(1)$ converges weakly to $2^{1/3}\mathcal{M}$, where $\mathcal{M}$ is the almost sure unique maximizer of* Airy_2 *process minus the parabola x^2.*

We now explain how the pointwise weak convergence result in Theorem 6.1.7 (b) can be upgraded to a process-level convergence modulo the following conjecture.

Conjecture 6.1.9 (KPZ sheet to Airy sheet)**.** *Set $\mathfrak{h}_t(x,y) := t^{-1/3}[\log \mathcal{Z}(t^{2/3}x, 0; t^{2/3}y, t) + \frac{t}{24}]$. As $t \to \infty$ we have the following convergence in law (as functions in (x,y))*

$$2^{1/3}\mathfrak{h}_t(2^{1/3}x, 2^{1/3}y) \xrightarrow{d} \mathcal{S}(x,y)$$

in the uniform-on-compact topology. Here $\mathcal{S}$ is the parabolic Airy sheet.

When either x or y is fixed, the above weak convergence as a function in one variable is proven in [280]. For zero-temperature models, such convergence has been shown recently in [141] for a large class of integrable models. It remains to show that their methods can be extended to prove the Airy sheet convergence for positive-temperature models such as above.

Assuming the validity of Conjecture 6.1.9, we can strengthen Theorem 6.1.7 (b) to the following statement.

Theorem 6.1.10 (Process annealed long-time convergence)**.** *Fix $\varepsilon > 0$. Let $V \sim \mathsf{CDRP}(0,0;0,\varepsilon^{-1})$. Define $L_t^{(\varepsilon)} := \varepsilon^{2/3}V(\varepsilon^{-1}t)$, $t \in [0,1]$. This scaling produces a measure on $C([0,1])$ for each $\varepsilon > 0$ conditioned on ξ. Assume Conjecture 6.1.9. For $t \in (0,1)$, $\varepsilon \downarrow 0$, the annealed law of $L_t^{(\varepsilon)}$ as a process in t converges weakly to $\Gamma(\sqrt{2}t)$, where $\Gamma(\cdot)$ is the geodesic of the directed landscape $\mathcal{L}$ from $(0,0)$ to $(0,\sqrt{2})$.*

6.1.3 Proof Ideas

Our main result on short-time and long-time tightness of CDRP (i.e., Theorems 6.1.4, 6.1.7 and 6.1.8) follows a host of efforts that attempts to unravel the geometry of CDRP paths. In [132], the authors showed that the quenched density of point-to-point long-time CDRP exhibit pointwise localization. In particular, they showed any particular point on a point-to-point CDRP of length t lives within a order 1 window of a 'favorite site' (depending only on the environment) and this favorite site varies in a $t^{2/3}$ window upon changing the environment. This suggests that the annealed law of polymers are within $t^{2/3}$ window *pointwise*. Our theorems on long-time tightness extend this result to the *full path* of the polymers.

One of the key ingredients behind our tightness proofs is a detailed probabilistic understanding of the log-partition function of CDRP. The log of the partition function of point-to-point CDRP, i.e.,

$$\mathcal{H}(x,s;y,t) := \log \mathcal{Z}(x,s;y,t) \tag{6.1.8}$$

solves the KPZ equation with narrow wedge initial data. Introduced in [218] as a model for random growth interfaces, KPZ equation has been extensively studied in both the mathematics and the physics communities (see [166, 278, 113, 192, 191, 281, 87, 124] and the references therein). In [6], the authors showed the one-point distribution of the KPZ equation $\mathcal{H}(x,t) := \mathcal{H}(0,0;x,t)$, has limiting Tracy-Widom GUE fluctuations of the order $t^{1/3}$ as $t \uparrow \infty$ (long-time regime), whereas fluctuations are Gaussian of the order $t^{1/4}$ as $t \downarrow 0$ (short-time regime). Detailed information of the one-point tails of $\mathcal{H}(x,t)$ as well as tail for the spatial process $\mathcal{H}(\cdot,t)$ are rigorously proved in the mathematics works [115, 116, 117, 310, 131] for long-time regime and in [128, 249, 238, 311] for short-time regime.

For brevity, we only sketch the proof for our long-time path tightness result. The proof of short-time path tightness uses a relation of annealed law of CDRP with that of Brownian counterparts (Lemma 6.4.1). The finite-dimensional convergence for the short-time case (Theorem 6.1.4) follows from chaos expansion and the same results for the long-time regime (Theorem 6.1.7 (b) and Theorem 6.1.8 (c)) follow from the localization results in [132]. Let us take a long-time polymer $V \sim \mathsf{CDRP}(0,0;0,\varepsilon^{-1})$ and scale it according to long-time scaling $L_t^{(\varepsilon)} = \varepsilon^{-2/3}V(\varepsilon^{-1}t)$ for $t \in [0,1]$. By the definition of the CDRP (Definition 6.1.1), we see that the joint law of $(L_s^{(\varepsilon)}, L_t^{(\varepsilon)})$ (where $0 < s < t < 1$) is proportional to

$$\varepsilon^{-4/3} \exp\left[\Lambda_{(s,t);\varepsilon}(x,y)\right]$$

where

$$\Lambda_{(s,t);\varepsilon}(x,y) := \mathcal{H}\big(0,0;x\varepsilon^{-\frac{2}{3}},\tfrac{s}{\varepsilon}\big) + \mathcal{H}\big(x\varepsilon^{-\frac{2}{3}},\tfrac{s}{\varepsilon};y\varepsilon^{-\frac{2}{3}},\tfrac{t}{\varepsilon}\big) + \mathcal{H}\big(y\varepsilon^{-\frac{2}{3}},\tfrac{t}{\varepsilon};0,\tfrac{1}{\varepsilon}\big) + \mathsf{Err}_{(s,t);\varepsilon}. \quad (6.1.9)$$

Here $\mathsf{Err}_{(s,t);\varepsilon}$ is a correction term free of x, y that one needs to add to extract meaningful fluctuation and tail results for the KPZ equation (see statement of Lemma 6.3.7). This correction term does not affect the joint density as it can be absorbed into the proportionality constant.

We next proceed to understand behaviors of the process $(x,y) \mapsto \Lambda_{(s,t);\varepsilon}(x,y)$. From [6], it is known that for each fixed $s < t$ and $y \in \mathbb{R}$, the process $x \mapsto [\mathcal{H}(x,s;y,t) + \frac{(x-y)^2}{2(t-s)}]$ is stationary. Naively speaking, $x \mapsto \mathcal{H}(x,s;y,t)$ looks like a negative parabola: $-\frac{(x-y)^2}{2(t-s)}$. Thus it is natural to expect

$$\varepsilon^{1/3}\Lambda_{(s,t);\varepsilon}(x,y) \approx -\frac{x^2}{2s} - \frac{(y-x)^2}{2(t-s)} - \frac{y^2}{2(1-t)}. \tag{6.1.10}$$

One of the technical contributions of this paper is to rigorously prove the above approximation holds for all x, y. Given any $\nu > 0$, we show with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^2)$,

$$\varepsilon^{1/3}\Lambda_{(s,t);\varepsilon}(x,y) \leq M - (1-\nu)\left[\frac{x^2}{2s} + \frac{(y-x)^2}{2(t-s)} + \frac{y^2}{2(1-t)}\right], \text{ for all } x, y \in \mathbb{R}.$$

The precise statement of the above result appears in Lemma 6.3.7. This multivariate process estimate allows us to conclude the quenched density of $(L_s^{(\varepsilon)}, L_t^{(\varepsilon)})$ at (x, y) is exponentially small, whenever $\frac{|x-y|}{\sqrt{t-s}} \to \infty$. Armed with this understanding of quenched density, in Proposition 6.3.1, we show that given any $\delta > 0$, with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^2)$ we have

$$|L_s^{(\varepsilon)} - L_t^{(\varepsilon)}| \leq M|t-s|^{\frac{1}{2}-\delta}.$$

In fact the sharp decay estimates of quenched density (Lemma 6.3.7) allows us to prove a quenched version of the above statement (Proposition 6.3.1). Due to exponentially tight probability bounds of the above two-point differences, Proposition 6.3.1 can be extended to quenched modulus of continuity estimates (Proposition 6.3.3) by standard methods. This leads to the path tightness of long-time CDRP.

Outline

The rest of the paper is organized as follows. Section 6.2 reviews some of the existing results related to the KPZ equation before proving a useful result on the short-time local fluctuations of the KPZ equation (Proposition 6.2.4). We then prove in Section 6.3 a multivariate spatial process tail bound (Lemma 6.3.7) and modulus of continuity results (Propositions 6.3.1 and 6.3.1-(point-to-line)) that culminate in the quenched modulus of continuity estimate in Proposition 6.3.3 and

Proposition 6.3.3-(point-to-line). In Section 6.4, we prove Theorems 6.1.4, 6.1.7, and 6.1.8, and Theorem 6.1.10 (modulo Conjecture 6.1.9). Lastly, proof of a technical lemma used in Section 6.2 appears in Appendix 7.6.

6.2 Short- and long-time tail results for KPZ equation

Throughout this paper we use $\mathrm{C} = \mathrm{C}(x, y, z, \ldots) > 0$ to denote a generic deterministic positive finite constant that may change from line to line, but dependent on the designated variables $x, y, z, \ldots$. We use sans serif fonts such as $\mathsf{A}, \mathsf{B}, \ldots$ to denote events and $\neg\mathsf{A}, \neg\mathsf{B}, \ldots$ to denote their complements.

In this section, we collect several estimates related to the short-time and long-time tails of the KPZ equation. We record existing estimates from the literature in Proposition 6.2.2 and Proposition 6.2.3. These estimates form crucial tools to our later proofs. For our analysis, we also require an estimate on the short-time local fluctuations of the KPZ equation which is not available in the literature. We present this new estimate in Proposition 6.2.4. Its proof appears at the end of this section.

Recall the four-parameter stochastic heat equation $\mathcal{Z}(x, s; y, t)$ from (6.1.3). We set

$$\mathcal{H}(x, s; y, t) := \log \mathcal{Z}(x, s; y, t). \tag{6.2.1}$$

When $x = s = 0$, we use the abbreviated notation $\mathcal{H}(y, t) := \mathcal{H}(0, 0; y, t)$. As mentioned in the introduction, fluctuation and scaling of the KPZ equation varies as $t \downarrow 0$ (short-time) and $t \uparrow \infty$ (long-time). For the two separate regimes we consider the following scalings:

$$\begin{aligned} \mathfrak{g}_{s,t}(x, y) &:= \frac{\mathcal{H}(\sqrt{\frac{\pi(t-s)}{4}}x, s; \sqrt{\frac{\pi(t-s)}{4}}y, t) + \log\sqrt{2\pi(t-s)}}{(\frac{\pi(t-s)}{4})^{1/4}} && \text{for the short-time regime,} \\ \mathfrak{h}_{s,t}(x, y) &:= \frac{\mathcal{H}((t-s)^{2/3}x, s; (t-s)^{2/3}y, t) + \frac{t-s}{24}}{(t-s)^{1/3}} && \text{for the long-time regime.} \end{aligned} \tag{6.2.2}$$

We will often refer to the above bivariate functions as short-time and long-time KPZ sheet. In particular, when both $s = 0$ and $x = 0$, we use the shorthands $\mathfrak{g}_t(y) := \mathfrak{g}_{0,t}(0, y)$, and $\mathfrak{h}_t(y) := \mathfrak{h}_{0,t}(0, y)$.

Remark 6.2.1. The above scalings satisfy several distributional identities. For fixed $s < t$ and $y \in \mathbb{R}$, from chaos representation for SHE it follows that

$$\mathcal{Z}(0, s; x, t) \stackrel{d}{=} \mathcal{Z}(0, s; -x, t), \quad \mathcal{Z}(x, s; y, t) \stackrel{d}{=} \mathcal{Z}(0, 0; y - x, t - s).$$

where the equality in distribution holds as processes in x. This leads to $\mathfrak{g}_{s,t}(x, y) \stackrel{d}{=} \mathfrak{g}_{t-s}(x - y)$ and $\mathfrak{h}_{s,t}(x, y) \stackrel{d}{=} \mathfrak{h}_{t-s}(x - y)$, as processes in x.

The following proposition collects several probabilistic facts for the long-time rescaled KPZ equation.

Proposition 6.2.2. *Recall $\mathfrak{h}_t(x)$ from* (6.2.2). *The following results hold:*

(a) For each $t > 0$, $\mathfrak{h}_t(x) + x^2/2$ is stationary in x.

(b) Fix $t_0 > 0$. There exists a constant $\mathrm{C} = \mathrm{C}(t_0) > 0$ such that for all $t \ge t_0$ and $s > 0$ we have

$$\mathbf{P}\left(|\mathfrak{h}_t(0)| \ge s\right) \le \mathrm{C} \exp\left(-\tfrac{1}{\mathrm{C}} s^{3/2}\right).$$

(c) Fix $t_0 > 0$. There exists a constant $\mathrm{C} = \mathrm{C}(t_0) > 0$ such that for all $x \in \mathbb{R}$, $s > 0$, $t \ge t_0$, and $\gamma \in (0, 1]$, we have

$$\mathbf{P}\left(\sup_{z \in [x, x+\gamma]} \left|\mathfrak{h}_t(z) + \tfrac{z^2}{2} - \mathfrak{h}_t(x) - \tfrac{x^2}{2}\right| \ge s\sqrt{\gamma}\right) \le \mathrm{C} \exp\left(-\tfrac{1}{\mathrm{C}} s^{3/2}\right).$$

The results in Proposition 6.2.2 are a culmination of results from several papers. Part (a) follows from [6, Corollary 1.3 and Proposition 1.4]. The one-point tail estimates for KPZ equation are obtained in [115, 116]. One can derive part (b) from those results or can combine the statements of Proposition 2.11 and 2.12 in [117] to get the same. Part (c) is Theorem 1.3 from [117].

The study of short-time tails was initiated in [128]. Below we recall some known results from the same paper.

Proposition 6.2.3. *Recall $\mathfrak{g}_t(x)$ from* (6.2.2)*. The following results hold:*

(a) For each $t > 0$, $\mathfrak{g}_t(x) + \frac{(\pi t/4)^{3/4}}{2t}x^2$ is stationary in x.

(b) There exists a constant $\mathrm{C} > 0$ such that for all $t \le 1$ and $s > 0$ we have

$$\mathbf{P}(|\mathfrak{g}_t(0)| > s) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^{3/2}\right).$$

Part (a) follows from [128, Lemma 2.11]. The one-point tail estimates for short-time rescaled KPZ equation are obtained in [128, Corollary 1.6, Theorem 1.7], from which one can derive part (b).

For convenience, we write $m_t(x) := (\frac{\pi t}{4})^{3/4}\frac{x^2}{2t}$ to denote the parabolic term associated to the short-time scaling. The following result concerns the short-time analogue of Proposition 6.2.2 (c).

Proposition 6.2.4 (Short-time local fluctuations of the KPZ equation)**.** *There exists a constant $\mathrm{C} > 0$ such that for all $t \in (0,1)$, $x \in \mathbb{R}$, $\gamma \in (0, \sqrt{t})$ and $s > 0$ we have*

$$\mathbf{P}\left(\sup_{z\in[x,x+\gamma]} |\mathfrak{g}_t(z) + m_t(z) - \mathfrak{g}_t(x) - m_t(x)| \ge s\sqrt{\gamma}\right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^{3/2}\right). \tag{6.2.3}$$

Remark 6.2.5. The parabolic term $m_t(x)$ is steeper (as $t \le 1$) than the usual parabola that appears in the long-time scaling. This is the reason why Proposition 6.2.4 requires $\gamma < \sqrt{t}$, whereas Proposition 6.2.2 (c) holds for all $\gamma \in (0,1]$.

The proof of Proposition 6.2.4 follows the same strategy as those of Proposition 4.3 and Theorem 1.3 in [117] which employ the Brownian Gibbs property of the KPZ line ensemble (see [**CH16**]). The same Brownian Gibbs property continues to hold for short-time $\mathfrak{g}_t(\cdot)$ process (see Lemma 2.5 (4) in [128]). We include the proof of Proposition 6.2.4 below for completeness after first describing its key proof ingredient.

We recall a property of $\mathfrak{g}_t(\cdot)$ under *monotone* events. Given an interval $[a,b]$, we denote $\mathcal{B}(C([a,b]))$ to be the Borel σ-algebra on $C([a,b])$ generated by the uniform norm topology. We call an event $A \in \mathcal{B}(C([a,b]))$ monotone w.r.t. $[a,b]$ if for every pair of functions $f,g \in [a,b] \to \mathbb{R}$ with $f(a)=g(a)$, $f(b)=g(b)$ and $f(x) \ge g(x)$ for all $x \in (a,b)$, we have

$$f(x) \in A \implies g(x) \in A. \tag{6.2.4}$$

We call $(\mathfrak{a},\mathfrak{b})$ a stopping domain for $\mathfrak{g}_t(\cdot)$ if $\{\mathfrak{a} \le a, \mathfrak{b} \ge b\}$ is measurable w.r.t. σ-algebra generated by $(\mathfrak{g}_t(x))_{x\notin(a,b)}$ for all $a,b\in\mathbb{R}$. A crucial property is the following:

Lemma 6.2.6. *Fix any $t>0$. For any $[a,b]\subset\mathbb{R}$, and a monotone set $A\in\mathcal{B}(C([a,b]))$ (w.r.t. $[a,b]$), we have*

$$\mathbf{P}\left[\mathfrak{g}_t(\cdot)\mid_{[a,b]} \in A \mid (\mathfrak{g}_t(x))_{x\notin(a,b)}\right] \le \mathbf{P}_{\text{free}}^{(a,b),(\mathfrak{g}_t(a),\mathfrak{g}_t(b))}(A) \tag{6.2.5}$$

where $\mathbf{P}_{\text{free}}^{(a,b),(y,z)}$ denotes the law of Brownian bridge on $[a,b]$ starting at y and ending at z. Furthermore (6.2.5) *continues to hold if (a,b) is a stopping domain for $\mathfrak{g}_t(\cdot)$.*

We will abuse our definition and call $\{\mathfrak{g}_t(\cdot)\mid_{[a,b]}\in A\}$ to be monotone w.r.t. $[a,b]$ if A is monotone w.r.t. $[a,b]$. The proof of the above lemma follows by utilizing the notion of the KPZ line ensemble and its Brownian Gibbs property [**CH16**, 128]. We defer its proof and the necessary background on the KPZ line ensemble to Appendix 7.6.

Proof of Proposition 6.2.4. Assume $s\ge 100$. For $s\le 100$, the constant $\mathrm{C}>0$ can be adjusted so that the proposition holds trivially. We fix a $t_0\in(0,1)$ such that for all $s\ge 100$, and $t\le t_0$ we have

$$\tfrac{1}{4}s \ge t^{1/4}(s+m_t(2)) = t^{1/4}s + 2(\pi/4)^{3/4}. \tag{6.2.6}$$

Let us first consider $t \in [t_0, 1]$. We use the scalings from (6.2.2) to get

$$\mathfrak{g}_t(x) + m_t(x) = \frac{1}{\sqrt{r_t}} \left(\mathfrak{h}_t(r_t x) + \frac{r_t^2 x^2}{2} \right) + c_t, \tag{6.2.7}$$

where $r_t := t^{-1/6}\sqrt{\pi/4}$ and $c_t := (\pi t/4)^{-1/4}(\sqrt{2\pi t} - t/24)$. Take any $x \in \mathbb{R}$ and $\gamma \in (0, \sqrt{t})$. We have $r_t\gamma \le 1$. Setting $y := r_t x$ and then applying Proposition 6.2.2 (c) with $x \mapsto y$ and $\gamma \mapsto r_t\gamma$ we get

$$\text{l.h.s. of (6.2.3)} = \mathbf{P}\left(\sup_{z\in[y,y+r_t\gamma]} \left| \mathfrak{h}_t(z) + \frac{z^2}{2} - \mathfrak{h}_t(y) - \frac{y^2}{2} \right| \ge s\sqrt{\gamma r_t} \right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}} s^{3/2}\right). \tag{6.2.8}$$

Let us now assume $t \le t_0$. By Proposition 6.2.3 (a), we know that the process $\mathfrak{g}_t(x) + m_t(x)$ is stationary in x. Thus it suffices to prove Proposition 6.2.4 with $x = 0$. Consider the following events

$$\mathsf{G}_{\gamma,s} := \bigcap_{x\in\{\gamma-2,0,\gamma,2\}} \left\{ -\tfrac{s}{4} \le \mathfrak{g}_t(x) + m_t(x) \le \tfrac{s}{4} \right\},$$

$$\mathsf{Fall}_{\gamma,s} := \left\{ \inf_{z\in[0,\gamma]} (\mathfrak{g}_t(z) + m_t(z)) \le \mathfrak{g}_t(0) - s\gamma^{1/2} \right\},$$

$$\mathsf{Rise}_{\gamma,s} := \left\{ \sup_{z\in[0,\gamma]} (\mathfrak{g}_t(z) + m_t(z)) \ge \mathfrak{g}_t(0) + s\gamma^{1/2} \right\}.$$

By one-point tail bounds from Proposition 6.2.3 (b) we have that $\mathbf{P}(\neg\mathsf{G}_{\gamma,s}) \le \mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$. Thus, to show the proposition, it suffices to verify the following two bounds:

$$\mathbf{P}\left(\mathsf{Fall}_{\gamma,s}, \mathsf{G}_{\gamma,s}\right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^2\right), \qquad \mathbf{P}\left(\mathsf{Rise}_{\gamma,s}, \mathsf{G}_{\gamma,s}\right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^2\right). \tag{6.2.9}$$

We begin with the $\mathsf{Fall}_{\gamma,s}$ bound in (6.2.9). Clearly $\mathsf{Fall}_{\gamma,s}$ event is monotone w.r.t. $[0, 2]$, by Lemma 6.2.6 we have

$$\mathbf{P}\left(\mathsf{Fall}_{\gamma,s} \mid (\mathfrak{g}_t(x))_{x\in(0,2)}\right) \le \mathbf{P}_{\text{free}}^{(0,2),(\mathfrak{g}_t(0),\mathfrak{g}_t(2))}\left(\mathsf{Fall}_{\gamma,s}\right)$$

where $\mathbf{P}_{\mathsf{free}}^{(a,b),(y,z)}$ denotes the law of Brownian bridge on $[a,b]$ starting at y and ending at z. Using this we have

$$\begin{aligned}
\mathbf{P}\left(\mathsf{Fall}_{\gamma,s},\mathsf{G}_{\gamma,s}\right) &\le \mathbf{P}\left(\mathsf{Fall}_{\gamma,s},\mathfrak{g}_t(0)\le \tfrac{s}{4},\mathfrak{g}_t(2)+m_t(2)\ge -\tfrac{s}{4}\right)\\
&\le \mathbf{E}\left[\mathbf{1}_{\mathfrak{g}_t(0)\le\frac{s}{4}}\cdot\mathbf{1}_{\mathfrak{g}_t(2)+m_t(2)\ge-\frac{s}{4}}\mathbf{P}_{\mathsf{free}}^{(0,2),(\mathfrak{g}_t(0),\mathfrak{g}_t(2))}\left(\mathsf{Fall}_{\gamma,s}\right)\right]\\
&\le \sup\left\{\mathbf{P}_{\mathsf{free}}^{(0,2),y,z}\left(\mathsf{Fall}_{\gamma,s}\right): y\le\tfrac{s}{4}, z+m_t(2)\ge-\tfrac{s}{4}\right\}\\
&= \mathbf{P}_{\mathsf{free}}^{(0,2),s/4,-s/4-m_t(2)}\left(\mathsf{Fall}_{\gamma,s}\right). \qquad (6.2.10)
\end{aligned}$$

Next, we write the final term in (6.2.10) as

$$\mathbf{P}_{\mathsf{free}}^{(0,2),s/4,-s/4-m_t(2)}\left(\mathsf{Fall}_{\gamma,s}\right)=\mathbf{P}\left(\inf_{z\in[0,\gamma]}\left\{B'(z)+m_t(z)\right\}\le -s\gamma^{1/2}\right)$$

where $B':[0,2]\to\mathbb{R}$ is a Brownian bridge with $B'(0)=0$ and $B'(2)=-m_t(2)-\frac{s}{2}$. Now, set $B(z):=B'(z)-\frac{z}{2}(-m_t(2)-\frac{s}{2})$. Then B is a Brownian bridge with $B(0)=B(2)=0$ and we obtain

$$\begin{aligned}
\mathbf{P}\left(\inf_{z\in[0,\gamma]}(B'(z)+m_t(z))\le -s\gamma^{1/2}\right) &\le \mathbf{P}\left(\inf_{z\in[0,\gamma]}B(z)\le -s\gamma^{1/2}-\tfrac{\gamma}{2}(-m_t(2)-\tfrac{s}{2})\right)\\
&\le \mathbf{P}\left(\inf_{z\in[0,\gamma]}B(z)\le -\tfrac{s}{2}\gamma^{1/2}\right). \qquad (6.2.11)
\end{aligned}$$

The latter inequality is due to $\gamma^{1/2}(m_t(2)+\frac{s}{2})\le s$ as $s\ge 100$ and $\gamma\le\sqrt{t}$. The right-hand probability can be estimated via Brownian calculations, which yields the desired bound of the form $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^2)$.

We next prove the $\mathsf{Rise}_{\gamma,s}$ bound in (6.2.9). Note that $\sup_{z\in[0,\gamma]}m_t(z)\le\frac{\gamma^2}{t^{1/4}}\le\frac{1}{2}s\gamma^{1/2}$ (as $\gamma\le\sqrt{t}\le 1$ and $s\ge 4$). Thus it suffices to show

$$\mathbf{P}\left(\mathsf{Rise}_{\gamma,s}^{(1)},\mathsf{G}_{\gamma,s}\right)\le\mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^2\right),\quad \mathsf{Rise}_{\gamma,s}^{(1)}:=\left\{\sup_{z\in[0,\gamma]}\mathfrak{g}_t(z)\ge\mathfrak{g}_t(0)+\tfrac{1}{2}s\sqrt{\gamma}\right\}. \qquad (6.2.12)$$

Set

$$\chi := \inf \left\{ x \in (0, \gamma] \mid \mathfrak{g}_t(x) - \mathfrak{g}_t(0) \geq \tfrac{1}{2} s \gamma^{1/2} \right\},$$

and set $\chi = \infty$ if no such points exist. Then we have $\mathbf{P}\left(\mathsf{Rise}_{\gamma,s}^{(1)}, \mathsf{G}_{\gamma,s}\right) = \mathbf{P}\left(\chi \leq \gamma, \mathsf{G}_{\gamma,s}\right)$ and we can write the right-hand probability as

$$\mathbf{P}\left(\chi \leq \gamma, \mathsf{G}_{\gamma,s}, \mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) < \tfrac{1}{4} s\sqrt{\gamma}\right) + \mathbf{P}\left(\chi \leq \gamma, \mathsf{G}_{\gamma,s}, \mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \geq \tfrac{1}{4} s\sqrt{\gamma}\right). \quad (6.2.13)$$

On the event $\{\chi \leq \gamma, \mathsf{G}_{\gamma,s}, \mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) < \frac{1}{4} s\sqrt{\gamma}\}$ we have that $\{\mathfrak{g}_t(\gamma) - \mathfrak{g}_t(0) \geq \frac{1}{4} s\sqrt{\gamma}\}$ holds as the continuity of $\mathfrak{g}_t(\cdot)$ implies that $\mathfrak{g}_t(\chi) = \mathfrak{g}_t(0) + \frac{1}{2} s\sqrt{\gamma}$ on $\{\chi \leq \gamma\}$ event. Now with the same argument of the $\mathsf{Fall}_{\gamma,s}$ event, we bound the probability of this occurrence by $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^2)$ for some constant $\mathrm{C} > 0$. This is why $\mathsf{G}_{\gamma,s}$ involves $\mathfrak{g}_t(-2+\gamma)$ and $\mathfrak{g}_t(\gamma)$. The parabolic term $m_t(z)$ again can be ignored as $\sup_{z\in[0,\gamma]} m_t(z) \leq \gamma^{3/2} \leq \frac{1}{8} s\gamma^{1/2}$ for $s \geq 8$.

Let us focus on the second term in (6.2.13). Note that $(\chi, 2)$ is a stopping domain and $\{\mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \geq \frac{1}{4} s\sqrt{\gamma}\}$ is a monotone event w.r.t. $[\chi, 2]$. Applying Lemma 6.2.6 one has

$$\mathbf{P}\left(\mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \geq \tfrac{1}{4} s\sqrt{\gamma} \mid (\mathfrak{g}_t(x))_{x\notin(\chi,2)}\right) \leq \mathbf{P}_{\text{free}}^{(\chi,2),(\mathfrak{g}_t(\chi),\mathfrak{g}_t(2))}\left(\mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \geq \tfrac{1}{4} s\sqrt{\gamma}\right).$$

Note that on $\{\chi \leq \gamma, \mathsf{G}_{\gamma,s}\}$ we have

$$|\mathfrak{g}_t(\chi) - \mathfrak{g}_t(2)| = |\mathfrak{g}_t(0) + \tfrac{1}{2} s\sqrt{\gamma} - \mathfrak{g}_t(2)| \leq s/4 + \tfrac{1}{2} s\sqrt{\gamma} + m_t(2) + s/4 = s + m_t(2). \quad (6.2.14)$$

As $2-\chi \geq 1$ on $\{\chi \leq \gamma\}$, we thus get that the absolute value of the slope of the linearly interpolated line joining $(\chi, \mathfrak{g}_t(\chi))$ and $(2, \mathfrak{g}_t(2))$ is at most $s + m_t(2)$. Note that $\frac{1}{4} s\sqrt{\gamma} \geq \gamma(s + m_t(2))$ due to (6.2.6). Thus the event $\{\mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \geq \frac{1}{4} s\sqrt{\gamma}\}$ entails that the $\mathfrak{g}_t(\gamma)$ lies below the linearly interpolated line. Under Brownian law, this has probability $1/2$. Thus,

$$\begin{aligned}
&\mathbf{P}\left(\chi \leq \gamma, \mathsf{G}_{\gamma,s}, \mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \geq \tfrac{1}{4} s\sqrt{\gamma}\right) \\
&\leq \mathbf{E}\left[\mathbf{1}_{\chi\leq\gamma,\mathsf{G}_{\gamma,s}} \mathbf{P}_{\text{free}}^{(\chi,2),(\mathfrak{g}_t(\chi),\mathfrak{g}_t(2))}\left(\mathfrak{g}_t(\chi) - \mathfrak{g}_t(\gamma) \geq \tfrac{1}{4} s\sqrt{\gamma}\right)\right] \leq \tfrac{1}{2}\mathbf{E}\left[\mathbf{1}_{\chi\leq\gamma,\mathsf{G}_{\gamma,s}}\right].
\end{aligned}$$

Hence we have shown that $\mathbf{P}\left(\chi \le \gamma, \mathsf{G}_{\gamma,s}\right) \le \mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^2) + \frac{1}{2}\mathbf{P}\left(\chi \le \gamma, \mathsf{G}_{\gamma,s}\right)$ which implies that $\mathbf{P}\left(\chi \le \gamma, \mathsf{G}_{\gamma,s}\right) \le 2\mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^2)$ which gives us the bound in (6.2.12), completing the proof. $\square$

6.3 Modulus of Continuity for rescaled CDRP measures

The main goal of this section is to establish quenched modulus of continuity estimates: Proposition 6.3.3 and Proposition 6.3.3-(point-to-line), for CDRP measures under long-time scalings. The proof of these propositions requires detailed study of the tail probabilities of two-point difference when scaled according to long-time. This is conducted in Proposition 6.3.1 and Proposition 6.3.1-(point-to-line) respectively. One of the key technical inputs in the proofs of Propositions 6.3.1 and 6.3.1-(point-to-line) is a parabolic decay estimate of a multivariate spatial process involving several long-time KPZ sheets. This estimate appears in Lemma 6.3.7 and is proved in Section 6.3.1. In the following text, we first state those Propositions 6.3.1 and 6.3.1-(point-to-line) and assuming their validity, we state and prove the modulus of continuity estimates. Proofs of Proposition 6.3.1 and 6.3.1-(point-to-line) are deferred to Section 6.3.2.

Proposition 6.3.1 (Long-time two-point difference)**.** *Fix any $\varepsilon \in (0,1]$, $\delta \in (0,\frac{1}{2})$, and $\tau \ge 1$. Take $x \in [-\tau\varepsilon^{-\frac{2}{3}}, \tau\varepsilon^{-\frac{2}{3}}]$. Let $V \sim \mathsf{CDRP}(0,0;x,\varepsilon^{-1})$. For $t \in [0,1]$, set $L_t^{(\varepsilon)} := \varepsilon^{\frac{2}{3}}V(\varepsilon^{-1}t)$. There exist two absolute constants $\mathrm{C}_1(\tau,\delta) > 0$ and $\mathrm{C}_2(\tau,\delta) > 0$ such that for all $m \ge 1$ and $t \ne s \in [0,1]$ we have*

$$\mathbf{P}\left[\mathbf{P}^{\xi}(|L_s^{(\varepsilon)} - L_t^{(\varepsilon)}| \ge m|s-t|^{\frac{1}{2}-\delta}) \ge \mathrm{C}_1\exp(-\tfrac{1}{\mathrm{C}_1}m^2)\right] \le \mathrm{C}_2\exp\left(-\tfrac{1}{\mathrm{C}_2}m^3\right).$$

We have the following point-to-line analogue.

Proposition 6.3.1-(point-to-line). *Fix any $\varepsilon \in (0,1]$, $\delta \in (0,\frac{1}{2})$. Let $V \sim \mathsf{CDRP}(0,0;*,\varepsilon^{-1})$. For $t \in [0,1]$, set $L_{t,*}^{(\varepsilon)} := \varepsilon^{\frac{2}{3}}V(\varepsilon^{-1}t)$. There exist two absolute constants $\mathrm{C}_1(\delta) > 0$ and $\mathrm{C}_2(\delta) > 0$ such that for all $m \ge 1$ and $t \ne s \in [0,1]$ we have*

$$\mathbf{P}\left[\mathbf{P}_*^{\xi}(|L_{s,*}^{(\varepsilon)} - L_{t,*}^{(\varepsilon)}| \ge m|s-t|^{\frac{1}{2}-\delta}) \ge \mathrm{C}_1\exp(-\tfrac{1}{\mathrm{C}_1}m^2)\right] \le \mathrm{C}_2\exp\left(-\tfrac{1}{\mathrm{C}_2}m^3\right).$$

Remark 6.3.2. In the above propositions, the quenched probability of the tail event of two-point difference of rescaled polymers is viewed as a random variable. The above propositions provide quantitative decay estimates of this random variable being away from zero for point-to-point and point-to-line polymers under long-time regime.

Proposition 6.3.3 (Quenched Modulus of Continuity)**.** *Fix $\varepsilon \in (0,1]$, $\delta \in (0, \frac{1}{2})$ and $\tau \geq 1$. Take $y \in [-\tau\varepsilon^{-\frac{2}{3}}, \tau\varepsilon^{-\frac{2}{3}}]$. Let $V \sim \mathsf{CDRP}(0,0;y,\varepsilon^{-1})$. Set $L_t^{(\varepsilon)} := \varepsilon^{\frac{2}{3}}V(\varepsilon^{-1}t)$ for $t \in [0,1]$. Then there exist two constants $\mathrm{C}_1(\tau,\delta) > 0$ and $\mathrm{C}_2(\tau,\delta) > 0$ such that for all $m \geq 1$ we have*

$$\mathbf{P}\left[\mathbf{P}^{\xi}\left(\sup_{t\neq s\in[0,1]} \frac{|L_s^{(\varepsilon)} - L_t^{(\varepsilon)}|}{|t-s|^{\frac{1}{2}-\delta}\log\frac{2}{|t-s|}} \geq m\right) \geq \mathrm{C}_1 \exp\left(-\tfrac{1}{\mathrm{C}_1}m^2\right)\right] \leq \mathrm{C}_2 \exp\left(-\tfrac{1}{\mathrm{C}_2}m^3\right). \tag{6.3.1}$$

Proposition 6.3.3-(point-to-line). *Fix $\varepsilon \in (0,1]$, $\delta \in (0,\frac{1}{2})$. Let $V \sim \mathsf{CDRP}(0,0;*,\varepsilon^{-1})$. For $t \in [0,1]$, set $L_{t,*}^{(\varepsilon)} := \varepsilon^{\frac{2}{3}}V(\varepsilon^{-1}t)$. Then there exist two constants $\mathrm{C}_1(\delta) > 0$ and $\mathrm{C}_2(\delta) > 0$ such that for all $m \geq 1$ we have*

$$\mathbf{P}\left[\mathbf{P}_*^{\xi}\left(\sup_{t\neq s\in[0,1]} \frac{|L_{s,*}^{(\varepsilon)} - L_{t,*}^{(\varepsilon)}|}{|t-s|^{\frac{1}{2}-\delta}\log\frac{2}{|t-s|}} \geq m\right) \geq \mathrm{C}_1 \exp\left(-\tfrac{1}{\mathrm{C}_1}m^2\right)\right] \leq \mathrm{C}_2 \exp\left(-\tfrac{1}{\mathrm{C}_2}m^3\right).$$

Remark 6.3.4. The paths of continuum directed random polymer are known to be Hölder continuous with exponent γ, for every $\gamma < 1/2$ (see [4, Theorem 4.3]). Our Theorem 6.3.3 corroborates this fact by giving quantitative tail bounds to the quenched modulus of continuity.

Before proving Propositions 6.3.3 and 6.3.3-(point-to-line), we present below a few important corollaries for point-to-point long-time polymer. Similar corollaries hold for point-to-line case as well.

Corollary 6.3.5. *Fix $\varepsilon \in (0,1]$, and $\tau \geq 1$. Take $x \in [-\tau\varepsilon^{-\frac{2}{3}}, \tau\varepsilon^{-\frac{2}{3}}]$. Let $V \sim \mathsf{CDRP}(0,0;x,\varepsilon^{-1})$. For $t \in [0,1]$ set $L_t^{(\varepsilon)} := \varepsilon^{\frac{2}{3}}V(\varepsilon^{-1}t)$. Then there exist two constants $\mathrm{C}_1(\tau) > 0$ and $\mathrm{C}_2(\tau) > 0$ such that for all $m \geq 1$ we have*

$$\mathbf{P}\left[\mathbf{P}^{\xi}\left(\sup_{t\in[0,1]} |L_t^{(\varepsilon)}| \geq m\right) \geq \mathrm{C}_1 \exp\left(-\tfrac{1}{\mathrm{C}_1}m^2\right)\right] \leq \mathrm{C}_2 \exp\left(-\tfrac{1}{\mathrm{C}_2}m^3\right). \tag{6.3.2}$$

Proof. Set $s = 0$ and $\rho = 1 + \sup_{t\in(0,1]} t^{1/4}\log\frac{2}{t} \in (1,\infty)$. By Proposition 6.3.3, with $\delta = \frac{1}{4}$ there exist $\mathrm{C}_1(\tau)$ and $\mathrm{C}_2(\tau)$ such that for all $m \geq 1$, (6.3.1) holds with $s = 0$. Replacing m with m/ρ in (6.3.1) yields that

$$\begin{aligned}
&\mathbf{P}\left[\mathbf{P}^{\xi}\left(\sup_{t\in[0,1]} |L_t^{(\varepsilon)}| \geq m\right) \geq \mathrm{C}_1 \exp\left(-\tfrac{1}{\mathrm{C}_1}m^2\right)\right] \\
&\leq \mathbf{P}\left[\mathbf{P}^{\xi}\left(\sup_{t\in[0,1]} \frac{|L_t^{(\varepsilon)}|}{t^{\frac{1}{4}}\log\frac{2}{t}} \geq \frac{m}{\rho}\right) \geq \mathrm{C}_1 \exp\left(-\tfrac{1}{\mathrm{C}_1}(\tfrac{m}{\rho})^2\right)\right] \leq \mathrm{C}_2 \exp\left(-\tfrac{1}{\mathrm{C}_2}(\tfrac{m}{\rho})^3\right).
\end{aligned}$$

Adjusting C_2 further we get the desired result. □

From Proposition 6.3.3, we also obtain the annealed modulus of continuity.

Corollary 6.3.6 (Annealed Modulus of Continuity)**.** *Fix* $\varepsilon \in (0,1]$*,* $\delta \in (0,\frac{1}{2})$ *and* $\tau \geq 1$*. Take* $y \in [-\tau\varepsilon^{-\frac{2}{3}}, \tau\varepsilon^{-\frac{2}{3}}]$*. Let* $V \sim \mathsf{CDRP}(0,0;y,\varepsilon^{-1})$*. Set* $L_t^{(\varepsilon)} := \varepsilon^{\frac{2}{3}}V(\varepsilon^{-1}t)$ *for* $t \in [0,1]$*. Then there exists a constant* $\mathrm{C}(\tau,\delta) > 0$ *such that for all* $m \geq 1$ *we have*

$$\mathbf{P}\left(\sup_{t\neq s\in[0,1]} \frac{|L_s^{(\varepsilon)} - L_t^{(\varepsilon)}|}{|t-s|^{\frac{1}{2}-\delta}\log\frac{2}{|t-s|}} \geq m\right) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}m^2\right). \tag{6.3.3}$$

Clearly one has similar corollaries for the point-to-line version which follow from Proposition 6.3.3-(point-to-line) instead. For brevity, we do not record them separately. We now assume Proposition 6.3.1 (Proposition 6.3.1-(point-to-line)) and complete the proof of Proposition 6.3.3 (Proposition 6.3.3-(point-to-line)).

Proof of Propositions 6.3.3 and 6.3.3-(point-to-line). Fix $\tau \geq 1$ and $m \geq 16\tau^2 + 1$. The main idea is to mimic Levy's proof of modulus of continuity of Brownian motion. Since our proposition deals with quenched versions, we keep the proof here for the sake of completeness. We only prove (6.3.1) using Proposition 6.3.1. Proof of Proposition 6.3.3-(point-to-line) follows in a similar manner using Proposition 6.3.1-(point-to-line). To prove (6.3.1), we first control the modulus of continuity on dyadic points of $[0,1]$. Fix $\delta > 0$ and set $\gamma = \frac{1}{2} - \delta$. Define

$$\|L^{(\varepsilon)}\|_n := \sup_{k=\{1,\dots,2^n\}} \left| L^{(\varepsilon)}_{k2^{-n}} - L^{(\varepsilon)}_{(k-1)2^{-n}} \right|, \quad \|L^{(\varepsilon)}\| := \sup_{n\geq 0} \frac{\|L^{(\varepsilon)}\|_n 2^{n\gamma}}{n+1}.$$

Observe that by union bound

$$\mathbf{P}^{\xi}\left(\|L^{(\varepsilon)}\| \geq m\right) \leq \sum_{n=0}^{\infty}\sum_{k=1}^{2^n} \mathbf{P}^{\xi}\left(\left|L^{(\varepsilon)}_{k2^{-n}} - L^{(\varepsilon)}_{(k-1)2^{-n}}\right| \geq m2^{-n\gamma}(n+1)\right).$$

Thus in light of Proposition 6.3.1 we see that with probability at least

$$1 - \sum_{n=0}^{\infty} \mathrm{C}_2 2^n \exp\left(-\tfrac{1}{\mathrm{C}_2} m^3 (n+1)^3\right) \geq 1 - \mathrm{C}_2' \exp\left(-\tfrac{1}{\mathrm{C}_2'} m^3\right)$$

we have

$$\mathbf{P}^{\xi}\left(\|L^{(\varepsilon)}\| \geq m\right) \leq \sum_{n=0}^{\infty} \mathrm{C}_1 2^n \exp\left(-\tfrac{1}{\mathrm{C}_1} m^2 (n+1)^2\right) \leq \mathrm{C}_1' \exp\left(-\tfrac{1}{\mathrm{C}_1'} m^2\right).$$

Finally one can extend the results to all points by continuity of $L^{(\varepsilon)}$ and observing the following string of inequalities that holds deterministically. For any $0 \leq s < t \leq 1$ we have

$$|L^{(\varepsilon)}_t - L^{(\varepsilon)}_s| \leq \sum_{n=1}^{\infty} \left| L^{(\varepsilon)}_{2^{-n}\lfloor 2^n t\rfloor} - L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} t\rfloor} - L^{(\varepsilon)}_{2^{-n}\lfloor 2^n s\rfloor} + L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} s\rfloor} \right|. \tag{6.3.4}$$

Note that we have

$$\begin{aligned}
&\left| L^{(\varepsilon)}_{2^{-n}\lfloor 2^n t\rfloor} - L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} t\rfloor} - L^{(\varepsilon)}_{2^{-n}\lfloor 2^n s\rfloor} + L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} s\rfloor} \right| \\
&\leq \left| L^{(\varepsilon)}_{2^{-n}\lfloor 2^n t\rfloor} - L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} t\rfloor} \right| + \left| L^{(\varepsilon)}_{2^{-n}\lfloor 2^n s\rfloor} - L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} s\rfloor} \right| \leq 2\|L^{(\varepsilon)}\|_n,
\end{aligned}$$

and

$$\left| L^{(\varepsilon)}_{2^{-n}\lfloor 2^n t\rfloor} - L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} t\rfloor} - L^{(\varepsilon)}_{2^{-n}\lfloor 2^n s\rfloor} + L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} s\rfloor} \right|$$

$$\leq \left|L^{(\varepsilon)}_{2^{-n}\lfloor 2^n t\rfloor} - L^{(\varepsilon)}_{2^{-n}\lfloor 2^n s\rfloor}\right| + \left|L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} t\rfloor} - L^{(\varepsilon)}_{2^{-n+1}\lfloor 2^{n-1} s\rfloor}\right| \leq 2(t-s)2^n \|L^{(\varepsilon)}\|_n.$$

Combining the above two inequalities we get

$$\begin{aligned}\text{r.h.s. of (6.3.4)} &\leq \sum_{n=1}^{\infty} 2\left(|t-s|2^n \wedge 2\right)\|L^{(\varepsilon)}\|_n \\ &\leq \|L^{(\varepsilon)}\| \sum_{n=1}^{\infty} (n+1)2^{-n\gamma}\left(|t-s|2^n \wedge 2\right) \leq c_2 \|L^{(\varepsilon)}\| \cdot |t-s|^{\gamma} \log \tfrac{2}{|t-s|}.\end{aligned}$$

where $c_2 > 0$ is an absolute constant. Combining this with the bound for $\mathbf{P}^{\xi}(\|L^{(\varepsilon)}\| \geq m)$, completes the proof. □

6.3.1 Tail bounds for multivariate spatial process

Recall the KPZ sheet $\mathcal{H}(\cdot,\cdot;\cdot,\cdot)$ defined in (6.2.1). The core idea behind the proof of Propositions 6.3.1 and 6.3.1-(point-to-line) is to establish parabolic decay estimates of sum of several KPZ sheets scaled according to long-time. We record this parabolic decay estimate in the following Lemma 6.3.7.

Lemma 6.3.7 (Long-time multivariate spatial process tail bound)**.** *Fix any $k \in \mathbb{Z}_{>0}$ and $\nu \in (0,1)$. Set $x_0 = 0$, and $\vec{x} := (x_1, \ldots, x_k)$. For any $\varepsilon \in (0,1)$ consider $0 = t_0 < t_1 < \cdots < t_k = 1$. Set $\vec{t} := (t_1, \ldots, t_k)$. Then there exists a constant $\mathrm{C} = \mathrm{C}(k,\nu)$ such that for all $s > 0$ we have*

$$\mathbf{P}\left(\sup_{\vec{x}\in\mathbb{R}^k}\left[F_{\vec{t};\varepsilon}(\vec{x}) + \sum_{i=0}^{k-1}\frac{(1-\nu)(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}\right] \geq s\right) \leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^{3/2}\right). \tag{6.3.5}$$

where

$$\begin{aligned}F_{\vec{t};\varepsilon}(\vec{x}) := \varepsilon^{1/3}\sum_{i=0}^{k-1}\Big[&\mathcal{H}(x_i\varepsilon^{-2/3}, \varepsilon^{-1}t_i; x_{i+1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{i+1}) + \tfrac{\varepsilon^{-1}(t_{i+1}-t_i)}{24} \\ &+\mathbf{1}\{t_{i+1}-t_i \leq \varepsilon\}\cdot \log\sqrt{2\pi\varepsilon^{-1}(t_{i+1}-t_i)}\Big].\end{aligned} \tag{6.3.6}$$

Proof. For clarity, we split the proof into three steps.

Step 1. Let us fix any $\varepsilon \in (0,1)$ consider $0 = t_0 < t_1 < \cdots < t_k = 1$. For brevity, we denote $F(\vec{x}) := F_{\vec{t};\varepsilon}(\vec{x})$ and set

$$\overline{F}(\vec{x}) := F(\vec{x}) + \sum_{i=0}^{k-1} \frac{(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}. \tag{6.3.7}$$

For any $\vec{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$, set $V_{\vec{a}} := [a_1, a_1+1] \times \cdots \times [a_k, a_k+1]$ and set

$$\|\vec{a}\|^2 := a_1^2 + \min_{\vec{x}\in V_{\vec{a}}} \sum_{i=1}^{k-1} (x_{i+1}-x_i)^2. \tag{6.3.8}$$

We claim that for any $\vec{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ and $\nu \in (0,1)$

$$\mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}}\left[F(\vec{x}) + \sum_{i=0}^{k-1} \frac{(1-\nu)(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}\right] \ge s\right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}(s^{3/2} + \|\vec{a}\|^3)\right) \tag{6.3.9}$$

for some $\mathrm{C} = \mathrm{C}(k,\nu) > 0$. Assuming (6.3.9) by union bound we obtain

$$\begin{aligned}
\text{l.h.s of (6.3.5)} &= \mathbf{P}\left(\sup_{\vec{x}\in\mathbb{R}^k}\left[F(\vec{x}) + \sum_{i=0}^{k-1} \frac{(1-\nu)(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}\right] \ge s\right) \\
&\le \sum_{\vec{a}\in\mathbb{Z}^k} \mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}}\left[F(\vec{x}) + \sum_{i=0}^{k-1} \frac{(1-\nu)(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}\right] \ge s\right) \\
&\le \sum_{\vec{a}\in\mathbb{Z}^k} \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}(s^{3/2} + \|\vec{a}\|^3)\right).
\end{aligned}$$

The r.h.s. of the above display is upper bounded by $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$ and proves (6.3.5). Thus it suffices to verify (6.3.9) in the rest of the proof.

Step 2. In this step, we prove the claim in (6.3.9). Note that

$$\mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}}\left[F(\vec{x}) + \sum_{i=0}^{k-1} \frac{(1-\nu)(x_{i+1}-x_i)^2}{2(t_{i+1}-t_i)}\right] \ge s\right) \le \mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}} |\overline{F}(\vec{x})| \ge s + \tfrac{\nu}{2}\|\vec{a}\|^2\right)$$

by the definition of $\overline{F}(\cdot)$ in (6.3.7) and the defintion of $\|\vec{a}\|^2$ from (6.3.8). Applying union bound yields

$$\begin{aligned}
&\mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}} |\overline{F}(\vec{x})| \ge s + \tfrac{\nu}{2}\|\vec{a}\|^2\right) \\
&\le \mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}} |\overline{F}(\vec{x}) - \overline{F}(\vec{a})| \ge \tfrac{s}{2} + \tfrac{\nu}{4}\|\vec{a}\|^2\right) + \mathbf{P}\left(|\overline{F}(\vec{a})| \ge \tfrac{s}{2} + \tfrac{\nu}{4}\|\vec{a}\|^2\right). \qquad (6.3.10)
\end{aligned}$$

In the rest of the proof, we bound both summands on the r.h.s of (6.3.10) from above by $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}(s^{3/2}+\|\vec{a}\|^3))$ individually. To control the first term, we first need an a priori estimate. We claim that for all $u \in [0,1]$, $i = 1, 2, \ldots, k$ and $s > 0$ we have

$$\mathbf{P}\left(\overline{F}(\vec{a} + e_i \cdot u) - \overline{F}(\vec{a}) \ge s u^{1/4}\right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}} s^{3/2}\right). \qquad (6.3.11)$$

for some absolute constant $\mathrm{C} > 0$. We will prove (6.3.11) in the next step. Given (6.3.11), appealing to Lemma 3.3 in [140] with $\alpha = \alpha_i = \frac{1}{4}, \beta = \beta_i = \frac{3}{2}, r = r_i = 1$, we get that for all $m > 0$

$$\mathbf{P}\left(\sup_{\vec{x}\in V_{\vec{a}}} |\overline{F}(\vec{x}) - \overline{F}(\vec{a})| \ge m\right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}} m^{3/2}\right).$$

Taking $m = \frac{s}{2} + \frac{\nu}{4}\|\vec{a}\|^2$ in above, this yields the desired estimate for the first term in (6.3.10).

For the second term in (6.3.10), via the definition of $\overline{F}$ in (6.3.7) applying union bounds we have

$$\begin{aligned}
&\mathbf{P}\left(|\overline{F}(\vec{a})| \ge \tfrac{s}{4} + \tfrac{\nu}{4}\|\vec{a}\|^2\right) \\
&\le \sum_{i=0}^{k-1} \mathbf{P}\left(\left|\varepsilon^{1/3}\mathcal{H}(a_i\varepsilon^{-2/3}, \varepsilon^{-1}t_i; a_{i+1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{i+1}) + \tfrac{\varepsilon^{-2/3}(t_{i+1}-t_i)}{24}\right.\right. \\
&\qquad\qquad \left.\left. + \tfrac{(a_{i+1}-a_i)^2}{2(t_{i+1}-t_i)} + \varepsilon^{1/3}\mathbf{1}\{t_{i+1} - t_i \le \varepsilon\} \cdot \log\sqrt{2\pi\varepsilon^{-1}(t_{i+1}-t_i)}\right| \ge \tfrac{s}{4k} + \tfrac{\nu}{4k}\|\vec{a}\|^2\right) \\
&\le \sum_{i=0}^{k-1} \mathbf{P}\left(\left|\varepsilon^{1/3}\mathcal{H}(0, \varepsilon^{-1}(t_{i+1}-t_i)) + \tfrac{\varepsilon^{-2/3}(t_{i+1}-t_i)}{24}\right.\right. \\
&\qquad\qquad \left.\left. + \varepsilon^{1/3}\mathbf{1}\{t_{i+1} - t_i \le \varepsilon\} \cdot \log\sqrt{2\pi\varepsilon^{-1}(t_{i+1}-t_i)}\right| \ge \tfrac{s}{4k} + \tfrac{\nu}{4k}\|\vec{a}\|^2\right) \qquad (6.3.12)
\end{aligned}$$

where the last line follows from stationarity of the shifted version of $\mathcal{H}$. Now if $\varepsilon^{-1}(t_{i+1}-t_i) > 1$, we may use long-time scaling to get

$$\varepsilon^{1/3}\mathcal{H}(0,\varepsilon^{-1}(t_{i+1}-t_i)) + \frac{\varepsilon^{-2/3}(t_{i+1}-t_i)}{24} = \frac{\mathfrak{h}_{\varepsilon^{-1}(t_{i+1}-t_i)}(0)}{(t_{i+1}-t_i)^{-1/3}}.$$

Using the fact that $\varepsilon < |t_{i+1}-t_i| \le 1$ along with the one-point long-time tail estimates from Proposition 6.2.2 (b) we get

$$\begin{aligned}\mathbf{P}\left(|\mathfrak{h}_{\varepsilon^{-1}(t_{i+1}-t_i)}(0)| \ge (t_{i+1}-t_i)^{-1/3}(\tfrac{s}{4k}+\tfrac{\nu}{4k}\|\vec{a}\|^2)\right) &\le \mathbf{P}\left(|\mathfrak{h}_{\varepsilon^{-1}(t_{i+1}-t_i)}(0)| \ge \tfrac{s}{4k}+\tfrac{\nu}{4k}\|\vec{a}\|^2\right) \\ &\le \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}(s+\|\vec{a}\|^2)^{3/2}) \\ &\le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}(s^{3/2}+\|\vec{a}\|^3)\right),\end{aligned}$$

for some constant $\mathrm{C}=\mathrm{C}(k,\nu)>0$. If $\varepsilon^{-1}(t_{i+1}-t_i)\le 1$, we may use short-time scaling to get

$$\begin{aligned}\varepsilon^{1/3}\mathcal{H}(0,\varepsilon^{-1}(t_{i+1}-t_i)) + \frac{\varepsilon^{-2/3}(t_{i+1}-t_i)}{24} + \varepsilon^{1/3}\log\sqrt{2\pi\varepsilon^{-1}(t_{i+1}-t_i)} \\ = \varepsilon^{1/3}\big(\tfrac{\pi\varepsilon^{-1}(t_{i+1}-t_i)}{4}\big)^{1/4}\mathfrak{g}_{\varepsilon^{-1}(t_{i+1}-t_i)}(0) + \frac{\varepsilon^{-2/3}(t_{i+1}-t_i)}{24}.\end{aligned}$$

The linear term above is uniformly bounded in this case. Furthermore,

$$\varepsilon^{1/3}\big(\tfrac{\pi\varepsilon^{-1}(t_{i+1}-t_i)}{4}\big)^{1/4} = \big(\tfrac{\pi(t_{i+1}-t_i)}{4}\big)^{1/4}\varepsilon^{1/12} \le 1.$$

Thus, in this case, appealing to one-point short-time tail estimates from Proposition 6.2.3 (b), we have

$$\text{r.h.s. of (6.3.12)} \le \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}(s+\|\vec{a}\|^2)^{3/2}) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}(s^{3/2}+\|\vec{a}\|^3)\right)$$

for some constant $\mathrm{C}=\mathrm{C}(k,\nu)>0$.

This proves the required bound for the second term in (6.3.10). Combining the bounds for the two terms in (6.3.11), we thus arrive at (6.3.9). Hence, all we are left to show is (6.3.11) which we do in the next step.

Step 3. Fix $\vec{a} \in \mathbb{Z}^k$, fix $i = 1, 2, \ldots, k$. The goal of this step is to show (6.3.11). Towards this end, note that for each coordinate vector $e_i, i = 1, \ldots, k-1$, and for $u \in [0,1]$ observe that

$$\begin{aligned}
&\overline{F}(\vec{a} + e_i \cdot u) - \overline{F}(\vec{a}) \\
&= \varepsilon^{1/3}\left[\mathcal{H}(a_{i-1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{i-1}; (a_i+u)\varepsilon^{-2/3}, \varepsilon^{-1}t_i) - \mathcal{H}(a_{i-1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{i-1}; a_i\varepsilon^{-2/3}, \varepsilon^{-1}t_i)\right] \\
&\qquad\qquad + \frac{(a_{i-1} - a_i - u)^2 - (a_{i-1} - a_i)^2}{2(t_i - t_{i-1})} \\
&\quad + \varepsilon^{1/3}\left[\mathcal{H}((a_i+u)\varepsilon^{-2/3}, \varepsilon^{-1}t_i; a_{i+1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{i+1}) - \mathcal{H}(a_i\varepsilon^{-2/3}, \varepsilon^{-1}t_i; a_{i+1}\varepsilon^{-2/3}, \varepsilon^{-1}t_{i+1})\right] \\
&\qquad\qquad + \frac{(a_{i+1} - a_i - u)^2 - (a_{i+1} - a_i)^2}{2(t_{i+1} - t_i)}.
\end{aligned}$$

Thus using distributional identities (see Remark 6.2.1) by union bound for all $s > 0$ we get that

$$\begin{aligned}
&\mathbf{P}\left(|\overline{F}(\vec{a} + e_i \cdot u) - \overline{F}(\vec{a})| \ge su^{\frac{1}{4}}\right) \\
&\le \mathbf{P}\left(\varepsilon^{\frac{1}{3}}\left|\overline{\mathcal{H}}_{\varepsilon^{-1}(t_i - t_{i-1})}((a_i + u - a_{i-1})\varepsilon^{-2/3}) - \overline{\mathcal{H}}_{\varepsilon^{-1}(t_i - t_{i-1})}((a_i - a_{i-1})\varepsilon^{-2/3})\right| \ge \tfrac{s}{2}u^{\frac{1}{4}}\right) \qquad (6.3.13) \\
&\quad + \mathbf{P}\left(\varepsilon^{\frac{1}{3}}\left|\overline{\mathcal{H}}_{\varepsilon^{-1}(t_{i+1} - t_i)}((a_i + u - a_{i+1})\varepsilon^{-2/3}) - \overline{\mathcal{H}}_{\varepsilon^{-1}(t_{i+1} - t_i)}((a_i - a_{i+1})\varepsilon^{-2/3})\right| \ge \tfrac{s}{2}u^{\frac{1}{4}}\right), \qquad (6.3.14)
\end{aligned}$$

where $\overline{\mathcal{H}}_t(x) := \mathcal{H}(x, t) + \frac{x^2}{2t}$. We now proceed to bound the second term on the r.h.s. of above display (that is the term in (6.3.14)); the bound for the first term follows analogously.

Case 1. $\varepsilon^{-1}(t_{i+1} - t_i) \ge 1$. We then use the long-time scaling to conclude

$$\begin{aligned}
&\varepsilon^{\frac{1}{3}}\left|\overline{\mathcal{H}}_{\varepsilon^{-1}(t_{i+1} - t_i)}((a_i + u - a_{i+1})\varepsilon^{-2/3}) - \overline{\mathcal{H}}_{\varepsilon^{-1}(t_{i+1} - t_i)}((a_i - a_{i+1})\varepsilon^{-2/3})\right| \\
&\qquad = \frac{\overline{\mathfrak{h}}_{\varepsilon^{-1}(t_{i+1} - t_i)}\left(\frac{a_i + u - a_{i+1}}{(t_{i+1} - t_i)^{2/3}}\right) - \overline{\mathfrak{h}}_{\varepsilon^{-1}(t_{i+1} - t_i)}\left(\frac{a_i - a_{i+1}}{(t_{i+1} - t_i)^{2/3}}\right)}{(t_{i+1} - t_i)^{-1/3}}
\end{aligned}$$

where $\overline{\mathfrak{h}}_s(x) := \mathfrak{h}_s(x) + \frac{x^2}{2}$. We now consider two cases depending on the value of u.

Case 1.1. Suppose $u \in [0, (t_{i+1} - t_i)^{2/3}]$. By Proposition 6.2.2 (c) with $\gamma \mapsto \frac{u}{(t_{i+1}-t_i)^{2/3}}$, and using the fact that $\sqrt{\gamma} \le u^{1/4}(t_{i+1} - t_i)^{-1/3}$, we see that (6.3.14) $\le \mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$ for some $\mathrm{C} > 0$ in this case.

Case 1.2. For $u \in [(t_{i+1}-t_i)^{2/3}, 1]$, we rely on one-point tail bounds. Indeed applying union bound we have

$$\begin{aligned}(6.3.13) &\le \mathbf{P}\left(\left|\frac{\overline{\mathfrak{h}}_{\varepsilon^{-1}(t_{i+1}-t_i)}\left(\frac{a_i+u-a_{i+1}}{(t_{i+1}-t_i)^{2/3}}\right)}{(t_{i+1}-t_i)^{-1/3}}\right| \ge \tfrac{s}{8}u^{1/4}\right) + \mathbf{P}\left(\left|\frac{\overline{\mathfrak{h}}_{\varepsilon^{-1}(t_{i+1}-t_i)}\left(\frac{a_i-a_{i+1}}{(t_{i+1}-t_i)^{2/3}}\right)}{(t_{i+1}-t_i)^{-1/3}}\right| \ge \tfrac{s}{8}u^{1/4}\right) \\ &\le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^{3/2}u^{3/8}(t_{i+1}-t_i)^{-1/2}\right) \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}s^{3/2}\right).\end{aligned}$$

The penultimate inequality above follows from Proposition 6.2.2 (a), (b) and the last one follows from the fact $u \ge (t_{i+1} - t_i)^{2/3}$ and $t_{i+1} - t_i \in (0, 1]$.

Case 2. $\varepsilon^{-1}(t_{i+1} - t_i) \le 1$. We here use the short-time scaling to conclude

$$\begin{aligned}&\varepsilon^{\frac{1}{3}}\left|\overline{\mathcal{H}}_{\varepsilon^{-1}(t_{i+1}-t_i)}((a_i+u-a_{i+1})\varepsilon^{-2/3}) - \overline{\mathcal{H}}_{\varepsilon^{-1}(t_{i+1}-t_i)}((a_i-a_{i+1})\varepsilon^{-2/3})\right| \\ &\quad = \left(\tfrac{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}{4}\right)^{\frac{1}{4}}\left[\overline{\mathfrak{g}}_{\varepsilon^{-1}(t_{i+1}-t_i)}\left(\frac{2(a_i+u-a_{i+1})}{\sqrt{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}}\right) - \overline{\mathfrak{g}}_{\varepsilon^{-1}(t_{i+1}-t_i)}\left(\frac{2(a_i-a_{i+1})}{\sqrt{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}}\right)\right]\end{aligned}$$

where $\overline{\mathfrak{g}}_s(x) := \mathfrak{g}_s(x) + \frac{(\pi s/4)^{3/4}x^2}{2s}$. We again consider two cases depending on the value of u.

Case 2.1. Suppose $u \in (0, \frac{\sqrt{\pi}}{2}\varepsilon^{-1/3}(t_{i+1} - t_i))$. Then $\frac{2u}{\sqrt{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}} < \sqrt{\varepsilon^{-1}(t_{i+1} - t_i)}$. This allows us to apply Proposition 6.2.4 with $\gamma \mapsto \frac{2u}{\sqrt{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}}$ and $t \mapsto \varepsilon^{-1}(t_{i+1} - t_i)$. Using the fact that $u^{1/2} \le u^{1/4}$ for $u \in [0, 1]$, we see that (6.3.14) $\le \mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$ for some $\mathrm{C} > 0$ in this case.

Case 2.2. For $u \in [\frac{\sqrt{\pi}}{2}\varepsilon^{-1/3}(t_{i+1} - t_i), 1]$, we rely on stationarity and one-point tail bounds (Proposition 6.2.3 (a), (b)). Indeed applying union bound we have

$$\begin{aligned}(6.3.14) &\le \mathbf{P}\left(\left(\tfrac{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}{4}\right)^{\frac{1}{4}}\left|\overline{\mathfrak{g}}_{\varepsilon^{-1}(t_{i+1}-t_i)}\left(\frac{2(a_i+u-a_{i+1})}{\sqrt{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}}\right)\right| \ge \tfrac{s}{8}u^{1/4}\right) \\ &\qquad + \mathbf{P}\left(\left(\tfrac{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}{4}\right)^{\frac{1}{4}}\left|\overline{\mathfrak{g}}_{\varepsilon^{-1}(t_{i+1}-t_i)}\left(\frac{2(a_i-a_{i+1})}{\sqrt{\pi\varepsilon^{1/3}(t_{i+1}-t_i)}}\right)\right| \ge \tfrac{s}{8}u^{1/4}\right)\end{aligned}$$

$$\leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}\left[su^{1/4}(t_{i+1}-t_i)^{-1/4}\varepsilon^{-\frac{1}{12}}\right]^{3/2}\right).$$

As $u \geq \frac{\sqrt{\pi}}{2}\varepsilon^{-1/3}(t_{i+1}-t_i)$, and $\varepsilon \in (0,1)$ we have $u^{1/4}(t_{i+1}-t_i)^{-1/4}\varepsilon^{-1/12} \geq \frac{\sqrt{\pi}}{2}$. Thus the last expression above is at most $\mathrm{C}\exp\left(-\frac{1}{\mathrm{C}}s^{3/2}\right)$.

Combining the above two cases we have (6.3.14) $\leq \mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$ uniformly for $u \in [0,1]$. By the same argument one can show the term in (6.3.13) is also upper bounded by $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$. This yields (6.3.11) for $i = 1, 2, \ldots, k-1$.

Finally for $i = k$, observe that

$$\begin{aligned}
&\overline{F}(\vec{a}+e_k\cdot u)-\overline{F}(\vec{a})\\
&=\varepsilon^{1/3}\left[\mathcal{H}(a_{k-1}\varepsilon^{-2/3},\varepsilon^{-1}t_{k-1};(a_k+u)\varepsilon^{-2/3},\varepsilon^{-1})-\mathcal{H}(a_{k-1}\varepsilon^{-2/3},\varepsilon^{-1}t_{k-1};a_k\varepsilon^{-2/3},\varepsilon^{-1})\right]\\
&\qquad\qquad\qquad+\frac{(a_{k-1}-a_k-u)^2-(a_{k-1}-a_k)^2}{2(1-t_{k-1})}.
\end{aligned}$$

Then (6.3.11) follows for $i = k$ by the exact same computations as above. This completes the proof of the lemma. □

6.3.2 Proof of Proposition 6.3.1 and 6.3.1-(point-to-line)

We now present the proofs of Proposition 6.3.1 and 6.3.1-(point-to-line).

Proof of Proposition 6.3.1. We assume $m \geq 16\tau^2 + 1$. Otherwise the constant C_1 can be chosen large enough so that the inequality holds trivially. Without loss of generality assume $s < t$. We first consider the case when $s, t \in (0,1)$. Note that

$$\begin{aligned}
&\mathbf{P}^{\xi}(|L_s^{(\varepsilon)}-L_t^{(\varepsilon)}|\geq m|s-t|^{\frac{1}{2}-\delta})\\
&=\iint\limits_{|u-v|\geq m\varepsilon^{-2/3}|s-t|^{\frac{1}{2}-\delta}}\frac{\mathcal{Z}(0,0;u,\varepsilon^{-1}s)\mathcal{Z}(u,\varepsilon^{-1}s;v,\varepsilon^{-1}t)\mathcal{Z}(v,\varepsilon^{-1}t;x,\varepsilon^{-1})}{\mathcal{Z}(0,0;x,\varepsilon^{-1})}\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

We make a change of variable $u = p\varepsilon^{-2/3}$, $v = q\varepsilon^{-2/3}$ and $x = z\varepsilon^{-2/3}$. Then

$$\begin{aligned}&\mathbf{P}^{\xi}(|L_s^{(\varepsilon)} - L_t^{(\varepsilon)}| \geq m|s-t|^{\frac{1}{2}-\delta})\\ =&\varepsilon^{-4/3} \iint\limits_{|p-q|\geq m|s-t|^{\frac{1}{2}-\delta}} \frac{\mathcal{Z}(0,0;p\varepsilon^{-2/3},\frac{s}{\varepsilon})\mathcal{Z}(p\varepsilon^{-2/3},\frac{s}{\varepsilon};q\varepsilon^{-2/3},\frac{t}{\varepsilon})\mathcal{Z}(q\varepsilon^{-2/3},\frac{t}{\varepsilon};z\varepsilon^{-2/3},\frac{1}{\varepsilon})}{\mathcal{Z}(0,0;z\varepsilon^{-2/3},\varepsilon^{-1})}\,\mathrm{d}q\,\mathrm{d}p.\end{aligned} \tag{6.3.15}$$

Recall the multivariate spatial process $F_{\vec{t};\varepsilon}(\vec{x})$ from (6.3.6). Take $k = 3$ and set $\vec{t} = (s,t,1)$, and $\vec{x} = (p,q,z)$. We also set

$$B(\vec{t}) := \mathbf{1}\{s \leq \varepsilon\} \log\sqrt{2\pi\frac{s}{\varepsilon}} + \mathbf{1}\{t-s \leq \varepsilon\}\log\sqrt{2\pi\frac{t-s}{\varepsilon}} + \mathbf{1}\{1-t \leq \varepsilon\}\log\sqrt{2\pi\frac{1-t}{\varepsilon}}.$$

For the numerator of the integrand in (6.3.15) observe that

$$\mathcal{Z}(0,0;p\varepsilon^{-\frac{2}{3}},\tfrac{s}{\varepsilon})\mathcal{Z}(p\varepsilon^{-\frac{2}{3}},\tfrac{s}{\varepsilon};q\varepsilon^{-\frac{2}{3}},\tfrac{t}{\varepsilon})\mathcal{Z}(q\varepsilon^{-\frac{2}{3}},\tfrac{t}{\varepsilon};z\varepsilon^{-\frac{2}{3}},\tfrac{1}{\varepsilon}) = \exp\left[\varepsilon^{-\frac{1}{3}}F_{\vec{t};\varepsilon}(\vec{x}) - \tfrac{\varepsilon^{-1}}{24} - B(\vec{t})\right]. \tag{6.3.16}$$

Set $M = \frac{m^2}{64}$. Applying Lemma 6.3.7 with $\nu = \frac{1}{2}$ and $s = M$, we see that with probability greater than $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^{3/2})$,

$$\text{r.h.s. of (6.3.16)} \leq \exp\left[\varepsilon^{-1/3}M - \varepsilon^{-1/3}\left(\tfrac{p^2}{4s} + \tfrac{(q-p)^2}{4(t-s)} + \tfrac{(z-q)^2}{4(1-t)}\right) - \tfrac{\varepsilon^{-1}}{24} - B(\vec{t})\right]. \tag{6.3.17}$$

On the other hand, for the denominator of the integrand in (6.3.15) by one-point long-time tail bound from Proposition 6.2.2 with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^{3/2})$ we have

$$\mathcal{Z}(0,0;z\varepsilon^{-2/3},\varepsilon^{-1}) \geq \exp\left(\varepsilon^{-1/3}\mathfrak{h}_{\varepsilon^{-1}}(z) - \tfrac{\varepsilon^{-1}}{24}\right) \geq \exp\left(-\varepsilon^{-1/3}(M + \tfrac{1}{2}\tau^2) - \tfrac{\varepsilon^{-1}}{24}\right).$$

Combining the previous equation with (6.3.17) we get that with probability at least $1-\mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^{3/2})$

we have

$$\text{r.h.s. of (6.3.15)} \le \varepsilon^{-\frac{4}{3}} \exp\left(\varepsilon^{-1/3}(2M + \tfrac{1}{2}\tau^2) - B(\vec{t})\right) \cdot$$
$$\iint_{|p-q|\ge m|s-t|^{\frac{1}{2}-\delta}} \exp\left[-\varepsilon^{-1/3}\left(\tfrac{p^2}{4s} + \tfrac{(q-p)^2}{4(t-s)} + \tfrac{(z-q)^2}{4(1-t)}\right)\right] dq\, dp$$
$$\le \varepsilon^{-\frac{4}{3}} \exp\left(\varepsilon^{-\frac{1}{3}}(2M + \tfrac{1}{2}\tau^2 - \tfrac{m^2}{4|t-s|^{2\delta}}) - B(\vec{t})\right) \iint_{\mathbb{R}^2} \exp\left[-\varepsilon^{-\frac{1}{3}}\left(\tfrac{p^2}{4s} + \tfrac{r^2}{4(1-t)}\right)\right] dr\, dp$$
$$= 4\pi\sqrt{s(1-t)}\varepsilon^{-1} \exp\left(\varepsilon^{-\frac{1}{3}}(2M + \tfrac{1}{2}\tau^2 - \tfrac{m^2}{4|t-s|^{2\delta}}) - B(\vec{t})\right). \tag{6.3.18}$$

Observe that

$$\sqrt{r}\exp\left(-\mathbf{1}\{r \le \varepsilon\}\log\sqrt{\tfrac{2\pi r}{\varepsilon}}\right) \le 1. \tag{6.3.19}$$

As $M = \frac{m^2}{64}$ we have $2M - \frac{m^2}{4|t-s|^{2\delta}} \le -\frac{m^2}{8|t-s|^{2\delta}}$. Furthermore $\frac{1}{2}\tau^2 \le \frac{m^2}{16|t-s|^{2\delta}}$ under the condition $m \ge 16\tau^2 + 1$. Thus,

$$\text{r.h.s. of (6.3.18)} \le 4\pi\varepsilon^{-1}\exp\left(-\varepsilon^{-\frac{1}{3}}\frac{m^2}{16|t-s|^{2\delta}} - \mathbf{1}\{t - s \le \varepsilon\}\log\sqrt{\frac{2\pi(t-s)}{\varepsilon}}\right).$$

Clearly the last expression is at most $\mathrm{C}_1\exp(-\frac{1}{\mathrm{C}_1}m^2)$ for some $\mathrm{C}_1 > 0$ depending on τ, δ. This bound holds uniformly over $t, s \in (0,1)$ with $t \neq s$ and $\varepsilon \in (0,1)$. This concludes the proof for $s, t \in (0,1)$.

Finally, when $s = 0$ we have

$$\mathbf{P}^{\xi}(|L_t^{(\varepsilon)}| \ge m|t|^{\frac{1}{2}-\delta}) = \iint_{|v|\ge m\varepsilon^{-2/3}|t|^{\frac{1}{2}-\delta}} \frac{\mathcal{Z}(0, ; v, \varepsilon^{-1}t)\mathcal{Z}(v, \varepsilon^{-1}t; x, \varepsilon^{-1})}{\mathcal{Z}(0, 0; x, \varepsilon^{-1})}\, \mathrm{d}v.$$

The proof can now be completed by following the argument for $s, t \in (0,1)$ case. Indeed, the denominator can be bounded by the exact same manner as above, whereas the numerator can be controlled with the $k = 2$ version of Lemma 6.3.7. The case $t = 1$ is analogous to the case $s = 0$. We have thus established Proposition 6.3.1. □

Proof of Proposition 6.3.1-(point-to-line). We now explain how the above proof can be modified to extend it to the point-to-line version. Fix any $m > 0$ and $M > 1$. Indeed observe that for $0 < s < t < 1$, one has

$$\begin{aligned}
&\mathbf{P}_*^{\xi}(|L_{s,*}^{(\varepsilon)} - L_{t,*}^{(\varepsilon)}| \geq m|s-t|^{\frac{1}{2}-\delta}) \\
=&\varepsilon^{-\frac{4}{3}} \int_{\mathbb{R}} \iint_{|p-q|\geq m|s-t|^{\frac{1}{2}-\delta}} \frac{\mathcal{Z}(0,0;p\varepsilon^{-\frac{2}{3}},\frac{s}{\varepsilon})\mathcal{Z}(p\varepsilon^{-\frac{2}{3}},\frac{s}{\varepsilon};q\varepsilon^{-\frac{2}{3}},\frac{t}{\varepsilon})\mathcal{Z}(q\varepsilon^{-\frac{2}{3}},\frac{t}{\varepsilon};z\varepsilon^{-\frac{2}{3}},\frac{1}{\varepsilon})}{\int_{\mathbb{R}}\mathcal{Z}(0,0;y\varepsilon^{-\frac{2}{3}},\varepsilon^{-1})\mathrm{d}y}\,\mathrm{d}q\,\mathrm{d}p\,\mathrm{d}z.
\end{aligned} \tag{6.3.20}$$

Since Lemma 6.3.7 is a process-level estimate that allows even the endpoint to vary, (6.3.17) continues to hold simultaneously for all $p, q, z \in \mathbb{R}$ with same high probability. However for the lower bound on the denominator, one-point lower-tail bound is not sufficient. Instead, for the denominator we use long-time process-level lower bound from Proposition 4.1 in [117] to get that with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^{3/2})$ we have

$$\int_{\mathbb{R}} \mathcal{Z}(0,0;y\varepsilon^{-2/3},\varepsilon^{-1})\mathrm{d}y \geq \int_{\mathbb{R}} \exp\left(-\tfrac{M+y^2}{\varepsilon^{1/3}} - \tfrac{\varepsilon^{-1}}{24}\right)\mathrm{d}y \geq \mathrm{C}\varepsilon^{\frac{1}{6}}\exp\left(-\varepsilon^{-1/3}M - \tfrac{\varepsilon^{-1}}{24}\right).$$

Combining the previous equation with (6.3.17) we get that with probability at least $1-\mathrm{C}\exp(-\frac{1}{\mathrm{C}}M^{3/2})$ we have

$$\begin{aligned}
\text{r.h.s. of (6.3.20)} \leq \varepsilon^{-\frac{3}{2}} \exp\left(2\varepsilon^{-1/3}M - B(\vec{t})\right) \cdot \\
\int_{\mathbb{R}} \iint_{|p-q|\geq m|s-t|^{\frac{1}{2}-\delta}} \exp\left[-\varepsilon^{-1/3}\left(\tfrac{p^2}{4s} + \tfrac{(q-p)^2}{4(t-s)} + \tfrac{(z-q)^2}{4(1-t)}\right)\right] dq\,dp\,dz.
\end{aligned} \tag{6.3.21}$$

On $|p-q| \geq m|s-t|^{\frac{1}{2}-\delta}$, we have $(q-p)^2/4(t-s) \geq (q-p)^2/8(t-s) + m^2/8|t-s|^{2\delta}$. Applying this inequality followed by expanding the range of integration we get

$$\begin{aligned}
\text{r.h.s. of (6.3.21)} \leq \varepsilon^{-\frac{3}{2}} \exp\left(\varepsilon^{-\frac{1}{3}}(2M - \tfrac{m^2}{8|t-s|^{2\delta}}) - B(\vec{t})\right) \\
\cdot \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp\left[-\varepsilon^{-\frac{1}{3}}\left(\tfrac{p^2}{4s} + \tfrac{r^2}{8(t-s)} + \tfrac{u^2}{4(1-t)}\right)\right] dq\,dr\,du
\end{aligned}$$

$$= \sqrt{2^7\pi^3 s(1-t)(t-s)} \cdot \varepsilon^{-1} \exp\left(\varepsilon^{-\frac{1}{3}}(2M - \tfrac{m^2}{8|t-s|^{2\delta}}) - B(\vec{t})\right).$$

Just as in the proof of Proposition 6.3.1, setting $M = \frac{m^2}{64}$, and using (6.3.19), the above expression can be shown to be at most $\mathrm{C}\exp(-\frac{1}{\mathrm{C}}m^2)$ uniformly over $\varepsilon \in (0,1)$ and $0 < s < t < 1$. This establishes the proposition. □

6.4 Annealed Convergence for short-time and long-time

In this section we prove our main results. In Section 6.4.1 we prove Theorems 6.1.4, 6.1.7, and 6.1.8. In Section 6.4.2, we show Theorem 6.1.10 assuming Conjecture 6.1.9.

6.4.1 Proof of Theorems 6.1.4, 6.1.7, and 6.1.8

In this section we prove results related to short-time and long-time tightness and related pointwise weak convergence. While the proof of long-time tightness relies on modulus of continuity estimates from Proposition 6.3.1 and Proposition 6.3.1-(point-to-line), the proof of short-time tightness relies on the following Brownian relation of annealed law of CDRP.

Lemma 6.4.1 (Brownian Relation)**.** *Let $X \sim \mathsf{CDRP}(0,0;0,t)$ and $Y \sim \mathsf{CDRP}(0,0;*,t)$. For any continuous functional $\mathcal{L} : C([0,t]) \to \mathbb{R}$ we have*

$$\mathbf{E}\left[\mathcal{Z}(0,0;0,t)\sqrt{2\pi t} \cdot \mathcal{L}(X)\right] = \mathbf{E}(\mathcal{L}(B)), \quad \mathbf{E}\left[\mathcal{Z}(0,0;*,t) \cdot \mathcal{L}(Y)\right] = \mathbf{E}(\mathcal{L}(B_*)) \qquad (6.4.1)$$

where B_* and B are standard Brownian motion and standard Brownian bridge on $[0,t]$ respectively.

Remark 6.4.2. Note that $\sqrt{2\pi t} = \frac{1}{p(0,t)}$ where $p(0,t)$ is the heat kernel. Since the Brownian bridge finite-dimensional densities are product of heat kernels divided by $p(0,t)$, this additional factor $\frac{1}{p(0,t)}$ is required in the point-to-point version for appropriate comparison to the the Brownian bridge law (see (6.4.2) below).

Proof. Take $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = t$. The Brownian motion identity appears as Lemma 4.2 in [4]. To show the bridge version note that by Definition 6.1.1, the quantity

$$\mathcal{Z}(0,0;0,t)\mathbf{P}^{\xi}(X(t_1) \in \mathrm{d}x_1, \ldots, X(t_k) \in \mathrm{d}x_k)$$

is product of independent random variables with mean $p(x_{j+1}-x_j, t_{j+1}-t_j)$ where $p(x,t)$ denotes the heat kernel. Noting that $p(0,t) = \frac{1}{\sqrt{2\pi t}}$, and recalling the finite-dimensional distribution of Brownian bridge (Problem 6.11 in [214]) we get that

$$\begin{aligned}\mathbf{E}\left[\mathcal{Z}(0,0;0,t)\sqrt{2\pi t}\cdot \mathbf{P}^{\xi}(X(t_1) \in \mathrm{d}x_1, \ldots, X(t_k) \in \mathrm{d}x_k)\right] &= \frac{1}{p(0,t)}\prod_{j=0}^{k} p(x_{j+1}-x_j, t_{j+1}-t_j) \\ &= \mathbf{P}\left(B(t_1) \in \mathrm{d}x_1, \ldots, B(t_k) \in \mathrm{d}x_k\right).\end{aligned} \tag{6.4.2}$$

(6.4.1) now follows from the above by approximation of $\mathcal{L}$ with simple functions. □

Proof of Theorem 6.1.4. We first show finite-dimensional convergence. Fix $0 = t_0 < t_1 < \cdots < t_{k+1} = 1$. Take $x_1, \ldots, x_k \in \mathbb{R}$. Set $x_0 = 0$ and $x_{k+1} = 0$. Note that the density for $(Y_{t_i}^{(\varepsilon)})_{i=1}^k$ at $(x_i)_{i=1}^k$ is given by

$$f_{\vec{t};\varepsilon}(\vec{x}) := \frac{\varepsilon^{k/2}}{\mathcal{Z}(0,0;0,\varepsilon)}\prod_{j=0}^{k}\mathcal{Z}(\sqrt{\varepsilon}x_j, \varepsilon t_j; \sqrt{\varepsilon}x_{j+1}, \varepsilon t_{j+1}).$$

For a Brownian bridge B on $[0,1]$ starting at 0 and ending at x, the density for $(B_{t_i})_{i=1}^k$ at $(x_i)_{i=1}^k$ is given by

$$g_{\vec{t}}(\vec{x}) := \frac{1}{p(0,1)}\prod_{j=0}^{k} p(x_{j+1}-x_j, t_{j+1}-t_j)$$

where $p(x,t) = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}$. Using the distributional identities for $\mathcal{Z}$ (see Remark 6.2.1) and using

Equation (8.11) in [99] and Brownian scaling, we deduce

$$\frac{\mathcal{Z}(\sqrt{\varepsilon}x_j, \varepsilon t_j; \sqrt{\varepsilon}x_{j+1}, \varepsilon t_{j+1})}{p(\sqrt{\varepsilon}(x_{j+1}-x_j), \varepsilon(t_{j+1}-t_j))} \stackrel{d}{=} \mathbf{E}_{0,0}^{t_{j+1}-t_j, x_{j+1}-x_j}\left[: \exp : \left\{\varepsilon^{1/4}\int_0^{t_{j+1}-t_j} \xi(s, B(s))ds\right\}\right]$$

where B is a Brownian bridge conditioned $B(0) = 0$ and $B(t_{j+1}-t_j) = x_{j+1}-x_j$. The expectation above is taken w.r.t. this Brownian bridge only. Here : exp : denotes the Wick exponential (see [99] for details). The right side of the above equation is a random variable (function of the noise ξ). We claim that this random variable converges to 1 in probability. Indeed using chaos expansion, and Lemma 2.4 in [105], it follows that for every fixed t, x we have

$$\mathbf{E}\left[\left\{\mathbf{E}_{0,0}^{t,x}\left[: \exp : \left\{\varepsilon^{1/4}\int_0^{t} \xi(s, B(s))ds\right\}\right] - 1\right\}^2\right] = \sqrt{\varepsilon}\sum_{k=1}^{\infty}\frac{\varepsilon^{(k-1)/2}t^{k/2}}{(4\pi)^{k/2}}\frac{(\Gamma(1/2)^k)}{\Gamma(k/2)}.$$

The above sum converges. Thus as $\varepsilon \downarrow 0$, the above expression goes to zero, proving the claim. As $p(\sqrt{\varepsilon}x, \varepsilon t) = \varepsilon^{-1/2}p(x,t)$, we thus have $f_{\vec{t};\varepsilon}(\vec{x}) \xrightarrow{p} g_{\vec{t}}(\vec{x})$. Thus the quenched finite-dimensional density of $Y^{(\varepsilon)}$ converges in probability to the finite-dimensional density of the Brownian Bridge. We now show that the same holds for the annealed law. Indeed, note that $\left|g_{\vec{t}}(\vec{x}) - f_{\vec{t};\varepsilon}(\vec{x})\right|^+$ converges to zero in probability and is bounded above by $g_{\vec{t}}(\vec{x})$. Thus by DCT and Jensen's inequality, we obtain

$$\left|g_{\vec{t}}(\vec{x}) - \mathbf{E}[f_{\vec{t};\varepsilon}(\vec{x})]\right|^+ \le \mathbf{E}_{\xi}\left|g_{\vec{t}}(\vec{x}) - f_{\vec{t};\varepsilon}(\vec{x})\right|^+ \to 0$$

as $\varepsilon \downarrow 0$. Now by Scheffe's theorem, it follows that the annealed finite-dimensional distribution of $Y^{(\varepsilon)}$ converges weakly to the finite-dimensional distribution of the Brownian bridge.

Let us now verify tightness. Recall that $X(\varepsilon t) = \sqrt{\varepsilon}Y_t^{(\varepsilon)}$. Observe that by union bound followed by Markov inequality we have

$$\mathbf{P}\left[\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}} |Y_t^{(\varepsilon)} - Y_s^{(\varepsilon)}| \ge \eta\right] \le \mathbf{P}\left[\mathcal{Z}(0,0;0,\varepsilon)\sqrt{2\pi\varepsilon}\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}} |Y_t^{(\varepsilon)} - Y_s^{(\varepsilon)}| \ge \eta\delta^{1/3}\right]$$

$$+\mathbf{P}\left[\mathcal{Z}(0,0;0,\varepsilon)\sqrt{2\pi\varepsilon} \le \delta^{1/3}\right]$$

$$\le \frac{\sqrt{2\pi}}{\eta\delta^{1/3}}\mathbf{E}\left[\mathcal{Z}(0,0;0,\varepsilon)\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}}|X(\varepsilon t)-X(\varepsilon s)|\right]$$

$$+\mathbf{P}\left[\mathfrak{g}_\varepsilon(0) \le (4\varepsilon/\pi)^{-1/4}\log(\delta^{1/3})\right].$$

Note that by one-point short-time tail bounds from Proposition 6.2.3 (b), the second expression above goes to zero as $\delta \downarrow 0$ uniformly in $\varepsilon \le 1$. For the first expression, by Lemma 6.4.1 we have

$$\mathbf{E}\left[\mathcal{Z}(0,0;0,\varepsilon)\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}}|X(\varepsilon t)-X(\varepsilon s)|\right] = \frac{1}{\sqrt{2\pi\varepsilon}}\mathbf{E}\left[\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}}|B'_{\varepsilon t}-B'_{\varepsilon s}|\right],$$

where B' is a Brownian bridge on $[0,\varepsilon]$. By scaling property of Brownian bridges we may write the last expression simply as

$$\frac{1}{\sqrt{2\pi}}\mathbf{E}\left[\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}}|B_t - B_s - (t-s)B_1|\right]$$

where B is a Brownian motion on $[0,1]$. This expression is free of ε and by [167, Lemma 1] this goes to zero with rate $O(\delta^{1/2-\gamma})$ for any $\gamma > 0$. Thus we have shown

$$\limsup_{\delta\downarrow 0}\sup_{\varepsilon\in(0,1)}\mathbf{P}\left[\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}}|Y_t^{(\varepsilon)} - Y_s^{(\varepsilon)}| \ge \eta\right] = 0.$$

Since $Y_0^{(\varepsilon)} = 0$, by standard criterion of tightness (see Theorem 4.10 in [214]) combined with finite-dimensional convergence shown before, we have weak convergence to Brownian Bridge. This completes the proof. □

Proof of Theorem 6.1.7. Let us first prove (a) using Corollary 6.3.6. Fix $\gamma \in (0,1)$. We consider $\beta \in (0,1)$ small enough so that $\gamma \ge \rho(\beta)$ where $\rho(\beta) := \sup_{t\in(0,\beta]} t^{\frac{1}{4}}\log\frac{2}{t}$. Taking $\delta = \frac{1}{4}$, the estimates in (6.3.3) ensure that for all $\varepsilon \in (0,1)$ we have

$$\mathbf{P}\left(\sup_{t\neq s\in[0,1],|t-s|<\beta}|L_s^{(\varepsilon)}-L_t^{(\varepsilon)}|\geq\gamma\right)\leq\mathbf{P}\left(\sup_{t\neq s\in[0,1],|t-s|<\beta}\frac{|L_s^{(\varepsilon)}-L_t^{(\varepsilon)}|}{|t-s|^{\frac14}\log\frac{2}{|t-s|}}\geq\frac{\gamma}{\rho(\beta)}\right)$$
$$\leq \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}\tfrac{\gamma^2}{\rho(\beta)^2}\right).$$

Note that as $\beta\downarrow 0$, we have $\rho(\beta)\downarrow 0$. Hence

$$\limsup_{\beta\downarrow 0}\sup_{\varepsilon\in(0,1)}\mathbf{P}\left(\sup_{t\neq s\in[0,1],|t-s|<\beta}|L_s^{(\varepsilon)}-L_t^{(\varepsilon)}|\geq\gamma\right)=0.$$

Since $L_0^{(\varepsilon)}=0$, the above modulus of continuity estimate yields tightness for the process $L_t^{(\varepsilon)}$.

For (b), let us fix $t\in(0,1)$ and consider $V\sim \mathsf{CDRP}(0,0;0,\varepsilon^{-1})$. Let $\mathcal{M}_{t,\varepsilon^{-1}}$ denote the unique mode of the quenched density of $V(\varepsilon^{-1}t)$. By [132, Theorem 1.4], we know $\mathcal{M}_{t,\varepsilon^{-1}}$ exists uniquely almost surely. By [132, Corollary 7.3] we have

$$\limsup_{K\to\infty}\limsup_{\varepsilon\downarrow 0}\mathbf{P}^{\xi}(|V(\varepsilon^{-1}t)-\mathcal{M}_{t,\varepsilon^{-1}}|\geq K)=0,\ \text{ in probability.}$$

Applying reverse Fatou's Lemma we have

$$\limsup_{K\to\infty}\limsup_{\varepsilon\downarrow 0}\mathbf{P}(|V(\varepsilon^{-1}t)-\mathcal{M}_{t,\varepsilon^{-1}}|\geq K)=0.$$

Thus in particular, $\varepsilon^{-\frac23}[V(\varepsilon^{-1}t)-\mathcal{M}_{t,\varepsilon^{-1}}]\xrightarrow{p}0$. However, $\varepsilon^{-2/3}\mathcal{M}_{t,\varepsilon^{-1}}\xrightarrow{d}\Gamma(t\sqrt{2})$ due to [132, Theorem 1.8]. This proves (b). $\square$

Proof of Theorem 6.1.8. Let us first prove part (a) which claims short-time process convergence. We first show finite-dimensional convergence. Fix $0=t_0<t_1<\cdots<t_{k+1}=1$. Take $x_1,\ldots,x_k\in$

$\mathbb{R}$. Set $x_0 = 0$ and $x_{k+1} = *$. Note that the density for $(Y_*^{(\varepsilon)}(t_i))_{i=1}^k$ at $(x_i)_{i=1}^k$ is given by

$$f^*_{\vec{t};\varepsilon}(\vec{x}) := \frac{\varepsilon^{k/2}}{\mathcal{Z}(0,0;*,\varepsilon)} \prod_{j=0}^{k} \mathcal{Z}(\sqrt{\varepsilon}x_j, \varepsilon t_j; \sqrt{\varepsilon}x_{j+1}, \varepsilon t_{j+1}).$$

From the finite-dimensional convergence argument in proof of Theorem 6.1.4 we know that

$$\varepsilon^{k/2} \prod_{j=0}^{k-1} \mathcal{Z}(\sqrt{\varepsilon}x_j, \varepsilon t_j; \sqrt{\varepsilon}x_{j+1}, \varepsilon t_{j+1}) \xrightarrow{p} \prod_{j=0}^{k-1} p(x_{j+1} - x_j, t_{j+1} - t_j) =: g^*_{\vec{t}}(\vec{x}). \tag{6.4.3}$$

Note that $g^*_{\vec{t}}(\vec{x})$ is the finite-dimensional density for the standard Brownian motion. We now claim that

$$\mathcal{Z}(0,0;*,\varepsilon) \xrightarrow{p} 1, \quad \mathcal{Z}(\sqrt{\varepsilon}x_{k-1}, \varepsilon t_{k-1}; *, \varepsilon t_k) \xrightarrow{p} 1. \tag{6.4.4}$$

Combining (6.4.3) and (6.4.4) we have that $f^*_{\vec{t};\varepsilon}(\vec{x}) \xrightarrow{p} g^*_{\vec{t}}(\vec{x})$ which implies quenched finite-dimensional density convergence. This convergence can then be upgraded to annealed finite-dimensional density convergence by the same argument of the proof of Theorem 6.1.4.

We thus focus on proving (6.4.4). To prove the first part of (6.4.4) we utilize the short-time scaling from (6.2.2) to get

$$\mathcal{Z}(0,0;*,\varepsilon) = \int_{\mathbb{R}} e^{\mathcal{H}(x,t)} dx = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}} \exp\left((\tfrac{\pi\varepsilon}{4})^{1/4} \mathfrak{g}_\varepsilon\big(x\sqrt{\tfrac{4}{\pi\varepsilon}}\big) \right) dx. \tag{6.4.5}$$

Fix any $\nu \in (0,1)$. Applying [128, Proposition 4.4] (with $s = \varepsilon^{-\frac{1}{6}}$) we get that with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}\varepsilon^{-\frac{1}{4}})$

$$-\frac{(\pi\varepsilon/4)^{3/4}(1+\nu)x^2}{2\varepsilon} - \varepsilon^{-1/6} \le \mathfrak{g}_\varepsilon\big(x\big) \le -\frac{(\pi\varepsilon/4)^{3/4}(1-\nu)x^2}{2\varepsilon} + \varepsilon^{-1/6}, \text{ for all } x \in \mathbb{R}, \tag{6.4.6}$$

where the constant C depends on ν. Inserting the above inequality in (6.4.5) we get that with probability at least $1 - \mathrm{C}\exp(-\frac{1}{\mathrm{C}}\varepsilon^{-\frac{1}{4}})$

$$\exp\left(-(\tfrac{\pi}{4})^{1/4}\varepsilon^{1/12}\right)\frac{1}{\sqrt{2\pi\varepsilon}}\int_{\mathbb{R}} e^{-\frac{(1+\nu)x^2}{2\varepsilon}}dx \le \mathcal{Z}(0,0;*,\varepsilon) \le \exp\left((\tfrac{\pi}{4})^{1/4}\varepsilon^{1/12}\right)\frac{1}{\sqrt{2\pi\varepsilon}}\int_{\mathbb{R}} e^{-\frac{(1-\nu)x^2}{2\varepsilon}}dx.$$

Thus

$$\mathbf{P}\left(\frac{\exp\left(-(\frac{\pi}{4})^{1/4}\varepsilon^{1/12}\right)}{\sqrt{1+\nu}} \le \mathcal{Z}(0,0;*,\varepsilon) \le \frac{\exp\left((\frac{\pi}{4})^{1/4}\varepsilon^{1/12}\right)}{\sqrt{1-\nu}}\right) \ge 1 - \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}\varepsilon^{-\frac{1}{4}}),$$

which implies

$$\limsup_{\varepsilon\to\infty}\mathbf{P}\left(\tfrac{1}{\sqrt{1+\nu}} \le \mathcal{Z}(0,0;*,\varepsilon) \le \tfrac{1}{\sqrt{1-\nu}}\right) = 1.$$

Taking $\nu \downarrow 0$, we get the first part of (6.4.4). The second part follows analogously.

Let us now verify tightness. Observe that by union bound followed by Markov inequality we have

$$\begin{aligned}
\mathbf{P}\left[\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}}|Y_*^{(\varepsilon)}(t) - Y_*^{(\varepsilon)}(t)| \ge \eta\right] &\le \mathbf{P}\left[\mathcal{Z}(0,0;*,\varepsilon)\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}}|Y_*^{(\varepsilon)}(t) - Y_*^{(\varepsilon)}(s)| \ge \eta\delta^{1/3}\right] \\
&\qquad + \mathbf{P}\left[\mathcal{Z}(0,0;*,\varepsilon) \le \delta^{1/3}\right] \\
&\le \frac{1}{\eta\delta^{1/3}}\mathbf{E}\left[\mathcal{Z}(0,0;*,\varepsilon)\frac{1}{\sqrt{\varepsilon}}\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}}|X(\varepsilon t) - X(\varepsilon s)|\right] \\
&\qquad + \mathbf{P}\left[\mathfrak{g}_\varepsilon(*) \le \varepsilon^{-1/4}\log(\delta^{1/3})\right],
\end{aligned} \tag{6.4.7}$$

where

$$\begin{aligned}
\mathfrak{g}_\varepsilon(*) &:= \varepsilon^{-1/4}\log\mathcal{Z}(0,0;*,\varepsilon) \\
&= \varepsilon^{-1/4}\left[-\log\sqrt{2\pi\varepsilon} + \log\int_{\mathbb{R}}\exp\left((\tfrac{\pi\varepsilon}{4})^{1/4}\mathfrak{g}_\varepsilon(\sqrt{\tfrac{4}{\pi\varepsilon}}x)\right)\mathrm{d}x\right]
\end{aligned}$$

with $\mathfrak{g}_\varepsilon(x)$ defined in (6.2.2). Let us now bound each term in the r.h.s. of (6.4.7) separately. For the second term we claim that

$$\limsup_{\delta\downarrow 0}\sup_{\varepsilon\in(0,1)}\mathbf{P}\left[\mathfrak{g}_\varepsilon(*)\le \varepsilon^{-1/4}\log(\delta^{1/3})\right]=0. \tag{6.4.8}$$

Note that by Proposition 4.4 in [128] (the infimum process bound with $\nu=1$) we have for any $s>0$ with probability at least $1-\mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$,

$$(\tfrac{\pi\varepsilon}{4})^{1/4}\mathfrak{g}_\varepsilon(\sqrt{\tfrac{4}{\pi\varepsilon}}x)\ge -(\tfrac{\pi\varepsilon}{4})^{1/4}\left[s+(\tfrac{\pi\varepsilon}{4})^{3/4}\cdot\tfrac{4}{\pi\varepsilon}\tfrac{x^2}{\varepsilon}\right]=-(\tfrac{\pi\varepsilon}{4})^{1/4}s-\tfrac{x^2}{\varepsilon},\ \text{for all } x\in\mathbb{R}.$$

Thus, with probability at least $1-\mathrm{C}\exp(-\frac{1}{\mathrm{C}}s^{3/2})$,

$$\begin{aligned}\mathfrak{g}_\varepsilon(*)&\ge \varepsilon^{-1/4}\left[-\log\sqrt{2\pi\varepsilon}+\log\left(\int_{\mathbb{R}}\exp\left(-(\tfrac{\pi\varepsilon}{4})^{1/4}s-\tfrac{x^2}{\varepsilon}\right)\mathrm{d}x\right)\right]\\&=\varepsilon^{-1/4}\left[-\log\sqrt{2\pi\varepsilon}+\log\left(\sqrt{\pi\varepsilon}\exp\left(-(\tfrac{\pi\varepsilon}{4})^{1/4}s\right)\right)\right]\\&=\varepsilon^{-1/4}\left[-\log\sqrt{2}-(\tfrac{\pi\varepsilon}{4})^{1/4}s\right]\ge -s-\varepsilon^{-1/4}\log 2.\end{aligned}$$

Now we take $s=-\varepsilon^{-1/4}\log(2\delta^{1/6})$ which is positive for δ small enough. Then $-s-\varepsilon^{-1/4}\log 2=\frac{1}{2}\varepsilon^{-1/4}\log(\delta^{1/3})>\varepsilon^{-1/4}\log(\delta^{1/3})$. Hence uniformly in all $\varepsilon\in(0,1)$, with probability at least $1-\mathrm{C}\exp(-\frac{1}{\mathrm{C}}[-\log(2\delta^{1/6})]^{3/2})$, we have $\mathfrak{g}_\varepsilon(*)\ge\varepsilon^{-1/4}\log(\delta^{1/3})$. This verifies (6.4.8).

Next for the first expression on r.h.s. of (6.4.7), by Lemma 6.4.1 we have

$$\mathbf{E}\left[\mathcal{Z}(0,0;*,\varepsilon)\frac{1}{\sqrt{\varepsilon}}\sup_{\substack{0\le t,s\le 1\\|t-s|\le\delta}}|X(\varepsilon t)-X(\varepsilon s)|\right]=\frac{1}{\sqrt{\varepsilon}}\mathbf{E}\left[\sup_{\substack{0\le t,s\le 1\\|t-s|\le\delta}}|B'_{\varepsilon t}-B'_{\varepsilon s}|\right],$$

where B' is a Brownian motion on $[0,\varepsilon]$. By scaling property of Brownian motion we may write the last expression simply as

$$\mathbf{E}\left[\sup_{\substack{0\le t,s\le 1\\|t-s|\le\delta}}|B_t-B_s|\right]$$

where B is a Brownian motion on $[0,1]$. This expression is free of ε and by [167, Lemma 1] this goes to zero with rate $O(\delta^{1/2-\gamma})$ for any $\gamma > 0$. Thus we have shown

$$\limsup_{\delta\downarrow 0} \sup_{\varepsilon\in(0,1)} \mathbf{P}\left[\sup_{\substack{0\le t,s\le 1\\ |t-s|\le\delta}} |Y_*^{(\varepsilon)}(t) - Y_*^{(\varepsilon)}(s)| \ge \eta\right] = 0.$$

Since $Y_*^{(\varepsilon)}(0) = 0$, this proves tightness. Along with finite-dimensional convergence, this establishes part (a).

The tightness results in part (b) follows via the same arguments as in the proof of Theorem 6.1.7 (a) utilizing the point-to-line modulus of continuity from Proposition 6.3.3-(point-to-line). For part (c), we rely on localization results from [132]. Indeed, by Theorem 1.5 in [132], we know the quenched density of $V(\varepsilon^{-1})$ (recall $V \sim \mathsf{CDRP}(0,0;*,\varepsilon^{-1})$) has a unique mode $\mathcal{M}_{*,\varepsilon^{-1}}$ almost surely. By the same argument as in the proof of Theorem 6.1.7 (b), the point-to-line version of Corollary 7.3 in [132] leads to the fact that $\varepsilon^{-2/3}[L_*^{(\varepsilon)}(1) - \mathcal{M}_{*,\varepsilon^{-1}}] \xrightarrow{p} 0$. Finally from Theorem 1.8 in [132] we have $\varepsilon^{-2/3}\mathcal{M}_{*,\varepsilon^{-1}} \xrightarrow{d} 2^{1/3}\mathcal{M}$. This establishes (c). $\square$

6.4.2 Proof of Theorem 6.1.10 modulo Conjecture 6.1.9

In this section we prove Theorem 6.1.10 assuming Conjecture 6.1.9. The proof also relies on a technical result which we first state below.

Lemma 6.4.3 (Deterministic convergence)**.** *Let $f(x) : \mathbb{R}^k \to \mathbb{R}$ be a continuous function with a unique maximizer $\vec{a} \in \mathbb{R}^k$ and $f_\varepsilon(x) : \mathbb{R}^k \to \mathbb{R}$ be a sequence of continuous functions that converges to $f(x)$ uniformly over compact subsets. Fix any $\delta > 0$ and take $M > 0$ so that $(a_i - \delta, a_i + \delta) \in [-M, M]$ for all i. For $x \in \mathbb{R}$, set*

$$g_\varepsilon(x) := \frac{\exp(\varepsilon^{-\frac{1}{3}} f_\varepsilon(\vec{x}))}{\int_{[-M,M]^k} \exp(\varepsilon^{-\frac{1}{3}} f_\varepsilon(\vec{y}))\mathrm{d}\vec{y}}.$$

For all $\vec{b} \in [-M, M]^k$, we have:

$$\limsup_{\varepsilon\downarrow 0}\int_{-M}^{b_1}\cdots\int_{-M}^{b_k} g_\varepsilon(\vec{x})\mathrm{d}\vec{x} \le \prod_{i=1}^k \mathbf{1}\{a_i \le b_i+\delta\}, \tag{6.4.9}$$

$$\liminf_{\varepsilon\downarrow 0}\int_{-M}^{b_1}\cdots\int_{-M}^{b_k} g_\varepsilon(\vec{x})\mathrm{d}\vec{x} \ge \prod_{i=1}^k \mathbf{1}\{a_i \le b_i-\delta\}. \tag{6.4.10}$$

Proof of this lemma follows via standard real analysis and hence we defer its proof to the end of this section. We now proceed to prove Theorem 6.1.10 assuming the above lemma.

Proof of Theorem 6.1.10. For clarity we split the proof into three steps.

Step 1. Fix $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$. For convenience set $\Gamma_{t_i} := \Gamma(t_i\sqrt{2})$ where $\Gamma(\cdot)$ is the geodesic of directed landscape from $(0,0)$ to $(0,\sqrt{2})$. Consider any $\vec{a} = (a_1,\ldots,a_k) \in \mathbb{R}^k$, which is a continuity point for the CDF of $(\Gamma_{t_i})_{i=1}^k$. For any $M \ge \sup_i |a_i| + 1$, define

$$V_{\vec{a}}(M) := [-M, a_i]\times\cdots\times[-M,a_k] \subset \mathbb{R}^k. \tag{6.4.11}$$

To show convergence in finite-dimensional distribution, it suffices to prove that as $\varepsilon \downarrow 0$

$$\mathbf{P}\left(\bigcap_{i=1}^k \{L_{t_i}^{(\varepsilon)} \le a_i\}\right) \to \mathbf{P}\left(\bigcap_{i=1}^k \{\Gamma_{t_i} \le a_i\}\right). \tag{6.4.12}$$

From Definition 6.1.1 and using the long-time scaling from (6.2.2), we obtain that the joint density of $(L_{t_1}^{(\varepsilon)}, L_{t_2}^{(\varepsilon)},\ldots,L_{t_k}^{(\varepsilon)})$ at $(x_i)_{i=1}^k$ is given by

$$\frac{g_{\vec{t};\varepsilon}(\vec{x})}{\int_{\mathbb{R}^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}, \qquad g_{\vec{t};\varepsilon}(\vec{x}) := \exp(\varepsilon^{-1/3}U_{\vec{t};\varepsilon}(\vec{x}))$$

where

$$U_{\vec{t};\varepsilon}(\vec{x}) := \sum_{i=1}^{k+1}(t_i-t_{i-1})^{1/3}\mathfrak{h}_{\varepsilon^{-1}t_{i-1},\varepsilon^{-1}t_i}((t_i-t_{i-1})^{-2/3}x_{i-1},(t_i-t_{i-1})^{-2/3}x_i) \tag{6.4.13}$$

Here $x_0 = x_{k+1} = 1$.

In this step, we reduce our computation to understanding the integral behavior of $g_{\vec{t};\varepsilon}$ on a compact set. More precisely, the goal of this step is to show there there exists a constant $\mathrm{C} > 0$ such that for all M large enough

$$\left|\mathbf{P}\left(\bigcap_{i=1}^{k}\{L_{t_i}^{(\varepsilon)} \le a_i\}\right) - \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right]\right| \le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}M^2\right) \tag{6.4.14}$$

where $V_{\vec{a}}(M)$ is defined in (6.4.11). We proceed to prove (6.4.14) by demonstrating appropriate lower and upper bounds. For upper bound observe that by union bound we have

$$\begin{aligned}
\mathbf{P}\left(\bigcap_{i=1}^{k}\{L_{t_i}^{(\varepsilon)} \le a_i\}\right) &\le \mathbf{P}\left(\bigcap_{i=1}^{k}\{L_{t_i}^{(\varepsilon)} \in [-M, a_i]\}\right) + \mathbf{P}\left(\sup_{t\in[0,1]} |L_t^{(\varepsilon)}| \ge M\right) \\
&\le \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right] + \mathbf{P}\left(\sup_{t\in[0,1]} |L_t^{(\varepsilon)}| \ge M\right) \\
&\le \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right] + \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}M^2\right)
\end{aligned} \tag{6.4.15}$$

where the last inequality follows from Corollary 6.3.5 for some constant $\mathrm{C} > 0$. For the lower bound we have

$$\begin{aligned}
\mathbf{P}\left(\bigcap_{i=1}^{k}\{L_{t_i}^{(\varepsilon)} \le a_i\}\right) &\ge \mathbf{P}\left(\bigcap_{i=1}^{k}\{L_{t_i}^{(\varepsilon)} \in [-M, a_i]\}\right) \\
&= \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}} \cdot \frac{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{\mathbb{R}^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right] \\
&\ge \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}} \cdot \mathbf{P}^{\xi}\left(\sup_{t\in[0,1]} |L_t^{(\varepsilon)}| \le M\right)\right].
\end{aligned} \tag{6.4.16}$$

By Corollary 6.3.5 we see that there exist two constants $\mathrm{C}_1, \mathrm{C}_2 > 0$ such that with probability at least $1 - \mathrm{C}_2\exp(-\frac{1}{\mathrm{C}_2}M^3)$, the random variable $\mathbf{P}^{\xi}\left(\sup_{t\in[0,1]} |L_t^{(\varepsilon)}| \le M\right)$ is at least $1 -$

$\mathrm{C}_1 \exp(-\frac{1}{\mathrm{C}_1}M^2)$. Thus,

$$\text{r.h.s. of (6.4.16)} \geq \left[1 - \mathrm{C}_2 \exp\left(-\tfrac{1}{\mathrm{C}_2}M^3\right)\right] \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}} \cdot \left[1 - \mathrm{C}_1 \exp\left(-\tfrac{1}{\mathrm{C}_1}M^2\right)\right]\right]$$
$$\geq \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right] - \mathrm{C}_1 \exp\left(-\tfrac{1}{\mathrm{C}_1}M^2\right). \tag{6.4.17}$$

In view of (6.4.15) and (6.4.17), we thus arrive at (6.4.14) by adjusting the constants. This completes our work for this step.

Step 2. In this step, we discuss how directed landscape and hence the geodesic appear in the limit. Recall the random function $U_{\vec{t};\varepsilon}(\vec{x})$ from (6.4.13). We exploit Conjecture 6.1.9, to show that as $\varepsilon \downarrow 0$, as $\mathbb{R}^k$-valued processes we have the following convergence in law

$$U_{\vec{t};\varepsilon}(\vec{x}) \xrightarrow{d} \mathbf{U}_{\vec{t}}(\vec{x}) := 2^{-\frac{1}{3}} \sum_{i=1}^{k+1} \mathcal{L}(x_{i-1}, t_{i-1}\sqrt{2}; x_i, t_i\sqrt{2}) \tag{6.4.18}$$

in the uniform-on-compact topology. Here $\mathcal{L}(x, s; y, t)$ denotes the directed landscape. Note that by Definition 6.1.6, $(\Gamma_{t_i})_{i=1}^k$ is precisely the almost sure unique k-point maximizer of $\mathbf{f}_{\vec{t}}(\vec{x})$.

To show (6.4.18), we rely on Conjecture 6.1.9 heavily. Indeed, assuming Conjecture 6.1.9, for each i, as $\varepsilon \downarrow 0$ we have

$$\begin{aligned}&\mathfrak{h}_{\varepsilon^{-1}t_{i-1},\varepsilon^{-1}t_i}((t_i - t_{i-1})^{-2/3}x, (t_i - t_{i-1})^{-2/3}y)\\ &\qquad \xrightarrow{d} 2^{-1/3}\mathcal{S}^{(i)}\big(2^{-1/3}(t_i - t_{i-1})^{-2/3}x, 2^{-1/3}(t_i - t_{i-1})^{-2/3}y\big)\end{aligned}$$

where the convergence holds under the uniform-on-compact topology. Here $\mathcal{S}^{(i)}$ are independent Airy sheets as $\mathfrak{h}_{\varepsilon^{-1}t_{i-1},\varepsilon^{-1}t_i}(\cdot,\cdot)$ are independent. Now by the definition of directed landscape we have

$$2^{-\frac{1}{3}} \sum_{i=1}^{k+1} \mathcal{L}(x_{i-1}, t_{i-1}\sqrt{2}; x_i, t_i\sqrt{2})$$

$$\stackrel{d}{=} 2^{-\frac{1}{3}} \sum_{i=1}^{k+1} (t_{i+1}-t_i)^{1/3} \mathcal{S}^{(i)}\big(2^{-1/3}(t_i-t_{i-1})^{-2/3}x_{i-1}, 2^{-1/3}(t_i-t_{i-1})^{-2/3}x_i\big)$$

with $x_0 = x_{k+1} = 1$. Here the equality in distribution holds as $\mathbb{R}^k$-valued processes in $\vec{x}$. This allow us to conclude the desired convergence for $U_{\vec{t};\varepsilon}(\vec{x})$ in (6.4.18), completing our work for this step.

Step 3. In this step, we complete the proof of (6.4.12) utilizing (6.4.14) and the weak convergence in (6.4.18). Using Skorokhod's representation theorem, given any fixed M, we may assume that we are working on a probability space where

$$\mathbf{P}(\mathsf{A}) = 1, \quad \text{for } \mathsf{A} := \left\{ \sup_{\vec{x}\in[-M,M]^k} \left| U_{\vec{t};\varepsilon}(\vec{x}) - \mathbf{U}_{\vec{t}}(\vec{x}) \right| \to 0 \right\}.$$

Let us define

$$(\Gamma_{t_i}(M))_{i=1}^k := \operatorname*{argmax}_{\vec{x}\in[-M,M]^k} \mathbf{f}_{\vec{t}}(\vec{x}),$$

where in case there are multiple maximizers we take the one whose sum of coordinates is the largest. We next define

$$\mathsf{B} := \left\{ \operatorname*{argmax}_{\vec{x}\in[-M,M]^k} \mathbf{U}_{\vec{t}}(\vec{x}) \text{ exists uniquely and } (\Gamma_{t_i}(M))_{i=1}^k \in [-\tfrac{M}{2}, \tfrac{M}{2}]^k \right\}.$$

Fix any $\delta \in (0, \frac{M}{2})$. By Lemma 6.4.3 we have

$$\begin{aligned}
\limsup_{\varepsilon\downarrow 0} \mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}} \right] &\le \mathbf{P}(\neg \mathsf{B}) + \mathbf{E}\left[\limsup_{\varepsilon\downarrow 0} \frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}} \mathbf{1}\{\mathsf{A}\cap\mathsf{B}\} \right] \\
&\le \mathbf{P}(\neg \mathsf{B}) + \mathbf{P}\left(\bigcap_{i=1}^k \{\Gamma_{t_i}(M) \le a_i + \delta\} \right) \\
&\le \mathbf{P}(\neg \mathsf{B}) + \mathbf{P}\left(\bigcap_{i=1}^k \{\Gamma_{t_i} \le a_i + \delta\} \right) + \mathbf{P}\left(\sup_{t\in[0,1]} |\Gamma_t| \ge M \right),
\end{aligned} \tag{6.4.19}$$

where the last inequality follows by observing that $\Gamma_{t_i}(M) = \Gamma_{t_i}$ for all i, whenever $\sup_{t\in[0,1]} |\Gamma_t| \le$

M (and the fact that $\Gamma(\cdot)$ exists uniquely almost surely via Theorem 12.1 in [138]). In the same manner we have

$$\begin{aligned}\liminf_{\varepsilon\downarrow 0}\mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right] &\ge \mathbf{E}\left[\liminf_{\varepsilon\downarrow 0}\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\mathbf{1}\{\mathsf{A}\cap\mathsf{B}\}\right] \\ &\ge \mathbf{P}\left(\bigcap_{i=1}^{k}\{\Gamma_{t_i}(M)\le a_i-\delta\},\mathsf{A}\cap\mathsf{B}\right) \\ &\ge \mathbf{P}\left(\bigcap_{i=1}^{k}\{\Gamma_{t_i}\le a_i-\delta\}\right)-\mathbf{P}(\neg\mathsf{B})-\mathbf{P}\left(\sup_{t\in[0,1]}|\Gamma_t|\ge M\right).\end{aligned} \tag{6.4.20}$$

By Proposition 12.3 in [138],

$$\mathbf{P}(\neg\mathsf{B})\le\mathbf{P}\left(\sup_{t\in[0,1]}|\Gamma_t|\ge M\right)\le \mathrm{C}\exp\left(-\tfrac{1}{\mathrm{C}}M^3\right).$$

Thus taking $M\uparrow\infty$, followed by $\delta\downarrow 0$, and using the fact that $\vec{a}$ is a continuity point of the density on both sides of (6.4.19) and (6.4.20) we have

$$\lim_{M\to\infty}\limsup_{\varepsilon\downarrow 0}\mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right]=\lim_{M\to\infty}\liminf_{\varepsilon\downarrow 0}\mathbf{E}\left[\frac{\int_{V_{\vec{a}}(M)} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}{\int_{[-M,M]^k} g_{\vec{t};\varepsilon}(\vec{y})d\vec{y}}\right]=\mathbf{P}\left(\bigcap_{i=1}^{k}\{\Gamma_{t_i}\le a_i\}\right)$$

Combining this with (6.4.14) we thus arrive at (6.4.12). This completes the proof. □

Proof of Lemma 6.4.3. We begin by proving (6.4.9). When $a_i\le b_i+\delta$ for all i, the r.h.s of (6.4.9) is 1 whereas the l.h.s of (6.4.9) is always less than 1. Thus we focus on when $a_j>b_j+\delta$ for some j. In that case $\vec{a}\notin[-M,b_1]\times\cdots\times[-M,b_k]$. As $\vec{a}$ is the unique maximizer of the continuous function $f(\vec{x})$, there exists $\eta>0$ such that

$$\sup_{y_i\in[-M,b_i],i=1,2,\ldots,k} f(\vec{y})<f(\vec{a})-\eta.$$

By uniform convergence over compacts, we can get ε_0 such that

$$\sup_{\varepsilon\le\varepsilon_0}\sup_{\vec{x}\in[-M,M]^k}|f_\varepsilon(\vec{x})-f(\vec{x})|<\tfrac{1}{4}\eta.$$

By continuity of f at $\vec{a}$, we can get $\delta_0<\delta$ such that for all $0\le\rho\le\delta$ we have

$$\sup_{x_i\in[a_i-\rho,a_i+\rho],i=1,\ldots,k}|f(\vec{x})-f(\vec{a})|<\tfrac{1}{4}\eta.$$

Thus for all $\varepsilon\le\varepsilon_0$ and $0\le\rho\le\delta_0$ we have $f_\varepsilon(\vec{x})\ge f(\vec{a})-\frac{1}{2}\eta$ for all $\vec{x}$ with $x_i\in[a_i-\rho,a_i+\rho]$. And for all $\varepsilon\le\varepsilon_0$, $f_\varepsilon(\vec{y})<f(\vec{a})-\frac{3}{4}\eta$ for all $\vec{y}$ with $y_i\in[-M,b_i]$. Thus in conclusion

$$\int_{-M}^{b_1}\cdots\int_{-M}^{b_k}\exp(\varepsilon^{-\frac{1}{3}}f_\varepsilon(\vec{x}))\mathrm{d}\vec{x}\le(2M)^k\exp(\varepsilon^{-1/3}[f(\vec{a})-\tfrac{3}{4}\eta)])$$

and

$$\int_{[-M,M]^k}\exp(\varepsilon^{-\frac{1}{3}}f_\varepsilon(\vec{x}))\mathrm{d}\vec{x}\ge\int_{a_1-\delta_0}^{a_1+\delta_0}\cdots\int_{a_k-\delta_0}^{a_k+\delta_0}\exp(\varepsilon^{-\frac{1}{3}}f_\varepsilon(\vec{x}))\mathrm{d}\vec{x}\ge(2\delta_0)^k\exp(\varepsilon^{-1/3}[f(\vec{a})-\tfrac{1}{2}\eta)]).$$

Combining the above two bounds we have

$$\int_{-M}^{b_1}\cdots\int_{-M}^{b_k}g_\varepsilon(\vec{x})\mathrm{d}\vec{x}\le(\tfrac{M}{\delta_0})^k\exp(-\tfrac{1}{4}\varepsilon^{-1/3}\eta),$$

which goes to zero as $\varepsilon\downarrow 0$. Thus, we conclude the proof of (6.4.9). The proof of (6.4.10) follows analogously. $\square$

6.5 Proof of Lemma 6.2.6

In this section, we prove Lemma 6.2.6. The idea is to view short-time scaled KPZ equation $\mathfrak{g}_t(\cdot)$ defined in (6.2.2) as the lowest index curve of an appropriate line ensemble and use certain stochastic monotonicity properties of the same. To make our exposition self-contained, below we briefly introduce the line ensemble machinery.

Fix $t > 0$ throughout this section and consider the convex function

$$\mathbf{G}_t(x) = (\pi t/4)^{1/2} e^{(\pi t/4)^{1/4} x}.$$

Recall the general notion of line ensembles from Section 2 in [109]. Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \ldots)$ be an $\mathbb{N} \times \mathbb{R}$ indexed line ensemble. Fix $k_1 \le k_2$ with $k_1, k_2 \in \mathbb{N}$ and an interval $(a, b) \in \mathbb{R}$ and two vectors $\vec{x}, \vec{y} \in \mathbb{R}^{k_2 - k_1 + 1}$. Let $\mathbb{P}_{\text{free}}^{k_1,k_2,(a,b),\vec{x},\vec{y}}$ denote the law of $k_2 - k_1 + 1$ many independent Brownian bridges taking values $\vec{x}$ at time a and $\vec{y}$ at time b. Given two measurable functions $f, g : (a, b) \to \mathbb{R} \cup \{\pm\infty\}$, the law $\mathbb{P}_{\mathbf{G}_t}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ on $\mathcal{L}_{k_1}, \ldots, \mathcal{L}_{k_2} : (a, b) \to \mathbb{R}$ has the following Radon-Nikodym derivative w.r.t. $\mathbb{P}_{\text{free}}^{k_1,k_2,(a,b),\vec{x},\vec{y}}$:

$$\frac{d\mathbb{P}_{\mathbf{G}_t}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}}{d\mathbb{P}_{\text{free}}^{k_1,k_2,(a,b),\vec{x},\vec{y}}}(\mathcal{L}_{k_1}, \ldots, \mathcal{L}_{k_2}) = \frac{\exp\left\{ -\sum_{i=k_1}^{k_2+1} \int \mathbf{G}_t\big(\mathcal{L}_i(x) - \mathcal{L}_{i-1}(x)\big) dx \right\}}{Z_{\mathbf{G}_t}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}}, \tag{6.5.1}$$

where $\mathcal{L}_{k_1-1} = f$, or ∞ if $k_1 = 1$; and $\mathcal{L}_{k_2+1} = g$. Here $Z_{\mathbf{G}_t}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}$ is the normalizing constant which produces a probability measure. We say $\mathcal{L}$ enjoys the $\mathbf{G}_t$*-Brownian Gibbs property* if, for all $K = \{k_1, \ldots, k_2\} \subset \mathbb{N}$ and $(a, b) \subset \mathbb{R}$, the following distributional equality holds:

$$\text{Law}\Big(\mathcal{L}_{K\times(a,b)} \text{ conditioned on } \mathcal{L}_{\mathbb{N}\times\mathbb{R}\setminus K\times(a,b)}\Big) = \mathbb{P}_{\mathbf{G}_t}^{k_1,k_2,(a,b),\vec{x},\vec{y},f,g}, \tag{6.5.2}$$

where $\vec{x} = (\mathcal{L}_{k_1}(a), \ldots, \mathcal{L}_{k_2}(a))$, $\vec{y} = (\mathcal{L}_{k_1}(b), \ldots, \mathcal{L}_{k_2}(b))$, and where again $\mathcal{L}_{k_1-1} = f$, or ∞ if $k_1 = 1$; and $\mathcal{L}_{k_2+1} = g$.

Similar to the Markov property, a *strong* version of the $\mathbf{G}_t$-Brownian Gibbs property that is valid with respect to *stopping domains* exists. A pair $(\mathfrak{a}, \mathfrak{b})$ of random variables is called a K-stopping domain if $\big\{\mathfrak{a} \le a, \mathfrak{b} \ge b\big\} \in \mathfrak{F}_{\text{ext}}\big(K \times (a, b)\big)$, the σ-field generated by $\mathcal{L}_{(\mathbb{N}\times\mathbb{R})\setminus(K\times(a,b))}$. $\mathcal{L}$ satisfies the strong $\mathbf{G}_t$-Brownian Gibbs property if for all $K = \{k_1, \ldots, k_2\} \subset \mathbb{N}$ and K-stopping domain if $(\mathfrak{a}, \mathfrak{b})$, the conditional distribution of $\mathcal{L}_{K\times(\mathfrak{a},\mathfrak{b})}$ given $\mathfrak{F}_{\text{ext}}\big(K \times (\mathfrak{a}, \mathfrak{b})\big)$ is $\mathbb{P}_{\mathbf{G}_t}^{k_1,k_2,(l,r),\vec{x},\vec{y},f,g}$, where $\ell = \mathfrak{a}$, $r = \mathfrak{b}$, $\vec{x} = (\mathcal{L}_i(\mathfrak{a}))_{i\in K}$, $\vec{y} = (\mathcal{L}_i(\mathfrak{b}))_{i\in K}$, and where again $\mathcal{L}_{k_1-1} = f$, or ∞ if $k_1 = 1$;

and $\mathcal{L}_{k_2+1} = g$.

The following lemma shows how the short-time scaled KPZ process $\mathfrak{g}_t(\cdot)$ fits into a line ensemble satisfying the $\mathbf{G}_t$-Brownian Gibbs property.

Lemma 6.5.1 (Lemma 2.5 in [128] and Lemma 2.5 of [**CH16**])**.** *For each $t > 0$, there exists an $\mathbb{N} \times \mathbb{R}$-indexed line ensemble $\{\mathfrak{g}_t^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$ satisfying the $\mathbf{G}_t$-Brownian Gibbs property and the lowest indexed curve $\mathfrak{g}_t^{(1)}(x)$ is equal in distribution (as a process in x) to $\mathfrak{g}_t(x)$ defined in* (6.2.2)*. Furthermore, the line ensemble $\{\mathfrak{g}_t^{(n)}(x)\}_{n\in\mathbb{N},x\in\mathbb{R}}$ satisfies the strong $\mathbf{G}_t$-Brownian Gibbs property.*

Before beginning the proof of Lemma 6.2.6 we recall one more property of line ensembles, i.e. the stochastic monotonicity, which is indispensable to the study of monotone events in Lemma 6.2.6.

Lemma 6.5.2 (Lemmas 2.6 and 2.7 of [**CH16**])**.** *Fix a finite interval $(a, b) \subset \mathbb{R}$ and $x, y \in \mathbb{R}$. For $i \in \{1, 2\}$, fix measurable functions $g_i : (a, b) \to \mathbb{R} \cup \{-\infty\}$ such that $g_2(s) \le g_1(s)$ for $s \in (a, b)$. For each $v \in \{1, 2\}$, let $\mathbf{P}_v$ denote the law $\mathbb{P}_{\mathbf{G}_t}^{1,1,(a,b),x,y,+\infty,g_v}$, so that a $\mathbf{P}_v$-distributed random variable $\mathcal{R}_i = \{\mathcal{R}_v(s)\}_{s\in(a,b)}$ is a random function on $[a, b]$ with endpoints x and y. Then a common probability space may be constructed on which the two measures are supported such that, almost surely, $\mathcal{R}_1(s) \ge \mathcal{R}_2(s)$ for all $s \in (a, b)$.*

Proof of Lemma 6.2.6. Fix an interval $[a, b]$ and a corresponding monotone set $A \in \mathcal{B}(C([a, b]))$. By Lemma 6.5.1 and tower property of expectation we may write

$$\begin{aligned}
\mathbf{P}\left[\mathfrak{g}_t(\cdot) \mid_{[a,b]} \in A \mid (\mathfrak{g}_t(x))_{x\notin(a,b)}\right] &= \mathbf{E}^{(\ge 2)}\left[\mathbf{P}\left[\mathfrak{g}_t^{(1)}(\cdot) \mid_{[a,b]} \in A \mid (\mathfrak{g}_t^{(n)}(\cdot))_{n\ge 2}, (\mathfrak{g}_t^{(1)}(x))_{x\notin(a,b)}\right]\right] \\
&= \mathbf{E}^{(\ge 2)}\left[\mathbb{P}_{\mathbf{G}_t}^{1,1,(a,b),\mathfrak{g}_t^{(1)}(a),\mathfrak{g}_t^{(1)}(b),+\infty,\mathfrak{g}_t^{(2)}(\cdot)}\left(\mathfrak{g}_t^{(1)}(\cdot) \mid_{[a,b]} \in A\right)\right]
\end{aligned} \tag{6.5.3}$$

where the last equality follows from (6.5.2). Here $\mathbf{E}^{(\ge 2)}$ denotes the expectation operator taken over all lower curves $\{\mathfrak{g}_t^{(n)}(\cdot)\}_{n\ge 2}$. Now by Lemma 6.5.2, decreasing $\mathfrak{g}_t^{(2)}(\cdot)$ pointwise on $[a, b]$ reduces the value of $\mathfrak{g}_t^{(1)}(\cdot)$ pointwise stochastically. But by the definition of monotone set A (see

(6.2.4)), we know decreasing $\mathfrak{g}_t^{(1)}(\cdot)\mid_{[a,b]}$ stochastically pointwise and keeping the endpoint fixed, only increases the conditional probability appearing above. Thus, we may drop $\mathfrak{g}_t^{(2)}(\cdot)$ all the way to $-\infty$, to obtain

$$\text{r.h.s. of (6.5.3)} \le \mathbf{E}^{(\ge 2)}\left[\mathbb{P}_{\mathbf{G}_t}^{1,1,(a,b),\mathfrak{g}_t^{(1)}(a),\mathfrak{g}_t^{(1)}(b),+\infty,-\infty}\left(\mathfrak{g}_t^{(1)}(\cdot)\mid_{[a,b]}\in A\right)\right]. \tag{6.5.4}$$

Under the above situation the Radon-Nikodym derivative appearing in (6.5.1) becomes constant, and thus

$$\mathbb{P}_{\mathbf{G}_t}^{1,1,(a,b),\mathfrak{g}_t^{(1)}(a),\mathfrak{g}_t^{(1)}(b),+\infty,-\infty}[\cdot] = \mathbb{P}_{\text{free}}^{1,1,(a,b),\mathfrak{g}_t^{(1)}(a),\mathfrak{g}_t^{(1)}(b)}[\cdot].$$

The measure on the right side above is a single Brownian bridge measure on $[a,b]$ starting at $\mathfrak{g}_t^{(1)}(a)$ and ending at $\mathfrak{g}_t^{(1)}(b)$ and hence free of $\{\mathfrak{g}_t^{(n)}(\cdot)\}_{n\ge 2}$. Thus r.h.s. of (6.5.4) can be viewed as $\mathbf{P}_{\text{free}}^{(a,b),(\mathfrak{g}_t^{(1)}(a),\mathfrak{g}_t^{(1)}(b))}(A)$. This establishes (6.2.5). The case when $[a,b]$ is a stopping domain follows from the same calculation and the fact that $\{\mathfrak{g}_t^{(n)}(\cdot)\}_{n\ge 1}$ satisfies the strong $\mathbf{G}_t$-Brownian Gibbs property via Lemma 6.5.1. $\square$

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Chapter 7: KPZ exponents for the half-space log-gamma polymer

7.1 Introduction

7.1.1 The model and the main results

Fix $\theta > 0$, $\alpha > -\theta$, and consider a family of independent variables $(W_{i,j})_{(i,j)\in\mathcal{I}}$ with $\mathcal{I} := \{(i,j) \in \mathbb{Z}^2_{>0} : j \le i\}$ such that

$$W_{i,j} \sim \text{Gamma}^{-1}(\alpha+\theta) \text{ for } i = j \qquad \text{and} \qquad W_{i,j} \sim \text{Gamma}^{-1}(2\theta) \text{ for } j < i, \tag{7.1.1}$$

where $X \sim \text{Gamma}^{-1}(\beta)$ means X is a random variable with density $\mathbf{1}\{x > 0\}\Gamma^{-1}(\beta)x^{-\beta-1}e^{-1/x}$. A directed lattice path $\pi = \big((x_i, y_i)\big)_{i=1}^{k}$ confined to the half-space index set $\mathcal{I}$ is an up-right path with all $(x_i, y_i) \in \mathcal{I}$, such that it only makes unit steps in the coordinate directions, that is, $(x_{i+1}, y_{i+1}) = (x_i, y_i) + (0,1)$ or $(x_{i+1}, y_{i+1}) = (x_i, y_i) + (1,0)$; see Figure 7.1. Given $(m,n) \in \mathcal{I}$, we denote $\Pi_{m,n}$ to be the set of all directed paths from $(1,1)$ to (m,n) confined to $\mathcal{I}$. Given the random variables from (7.1.1), we define the weight of a path π and the point-to-point partition function of the half-space log-gamma ($\mathcal{HSLG}$) polymer as

$$w(\pi) := \prod_{(i,j)\in\pi} W_{i,j}, \qquad Z_{(\alpha,\theta)}(m,n) := \sum_{\pi\in\Pi_{m,n}} w(\pi).$$

The parameter α controls the strength of the boundary weights and there is a phase transition in the behavior of this model at $\alpha = 0$. In our current work we will probe the behavior in the critical regime where α is in a scaling window of order $N^{-1/3}$ of 0, as well as in the supercritical regime when α is strictly positive. The subcritical regime may be probed in subsequent work as described in Section 7.1.4. This phase transition has been the subject of quite a lot of previous

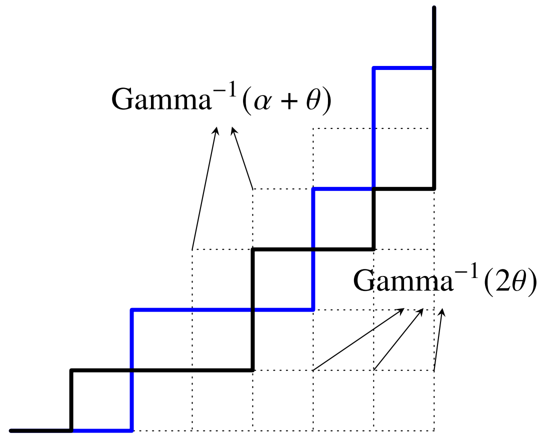

Figure 7.1: Vertex weights for the half-space log-gamma polymer and two possible paths (one marked in blue and the other in black) in $\Pi_{8,8}$.

work, some which we review in Section 7.1.4. The basic picture (some as of yet unproved) is as follows. For $\alpha \geq 0$ the free energy (i.e., log of the partition function) should demonstrate the KPZ $1/3$ fluctuation and $2/3$ transversal scaling exponents as well as certain universal limiting distributions. Here the transversal scaling references both the $N^{2/3}$ fluctuations of the endpoint of the length $2N$ half-space polymer as well as the $N^{2/3}$ correlation length of the free energy as a function of (m, n) subject to $m + n = 2N$. For $\alpha < 0$ the situation is different – the free energy fluctuations should be of order $N^{1/2}$, the endpoint should fluctuate transversally in an order one scale (i.e., not growing with N), while the free energy correlation length should be of order N and the limiting distributions should be Gaussian. To be clear, in terms of the polymer measure, this phase transition relates to the pinning ($\alpha < 0$) or unpinning ($\alpha \geq 0$) of the path from the diagonal.

Our main result captures the KPZ scaling exponents in the critical and subcritical regimes.

Theorem 7.1.1. *Fix $\theta, r > 0$. For each $\alpha > -\theta$, $s \in [0, r]$, and $N \geq \max\{3, r^3\}$ define the centered and scaled $\mathcal{HSLG}$ free energy process*

$$f_N^{\alpha}(s) := \frac{\log Z_{(\alpha,\theta)}(N + sN^{2/3}, N - sN^{2/3}) + 2N\Psi(\theta)}{N^{1/3}}.$$

Here Ψ denotes the digamma function defined on $\mathbb{R}_{>0}$ by

$$\Psi(z) := \partial_z \log \Gamma(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right), \tag{7.1.2}$$

where γ is the Euler-Mascheroni constant. The function $f_N^{\alpha}(\cdot)$ is linearly interpolated in between points where $Z_{(\alpha,\theta)}$ is defined. Let $\mathbf{P}_N^{\alpha}$ denotes the law of $f_N^{\alpha}(\cdot)$ as a random variable in $(C[0,r], C)$ – the space of continuous functions on $[0,r]$ equipped with uniform topology and Borel σ-algebra C. Then the following holds.

(a) The sequence $\mathbf{P}_N^{\alpha}$ is tight for each $\alpha \in (0,\infty)$.

(b) For $\alpha_N = N^{-1/3}\mu$ with $\mu \in \mathbb{R}$ fixed (noting that for large enough N, $\alpha_N > -\theta$, and thus $f_N^{\alpha_N}(\cdot)$ is well defined), the sequence $\mathbf{P}_N^{\alpha_N}$ is tight.

As discussed below, it is possible to show (e.g. using the ideas of [29]) absolute continuity of the limit points in Theorem 7.1.1 with respect to certain Brownian measures. We do not pursue this here, but remark further about this and related directions below.

The rest of this introduction is structured as follows. Section 7.1.2 introduces the idea of a half-space Gibbsian line ensemble, the study of which constitutes the key technical innovation responsible for the above theorem. Section 7.1.3 provides a precise definition of the half-space log-gamma line ensemble and Gibbs property, the key input from [34] and then a sketch of the steps to proving Theorem 7.1.1. Finally, Section 7.1.4 reviews some related work in studying half-space polymer and related models (Section 7.1.2 contains extensive literature review on the topic of Gibbsian line ensembles).

7.1.2 Half-space Gibbsian line ensembles

In order to prove Theorem 7.1.1 we develop a new probabilistic structure – half space Gibbsian line ensembles – and introduce a toolbox through which to study limits of such ensembles. A remarkable fact, due to the geometric RSK correspondence [121, 263, 260, 59] and the half-space

Whittaker process [25], is that the free energy process $\log Z_{(\alpha,\theta)}(N+m, N-m)$ for the log-gamma polymer can be embedded as the top labeled curve of an ensemble of log-gamma increment random walks interacting through a soft version of non-intersection conditioning and subject to an energetic interaction at the left boundary (where $m = 0$) depending on the value of α. In particular, when $\alpha > 0$ the $2i-1$ and $2i$ labeled curves of the line ensemble are attracted for each i, while for $\alpha < 0$ they are repulsed (and $\alpha = 0$ corresponds to no interaction). We briefly describe this embedding in Section 7.1.3 and 7.1.3 (see Section 7.2.2 for further discussion).

The basic premise of Gibbsian line ensembles, as initiated in the study of full-space models in [109], is to use the resampling invariance of a sequence of such ensembles to propagate one-point tightness information (generally for the top curve of the ensemble) into tightness of the entire sequence of ensembles. In particular once the scale of one-point fluctuations (in this case $N^{1/3}$) is known, the Gibbs property implies transversal fluctuations are correlated in a diffusive scale (in this case $N^{2/3}$) and that lower curves also all fluctuate with these exponents in the same scale. In other words, one point tightness of the top curve translates into spatial tightness of the entire ensemble. Moreover, all subsequential limits of these line ensembles enjoy, themselves, a Gibbs property corresponding to the diffusive limit of that of the pre-limiting ensembles. This general approach has been applied widely in studying a variety of different Gibbs properties related to probabilistic models, e.g. [110, 114, 137, 322, 29, 154, 155, 295, 319]. Moreover, it has been leveraged to give fine information about the local behavior of these models [197, 199, 196, 198, 81, 172, 171, 82, 83, 117, 128, 318, 132, 133, 174] and in studying related scaling limits such as the Airy sheet and directed landscape [**dov18**, 140, 37, 45, 139, 288, 119, 173, 283].

In this work we initiate the study of half-space Gibbsian line ensembles. These are measures on collections of curves in which there exists a left boundary around which the Gibbs property differs from its behavior in the bulk. As an illustrative example, consider curves ${}_1(s) \geq {}_2(s) \geq \cdots$ for $s \geq 0$ which enjoy the following resampling invariance. In the bulk, for $0 < a < b$ and $1 \leq k_1 \leq k_2$ the law of ${}_{[\![k_1,k_2]\!]}([a,b])$ (i.e., curves k_1 through k_2 on the interval $[a,b]$) conditioned on the values of ${}_{[\![k_1,k_2]\!]}(a)$, ${}_{[\![k_1,k_2]\!]}(b)$, ${}_{k_1-1}([a,b])$ (if $k_1 = 1$ then ${}_0 \equiv +\infty$) and ${}_{k_2+1}([a,b])$ is that of Brownian motions conditioned to start and end at the correct boundary values and to not intersect each other or the curve $_{k_1-1}([a,b])$ above and $_{k_2+1}([a,b])$ below. At the boundary, for $c > 0$ and $1 \leq k_1 \leq k_2$ the law of $_{\llbracket k_1,k_2 \rrbracket}([0,c])$ conditioned on the values of $_{\llbracket k_1,k_2 \rrbracket}(c)$, $_{k_1-1}([0,c])$ and $_{k_2+1}([0,c])$ is the law of Brownian motions conditioned to end at values $_{\llbracket k_1,k_2 \rrbracket}(c)$ at time c, not intersect with each other or the $_{k_1-1}$ and $_{k_2+1}$ curves on the interval $[0,c]$ and to have values at zero such that $_{2i-1}(0) = {}_{2i}(0)$ for all i. It is this last condition that is quite novel to the half-space models. An example of such an ensemble is illustrated in Figure 7.2 (B).

Half-space Gibbsian line ensembles have not previously been studied. However, this structure exists implicitly in some previous literature studying half-space integrable probabilistic models. For instance, the half-space (or Pfaffian) Schur processes [**sis**, 74, 12] have such a structure where the Brownian resampling is replaced by certain discrete random walks (geometric, exponential or Bernoulli), the non-intersection conditioning persists, and where the odd/even pairing at the boundary is replaced by an exponential interaction in the spirit of $e^{-\alpha({}_{2i-1}(0)-{}_{2i}(0))}$. Half-space Whittaker processes [**hbigmac**] have a more complicated Gibbs property which is the one relevant to our current work. Essentially, the Brownian motion is replaced by log-gamma random walks, the non-intersection by a soft exponential energy reweighing, and the interaction at zero by the same sort of $e^{-\alpha({}_{2i-1}(0)-{}_{2i}(0))}$ reweighing. There are other half-space Gibbs properties that should be studied such as related to half-space version of Hall-Littlewood processes, q-Whittaker processes and their spin generalizations. Furthermore, periodic or two-sided boundary versions of Gibbsian line ensembles (for instance related to periodic or two-sided boundary versions of Schur processes as in [66, 50] will also likely play a key role in study of related integrable probabilistic models and hence warrant study in the spirit of what is done here.

The core technical purpose of this paper is to extend the Gibbsian line ensemble methodology to address half-space models. We do this for the type of Gibbs property mentioned above that relates to half-space Whittaker processes which, owing to its relation to the log-gamma polymer, we call the half-space log-gamma Gibbs property. However, the ideas and tools developed here should be useful in studying more general line ensembles and related probabilistic models mentioned above.

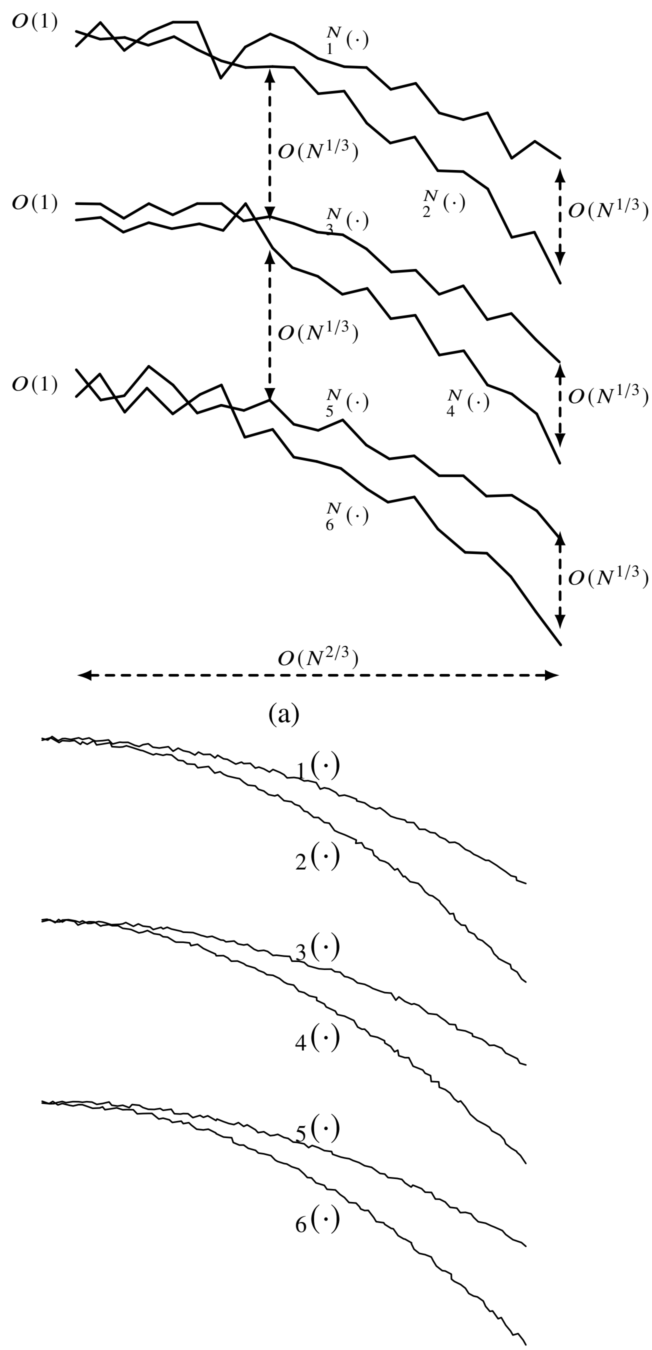

Figure 7.2: (A) depicts the half-space log-gamma line ensemble for large N along with the type of scalings that are deduced in proving Theorem 7.1.1. This ensemble enjoys a half-space log-gamma Gibbs property. (B) depicts a potential limiting line ensemble which should enjoy a half-space non-intersecting Brownian Gibbs property.

As in the full-space setting, the challenge is to develop a route to take one-point fluctuation information about the top curve $^N_1$ of a sequence of line ensembles ^{N} and propagate that into fluctuation information about the whole ensemble. (Figure 7.2 (A) illustrates the scalings that we prove to be associated with this sequence of line ensembles.) One-point information about the top curve for the half-space log-gamma polymer (and hence the top curve of our line ensemble) is in short supply with only two result due to (chronologically) [34] and then [205].

As explained in Section 7.1.3, we rely only on the work of [34]. From [34] we are able to extract two vital pieces of information: after proper centering the process $s \mapsto N^{-1/3}{}^{N}_{1}(sN^{2/3})$ stays bounded from positive infinity at $N \to \infty$, and at a random sequence of growing times $s_1^N, s_2^N, \ldots$ that stay tight as $N \to \infty$, the process has tight (bounded from positive and negative infinity) fluctuations around the parabola $-\nu s^2$ (for some explicit $\nu > 0$). The slightly odd nature of these inputs come from the fact that [34] studies a point-to-(partial)line partition function and not point-to-point directly. The work of [205] does provide tightness (and a limit theorem) for the point-to-point free energy, but is restricted to precisely the left boundary of ${}^{N}_{1}$ which is insufficient information for our approach. Currently, there are no limit theorems proved for the point-to-point free energy process away from the left-boundary.

With the above input we proceed to show how the Gibbs property propagates tightness to the whole ensemble. The idea is to first argue that (with proper centering) $N^{-1/3}{}^{N}_{2}(sN^{2/3})$ must be tight at some random time s. If not, the first curve would not follow a parabolic decay but rather a linear one in contradiction our parabolic decay input. Now, we know that the (scaled) first and second curves are tight at some random times (not necessarily the same). The next step is to argue that this pair of scaled curves to the left of the random times (including the left-boundary) are likewise tight. This relies on showing (using the Gibbs property and some a priori bounds) that the third curve cannot rise much beyond the first two curves, and that the first two curves remain bounded from infinity (as follows from [34]). With this and a form of stochastic monotonicity associated to this Gibbs property, the control over the first two curves can be established by a fine analysis of the behavior of a pair of log-gamma random walks subject to soft non-intersection conditioning and attractive energetic pinning at zero. We call these *weighted paired random walks* and a substantial amount of work is needed to develop tools and estimates regarding them. We give a more detailed overview of the steps of our proof in Section 7.1.3. The attractive nature of the boundary is directly linked to the choice here that $\alpha \geq 0$.

In this paper we do not pursue showing that the tightness propagation process extends to the entire line ensemble, though it very likely can be done, e.g. in the spirit of [154]. Any subsequential limit should enjoy the type of half-space Brownian Gibbs property discussed earlier. This would show that any such subsequential limit should also enjoy local comparison to Brownian motions when looking away from the boundary, or two 2-particle Dyson Brownian motions started paired together when looking near the boundary (provided $\alpha > 0$; for $\alpha = N^{-1/3}\mu$ the paired particles start in an attractive potential but are not equal). In fact, for the top two curves we can extract (though do not explicitly record here) such absolute continuity results without showing tightness of the whole ensemble, e.g. as in [29]. Note, the Gibbs property in [154, 29] differs slightly from that here since they consider point-to-point polymer endpoints varying along a horizontal line, while we consider endpoints varying along a down-right zigzag path.

Besides the directions alluded to above, we mention here a few more natural points of inquiry spurred by our work. Our analysis is restricted to $\alpha \geq 0$. When $\alpha < 0$, the pair interaction at the boundary becomes repulsive and thus curves separate and behave quite differently. In particular, the log-gamma free energy (i.e., top curve) is expected to have $O(\sqrt{N})$ Gaussian fluctuations and $O(1)$ transversal fluctuation around (N, N). The Gaussian fluctuations on the diagonal was recently proven in [205], while the $O(1)$ transversal fluctuations result will appear in the upcoming work [134]. The behavior in this $O(1)$ scale relates to a portion of the phase diagram for the half-space log-gamma stationary measure [27]. Using our Gibbsian line ensemble techniques and modifications of the log-gamma polymer (i.e., adding a boundary condition on the first row too), it should be possible to access and re-derive the description of the entire phase diagram.

Beyond tightness, the half-space log-gamma line ensemble should converge to a universal limit, the half-space Airy line ensemble. This object, which should enjoy the type of Brownian Gibbs property discussed earlier, has not been constructed. While the log-gamma convergence result is currently out of reach, it should be possible to construct this from solvable last passage percolation, i.e. half-space Schur processes [12]. This should enjoy uniqueness characterization in the spirit of [152, 149] and may even relate to a half-space Airy sheet in the spirit of [**dov18**].

A different scaling regime for the half-space log-gamma line ensemble involves weak-noise scaling in which θ goes to infinity while α remains fixed. In the full-space setting, [320] proved tightness of the full-space line ensemble and (via [120]) convergence to the KPZ line ensemble [264, 110]. A half-space analog of this result should hold and help in exploring questions related to the half-space KPZ equation and the corresponding half-space continuous directed random polymer.

7.1.3 Ideas in the proof of Theorem 7.1.1

We start in Section 7.1.3 by precisely defining the half-space log-gamma Gibbs measure and line ensemble. In Section 7.1.3 we record the key input from [34] which we then combine with the Gibbs line ensemble structure in Section 7.1.3 to give the key deductions in the course of proving Theorem 7.1.1 (see Section 7.5 for the full proof of this theorem).

Though the Gibbs measure and line ensemble definition holds for general α, most of our discussion, especially around the proof, will focus on the case $\alpha > 0$ which is harder than the $\alpha = N^{-1/3}\mu$ case. As noted earlier, we do not address the case of $\alpha < 0$ here.

$\mathcal{HSLG}$ Gibbs measures and the $\mathcal{HSLG}$ line ensemble

The main technique that goes into the proof of Theorem 7.1.1 is our construction of the half-space log-gamma ($\mathcal{HSLG}$) line ensemble whose law enjoys a property that we call the half-space log-gamma ($\mathcal{HSLG}$) Gibbs measures. In what follows we construct these objects and describe how they relate to the $\mathcal{HSLG}$ polymer free energy.

We will start by defining the $\mathcal{HSLG}$ Gibbs measure whose state-space and associated weight function is indexed by the following directed/colored graph. Define the graph G with vertices $V(G) := \{(m,n) : m \in \mathbb{Z}_{>0}, n \in \mathbb{Z}_{<0} + \frac{1}{2}\mathbf{1}_{m\in 2\mathbb{Z}}\}$ and with the following directed/colored edges:

- For each $(m,n) \in \mathbb{Z}_{>0}^2$, we put two **blue** edges:

$$(2m-1,-n) \to (2m,-n+0.5) \text{ and } (2m+1,-n) \to (2m,-n+0.5).$$

- For each $(m,n) \in \mathbb{Z}_{>0}^2$, we put two **black** edges:

$$(2m,-n-0.5) \to (2m-1,-n) \text{ and } (2m,-n+0.5) \to (2m+1,-n).$$

- For each $m \in \mathbb{Z}_{>0}$, we put one **red** edge: $(1,-2m+1) \to (1,-2m)$.

The portion of the corresponding graph is shown in Figure 7.3 (A). We write $E(G)$ for the set of edges of graph G and $e = \{v_1 \to v_2\}$ for a generic directed edge from v_1 to v_2 in $E(G)$ (the color of the edge is suppressed from the notation).

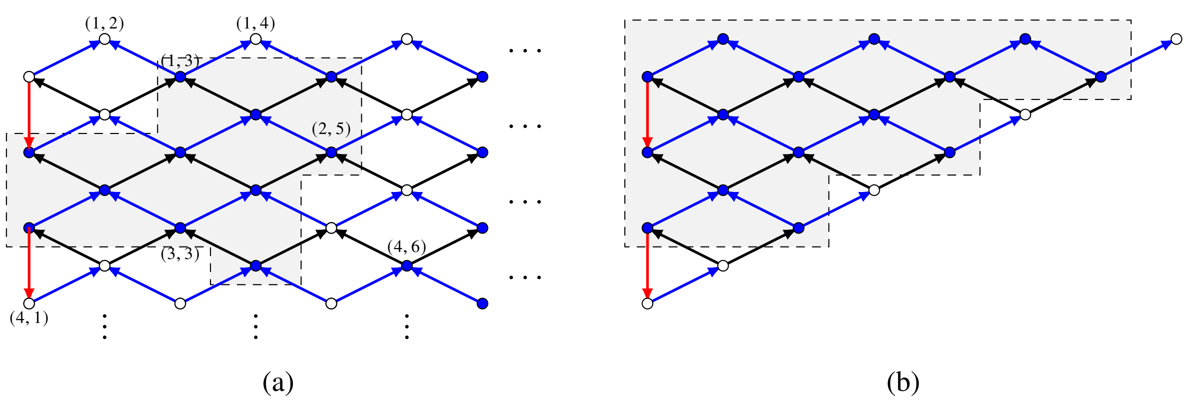

Figure 7.3: (A) The graph G associated to half-space log-gamma Gibbs measures. In the figure, a few of the vertices of G are labeled by ϕ-induced labeling. A generic bounded connected domain Λ is shown in the figure which contains all vertices in the shaded region. $\partial\Lambda$ consists of white vertices in the figure. (B) The domain K_N considered in Theorem 7.1.3. Λ_N^* consists of vertices in the shaded region.

We next define a bijection $\phi : V(G) \to \mathbb{Z}_{>0}^2$ by $\phi((m,n)) = (-\lfloor n \rfloor, m)$. This pushes the directed/colored edges in G onto directed/colored edges on $\mathbb{Z}_{>0}^2$ which we denote by $E(\mathbb{Z}_{>0}^2)$. We will always view G as in Figure 7.3 and will use the ϕ-induced labeling when describing this graph.

We associate to each $e \in E(\mathbb{Z}^2)$ a weight function based on the color of edge defined as follows:

$$W_e(x) := \begin{cases} \exp(\theta x - e^x) & \text{if } e \text{ is } \textbf{blue}, \\ \exp(-e^x) & \text{if } e \text{ is } \textbf{black}, \\ \exp(-\alpha x) & \text{if } e \text{ is } \textbf{red}. \end{cases} \tag{7.1.3}$$

Definition 7.1.2 (Half-space log-gamma Gibbs measure)**.** Consider the graph $\mathbb{Z}^2_{>0}$ endowed with directed/colored edges $E(\mathbb{Z}^2_{>0})$ as above. Let Λ be a bounded connected subset of $\mathbb{Z}^2_{>0}$. Set

$$\partial\Lambda := \left\{v \in \mathbb{Z}^2_{>0} \cap \Lambda^c : \{v' \to v\} \in E(\mathbb{Z}^2_{>0}) \text{ or } \{v \to v'\} \in E(\mathbb{Z}^2_{>0}), \text{ for some } v' \in \Lambda\right\}.$$

The half-space log-gamma ($\mathcal{HSLG}$) Gibbs measure for the domain Λ, with boundary condition $\big(u_{i,j} \in \mathbb{R} : (i,j) \in \partial\Lambda\big)$, is a measure on $\mathbb{R}^{|\Lambda|}$ with density at $(u_{i,j})_{(i,j)\in\Lambda}$ proportional to

$$\prod_{e=\{v_1\to v_2\}\in E(\Lambda\cup\partial\Lambda)} W_e(u_{v_1} - u_{v_2}).$$

Lemma 7.6.2 shows that the $\mathcal{HSLG}$ Gibbs measure is well defined.

The following result shows how the $\mathcal{HSLG}$ free energy process can be embedded in a $\mathcal{HSLG}$ Gibbs measure. Its proof, given in Section 7.2.2, relies directly on results of [34] that build on the analysis of the log-gamma polymer via the geometric RSK correspondence [121] on symmetrized domains [263, 260, 59]. In Section 7.2.2, for each $N > 0$, we will define explicitly such a choice for $\big({}^N_i(j) : (i,j) \in \mathcal{K}_N\big)$ that will satisfy the two criterion of the above theorem. We will call this the half-space log-gamma ($\mathcal{HSLG}$) line ensemble.

Theorem 7.1.3. *Fix $\theta > 0$, $\alpha > -\theta$, and $N \in \mathbb{Z}_{>0}$. Set $\mathcal{K}_N := \{(i,j) \in \mathbb{Z}^2_{>0} : i \in [1,N], j \in [1, 2N-2i+2]\}$. There exists a collection of random variables $\big({}^N_i(j) : (i,j) \in \mathcal{K}_N\big)$ defined on the same probability space such that the following holds.*

(i) We have the following equality in distribution

$$\left({}_{1}^{N}(2j+1)\right)_{j\in[\![0,N-1]\!]} \stackrel{d}{=} \Big(\log Z_{(\alpha,\theta)}(N+j,N-j)+2N\Psi(\theta)\Big)_{j\in[\![0,N-1]\!]}. \tag{7.1.4}$$

(ii) Let $\Lambda_N^* := \{(i,j)\in\mathbb{Z}_{>0}^2 : i\in[1,N-1], j\in[1,2N-2i+1]\}$. *The law of* $\big({}_{i}^{N}(j) : (i,j)\in\Lambda_N^*\big)$ *conditioned on* $\big({}_{i}^{N}(j) : (i,j)\in(\Lambda_N^*)^c\big)$ *is given by the* $\mathcal{HSLG}$ *Gibbs measure for the domain* Λ_N^* *with boundary condition* $\big({}_{i}^{N}(j) : (i,j)\in\partial\Lambda_N^*\big)$.

Deductions from the $\mathcal{HSLG}$ Gibbs measures

It is useful to visualize the $\mathcal{HSLG}$ Gibbs measures from Theorem 7.1.3 in terms of the language of Gibbsian line ensembles. Consider k and T fixed and N sufficiently large so that all of the random variables ${}_{1}^{N}[\![1,T]\!], {}_{2}^{N}[\![1,T]\!], \ldots, {}_{2k}^{N}[\![1,T]\!]$ are defined. We will think of ${}_{i}^{N}$ as the label i line in the ensemble. The values of $\big({}_{i}^{N}(2T+1) : i\in[\![1,2k]\!]\big)$ and ${}_{2k+1}^{N}(\cdot)$ constitute boundary data which, once known, uniquely identify (via the Gibbs description) the laws of ${}_{1}^{N}[\![1,T]\!], {}_{2}^{N}[\![1,T]\!], \ldots, {}_{2k}^{N}[\![1,T]\!]$.

Let us consider the three types of weights in the Gibbs measure. The weights corresponding to **black** edges $v_1 \longrightarrow v_2$ contribute a factor of $e^{-e^{u_{v_1}-u_{v_2}}}$ (here u_v is the dummy variable in the Gibbs density corresponding to a vertex v) in the Gibbs measure. Whenever $u_{v_1} \gg u_{v_2}$, this weight is very close to 0, whereas otherwise it is close to 1. Thus, this weight produces a soft version of conditioning on the event that ${}_{v_2}^{N} \geq {}_{v_1}^{N}$. **Black** edges arise between consecutive lines thus we expect that our measure will strongly favor configurations where ${}_{1}^{N}(\cdot) \geq {}_{2}^{N}(\cdot) \geq {}_{3}^{N}(\cdot) \geq \cdots$, i.e., the curves are non-intersecting (Theorem 7.3.1 provides a precise statement substantiating this). Of course, the soft nature of this conditioning will not rule out crossing, but a heavy penalty will be incurred so at a heuristic level it is useful to think in terms of non-intersecting lines.

The **red** edges are $(2i-1,1) \longrightarrow (2i,1)$ and come with a weight $e^{-\alpha(u_{2i-1,1}-u_{2i,1})}$. This weight is close to 0 when $u_{2i-1,1} \gg u_{2i,1}$ (since $\alpha > 0$). This creates an attractive force between ${}_{2i-1}^{N}(1)$ and ${}_{2i}^{N}(1)$ which tries to establish the ordering ${}_{2i-1}^{N} \leq {}_{2i}^{N}$. Of course, this is in opposition to the soft non-intersecting influence already discussed. Combined, these forces ultimately (through our analysis of weighted paired random walks) result in the difference ${}^{N}_{2i-1}(1) - {}^{N}_{2i}(1)$ remaining $O(1)$ as $N \to \infty$. In contrast, in the critical regime, when $\alpha_N = N^{-1/3}\mu$, the attraction weakens with N and the forces result in ${}^{N}_{2i-1}(1) - {}^{N}_{2i}(1)$ like $O(N^{1/3})$. It is the $O(1)$ distance between ${}^{N}_{2i-1}(1)$ and ${}^{N}_{2i}(1)$ that makes the supercritical case harder than the critical case.

Finally, consider the **blue** edges that encode the Gibbs weights between consecutive values of a given line, i.e. between ${}^{N}_{i}(j)$ and ${}^{N}_{i}(j+1)$. Alone, these weights define log-gamma increment random walks (with two-step periodicity in the law of the increments). Thus, putting these three factors together one arrives at the picture illustrated in Figure 7.2 (A) – an ensemble of softly non-intersecting log-gamma random walks whose $2i-1$ and $2i$ left starting points are $O(1)$ distance apart for each i. In order to prove Theorem 7.1.1 we essentially need to justify the distance scales in Figure 7.2 (A). To do that, we use the Gibbs property (in the spirit of the line ensemble language described above) along with some one-point control over the behavior of ${}^{N}_{1}$ that we describe now.

Point-to-line free energy fluctuations

The $\mathcal{HSLG}$ Gibbs measures machinery gives us access to the behavior of the $\mathcal{HSLG}$ line ensemble conditioned on the boundary data. However, we still need to understand the behavior of the boundary data. The theory of (full-space) Gibbsian line ensembles that has been developed over the last decade has become proficient at taking very minimal seed information, such as the scale in which tightness occurs for the one-point fluctuations of the top curve of a Gibbsian line ensemble, and outputting the scaling and tightness for the entire edge of the line ensemble. We take the first step in developing such a half-space theory.

There are currently only two fluctuation results about the $\mathcal{HSLG}$ polymer. The first (chronologically) is a result of [34] that we will recall below and appeal to, while the second is the slightly more recent work of [205] that proves a limit theorem for $N^{-1/3}{}^{N}_{1}(1)$ (i.e. $f_N^{\alpha}(0)$). Our work began prior to the release of [205] and thus we rely only on the work of [34]. The control [205] provides is for ${}^{N}_{1}(1)$ only and since we need some information away from the boundary too, most of the work herein is unavoidable and not really simplified by using [205]. It is natural to wonder if [205] could have been used along, in place of [34], at the seed for our analysis. While we do not rule this out, it would certainly require a very different type of argument since we rely heavily on the fact that [34] provides some information about ${}^N_1(j)$ as j varies.

Let us recall the result of [34]. For each $k > 0$, define the point-to-(partial)line partition function

$$Z_N^{\mathrm{line}}(k) := \sum_{j=\lceil k \rceil}^{N} Z_{(\alpha,\theta)}(N+j, N-j). \tag{7.1.5}$$

This sum is restricted to endpoints at least distance $2k$ from the boundary. Set $p = \frac{N+k}{N-k}$, Let θ_c be the unique solution to $\Psi'(\theta_c) = p\Psi'(2\theta - \theta_c)$ and set (recall the digamma function Ψ from (7.1.2))

$$f_{\theta,p} := -\Psi(\theta_c) - p\Psi(2\theta - \theta_c), \quad \sigma_{\theta,p} := \left(\tfrac{1}{2}(-\Psi''(\theta_c) - p\psi''(2\theta - \theta_c)\right)^{1/3}.$$

Theorem 7.1.4 (Theorem 1.10 in [34])**.** *Suppose $(k_N)_{N\in\mathbb{Z}_{>0}}$ is such that for some $y \in \mathbb{R} \cup \{\infty\}$, $\lim_{N\to\infty}(N - k_N)^{1/3}\sigma_{\theta,p}(\alpha + \theta - \theta_c) = y$. Then, as $N \to \infty$*

$$\frac{\log Z_N^{\mathrm{line}}(k_N) - (N - k_N) f_{\theta,p}}{(N - k_N)^{1/3}\sigma_{\theta,p}} \to U_{-y}.$$

where for $y \in \mathbb{R}$, U_{-y} is distributed as the Baik-Ben Arous-Péché distribution with parameter y (see Eq. (5.2) in [34]). When $y = \infty$, $U_{-\infty}$ is distributed as the GUE Tracy-Widom distribution.

The crucial deduction from Theorem 7.1.4 is that there exists $\nu > 0$ such that for each $M > 0$,

$$V_N(M) + M^2 \xrightarrow{d} X_M, \quad \text{where} \quad V_N(M) := \frac{\log Z_N^{\mathrm{line}}(MN^{2/3}) + 2\Psi(\theta)N}{N^{1/3}\nu}. \tag{7.1.6}$$

Here the BBP distributions of the limiting random variables $(X_M)_{M>0}$ form a tight sequence in M, in particular they converge in law to the GUE Tracy-Widom distribution as $M \to \infty$. A precise version of this deduction in given later in Lemma 7.3.6. Essentially, the rescaled point-to-(partial)line free energy process $V_N(M)$ looks like an inverted parabola $-M^2$ with tight fluctuations around it.

Using the Gibbs line ensemble structure to prove Theorem 7.1.1

We now give a brief overview of the steps of our proof and how it follows from combining the seed information from [34], i.e. (7.1.6), and the $\mathcal{HSLG}$ line ensemble Gibbs property.

Fix any $r > 0$ and N_0 large enough so everything below is well defined for $N \geq N_0$. Let us set $T = 8\lfloor rN^{2/3} \rfloor$ (the key point is that this time window scales like $N^{2/3}$). We will say that a sequence of random variables X_N is *upper-tight* if $\max(X_N, 0)$ is tight, and *lower-tight* if $\min(X_N, 0)$ is tight. Recall that X_N is tight if for all $\varepsilon > 0$ there exists $K = K(\varepsilon) > 0$ such that $\mathbf{P}(|X_N| \geq K) < \varepsilon$ for all $N \geq N_0$. If X_N is both upper and lower tight, then it is tight. The broad steps of establishing our main theorem can be summarized as follows. Note that we consider staggered (i.e., even and odd) arguments for ${}^{N}_{1}$ and ${}^{N}_{2}$ below due to the diagonal interaction of the Gibbs property. This is a technical point which can be ignored currently.

(i) Prove that $N^{-\frac{1}{3}}{}^{N}_{1}(2p^* - 1)$ and $N^{-\frac{1}{3}}{}^{N}_{2}(2p^*)$ are tight for some random $p^* \in [M_1 N^{\frac{2}{3}}, M_2 N^{\frac{2}{3}}]$.

(ii) Assuming (i) and ${}^{N}_{3} \equiv -\infty$, prove that $N^{-\frac{1}{3}}{}^{N}_{1}(1)$ and $N^{-\frac{1}{3}}{}^{N}_{2}(2)$ are lower-tight.

(iii) Assuming (i) and ${}^{N}_{3} \equiv -\infty$, prove that any $M^* > 0$, with positive probability (depending on M^* and r but not of N) that

$$ {}^{N}_{1}(p) \geq M^* N^{1/3}, \quad \text{and} \quad {}^{N}_{2}(p) \geq M^* N^{1/3}, \quad \text{for all } p \in [\![1, T]\!]. $$

(iv) Assuming (i) and ${}^{N}_{3}(\cdot) \equiv -\infty$, prove process-level tightness of

$$ (N^{-1/3}{}^{N}_{1}(xN^{2/3}), N^{-1/3}{}^{N}_{2}(xN^{2/3}))_{x \in [0,2r]}. $$

We shall describe how we establish the above broad steps in a moment. Let us first sketch how the above steps work together to yield our main theorem, Theorem 7.1.1.

We first argue that $N^{-\frac{1}{3}}{}^{N}_{1}(1)$ and $N^{-\frac{1}{3}}{}^{N}_{2}(2)$ are tight without any conditioning on ${}^{N}_{3} \equiv -\infty$. Indeed, since point-to-line free energy is an upper bound for the point-to-point free energy process, utilizing (7.1.6) it follows immediately that $N^{-\frac{1}{3}}{}^{N}_{1}(1)$ and $N^{-\frac{1}{3}}{}^{N}_{2}(2)$ are upper-tight. Since the lower-tightness event is increasing with respect to the boundary data, by stochastic monotonicity of $\mathcal{HSLG}$ Gibbs line ensembles (Proposition 8.2.3) it suffices to show the lower-tightness under ${}^{N}_{3} \equiv -\infty$, which is precisely established in item (ii).

We next argue that with positive probability there is a uniform separation of order $N^{1/3}$ between the first two curves ${}^{N}_{1}$ and ${}^{N}_{2}$ and the third curve ${}^{N}_{3}$. Indeed, once we have tightness at the left boundary, it is not hard to show that $N^{-\frac{1}{3}}{}^{N}_{1}(2v-1)$ and $N^{-\frac{1}{3}}{}^{N}_{2}(2v)$ are tight for any $v \in [\![1, p^*]\!]$. Combining this with the soft non-intersection property of the line ensembles and (ii), we deduce in Theorem 7.3.7 that with high probability $\sup_{p\in[\![1,2T]\!]} N^{-1/3}{}^{N}_{3}(p)$ is upper tight. As the event considered in (iii) is increasing with respect to the boundary data, using (iii) we establish the desired uniform separation of order $N^{1/3}$.

Finally, to prove the process-level tightness of the top two curves of our ensemble, we combine the uniform separation deduced above with the stochastic monotonicity of $\mathcal{HSLG}$ Gibbs line ensembles (Proposition 8.2.3) and the process-level tightness of the first two curves with the third curve moved to $-\infty$ (as shown in item (iv)). This establishes tightness of the first two curves which, through identifying the first curve with the free energy process, yields Theorem 7.1.3.

Remark 7.1.5. We mention that in a recent work, [205] established fluctuation results for $\log Z_{(\alpha,\theta)}(N, N)$ (equivalently ${}^{N}_{1}(1)$). Their result immediately implies the tightness of $N^{-1/3}{}^{N}_{1}(1)$. However, to carry our proof outlined above we need tightness of ${}^{N}_{2}(1)$, and other fine information about ${}^{N}_{1}$ and ${}^{N}_{2}$ away from the boundary, as described in item (ii) and (iv), which to our best understanding is beyond the scope of [205]. Our argument does not need the input from [205] and since there is little reduction in length or complexity gained from using [205], we have opted to rely only on the results from [34].

We return to steps (i)-(iv) stated above and describe the main ideas in achieving them.

Proof idea for (i):

(a) We start by proving (Theorem 7.3.1) that the curves ${}^{N}_{i}$ are typically non-intersecting, or at least do not overlap by much. Owing to this and the fact that the point-to-line partition function controlled in [34] dominates the point-to-point partition function for any point along the line, it follows that $\sup_{i,j} N^{-1/3}{}^{N}_{i}(j)$ is upper-tight. Lower-tightness is more difficult.

(b) From the parabolic decay of the point-to-(partial)line free energy (7.1.6), we deduce that the point-to-point free energy process has to be in the $N^{1/3}$ fluctuation scale at some random point p_1^* in a $O(N^{2/3})$ window. We essentially (see Proposition 7.3.4) we show that for M_0 large enough

$$\sup_{p \in [\![SN^{2/3}, (M_0+2S)N^{2/3}]\!]} \frac{{}^{N}_{1}(2p+1)}{N^{1/3}\nu} + S^2 \tag{7.1.7}$$

is uniformly tight over all $S > 0$. The parameter ν is an explicit function of θ, see (7.3.13). Here the crucial point is the uniformity, i.e., the constant $K(\varepsilon)$ in the definition of tightness can be chosen independent of $S > 0$. Thus, in $N^{1/3}$ and $N^{2/3}$ scaling ${}^{N}_{1}$ follows an inverted parabola.

(c) We next show essentially (see Proposition 7.3.3) that there exists M_1 and M_2 large enough so that $\sup_{p_2 \in [\![M_1N^{2/3}, M_2N^{2/3}]\!]} N^{-1/3}{}^{N}_{2}(2p_2)$ is tight. The idea is if L_2^N is uniformly low in $[M_1N^{2/3}, M_2N^{2/3}]$, then, due to the Gibbs property of the line ensemble, the first curve L_1^N behaves like a random bridge, i.e., linearly, in that interval. However, as we show in the proof of Proposition 7.3.3, this violates the inverted parabolic trajectory (7.1.7) for some S. This leaves us with a random point $p_2^* \in [M_1N^{2/3}, M_2N^{2/3}]$, so that $N^{-1/3}{}^{N}_{2}(2p_2^*)$ is tight. Owing to the typical non-intersection (Proposition 7.3.1) we have that $N^{-1/3}{}^{N}_{1}(2p_2^* - 1)$ is tight as well.

<u>**Proof idea for (ii) and (iii)**:</u> Our proof relies on understanding the

$$\text{law of } {}^{N}_{1}, {}^{N}_{2} \text{ conditioned on } {}^{N}_{1}(2T-1) = a_T, {}^{N}_{2}(2T) = b_T \text{ and } {}^{N}_{3} \equiv -\infty \tag{7.1.8}$$

for T, as above, of order $N^{2/3}$ and a_T, b_T of order $\sqrt{T}$. Since the events in (ii) and (iii) are increasing w.r.t. the boundary data, we may further assume $a_T = 0$ and $b_T = -\sqrt{T}$ by stochastic monotonicity

(Proposition 8.2.3). We focus on the proof idea of (ii). The proof idea for (iii) is quite similar and done parallelly in Section 7.4.

As alluded to in Section 7.1.2, (7.1.8) has connections to *weighted paired random walk* (WPRW) law. We now briefly introduce them here. Let f denote the density of $\log Y_1 - \log Y_2$ were Y_1, Y_2 are independent Gamma(θ) random variables. Let $g(z) := [\Gamma(\alpha)]^{-1} e^{\alpha z - e^z}$. A paired random walk (PRW) $(R_k, S_k)_{k=1}^T$ with endpoints $R_T = a_T$ and $S_T = b_T$ is a measure on $\mathbb{R}^{2T-2}$ with density

$$\mathbf{P}\left(\bigcap_{k=0}^{T-1}\{R_k \in dx_k, S_k \in dy_k\}\right) \propto g(y_1 - x_1) \prod_{i=2}^{T}[f(x_i - x_{i-1}) f(y_i - y_{i-1})] \prod_{k=1}^{T-1} dx_k\, dy_k. \quad (7.1.9)$$

We define the random variable:

$$W_{\mathrm{sc}} := \exp\left(- e^{S_1 - R_1} - \sum_{k=2}^{T-1} \left(e^{S_k - R_{k+1}} + e^{S_k - R_k} \right) \right).$$

Using W_{sc} we define a new measure on $\mathbb{R}^{2T-2}$ as:

$$\mathbf{P}_{W_{\mathrm{sc}}}(\mathsf{A}) = \frac{\mathbf{E}[W_{\mathrm{sc}} \mathbf{1}_{\mathsf{A}}]}{\mathbf{E}[W_{\mathrm{sc}}]} \quad (7.1.10)$$

We call the above measure as weighted paired random walk (WPRW). The measure depends on a_T and b_T as well but we have hide its dependency from the notation.

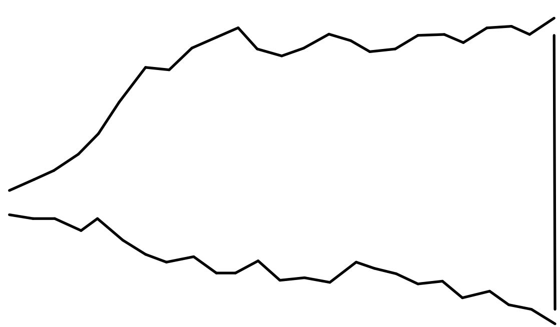

Figure 7.4: The above is a realization of the PRW law having non-intersection where we have assumed $a_T = 0$ and $b_T = -\sqrt{T}$.

The law of $(R_k, S_k)_{k=1}^T$ under $\mathbf{P}_{W_{\mathrm{sc}}}$ is same as the law of $({}^{N}_{1}(2k-1), {}^{N}_{2}(2k))_{k=1}^T$ under (7.1.8).

Thus to establish (ii), it suffices to show $\mathbf{P}_{W_{\mathrm{sc}}}(R_1 \le -M\sqrt{T})$ and $\mathbf{P}_{W_{\mathrm{sc}}}(S_1 \le -M\sqrt{T})$ can be made arbitrarily small by choosing M large enough. Towards this end, we shall utilize the formula in (7.1.10) for $\mathsf{A} := \{R_1 \le -M\sqrt{T}\}$ (and $\mathsf{A} := \{S_1 \le -M\sqrt{T}\}$). Under this setting we show that

$$\begin{aligned} &\tfrac{1}{\mathrm{C}}\cdot T^{-1/2} \le \mathbf{E}[W_{\mathrm{sc}}] \le \mathrm{C}\cdot T^{-1/2}, \\ &\tfrac{1}{\mathrm{C}}\cdot T^{-1/2}\cdot\mathbf{E}[\mathbf{1}_{\mathsf{A}}] \le \mathbf{E}[W_{\mathrm{sc}}\mathbf{1}_{\mathsf{A}}] \le \mathrm{C}\cdot T^{-1/2}\cdot\mathbf{E}[\mathbf{1}_{\mathsf{A}}]. \end{aligned} \tag{7.1.11}$$

for some universal constant $\mathrm{C} > 0$. The inequalities in (7.1.11) are established in the proof of Lemma 7.4.1, Lemma 7.4.11, and Corollary 7.4.12. The $T^{-1/2}$ behavior here is particular to $(a_T, b_T) = (0, -\sqrt{T})$. For general boundary values of a_T, b_T, this estimates may not be true.

We now briefly explain the proof idea first for the upper bound of $\mathbf{E}[W_{\mathrm{sc}}]$. Along the way, we shall explain why $T^{-1/2}$ factor shows up. The starting point of our proof is to first convert the soft non-intersection reweighing (W_{sc}) to hard non-intersection conditioning by the following inequality

$$W_{\mathrm{sc}} \le \sum_{p=0}^{\infty} e^{-e^p}\cdot\mathbf{1}\{R_k \ge S_k - p - 1, \forall k \in [\![2, T]\!]\}.$$

Let us write $\mathsf{NI} := \{R_k \ge S_k, \forall k \in [\![2, T]\!]\}$ for the non-intersection event. In Lemma 7.7.5, we show that there is an absolute constant $\mathrm{C} > 0$, such that $\mathbf{P}(R_k \ge S_k - p - 1, \forall k \in [\![2, T]\!]) \le e^{\mathrm{C}p}\cdot\mathbf{P}(\mathsf{NI})$ for all $p \ge 0$. Thus plugging this inequality in the above display yields $\mathbf{E}[W_{\mathrm{sc}}] \le \mathrm{C}\cdot\mathbf{P}(\mathsf{NI})$. It is thus suffices to understand the order of $\mathbf{P}(\mathsf{NI})$.

Due to the presence of g factor in (7.1.9), under PRW we expect a pinning effect in the left boundary, i.e., $R_1 - S_1 = O(1)$. Thus we expect the large scale behavior under PRW should be similar to that of two independent random walks with zero as a starting point. It is well known (see for example [297, 228]) that when R_k, S_k are independent random walks with $R_1 = S_1 = 0$, the non-intersection probability $\mathbf{P}(\mathsf{NI})$ is precisely of the order $T^{-1/2}$. This is why we expect $T^{-1/2}$ behavior of the non-intersection probability under PRW law as well. We show this fact by using the following two lemmas:

- Lemma 7.4.7: $|R_1 - S_1|$, $R_1/\sqrt{T}$, and $S_1/\sqrt{T}$ have exponential tails.
- Lemma 7.7.6: $\mathbf{P}\big(\mathsf{NI} \cdot \mathbf{1}_{|R_1|+|S_1| \le \sqrt{T}(\log T)^{3/2}}\big) \le \frac{\mathrm{C}}{\sqrt{T}} \mathbf{E}\left[(\max\{R_1 - S_1, 0\} + 1) \cdot \max\left\{\frac{1}{\sqrt{T}}|R_1|, 2\right\}^{\frac{3}{2}}\right]$.

The first bullet point follows from the description of the PRW law in (7.1.9) and the nature of the densities f and g. The second bullet point is more subtle and requires various estimates under random walk bridge law that are uniform over a specified set of starting and ending points. The details are presented in Appendix 7.7. Clearly from the above two bullet points, it is not hard to deduce that $\mathbf{P}(\mathsf{NI}) \le CT^{-1/2}$. The upper bound for $\mathbf{E}[W_{\mathrm{sc}} \mathbf{1}_{\mathsf{A}}]$ also follows in a similar manner.

The lower bound argument for $\mathbf{E}[W_{\mathrm{sc}}]$ is more involved. Here we define a particular event, called Gap_β (see (7.4.20) for definition), and show in Lemma 7.4.8 that $W_{\mathrm{sc}} \ge a_\beta \cdot \mathbf{1}_{\mathsf{Gap}_\beta \cap \{R_1 - S_1 \in [0,1]\}}$, for some deterministic constant $a_\beta > 0$. To estimate $\mathbf{P}(\mathsf{Gap}_\beta \cap \{R_1 - S_1 \in [0,1]\})$. Note that conditioned on R_1, S_1, PRW are two independent random walk bridges started from R_1, S_1 and ending at a_T, b_T. In Lemma 7.4.10, we show that random walk bridge can be compared to modified random bridge which is a certain concatenation of random walks and random bridges (see Definition 7.4.9). In particular, this leads to

$$\mathbf{P}(\mathsf{Gap}_\beta \cap \{R_1 - S_1 \in [0,1]\}) \ge \mathbf{E}\left[\mathbf{1}_{R_1 - S_1 \in [0,1]} \cdot \widetilde{\mathbf{P}}_{R_1,S_1}(\mathsf{Gap}_\beta)\right] \tag{7.1.12}$$

where $\widetilde{\mathbf{P}}_{R_1,S_1}$ is the law of two independent modified random bridges started from R_1, S_1 and ending at a_T, b_T. The key point is that the modified random bridge has a true random walk portion, and hence one can rely on standard non-intersecting random walk estimates to eventually obtain estimates for the above quantity. In Appendix 7.7, we establish various uniform estimates and in particular show that $\widetilde{\mathbf{P}}_{R_1,S_1}(\mathsf{NI}) \ge CT^{-1/2}$ and $\widetilde{\mathbf{P}}_{R_1,S_1}(\mathsf{Gap}_\beta \mid \mathsf{NI}) \ge \frac{1}{2}$ uniformly over $R_1, S_1 \le M\sqrt{T}$ and $R_1 - S_1 \in [0,1]$. This leads to a $T^{-1/2}$ order lower bound for the right hand side of (7.1.12).

There is an extensive literature in studying non-intersecting random walks and random bridges that are pinned at the starting and/or ending points (see [18, 161, 144] and the reference therein). In particular, non-intersecting random walks and non-intersecting random bridges under diffusive scaling are known to converge to Dyson Brownian motion and non-intersecting Brownian bridges. Our work establishes *uniform* estimates (uniform over starting and ending points) for non-intersection probabilities of random walks and random bridges. Indeed in our technical arguments, we require our estimates of probability of events (such as the Gap_β event and others) under random bridges to be *uniform* over *all possible* $O(1)$ apart ending points that lies in a diffusive $O(T^{1/2})$ window. In Appendix 7.7, we thus develop the machinery to establish such uniform estimates.

Proof idea for (iv):

The argument to prove (iv) also uses the machinery developed in the proof of (ii) and (iii). However, one caveat is that to show (iv), one needs to consider events related to modulus of continuity of the processes which are not increasing events. Thus one can not assume $a_T = 0, b_T = -\sqrt{T}$. Instead, here we can only use the soft non-intersection property to deduce that $a_T - b_T \ge -(\log T)^{7/6}$. Under this boundary conditions the estimates in (7.1.11) may no longer be true. However note that (iv) claims tightness for the processes on the range $[\![1, T/4]\!]$, i.e., the first quarter of the total points. Hence here the strategy is to first perform a decomposition:

$$\frac{\mathbf{E}[W_{\mathrm{sc}}\mathbf{1}_{\mathsf{A}}]}{\mathbf{E}[W_{\mathrm{sc}}]} \asymp \frac{\mathbf{E}[W_{\mathrm{sc}}^1\mathbf{1}_{\mathsf{A}}]\cdot\mathbf{E}[W_{\mathrm{sc}}^2]}{\mathbf{E}[W_{\mathrm{sc}}^1]\mathbf{E}[W_{\mathrm{sc}}^2]} = \frac{\mathbf{E}[W_{\mathrm{sc}}^1\mathbf{1}_{\mathsf{A}}]}{\mathbf{E}[W_{\mathrm{sc}}^1]}, \tag{7.1.13}$$

where A corresponds to the tightness event associated to the modulus of continuity of the process. Here W_{sc}^1 (resp. W_{sc}^2) are super-exponential factors associated to the first (resp. last) quarter of the points of the walk (the point before the first dashed line and points after the second dashed line in Figure 7.5).

Let us briefly explain why the approximation in (7.1.13) is true. Recall from our discussion in the proof idea of (ii) and (iii) that W_{sc} is close to the indicator of non-intersection event. Since the walks have pinning effect in either of the boundaries, the non-intersection probabilities essentially depend on the initial part of walks and final part of the walks. The non-intersection probability for the middle portion does not decay to zero. Thus we expect, $\mathbf{E}[W_{\mathrm{sc}}\mathbf{1}_{\mathsf{A}}] \asymp \mathbf{E}[W_{\mathrm{sc}}^1 W_{\mathrm{sc}}^2\mathbf{1}_{\mathsf{A}}]$.

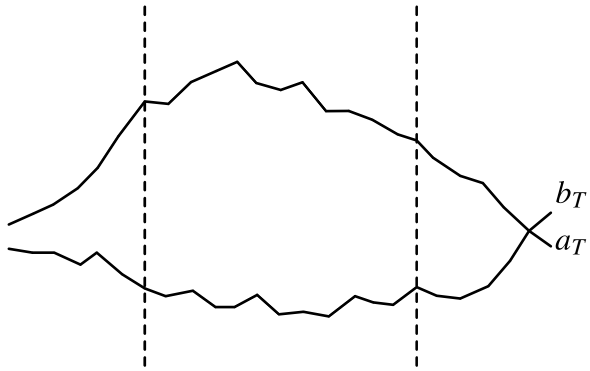

Figure 7.5: The above figure corresponds to a realization of PRW law with non-intersection, where we can only assume $a_T, b_T = O(\sqrt{T})$ and $a_T - b_T \geq -(\log T)^{7/6}$. We utilize the fact that the event associated to (iv) depends only on the first-quarter of points of the walk.

As mentioned before, one can compare the PRW law to modified random bridges. In a modified random bridge the first quarter and the last quarter of the walk are completely independent. Thus, via the comparison between PRW and modified random bridges, one eventually gets to $\mathbf{E}[W_{\mathrm{sc}}^1 W_{\mathrm{sc}}^2 \mathbf{1}_{\mathsf{A}}] \asymp \mathbf{E}[W_{\mathrm{sc}}^1 \mathbf{1}_{\mathsf{A}}]\mathbf{E}[W_{\mathrm{sc}}^2]$. The last term in (7.1.13) is then estimated by following similar ideas to what described in the proof of (ii) and (iii). The full technical details are presented in the proofs of Lemmas 7.5.2 and 7.5.6.

7.1.4 Related works on half-space polymers

Half-space polymers are a particular variant of full-space polymers that are well-studied in the literature (introduced in [202, 206, 61]). Full-space polymers are widely believed to be in the KPZ universality class in the sense that they are expected to have $1/3$ fluctuation exponent and $2/3$ transversal exponent. However, besides a few solvable models, these exponents are not proven rigorously for general polymers. We refer to [99, 294, 44, 29, 132, 133] and references therein for more details.

Half-space polymer models have been studied in the physics literature since the work of Kardar [215]. They arise naturally in the context of modeling wetting phenomena [269, 1, 76] where one studies directed polymers in the presence of a wall. They have been of great interest due to the presence of phase transition (called the 'depinning transition') and a rich phase diagram for limiting distributions based on the diagonal strength. This phase diagram was first rigorously proven for geometric last passage percolation (LPP), i.e., polymers with zero temperature, in a series of works by Baik and Rains [15, 17, 16]. Multi-point fluctuations were studied then in [289] and similar results were later proven for exponential LPP in [11, 12] using Pfaffian Schur processes. For further recents works on half-space LPP, we refer to [**bete**, 51, 52, 165].

Positive temperature models such as polymers resisted rigorous treatment for longer compared to LPP since they are no longer directly related to Pfaffian point processes. For such class of models in the half-space geometry, the first rigorous proof of depinning transition appeared in [34] where the authors proved precise fluctuation results including the BBP phase transition [10] for the point-to-line log-gamma free energy. For the point-to-point log-gamma free energy, the limit theorem along with Baik-Rains phase transition was conjectured in [**hbigmac**] based on an uncontrolled steepest descent analysis of certain formulas coming from half-space Macdonald processes. This result was proved recently in [205] using a new set of ideas, relating the half-space model to a free boundary version of the Schur process. In fact, [205] also proves analogous results for the half-space KPZ equation which is the free energy of the continuum directed random polymer in half-space. The half-space KPZ equation arises as a limit of free energy of $\mathcal{HSLG}$ polymer [320, 27]. Since the early work by Kardar [215], the half-space KPZ equation has received significant attention, with a flurry of new results recently in in both mathematics [123, 26, 25, 272, 271, 27, 205] and physics literature [189, 67, 207, 142, 231, 30, 32, 31]. Apart from log-gamma and continuum polymer, a half-space version of the beta polymer was recently introduced and studied in [33].

Organization

In Section 7.2, we study several properties of $\mathcal{HSLG}$ Gibbs measures and Gibbsian line ensemble, and prove Theorem 7.1.3. Section 7.3 is divided into three subsections that discuss three important probabilistic results for the line ensemble. In Section 7.3.1, we show a certain ordering of points on the line ensemble (Theorem 7.3.1). This is the precise technical form of the typical non-intersection property discussed in Section 7.1.3. In Section 7.3.2, we show that there is 'high point on the second curve' (Theorem 7.3.3) that is discussed in item (i)(c) from Section 7.1.3. In Section 7.3.2, we provide high probability uniform upper bounds for the second and third curves (Theorem 7.3.7). These bounds are used later in proving item (ii) from Section 7.1.3. In Section 7.4, we prove one-point tightness on the left boundary and study the probability of a certain 'region pass event'. The study of the region pass event is utilized in proving the lower bound on the uniform separation between the first two curves and the third curve (described earlier in (ii) from Section 7.1.3). Finally, in Section 7.5, we study the modulus of continuity under the WPRW law and prove Theorem 7.1.1. Appendix 7.6 collects several basic facts about log-gamma random variables and related measures. Appendix 7.7 is devoted to proving several technical estimates related to non-intersecting random walk bridges which are required in studying the WPRW law. Appendix 7.8 includes the proof of stochastic monotonicity for $\mathcal{HSLG}$ Gibbsian line ensembles.

Notations and Conventions

For $a, b \in \mathbb{R}$, we denote $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$, $a \wedge b = \min(a, b)$, and $a \vee b = \max(a, b)$. Throughout this paper we work with three fixed parameters: $\theta > 0$ (bulk parameter), $\zeta > 0$ (supercritical boundary parameter), and $\mu \in \mathbb{R}$ (critical boundary parameter). All our constants appearing in the rest of the paper may depend on θ, ζ, μ and possibly other specified variables. We will only specify the dependency of the constants on the variables besides θ, ζ, μ by writing $\mathrm{C} = \mathrm{C}(a, b, c, \cdots) > 0$ to denote a generic deterministic positive finite constant that may change from line to line, but is dependent on the designated variables $a, b, c, \cdots$. We will often write $f(x) \gtrsim_{\square} g(x)$ to mean that for all x, $f(x) \geq \mathrm{C}g(x)$ for some $\mathrm{C} > 0$ depending on the subscript parameters. If $f(x) \gtrsim_{\square} g(x)$ and $g(x) \gtrsim_{\square} f(x)$, we write $f(x) \asymp_{\square} g(x)$. Given a density function f, $X \sim f$ denotes a random variable X whose distribution function has a density given by f. For two densities f and g, we write $f * g(x) = \int_{\mathbb{R}} f(z)g(x-z)dz$ for the convolution density.

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7.2 Half-space log-gamma objects and proof of Theorem 7.1.3

In Section 8.2.1, we gather several useful properties of $\mathcal{HSLG}$ Gibbs measures from Definition 7.1.2 including stochastic monotonicity (Proposition 8.2.3). The $\mathcal{HSLG}$ line ensemble is defined in Section 7.2.2 which includes the proof of Theorem 7.1.3.

7.2.1 Properties of $\mathcal{HSLG}$ Gibbs measures

We start by writing down several immediate observations that all follow directly from the definition of $\mathcal{HSLG}$ Gibbs measures (Definition 7.1.2).

Observation 7.2.1. *Fix a bounded connected subset Λ. For each $(i,j) \in \partial\Lambda$ fix some $u_{i,j} \in \mathbb{R}$. Fix any $\mathrm{C} > 0$. Let $\big(L(v) : v \in \Lambda\big)$ be a collection of random variables that are distributed as the $\mathcal{HSLG}$ Gibbs measure on the domain Λ with boundary condition $\big(u_{i,j} : (i,j) \in \partial\Lambda\big)$.*

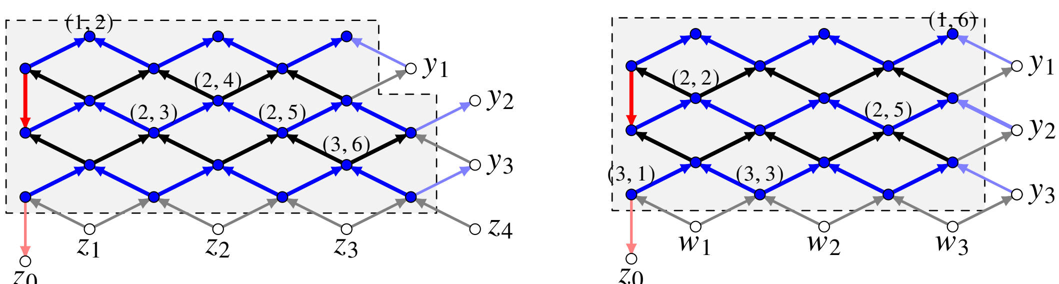
(a) (Translation invariance) The law of $\big(L(v) + \mathrm{C} : v \in \Lambda\big)$ is given by the $\mathcal{HSLG}$ Gibbs measure on the domain Λ with boundary condition $\big(u_{i,j} + \mathrm{C} : (i,j) \in \partial\Lambda\big)$.

(b) *(Gibbs property on smaller domain) Take a bounded connected $\Lambda' \subset \Lambda$. The law of $\big(L(v) : v \in \Lambda'\big)$ conditioned on $\big(L(v) : v \in \Lambda \setminus \Lambda'\big)$ is given by the $\mathcal{HSLG}$ Gibbs measure on the domain Λ' with the boundary condition $\big(L(v) : v \in \partial\Lambda'\big)$ where we set $L(v) = u_v$ for $v \in \partial\Lambda$.*

Although $\mathcal{HSLG}$ Gibbs measures are defined for any bounded connected subset Λ, we will be mainly concerned with two kinds of domains Λ. Given $k \geq 1$ and $T \geq 2$, we define

$$\mathcal{K}_{k,T} := \left\{(i,j) : i \in [\![1,k]\!], j \in [\![1, 2T-1-\mathbf{1}_{i=1}]\!]\right\}, \quad \mathcal{K}'_{k,T} := [\![1,k]\!] \times [\![1, 2T-2]\!]. \tag{7.2.1}$$

The domains $\mathcal{K}_{k,T}$ and $\mathcal{K}'_{k,T}$ are shown as shaded regions in Figure 7.6.



(a) $\mathcal{K}_{k,T}$ (b) $\mathcal{K}'_{k,T}$

Figure 7.6: Two domains $\mathcal{K}_{k,T}$ and $\mathcal{K}'_{k,T}$ are shown in (A) and (B) with $k = 3, T = 4$ and boundary conditions $(\vec{y}, \vec{z})$ and $(\vec{y}, \vec{w})$ respectively. They include all the vertices within the gray dashed box as well some labels for the points. The directed edges with lighter colors are edges connecting vertices from Λ to $\partial\Lambda$ or viceversa ($\Lambda = \mathcal{K}_{k,T}$ or $\Lambda = \mathcal{K}'_{k,T}$). The boundary variable z_0 does not actually play any role in the density of the corresponding $\mathcal{HSLG}$ Gibbs measure after normalizing it to be a probability density. This point is explained after the statement of Observation 7.2.2.

Observation 7.2.2. *Fix $k, T \in \mathbb{Z}_{\geq 2}$ and $\alpha > -\theta$. Fix $\vec{y} \in \mathbb{R}^k$, $\vec{z} \in \mathbb{R}^T$, and $\vec{w} \in \mathbb{R}^{T-1}$.*

(a) *The $\mathcal{HSLG}$ Gibbs measure on the domain $\mathcal{K}_{k,T}$ with boundary condition $(\vec{y}, \vec{z})$ is a probability measure on $\mathbb{R}^{|\mathcal{K}_{k,T}|}$ whose density at $\mathbf{u} = (u_{i,j})_{(i,j)\in\mathcal{K}_{k,T}}$ is proportional to*

$$f_{k,T}^{\vec{y},\vec{z}}(\mathbf{u}) := \prod_{i=1}^{k} \left[e^{(-1)^i \alpha u_{i,1}} \prod_{j=1}^{T-\mathbf{1}_{i=1}} W(u_{i+1,2j}; u_{i,2j+1}, u_{i,2j-1}) \prod_{j=1}^{2T-1-\mathbf{1}_{i=1}} G_{\theta,(-1)^{j+1}}(u_{i,j} - u_{i,j+1}) \right] \tag{7.2.2}$$

where $W(a;b,c) := \exp(-e^{a-b} - e^{a-c})$ *and (owing to the two-step periodicity of the measures)*

$$G_{\theta,(-1)^m}(y) := e^{\theta(-1)^m y - e^{(-1)^m y}}/\Gamma(\theta). \tag{7.2.3}$$

Here $u_{k+1,2j} = z_j$ *for each* $j \in [\![1,T]\!]$, $u_{1,2T-1} = y_1$, *and* $u_{i,2T} = y_i$, $u_{i,2T+1} := +\infty$ *(so that the factor* $\exp(-e^{u_{i+1,2T}-u_{i,2T+1}}) = 1$*) for each* $i \in [\![2,k]\!]$.

(b) The $\mathcal{HSLG}$ *Gibbs measure on the domain* $\mathcal{K}'_{k,T}$ *with boundary condition* $(\vec{y},\vec{w})$ *is a probability measure on* $\mathbb{R}^{|\mathcal{K}'_{k,T}|}$ *whose density at* $\mathbf{u} = (u_{i,j})_{(i,j)\in\mathcal{K}'_{k,T}}$ *is proportional to*

$$Q^{\vec{y}',\vec{z}}_{k,T}(\mathbf{u}) := \prod_{i=1}^{k}\left[e^{(-1)^i \alpha u_{i,1}} \prod_{j=1}^{T-1} W(u_{i+1,2j}; u_{i,2j+1}, u_{i,2j-1}) \prod_{j=1}^{2T-2} G_{\theta,(-1)^{j+1}}(u_{i,j} - u_{i,j+1})\right]. \tag{7.2.4}$$

Here $u_{k+1,2j} = w_j$ *for each* $j \in [\![1,T-1]\!]$, *and* $u_{i,2T-1} = y_i$ *for each* $i \in [\![1,k]\!]$.

Let us explain how we arrived at the above formulas (see Figure 7.6 for a visual representation of the measures). Recall the edge weights from (7.1.3). The blue edges in the figure corresponds to $G_{\theta,(-1)^{j+1}}(\cdot)$ factors that appear in (7.2.2) and (7.2.4). The $(-1)^{j+1}$ factor is due to the alternate switching of the direction of blue weights as we read off from left to right. Here we have obtained the G function from the blue edge weights by multiplying by a constant. This is done so that the G function becomes density (i.e., integrates to 1), a fact that will be useful in the later analysis. The black edge weights from (7.1.3) corresponds to the W factor in (7.2.2) and (7.2.4). Finally the red edge weights are of type $e^{-\alpha u_{2i-1,1} - u_{2i,1}} = e^{-\alpha u_{2i-1,1}} \cdot e^{\alpha u_{2i,1}}$. Note that only for odd k is $(k+1,1) \in \partial\mathcal{K}_{k,T}, \partial\mathcal{K}'_{k,T}$. In that case, the factor $e^{-\alpha u_{k+1,1}}$ can be absorbed into the proportionality constant. Thus, overall, the red weights contributes the factor $\prod_{i=1}^{k} e^{(-1)^i \alpha_{i,1}}$ in the above densities. This also explains why the z_0 value does not play any role in the definition of these densities.

We will mostly be concerned with the $\mathcal{HSLG}$ Gibbs measure on $\mathcal{K}_{k,T}$ with boundary condition $(\vec{y},\vec{z})$. We will denote the probability and the expectation operator under this law as $\mathbf{P}^{\vec{y},\vec{z};k,T}_{\alpha}$ and $\mathbf{E}^{\vec{y},\vec{z};k,T}_{\alpha}$ respectively and a random variable with this law by $L := \big(L(i,j) := L_i(j) : (i,j) \in \mathcal{K}_{k,T}\big)$.

We may drop α and write $\mathbf{P}^{\vec{y},\vec{z};k,T}$ and $\mathbf{E}^{\vec{y},\vec{z};k,T}$ when clear from the context. We will also use the $\mathcal{HSLG}$ Gibbs measure on $\mathcal{K}_{k,T}$ with boundary condition $\vec{y} \in \mathbb{R}^k$, $\vec{z} := (-\infty)^T$, as now defined.

Definition 7.2.3. The *bottom-free* measure on the domain $\mathcal{K}_{k,T}$ with boundary condition $\vec{y}$ is the $\mathcal{HSLG}$ Gibbs measure on the domain $\mathcal{K}_{k,T}$ with boundary condition $(\vec{y}, (-\infty)^T)$. We shall see in Observation 7.4.2 that the corresponding density $f_{k,T}^{\vec{y},(-\infty)^T}$ is integrable when k is even and $\alpha \in \mathbb{R}$ (in fact, in that case the measure does not even depend on α) or when k is odd and $\alpha \in (-\theta, \theta)$. Note that whenever the bottom-free measure is a valid probability measure, we have

$$\mathbf{P}_{\alpha}^{\vec{y},\vec{z};k,T}(\mathsf{A}) = \frac{1}{V_k^T(\vec{y},\vec{z})}\mathbf{E}_{\alpha}^{\vec{y};(-\infty)^T;k,T}\left[\mathbf{1}_{\mathsf{A}} \cdot \prod_{j=1}^{T} W(z_{2j}; L_k(2j+1), L_k(2j-1))\right], \tag{7.2.5}$$

for any event A, where we set $L_k(2T+1) = +\infty$ and the normalization is given by

$$V_k^T(\vec{y},\vec{z}) := \mathbf{E}_{\alpha}^{\vec{y};(-\infty)^T;k,T}\left[\prod_{j=1}^{T} W(z_{2j}; L_k(2j+1), L_k(2j-1))\right]. \tag{7.2.6}$$

In other words, we can build the full Gibbs measure $\mathbf{P}_{\alpha}^{\vec{y},\vec{z};k,T}$ by reweighting the bottom-free measure by a Radon-Nikodym derivative given by the expression (except $\mathbf{1}_{\mathsf{A}}$) inside the expectation in (7.2.5), normalized by dividing by $V_k^T(\vec{y},\vec{z})$.

Next, we record how $\mathcal{HSLG}$ Gibbs measures are absolutely continuous with certain random walks.

Remark 7.2.4 (Absolute continuity with random walks)**.** The bottom-free measure can be described as a reweighting of certain log-gamma random walks subject to a Radon-Nikoydym derivative. This type of absolute continuity with respect to random walks is a common feature in Gibbsian line ensembles, see [109, 110, 155, 29, 154] for example. However, the key difference from the existing works is that now we may condition only on the right-side boundary and consider the law up to and including at zero. This is essential in understanding the effect of the boundary parameter on the $\mathcal{HSLG}$ Gibbs measures.

Consider a collection of independent random variables $(X_{i,v})_{(i,v)\in\mathcal{K}_{k,T}}$ where $X_{i,v}$ has probability density function $G_{\theta,(-1)^{v+1}}(\cdot)$ from (7.2.3). For $(i,j)\in\mathcal{K}_{k,T}$ set

$$V_i(j) := y_i + \sum_{v=1}^{2T-\mathbf{1}_{i=1}-j} X_{i,2T-\mathbf{1}_{i=1}-v}.$$

We denote the probability and the expectation operator for $(V_i(j))_{(i,j)\in\mathcal{K}_{k,T}}$ as $\mathbf{P}_{\text{free}}^{\vec{y};k,T}$ and $\mathbf{E}_{\text{free}}^{\vec{y};k,T}$ respectively. In words, $\mathbf{P}_{\text{free}}^{\vec{y};k,T}$ is the law of k independent 'zigzag' random walks starting at $\vec{y}$ ('zigzag' as the increments alternate their signs). For each $\alpha > -\theta$, the $\mathcal{HSLG}$ Gibbs measure $\mathbf{P}_{\alpha}^{\vec{y},\vec{z};k,T}$ is absolutely continuous with respect to $\mathbf{P}_{\text{free}}^{\vec{y};k,T}$ with a Radon-Nikodym derivative

$$\frac{d\mathbf{P}_{\alpha}^{\vec{y},\vec{z};k,T}}{d\mathbf{P}_{\text{free}}^{\vec{y};k,T}}(L_1,\ldots,L_k) \propto \exp\left(-\mathbf{H}_\alpha\big((L_j)_{j=1}^k;\vec{z}\big)\right),$$

where $\mathbf{H}_\alpha$ (sometimes called the interaction Hamiltonian in the literature) is given by

$$\mathbf{H}_\alpha\big((L_j)_{j=1}^k;\vec{z}\big) = \sum_{i=1}^k \alpha(-1)^{i+1}L_i(1) + \sum_{i=1}^k \sum_{j=1}^{T-\mathbf{1}_{i=1}} \left[e^{L_{i+1}(2j)-L_i(2j+1)} + e^{L_{i+1}(2j)-L_i(2j-1)}\right]$$

with $L_{k+1}(j) = z_j$ for $j\in[\![1,2T]\!]$, $L_i(2T-\mathbf{1}_{i=1}) = y_i$ and $L_i(2T+1)=\infty$ for $i\in[\![1,k]\!]$. The above observation follows immediately from the form of the density given in (7.2.2).

Besides one-sided conditioning as in the above remark, we can also use the Gibbs property when conditioning on boundary data on both sides as is standard full-space discrete line ensembles [155, 29, 154]. We record here one such result that will be useful in our later proofs.

Observation 7.2.5 (Two-sided boundaries). *Fix $1 \le T_1 < T_2 - 1$. Suppose L is distributed as $\mathbb{P}^{\vec{y},\vec{z};1,T_2}$. Let $(X_j)_{j=T_1-1}^{T_2-1}$ be a random bridge from $X_{T_1-1} = a$ to $X_{T_2-1} = b$ with i.i.d. increments from the density $G_{\theta,1} * G_{\theta,-1}$. The law of $\big(L_1(2j+1) : T_1 \le j \le T_2-2\big)$ conditioned on $\{L_1(2T_1-1) = a, L_1(2T_2-1) = b\}$ is absolutely continuous with respect to the law of $(X_j)_{j=T_1}^{T_2-2}$*

with Radon-Nikodym derivative proportional to

$$\widetilde{W} := \exp\left(- \sum_{j=T_1}^{T_2-1} (e^{z_j - X_j} + e^{z_j - X_{j-1}}) \right).$$

The $G_{\theta,1} * G_{\theta,-1}$ appearing in the statement of Observation 7.2.5 is due to the fact that we focus on the marginal distribution of the odd points only (see Figure 7.7). The proof follows by utilizing the explicit form of the density given in (7.2.2).

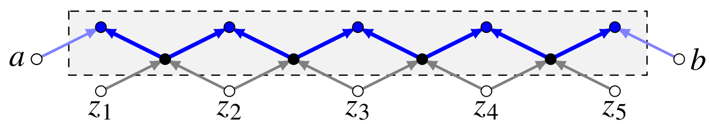


Figure 7.7: The marginal distribution of the odd (black) points of the $\mathcal{HSLG}$ Gibbs measure shown above with $T_1 = 1, T_2 = 6$ is described in Observation 7.2.5.

As with full space line ensemble Gibbs measures [109, 110, 320, 29, 154], the $\mathcal{HSLG}$ Gibbs measures satisfy *stochastic monotonicity* with respect to the boundary data.

Proposition 7.2.6 (Stochastic monotonicity)**.** *Fix $k_1 \le k_2$, $a_i \le b_i$ for $k_1 \le i \le k_2$ and $\alpha > -\theta$. Let*

$$\Lambda := \{(i,j) : k_1 \le i \le k_2, a_i \le j \le b_i\}.$$

There exists a probability space consisting of a collection of random variables

$$\left(L(v; (u_w)_{w\in\partial\Lambda}) : v \in \Lambda, (u_w)_{w\in\partial\Lambda} \in \mathbb{R}^{|\partial\Lambda|}\right)$$

such that

1. *For each $(u_w)_{w\in\partial\Lambda} \in \mathbb{R}^{|\partial\Lambda|}$, the marginal law of $\left(L(v; (u_w)_{w\in\partial\Lambda}) : v \in \Lambda\right)$ is given by the $\mathcal{HSLG}$ Gibbs measure for the domain Λ with boundary condition $(u_w)_{w\in\partial\Lambda} \in \mathbb{R}^{|\partial\Lambda|}$.*

2. *With probability* 1, *for all* $v \in \Lambda$ *we have*

$$L\big(v; (u_w)_{w\in\partial\Lambda}\big) \le L\big(v; (u'_w)_{w\in\partial\Lambda}\big) \textit{ whenever } u_w \le u'_w \textit{ for all } w \in \partial\Lambda.$$

The proof of the above proposition follows a similar strategy as in [29, 154] and is provided in Appendix 7.8 for completeness.

7.2.2 The $\mathcal{HSLG}$ line ensemble and Proof of Theorem 7.1.3

In this section we define the half-space log-gamma ($\mathcal{HSLG}$) line ensemble and prove Theorem 7.1.3. The construction of the line ensemble is based on the multipath point-to-point partition functions. These are defined in (7.2.8) as sums over multiple non-intersecting paths on the full quadrant $\mathbb{Z}^2_{>0}$ (not just half-quadrant) of products of the symmetrized versions of the weights from (7.1.1):

$$\widetilde{W}_{i,j} \sim \begin{cases} \frac{1}{2}W_{i,j} & \text{when } i = j, \\ W_{j,i} & \text{when } j > i, \\ W_{i,j} & \text{when } j < i. \end{cases} \tag{7.2.7}$$

For $m, n, r \in \mathbb{Z}_{\ge 1}$ with $n \ge r$, let $\Pi^{(r)}_{m,n}$ be the set of r-tuples of non-intersecting upright paths in $\mathbb{Z}^2_{>0}$ starting from $(1, r), (1, r-1), \cdots, (1, 1)$ and going to $(m, n), (m, n-1), \ldots, (m, n-r+1)$ respectively. We define the multipath point-to-point symmetrized partition function as

$$Z^{(r)}_{\mathrm{sym}}(m, n) := \sum_{(\pi_1,\ldots,\pi_r)\in\Pi^{(r)}_{m,n}} \prod_{(i,j)\in\pi_1\cup\cdots\cup\pi_r} \widetilde{W}_{i,j}, \tag{7.2.8}$$

with the convention that $Z^{(0)}_{\mathrm{sym}}(m, n) \equiv 1$ for all $m, n \in \mathbb{Z}_{\ge 1}$.

Definition 7.2.7 (Half-space log-gamma line ensemble)**.** Fix $N \in \mathbb{Z}_{\ge 1}$. For each $i \in [\![1, N]\!]$ and

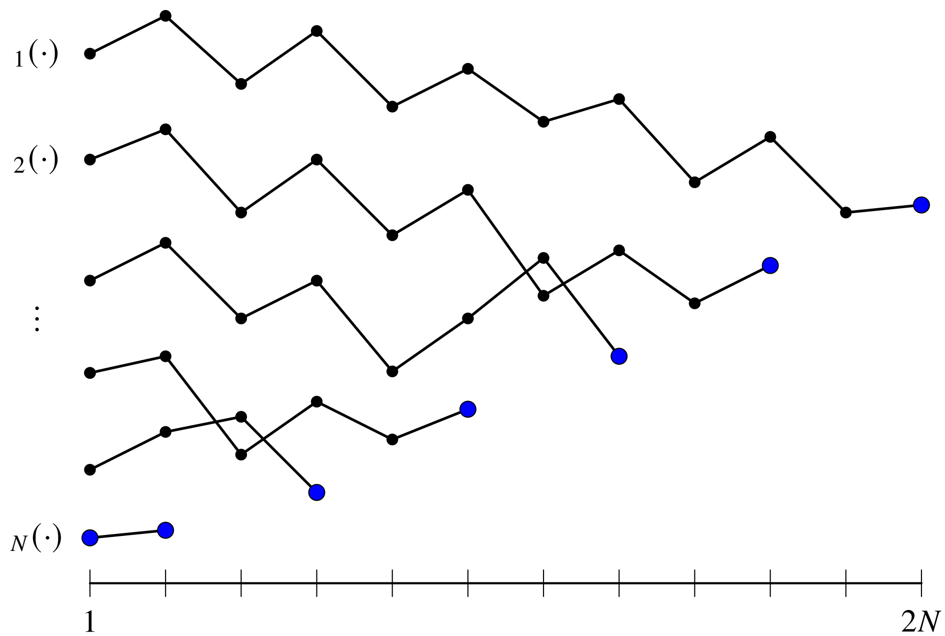


Figure 7.8: Half-space log gamma line ensemble $= (_i(\cdot))_{i=1}^N$ ($N = 6$ in above figure). Each curve $_i(\cdot)$ has $2N - 2i + 2$ many coordinates. Λ_N^* in Theorem 7.1.3 is the set of all black points in the above figure. Theorem 7.1.3 tells us that conditioned on the blue points, the law of the black points is given by the $\mathcal{HSLG}$ Gibbs measures.

$j \in [\![1, 2N - 2i + 2]\!]$, we set

$$_i^N(j) = \log\left(\frac{2Z_{\text{sym}}^{(i)}(p,q)}{Z_{\text{sym}}^{(i-1)}(p,q)}\right) + 2\Psi(\theta)N.$$

where $p := N + \lfloor j/2 \rfloor$ and $q := N - \lceil j/2 \rceil + 1$. We call the collection of random variables

$$\left(_i^N(j) : i \in [\![1, N]\!], j \in [\![1, 2N - 2i + 2]\!]\right)$$

the half-space log-gamma ($\mathcal{HSLG}$) line ensemble with parameters (α, θ), see Figure 7.8.

Proof of Theorem 7.1.3. Recalling the convention $Z_{\text{sym}}^{(0)}(m,n) \equiv 1$, we can write

$$_1^N(j) = \log\left(2Z_{\text{sym}}^{(1)}(N + \lfloor j/2 \rfloor, N - \lceil j/2 \rceil + 1)\right) + 2\Psi(\theta)N.$$

Assuming Part (ii) of Theorem 7.1.3 (verified below), Part (i) follows from the easily verified identity $2Z^{(1)}_{\mathrm{sym}}(p,q) = Z(p,q)$. The above identity is noted in Section 2.1 of [34] and follows easily due to symmetry of the weights (the factor of 2 comes from a lack of double-counting the weight at $(1,1)$). This is an equality (not just in distribution). In any case, from it immediately follows the claim of Part (i).

Part (ii) is a highly non-trivial deduction from first principles. However, the works of [121, 263, 260, 59, 34] have built a rich theory using the geometric RSK correspondence from which this part follows in a rather straightforward manner, as now described. We seek to determine the joint density of the $\mathcal{HSLG}$ line ensemble defined above. Let us start by defining

$$K_N := \{(i,j) : i \in [\![1,N]\!], j \in [\![1, 2N-2i+2]\!]\}, \quad \mathcal{I}^{(N)} := \{(i,j) \in \mathbb{Z}^2_{\geq 1} : i+j \leq 2N+1\}.$$

Note that the map $(i,j) \mapsto (N + \lfloor j/2 \rfloor - i + 1, N - \lceil j/2 \rceil - i + 2)$ is a bijection from K_N to $\mathcal{I}^{(N)} \cap \{i \geq j\}$. For any $(i,j) \in K_N$, we then define

$$T_{N+\lfloor j/2\rfloor - i+1, N-\lceil j/2\rceil - i+2} := \frac{Z^{(i)}_{\mathrm{sym}}(N + \lfloor j/2 \rfloor, N - \lceil j/2 \rceil + 1)}{Z^{(i-1)}_{\mathrm{sym}}(N + \lfloor j/2 \rfloor, N - \lceil j/2 \rceil + 1)},$$

and then set $T_{j,i} := T_{i,j}$ for $i \geq j$. From Proposition 2.6 in [260], $(T_{i,j})_{(i,j)\in\mathcal{I}^{(N)}}$ is precisely the image under the geometric RSK map of the symmetrized weights (7.2.7) with indices restricted to the $\mathcal{I}^{(N)}$ array. The density of this image has been computed in [34]. Indeed, setting $m = 0, n = N$, $\alpha_i = \theta$ and $\alpha_0 = \alpha$ in the final two (unnumbered) equations on page 28 in [34] (in the arXiv version see the second unnumbered equation on page 20), we see that the density of $(2T_{i,j})_{i\geq j}$ at $(t_{i,j})_{i\geq j}$ is given by

$$\frac{e^{-\frac{1}{t_{1,1}}} \prod_{i=1}^N t_{i,i}^{(-1)^{N-i+1}\alpha}}{(\Gamma(\alpha+\theta))^N (\Gamma(2\theta))^{N^2}} \prod_{j=1}^N \left(\frac{\tau_{2N-2j+2} \cdot \tau_{2N-2j}}{\tau^2_{2N-2j+1}}\right)^{\theta} \exp\left(-\sum_{i\geq j>1} \frac{t_{i,j-1}}{t_{i,j}} - \sum_{i>j} \frac{t_{i-1,j}}{t_{i,j}}\right) \prod_{(i,j)\in\mathcal{I}^{(N)}} t_{i,j}^{-1} \mathbf{1}_{t_{i,j}>0} \tag{7.2.9}$$

where the τ variables are defined as $\tau_k = \prod\left(t_{i,j} : (i,j) \in \mathcal{I}^{(N)}, i-j = k\right) = \prod\left(t_{i+k,i} : 1 \leq i \leq N - \frac{k-1}{2}\right)$. In fact, the density formula in [34] is for $(2T_{i,j})_{i\leq j}$ at $(t_{i,j})_{i\leq j}$, thus we needed to permute the indices in that formula to arrive at the above formula. The line ensemble ${}_i^N(j)$ defined in Definition 7.2.7 is related to $(2T_{i,j})_{(i,j)\in\mathcal{I}^{(N)}}$ via the relation

$$ {}_i^N(j) - 2\Psi(\theta)N = \log\left(T_{N+\lfloor j/2\rfloor-i+1,N-\lceil j/2\rceil-i+2}\right). $$

Thus, under the change of variables $u_{i,j} = \log\left(t_{N+\lfloor j/2\rfloor-i+1,N-\lceil j/2\rceil-i+2}\right)$ for $(i,j)\in K_N$ and after considerable rewriting of (7.2.9), we find that the density of $({}_i^N(j) - 2\Psi(\theta)N)$ at $(u_{i,j})_{(i,j)\in K_N}$ is given by

$$ \frac{e^{-e^{-u_{N,1}}}\prod_{i=1}^{N} e^{(-1)^i u_{i,1}\alpha}}{(\Gamma(\alpha+\theta))^N(\Gamma(2\theta))^{N^2}} \prod_{i=1}^{N}\left(e^{-\theta u_{i,2N-2i+2}} \prod_{j=1}^{N-i+1} e^{\theta[u_{i,2j-1}-u_{i,2j}]} \prod_{j=1}^{N-i} e^{-\theta[u_{i,2j}-u_{i,2j+1}]}\right) \tag{7.2.10} $$

$$ \cdot\exp\left(-\sum_{i=1}^{N}\sum_{j=1}^{N-i+1} e^{u_{i,2j-1}-u_{i,2j}} - \sum_{i=1}^{N-1}\sum_{j=1}^{N-i} e^{u_{i+1,2j}-u_{i,2j+1}}\right) \tag{7.2.11} $$

$$ \cdot\exp\left(-\sum_{i=1}^{N-1}\sum_{j=1}^{N-i} e^{u_{i,2j+1}-u_{i,2j}} - \sum_{i=1}^{N-1}\sum_{j=1}^{N-i} e^{u_{i+1,2j}-u_{i,2j-1}}\right). \tag{7.2.12} $$

This follows from the fact that the factor $\prod t_{i,j}^{-1}$ in (7.2.9) is absorbed as the Jacobian of the change of variables, as well as the following four relations:

$$ \prod_{j=1}^{N}\left(\frac{\tau_{2N-2j+2}\tau_{2N-2j}}{\tau_{2N-2j+1}^2}\right)^{\theta} = \prod_{i=1}^{N}\left(e^{-\theta u_{i,2N-2i+2}} \prod_{j=1}^{N-i+1} e^{\theta[u_{i,2j-1}-u_{i,2j}]} \prod_{j=1}^{N-i} e^{\theta[u_{i,2j+1}-u_{i,2j}]}\right) \tag{7.2.13} $$

$$ \sum_{i>j}\frac{t_{i-1,j}}{t_{i,j}} = \sum_{i=1}^{N}\sum_{j=1}^{N-i+1} e^{u_{i,2j-1}-u_{i,2j}} + \sum_{i=1}^{N-1}\sum_{j=1}^{N-i} e^{u_{i+1,2j}-u_{i,2j+1}} \tag{7.2.14} $$

$$ \sum_{i\geq j>1}\frac{t_{i,j-1}}{t_{i,j}} = \sum_{i=1}^{N-1}\sum_{j=1}^{N-i} e^{u_{i,2j+1}-u_{i,2j}} + \sum_{i=1}^{N-1}\sum_{j=1}^{N-i} e^{u_{i+1,2j}-u_{i,2j-1}} \tag{7.2.15} $$

$$ \prod_{j=1}^{N} t_{j,j}^{(-1)^{N-j+1}\alpha} = \prod_{j=1}^{N} t_{N-j+1,N-j+1}^{(-1)^j\alpha} = \prod_{i=1}^{N} e^{(-1)^i u_{i,1}\alpha}. \tag{7.2.16} $$

While (7.2.16) is obvious, (7.2.13), (7.2.14) and (7.2.15) are proved in Appendix 7.9.

Recall now that we are interested in the density conditioned on $({}_i^N(j)-2\Psi(\theta)N)$ at $(u_{i,j})_{(i,j)\in K_N\setminus\Lambda_N^*}$.

To compute this conditional density we may absorb all the $u_{i,j}$ terms with $(i,j) \in K_N \setminus \Lambda_N^*$ into the proportionality constant. Thus in (7.2.10), we may absorb the $e^{-u_{N,1}}$ term and $e^{-\theta u_{i,2N-2i+2}}$ terms and observe

$$\prod_{i=1}^{N} e^{(-1)^i u_{i,1}\alpha} \propto \prod_{i \in [\![1,N/2]\!]} e^{-\alpha(u_{2i-1,1}-u_{2i,1})}.$$

Upon a quick inspection of the form of the weight function in (7.1.3), one sees that these factors are precisely the red edge weights functions in the $\mathcal{HSLG}$ Gibbs measure on the domain Λ_N^*; see Figure 7.3 (B) and Definition 7.1.2. Combining the terms which have $(u_{i,2j-1} - u_{i,2j})$ and $(u_{i,2j+1} - u_{i,2j})$ in (7.2.10), (7.2.11), (7.2.12) give rise to the following factor

$$\prod_{i=1}^{N}\prod_{j=1}^{N-i+1} \exp\left(\theta(u_{i,2j-1} - u_{i,2j}) - e^{u_{i,2j-1}-u_{i,2j}}\right) \prod_{i=1}^{N-1}\prod_{j=1}^{N-i} \exp\left(\theta(u_{i,2j+1} - u_{i,2j}) - e^{u_{i,2j+1}-u_{i,2j}}\right).$$

The above factor corresponds to the blue edge weight functions in the $\mathcal{HSLG}$ Gibbs measure on the domain Λ_N^*. Finally, the remaining terms in (7.2.11) and (7.2.12) corresponds to black edge weight function in the $\mathcal{HSLG}$ Gibbs measure on the domain Λ_N^*. Thus the density of $\{{}_i^N(j) - 2\Psi(\theta)N : (i,j) \in \Lambda_N^*\}$ conditioned on $\{{}_i^N(j) - 2\Psi(\theta)N : (i,j) \in K_N \setminus \Lambda_N^*\}$ is precisely given by the $\mathcal{HSLG}$ Gibbs measure with boundary condition $\{{}_i^N(j) - 2\Psi(\theta)N : (i,j) \in K_N \setminus \Lambda_N^*\}$ as in Definition 7.1.2. By the translation invariance of the Gibbs measures, we obtained Part (ii) of Theorem 7.1.3. $\square$

7.3 Properties of the first three curves

In this section we extract probabilistic information about the first few curves of $\mathcal{HSLG}$ line ensemble ^{N} (Definition 7.2.7). The section is divided into three parts. In Section 7.3.1 we prove Theorem 7.3.1, which claims that there is a certain high probability ordering among the points of the curve. Section 7.3.2 contains Theorem 7.3.3 which asserts that with high probability there is a point $p = O(N^{2/3})$ such that ${}_2^N(p)$ is reasonably large. Finally in Section 7.3.3, we show Theorem 7.3.7 which argues that with high probability $({}_2^N(s))_{s\in[\![1,kN^{2/3}]\!]}$ and $({}_3^N(s))_{s\in[\![1,kN^{2/3}]\!]}$ always lies below $MN^{1/3}$ for large enough M.

7.3.1 Ordering of the points in the line ensemble

In this section we show that there is ordering among the points of the $\mathcal{HSLG}$ line ensemble. Throughout this subsection we shall assume $\alpha \in (-\theta, \infty)$ is a fixed parameter. The results can be easily extended to the case where $\alpha = \alpha(N)$ satisfying

$$-\theta < \liminf_{N\to\infty} \alpha(N) \le \limsup_{N\to\infty} \alpha(N) < \infty.$$

We consider the $\mathcal{HSLG}$ line ensemble ^{N} from Definition 7.2.7 with parameter (α, θ).

Theorem 7.3.1. *Fix any $k \in \mathbb{Z}_{>0}$ and $\rho \in (0,1)$. There exists $N_0 = N_0(\rho, k) > 0$ such that for all $N \ge N_0$, $i \in [\![1, k]\!]$ and $p \in [\![1, N-k-2]\!]$ the following inequalities holds:*

$$\begin{aligned}
\mathbf{P}({}_i^N(2p+1) \le {}_i^N(2p) + (\log N)^{7/6}) \ge 1 - \rho^N, \\
\mathbf{P}({}_i^N(2p-1) \le {}_i^N(2p) + (\log N)^{7/6}) \ge 1 - \rho^N, \\
\mathbf{P}({}_{i+1}^N(2p) \le {}_i^N(2p+1) + (\log N)^{7/6}) \ge 1 - \rho^N, \\
\mathbf{P}({}_{i+1}^N(2p) \le {}_i^N(2p-1) + (\log N)^{7/6}) \ge 1 - \rho^N.
\end{aligned} \tag{7.3.1}$$

We refer to the caption of Figure 7.9 for a visual interpretation of the above Theorem. In order to prove the above theorem, we first provide an apriori loose bound for the entries of the first k curves of the line ensemble ^{N}.

Proposition 7.3.2. *Fix any $\rho \in (0,1)$ and $k \in \mathbb{Z}_{>0}$. There exists a constant* $\mathrm{C} = \mathrm{C}(\rho, k) > 0$ *and $N_0(\rho, k) > 0$ such that for all $N \ge N_0$, $i \in [\![1, k]\!]$, $j \in [\![1, 2N-2i+2]\!]$ we have*

$$\mathbf{P}\left(|{}_i^N(j)| \le \mathrm{C} \cdot N\right) \ge 1 - \rho^N. \tag{7.3.2}$$

We first prove Theorem 7.3.1 assuming Proposition 7.3.2.

Proof of Theorem 7.3.1. Fix any $\rho \in (0,1)$ and $k \in \mathbb{Z}_{>0}$. Set $T := N - k$. Fix $i_0 \in [\![1, k]\!]$ and $p \in [\![1, T-2]\!]$. We will show only the first of the inequalities in (7.3.1), as the rest are all proved

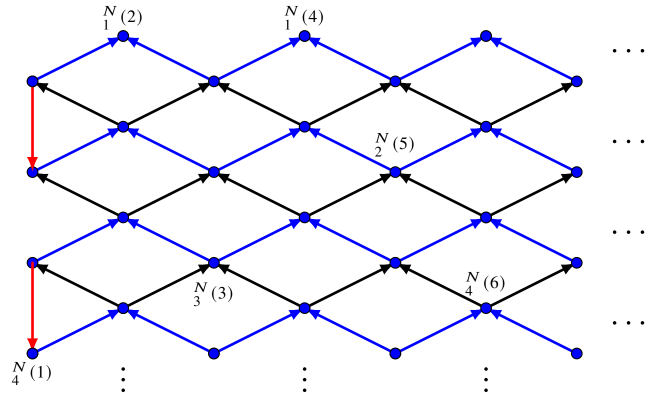


Figure 7.9: Ordering of points within Half-space log gamma line ensemble: The above figure consists of first 3 curves of the line ensemble . A black or blue arrow from $a \to b$ signifies $a \le b - (\log N)^{7/6}$ with exponential high probability. The blue arrows depicts the ordering within a particular indexed curve (inter-ordering). The black arrow indicates ordering between the two consecutive curves (intra-ordering).

analogously. For simplicity, we write for ^{N}. Consider the event

$$\mathsf{V} := \left\{ _{i_0}(2p+1) \ge {}_{i_0}(2p) + (\log N)^{7/6} \right\}.$$

We apply Proposition 7.3.2 with $k \mapsto k+1$ and $\rho \mapsto \rho/2$ to get $\mathrm{C} > 0$ so that for all large enough N, by union bound we have $\mathbf{P}(\mathsf{A}) \ge 1 - 2Nk \cdot (\rho/2)^N$ where

$$\mathsf{A} := \left\{ |_{k+1}(j)|, |_i(2T-1)| \le \mathrm{C} \cdot N,\ j \in [\![1, 2T]\!], i \in [\![1, k]\!] \right\}.$$

Thus if we consider the σ-field

$$\mathcal{F} := \sigma\left(_{k+1}(j), {}_i(2T-1) :\ j \in [\![1, 2T]\!], i \in [\![1, k]\!] \right),$$

by union bound and tower property of the conditional expectation we have

$$\mathbf{P}(\mathsf{V}) \le \mathbf{P}(\neg\mathsf{A}) + \mathbf{P}(\mathsf{V} \cap \mathsf{A}) \le 2Nk \cdot (\rho/2)^N + \mathbf{E}\left[\mathbf{1}_{\mathsf{A}} \mathbf{E}[\mathbf{1}_{\mathsf{V}} \mid \mathcal{F}]\right]. \tag{7.3.3}$$

Recall $\mathcal{K}'_{k,T}$ from (7.2.1). From Theorem 7.1.3 and Observation 7.2.1 (b), the law of $\{(v) : v \in$

$\mathcal{K}'_{k,T}\}$ conditioned on $\mathcal{F}$ is given by the $\mathcal{HSLG}$ Gibbs measure on the domain $\mathcal{K}'_{k,T}$ with boundary condition $\vec{y} := \{{}_j(2T-1)\}_{j=1}^{k}$. $\vec{z} := \{{}_{k+1}(2i)\}_{i=1}^{T-1}$. In view of Observation 7.2.2 (b) we see that

$$\mathbf{E}[\mathbf{1}_{\mathsf{V}} \mid \mathcal{F}] = \frac{\int_{\mathsf{V}} Q_{k,T}^{\vec{y},\vec{z}}(\mathbf{u})d\mathbf{u}}{\int_{\mathbb{R}^{|\mathcal{K}'_{k,T}|}} Q_{k,T}^{\vec{y},\vec{z}}(\mathbf{u})d\mathbf{u}} \tag{7.3.4}$$

where $Q_{k,T}^{\vec{y},\vec{z}}(\mathbf{u})$ is defined in (7.2.4). We will now bound the numerator and denominator of (7.3.4) respectively. Towards this end, we claim that there exists $R, \tau > 0$ depending only on $k, \alpha, \theta, \mathrm{C}$ such that

$$\mathbf{1}_{\mathsf{A}} \cdot \int_{\mathsf{V}} Q_{k,T}^{\vec{y},\vec{z}}(\mathbf{u})d\mathbf{u} \le \mathbf{1}_{\mathsf{A}} \exp(-\tfrac{1}{2}e^{(\log N)^{7/6}}) \cdot R^N, \quad \text{and} \quad \mathbf{1}_{\mathsf{A}} \int_{\mathbb{R}^{|\mathcal{K}'_{k,T}|}} Q_{k,T}^{\vec{y},\vec{z}}(\mathbf{u})d\mathbf{u} \ge \mathbf{1}_{\mathsf{A}} \cdot \tau^N. \tag{7.3.5}$$

Clearly plugging this bounds back in (7.3.4) and then back in (7.3.3) leads to $\mathbf{P}(\mathsf{V}) \le \rho^N$ for all large enough N, which is precisely what we wanted to show. Thus we focus on proving the two inequalities in (7.3.5).

Proof of the first inequality in (7.3.5)**.** Recall G defined in (7.2.3). Set

$$H_{\theta,(-1)^k}(y) := e^{\frac{1}{2}e^{(-1)^k y}} G_{\theta,(-1)^k}(y) = \frac{1}{\Gamma(\theta)} \exp(\theta(-1)^k y - \tfrac{1}{2}e^{(-1)^k y}), \quad W(a;b,c) := \exp(-e^{a-b} - e^{a-c}).$$

Set $\sqrt{W}(a;b,c) := \sqrt{W(a;b,c)}$. From (7.2.4) we have

$$\begin{aligned} Q_{k,T}^{\vec{y},\vec{z}}(\mathbf{u}) = \prod_{i=1}^{k} \Bigg[e^{(-1)^i \alpha u_{i,1}} \prod_{j=1}^{T-1} \sqrt{W}(u_{i+1,2j}; u_{i,2j+1}, u_{i,2j-1}) \prod_{j=1}^{2T-2} H_{\theta,(-1)^{j+1}}(u_{i,j} - u_{i,j+1}) \Bigg] \\ \cdot \prod_{i=1}^{k} \Bigg[\prod_{j=1}^{T-1} \sqrt{W}(u_{i+1,2j}; u_{i,2j+1}, u_{i,2j-1}) \prod_{j=1}^{2T-2} \exp(-\tfrac{1}{2}e^{(-1)^{j+1}(u_{i,j} - u_{i,j+1})}) \Bigg]. \end{aligned}$$

Now on V, among the terms appearing in the last line of the above equation, the term $\exp(-\frac{1}{2}e^{u_{2p+1,i_0} - u_{2p,i_0}})$ is at most $\exp(-\frac{1}{2}e^{(\log N)^{7/6}})$. We bound the rest of the terms in the above last line just by 1, so that on V, we have $Q^{\vec{y},\vec{z}}_{k,T}(\mathbf{u}) \leq e^{-\frac{1}{2}e^{(\log N)^{7/6}}} \widetilde{Q}^{\vec{y},\vec{z}}_{k,T}(\mathbf{u})$ where

$$\widetilde{Q}^{\vec{y},\vec{z}}_{k,T}(\mathbf{u}) := \prod_{i=1}^{k} \left[e^{(-1)^i \alpha u_{i,1}} \prod_{j=1}^{T-1} \sqrt{W}(u_{i+1,2j}; u_{i,2j+1}, u_{i,2j-1}) \prod_{j=1}^{2T-2} H_{\theta,(-1)^{j+1}}(u_{i,j} - u_{i,j+1}) \right].$$

By Lemma 7.6.2 it follows that $\int_{\mathbb{R}^{|\mathcal{K}'_{k,T}|}} \widetilde{Q}^{\vec{y},\vec{z}}_{k,T}(\mathbf{u}) d\mathbf{u} \leq R^N$ for some $R > 0$ depending on k, α, θ and C only. This verifies the first inequality in (7.3.5).

Proof of the second inequality in (7.3.5)**.** We define the event

$$\mathsf{D} := \bigcap_{i=1}^{k} \bigcap_{j=1}^{2T-2} \{_i(j) - CN - 2N + 2i \in [0, 1]\}.$$

Note that on D, $|_i(1)| \leq CN + 2N + 3$ and $_{i+1}(2j) \leq L_i(2j+1), _i(2j-1)$. Hence on D we have

$$W(_{i+1}(2j); _i(2j+1), _i(2j-1)) = \exp\left(-e^{_{i+1}(2j) - _i(2j+1)} - e^{_{i+1}(2j) - _i(2j-1)} \right) \geq e^{-2}.$$

Hence on D we have

$$\begin{aligned} Q^{\vec{y},\vec{z}}_{k,T}(\mathbf{u}) &= \prod_{i=1}^{k} \left[e^{(-1)^i \alpha u_{i,1}} \prod_{j=1}^{T-1} W(u_{i+1,2j}; u_{i,2j+1}, u_{i,2j-1}) \prod_{j=1}^{2T-2} G_{\theta,(-1)^{j+1}}(u_{i,j} - u_{i,j+1}) \right] \\ &\geq e^{-\alpha k(CN+2N)} e^{-2kT} \prod_{i=1}^{k} \prod_{j=1}^{2T-2} G_{\theta,(-1)^{j+1}}(u_{i,j} - u_{i,j+1}). \end{aligned}$$

Again note that on D, $|_i(j) - _i(j+1)| \leq 2$ for all $i \in [\![1, k]\!]$ and $j \in [\![1, 2T-3]\!]$, whereas on $\mathsf{A} \cap \mathsf{D}$,

$$_i(2T-2) - _i(2T-1) \in [0, 2CN + 2N].$$

Thus, on D

$$Q^{\vec{y},\vec{z}}_{k,T}(\mathbf{u}) \geq e^{-\alpha k(CN+2N) - 2kT} \left[\inf_{|x| \leq 2} G_{\theta,1}(x) \right]^{k(2T-3)} \left[\inf_{x \in [0, 2CN+2N]} G_{\theta,1}(-x) \right]^{k}.$$

Note that the lower tail of $G_{\theta,1}(x)$ is exponential. Thus $\inf_{x\in[0,2CN+2N]} G_{\theta,1}(-x) \geq \tau_1^N$ for some $\tau_1 > 0$ depending on α, θ, and C. Thus overall on $\mathsf{A} \cap \mathsf{D}$, $Q_{k,T}^{\vec{y},\vec{z}}(\mathbf{u}) \geq \tau^N$ for some τ depending on α, θ, k, and C. Since the Lebesgue measure of D is 1 we have

$$\mathbf{1}_{\mathsf{A}} \int_{\mathbb{R}^{|\mathcal{K}'_{k,T}|}} Q_{k,T}^{\vec{y},\vec{z}}(\mathbf{u})d\mathbf{u} \geq \mathbf{1}_{\mathsf{A}} \int_{\mathsf{D}} Q_{k,T}^{\vec{y},\vec{z}}(\mathbf{u})d\mathbf{u} \geq \mathbf{1}_{\mathsf{A}} \cdot \tau^N \int_{\mathsf{D}} d\mathbf{u} = \mathbf{1}_{\mathsf{A}} \cdot \tau^N.$$

This proves the second inequality in (7.3.5) completing the proof. $\square$

Proof of Proposition 7.3.2. Recall ${}_i^N(j)$ from Definition 7.2.7. Fix any $k \in \mathbb{Z}_{>0}$ and $\rho \in (0,1)$. For all $r \in [\![1,k]\!]$ and $j \in [\![1, 2N-2i+2]\!]$ set

$$\mathcal{B}_r(j) := \sum_{i=1}^{r} {}_i^N(j) = r\log 2 + 2r\Psi(\theta)N + \log Z_{\mathrm{sym}}^{(r)}(N + \lfloor j/2 \rfloor, N - \lceil j/2 \rceil + 1),$$

where recall $Z_{\mathrm{sym}}^{(r)}(\cdot,\cdot)$ defined in (7.2.8). Set $\mathcal{B}_0(j) \equiv 0$. We claim that there exist $\mathrm{C} = \mathrm{C}(\rho, k) > 0$ and $N_0 = N_0(\rho, k) > 0$, such that for all $N \geq N_0$ and $r \in [\![1,k]\!]$

$$\mathbf{P}\left(\left|\log Z_{\mathrm{sym}}^{(r)}(N + \lfloor j/2 \rfloor, N - \lceil j/2 \rceil + 1)\right| \leq \mathrm{C} \cdot N\right) \geq 1 - \rho^N. \tag{7.3.6}$$

In view of the above bound, setting $\mathrm{C}' = \mathrm{C} + 2k|\Psi(\theta)| + k\log 2$ we see that, by triangle inequality and union bound

$$\mathbf{P}(|{}_r(j)| \leq 2\mathrm{C}' \cdot N) \geq \mathbf{P}(|\mathcal{B}_{r-1}(j)| \leq \mathrm{C}' \cdot N) + \mathbf{P}(|\mathcal{B}_r(j)| \leq \mathrm{C}' \cdot N) - 1 \geq 1 - 2 \cdot \rho^N.$$

Adjusting ρ, N_0 the above inequality yields (7.3.2). The rest of the proof is devoted in proving (7.3.6).

Recall that $Z_{\mathrm{sym}}^{(r)}(\cdot,\cdot)$, defined in (7.2.8), can be viewed as sum of weights of r-tuple of non-intersecting paths. We first provide concentration bound for weight of a given path π with endpoints in $\mathcal{I}_{\mathbf{sym}}^{(N)} := \{(i,j) : i + j \leq 2N + 1\}$ via standard Chernoff bound for i.i.d. random variables. Then we provide an upper bound on the number of r-tuple of non-intersecting paths. Via union bound, this gives a concentration bound of type (7.3.6) for $Z_{\mathrm{sym}}^{(r)}(\cdot,\cdot)$.

Recall the symmetric weight $\widetilde{W}_{i,j}$ from (7.2.7). Note that for an upright path π, $(i,j)\in\pi$ and $(j,i)\in\pi$ cannot happen simultaneously provided $i\neq j$. Thus $(\widetilde{W}_{i,j})_{(i,j)\in\pi}$ forms an independent collection. Set

$$R_1 := \max\{\log\Gamma(\theta)-\log\Gamma(2\theta), \log\Gamma(\alpha)-\theta\log 2-\log\Gamma(\alpha+\theta)\},$$
$$R_2 := \max\{\log\Gamma(3\theta)-\log\Gamma(2\theta), \log\Gamma(\alpha+2\theta)+\theta\log 2-\log\Gamma(\alpha+\theta)\}.$$

Using moments of Gamma distribution and Markov inequality for each $s>0$ we have

$$\begin{aligned}\mathbf{P}\Bigg(\sum_{(i,j)\in\pi}\log\widetilde{W}_{i,j}\geq \tfrac{s+R_1}{\theta}|\pi|\Bigg) &\leq e^{-(s+R_1)|\pi|}\prod_{(i,j)\in\pi}\mathbf{E}[\widetilde{W}_{i,j}^{\theta}] \\ &= e^{-(s+R_1)|\pi|}\prod_{(i,j)\in\pi, i\neq j}\frac{\Gamma(\theta)}{\Gamma(2\theta)}\prod_{(i,i)\in\pi}\frac{\Gamma(\alpha)}{2^{\theta}\Gamma(\alpha+\theta)}\leq e^{-s|\pi|},\end{aligned}$$

and

$$\begin{aligned}\mathbf{P}\Bigg(\sum_{(i,j)\in\pi}\log\widetilde{W}_{i,j}\leq -\tfrac{s+R_2}{\theta}|\pi|\Bigg) &\leq e^{-(s+R_2)|\pi|}\prod_{(i,j)\in\pi}\mathbf{E}[\widetilde{W}_{i,j}^{-\theta}] \\ &= e^{-(s+R_2)|\pi|}\prod_{(i,j)\in\pi, i\neq j}\frac{\Gamma(3\theta)}{\Gamma(2\theta)}\prod_{(i,i)\in\pi}\frac{2^{\theta}\Gamma(\alpha+2\theta)}{\Gamma(\alpha+\theta)}\leq e^{-s|\pi|}.\end{aligned}$$

This leads to the following concentration bound

$$\mathbf{P}\Bigg(\Bigg|\sum_{(i,j)\in\pi}\log\widetilde{W}_{i,j}\Bigg|\leq \tfrac{s+R_1+R_2}{\theta}|\pi|\Bigg)\geq 1-2e^{-s|\pi|}. \tag{7.3.7}$$

To upgrade the above bound to (7.3.6), we need an upper bound for the number of r-tuples of non-intersecting upright paths. To do this, we introduce a few notations. Set $m := N+\lfloor j/2\rfloor$, $n := N-\lceil j/2\rceil+1$. Given two points $\mathbf{x},\mathbf{y}\in\mathcal{I}_{\mathbf{sym}}^{(N)}$, let $F_N(\mathbf{x}\to\mathbf{y})$ be the set of all upright paths from $\mathbf{x}$ to $\mathbf{y}$. For any $\pi \in \Pi_{(m,n)}^{(r)}$ we have $N \le |\pi| \le 2N$. Furthermore, $|F_N(\mathbf{x} \to \mathbf{y})| \le 4^N$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{I}_{\mathbf{sym}}^{(N)}$. Thus $|\Pi_{(m,n)}^{(r)}| \le 4^{kN}$ as $r \le k$. Fix $s = s(\rho, k) > 0$ such that $4^{kN} \cdot 2e^{-sN} \le \rho^N$ and consider the event

$$\mathsf{A} := \left\{ \left| \log \prod_{(i,j) \in \pi_1 \cup \cdots \cup \pi_r} \widetilde{W}_{i,j} \right| \le \tfrac{s+R_1+R_2}{\theta} \cdot 2rN \text{ for all } (\pi_q)_{q=1}^r \in \Pi_{(m,n)}^{(r)} \right\}.$$

Applying the concentration bound (7.3.7) for each path in $\Pi_{(m,n)}^{(r)}$, an union bound yields

$$\mathbf{P}(\mathsf{A}) \ge 1 - 4^{kN} \cdot 2e^{-sN} \ge 1 - \rho^N. \tag{7.3.8}$$

Next set $\mathrm{C} = \mathrm{C}(\rho, k) := k \log 4 + \frac{s+R_1+R_2}{\theta} 2k$. Note that on A we have

$$\begin{aligned} \log Z_{\mathrm{sym}}^{(r)}(m,n) &\le \log \left[\sum_{(\pi_1,\ldots,\pi_r) \in \Pi_{(m,n)}^{(r)}} \prod_{(i,j) \in \pi_1 \cup \cdots \cup \pi_r} \widetilde{W}_{i,j} \right] \\ &\le \log \left[4^{kN} \cdot e^{\frac{s+R_1+R_2}{\theta} 2rN} \right] \le kN \log 4 + \tfrac{s+R_1+R_2}{\theta} 2kN \le \mathrm{C} \cdot N. \end{aligned} \tag{7.3.9}$$

Similarly for the lower bound we consider any $(\pi_1, \ldots, \pi_r) \in \Pi_{(m,n)}^{(r)}$ which forms a disjoint collection of paths. Then on A we have

$$\log Z_{\mathrm{sym}}^{(r)}(m,n) \ge \log \prod_{(i,j) \in \pi_1 \cup \cdots \cup \pi_r} \widetilde{W}_{i,j} \ge -\tfrac{s+R_1+R_2}{\theta} 2kN \ge -\mathrm{C} \cdot N. \tag{7.3.10}$$

(7.3.6) now follows from (7.3.9) and (7.3.10) and the bound in (7.3.8). This completes the proof.

$\square$

7.3.2 High point on the second curve

The goal of this section is to show there is a point $p = O(N^{2/3})$ such that with high probability ${}^{N}_{2}(2p) \ge -\mathrm{C}N^{1/3}$ where ^{N} is the $\mathcal{HSLG}$ line ensemble defined in Definition 7.2.7. For the rest of this section we work with the boundary parameter fixed in critical or supercritical phase. We assume α equals α_1 or α_2 where

$$\begin{cases} \alpha_1 := \alpha_1(N) := N^{-1/3}\mu & \text{(Critical)} \\ \alpha_2 := \zeta & \text{(Super-Critical)} \end{cases} \tag{7.3.11}$$

where $\mu \in \mathbb{R}$ and $\zeta > 0$ are fixed numbers. The labeling of the parameter might seem a bit unnatural at this moment. It is related to the technical arguments in Section 7.4. Broadly speaking, when the boundary parameter is α_i, we shall resample top i curves of the $\mathcal{HSLG}$ line ensemble in Section 7.4.

Theorem 7.3.3 (High point on the second curve)**.** *Fix any $\varepsilon \in (0,1)$ and $k > 0$. There exist $R_0(k,\varepsilon) > 0$ such that for all $R \geq R_0$*

$$\liminf_{N\to\infty} \mathbf{P}\left(\sup_{p\in[kN^{2/3},RN^{2/3}]} \mathcal{L}_2^N(2p) \geq -[\tfrac{1}{8}R^2\nu + 2\sqrt{R}]N^{1/3} \right) > 1-\varepsilon. \tag{7.3.12}$$

where

$$\nu := \frac{(\Psi'(\theta))^2}{(-\Psi''(\theta))^{4/3}}. \tag{7.3.13}$$

The proof of Theorem 7.3.3 relies on two probabilistic information related to the first curve which we record below.

Proposition 7.3.4 (High point on the first curve)**.** *Fix any $\varepsilon \in (0,1)$. There exists $M_0(\varepsilon) > 0$ such that for all $M_1, M_2 \geq M_0$ and $k > 0$ we have*

$$\liminf_{N\to\infty} \mathbf{P}\left(\sup_{p\in[\![kN^{2/3},(M_1+2k)N^{2/3}]\!]} \frac{\mathcal{L}_1^N(2p+1)}{N^{1/3}} + k^2\nu \leq M_2 \right) > 1-\varepsilon, \tag{7.3.14}$$

$$\liminf_{N\to\infty} \mathbf{P}\left(\sup_{p\in[\![kN^{2/3},(M_1+2k)N^{2/3}]\!]} \frac{\mathcal{L}_1^N(2p+1)}{N^{1/3}} + k^2\nu \geq -M_2 \right) > 1-\varepsilon. \tag{7.3.15}$$

where ν is defined in (7.3.13).

Figure 7.10 depicts the high probability events considered in Proposition 7.3.4.

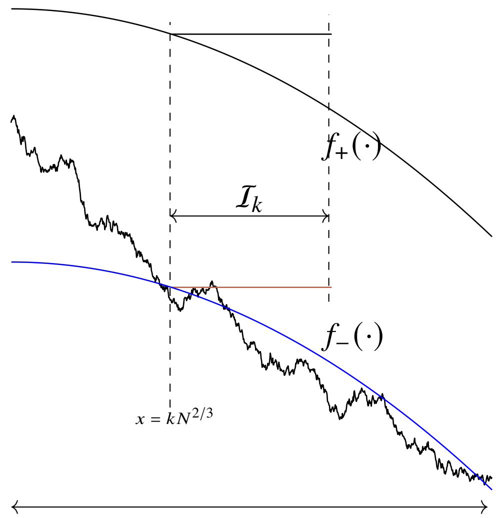


Figure 7.10: Diagram for Proposition 7.3.4. In the above figure ${}_1^N(2p+1)$ is given by the black rough curve. $f_\pm(x) := -(N\nu)^{-1}x^2 \pm M_2N^{1/3}$ are the parabolic curves drawn above. The horizontal lines are drawn in such a way that they meet the parabolas at $x = kN^{2/3}$. The event in (7.3.14) tells us that on $\mathcal{I}_k := [\![kN^{2/3}, (M_1+2k)N^{2/3}]\!]$ the black rough curve stays entirely below the black horizontal line. The event in (7.3.15) asserts that there is a point on $\mathcal{I}_k$ such that the black rough curve is above the red horizontal curve at that point.

Proposition 7.3.5 (Low point on the first curve)**.** *Fix any $\varepsilon \in (0,1)$. There exists $M_0(\varepsilon)$ such that for all $M \geq M_0$,*

$$\liminf_{N\to\infty} \mathbf{P}\left(\mathcal{L}_1^N(2MN^{2/3}+1) \leq -\tfrac{1}{8}M^2N^{1/3}\nu\right) > 1-\varepsilon, \tag{7.3.16}$$

where ν is defined in (7.3.13).

The proofs of Propositions 7.3.4 and 7.3.5 rely on the fluctuation results from [34], namely Theorem 7.1.4, and are postponed to the next subsection. Assuming their validity, we complete the proof of Theorem 7.3.3.

Proof of Theorem 7.3.3. For clarity we divide the proof into two steps.

Step 1. In this step we define notation and events used in the proof. For simplicity we write ${}_i(j)$ for ${}^N_i(j)$. Fix $\varepsilon \in (0,1)$ and $k > 0$. Take M_0 from Proposition 7.3.4. We set R large enough so that

$$2^{-5}R \geq 2k+1, \qquad M_0 - 2^{-5}(\tfrac{1}{8}R^2\nu + M_0) + R^{3/2} \leq -M_0 - 2^{-10}R^2\nu \tag{7.3.17}$$

and $S := 2^{-5}R$. The precise choice of R will depend on certain probability bounds that will be specified in the next step. We set

$$a := M_0 N^{1/3}, \;\; b := -\tfrac{1}{8}R^2 N^{1/3}\nu, \;\; n := RN^{2/3} - kN^{2/3}, \;\; v := -[\tfrac{1}{8}R^2\nu + 2\sqrt{R}]N^{1/3}.$$

Let us define the sets $\mathcal{I} := [\![SN^{2/3}, (M_0+2S)N^{2/3}]\!]$ and $\mathcal{J} := [\![kN^{2/3}, RN^{2/3}]\!]$. Next we define the following events:

$$\mathsf{A} := \left\{ \sup_{p\in\mathcal{J}} \mathcal{L}_2(2p) \leq v \right\}, \quad \mathsf{B} := \left\{ \mathcal{L}_1(2kN^{2/3}+1) \leq a, \mathcal{L}_1(2RN^{2/3}+1) \leq b \right\}.$$

The A event demands that the second curve ${}_2(2p)$ does not rise above v for any $p \in \mathcal{J}$. The B event requires both ${}_1(2kN^{2/3}+1)$ and ${}_1(2RN^{2/3}+1)$ to be less than a and b respectively. Finally we set

$$\mathsf{C} := \left\{ \sup_{p\in\mathcal{I}} {}_1(2p+1) + S^2\nu N^{1/3} \geq -a \right\}$$

In words, the event C ensures there exists some $p \in \mathcal{I}$ such that ${}_1(2p+1)$ is greater than $-a - S^2\nu N^{1/3}$.

Note that by Proposition 7.3.4 we have $\mathbf{P}(\mathsf{C}) \geq 1-\varepsilon$. Furthermore, by Proposition 7.3.4 and Proposition 7.3.5 for large enough R we also have $\mathbf{P}(\neg\mathsf{B}) \leq 2\varepsilon$. We claim that for all large enough R we have

$$\mathbf{P}(\mathsf{A} \cap \mathsf{B} \cap \mathsf{C}) \leq \varepsilon. \tag{7.3.18}$$

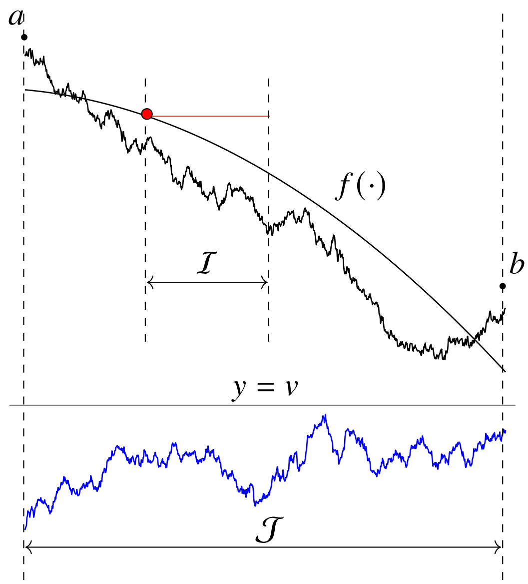


Figure 7.11: In the above figure ${}_1(2p+1)$ (black curve) and ${}_2(2p)$ (blue curve) are plotted for $p \in \mathcal{J}$. A denotes the event that the blue curve lies below the horizontal line $y = v$. B denotes the event that the black curve starts below a and ends below b. The curve f in the figure is given by $f(x) = -(Nv)^{-1}x^2 - a$. The event C denotes that there is a point $p' \in \mathcal{I}$ on black curve such that ${}_1(2p'+1) \geq f(SN^{2/3})$ (this event does not occur in the above figure). The key idea is that on $\mathsf{A} \cap \mathsf{B}$, the blue curve lies below $y = v$ completely, and the black curve behaves like a simple random bridge and follows a linear trajectory with starting and ending points less than a and b respectively. As a result, the event C (which requires the black curve to follow parabolic trajectory) does not occur with high probability. But we know both B and C occurs with high probability. Thus the event A occurs with low probability.

We prove (7.3.18) in next step. Assuming this, note that by union bound we have

$$\mathbf{P}(\neg\mathsf{A}) \geq \mathbf{P}(\mathsf{C}) - \mathbf{P}(\neg\mathsf{B}) - \mathbf{P}(\mathsf{A} \cap \mathsf{B} \cap \mathsf{C}) \geq 1 - 4\varepsilon.$$

Changing $\varepsilon \mapsto \varepsilon/4$ we arrive at (7.3.12). This completes the proof modulo (7.3.18).

Step 2. In this step we will prove (7.3.18). The readers are encouraged to consult with Figure 7.11 and its caption to get an overview of the key idea behind the proof.

We consider the σ-field:

$$\mathcal{F} := \sigma\big\{{}_2[\![1, 2N - 2i]\!], {}_1([\![1, 2kN^{2/3} + 1]\!] \cup [\![2RN^{2/3} + 1, 2N]\!])\big\}.$$

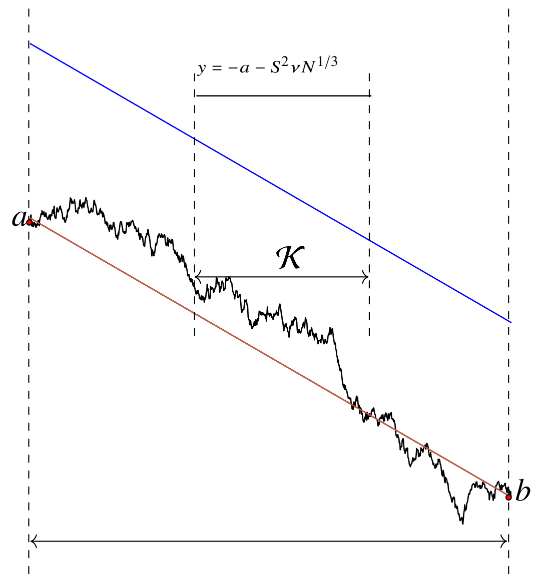


Figure 7.12: In the above figure the random bridge X_i from a to b is depicted by the black curve. The event D ensures the random bridge lies below the blue line $y = a + \frac{x}{n}(b-a) + \sqrt{Rn}$. The event C requires $X_i \geq -[M_0 + S^2\nu]N^{1/3}$ for some $i \in \mathcal{K} := [(S-k)N^{2/3}, (M_0 + 2S - k)N^{2/3}]$. One can choose R large enough so that the horizontal black line $y = -[M_0 + S^2\nu]N^{1/3}$ lies above the blue line $y = a + \frac{x}{n}(b-a) + \sqrt{Rn}$ for all $x \geq (S-k)N^{2/3}$. This forces $\mathsf{D} \subset \neg\mathsf{C}$.

Note that $\mathsf{A} \cap \mathsf{B}$ is measurable with respect to $\mathcal{F}$. Hence

$$\mathbf{P}(\mathsf{A} \cap \mathsf{B} \cap \mathsf{C}) = \mathbf{E}\left[\mathbf{1}_{\mathsf{A}\cap\mathsf{B}}\mathbf{E}\left[\mathbf{1}_{\mathsf{C}} \mid \mathcal{F}\right]\right].$$

Using the Gibbs property for two-sided boundaries (see Observation 7.2.5), the conditional law is determined by the boundary data and is monotone with respect to the boundary data (see Proposition 8.2.3). On the event $\mathsf{A} \cap \mathsf{B}$, $_2$ (on even points) is at most ν, $_1(2kN^{2/3} + 1)$ is at most a and $_1(2RN^{2/3} + 1)$ is at most b. Thus by stochastic monotonicity on $\mathsf{A} \cap \mathsf{B}$ we have

$$\mathbf{1}_{\mathsf{A}\cap\mathsf{B}} \cdot \mathbf{E}\big(\mathbf{1}_{\mathsf{C}} \mid \mathcal{F}\big) \leq \mathbf{1}_{\mathsf{A}\cap\mathsf{B}} \cdot \frac{\mathbf{E}_{\text{free}}^{a,b}\left(W(\vec{X},\nu)\mathbf{1}\{\mathsf{C}\}\right)}{\mathbf{E}_{\text{free}}^{a,b}\left(W(\vec{X},\nu)\right)} \leq \mathbf{1}_{\mathsf{A}\cap\mathsf{B}} \cdot \frac{\mathbf{P}_{\text{free}}^{a,b}\left(\mathsf{C}\right)}{\mathbf{E}_{\text{free}}^{a,b}\left(W(\vec{X},\nu)\right)}. \tag{7.3.19}$$

Here $\vec{X} = (X_0, \ldots, X_n)$ is a random bridge with $X_0 = a$ and $X_n = b$ with i.i.d. increments from $G_{\theta,+1} * G_{\theta,-1}$, and n. $\mathbf{P}_{\text{free}}^{a,b}$ and $\mathbf{E}_{\text{free}}^{x,y}$ denotes the probability and the expectation operator of the random walk, and $W(\vec{X},v) := \exp\left(-2\sum_{i=1}^{n-1} e^{v-X_i}\right)$. The event C is now interpreted as

$$\mathsf{C} = \left\{ \sup_{p\in[\![SN^{2/3},(M_0+2S)N^{2/3}]\!]} X_{p-kN^{2/3}} + S^2 vN^{1/3} \geq -a \right\}.$$

Note that

$$\begin{aligned}\mathbf{E}_{\text{free}}^{a,b}\left(W(\vec{X},v)\right) &\geq \exp\left(-2ne^{-\sqrt{n}}\right)\mathbf{P}_{\text{free}}^{x,y}\left(X_i \geq v+\sqrt{n},\ \text{for all } i\in[\![0,n]\!]\right) \\ &\geq \exp\left(-2ne^{-\sqrt{n}}\right)\mathbf{P}_{\text{free}}^{a,b}\left(X_i - a - \tfrac{i(b-a)}{n} \geq -\sqrt{n},\ \text{for all } i\in[\![0,n]\!]\right). \qquad (7.3.20)\end{aligned}$$

where the last inequality follows by noting that $X_i - a - \frac{i(b-a)}{n} \geq -\sqrt{n}$ implies $X_i \geq b-\sqrt{n} \geq v+\sqrt{n}$. Now by the KMT coupling for Brownian bridges and estimates for Brownian bridges, r.h.s. of (7.3.20) is uniformly bounded below by some absolute constant δ. We now claim that for all large enough R we have

$$\mathsf{D}\subset\neg\mathsf{C}, \quad \mathbf{P}_{\text{free}}^{a,b}(\mathsf{D}) \geq 1-\varepsilon\delta, \quad \text{where } \mathsf{D} := \left\{\sup_{i\in[\![0,n]\!]}\left(X_i - a - \tfrac{i(b-a)}{n}\right) \leq \sqrt{R}\sqrt{n}\right\}. \qquad (7.3.21)$$

Note that (7.3.21) implies $\mathbf{P}_{\text{free}}^{x,y}(\mathsf{C}) \leq \varepsilon\delta$. Plugging this back in (7.3.19) along with the bound $\mathbf{E}_{\text{free}}^{x,y}\left(W(\vec{X},z)\right) \geq \delta$, yields that r.h.s. of (7.3.19) is at most ε. This proves (7.3.18).

Let us now verify (7.3.21). Indeed, $\mathbf{P}_{\text{free}}^{a,b}(\mathsf{D})$ can be made arbitrarily close to 1 by choosing R large enough due to the KMT coupling for Brownian bridges. We choose R so large that $\mathbf{P}_{\text{free}}^{a,b}(\mathsf{D})$ is at least $1-\varepsilon\delta$. Let us now verify $\mathsf{D}\subset\neg\mathsf{C}$ (see also Figure 7.12 and its caption). For $q \geq S$ we see that

$$\begin{aligned}a + \tfrac{(q-k)(b-a)}{R-k} + \sqrt{R}\sqrt{n} &\leq \left[M_0 - \tfrac{S-k}{R-k}\left(\tfrac{1}{8}R^2 v + M_0\right) + R^{3/2}\right]N^{1/3} \\ &\leq \left[M_0 - 2^{-5}\left(\tfrac{1}{8}R^2 v + M_0\right) + R^{3/2}\right]N^{1/3} \leq -\left[M_0 + S^2 v\right]N^{1/3}\end{aligned}$$

The penultimate inequality follows by observing that as $S = 2^{-5}R$, we have $S-k \geq 2^{-5}(R-k) > 0$.

Finally the last inequality follows from (7.3.17). Thus for all $p \geq SN^{2/3}$,

$$x + \tfrac{(p-kN^{2/3})(y-x)}{(R-k)N^{2/3}} + \sqrt{R}\sqrt{n} \leq M_0 N^{1/3} - S^2 \nu N^{1/3}$$

Clearly this implies $\mathsf{D} \subset \neg\mathsf{C}$, completing the proof (7.3.21). □

Proof of Propositions 7.3.4 and 7.3.5

Recall $Z_N^{\text{line}}(m)$, the point-to-(partial)line partition function defined in (7.1.5). The proofs of Proposition 7.3.4 and Proposition 7.3.5 rely on the following lemma.

Lemma 7.3.6 (Uniform tightness)**.** *Fix $\varepsilon \in (0,1)$. There exists $K_0 = K_0(\varepsilon) > 0$, such that for all $M > 0$ and $K \geq K_0$ we have*

$$\liminf_{N\to\infty} \mathbf{P}\left(-K \leq \frac{\log Z_N^{\text{line}}(MN^{2/3}) + 2\Psi(\theta)N}{N^{1/3}} + M^2\nu \leq K\right) > 1 - \varepsilon$$

where ν is defined in (7.3.13)*.*

We remark that the above lemma was alluded in the introduction in the form of (7.1.6).

Proof. We recall the notations introduced in Section 7.1.3. Fix any $M > 0$. Set $k = MN^{2/3}$ and $p := 1 + \frac{2k}{N-k}$. Let θ_c be the unique solution to $\Psi'(\theta_c) - p\Psi'(2\theta - \theta_c) = 0$. Set $f_{\theta,p} = -\Psi(\theta_c) - p\Psi(2\theta - \theta_c)$ and $\sigma_{\theta,p}^3 = \frac{1}{2}(-\Psi''(\theta_c) - \Psi''(2\theta - \theta_c))$ where Ψ is the digamma function defined in (7.1.2). It is not hard to check that

$$(N-k)f_{\theta,p} = -2N\Psi(\theta) + M^2N^{1/3}(\Psi'(\theta))^2/\Psi''(\theta) + O(1), \text{ and } \sigma_{\theta,p} = (-\Psi''(\theta))^{1/3} + O(1),$$

where $O(1)$ terms depend on M, θ, but are bounded in N. When $\alpha = \alpha_2 > 0$, we have that $\lim_{N\to\infty}(N-k)^{1/3}\sigma_{\theta,p}(\alpha_2 + \theta - \theta_c) = \infty$ for each fixed $M > 0$. Thus by Theorem 7.1.4 we get

$$\frac{\log Z_N^{\text{line}}(MN^{2/3}) + 2\Psi(\theta)N}{(-N\Psi''(\theta))^{1/3}} + M^2\nu \xrightarrow{d} \mathrm{TW}_{\mathrm{GUE}},$$

where $\mathrm{TW}_{\mathrm{GUE}}$ is the GUE Tracy-Widom distribution [308] and ν is defined in (7.3.13). For $\alpha = \alpha_1 = N^{-1/3}\mu$, we have $\lim_{N\to\infty}(N-k)^{1/3}\sigma_{\theta,p}(\alpha_1+\theta-\theta_c) = y := \sigma_{\theta,1}(\mu - M\Psi'(\theta)/\Psi''(\theta))$. Another application of Theorem 7.1.4 yields

$$\frac{\log Z_N^{\mathrm{line}}(MN^{2/3}) + 2\Psi(\theta)N}{(-N\Psi''(\theta))^{1/3}} + M^2\nu \xrightarrow{d} U_{-y}.$$

where U_{-y} is the Baik-Ben Arous-Péché distribution [10] (see [34, (5.2)] for definition). As $M \to \infty$, $y \to \infty$. Since $U_{-y} \to \mathrm{TW}_{\mathrm{GUE}}$ as $y \to \infty$ (see [17, (2.36)]), we can thus get tightness uniformly in M. This completes the proof. $\square$

Proof of Proposition 7.3.4. Fix $k > 0$, $\varepsilon \in (0,1)$. Since for any $M_1 > 0$

$$\sup_{j\in[\![kN^{2/3},(M_1+2k)N^{2/3}]\!]} Z(N+j,N-j) \le Z_N^{\mathrm{line}}(kN^{2/3}).$$

Appealing to Lemma 7.3.6 with $M \mapsto k$ we see that

$$\mathbf{P}\left(\sup_{j\in[\![kN^{2/3},(M_1+2k)N^{2/3}]\!]} \frac{\log Z(N+j,N-j) + 2\Psi(\theta)N}{N^{1/3}} + k^2\nu \le M_2\right) \ge 1-\varepsilon,$$

where M_2 can be chosen to be any $M \ge K_0$ where $K_0(\varepsilon)$ comes from Lemma 7.3.6. Recalling that ${}_1(2j+1) = \log Z(N+j,N-j) + 2\Psi(\theta)N$ from (7.1.4), we get (7.3.14).

The remainder of the proof is now devoted in proving (7.3.15). Towards this end, set $K_1 = \frac{1}{2}(M_1+2k)^2\nu$. Choose M_1 large enough so that $K_1 \ge K_0(\varepsilon/4)$ where K_0 comes from Lemma 7.3.6. Applying Lemma 7.3.6 with $M \mapsto M_1+2k$, $K \mapsto K_1$, and $\varepsilon \mapsto \varepsilon/4$ we have

$$\liminf_{N\to\infty} \mathbf{P}\left(\frac{\log Z_N^{\mathrm{line}}((M_1+2k)N^{2/3}) + 2\Psi(\theta)N}{N^{1/3}} \le -\tfrac{1}{2}(M_1+2k)^2\nu\right) > 1 - \tfrac{1}{4}\varepsilon. \tag{7.3.22}$$

Now we take $K_2 = (\frac{(M_1+2k)^2}{4} - k^2)\nu - \log 2 \ge \frac{1}{4}M_1^2\nu$. We again choose M_1 large enough so that

$K_2 \geq K_0(\varepsilon/4)$. Then applying Lemma 7.3.6 with $M \mapsto k$, $K \mapsto K_2$, and $\varepsilon \mapsto \varepsilon/4$ we have

$$\liminf_{N\to\infty} \mathbf{P}\left(\frac{\log Z_N^{\mathrm{line}}(kN^{2/3}) + 2\Psi(\theta)N}{N^{1/3}} \geq -\tfrac{1}{4}(M+2k)^2\nu + \log 2\right) > 1 - \tfrac{1}{4}\varepsilon. \tag{7.3.23}$$

By union bound the above two estimates implies for all large enough M_1 we have

$$\liminf_{N\to\infty} \mathbf{P}\left(Z_N^{\mathrm{line}}(kN^{2/3}) > 2\cdot Z_N^{\mathrm{line}}((M_1+2k)N^{2/3})\right) > 1 - \tfrac{1}{2}\varepsilon. \tag{7.3.24}$$

Let us temporarily set $A = Z_N^{\mathrm{line}}(kN^{2/3}) - Z_N^{\mathrm{line}}((M_1+2k)N^{2/3}$ and $B = Z_N^{\mathrm{line}}((M_1+2k)N^{2/3}$. Observe that $A + B > 2B$ implies $2A > A + B$. Recall from (7.1.5) that

$$A = \sum_{\lceil kN^{2/3}\rceil}^{\lceil (M_1+2k)N^{2/3}\rceil - 1} Z(N+j, N-j) \leq (M_1+k)N^{\frac{2}{3}} \sup_{j\in[\![kN^{\frac{2}{3}}, (M_1+2k)N^{\frac{2}{3}}]\!]} Z(N+j, N-j).$$

We thus have

$$\begin{aligned}&\left\{Z_N^{\mathrm{line}}(kN^{2/3}) > 2\cdot Z_N^{\mathrm{line}}((M_1+2k)N^{2/3})\right\}\\ &\subset \left\{\sup_{j\in[\![kN^{\frac{2}{3}}, (M_1+2k)N^{\frac{2}{3}}]\!]} \log Z(N+j, N-j) > \log Z_N^{\mathrm{line}}(kN^{\frac{2}{3}}) - \log(2(M_1+k)N^{\frac{2}{3}})\right\}.\end{aligned} \tag{7.3.25}$$

By Lemma 7.3.6, one can choose M_2 large enough (but free of k) so that

$$\liminf_{N\to\infty} \mathbf{P}(\log Z_N^{\mathrm{line}}(kN^{2/3}) + 2\Psi(\theta)N + k^2\nu N^{1/3} \geq -M_2 N^{\frac{1}{3}} + \log(2(M_1+k)N^{\frac{2}{3}}) > 1 - \tfrac{1}{2}\varepsilon.$$

Using this, in view of (7.3.25) and (7.3.24), and using ${}_1(2j+1) = \log Z(N+j, N-j) + 2\Psi(\theta)N$ (see (7.1.4)) we arrive at (7.3.15). This proves Proposition 7.3.4. $\square$

Proof of Proposition 7.3.5. We use the same notations from proof of Proposition 7.3.4 and utilize (7.3.22) and (7.3.23) obtained there with $k = 1$. Let us set $M = M_1 + 2$. Indeed combining (7.3.22) and (7.3.23) we have

$$\liminf_{N\to\infty} \mathbf{P}\left(\log Z_N^{\text{line}}(N^{2/3}) > \tfrac{1}{4}M^2N^{1/3}\nu + \log Z_N^{\text{line}}(MN^{2/3})\right) \geq 1 - \tfrac{1}{2}\varepsilon.$$

As $Z_N^{\text{line}}(MN^{2/3}) \geq Z(N + MN^{2/3}; N - MN^{2/3})$, this leads to

$$\liminf_{N\to\infty} \mathbf{P}\left(\log Z_N^{\text{line}}(N^{2/3}) > \tfrac{1}{4}M^2N^{1/3}\nu + \log Z(N + MN^{2/3}, N - MN^{2/3})\right) \geq 1 - \tfrac{1}{2}\varepsilon.$$

Again by Lemma 7.3.6, one can choose M large enough so that

$$\liminf_{N\to\infty} \mathbf{P}\Bigg(\log Z_N^{\text{line}}(N^{2/3}) \leq \tfrac{1}{8}M^2N^{1/3}\nu - 2\Psi(\theta)N\Bigg) > 1 - \tfrac{1}{2}\varepsilon,$$

which forces

$$\liminf_{N\to\infty} \mathbf{P}\left(\log Z(N + MN^{2/3}; N - MN^{2/3}) < -2N\Psi(\theta) - \tfrac{1}{8}M^2N^{1/3}\nu\right) \geq 1 - \varepsilon.$$

Recalling that $_1(2MN^{2/3} + 1) = \log Z(N + MN^{2/3}; N - MN^{2/3}) - 2\Psi(\theta)N$ from (7.1.4), we get (7.3.16). This completes the proof of Proposition 7.3.5. □

7.3.3 Spatial properties of the lower curves

In this section, we study spatial properties of the lower curves of the $\mathcal{HSLG}$ line ensemble. The main result of this section is the following.

Theorem 7.3.7. *Fix any $p \in \{1, 2\}$. Set $\alpha := \alpha_p$ according to* (7.3.11)*. Consider the $\mathcal{HSLG}$ line ensemble from Definition 7.2.7 with parameters (α, θ). Given any $k, \varepsilon > 0$, there exist constants $M = M(k, \varepsilon) \geq 1$ and $N_0(k, \varepsilon) \geq 1$ such that for all $N \geq N_0(k, \varepsilon)$ and $v \in \{2, 3\}$ we have*

$$\mathbf{P}\left(\sup_{s\in[\![1,kN^{2/3}]\!]} {}^N_v(s) \geq MN^{1/3}\right) \leq \varepsilon. \tag{7.3.26}$$

In plain words, Theorem 7.3.7 argues that with high probability on the domain $[\![1, kN^{2/3}]\!]$, the entire second curve and third curve lies below a threshold $MN^{1/3}$. The proof of Theorem 7.3.7 can be easily extended to include other lower indexed curves as well. However, for the proofs of our main results, it suffices to consider the first three curves.

Recall from Theorem 7.1.3 that the conditional laws of the $\mathcal{HSLG}$ line ensemble are given by $\mathcal{HSLG}$ Gibbs measures introduced in Definition 7.1.2. The key technical ingredient in proving Theorem 7.3.7 is the tightness of left boundary points of the first two curves under bottom-free measure, defined in Definition 7.2.3.

Lemma 7.3.8. *Fix any $p \in \{1,2\}$. Set $\alpha := \alpha_p$ according to (7.3.11). Fix any $r \geq 1$ and $\varepsilon > 0$. Set $T = \lfloor rN^{2/3} \rfloor$. Define*

$$A := \begin{cases} 1 + \sqrt{r}|\mu|\Psi'(\tfrac{1}{2}\theta) & \text{if } p = 1, \\ 1 & \text{if } p = 2. \end{cases} \tag{7.3.27}$$

There exists $M = M(\varepsilon) > 0$ and $N_0(\varepsilon) > 0$ such that for all $N \geq N_0$ we have

$$\mathbf{P}_{\alpha_p}^{(0,-A\sqrt{T}),(-\infty)^T;2,T}\big(|L_1(1)| + |L_2(2)| \geq M\sqrt{T}\big) \leq \varepsilon. \tag{7.3.28}$$

where the law $\mathbf{P}_{\alpha_p}^{\vec{y},(-\infty)^{2T};2,T}$ is defined in Definition 7.2.3. Furthremore, there exists $\widetilde{M} = \widetilde{M}(\varepsilon) > 0$ and $\widetilde{N}_0(\varepsilon) > 0$ such that for all $N \geq \widetilde{N}_0$ we have

$$\mathbf{P}_{\alpha_1}^{0,(-\infty)^T;1,T}\big(|L_1(1)| \geq \widetilde{M}\sqrt{T}\big) \leq \varepsilon. \tag{7.3.29}$$

As we shall see in the next section, the proof of the above lemma can be extended to include $L_2(1)$ instead of $L_2(2)$. For technical reasons we work with $L_2(2)$.

As mentioned in the introduction, the proof of Lemma 7.3.8 relies on several ingredients related to non-intersecting random walks. We postpone its proof to Section 7.4. We now complete the proof of Theorem 7.3.7 assuming Lemma 7.3.8.

Proof of Theorem 7.3.7. We first prove Theorem 7.3.7 for the $v = 2$ case and then use it to show

$v = 3$ case.

Part I: $v = 2$ **case.** For clarity we divide the proof into two steps.

Step 1. We shall write ${}_i(j)$ for the $\mathcal{HSLG}$ line ensemble ${}_i^N(j)$. Recall that the points in the line ensemble satisfy certain high probability ordering due to Theorem 7.3.1. In particular, if we know the even points on ${}_2$ are not too high, Theorem 7.3.1 will force that with high probability the odd points are not too high as well. Thus it suffices to control the even points on ${}_2$. In this step, we flesh out the details of the above idea. The proof of control on even points on ${}_2$ appears in the second step of the proof.

We begin by defining a few events that will appear in the rest of the proof. Fix $k, \varepsilon > 0$. For any $r \in [\![1, kN^{2/3}]\!] \cap 2\mathbb{Z}$, define

$$\mathsf{A}_r(M) := \left\{{}_2(r) \geq MN^{1/3}\right\}, \quad \mathsf{F}_r(M) := \{{}_1(r-1) \geq \tfrac{3M}{4}N^{1/3}\}.$$

Define

$$\mathsf{B}_r(M) = \mathsf{A}_r(M) \cap \bigcap_{s \in [\![r+2, kN^{2/3}]\!] \cap 2\mathbb{Z}} \neg \mathsf{A}_s(M),$$

so that $(\mathsf{B}_r(M))_{r \in [\![1, kN^{2/3}]\!]}$ forms a disjoint collections of events. We next define

$$\mathsf{G}^+(M) := \bigsqcup_{r \in [\![1, kN^{2/3}]\!] \cap 2\mathbb{Z}} \mathsf{B}_r(M) \cap \mathsf{F}_r(M), \quad \mathsf{G}^-(M) := \bigsqcup_{r \in [\![1, kN^{2/3}]\!] \cap 2\mathbb{Z}} \mathsf{B}_r(M) \cap \neg \mathsf{F}_r(M).$$

In the above equation, we use $\sqcup$ instead of $\cup$ to stress on the fact that it is an union of disjoint events. Finally set $\mathsf{G}(M) := \mathsf{G}^+(M) \sqcup \mathsf{G}^-(M)$. Observe that the event

$$\neg \mathsf{G}(M) = \left\{ \sup_{s \in [\![1, kN^{2/3}]\!] \cap 2\mathbb{Z}} {}_2(s) < MN^{1/3} \right\}$$

controls the supremum of the second curve over the even points. Take $0 < k' < k$. By the union bound we get that

$$\mathbf{P}\left(\sup_{s\in[\![1,k'N^{2/3}]\!]} {}_2(s) \geq 3MN^{\frac{1}{3}}\right) \leq \mathbf{P}(\mathsf{G}(2M)) + \mathbf{P}\Bigg(\sup_{\substack{s\in[\![1,k'N^{2/3}]\!]\\ s\in(2\mathbb{Z}+1)}} {}_2(s) \geq 3MN^{\frac{1}{3}}, \neg\mathsf{G}(2M)\Bigg). \quad (7.3.30)$$

Note that on $-\mathsf{G}(2M)$ the supremum of ${}_2(s)$ over all $s \in [\![1, kN^{2/3}]\!]$ is at most $2MN^{1/3}$. Then by the ordering of the line ensemble (Theorem 7.3.1) on $\neg\mathsf{G}(2M)$ it is exponentially unlikely that any odd point within $[\![1, k'N^{2/3}]\!]$ will exceed $2MN^{1/3} + (\log N)^{7/6}$. In particular the second term on the r.h.s. of (7.3.30) can be made smaller than $\frac{\varepsilon}{2}$ by choosing N large enough and taking $M \geq 1$. For the first term we claim that there exists M_0, N_0 depending on k, ε such that for all $N \geq N_0$ and $M \geq M_0$ we have

$$\mathbf{P}(\mathsf{G}(2M)) \leq \tfrac{\varepsilon}{2}. \quad (7.3.31)$$

Clearly plugging this bound back in r.h.s. of (7.3.30) proves (7.3.26) with $M \mapsto 3M$ and $k' \mapsto k$. For the remainder of the proof we focus on proving (7.3.31).

Step 2. In this step we prove (7.3.31). Observe that from the definition of $\mathsf{G}^-(2M)$ we have

$$\mathbf{P}(\mathsf{G}^-(2M)) \leq \mathbf{P}\big({}_1(r-1) - {}_2(r) \geq -\tfrac{M}{2}N^{1/3}, \text{ for some } r \in [\![1, kN^{1/3}]\!] \cap 2\mathbb{Z}\big).$$

However by Theorem 7.3.1 the r.h.s. of the above equation can be made smaller that $\frac{\varepsilon}{4}$ for all $N \geq N_0$ and $M \geq 1$, by choosing $N_0 := N_0(k, \varepsilon) > 0$ appropriately. We next claim that

$$\mathbf{P}(\mathsf{G}^+(2M)) \leq 2\mathbf{P}(\mathsf{A}_2(M)) \leq \tfrac{\varepsilon}{4}. \quad (7.3.32)$$

As $\mathsf{G}(2M) = \mathsf{G}^-(2M) \cup \mathsf{G}^+(2M)$, in view of the above claim, (7.3.31) follows via a union bound.

Let us now prove (7.3.32). Observe that by definition of $G^+(2M)$ we have

$$\mathbf{P}\big(\mathsf{A}_2(M)\big) \geq \mathbf{P}\big(\mathsf{G}^+(2M) \cap \mathsf{A}_2(M)\big) = \sum_{r\in[\![1,kN^{2/3}]\!]\cap 2\mathbb{Z}} \mathbf{P}\big(\mathsf{B}_r(2M) \cap \mathsf{F}_r(2M) \cap \mathsf{A}_2(M)\big). \quad (7.3.33)$$

We focus on each of the terms in the above sum. Using the tower property of the expectation we have

$$\begin{aligned} &\mathbf{P}\big(\mathsf{B}_r(2M) \cap \mathsf{F}_r(2M) \cap \mathsf{A}_2(M)\big) \\ &= \mathbf{E}\left[\mathbf{1}_{\mathsf{B}_r(2M)\cap\mathsf{F}_r(2M)}\mathbf{E}\left(\mathbf{1}_{\mathsf{A}_2(M)} \mid {}_3, {}_1[\![r-1, kN^{2/3}]\!], {}_2[\![r, kN^{2/3}]\!]\right)\right]. \end{aligned} \quad (7.3.34)$$

Using the Gibbs property (see Theorem 7.1.3 and Observation 7.2.2 (a)) we have almost surely that

$$\begin{aligned} &\mathbf{1}_{\mathsf{B}_r(2M)\cap\mathsf{F}_r(2M)}\mathbf{E}\left(\mathbf{1}_{\mathsf{A}_2(M)} \mid {}_3, {}_1[\![r-1, kN^{2/3}]\!], {}_2[\![r, kN^{2/3}]\!]\right) \\ &= \mathbf{1}_{\mathsf{B}_r(2M)\cap\mathsf{F}_r(2M)}\mathbf{P}_{\alpha_p}^{\vec{y},\vec{z};2,r/2}(L_2(2) > MN^{1/3}) \\ &\geq \mathbf{1}_{\mathsf{B}_r(2M)\cap\mathsf{F}_r(2M)}\mathbf{P}_{\alpha_p}^{\vec{w},(-\infty)^r;2,r/2}(L_2(2) > MN^{1/3}), \end{aligned} \quad (7.3.35)$$

where $\vec{y} = ({}_1(r-1), {}_2(r))$, $\vec{z} = ({}_3(2v))_{v=1}^{r/2}$ and $\vec{w} := (\frac{3M}{2}N^{1/3}, \frac{3M}{2}N^{1/3} - A\sqrt{r/2})$ ($A \geq 1$ is defined in (7.3.27)). The last inequality above follows by stochastic monotonicity (Proposition 8.2.3). We now briefly explain how stochastic monotonicity works here. Not that the event $\{L_2(2) > MN^{1/3}\}$ is decreasing as the boundary data decreases. Thus to acheive a lower bounda, we can reduce the boundary $\vec{z}$ to $(-\infty)^r$. Furthermore, on $\mathsf{B}_r(2M) \cap \mathsf{F}_r(2M)$, we may reduce $\vec{y}$ to $\vec{w}$ as $y_i \geq w_i$ on $\mathsf{B}_r(2M) \cap \mathsf{F}_r(2M)$.

Note that $MN^{1/3} \geq Mk^{-\frac{1}{2}}\sqrt{r/2}$. Now by translation invariance (Observation 7.2.1 (a)) and Lemma 7.3.8, we may choose $M_0(k,\varepsilon)$ large enough so that for all $M \geq M_0$ and for all $r \in [\![1, kN^{2/3}]\!] \cap 2\mathbb{Z}$ we have

$$\mathbf{P}_{\alpha_p}^{\vec{w},(-\infty)^r;2,r/2}(L_2(2) > MN^{1/3}) = \mathbf{P}_{\alpha_p}^{(0,-A\sqrt{r/2}),(-\infty)^r;2,r/2}\big(L_2(2) > -\tfrac{1}{2}MN^{1/3}\big) \geq \tfrac{1}{2}.$$

Inserting the above bound in (7.3.35) and then going back to (7.3.34) we get

$$\text{r.h.s. of (7.3.34)} \geq \tfrac{1}{2}\mathbf{P}\big(\mathsf{B}_r(2M) \cap \mathsf{F}_r(2M)\big).$$

Recall that $\mathsf{B}_r(2M) \cap \mathsf{F}_r(2M)$ are all disjoint events whose union over $r \in [\![1, kN^{2/3}]\!] \cap 2\mathbb{Z}$ is given by $\mathsf{G}^+(2M)$. Summing the above inequality over $r \in [\![1, kN^{2/3}]\!] \cap 2\mathbb{Z}$, in view of (7.3.33), we thus arrive at $\mathbf{P}(\mathsf{A}_2(M)) \geq \frac{1}{2}\mathbf{P}(\mathsf{G}^+(2M))$. This proves the first inequality in (7.3.32). For the second one observe that by union bound

$$\mathbf{P}(\mathsf{A}_2(M)) \leq \mathbf{P}({}_1(3) - {}_2(2) \leq -N^{1/3}) + \mathbf{P}({}_1(3) \geq (M-1)N^{1/3}).$$

By Theorem 7.3.1 the first term on the r.h.s. of the above equation can be made arbitrarily small by choosing N large enough. As for the second term, recall the point-to-line partition function $Z_N^{\text{line}}(\cdot)$ from (7.1.5). From Theorem 7.1.4 we know $N^{-1/3}[\log Z_N^{\text{line}}(1) + 2\Psi(\theta)N]$ is tight. Since ${}_1^N(3) \leq \log Z_N^{\text{line}}(1) + 2\Psi(\theta)N$ (see (7.1.4)), one can make the second term arbitrarily small enough by choosing M, N large enough. This completes the proof of (7.3.32).

Part II: $v = 3$ **case.** Fix $k > 0$. Let us define

$$\mathsf{E} := \Big\{ \sup_{s \in [\![1, kN^{2/3}]\!]} {}_3(s) \geq MN^{1/3} \Big\}, \quad \mathsf{F} := \Big\{ \sup_{s \in [\![1, kN^{2/3}]\!]} {}_2(s) \geq \tfrac{1}{2}MN^{1/3} \Big\}.$$

By repeated application of the union bound we have

$$\begin{aligned}
\mathbf{P}(\mathsf{E}) &\leq \mathbf{P}(\mathsf{F}) + \mathbf{P}(\mathsf{E} \cap \neg\mathsf{F}) \\
&\leq \mathbf{P}(\mathsf{F}) + \mathbf{P}\Big({}_2(s) - {}_3(s) \leq -\tfrac{1}{2}MN^{1/3}, \text{ for some } s \in [\![1, kN^{2/3}]\!]\Big) \\
&\leq \mathbf{P}(\mathsf{F}) + \sum_{[\![1, kN^{2/3}]\!]} \mathbf{P}\big({}_2(s) - {}_3(s) \leq -\tfrac{1}{2}MN^{1/3}\big). \qquad (7.3.36)
\end{aligned}$$

By Theorem 7.3.1, there exists an absolute constant N_0 such that for all $s \geq 1$, and $M \geq 1$, we have $\mathbf{P}\left({}_2(s) - {}_3(s) < -\frac{1}{2}MN^{1/3}\right) \leq 2^{-N}$. Since we have established $v = 2$ case of Theorem 7.3.7, we may directed use (7.3.26) with $v \mapsto 2$, $M \mapsto \frac{1}{2}M$ and $\varepsilon \mapsto \frac{1}{2}\varepsilon$, to get that $\mathbf{P}(\mathsf{F}) \le \frac{1}{2}\varepsilon$ for all large enough N, M. Thus for all N, M large enough we have (7.3.36) $\le \frac{1}{2}\varepsilon + kN^{2/3}2^{-N} \le \varepsilon$. $N \ge N_0$. This completes the proof. $\square$

Theorem 7.3.7 and Lemma 7.3.8 can be used to deduce left boundary tightness for the $\mathcal{HSLG}$ line ensemble. We shall refer to this property as *endpoint tightness*.

Theorem 7.3.9 (Endpoint Tightness)**.** *Fix any $p \in \{1, 2\}$. Set $\alpha := \alpha_p$ according to* (7.3.11)*. Recall the $\mathcal{HSLG}$ line ensemble from Definition 7.2.7 with parameters (α, θ). The sequences $\{N^{-1/3}{}^N_1(1)\}_N$ and $\{N^{-1/3}{}^N_2(2)\}_N$ are tight.*

Again the proof can be extended to include tightness of $N^{-1/3}{}^N_2(1)$ as well, once we have the corresponding version in Lemma 7.3.8. We again refrain from doing so, as it is inconsequential to the proofs of our main theorem.

Remark 7.3.10. In [205], the authors computed the distributional limit of $N^{-1/3}{}^N_1(1)$ which implies tightness as well. Currently, their approach does not give access to information we need about the behavior of ${}^N_2(2)$.

Proof of Theorem 7.3.9. Fix an $\varepsilon > 0$. We shall show that for all large enough N, M we have

$$\mathbf{P}({}^N_1(1) \le MN^{1/3}) \ge 1 - 3\varepsilon, \quad \mathbf{P}({}^N_2(2) \le -MN^{1/3}) \le 3\varepsilon. \tag{7.3.37}$$

In view of the ordering of points in the line ensemble (Theorem 7.3.1), we know ${}^N_1(1) \ge {}^N_2(2) - (\log N)^{7/6}$ with probability at least $1 - 2^{-N}$. This along with the above equation ensures endpoint tightness. We thus focus on proving (7.3.37).

Proof of the first inequality in (7.3.37)**.** Recall the point-to-line partition function $Z_N^{\text{line}}(\cdot)$ from (7.1.5). From Theorem 7.1.4, we know $N^{-1/3}[\log Z_N^{\text{line}}(1) + 2\Psi(\theta)N]$ is tight. Since ${}^N_1(3) \le \log Z_N^{\text{line}}(1) + 2\Psi(\theta)N$, there exists $M_1(\varepsilon) > 0$ such that for all $N \ge 3$ we have $\mathbf{P}({}^N_1(3) \le$

$M_1N^{1/3}) \geq 1 - \varepsilon$. Thanks to Theorem 7.3.1, there exists $M_2(\varepsilon) > M_1(\varepsilon)$ such that for all $N \geq 3$

$$\mathbf{P}(\mathsf{A}) \geq 1 - 2\varepsilon, \quad \mathsf{A} := \left\{ {}^{N}_{1}(3) \leq M_1 N^{1/3}, \sup_{j \in [\![1,4]\!]} {}^{N}_{2}(j) \leq M_2 N^{1/3} \right\}.$$

Define $\mathcal{F} := \sigma\{({}^{N}_{1}(j))_{j\geq 3}, (({}^{N}_{i}[\![1, 2N-2i+2]\!])_{i\geq 2}\}$. By the union bound and tower property of the conditional expectation, for any $M_3 > 0$ we have

$$\mathbf{P}({}^{N}_{1}(1) \geq M_2N^{1/3} + M_3) \leq 2\varepsilon + \mathbf{E}\left[\mathbf{1}_{\mathsf{A}}\mathbf{E}\big(\mathbf{1}_{{}^{N}_{1}(1)\geq M_2N^{1/3}+M_3} \mid \mathcal{F}\big)\right]$$

Using Theorem 7.1.3 we have

$$\mathbf{E}\big(\mathbf{1}_{{}^{N}_{1}(1)\geq M_2N^{1/3}+M_3} \mid \mathcal{F}\big) = \mathbf{P}_{\alpha_p}^{{}^{N}_{1}(3),({}^{N}_{2}(2),{}^{N}_{2}(4));1,2}(L_1(1) \geq M_2N^{1/3} + M_3)$$

On event A, the boundary data are at most $M_2N^{1/3}$. By stochastic monotonicity (Proposition 8.2.3) and translation invariance of the Gibbs measure (Observation 7.2.1 (a)), under event A we have

$$\mathbf{1}_{\mathsf{A}} \cdot \mathbf{P}_{\alpha_p}^{{}^{N}_{1}(3),({}^{N}_{2}(j))_{j\in[\![1,4]\!]};1,2}(L_1(1) \geq M_2N^{1/3} + M_3) \leq \mathbf{1}_{\mathsf{A}} \cdot \mathbf{P}_{\alpha_p}^{0,(0,0,0,0);1,2}(L_1(1) \geq M_3).$$

The last probability can be made less than ε by taking M_3 large enough. Thus setting $M_4 = M_4(\varepsilon) := M_3 + M_2$, we see that for all $N \geq 3$, the first inequality in (7.3.37) holds with $M = M_4$.

Proof of the second inequality in (7.3.37). We start by defining two high probability events B_1 and B_2. The idea is to then show $\mathbf{P}\big(\{{}^{N}_{2}(2) \leq -MN^{1/3}\} \cap \mathsf{B}_1 \cap \mathsf{B}_2\big)$ can be made arbitrarily small by choosing N, M large enough.

We shall use Theorem 7.3.3 (high point on the second curve) with $k \mapsto 1$. Consider $R_0 = R_0(1, \varepsilon) > 0$ from Theorem 7.3.3. Set $R = \max\{R_0, 1\}$. By Theorem 7.3.3 with $k \mapsto 1$, there exists $M_5(\varepsilon) > 0$ such that for all large enough N

$$\mathbf{P}(\mathsf{B}_1) \geq 1 - \varepsilon, \quad \mathsf{B}_1 := \bigcup_{q=N^{2/3}}^{RN^{2/3}} \mathsf{B}_1(p), \quad \mathsf{B}_1(q) := \{{}^{N}_{2}(2q) \geq -M_5N^{1/3}\}.$$

We write the set B_1 as union of disjoint sets as follows:

$$\mathsf{C}_1(q) := \mathsf{B}_1(q) \cap \bigcap_{s=q+1}^{RN^{2/3}} \neg \mathsf{B}_1(s), \quad \mathsf{C}_1 := \bigsqcup_{q=N^{2/3}}^{RN^{2/3}} \mathsf{C}_1(q) = \mathsf{B}_1.$$

By Theorem 7.3.1, for large enough N we have

$$\mathbf{P}(\mathsf{B}_2) \geq 1-\varepsilon, \quad \mathsf{B}_2 := \bigcap_{q=N^{2/3}}^{RN^{2/3}} \mathsf{B}_2(q), \quad \mathsf{B}_2(q) := \left\{{}^{N}_{2}(2q) - {}^{N}_{1}(2q-1) \leq N^{1/3}\right\}.$$

Set $\mathcal{F}_q := \sigma\{({}^{N}_{1}(j-1), {}^{N}_{2}(j))_{j\geq 2q}, ({}^{N}_{i}[\![1, 2N-2i+2]\!])_{i\geq 3}\}$. Observe that $\mathsf{B}_2(q) \cap \mathsf{C}_1(q)$ is measurable with respect to $\mathcal{F}_q$. Note that for any $M_6 > 0$ we have

$$\begin{aligned}
\mathbf{P}\left(\left\{{}^{N}_{2}(2) \leq -M_6 N^{1/3}\right\} \cap \mathsf{B}_1 \cap \mathsf{B}_2 \right) &\leq \sum_{q=N^{2/3}}^{RN^{2/3}} \mathbf{P}\left(\left\{{}^{N}_{2}(2) \leq -M_6 N^{1/3}\right\} \cap \mathsf{C}_1(p) \cap \mathsf{B}_2(p) \right) \\
&= \sum_{q=N^{2/3}}^{RN^{2/3}} \mathbf{E}\left[\mathbf{1}_{\mathsf{B}_2(q)\cap\mathsf{C}_1(q)} \mathbf{E}\left[\mathbf{1}_{{}^{N}_{2}(2)\leq -M_6 N^{1/3}} \mid \mathcal{F}_q \right] \right]. \qquad (7.3.38)
\end{aligned}$$

By the Gibbs property (Theorem 7.1.3) we have

$$\begin{aligned}
\mathbf{1}_{\mathsf{B}_2(q)\cap\mathsf{C}_1(q)} \cdot \mathbf{E}\left[\mathbf{1}_{{}^{N}_{2}(2)\leq -M_6 N^{1/3}} \mid \mathcal{F}_q \right] &= \mathbf{1}_{\mathsf{B}_2(q)\cap\mathsf{C}_1(q)} \cdot \mathbf{P}_{\alpha_p}^{({}^{N}_{1}(2q-1), {}^{N}_{2}(2q)), ({}^{N}_{3}(2i))_{i=1}^{q}; 2, q}(L_2(2) \leq -M_6 N^{1/3}) \\
&\leq \mathbf{1}_{\mathsf{B}_2(q)\cap\mathsf{C}_1(q)} \cdot \mathbf{P}_{\alpha_p}^{(y_1,y_2),(-\infty)^q;2,q}(L_2(2) \leq -M_6 N^{1/3}),
\end{aligned}$$

where $y_1 = -(M_5+1)N^{1/3}$, $y_2 = -M_5 N^{1/3}$. The last inequality follows due to stochastic monotonicity (Proposition 8.2.3) as on the event $\mathsf{B}_2(q) \cap \mathsf{C}_1(q)$ we have ${}^{N}_{2}(2q) \geq -M_5 N^{1/3}$ and ${}^{N}_{1}(2q-1) \geq -(M_5+1)N^{1/3}$. By translation invariance and stochastic monotonicity we have

$$\mathbf{P}_{\alpha_p}^{(y_1,y_2),(-\infty)^q;2,q}(L_2(2) \leq -M_6 N^{1/3}) \leq \mathbf{P}_{\alpha_p}^{(0,-A\sqrt{q}),(-\infty)^q;2,q}(L_2(2) \leq (M_5+1-M_6)N^{1/3}) \leq \varepsilon,$$

where the last inequality is uniform over $q \in [\![N^{2/3}, RN^{2/3}]\!]$ and follows from Lemma 7.3.8 by taking M_6 large enough ($A \geq 1$ is defined in (7.3.27)). Plugging the above bound back in (7.3.38), and noting that $(\mathsf{B}_2(q))_{q=N^{2/3}}^{RN^{2/3}}$ forms a disjoint collection of events we have that (7.3.38) $\le \varepsilon$. Using the fact that $\mathbf{P}(\neg \mathsf{B}_i) \le \varepsilon$ for $i = 1, 2$, an application of the union bound yields the second inequality in (7.3.37) with $M = M_6$. This completes the proof. $\square$

7.4 Proof of Lemma 7.3.8

In this section, we prove Lemma 7.3.8 that asserts endpoint tightness of bottom-free measures defined in Definition 7.2.3. Along with Lemma 7.3.8, we also study probabilities of a certain event which we call *region pass event* under bottom-free measure.

Fix any $r, M > 0$ and $p \in \{1, 2\}$. Set $T = \lfloor rN^{2/3} \rfloor$. We define the region pass event as

$$\mathsf{RP}_{p,M} := \left\{ \inf_{i \in [\![1, 2T+p-2]\!]} L_p(i) \ge 2MN^{1/3} \right\}. \tag{7.4.1}$$

Informally speaking, region pass event requires the first $2T + p - 2$ points of the p-th curve to lie above $2MN^{1/3}$. Although this is a low probability event, in the following lemma we claim that one has a uniform lower bound on this event.

Lemma 7.4.1. *Fix any $r, M > 0$ and $p \in \{1, 2\}$. Set $T = \lfloor rN^{2/3} \rfloor$. We set $\alpha = \alpha_p$ according to* (7.3.11). *Recall the bottom-free measure from Definition 7.2.3. Let $\vec{y} \in \mathbb{R}^p$ with $y_i = -(M + i - 1)N^{1/3}$. There exists $\phi = \phi(r, M) > 0$ and $N_0(r, M) > 0$ such that for all $N \ge N_0$ we have*

$$\mathbf{P}_{\alpha_p}^{\vec{y},(-\infty)^{2T};p,2T}(\mathsf{RP}_{p,M}) \ge \phi. \tag{7.4.2}$$

In plain words, Lemma 7.4.1 says there is always positive probability that the first half of the points in second curve are higher than a given threshold (see Figure 7.13).

Recall that α_1 and α_2 are the boundary parameters corresponding to critical and supercritical phases respectively. Depending on the phase being supercritical or critical, the arguments for proving Lemma 7.3.8 and Lemma 7.4.1 are markedly different. We first give interpretation of the bottom-free laws under the two phases in Section 7.4.1. In Section 7.4.2 and 7.4.3, we provide

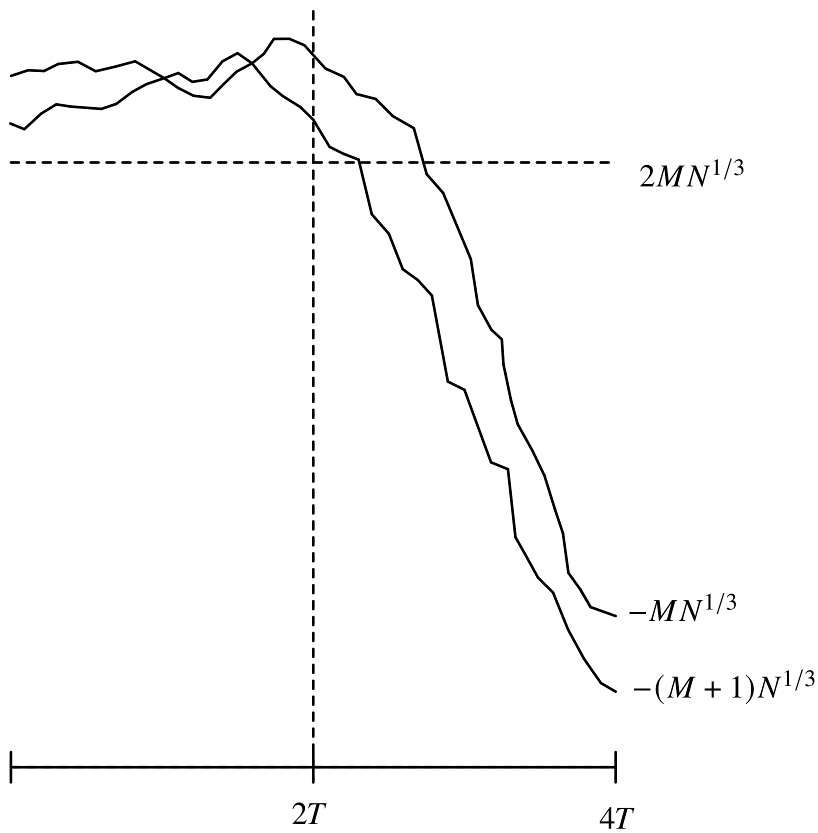


Figure 7.13: The above figure depicts the event $\mathsf{RP}_{2,M}$ under the law $\mathbf{P}_{\alpha_2}^{\vec{y},(-\infty)^{2T};2,2T}$.

proofs of the aforementioned lemmas for critical and supercritical phases respectively.

7.4.1 Interpretation of the bottom-free laws under critical and supercritical phase

In this section, we provide convenient interpretations of the bottom-free laws. We begin with the following observation where we mention how the bottom-free measure on the domain $\mathcal{K}_{k,T}$ with boundary condition $\vec{y}$ is well defined under certain cases.

Observation 7.4.2 (Well-definedness of bottom-free measures)**.** *Take $\vec{y} \in \mathbb{R}^k$. When k is even and $\alpha > 0$ or When $k \geq 1$ and $\alpha \in (-\theta, \theta)$, the bottom-free measure on the domain $\mathcal{K}_{k,T}$ with boundary condition $\vec{y}$ is well defined. Indeed, for $k \geq 1$ and $\alpha \in (-\theta, \theta)$, $f_{k,T}^{\vec{y},(-\infty)^T}(\mathbf{u})$ is proportional to*

$$\prod_{i=1}^{k-1} \prod_{j=1}^{T-\mathbf{1}_{i=1}} W(u_{i+1,2j}; u_{i,2j+1}, u_{i,2j-1}) \prod_{i=1}^{k} \prod_{j=1}^{2T-1-\mathbf{1}_{i=1}} G_{\theta+(-1)^{i+j-1}\alpha,(-1)^{j+1}}(u_{i,j} - u_{i,j+1}). \tag{7.4.3}$$

where $W(a;b,c) := \exp(-e^{a-b} - e^{a-c})$ *and* G *is defined in* (7.2.3)*. The above form follows from* (7.2.2) *by redistributing the edge weights cleverly. See Figure 7.14 A. For the case when* k *is even and* $\alpha > 0$*, we redistribute according to Figure 7.14 B. One can compute the explicit density for the corresponding bottom-free measure from the figure. For our later proofs, we record it only for* $k = 2$*.* $f_{2,T}^{\vec{y},(-\infty)^{2T}}(\mathbf{u})$ *is proportional to*

$$\exp\left(-e^{u_{2,2}-u_{1,3}}\right) G_{\alpha,1}(u_{2,2}-u_{1,1}) G_{\theta,1}(u_{1,1}-u_{1,2}) G_{\alpha+\theta,1}(u_{2,1}-u_{2,2}) \\ \prod_{j=2}^{T-1} W(u_{2,2j}; u_{1,2j+1}, u_{1,2j-1}) \prod_{i=1}^{2} \prod_{j=2}^{2T-1-\mathbf{1}_{i=1}} G_{\theta,(-1)^{j+1}}(u_{i,j}-u_{i,j+1}). \tag{7.4.4}$$

From the form of the densities in (7.4.3) *and* (7.4.4)*, it is clear that they are integrable.*

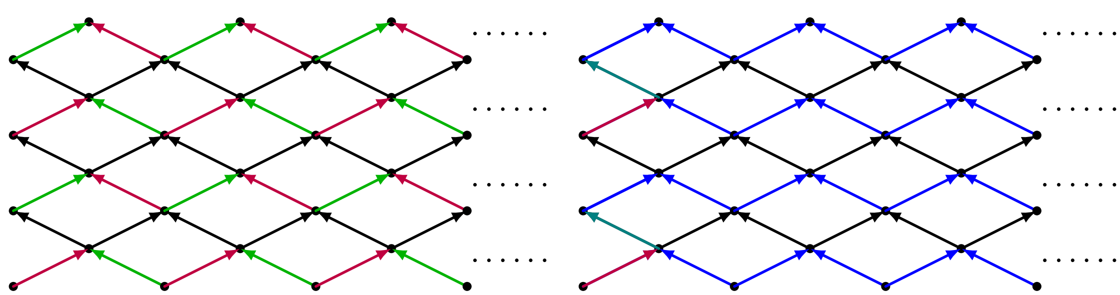


Figure 7.14: Redistribution of edge weights for $\alpha \in (-\theta, \theta)$ (Figure A) and for $\alpha > 0$ and k even (Figure B). The weights of **green**, **teal**, and **purple** edges are $e^{(\theta-\alpha)x-e^x}$, $e^{\alpha x - e^x}$, and $e^{(\theta+\alpha)x-e^x}$ respectively.

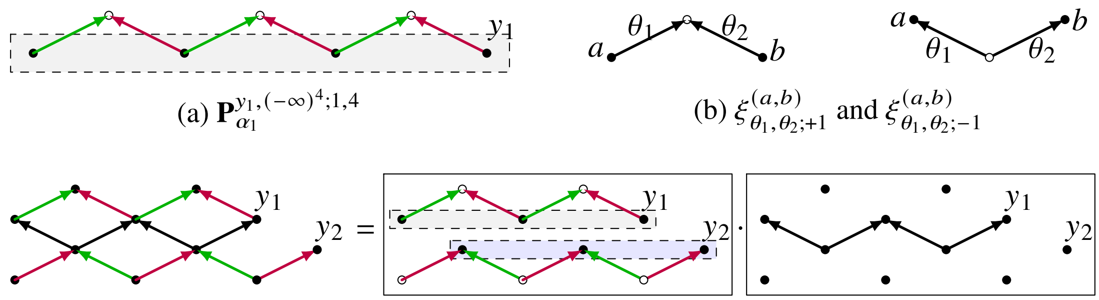
This observation allows us to interpret the bottom free laws in terms of random walks.

Critical phase

In this section, we give the interpretation of the bottom-free law under critical phase. We first give an informal interpretation based on the Figure 7.15. Towards this end, we introduce ξ-distributions. Given $\theta_1, \theta_2 > 0$ and $a, b \in \mathbb{R}$, we consider the following two probability density functions

$$\xi^{(a,b)}_{\theta_1,\theta_2;\pm1}(x) \propto G_{\theta_1,\pm1}(a-x)G_{\theta_2,\pm1}(b-x). \tag{7.4.5}$$

The graphical representation of the above two distributions are given in Figure 7.15 B.



(a) $\mathbf{P}^{y_1,(-\infty)^4;1,4}_{\alpha_1}$ (b) $\xi^{(a,b)}_{\theta_1,\theta_2;+1}$ and $\xi^{(a,b)}_{\theta_1,\theta_2;-1}$ (c) $\mathbf{P}^{(y_1,y_2),(-\infty)^3;2,3}_{\alpha_1}$

Figure 7.15: Figures (A) and (B) are graphical representations of probability distributions $\mathbf{P}^{y_1,(-\infty)^4;1,4}_{\alpha_1}$ and $\xi^{(a,b)}_{\theta_1,\theta_2;\pm1}$ respectively. Figure (C) shows decomposition of $\mathbf{P}^{(y_1,y_2),(-\infty)^4;2,4}_{\alpha_1}$ into $\widetilde{\mathbf{P}}^{(y_1,y_2)}$ (middle figure) and W_{cr} (right figure). The marginal law of the gray (blue resp.) shaded region is a random walk started at y_1 (y_2 resp.) with increment $G_{\theta+\alpha_1,-1}*G_{\theta-\alpha_1,+1}$ ($G_{\theta+\alpha_1,+1}*G_{\theta-\alpha_1,-1}$ resp.).

- Let us consider $(X(i))_{i\in[\![1,2S-1]\!]} \sim \mathbf{P}^{y_1,(-\infty)^S;1,S}_{\alpha_1}$. See Figure 7.15 (A) for the graphical representation of the law. We focus on the odd points (shaded inside the gray box in the figure). Note that $(X(2S-1-2k))_{k=0}^{S-1}$ is a random walk starting at $X(2S-1)=y_1$ with increments distributed as $G_{\theta+\alpha,1}*G_{\theta-\alpha,-1}$. Conditioned on the odd points, we have $X(2k)\sim\xi^{(X(2k-1),X(2k+1))}_{\theta-\alpha_1,\theta+\alpha_1;1}$.

- Let us now consider the $\mathbf{P}^{(y_1,y_2),(-\infty)^S;2,S}_{\alpha_1}$ law whose graphical representation is given in Figure 7.15 (C). We view the graph as superimposition of two graphs where in one graph we collect all the non-black edges and the other graph we include only the black edges (see Figure 7.15 (C)). We denote the law of the Gibbs measure formed by deleting the black edges as $\widetilde{\mathbf{P}}^{(y_1,y_2)}$ (middle figure in Figure 7.15 (C)). The law $\mathbf{P}^{(y_1,y_2),(-\infty)^S;2,S}_{\alpha_1}$ can be recovered from $\widetilde{\mathbf{P}}^{(y_1,y_2)}$ by viewing the black edges as a Radon-Nikodym derivative.

- If $(X_1(i))_{i\in[\![1,2S-1]\!]},(X_2(i))_{i\in[\![1,2S]\!]}\sim\widetilde{\mathbf{P}}^{(y_1,y_2)}$, we have $X_1(\cdot)$ independent of $X_2(\cdot)$ and X_1 is distributed as $\mathbf{P}_{\alpha_1}^{y_1,(-\infty)^S;1,S}$. X_2 has a similar representation with even points $(X_2(2S-2k))_{k=0}^{S-1}$ forming a random walk starting at y_2 with increments distributed as $G_{\theta+\alpha,-1} * G_{\theta-\alpha,1}$. Conditioned on the even points, we have $X_2(2k+1) \sim \xi_{\theta-\alpha_1,\theta+\alpha_1;-1}^{(X(2k),X(2k+2))}$ and $X_2(1) \sim G_{\theta+\alpha_1,1} + X_2(2)$.

All the above bullet points are direct consequences of the graphical representations of distributions in Figure 7.15. We now summarize our findings in the following observation.

Observation 7.4.3. *Consider an independent collection of random variables* $Y_{i,j} \overset{i.i.d.}{\sim} G_{\theta+\alpha_1,1}$ *and* $U_{i,j} \overset{i.i.d.}{\sim} \mathrm{Beta}(\theta-\alpha_1, 2\alpha_1)$ *for* $i = 1, 2$ *and* $j \in \mathbb{Z}_{>0}$. *Define*

$$V_{i,j} := Y_{i,2j} + \log U_{i,2j} - \mathbf{E}[\log U_{i,2j}] - Y_{i,2j-1}. \tag{7.4.6}$$

so that $V_{i,j}$ *form an i.i.d. collection of mean zero random variables. Set* $X_i(2S+i-2) = y_i$ *and for* $k \in [\![1, S-1]\!]$ *define*

$$X_i(2S+i-2k-2) := \Big(y_i + (-1)^{i+1}(k-1)\mathbf{E}[\log U_{1,1}]\Big) + (-1)^{i+1}\sum_{j=1}^{k-1} V_{i,j}, \tag{7.4.7}$$

and set

$$W_{\mathrm{cr}} := \exp\left(-\sum_{k=1}^{S-1}\left(e^{X_2(2k)-X_1(2k-1)} + e^{X_2(2k)-X_1(2k+1)}\right)\right). \tag{7.4.8}$$

Conditioned on $(X_i(2j+i-2))_{i\in\{1,2\}, j\in[\![1,S]\!]}$, *we set* $X_i(2k+i-1) \sim \xi_{\theta-\alpha_1,\theta+\alpha_1,(-1)^{i+1}}^{X_i(2k+i-2),X_i(2k+i)}$ *for* $i \in \{1,2\}$, $k \in [\![1, S-1]\!]$ *and* $X_2(1) \sim G_{\theta+\alpha,1} + X_2(2)$. *We have*

(a) $(X_1(i))_{i\in[\![1,2S-1]\!]}$ *is distributed as* $\mathbf{P}_{\alpha_1}^{y_1,(-\infty)^S;1,S}$.

(b) Let $\widetilde{\mathbf{P}}^{(y_1,y_2)}$ *denotes the joint law of* $\{(X_1(i))_{i\in[\![1,2S-1]\!]}, (X_2(i))_{i\in[\![1,2S]\!]}\}$. *This law has graphical representation given by the middle figure in Figure 7.15 (C). The law* $\mathbf{P}_{\alpha_1}^{(y_1,y_2),(-\infty)^S;2,S}$ *is*

absolutely continuous with respect to $\widetilde{\mathbf{P}}^{(y_1,y_2)}$ with

$$\frac{d\mathbf{P}_{\alpha_1}^{(y_1,y_2),(-\infty)^S;2,S}}{d\widetilde{\mathbf{P}}^{(y_1,y_2)}} = W_{\mathrm{cr}}.$$

Observation 7.4.3 follows from the three bullet points above and noting that $V_{i,j}+\mathbf{E}[\log U_{1,1}] \sim G_{\theta+\alpha,1} * G_{\theta-\alpha,-1}$. Note that W_{cr} precisely contains all the effect of the black edges in the Gibbs measure.

Supercritical phase

In the supercritical phase, the $\mathbf{P}_{\alpha_2}^{\vec{y},(-\infty)^S;1,S}$ law is a bit more complicated. To describe it, we first introduce paired random walk and weighted paired random walk (WPRW) below.

Definition 7.4.4 (Paired Random Walk and Weighted Paired Random Walk)**.** We fix two densities f and g. A paired random walk (PRW) $(S_k^{(n,1)}, S_k^{(n,2)})_{k=0}^n$ with endpoints $S_n^{(n,1)} = x_n$ and $S_n^{(n,2)} = y_n$ is a distribution on $2n$ points with density

$$\mathbf{P}\left(\bigcap_{k=0}^{n-1}\{S_k^{(n,1)} \in \mathrm{d}x_k, S_k^{(n,2)} \in \mathrm{d}y_k\}\right) \propto g(y_0 - x_0)\prod_{i=1}^{n}[f(x_i - x_{i-1})f(y_i - y_{i-1})]\prod_{k=0}^{n-1} dx_k\, dy_k.$$

We will denote the law of the above measure as $\mathbf{P}^{n;(x_n,y_n);f,g}$. We define the random variable:

$$W_{\mathrm{sc}} := \exp\left(-e^{S_0^{(n,2)}-S_1^{(n,1)}} - \sum_{k=1}^{n-1}\left(e^{S_k^{(n,2)}-S_{k+1}^{(n,1)}} + e^{S_k^{(n,2)}-S_k^{(n,1)}}\right)\right) \tag{7.4.9}$$

Using W_{sc} we define a new measure on $2n$ points as follows:

$$\mathbf{P}_{W_{\mathrm{sc}}}^{(x_n,y_n);f,g}(\mathsf{A}) = \frac{\mathbf{E}[W_{\mathrm{sc}}\mathbf{1}_{\mathsf{A}}]}{\mathbf{E}[W_{\mathrm{sc}}]}$$

We call the above measure as weighted paired random walk (WPRW).

With the above definition in place we now give the following interpretation of the $\mathbf{P}_{\alpha_2}^{\vec{y},(-\infty)^S;1,S}$ law.

Observation 7.4.5. *Fix any $\vec{y} \in \mathbb{R}^2$ and $S \in \mathbb{Z}_{>0}$. Suppose $(L_1(2j-1), L_2(2j))_{j\in[\![1,S]\!]} \sim \mathbf{P}_{W_{\mathrm{sc}}}^{\vec{y};f,g}$ with $f := G_{\theta,1} * G_{\theta,-1}$ and $g := G_{\alpha_2}$, with G defined in (7.2.3). Conditioned on $(L_1(2i-1), L_2(2i))_{i\in[\![1,S]\!]}$, set $L_2(1) \sim G_{\alpha_2+\theta,1} + L_2(2)$ and $L_1(2k) \sim \xi_{\theta,\theta;1}^{(L_1(2k-1),L_1(2k+1))}$, $L_2(2k+1) \sim \xi_{\theta,\theta;-1}^{(L_2(2k),L_2(2k+2))}$ for $k = 1, 2, \ldots, S-1$. Then $(L_i(j))_{i\in[\![1,2]\!],j\in[\![1,2S+i-2]\!]}$ is distributed as $\mathbf{P}_{\alpha_2}^{\vec{y},(-\infty)^S;2,S}$.*

To see that the above observation holds, we again decompose its graph into two graphs: one with black edges, say G_1, and one without black edges, say G_2 (see Figure 7.16). However, unlike the critical phase, the Gibbs measure corresponding to G_2 does not split into two independent parts because of the **teal** edge. For this measure, the marginal law of the odd points of the first curve and even points of the second curve together form the paired random walk. Upon taking the black edges into consideration, the odd points of the first curve and even points of the second curve jointly follows precisely the WPRW law.

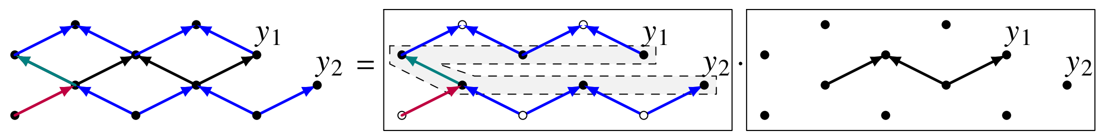


Figure 7.16: $\mathbf{P}_{\alpha_2}^{(y_1,y_2),(-\infty)^3;2,3}$ law is decomposed into two parts. The first part (middle figure) shaded region corresponds to a paired random walk. The second part (right figure) corresponds to W_{sc}.

7.4.2 Proof of Lemma 7.3.8 and Lemma 7.4.1 in critical phase

We continue with the notations from Observation 7.4.3. By KMT coupling for random walks [227] there exists an absolute constant C depending only on θ and μ such that for all S large enough,

$$\widetilde{\mathbf{P}}\left(\max_{k\le S-2}\Big|\sum_{j=1}^{k} V_{i,j} - \sigma B_k^{(i)}\Big| \ge \mathrm{C}\log S\right) \le 1/S. \tag{7.4.10}$$

where $\sigma^2 = \mathrm{Var}(V_{1,1})$ with $V_{1,1}$ defined in (7.4.6). Here $B^{(i)}$ are independent standard Brownian motions. Recall that $\alpha_1 = N^{-1/3}\mu$. Set $\kappa := \frac{1}{4}|\mu|\Psi'(\frac{1}{2}\theta) \ge 0$. As Ψ' is a decreasing nonnegative function on $[0, \infty)$, for large enough N we have

$$|\mathbf{E}[\log U_1]| = |\Psi(\theta - \alpha_1) - \Psi(\theta + \alpha_1)| \leq 2|\alpha_1|\Psi'(\tfrac{1}{2}\theta) = \tfrac{1}{2}\kappa N^{-1/3}. \tag{7.4.11}$$

Lemma 7.3.8 and Lemma 7.4.1 can now be proven using the above coupling and the estimate for $|\mathbf{E}[\log U_1]|$.

Proof of Lemma 7.3.8 in the case $p = 1$. Fix $\varepsilon \in (0, 1)$. Set

$$\beta_1 := \mathbf{P}(\sup_{x \in [0,1]} B_x^{(1)} \leq \tfrac{1}{8}) > 0, \quad \beta_2 := \inf_{n \in \mathbb{N}} \exp(-2(n-1)e^{-\frac{1}{2}\sqrt{n}}) > 0.$$

Set $S = T := \lfloor rN^{2/3} \rfloor$. Continuing with the notations from Observation 7.4.3, let us assume $(X_1(i))_{i \in [\![1, 2T-1]\!]}, (X_2(i))_{i \in [\![1, 2T]\!]}$ has the law $\widetilde{\mathbf{P}}^{(0, -A\sqrt{T})}$. Observe that $|(T-1)\mathbf{E}[\log U_1]| \leq \sqrt{r}\kappa\sqrt{T}$. Following the relation in (7.4.7) and the estimate in (7.4.10) we get that with probability at least $\beta_1^2 - \frac{2}{T}$ we have

$$\begin{aligned} X_1(2k-1) &\geq -\tfrac{1}{8}\sqrt{T} - \sqrt{r}\kappa\sqrt{T} - \mathrm{C}\log T, \text{ for all } k \in [\![1, T]\!], \text{ and} \\ X_2(2k) &\leq -A\sqrt{T} + \tfrac{1}{8}\sqrt{T} + \sqrt{r}\kappa\sqrt{T} + \mathrm{C}\log T, \text{ for all } k \in [\![1, T]\!]. \end{aligned}$$

Recall that $A = 1 + 2\sqrt{r}\kappa$ from (7.3.27). Thus for large enough T we have

$$\widetilde{\mathbf{P}}^{(0, -A\sqrt{T})}(X_1(2k-1) \wedge X_1(2k+1) \geq X_2(2k) + \tfrac{1}{2}\sqrt{T}, \text{ for all } k \in [\![1, T-1]\!]) \geq \tfrac{1}{2}\beta_1^2.$$

Following the definition of W_{cr} from (7.4.8) we thus get

$$\widetilde{\mathbf{E}}[W_{\mathrm{cr}}] \geq \tfrac{1}{2}\beta_1^2 \cdot \exp(-2(T-1)e^{-\frac{1}{2}\sqrt{T}}) \geq \tfrac{1}{2}\beta_1^2\beta_2.$$

By Observation 7.4.3, this forces

$$\mathbf{P}_{\alpha_1}^{(0,-A\sqrt{T}),(-\infty)^T;2,T}(|L_2(2)| \geq M\sqrt{T}) = \frac{\widetilde{\mathbf{E}}^{(0,-A\sqrt{T})}[W_{\mathrm{cr}}\mathbf{1}_{|X_2(2)|\geq M\sqrt{T}}]}{\widetilde{\mathbf{E}}^{(0,-A\sqrt{T})}[W_{\mathrm{cr}}]}$$
$$\leq 2\beta_1^{-2}\beta_2^{-1} \cdot \widetilde{\mathbf{P}}^{(0,-A\sqrt{T})}(|X_2(2)| \geq M\sqrt{T}).$$

Under $\widetilde{\mathbf{P}}^{(0,-A\sqrt{T})}$, $X_2(2)$ has variance $T \cdot \mathrm{Var}(V_{1,1})$ and mean $-A\sqrt{T} + (T-1)\mathbf{E}[\log U_{1,1}]$. One can thus choose M large enough so that the last term in the above equation is at most ε. Similarly one can show $\mathbf{P}_{\alpha_2}^{(0,-A\sqrt{T}),(-\infty)^T;2,T}(|L_1(1)| \geq M\sqrt{T}) \leq \varepsilon$ for all large enough M. This proves (7.3.28) for $p = 1$. For (7.3.29), observe that by Observation 7.4.3 (a) and Markov inequality one can take $\widetilde{M}$ large enough so that have

$$\mathbf{P}_{\alpha_1}^{0,(-\infty)^T;1,T}\big(|L_1(1)| \geq \widetilde{M}\sqrt{T}\big) = \mathbf{P}\big(|X_1(1)| \geq \widetilde{M}\sqrt{T}\big)$$
$$\leq \tfrac{1}{\widetilde{M}^2 T}\big(T \cdot \mathrm{Var}(V_{1,1}) + (|(T-1)\mathbf{E}[\log U_1]|)^2\big) \leq \varepsilon.$$

This completes the proof. □

Proof of Lemma 7.4.1 in the case $p = 1$. We continue with the same notations as in Observation 7.4.3. Set $S = 2T := 2\lfloor rN^{2/3}\rfloor$. Let us take $L(\cdot) = X_1(\cdot)$ where X_1 is defined in Observation 7.4.3. By Observation 7.4.3, we get that $(L(i))_{i=1}^{4T-1} \sim \mathbf{P}_{\alpha_1}^{-MN^{1/3};(-\infty)^{2T};1,2T}$. We may assume $V_{1,j}$ are defined in a probability space that includes a Brownian motion $B = B^{(1)}$ such that (7.4.10) holds. Recall that given a standard Brownian motion B and an open set $\mathcal{U} \subset C([0,1])$ with $\{f : f(0) = 0\} \subset \mathcal{U}$, we have $\mathbf{P}(B|_{[0,1]} \in \mathcal{U}) > 0$. Thus by the scale invariance of Brownian motion, there exists $\phi(\theta, \mu, r, M) > 0$ such that

$$\mathbf{P}\big(0 \leq \sigma B_x - (16M + 5\kappa r)N^{1/3} \leq MN^{1/3} \text{ for all } x \in [\tfrac{T}{2}, 2T]\big) \geq 2\phi.$$

Here $\kappa = \frac{1}{4}|\mu|\Psi'(\frac{1}{2}\theta) \geq 0$. Now for $y = -MN^{1/3}$ we have $|y + (k-1)\mathbf{E}[\log U_1]| \leq (M + \kappa r)N^{1/3}$ for all $k \leq 2T$. For large enough N we also have $\mathrm{C}\log 2T \leq MN^{1/3}$ where C comes from (7.4.10).

Thus in view of (7.4.7) and (7.4.10) we have

$$\mathbf{P}_1\Bigg[(14M+4\kappa r)N^{\frac{1}{3}} \le L(4T-1-2k) \\ \le (19M+6\kappa r)N^{\frac{1}{3}} \text{ for all } k \in [\![\tfrac{T}{2}, 2T-1]\!]\Bigg] \ge 2\phi - \tfrac{1}{2T}, \tag{7.4.12}$$

where for simplicity we use $\mathbf{P}_1 := \mathbf{P}_{\alpha_1}^{-MN^{\frac{1}{3}},(-\infty)^{2T};1,2T}$. Let us set

$$\mathsf{A} := \left\{(14M+4\kappa r)N^{\frac{1}{3}} \le L(4T-1-2k) \le (19M+6\kappa r)N^{\frac{1}{3}} \text{ for all } k \in [\![\tfrac{T}{2}, 2T-1]\!]\right\},$$
$$\mathsf{B}(k) := \left\{|L(2k-1)-L(2k)|, |L(2k+1)-L(2k)| \ge 2(5M+2\kappa r)N^{\frac{1}{3}}\right\}.$$

Recall the event $\mathsf{RP}_{1,M}$ from (7.4.1). Observe that

$$\mathsf{RP}_{1,M} \supset \mathsf{A} \cap \bigcap_{k\in[\![1,3T/2-1]\!]} \mathsf{B}(k)$$

Thus by applying the union bound we get

$$\mathbf{P}_1(\mathsf{RP}_{1,M}) \ge \mathbf{P}_1\Bigg(\mathsf{A} \cap \bigcap_{k\in[\![1,3T/2-1]\!]} \mathsf{B}(k)\Bigg) \ge \mathbf{P}_1(\mathsf{A}) - \sum_{k\in[\![1,3T/2-1]\!]} \mathbf{P}_1\Big(\mathsf{A} \cap \neg\mathsf{B}(k)\Big) \tag{7.4.13}$$

Let us denote $\mathcal{F}_{\text{odd}} := \sigma\{(L(2k-1))_{k=1}^{2T}\}$. Note that the event A is measurable with respect to $\mathcal{F}$. On the event A, $|L(2k+1)-L(2k-1)| \le (5M+2\kappa r)N^{\frac{1}{3}}$ for all $k \in [\![1, 3T/2-1]\!]$. Recall that the distribution of even points of L conditioned on $\mathcal{F}_{\text{odd}}$ are given by the ξ-distributions (see (7.4.5) and Observation 7.4.3). Applying the tail bound for ξ-distribution from Lemma 7.6.5 we have

$$\mathbf{1}_{\mathsf{A}}\mathbf{E}_1\big(\mathbf{1}_{\neg\mathsf{B}(k)} \mid \mathcal{F}_{\text{odd}}\big) \le \mathbf{1}_{\mathsf{A}} \cdot \exp\Big(-\mathrm{C}(5M+2\kappa r)N^{\frac{1}{3}}\Big),$$

for all $k \in [\![1, 3T/2-1]\!]$. Taking another expectation above and then plugging the bound back in

(7.4.13), along with the lower bound of $\mathbf{P}_1(\mathsf{A})$ from (7.4.12) we get that

$$\mathbf{P}_1(\mathsf{RP}_{1,M}) \geq 2\phi - \tfrac{1}{2T} - 3rN^{\frac{2}{3}}\exp\left(-\mathrm{C}(5M+2\kappa r)N^{\frac{1}{3}}\right).$$

Clearly for large enough N, the right side of above equation is always larger than ϕ. This completes the proof. $\square$

7.4.3 Proof of Lemma 7.4.1 in the Supercritical phase

Recall that Observation 7.4.5 establishes that the law of $\mathbf{P}_{\alpha_2}^{\vec{y},(-\infty)^S;1,S}$ is related to the law of weighted paired random walk (WPRW) defined in Definition 7.4.4. We thus first discuss few important properties of paired random walk and weighted paired random walk before going into the proof of Lemmas 7.3.8 and 7.4.1 in the supercritical phase.

Basic properties for paired random walk

In this subsection we study the law of paired random walk defined in Definition 7.4.4. We will work with PRWs whose increments are given by $f := G_{\theta,+1} * G_{\theta,-1}$ and $g := G_{\alpha_2}$. We record some of its key properties below.

Lemma 7.4.6 (Properties of the increments)**.** *f and g enjoy the following properties.*

1. *The density f is symmetric.*

2. *Let ψ denote the characteristic function corresponding to f. Given any $\delta > 0$, there exists η such that* $\sup_{t\geq\delta}|\psi(t)| = \eta < 1$.

3. *For any $a < b$,* $\inf_{x\in[a,b]} f(x) > 0$ *and* $\inf_{x\in[a,b]} g(x) > 0$.

4. *There exists a constant* $\mathrm{C} > 0$ *such that* $f(x) \leq \mathrm{C}e^{-|x|/\mathrm{C}}$ *and* $g(x) \leq \mathrm{C}e^{-|x|/\mathrm{C}}$. *In particular, this implies that if $X \sim f$ and $Y \sim g$, there exists $v > 0$ such that and*

$$\sup_{|t|\leq v}\left[\mathbf{E}[e^{tX}] + \mathbf{E}[e^{tY}]\right] < \infty.$$

In other words X and Y are subexponential random variables.

For the characteristic function from [2, Formula 6.1.25] one has

$$\psi(t) = \left|\frac{\Gamma(\theta + it)}{\Gamma(\theta)}\right|^2 = \prod_{n=0}^{\infty}\left[1 + \frac{t^2}{(\theta+n)^2}\right]^{-1}.$$

From here, one can verify part (2) of the above lemma. Remaining parts of the lemma are all standard to check and hence its proof is skipped. For the rest of this section, we reserve the notation f and g for $G_{\theta,+1} * G_{\theta,-1}$ and G_{α_2} respectively.

Fix any $M > 0$, $n \geq 1$, and consider $x_n, y_n \in \mathbb{R}$ with $|x_n|, |y_n| \leq M$. Suppose

$$(S_k^{(n,i)})_{k\in[\![0,n]\!],i=1,2} \sim \mathbf{P}^{n;(\sqrt{n}x_n,\sqrt{n}y_n);f,g}$$

be a PRW. Let f_n be the density of $\frac{X_1+\cdots+X_n}{\sqrt{n}}$ where X_i's are i.i.d. drawn from f. Assume $U_n, V_n \overset{i.i.d.}{\sim} f_n$. Observe that any event based on $(S_0^{(n,1)}, S_0^{(n,2)})$ can be written as

$$\mathbf{P}(S_0^{(n,1)}, S_0^{(n,2)} \in A) = \frac{\mathbf{E}[g(\sqrt{n}(V_n - U_n - x_n + y_n))\mathbf{1}_{(U_n+x_n,V_n+y_n)\in n^{-1/2}A}]}{\mathbf{E}[g(\sqrt{n}(V_n - U_n - x_n + y_n))]}. \tag{7.4.14}$$

The above formula is the guiding principle for extracting tail estimates of various kinds of functions of $(S_0^{(n,1)}, S_0^{(n,2)})$. We list few of them that are indispensable for our later analysis.

Lemma 7.4.7 (Tail estimates for the Entrance Law)**.** *Fix two open intervals $I_1, I_2 > 0$. Under the above setup, there exists a constant* $\mathrm{C} = \mathrm{C}(M) > 1$ *such that for all $n \geq 1$ and $\tau \geq 1$, we have*

$$\mathbf{P}(|S_0^{(n,1)}| \geq \tau\sqrt{n}) \leq \mathrm{C}e^{-\frac{1}{\mathrm{C}}\tau}, \tag{7.4.15}$$

$$\mathbf{P}(|S_0^{(n,1)} - S_0^{(n,2)}| \geq \tau) \leq \mathrm{C}e^{-\frac{1}{\mathrm{C}}\tau}, \tag{7.4.16}$$

$$\mathbf{P}(S_0^{(n,1)} - S_0^{(n,2)} \in I_1, S_0^{(n,1)} \in \sqrt{n}I_2) \geq \tfrac{1}{\mathrm{C}}. \tag{7.4.17}$$

Proof of Lemma 7.4.7. For simplicity let us write $z_n := x_n - y_n$. It is enough to prove the Lemma 7.4.7 for large enough n. So, throughout the proof we will assume n is large enough. We first claim that the denominator of the r.h.s. of (7.4.14), $\mathbf{E}[g(\sqrt{n}(V_n - U_n - x_n + y_n))]$, is of the order $n^{-1/2}$. To show this, we rely on certain Gaussian approximation from Lemma 7.6.3. Indeed, the estimate from Lemma 7.6.3 ensures that given any interval $B := [\frac{p}{\sqrt{n}}, \frac{p+1}{\sqrt{n}}] \subset [-2, 2]$, for all large enough n, we have $\sqrt{n}\mathbf{P}((V_n - U_n - x_n + y_n) \in B) \in [R^{-1}, R]$ for some $R > 1$ depending only on M. Thus, using the exponential tails of g we have

$$\begin{aligned}\mathbf{E}[g(\sqrt{n}(V_n - U_n - z_n))] &\le \mathrm{C}e^{-\frac{1}{\mathrm{C}}\sqrt{n}} + \sum_{p=0}^{\sqrt{n}} \mathrm{C}e^{-\frac{1}{\mathrm{C}}p}\mathbf{P}\big(\tfrac{p}{\sqrt{n}} \le |V_n - U_n - z_n| \le \tfrac{p+1}{\sqrt{n}}\big) \\ &\le \mathrm{C}e^{-\frac{1}{\mathrm{C}}\sqrt{n}} + \frac{R}{\sqrt{n}}\sum_{p=0}^{\sqrt{n}} \mathrm{C}e^{-\frac{1}{\mathrm{C}}p} \le \tfrac{\mathrm{C}_1}{\sqrt{n}}.\end{aligned} \tag{7.4.18}$$

On the other hand,

$$\begin{aligned}\mathbf{E}[g(\sqrt{n}(V_n - U_n - z_n))] &\ge \mathbf{E}[g(\sqrt{n}(V_n - U_n - z_n))\mathbf{1}_{V_n - U_n - x_n + y_n \in (\frac{1}{\sqrt{n}}, \frac{2}{\sqrt{n}})}] \\ &\ge \mathbf{P}\big(V_n - U_n - z_n \in (\tfrac{1}{\sqrt{n}}, \tfrac{2}{\sqrt{n}})\big) \cdot \inf_{x \in [1,2]} g(x) \\ &\ge \tfrac{R^{-1}}{\sqrt{n}} \cdot \inf_{x \in [1,2]} g(x),\end{aligned} \tag{7.4.19}$$

which is bounded below by $c'/\sqrt{n}$, by the property of g from Lemma 7.4.6. This proves the $n^{-1/2}$ order of the $\mathbf{E}[g(\sqrt{n}(V_n - U_n - z_n))]$.

Let us now prove the inequalities in Lemma 7.4.7 one by one. Clearly (7.4.18) can be modified to show $\mathbf{E}[g(\sqrt{n}(V_n - U_n - z_n))\mathbf{1}_{|\sqrt{n}(V_n - U_n - z_n)| \ge \tau}] \le \frac{\mathrm{C}}{\sqrt{n}}e^{-\frac{1}{\mathrm{C}}\tau}$. In view of (7.4.14) and (7.4.19), this leads to (7.4.16). For (7.4.17) notice that due to (7.4.14) and (7.4.19) we have

$$\begin{aligned}&\mathbf{P}(S_0^{(n,1)} - S_0^{(n,2)} \in I_1, S_0^{(n,1)} \in \sqrt{n}I_2) \\ &\ge \mathrm{C}_1^{-1}\sqrt{n} \cdot \inf_{x \in I_1} g(-x) \cdot \mathbf{P}(U_n + x_n \in \sqrt{n}I_2, U_n + x_n - V_n - y_n \in n^{-1/2}I_1).\end{aligned}$$

Using Lemma 7.6.3, the probability above can be shown lower bounded by $\frac{\mathrm{C}_2^{-1}}{\sqrt{n}}$ for some C_2 depending on M, I_1, I_2 but free of n. This proves (7.4.17). Finally, for (7.4.15) we observe

$$\mathbf{E}[g(\sqrt{n}(V_n - U_n - z_n))\mathbf{1}_{|U_n|\ge\tau}] \le \mathrm{C}e^{-\frac{\sqrt{n}}{\mathrm{C}}} + \sum_{p=1}^{\sqrt{n}} \mathrm{C}e^{-\frac{p}{\mathrm{C}}}\mathbf{P}\big(\tfrac{p}{\sqrt{n}} \le |V_n - U_n - z_n| \le \tfrac{p+1}{\sqrt{n}}, |U_n| \ge \tau\big).$$

We focus on the estimation of the probability that appears on r.h.s. of the above equation. We have

$$\begin{aligned}\mathbf{P}(\tfrac{p}{\sqrt{n}} \le |V_n - U_n - z_n| \le \tfrac{p+1}{\sqrt{n}}, |U_n| \ge \tau) \le \mathbf{E}\left[\mathbf{1}_{\tau\le|U_n|\le(\log n)^{3/2}}\mathbf{P}\left(\tfrac{p}{\sqrt{n}} \le |V_n - U_n - z_n| \le \tfrac{p+1}{\sqrt{n}} \mid U_n\right)\right] \\ + \mathbf{P}(|U_n| \ge (\log n)^{3/2}).\end{aligned}$$

By Lemma 7.6.3, the conditional probability above can be uniformly bounded above by $\frac{\mathrm{C}_3}{\sqrt{n}}$ for some C_3 free of p and n. Tail estimates of U_n, which follows from sub-exponential property of f (see Theorem 2.8.1 from [314]), show that the r.h.s. of the above equation is at most $\frac{\mathrm{C}}{\sqrt{n}}e^{-\frac{1}{\mathrm{C}}\tau}$. Plugging all the estimates back we get

$$\mathbf{E}\left[g(\sqrt{n}(V_n - U_n - z_n))\mathbf{1}_{|U_n|\ge\tau}\right] \le \tfrac{\mathrm{C}}{\sqrt{n}}e^{-\frac{1}{\mathrm{C}}\tau}.$$

Using the lower bound for the denominator from (7.4.19), in view of (7.4.14), we get (7.4.15). This completes the proof. □

In order to deal with WPRW law, the weighted version of the PRW law (see Definition 7.4.4), we next analyze W_{sc} weight defined in (7.4.9). We record a convenient lower bound for W_{sc} that will be useful in our later analysis. Fix any $p, q \ge 1$ with $p + q \le n - 1$. Given any $\beta > 0$, we consider several 'Gap' events:

$$\begin{aligned}\mathsf{Gap}_{1,\beta} &:= \{S_k^{(n,1)} - S_k^{(n,2)} \ge \beta k^{1/4} \text{ for all } k \in [\![1, p]\!]\}, \\ \mathsf{Gap}_{2,\beta} &:= \{S_k^{(n,1)} - S_k^{(n,2)} \ge \beta (n-k)^{1/4} \text{ for all } k \in [\![n-q, n-1]\!]\}, \\ \mathsf{Gap}_{3,\beta} &:= \{S_k^{(n,1)} - S_k^{(n,2)} \ge n^{1/4} \text{ for all } k \in [\![p+1, n-q]\!]\},\end{aligned}$$

$$\mathsf{Gap}_{4,\beta} := \{S_k^{(n,1)} - S_{k-1}^{(n,1)} \le \beta^{-1} k^{1/8} \text{ for all } k \in [\![1,p]\!]\},$$

$$\mathsf{Gap}_{5,\beta} := \{S_k^{(n,1)} - S_{k-1}^{(n,1)} \le \beta^{-1}(n-k+1)^{1/8} \text{ for all } k \in [\![n-q+1,n]\!]\},$$

$$\mathsf{Gap}_{6,\beta} := \{|S_k^{(n,1)} - S_{k-1}^{(n,1)}| \le \beta^{-1}(\log n) \text{ for all } k \in [\![p+1,n-q]\!]\}.$$

$\mathsf{Gap}_{1,\beta}, \mathsf{Gap}_{2,\beta}$, and $\mathsf{Gap}_{3,\beta}$ requires the first walk of the PRW, $S_k^{(n,1)}$, to be bigger than a threshold plus the second walk of the PRW, $S_k^{(n,2)}$ pointwise in the left ($[\![1,p]\!]$), right ($[\![n-q,n-1]\!]$), and middle ($[\![p+1,n-q]\!]$) region respectively. The type of threshold depends on the region. $\mathsf{Gap}_{4,\beta}, \mathsf{Gap}_{5,\beta}$, and $\mathsf{Gap}_{6,\beta}$ controls the increments of first walk of the PRW. Set

$$\mathsf{Gap}_{\beta} := \bigcap_{i=1}^{6} \mathsf{Gap}_{i,\beta}. \tag{7.4.20}$$

We have the following deterministic inequality for W_{sc}.

Lemma 7.4.8. *Recall W_{sc} from* (7.4.9)*. Given any $\beta > 0$, there exists $a_\beta > 0$ such that for all $n \ge 1$,*

$$W_{\mathrm{sc}} \ge a_\beta \cdot \mathbf{1}_{\mathsf{Gap}_\beta \cap \{|S_0^{(n,1)} - S_0^{(n,2)}| \le \beta^{-1}\}}.$$

where W_{sc} is defined in (7.4.9)*.*

Proof. Assume Gap_β holds. For $k \in [\![1,n-1]\!]$ we have

$$S_k^{(n,2)} - S_k^{(n,1)} \le -\min(\beta k^{1/4}, \beta(n-k)^{1/4}, n^{1/4}) =: \tau_n^{(1)}(k).$$

Clearly $\sum_{k=1}^{n-1} e^{S_k^{(n,2)} - S_k^{(n,1)}} \le \sum_{k=1}^{n-1} \exp(\tau_n^{(1)}(k))$ is uniformly bounded in n and hence can be bounded by some constant $T_1(\beta) \in (0,\infty)$. Similarly for $k \in [\![1,n-1]\!]$ we have

$$\begin{aligned} S_k^{(n,2)} - S_{k+1}^{(n,1)} &= S_k^{(n,2)} - S_k^{(n,1)} + S_k^{(n,1)} - S_{k+1}^{(n,1)} \\ &\le -\min(\beta k^{1/4}, \beta(n-k)^{1/4}, n^{1/4}) + \beta^{-1}\max((k+1)^{1/8}, (n-k)^{1/8}, (\log n)) \\ &=: \tau_n^{(2)}(k). \end{aligned}$$

Clearly $\sum_{k=1}^{n-1} e^{S_k^{(n,2)}-S_{k+1}^{(n,1)}} \le \sum_{k=1}^{n-1} \exp(\tau_n^{(2)}(k))$ is uniformly bounded in n and hence can be bounded by some constant $T_2(\beta) \in (0,\infty)$. Thus from the definition of W_{sc} in (7.4.9) we have

$$W_{\mathrm{sc}} \ge \mathbf{1}_{\mathsf{Gap}_\beta \cap \{|S_0^{(n,1)}-S_0^{(n,2)}| \le \beta^{-1}\}} \cdot \exp(-e^{2\beta^{-1}} - T_1(\beta) - T_2(\beta)).$$

Taking $a_\beta := \exp(-e^{2\beta^{-1}} - T_1(\beta) - T_2(\beta))$ completes the proof. □

We end this section by recording a technical lemma that allows us to compare random bridges with *modified random bridges*. In what follows, we use the notation $\mathbf{P}^n_{a\to b}$ to denote the law of a n-step random bridge starting at a and ending at b with increments drawn from f.

Definition 7.4.9 ($(n;p,q)$-modified random bridge)**.** Fix $n \ge 1$, and $p,q \in [\![0,n]\!]$ with $p+q \le n$. Take any $a,b \in \mathbb{R}$. Let $X_i, Y_i \overset{i.i.d.}{\sim} f$. Set $S_0^{(n)} := a$ and $S_n^{(n)} := b$. For $k \in [\![1,p]\!]$, set $S_k^{(n)} := a + \sum_{j=1}^k X_j$, and for $k \in [\![1,q]\!]$, $S_{n-k}^{(n)} = b - \sum_{j=1}^k Y_j$. Conditioned on $(S_k^{(n)})_{k\in[\![1,p]\!]}$ and $(S_{n-k}^{(n)})_{k\in[\![1,q]\!]}$, set $(S_k^{(n)})_{k=p}^{n-q} \sim \mathbf{P}^{n-p-q}_{\widetilde{a}\to\widetilde{b}}$ where $\widetilde{a} := S_p^{(n)}$ and $\widetilde{b} := S_{n-q}^{(n)}$. We call the $(S_k^{(n)})_{k\in[\![0,n]\!]}$ as $(n;p,q)$-modified random bridge of length n starting at a and ending at b.

The usual random bridge from a to b, is a random walk of length n started from a conditioned to end at b. In case of the modified random bridge, we start two random walks of length p and q from a and b respectively where the second one is viewed in reverse direction. Conditioned on these two walks, we connect their endpoints by a random bridge of length $n-p-q$. See Figure 7.17.

The laws of random bridge and modified random bridge can be compared with the help of the following lemma.

Lemma 7.4.10 (Comparison Lemma)**.** *Fix any $M > 0$ and $\delta \in (0,1/2)$, and $n \ge 1$. Set $p = \lfloor n\delta \rfloor$ and $q = \lfloor n - n\delta \rfloor$. Suppose $a,b \in \mathbb{R}$ with $|a-b| \le M\sqrt{n}$. Let $V(\vec{x})$ be the joint density of $(S_k^{(n)})_{k\in[\![0,n]\!]}$ where $(S_k^{(n)})_{k\in[\![0,n]\!]} \sim \mathbf{P}^n_{a\to b}$. For all $\vec{x} \in \mathbb{R}^{n-1}$ we have*

$$V(\vec{x}) \lesssim_{M,\delta} \widetilde{V}(\vec{x}). \tag{7.4.21}$$

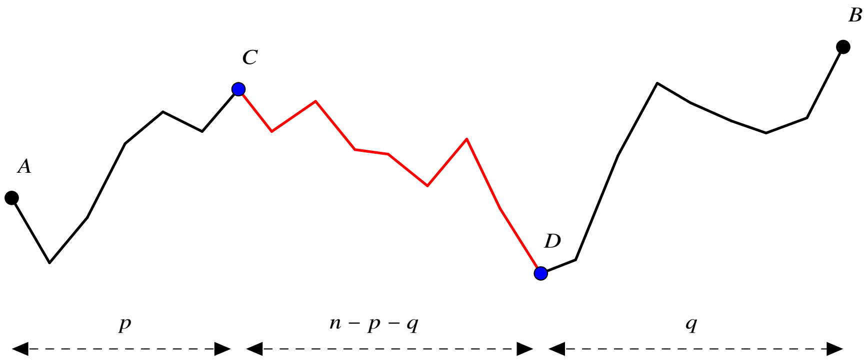


Figure 7.17: Modified random bridge. We start random walks of length p and q from A and B with the second one viewed in the reverse direction. From their endpoints C and D we then consider a random bridge of length $n-p-q$.

where $\widetilde{V}(\vec{x})$ is the joint density of the $(n;p,q)$-modified random bridge starting at a and ending at b. Furthermore, whenever $|x_p - x_{n-q}| \le \widetilde{M}\sqrt{n}$ we also have

$$V(\vec{x}) \gtrsim_{\widetilde{M},M,\delta} \widetilde{V}(\vec{x}). \tag{7.4.22}$$

Proof. We have

$$V(\vec{x}) := \frac{\prod_{j=0}^{n-1} f(x_{j+1}-x_j)}{f^{*n}(b-a)}, \quad \widetilde{V}(\vec{x}) := \frac{\prod_{j=0}^{n-1} f(x_{j+1}-x_j)}{f^{*(n-2\lfloor n\delta\rfloor)}(x_{\lfloor n\delta\rfloor} - x_{n-\lfloor n\delta\rfloor})}.$$

where $x_0 := a$ and $x_n := b$. By [163, Theorem 2, Chapter XV.5]

$$\sup_{z\in\mathbb{R}} \left| \sqrt{k} f^{*k}(z) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2k\sigma^2}} \right| \xrightarrow{k\to\infty} 0.$$

The above fact yields the desired result. □

Proof of Lemma 7.4.1 : $p = 2$ case

For clarity we split the proof into several steps.

Step 1. In this step, we reduce our work in showing (7.4.23). Fix $r > 0$. Set $T := \lfloor rN^{2/3} \rfloor$ and $n = 2T$. Recall y_i's and the event $\mathsf{RP}_{2,M}$ from the statement of the lemma. Observe that the lemma is clearly true for all small values of N. Hence it suffices to show (7.4.2) under $\liminf_{N\to\infty}$. Since $\mathsf{RP}_{2,M}$ is a monotone event, by Proposition 8.2.3 we have

$$\mathbf{P}_{\alpha_2}^{\vec{y},(-\infty)^{2T};2,2T}(\mathsf{RP}_{2,M}) \geq \mathbf{P}_{\alpha_2}^{\vec{x},(-\infty)^{2T};2,2T}(\mathsf{RP}_{2,M}).$$

where $x_1 = -2MN^{1/3}$, $x_2 = -2MN^{1/3} - \sqrt{n}$. By translation, it thus suffices to show that there exists $\phi = \phi(r, M) > 0$ such that

$$\liminf_{n\to\infty} \mathbf{P}_{\alpha_2}^{(0,-\sqrt{n}),(-\infty)^n;2,n}\left(\inf_{i\in[\![1,n]\!]} L_2(i) \geq 8Mr^{-1/2}\sqrt{n}\right) \geq \phi.$$

Towards this end we claim that there exists $\phi = \phi(r, M) > 0$ such that

$$\liminf_{n\to\infty} \mathbf{P}_{\alpha_2}^{(0,-\sqrt{n}),(-\infty)^{2n};2,n}(\mathsf{D}_m) \geq 2\phi, \tag{7.4.23}$$

where

$$\mathsf{D}_m := \left\{(L_1(2i-1), L_2(2i)) \in (10m\sqrt{n}, 11m\sqrt{n})^2 \text{ for all } i \in [\![1, n/2]\!]\right\},$$

and $m := Mr^{-1/2}$. Let us complete the proof assuming (7.4.23). Note that (7.4.23) controls the even points of the second curve. By Observation 7.4.5, we know conditioned on the even points, $L_2(2k+1) \sim \xi_{\theta,\theta;-1}^{L_2(2k),L_2(2k+2)}$ for $k = 1, 2, \ldots, 2n-1$. In view of Lemma 7.6.5, on the event D_m we have

$$\mathbf{E}[\mathbf{1}_{L_2(2k+1)\leq 8m\sqrt{n}} \mid \sigma(L_2(2k), L_2(2k+2))] \leq \mathrm{C}e^{-\frac{1}{\mathrm{C}}m\sqrt{n}}.$$

By Observation 7.4.5, $L_2(1) \sim G_{\alpha_2+\theta,1} + L_2(2)$. Thus by tail estimates for $G_{\alpha_2+\theta,1}$, we see that on the event D_m we have $\mathbf{P}(L_2(1) \leq 8m\sqrt{n} \mid L_2(2)) \leq \mathrm{C}e^{-\frac{1}{\mathrm{C}}m\sqrt{n}}$. Thus by union bound we have

$$\mathbf{P}_{\alpha_2}^{(0,-\sqrt{n}),(-\infty)^{2n};2,n}\left(\inf_{i\in[\![1,n]\!]} L_2(i) \geq 8m\sqrt{n}\right) \geq \mathbf{P}_{\alpha_2}^{(0,-\sqrt{n}),(-\infty)^{2n};2,n}(\mathsf{D}_m) - \mathrm{C}\cdot ne^{-\frac{1}{\mathrm{C}}m\sqrt{n}} \geq \phi,$$

for large enough n. This establishes Lemma 7.4.1 for $p = 2$ modulo (7.4.23).

Step 2. In this and subsequent steps we prove (7.4.23). Recall the PRW and WPRW laws from Definition 7.4.4. Recall from Observation 7.4.5 that $(L_1(2i-1), L_2(2i))_{i\in[\![1,n]\!]} \sim \mathbf{P}_{W_{\rm sc}}^{n;(0,-\sqrt{n});f,g}$ with $f := G_{\theta,1} * G_{\theta,-1}$, and $g := G_{\alpha+\theta}$. Let us take $(S_k^{(n,i)})_{k\in[\![0,n]\!],i=1,2} \sim \mathbf{P}^{n;(0,-\sqrt{n});f,g}$. We can write

$$\mathbf{P}_{W_{\rm sc}}^{n;(0,-\sqrt{n});f,g}(\mathsf{D}_m) = \frac{\mathbf{E}[W_{\rm sc}\mathbf{1}_{\mathsf{D}_m}]}{\mathbf{E}[W_{\rm sc}]} \tag{7.4.24}$$

where $W_{\rm sc}$ is defined in (7.4.9) and D_m is interpreted as

$$\mathsf{D}_m := \{(S_k^{(n,1)}, S_k^{(n,2)}) \in (10m\sqrt{n}, 11m\sqrt{n})^2 \text{ for all } k \in [\![1, n/2]\!]\}.$$

We will now provide appropriate lower and upper bounds for the numerator and denominator of r.h.s. of (7.4.24) respectively. For the upper bound we use the following general lemma.

Lemma 7.4.11. *There exists an absolute constant* $\mathrm{C} > 0$ *such that for all Borel sets* $A \subset \mathbb{R}^2$ *we have*

$$\mathbf{E}\left[W_{\rm sc}\mathbf{1}_{(S_0^{(n,1)},S_0^{(n,2)})\in A}\right] \le \frac{\mathrm{C}}{n} + \frac{\mathrm{C}}{\sqrt{n}}\mathbf{E}\left[\mathbf{1}_{(S_0^{(n,1)},S_0^{(n,2)})\in A}\left[(S_0^{(n,1)} - S_0^{(n,2)} + 1) \vee 1\right]\left[\tfrac{|S_0^{(n,1)}|}{\sqrt{n}} \vee 2\right]^3\right].$$

Proof of Lemma 7.4.11. Set $\mathsf{A} := \{(S_0^{(n,1)}, S_0^{(n,2)}) \in A\}$ and define

$$\mathsf{NI}_p := \left\{S_k^{(n,1)} - S_k^{(n,2)} \ge -p, \text{ for all } k \in [\![0, n]\!]\right\}. \tag{7.4.25}$$

We set $\mathsf{NI} := \mathsf{NI}_0$. Here NI stands for non-intersection. Observe that

$$\begin{aligned}\mathbf{E}[W_{\rm sc}\mathbf{1}_{\mathsf{A}}] &= \mathbf{E}[W_{\rm sc}\mathbf{1}\{\mathsf{A}\cap\mathsf{NI}_{\log\log n}^c\}] + \sum_{p=0}^{\log\log n-1}\mathbf{E}[W_{\rm sc}\mathbf{1}\{\mathsf{A}\cap\mathsf{NI}_{p+1}\cap\mathsf{NI}_p^c\}] \\ &\le \tfrac{1}{n} + \sum_{p=0}^{\log\log n}\exp(-e^p)\mathbf{P}\left(\mathsf{A}\cap\mathsf{NI}_{p+1}\right) \le \tfrac{1}{n} + \sum_{p=0}^{\log\log n}\exp(-e^p)e^{\mathrm{C}p}\cdot\mathbf{P}(\mathsf{A}\cap\mathsf{NI}).\end{aligned}$$

The first inequality above follows by noting that on NI_p^c we have $W \le \exp(-e^p)$. The second inequality is due to Lemma 7.7.5. Thus to suffices to bound $\mathbf{P}(\mathsf{A}\cap\mathsf{NI})$. Towards this end, we first define the event

$$\mathsf{B} := \left\{|S_0^{(n,i)}| \le (\log n)^{3/2}\sqrt{n} \text{ for } i=1,2, |S_0^{(n,1)} - S_0^{(n,2)}| \le (\log n)^{3/2}\right\}.$$

By union bound we have $\mathbf{P}(\mathsf{A}\cap\mathsf{NI}) \le \mathbf{P}(\mathsf{A}\cap\mathsf{B}\cap\mathsf{NI}) + \mathbf{P}(\mathsf{B}^c)$. For the second term note that by tail estimates from Lemma 7.4.7 we have

$$\mathbf{P}(\mathsf{B}^c) \le \sum_{i=1}^{2} \mathbf{P}(|S_0^{(n,i)}| \ge (\log n)^{3/2}) + \mathbf{P}\left(|S_0^{(n,1)} - S_0^{(n,2)}| \ge (\log n)^{3/2}\right) \le \tfrac{\mathrm{C}}{n}.$$

For the first term we condition on $\mathcal{F} := \sigma(S_0^{(n,1)}, S_0^{(n,2)})$ to get

$$\mathbf{P}(\mathsf{A}\cap\mathsf{B}\cap\mathsf{NI}) = \mathbf{E}\left[\mathbf{1}\{\mathsf{A}\cap\mathsf{B}\}\mathbf{E}[\mathbf{1}_{\mathsf{NI}} \mid \mathcal{F}]\right].$$

Now by Lemma 7.7.6, uniformly on the event B we have

$$\mathbf{E}[\mathbf{1}_{\mathsf{NI}} \mid \mathcal{F}] \le \tfrac{\mathrm{C}}{\sqrt{n}}\left[(S_0^{(n,1)} - S_0^{(n,2)} + 1) \vee 1\right]\left[\tfrac{|S_0^{(n,1)}|}{\sqrt{n}} \vee 2\right]^3$$

All the above estimates together establishes the lemma. $\square$

Note that taking $A = \mathbb{R}^2$, and utilizing the exponential tail estimates from Lemma 7.4.7 it follows that

$$\mathbf{E}[W_{\mathrm{sc}}] \le \tfrac{\mathrm{C}}{\sqrt{n}}. \tag{7.4.26}$$

This provides an upper bound for the denominator of r.h.s. of (7.4.24).

Step 3. In this step we prove an appropriate lower bound of the numerator in (7.4.24). Towards this end, consider the event

$$\mathsf{E}_m := \{1 \le S_0^{(n,1)} - S_0^{(n,2)} \le 2, S_0^{(n,1)}, S_0^{(n,2)} \in (\tfrac{41}{4} m\sqrt{n}, \tfrac{43}{4} m\sqrt{n})\},$$

and the σ-field $\mathcal{F} := \sigma(S_0^{(n,1)}, S_0^{(n,2)})$. Fix any $\beta > 0$ and consider the Gap_β event from (7.4.20). We have

$$\mathbf{E}[W_{\mathrm{sc}} \mathbf{1}_{\mathsf{D}_m}] \ge \mathbf{E}[W_{\mathrm{sc}} \mathbf{1}_{\mathsf{E}_m} \mathbf{1}_{\mathsf{D}_m} \mathbf{1}_{\mathsf{Gap}_\beta}] \ge a_\beta \mathbf{E}[\mathbf{1}_{\mathsf{E}_m} \mathbf{E}[\mathbf{1}_{\mathsf{D}_m \cap \mathsf{Gap}_\beta} \mid \mathcal{F}]] \tag{7.4.27}$$

where the second inequality above follows by noting that $W \ge a_\beta$ on $\mathsf{Gap}_\beta \cap \mathsf{E}$ (Lemma 7.4.8). Note that $\mathbf{E}[\mathbf{1}. \mid \mathcal{F}]$ is the law of two independent random bridges starting at $(S_0^{(n,1)}, S_0^{(n,2)})$ and ending at $(0, -\sqrt{n})$ with increments drawn from f. For simplicity set $b_1 = 0$, $b_2 = -\sqrt{n}$. Set $\rho := \rho(\frac{1}{16}, \frac{m\wedge 1}{8})$ from Corollary 7.7.3. By Lemma 7.4.10

$$\begin{aligned} \mathbf{1}_{\mathsf{E}_m} \cdot \mathbf{E}[\mathbf{1}_{\mathsf{D}_m \cap \mathsf{Gap}_\beta} \mid \mathcal{F}] &\gtrsim_m \mathbf{1}_{\mathsf{E}_m} \cdot \widetilde{\mathbf{P}}_{(S_0^{(n,1)}, S_0^{(n,2)})}(\mathsf{D}_m \cap \mathsf{Gap}_\beta) \\ &= \mathbf{1}_{\mathsf{E}_m} \cdot \widetilde{\mathbf{P}}_{(S_0^{(n,1)}, S_0^{(n,2)})}(\mathsf{D}_m \cap \mathsf{Gap}_\beta \mid \mathsf{NI}) \widetilde{\mathbf{P}}_{(S_0^{(n,1)}, S_0^{(n,2)})}(\mathsf{NI}) \end{aligned} \tag{7.4.28}$$

where $\widetilde{\mathbf{P}}_{(S_0^{(n,1)}, S_0^{(n,2)})}$ is the joint law of two independent $(n; n\rho, 0)$-modified random bridge from $S_0^{(n,i)}$ to b_i defined in Definition 7.4.9. Let us consider

$$\mathcal{P}_1 := \left\{(z_1, z_2) \in (\tfrac{41}{4} m\sqrt{n}, \tfrac{43}{4}\sqrt{n}) : 1 \le z_1 - z_2 \le 2\right\}$$

We now claim that there exists $\widetilde{\phi} = \widetilde{\phi}(m) > 0$ such that

$$\widetilde{\mathbf{P}}_{(a_1, a_2)}(\mathsf{D}_m \mid \mathsf{NI}) \ge 2\widetilde{\phi}. \tag{7.4.29}$$

uniformly over all $(a_1, a_2) \in \mathcal{P}_1$. We postpone its proof to next step. Let us complete the proof of the lemma assuming it. Note that by Lemma 7.7.1 and Lemma 7.7.2 we can get constants

$\delta > 0$, $M_2 > 0$ and $\mathrm{C}_1 > 0$ all depending on m such that uniformly over $(a_1, a_2) \in \mathcal{P}_1$ we have

$$\widetilde{\mathbf{P}}_{(a_1,a_2)}\big(S_k^{(n,1)} \ge S_k^{(n,2)} \text{ for all } k \in [\![1, n\rho]\!], S_{n\rho}^{(n,1)} - S_{n\rho}^{(n,2)} \ge \delta\sqrt{n}, |S_{n\rho}^{(n,i)}| \le M_2\sqrt{n}\big) \ge \tfrac{\mathrm{C}_1^{-1}}{\sqrt{n}}.$$

Set $\mathsf{G} := S_{n\rho}^{(n,1)} - S_{n\rho}^{(n,2)} \ge \delta\sqrt{n}, |S_{n\rho}^{(n,i)}| \le M_2\sqrt{n}$. Recall from the definition of $(n; n\rho, 0)$-modified random bridge that on $[\![n\rho, n]\!]$ the modified random bridge is just a random bridge from $S_{n\rho}^{(n,i)}$ to b_i. Applying Lemma 7.7.4 it follows that

$$\mathbf{1}_{\mathsf{G}} \cdot \widetilde{\mathbf{P}}_{(a_1,a_2)}\big(S_k^{(n,1)} \ge S_k^{(n,2)} \text{ for all } k \in [\![n\rho, n]\!]\big) \ge \mathbf{1}_{\mathsf{G}} \cdot \mathrm{C}_2^{-1},$$

for some constant $\mathrm{C}_2 > 0$ depending on m only. Thus we get $\widetilde{\mathbf{P}}(\mathsf{NI}) \ge \frac{\mathrm{C}_4^{-1}}{\sqrt{n}}$ uniformly on E_m for some deterministic constant C_4 depending on m only. By Lemma 7.7.8, we may choose β small enough depending on m such that $\widetilde{\mathbf{P}}_{(a_1,a_2)}(\mathsf{Gap}_\beta \mid \mathsf{NI}) \ge 1 - \widetilde{\phi}$ uniformly over $(a_1, a_2) \in \mathcal{P}_1$. Plugging this estimates back in (7.4.28), we see that $\mathbf{1}_{\mathsf{E}_m} \cdot \mathbf{E}[\mathbf{1}_{\mathsf{D}_m \cap \mathsf{Gap}_\beta} \mid \mathcal{F}] \ge \mathbf{1}_{\mathsf{E}_m} \widetilde{\phi} \frac{\mathrm{C}_4^{-1}}{\sqrt{n}}$. Now, by Lemma 7.4.7 ((7.4.17) in particular) we know that $\mathbf{P}(\mathsf{E}_m) \ge \mathrm{C}_5^{-1} > 0$ for some C_5 depending on m. Plugging this back in (7.4.27) we see that

$$\mathbf{E}[W_{\mathrm{sc}} \mathbf{1}_{\mathsf{D}_m}] \ge a_\beta \cdot \mathbf{P}(\mathsf{E}_m) \cdot \widetilde{\phi} \cdot \tfrac{\mathrm{C}_4^{-1}}{\sqrt{n}} =: \tfrac{\widetilde{\mathrm{C}}^{-1}}{\sqrt{n}}. \tag{7.4.30}$$

where $\widetilde{\mathrm{C}} > 0$ is a constant depending only on m. In view of the formula (7.4.24) and the upper bound from (7.4.26), setting $\phi := \frac{1}{2}\mathrm{C}^{-1} \cdot \widetilde{\mathrm{C}}^{-1}$, we thus arrive at (7.4.23).

Step 4. In this step we prove (7.4.29). By Lemma 7.7.2 ((7.7.1) in particular), we know there exists $\delta \in (0, \frac{1}{8}(m \wedge 1))$ small enough depending only on ρ such that $\widetilde{\mathbf{P}}_{(a_1,a_2)}(S_{n\rho}^{(n,1)} - S_{n\rho}^{(n,2)} \ge \delta\sqrt{n} \mid \mathsf{NI}) \ge \frac{15}{16}$ uniformly over $(a_1, a_2) \in \mathcal{P}_1$. Recall that $\rho = \rho(\frac{1}{16}, \frac{m\wedge 1}{8})$ comes from Corollary 7.7.3. In view of this choice of ρ from Corollary 7.7.3, uniformly over $(a_1, a_2) \in \mathcal{P}_1$ we have

$$\widetilde{\mathbf{P}}_{(a_1,a_2)}\left(\sup_{k \in [\![0, n\rho]\!]} |S_k^{(n,i)} - S_0^{(n,i)}| \le \tfrac{m\wedge 1}{8}\sqrt{n} \mid \mathsf{NI}\right) \ge \tfrac{15}{16}.$$

Since on $\mathcal{P}_1$ we also have $S_0^{(n,1)}, S_0^{(n,2)} \in (\frac{41}{4}m\sqrt{n}, \frac{43}{4}m\sqrt{n})$, combining the last two probability estimates we get

$$\widetilde{\mathbf{P}}_{(a_1,a_2)}\left(\left\{S_{n\rho}^{(n,1)} - S_{n\rho}^{(n,2)} \geq \delta\sqrt{n}\right\} \cap \mathsf{K}_1 \mid \mathsf{NI}\right) \geq \tfrac{7}{8},$$

where

$$\mathsf{K}_1 := \left\{S_k^{(n,1)}, S_k^{(n,2)} \in (\tfrac{81}{8}m\sqrt{n}, \tfrac{87}{8}m\sqrt{n}) \text{ for all } k \in [\![1, n\rho]\!], |S_{n\rho}^{(n,1)} - S_{n\rho}^{(n,2)}| \leq \tfrac{\sqrt{n}}{2}\right\}.$$

Following the definition of $(n; n\rho, 0)$-modified random bridge, to prove (7.4.29) it suffices to show

$$\mathbf{P}_{(c_1,c_2)}^{n-n\rho;(b_1,b_2)}\left(\left\{V_k^{(1)}, V_k^{(2)} \in (10m\sqrt{n}, 11m\sqrt{n}) \text{ for all } k \in [\![1, n/2]\!]\right\} \cap \mathsf{NI}\right) \geq \tfrac{16}{7}\widetilde{\phi}, \tag{7.4.31}$$

uniformly over $(c_1, c_2) \in \mathcal{P}_2$ where

$$\mathcal{P}_2 := \{(z_1, z_2) \in \mathbb{R}^2 : z_i \in (\tfrac{81}{8}m\sqrt{n}, \tfrac{87}{8}m\sqrt{n}), \tfrac{1}{2}\sqrt{n} \geq z_1 - z_2 \geq \delta\sqrt{n}\}.$$

Here $V_k^{(1)}, V_k^{(2)})_{k=0}^{n-n\rho}$ are two independent random bridges from c_i to b_i. Its law is denoted as $\mathbf{P}_{(c_1,c_2)}^{n-n\rho;(b_1,b_2)}$. For simplicity set $u := n - n\rho \geq \frac{3}{4}n$. By KMT coupling of random bridges [153] we may assume $V_k^{(i)}$'s are defined on a common probability space that supports two independent Brownian bridges (with certain explicit variance depending on the distribution of the increment $f = G_{\theta,1} * G_{\theta,-1}$) such that

$$\mathbf{P}_{(c_1,c_2)}^{u;(d_1,d_2)}\left(\sup_{k\in[\![0,u]\!],i=1,2} |V_k^{(i)} - \sqrt{u}B_{k/u}^{(i)} - c_i - \tfrac{k}{u}(d_i - c_i)| \geq \mathrm{C}\log n\right) \leq \tfrac{1}{n}. \tag{7.4.32}$$

Let $r_{n,i}(x)$ be the piece-wise linear function interpolated by three points: $r_{n,i}(0) = r_{n,i}(1) = 0$ and $r_{n,i}(3/4) = \frac{3}{4\sqrt{u}}(d_i - c_i)$. Let $\mathcal{U}_i := \mathcal{B}(r_{n,i}, \frac{1}{4}\delta)$ be the L^∞ open ball of $r_{n,i}(x)$ of radius $\frac{1}{4}\delta$. By properties of Brownian bridge, there exists a $\widetilde{\phi} = \widetilde{\phi}(m) > 0$ such that for all $(c_1, c_2) \in \mathcal{P}_2$, we have

$$\mathbf{P}(B^{(i)} \in \mathcal{U}_i \text{ for } i = 1, 2) \ge \tfrac{32}{7}\widetilde{\phi}.$$

Note that the above equation along with (7.4.32) implies that with probability $\frac{32}{7}\widetilde{\phi} - \frac{1}{n}$, for all n large enough (and hence u large enough) we have the following items simultaneously.

- For all $k \in [\![0, 3/4u]\!]$

$$|V_k^{(i)} - c_i| \le \mathrm{C}\log n + \tfrac{1}{4}\sqrt{u}\delta \le \tfrac{m}{8}\sqrt{u} < \tfrac{m}{8}\sqrt{n},$$

- For all $k \in [\![0, u]\!]$ we have

$$\begin{aligned} S_k^{(u,1)} &\ge \sqrt{u}r_{n,1}(\tfrac{k}{u}) + c_1 + \tfrac{k}{u}(d_1 - c_1) - \tfrac{1}{4}\sqrt{u}\delta - \mathrm{C}\log n \\ &\ge S_k^{(u,2)} + \sqrt{u}(r_{n,1}(\tfrac{k}{u}) - r_{n,2}(\tfrac{k}{u})) - \tfrac{1}{2}\sqrt{u}\delta + c_1 - c_2 + \tfrac{k}{u}(d_1 - d_2 - c_1 + c_2) - 2\mathrm{C}\log n \\ &\ge \sqrt{u}(r_{n,1}(k/u) - r_{n,2}(k/u)) + \sqrt{u}/2\delta - 2\mathrm{C}\log n + S_k^{(u,2)} \end{aligned}$$

  We have $r_{n,1}(x) \ge r_{n,2}(x)$ by construction, and $c_1 - c_2 + \frac{k}{u}(d_1 - d_2 - c_1 + c_2) \ge \sqrt{u}\delta$ for all $(c_1, c_2) \in \mathcal{P}$. Thus for all large enough n, $S_k^{(u,1)} > S_k^{(u,2)}$ for all $k \in [\![0, u]\!]$.

Thus, taking n large enough we have $\frac{32}{7}\widetilde{\phi} - \frac{1}{n} \ge \frac{16}{7}\widetilde{\phi}$. This establishes (7.4.31) completing the proof of Lemma 7.4.1.

Corollary 7.4.12. *There exists an absolute constant* $\mathrm{C} > 0$ *such that for all* $n \ge 1$.

$$\mathbf{E}^{n;(0,-\sqrt{n});f,g}[W_{\mathrm{sc}}] \ge \tfrac{\mathrm{C}^{-1}}{\sqrt{n}}.$$

The above corollary follows from (7.4.30) as $\mathbf{E}^{n;(0,-\sqrt{n});f,g}[W_{\mathrm{sc}}] \ge \mathbf{E}^{n;(0,-\sqrt{n});f,g}[W_{\mathrm{sc}}\mathbf{1}_{\mathsf{D}_1}]$. We remark that here it is important that the endpoints are $O(\sqrt{n})$ apart to get the precise order of $\mathbf{E}[W_{\mathrm{sc}}]$. We expect a different order if the endpoints are closer or lie in a reversed order. Later, in

Lemma 7.5.6, we shall prove a different lower bound for $\mathbf{E}[W_{\mathrm{sc}}]$ that is uniform over all possible endpoints in a specific window.

Proof of Lemma 7.3.8: $p = 2$ case

Given the machinery developed in the above proof, proof of Lemma 7.3.8 follows easily. By Observation 7.4.5 we have

$$\begin{aligned}\mathbf{P}_{\alpha_2}^{(0,-\sqrt{T}),(-\infty)^T;2,T}(|L_i(i)| \geq M\sqrt{T}) &= \mathbf{P}_W^{T;(0,-\sqrt{T});f,g}(|S_0^{(T,i)}| \geq M\sqrt{T}) \\ &= \frac{\mathbf{E}[W_{\mathrm{sc}}\mathbf{1}\{|S_0^{(T,i)}| \geq M\sqrt{T}\}]}{\mathbf{E}[W_{\mathrm{sc}}]}. \end{aligned} \tag{7.4.33}$$

Now by Corollary 7.4.12 we have $\mathbf{E}[W_{\mathrm{sc}}] \geq \frac{\mathrm{C}}{\sqrt{T}}$ and by Lemma 7.4.11 we have

$$\begin{aligned}\mathbf{E}[W_{\mathrm{sc}}\mathbf{1}_{|S_0^{(T,i)}|\geq M\sqrt{T}}] &\leq \tfrac{1}{T} + \tfrac{\mathrm{C}}{\sqrt{T}}\mathbf{E}\left[\mathbf{1}_{|S_0^{(T,i)}|\geq M\sqrt{T}}[(S_0^{(T,1)} - S_0^{(T,2)} + 1) \vee 1]\big[\tfrac{|S_0^{(T,1)}|}{\sqrt{T}} \vee 2\big]^3\right] \\ &\leq \tfrac{1}{T} + \tfrac{\mathrm{C}}{\sqrt{T}}\sqrt{\mathbf{E}[(S_0^{(T,1)} - S_0^{(T,2)} + 1) \vee 1]^2]}\sqrt{\mathbf{E}\left[\mathbf{1}_{|S_0^{(T,i)}|\geq M\sqrt{T}}\big[\tfrac{|S_0^{(T,1)}|}{\sqrt{T}} \vee 2\big]^6\right]}.\end{aligned}$$

Taking T and M large enough, in view of the tail estimates from Lemma 7.4.7, it follows that (7.4.33) can be made arbitrarily small. This completes the proof.

7.5 Modulus of continuity: proof of Theorem 7.1.1

In this section we prove our main theorem, Theorem 7.1.1, about spatial tightness of $\mathcal{HSLG}$ polymers. Due to the relation in (7.1.4), Theorem 7.1.1 essentially follows by controlling modulus of continuity of the first curve of log-gamma line ensemble. Towards this end, we recall the definition of modulus of continuity function.

Given continuous functions $f : \mathbb{Z}_{>0} \to \mathbb{R}$ and $U > 1$, we define the modulus of continuity function as

$$\omega_\delta^N(f;\llbracket 1,U\rrbracket) := \sup_{\substack{i_1,i_2\in\llbracket 1,U\rrbracket \\ |i_1-i_2|\le \delta N^{2/3}}} \sup_{i\in\llbracket 1,k\rrbracket} |f(i_1)-f(i_2)|.$$

We have the following result.

Proposition 7.5.1. *Fix $r,\gamma>0$ and $p\in\{1,2\}$. Set $\alpha=\alpha_p$ according to* (7.3.11)*. We have*

$$\lim_{\delta\downarrow 0}\limsup_{N\to\infty}\mathbf{P}_{\alpha_p}\left(\omega_\delta^N({}^N_1,\llbracket 1,2\lfloor rN^{2/3}\rfloor-1\rrbracket)\ge \gamma N^{1/3}\right)=0. \tag{7.5.1}$$

By standard criterion for functional tightness (see [58, Theorem 7.3]), the above result along with endpoint tightness from Theorem 7.3.9 leads to the tightness of $N^{-1/3}{}^N_1(\llbracket 1,2\lfloor rN^{2/3}\rfloor-1\rrbracket)$. This proves Theorem 7.1.1. The rest of this section is devoted to proving Proposition 7.5.1.

Proof of Proposition 7.5.1 relies on the following technical lemma which deals with the modulus of continuity for the bottom-free measure.

Lemma 7.5.2. *Fix any $M,S,k_1,k_2,\gamma>0$ with $k_2>k_1$. For each $N>0$, define the sets $I_{1,M}:=\{y\in\mathbb{R},|y|\le 2MN^{1/3}\}$, and*

$$I_{2,M}:=\{(y_1,y_2)\in\mathbb{R}^2: y_i\in I_{1,M/2}, y_1-y_2\ge -(\log N)^{7/6}\}$$

For each $p\in\{1,2\}$, there exist $\delta=\delta(M,S,k_1,k_2,\gamma,\varepsilon)>0$ and $N_0=N_0(M,S,k_1,k_2,\gamma,\varepsilon)>0$ such that for all $\vec{x}\in I_{p,M}$, $T\in\llbracket k_1N^{2/3},k_2N^{2/3}\rrbracket$, and $N\ge N_0$ we have

$$\sum_{i=1}^{p}\mathbf{P}_{\alpha_p}^{\vec{x},(-\infty)^T;p,T}\left(\sum_{k=1}^{i}|L_k(k)|\le SN^{1/3},\omega_\delta^N(L_i,\llbracket 1,T/4+i-2\rrbracket)\ge\gamma N^{1/3}\right)\le\varepsilon.$$

We postpone the proof of Lemma 7.5.2 to Section 7.5.3. In the next subsection, Section 7.5.1, we prepare a few lemmas that are used in the proof of Proposition 7.5.1. Proof of Proposition 7.5.1 appears in Section 7.5.2

7.5.1 Preparatory Lemmas

We first discuss a few consequences of Lemma 7.3.8 that form preparatory tools for our modulus of continuity analysis.

Lemma 7.5.3. *Fix any $\varepsilon \in (0, \frac{1}{2})$ and $T \geq 2$. Let $(X(i))_{i=1}^{2T-1}$ and $Y(i))_{i=1}^{2T-1}$ be two independent random vectors with density proportional to*

$$\prod_{i=1}^{2T-2} G_{\theta,(-1)^{i+1}}(u_i - u_{i+1}) \text{ and } \prod_{i=1}^{2T-2} G_{\theta,(-1)^{i}}(u_i - u_{i+1})$$

respectively where $u_1 = 0$ and $u_{2T-1} = 0$. There exists $M_0(\varepsilon) > 0$ such that for all $T \geq 2$ we have

$$\mathbf{P}\bigg(\sup_{i \in [\![1,2T-1]\!]} [|X(i)| + |Y(i)|] \geq M_0\sqrt{T}\bigg) \leq \varepsilon. \tag{7.5.2}$$

We refer to Figure 7.18 for graphical representation of the distributions appearing in Lemma 7.5.3.

Figure 7.18: Graphical representation of X (left) and Y (right) distribution from Lemma 7.5.3.

Proof. Fix $\varepsilon \in (0,1)$. Note that $(X(2i-1))_{i=1}^{T}$ forms a random bridge from 0 to 0 with increment from $G_{\theta,+1} * G_{\theta,-1}$. By KMT coupling for random bridges [153] along with Brownian bridge estimates, one can ensure there exists a constant $M > 0$ such that

$$\mathbf{P}(\mathsf{A}) \leq \tfrac{\varepsilon}{4}, \text{ where } \mathsf{A} := \bigg\{\sup_{i \in [\![1,T]\!]} |X(2i-1)| \geq M\sqrt{T}\bigg\}.$$

Let us write $\mathcal{F} := \sigma((X(2i-1))_{i=1}^{T})$. By a union bound we have

$$\mathbf{P}(\sup_{i \in [\![1,2T-1]\!]} |X(i)| \geq 5M\sqrt{T}) \leq \frac{\varepsilon}{4} + \sum_{i=1}^{T-1} \mathbf{E}\left[\mathbf{1}_{\neg\mathsf{A}} \cdot \mathbf{E}\left[\mathbf{1}_{|X(2i)| \geq 5M\sqrt{T}} \mid \mathcal{F}\right]\right] \tag{7.5.3}$$

Note that the distribution of even points given the odd points are given by the ξ-distribution introduced in (7.4.5). Observe that by Lemma 7.6.5,

$$\mathbf{1}_{X(2i-1),X(2i+1)\in(-M\sqrt{T},M\sqrt{T})}\cdot\mathbf{E}\left[\mathbf{1}_{|X(2i)|\ge 5M\sqrt{T}}\mid\mathcal{F}\right]\le \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}\sqrt{T}),$$

for some absolute constant $\mathrm{C}>0$. Plugging the above bound back in (7.5.3) and taking T large enough we get the r.h.s. of (7.5.3) is at most $\frac{\varepsilon}{2}$. Similarly one can show $\mathbf{P}(\sup_{i\in[\![1,2T-1]\!]}|Y(i)|\ge 5M\sqrt{T})\le\frac{\varepsilon}{2}$. Adjusting M, we arrive at (7.5.6). This completes the proof. $\square$

Lemma 7.5.4. *Fix any $p\in\{1,2\}$. Set $\alpha:=\alpha_p$ according to* (7.3.11). *Fix any $r\ge 1$ and $\varepsilon>0$. Set $T=\lfloor rN^{2/3}\rfloor$. There exists $M=M(\varepsilon)>0$ and $N_0(\varepsilon)>0$ such that for all $N\ge N_0$ we have*

$$\mathbf{P}_{\alpha_1}^{0,(-\infty)^T;1,T}\left(\sup_{i\in[\![1,2T-1]\!]}|L_1(i)|\ge M\sqrt{T}\right)\le\varepsilon, \tag{7.5.4}$$

$$\mathbf{P}_{\alpha_2}^{(0,-\sqrt{T}),(-\infty)^T;2,T}\left(\sup_{i\in[\![1,2T-1]\!]}|L_1(i)|+\sup_{j\in[\![1,2T}|L_2(j)|\ge M\sqrt{T}\right)\le\varepsilon, \tag{7.5.5}$$

where the law $\mathbf{P}_{\alpha_p}^{\vec{x},(-\infty)^{2T};2,T}$ is defined in Definition 7.2.3.

Proof. For clarity we divide the proof into two steps.

Step 1. Fix any $\varepsilon\in(0,\frac{1}{2})$ and consider $M_0(\varepsilon)$ from Lemma 7.5.3. In this step we prove (7.5.4). From Lemma 7.3.8 choose $M_1(\varepsilon)>0$ such that for all large enough T we have

$$\mathbf{P}_{\alpha_1}^{0,(-\infty)^T;1,T}(|L_1(1)|\ge M_1\sqrt{T})\ge\varepsilon,\quad \mathbf{P}_{\alpha_2}^{(0,-\sqrt{T}),(-\infty)^T;2,T}(|L_1(1)|+|L_2(2)|\ge M_1\sqrt{T})\ge\varepsilon. \tag{7.5.6}$$

Set $M_3:=2M_0+M_1+1$. Consider the events

$$\mathsf{A}:=\left\{\sup_{i\in[\![2,2T]\!]}L_2(i)\ge(M_0+M_1)\sqrt{T}\right\},\quad \mathsf{B}:=\left\{\sup_{i\in[\![1,2T-1]\!]}L_1(i)\ge(M_3+M_0)\sqrt{T}\right\}.$$

In view of (7.5.6), by a union bound we have

$$\mathbf{P}_{\alpha_1}^{0,(-\infty)^T;1,T}(\mathsf{B}) \leq \varepsilon + \mathbf{E}\left[\mathbf{1}_{L_1(1)\leq M_1\sqrt{T}}\mathbf{E}\left[\mathbf{1}_{\mathsf{B}} \mid \sigma(L_1(1))\right]\right]$$

As B is an increasing event with respect to the boundary data, to get an upper bound, we may assume $L_1(1) = L_1(2T-1) = M_1\sqrt{T}$. But note that under this boundary condition we have $(L_1(i) - M_1\sqrt{T})_{i=1}^{2T-1} \stackrel{d}{=} (X(i))_{i=1}^{2T-1}$. Thus, owing to (7.5.2), almost surely we have

$$\mathbf{1}_{L_1(1)\leq M_1\sqrt{T}}\mathbf{E}\left[\mathbf{1}_{\mathsf{B}} \mid \sigma(L_1(1))\right] \leq \mathbf{P}\left(\sup_{i\in[\![1,2T-1]\!]}|X(i)| \geq (2M_0+M)\sqrt{T}\right) \leq \varepsilon.$$

This implies $\mathbf{P}_{\alpha_1}^{0,(-\infty)^T;1,T}(\mathsf{B}) \leq 2\varepsilon$. Following similar calculations one can show

$$\mathbf{P}_{\alpha_1}^{0,(-\infty)^T;1,T}\left(\inf_{i\in[\![1,2T-1]\!]} L_1(i) \leq -(M_3+M_0)\sqrt{T}\right) \leq 2\varepsilon.$$

This proves (7.5.4) with $M = M_3$ for $\varepsilon \mapsto 2\varepsilon$.

Step 2. In this step we prove (7.5.5). At this point we encourage the readers to look at Figure 7.19 and its caption for an overview of the proof idea.

Let us set $\mathcal{F}_1 = \sigma(L_2(2), (L_1(i))_{i=1}^{2T-1})$ and $\mathcal{F}_2 = \sigma(L_1(1), (L_2(i))_{i=2}^{2T})$. We use the shorthand notation $\mathbf{P}_2$ for $\mathbf{P}_{\alpha_2}^{(0,-\sqrt{T}),(-\infty)^T;2,T}$. In view of (7.5.6), by a union bound

$$\mathbf{P}_2(\mathsf{A}) \leq \varepsilon + \mathbf{P}_2\left(\{L_2(2) \leq M_0\sqrt{T}\} \cap \mathsf{A}\right) \leq \varepsilon + \mathbf{E}_2\left[\mathbf{1}_{L_2(2)\leq M_0\sqrt{T}}\mathbf{E}_2\left[\mathbf{1}_{\mathsf{A}} \mid \mathcal{F}_1\right]\right].$$

As A is an increasing event with respect to the boundary data, to get an upper bound, we may assume $L_2(2T) = L_2(2) = M_1\sqrt{T}$, and $L_1(i) = +\infty$ for all $i \in [\![1, 2T-1]\!]$. But note that under this boundary condition we have $(L_2(i+1) - M_1\sqrt{T})_{i=1}^{2T-1} \stackrel{d}{=} (Y(i))_{i=1}^{2T-1}$. Thus, almost surely we have

$$\mathbf{1}_{L_2(2)\leq M_0\sqrt{T}} \cdot \mathbf{E}_2\left[\mathbf{1}_{\mathsf{A}} \mid \mathcal{F}_1\right] \leq \mathbf{P}\left(\sup_{i\in[\![1,2T-1]\!]}|X(i)| \geq M_0\sqrt{T}\right) \leq \varepsilon.$$

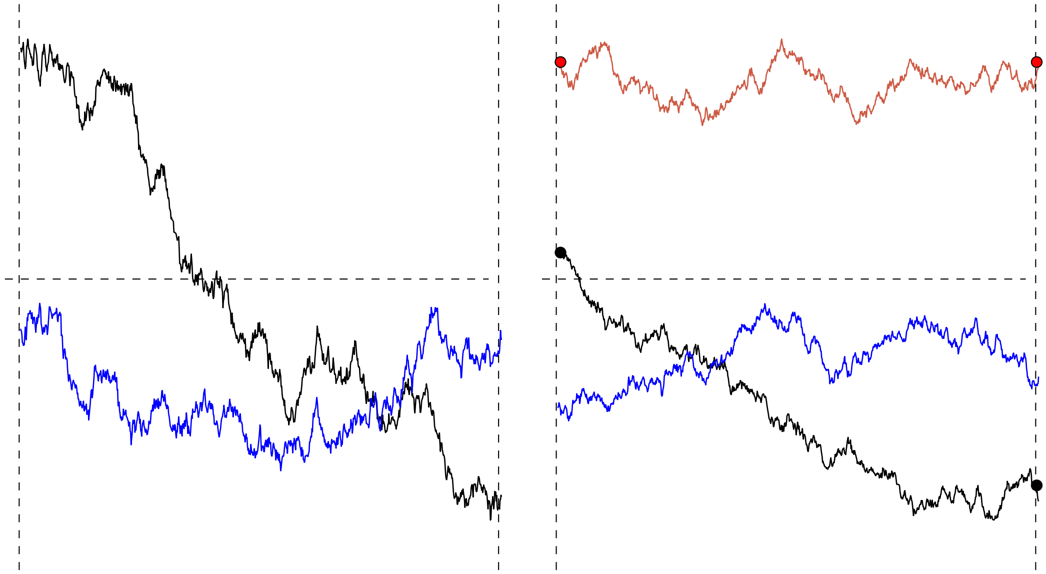


Figure 7.19: In the above figure, we have plotted $L_1[\![1, 2T-1]\!]$ (black curve) and $L_2[\![2, 2T]\!]$ (blue curve).Due to endpoint tightness, Lemma 7.3.8 ensure $L_1(1), L_2(2) \in (-M_0\sqrt{T}, M_0\sqrt{T})$. Assuming this, in order to seek an uniform upper bound for the blue curve, by stochastic monotonicity we may push the black curve all the way to $+\infty$. The resulting law for the blue curve is given by Y introduced in Lemma 7.5.3. Uniform upper bound for the resulting law for the blue curve law can then be estimated by Lemma 7.5.3. The upper bound is shown in the dashed line above. Once we have an uniform upper bound for the blue curve, we may elevate the endpoints of black curve much higher (from black points to red points in the above right figure) so that the curve no longer feels the effect of the blue curve. The red curve above denotes a sample for L_1 from this elevated end points. Without the blue curve its law (upto a translation) equals to X in Lemma 7.5.3. An uniform upper bound for the red curve can then be estimated by Lemma 7.5.3.

Thus $\mathbf{P}_2(\mathsf{A}) \le 2\varepsilon$. In view of this bound, applying a union bound we have

$$\mathbf{P}_2(\mathsf{B}) \le 3\varepsilon + \mathbf{E}_2\left[\mathbf{1}_{\{L_1(1)\le M_1\sqrt{T}\}\cap\neg\mathsf{A}}\mathbf{E}_2\left[\mathbf{1}_{\mathsf{B}} \mid \mathcal{F}_2\right]\right]$$

As B is an increasing event with respect to the boundary data, by stochastic monotonicity, to get an upper bound we may assume $L_1(1) = L_1(2T-1) = M_3\sqrt{T}$ and $L_2(i) = (M_0 + M_1)\sqrt{T}$ for all $i \in [\![2, 2T]\!]$. From the definition of the Gibbs measure, almost surely we have

$$\mathbf{1}_{\{L_1(1)\le M_1\sqrt{T}\}\cap\neg\mathsf{A}}\mathbf{E}_2\left[\mathbf{1}_{\mathsf{B}} \mid \mathcal{F}_2\right] \le \frac{1}{\mathbf{E}[\Delta]}\mathbf{E}\left[\Delta\cdot\mathbf{1}_{\mathsf{B}}\right].$$

where B is now interpreted as $\{\sup_{i\in[\![1,2T-1]\!]} X(i) \geq M_0\sqrt{T}\}$ and

$$\Delta = \exp\left(-\sum_{i=1}^{T-1}[e^{-(M_0+1)\sqrt{T}-X_1(2i-1)} + e^{-(M_0+1)\sqrt{T}-X_1(2i+1)}]\right).$$

As $\Delta \leq 1$, by (7.5.2), $\mathbf{E}[\Delta \cdot \mathbf{1}_{\mathsf{B}}] \leq \mathbf{E}[\mathbf{1}_{\mathsf{B}}] \leq \varepsilon$. By (7.5.2) we have $\mathbf{E}[\Delta] \geq (1-\varepsilon)\cdot\exp(-2(T-1)e^{-\sqrt{T}}) \geq \beta$ for some absolute constant $\beta > 0$. Thus, $\mathbf{P}_2(\mathsf{B}) \leq (3+\beta^{-1})\varepsilon$. Similarly one can show

$$\mathbf{P}_2\left(\inf_{i\in[\![2,2T]\!]} L_2(i) \leq -(M_3+M_0)\sqrt{T}\right) \leq (3+\beta^{-1})\varepsilon$$
$$\mathbf{P}_2\left(\inf_{i\in[\![1,2T-1]\!]} L_1(i) \leq -(M_0+M_1)\sqrt{T}\right) \leq 2\varepsilon.$$

Thus adjusting the constants we can find $\widetilde{M}$ such that

$$\mathbf{P}_2\left(\sup_{i\in[\![1,2T-1]\!]} |L_1(i)| + \sup_{j\in[\![2,2T]\!]} |L_2(j)| \geq (M-1)\sqrt{T}\right) \leq \varepsilon/3.$$

Finally via Observation 7.4.5 we know $L_2(1) - L_2(2) \sim G_{\alpha_2}$. Thus, by a union bound, for all T large enough we have $\mathbf{P}_2(|L_2(1)| \geq M\sqrt{T}) \leq \varepsilon/3 + \mathbf{P}_2(|L_2(1)-L_2(2)| \geq \sqrt{T}) \leq 2\varepsilon/3$. By another union bound, we arrive at (7.5.5). $\square$

Recall the normalizing constant $V_p^T(\vec{y},\vec{z})$ from (7.2.5) and (7.2.6). One can easily obtain a lower bound for this normalizing constant as a consequence of the Lemma 7.5.4.

Corollary 7.5.5. *Fix any $r > 0$ and for each $N > 0$ set $T = \lfloor rN^{2/3}\rfloor$. Fix any $p \in \{1,2\}$ and set $\alpha = \alpha_p$ according to* (7.3.11). *There exists $Q_0 = Q_0(r) > 0, N = N_0(r) > 0$ such that for all $Q \geq Q_0$ and $N \geq N_0$*

$$V_p^T(\vec{y},\vec{z}) := \mathbf{E}_{\alpha_p}^{\vec{y},(-\infty)^T;p,T}\left(\prod_{j=1}^{T} W(z_j; L_p(2j+1), L_p(2j-1))\right) \geq \tfrac{1}{2},$$

for all $\vec{z} \in \mathbb{R}^T$ with $z_i \leq QN^{1/3}$ and $\vec{y} \in \mathbb{R}^p$ with $y_i \geq (2Q-1)N^{1/3}$. Here $W(a;b,c) := e^{-e^{a-b}-e^{a-c}}$ and in above equation $L_p(2T+1) = \infty$.

Proof. Consider the event

$$\mathsf{A} := \left\{ \inf_{j \in [\![1,T]\!]} L_p(2j-1) \geq (Q+1)N^{1/3} \right\}.$$

Observe that

$$V_p^T \geq \mathbf{E}_{\alpha_p}^{\vec{y},(-\infty)^T;p,T} \left(\mathbf{1}_{\mathsf{A}} \prod_{j=1}^{T} W(z_j; L_p(2j+1), L_p(2j-1)) \right) \geq \exp(-2Te^{-N^{1/3}}) \cdot \mathbf{P}_{\alpha_p}^{\vec{y},(-\infty)^T;p,T}(\mathsf{A})$$

Taking N large enough ensures $\exp(-2Te^{-N^{1/3}}) \geq 1/\sqrt{2}$. Since A is an increasing event with respect to the boundary data, applying stochastic monotonicity and translation invariance we have

$$\mathbf{P}_{\alpha_p}^{\vec{y},(-\infty)^T;p,T}(\mathsf{A}) = \mathbf{P}_{\alpha_p}^{\vec{x},(-\infty)^T;p,T} \left(\inf_{j \in [\![1,T]\!]} L_p(2j-1) \geq -(Q-2)N^{1/3} \right)$$

where $\vec{x} = 0$ if $p = 1$ and $\vec{x} = (0, -\sqrt{T})$ if $p = 2$. Appealing to Lemma 7.5.4 we may choose Q large enough so that the above probability is at least $1/\sqrt{2}$. This completes the proof. $\square$

7.5.2 Proof of Proposition 7.5.1

For clarity we divide the proof into three steps.

Step 1. In this step, we give the roadmap of the proof of (7.5.1) leaving the technical details to later steps.

Fix $r, \varepsilon, \delta > 0$ and $p \in \{1, 2\}$. Fix $N \geq 3$ large enough so that $T = 8\lfloor rN^{2/3} \rfloor \geq 24$. Set $\alpha = \alpha_p$ according to (7.3.11) and consider the $\mathcal{HSLG}$ line ensemble ^{N} from Definition 7.2.7 with parameters (α, θ). Consider the event

$$\mathsf{MC}_\delta := \left\{ \omega_\delta^N({}_1^N, [\![1, T/4-1]\!]) \geq \gamma N^{1/3} \right\}.$$

By Theorem 7.3.9, there exists $S(\varepsilon) > 0$ such that

$$\mathbf{P}(\mathsf{A}_1) \geq 1 - \varepsilon, \text{ where } \mathsf{A}_1 := \left\{ N^{-1/3} |{}_1^N(1)| + N^{-1/3} |{}_2^N(2)| \leq S \right\}. \tag{7.5.7}$$

By Proposition 7.3.4, there exists $M_1(\varepsilon) > 0$ such that for all large enough N

$$\mathbf{P}\left({}_1^N(2T - 1) \geq M_1 N^{1/3} \right) \leq \varepsilon. \tag{7.5.8}$$

We claim that there exists $M_2(r, \varepsilon) > 0$ such that for all large enough N

$$\mathbf{P}\left({}_p^N(2T + p - 2) \leq -M_2 N^{1/3} \right) \leq \varepsilon. \tag{7.5.9}$$

We shall prove (7.5.9) in **Step 2**. Let us assume it for now. Set $M = \max\{M_1, M_2, 4\}$ and consider the event

$$\mathsf{B}_1 := \{ |{}_1^N(2T - 1)| \leq 2MN^{1/3} \},$$
$$\mathsf{B}_2 := \{ {}_2^N(2T) \geq -MN^{\frac{1}{3}}, {}_1^N(2T - 1) \leq MN^{\frac{1}{3}}, {}_1^N(2T - 1) \geq {}_1^N(2T) - (\log N)^{7/6} \},$$

For each $\beta > 0$ we define

$$\mathsf{C}(p, \beta) := \{ V_p^T \left(({}_j^N(2T + j - 2))_{j \in [\![1,p]\!]}; ({}_{p+1}^N(2k))_{k=1}^T \right) \geq \beta \}, \tag{7.5.10}$$

where $V_p^T(\cdot, \cdot)$ is defined in (7.2.6). We now claim that there exists $\beta(r, \varepsilon) > 0$ such that

$$\mathbf{P}\left(\neg \mathsf{C}(p, \beta) \right) \leq \varepsilon. \tag{7.5.11}$$

We work with this choice of β for the rest of this step. We postpone the proof of (7.5.11) to **Step 3**.

Let us now complete the proof of Proposition 7.5.1 assuming it. Consider the following σ-algebra:

$$\mathcal{F}_{p,k} := \sigma\left\{({}^{N}_{i}[\![1, 2N-2i+2]\!])_{i\ge p+1}, ({}^{N}_{i}(j))_{j\ge 2k+i-2, i\in[\![1,p]\!]}\right\}. \tag{7.5.12}$$

Clearly $\mathsf{B}_p \cap \mathsf{C}(p,\beta)$ is measurable with respect to $\mathcal{F}_{p,T}$. By union bound and tower property of conditional expectation we have

$$\begin{aligned}\mathbf{P}(\mathsf{MC}_\delta) \le \mathbf{P}(\neg\mathsf{A}_1) + \mathbf{P}\left(\neg\mathsf{B}_p\right) + \mathbf{P}\left(\neg\mathsf{C}(p,\beta)\right) \\ + \mathbf{E}\left[\mathbf{1}_{\mathsf{B}_p\cap\mathsf{C}(p,\beta)}\mathbf{E}\left(\mathbf{1}_{\mathsf{A}_1\cap\mathsf{MC}_\delta} \mid \mathcal{F}_{p,T}\right)\right].\end{aligned} \tag{7.5.13}$$

Let us bound the four terms on the r.h.s. of the above equation separately.

(a) A_1 **event:** We have $\mathbf{P}(\neg\mathsf{A}_1) \le \varepsilon$ due to (7.5.7).

(b) B_p **event:** Note that for large enough N, $\mathsf{B}_2 \subset \mathsf{B}_1$. Combining (7.5.8), (7.5.9), and Theorem 7.3.1 (with $\rho \mapsto \frac{1}{2}, M \mapsto M$), by a union bound we see that for all large enough N,

$$\begin{aligned}\mathbf{P}(\neg\mathsf{B}_p) &\le \mathbf{P}(\neg\mathsf{B}_2) \\ &\le \mathbf{P}(\mathsf{A}_1, {}^{N}_{2}(2T) \le -MN^{1/3}) + \mathbf{P}({}^{N}_{1}(2T-1) \ge MN^{1/3}) \\ &\qquad + \mathbf{P}\left({}^{N}_{1}(2T-1) \le {}^{N}_{1}(2T) - (\log N)^{7/6}\right) \\ &\le 2\varepsilon + 2^{-N} \le 3\varepsilon.\end{aligned}$$

(c) $\mathsf{C}(p,\beta)$ **event:** We have $\mathbf{P}(\neg\mathsf{C}(p,\beta)) \le \varepsilon$ due to (7.5.11).

(d) **Conditional probability:** By Theorem 7.1.3 and (7.2.5) we have

$$\mathbf{E}\left(\mathbf{1}_{\mathsf{A}_1\cap\mathsf{MC}_\delta} \mid \mathcal{F}_{p,T}\right) = \frac{\mathbf{E}_{\alpha_p}^{\theta;\vec{y},(-\infty)^{2T};p,T}\left[V_p^T\left(\vec{y}; ({}^{N}_{p+1}(2i))_{i=1}^T\right)\cdot \mathbf{1}_{\mathsf{A}_1\cap\mathsf{MC}_\delta}\right]}{V_p^T\left(\vec{y}; ({}^{N}_{p+1}(2i))_{i=1}^T\right)} \tag{7.5.14}$$

where $\vec{y} := ({}^{N}_{j}(2T+j-2))_{j\in[\![1,p]\!]}$ and $V_p^T(\cdot;\cdot)$ is defined in (7.2.6). From definition we have

$V_p^T\left(\vec{y};({}^{N}_{p+1}(2i))_{i=1}^T\right) \in [0,1]$. On $\mathsf{C}(p,\beta)$ we have

$$\mathbf{1}_{\mathsf{C}(p,\beta)} \cdot \text{r.h.s. of (7.5.14)} \le \mathbf{1}_{\mathsf{C}(p,\beta)} \cdot \beta^{-1} \cdot \mathbf{P}_{\alpha_p}^{\theta;\vec{y},(-\infty)^{2T};p,T}\left(\mathsf{A}_1 \cap \mathsf{MC}_\delta\right).$$

Observe that the event B_p ensures $\vec{y} \in I_{p,M}$ where the set $I_{p,M}$ is defined in the statement of Lemma 7.5.2. We can thus apply Lemma 7.5.2 with $M \mapsto M, u \mapsto S, k \mapsto k, \gamma \mapsto \gamma, \varepsilon \mapsto \beta \cdot \varepsilon$, to get a $\delta > 0$ such that

$$\mathbf{1}_{\mathsf{B}_p} \cdot \mathbf{P}_{\alpha_p}^{\theta;\vec{y},(-\infty)^{2T};p,T}\left(\mathsf{A}_1 \cap \mathsf{MC}_\delta\right) \le \mathbf{1}_{\mathsf{B}_p} \cdot \varepsilon,$$

for all large enough N. Thus overall we have

$$\mathbf{E}\left[\mathbf{1}_{\mathsf{B}_p \cap \mathsf{C}(p,\beta)} \mathbf{E}\left(\mathbf{1}_{\mathsf{A}_1 \cap \mathsf{MC}_\delta} \mid \mathcal{F}_{p,T}\right)\right] \le \varepsilon.$$

Plugging in the above four estimates back in r.h.s. of (7.5.13) and taking limit superior $N \to \infty$, followed by $\delta \downarrow 0$, yields

$$\limsup_{\delta \downarrow 0} \limsup_{N \to \infty} \mathbf{P}(\mathsf{MC}_\delta) \le 6\varepsilon.$$

As ε is arbitrary, we thus have (7.5.1), completing the proof.

Step 2. In this step we prove (7.5.9). We write $\mathbf{P}_{\alpha_p}$ instead of $\mathbf{P}$ to stress on the fact that the $\mathcal{HSLG}$ line ensemble has boundary parameter α_p, defined in (7.3.11). We claim that there exists $M_2(r,\varepsilon)$ such that for all large enough N

$$\mathbf{P}_{\alpha_p}\left(\mathsf{Fall}_{M_2}^{(p)}\right) \le \tfrac{\varepsilon}{4}, \quad \mathsf{Fall}_{M_2}^{(p)} := \left\{\inf_{j \in [\![1,4T+4]\!], i \in [\![1,p]\!]} {}^{N}_{i}(j) \le -M_2 N^{1/3}\right\}. \tag{7.5.15}$$

Note that as $\{{}^{N}_{p}(2T+p-2) \le -M_2 N^{1/3}\} \subset \mathsf{Fall}_{M_2}^{(p)}$, (7.5.15) implies (7.5.9). To show (7.5.15), we first define a few more necessary events. For each $R \geq 32r + 1$ we define

$$
\begin{aligned}
&\mathsf{B}_{R,j}^{(i)} := \left\{ {}^{N}_{i}(2j+i-2) \geq -R^2 N^{1/3} \right\}, \quad \widetilde{\mathsf{B}}_{R,j}^{(i)} := \mathsf{B}_{R,j}^{(i)} \cap \bigcup_{k \in [\![j+1, RN^{2/3}]\!]} \neg \mathsf{B}_{R,k}^{(i)}, \\
&\mathsf{B}_{R}^{(i)} := \bigcup_{j \in [\![4T+4, RN^{2/3}]\!]} \mathsf{B}_{R,j}^{(i)} = \bigsqcup_{j \in [\![4T+4, RN^{2/3}]\!]} \widetilde{\mathsf{B}}_{R,j}^{(i)} = \left\{ \sup_{j \in [\![4T+4, RN^{2/3}]\!]} {}^{N}_{i}(2j+i-2) \geq -R^2 N^{1/3} \right\}, \\
&\mathsf{Dif}_R := \left\{ {}^{N}_{1}(2j-1) \geq {}^{N}_{2}(2j) + (\log N)^2 \text{ for all } j \in [\![1, RN^{2/3}]\!] \right\}.
\end{aligned}
$$

By Theorem 7.3.1, Theorem 7.3.3, and Proposition 7.3.4, we can find a $R = R(r, \varepsilon) \geq 1$ such that for all large enough N, and for $v \in \{1, 2\}$

$$
\mathbf{P}_{\alpha_v}\left(\neg \mathsf{B}_{R}^{(1)}\right) + \mathbf{P}_{\alpha_v}\left(\neg \mathsf{B}_{R}^{(2)}\right) + \mathbf{P}_{\alpha_v}\left(\neg \mathsf{Dif}_R\right) \leq \tfrac{\varepsilon}{8}. \tag{7.5.16}
$$

We fix this choice of R. Observe that for large enough N, we have

$$
\widetilde{\mathsf{B}}_{R,i}^{(2)} \cap \mathsf{Dif}_R \subset \widetilde{\mathsf{B}}_{R,i}^{(2)} \cap \mathsf{B}_{2R,i}^{(1)},
$$

uniformly for all $i \in [\![4T+4, RN^{2/3}]\!]$. For $p = 2$, by the union bound and the tower property of conditional expectation, in view of (7.5.16), we have

$$
\begin{aligned}
\mathbf{P}_{\alpha_2}\left(\mathsf{Fall}_{M_2}^{(2)}\right) &\leq \mathbf{P}_{\alpha_2}\left(\neg \mathsf{B}_{R}^{(2)}\right) + \mathbf{P}\left(\neg \mathsf{Dif}_R\right) + \sum_{j \in [\![4T+4, RN^{2/3}]\!]} \mathbf{P}_{\alpha_2}\left(\widetilde{\mathsf{B}}_{R,j}^{(2)} \cap \mathsf{B}_{2R,j}^{(1)} \cap \mathsf{Fall}_{M_2}^{(2)}\right) \\
&\leq \tfrac{\varepsilon}{8} + \sum_{j \in [\![4T+4, RN^{2/3}]\!]} \mathbf{E}\left[\mathbf{1}_{\widetilde{\mathsf{B}}_{R,j}^{(2)} \cap \mathsf{B}_{2R,j}^{(1)}} \mathbf{E}_{\alpha_2}\left(\mathbf{1}_{\mathsf{Fall}_{M_2}^{(2)}} \mid \mathcal{F}_{2,j}\right)\right],
\end{aligned} \tag{7.5.17}
$$

where $\mathcal{F}_{p,k}$ is defined in (7.5.12). For $p = 1$, applying union bound and using (7.5.16) we have

$$
\begin{aligned}
\mathbf{P}_{\alpha_1}\left(\mathsf{Fall}_{M_2}^{(1)}\right) &\leq \mathbf{P}_{\alpha_1}\left(\neg \mathsf{B}_{R}^{(1)}\right) + \sum_{j \in [\![4T, RN^{2/3}]\!]} \mathbf{P}_{\alpha_1}\left(\widetilde{\mathsf{B}}_{R,j}^{(1)} \cap \mathsf{Fall}_{M_2}^{(1)}\right) \\
&\leq \tfrac{\varepsilon}{8} + \sum_{j \in [\![4T+4, RN^{2/3}]\!]} \mathbf{E}\left[\mathbf{1}_{\widetilde{\mathsf{B}}_{R,j}^{(1)}} \mathbf{E}_{\alpha_1}\left(\mathbf{1}_{\mathsf{Fall}_{M_2}^{(1)}} \mid \mathcal{F}_{1,j}\right)\right].
\end{aligned} \tag{7.5.18}
$$

We now proceed to control the conditional expectation $\mathbf{P}_{\alpha_p}\left(\mathsf{Fall}_{M_2}^{(p)} \mid \mathcal{F}_{p,j}\right)$ separately for $p = 1$ and $p = 2$. Applying the Gibbs property (Theorem 7.1.3), we have

$$\begin{aligned}\mathbf{1}_{\widetilde{\mathsf{B}}_{R,j}^{(2)}\cap\mathsf{B}_{2R,j}^{(1)}}\cdot\mathbf{E}_{\alpha_2}\left(\mathsf{Fall}_{M_2}^{(2)} \mid \mathcal{F}_{2,j}\right) &= \mathbf{1}_{\widetilde{\mathsf{B}}_{R,j}^{(2)}\cap\mathsf{B}_{2R,j}^{(1)}}\cdot\mathbf{E}_{\alpha_2}^{\vec{y},\vec{z};2,j}(\mathsf{Fall}_{M_2}^{(2)})\\ &\le \mathbf{1}_{\widetilde{\mathsf{B}}_{R,j}^{(2)}\cap\mathsf{B}_{2R,j}^{(1)}}\cdot\mathbf{P}_{\alpha_2}^{(0,-\sqrt{j}),(-\infty)^j;2,j}(\mathsf{Fall}_{M_2-4R^2}^{(2)}).\end{aligned}$$

Here $\vec{y} = (\mathcal{L}_1^N(2j-1), \mathcal{L}_2^N(2j))$ and $\vec{z} = (\mathcal{L}_3^N(2m))_{m=1}^j$. Let us briefly explain the above inequality. Note that on $\widetilde{\mathsf{B}}_{R,j}^{(2)} \cap \mathsf{B}_{2R,j}^{(1)}$ we have $y_i \ge (-4R^2N^{1/3} - (i-1)\sqrt{j})$ for $i = 1, 2$. Furthermore $\mathsf{Fall}_{M_2}^{(2)}$ is an event which decreases with respect to boundary data. Thus to obtain an upper bound, by stochastic monotonicty, we may take the boundary data from (y_1, y_2) to $(-4R^2N^{1/3}, -4R^2N^{1/3} - \sqrt{j})$ and $\vec{z}$ to $(-\infty)^j$. The above inequality then follows by translation invariance (see Observation 7.2.1 (a)). Similar applications of the Gibbs property and stochastic monotonicity yield that on $\widetilde{\mathsf{B}}_{R,j}^{(1)}$ we have

$$\mathbf{E}_{\alpha_1}\left(\mathbf{1}_{\mathsf{Fall}_{M_2}^{(1)}} \mid \mathcal{F}_{2,j}\right) \le \mathbf{P}_{\alpha_1}^{0,(-\infty)^j;1,j}(\mathsf{Fall}_{M_2-4R^2}^{(1)}).$$

We now claim that one can choose $M_2(r,\varepsilon) > 0$ large enough such that for all $j \in [\![4T+4, RN^{2/3}]\!]$,

$$\mathbf{P}_{\alpha_p}^{\vec{x},(-\infty)^j;p,j}(\mathsf{Fall}_{M_2-4R^2}^{(p)}) \le \tfrac{\varepsilon}{8}, \tag{7.5.19}$$

where $\vec{x} := 0$ (if $p = 1$) or $\vec{x} := (0, -\sqrt{j})$ (if $p = 2$). Plugging the above bound back in (7.5.18) and (7.5.17) and using the fact that $\{\widetilde{\mathsf{B}}_{R,j}^{(p)}\}_{j\in[\![4T+4,RN^{2/3}]\!]}$ is a disjoint collection of events we arrive at the bound in (7.5.15). Thus we are left to verify (7.5.19) in this step. But observe that

$$\mathbf{P}_{\alpha_p}^{\vec{x},(-\infty)^j;p,j}(\mathsf{Fall}_{M_2-4R^2}^{(p)}) \le \mathbf{P}_{\alpha_p}^{\vec{x},(-\infty)^j;p,j}\left(\inf_{k\in[\![1,2j+i-2]\!],i\in[\![1,p]\!]} L_i(k) \le -(M_2-4R^2)N^{1/3}\right)$$

By Lemma 7.5.4, one can choose M_2 large enough such that the above expression is bounded above by $\varepsilon/8$ for all $j \in [\![4T, RN^{2/3}]\!]$. This proves (7.5.19) completing our work for this step.

Step 3. In this step we prove (7.5.11). For each $Q > 0$ consider the event

$$\mathsf{D}_Q := \left\{ \sup_{i \in [\![1,4T+4]\!]} {}^{N}_{p+1}(i) \le QN^{1/3}, \ \inf_{j \in [\![1,p]\!]} {}^{N}_{j}(4T+j+2) \ge -QN^{1/3} + \sqrt{2T+1} \right\}. \tag{7.5.20}$$

By Theorem 7.3.1, Proposition 7.3.7, and (7.5.15) there exists $Q(r,\varepsilon) > 0$ large enough such that $\mathbf{P}(\neg \mathsf{D}_{p,Q}) \le \frac{\varepsilon}{2}$. Consider $\mathcal{F}_{p,2T+2}$ from (7.5.12). Recall the event $\mathsf{C}(p,\beta)$ from (7.5.10). By union bound and the tower-property of the expectation, we have

$$\mathbf{P}(\neg \mathsf{C}(p,\beta)) \le \mathbf{P}(\neg \mathsf{C}(p,\beta) \cap \mathsf{D}_Q) + \tfrac{\varepsilon}{2} = \mathbf{E}\left[\mathbf{1}_{\mathsf{D}_Q} \mathbf{E}[\mathbf{1}_{\mathsf{C}(p,\beta)} \mid \mathcal{F}_{p,2T+2}]\right] + \tfrac{\varepsilon}{2}. \tag{7.5.21}$$

Applying the Gibbs property and (7.2.5) we have

$$\mathbf{E}[\mathbf{1}_{\neg \mathsf{C}(p,\beta)} \mid \mathcal{F}_{p,2T+2}] = \mathbf{P}_{\alpha_p}^{\vec{y};\vec{z};p,2T+2}\big(\neg \mathsf{C}(p,\beta)\big)$$

with $\vec{y} = (y_1, \ldots, y_p)$ and $y_j = {}^{N}_{j}(4T+j+2)$ for $j \in [\![1,p]\!]$, and $\vec{z} = ({}^{N}_{p+1}(2k))_{k=1}^{2T+2}$. Let us set $\vec{x} = (-QN^{1/3} + \sqrt{2T+1})^p$. We now claim that there exists $Q_0(r,\varepsilon) > 0$, $N_0(r,\varepsilon) > 0$ and $\beta(r,\varepsilon) > 0$, such that for all $N \ge N_0$, $Q \ge Q_0$, $y_i \ge x_i$ and $\vec{z} \in \mathbb{R}^{2T+2}$ with $\sup_{i \in [\![1,2T+2]\!]} z_i \le QN^{1/3}$ we have

$$\mathbf{P}_{\alpha_p}^{\vec{y};\vec{z};p,2T+2}(\widetilde{\mathsf{C}}(p,\beta)) \le \tfrac{\varepsilon}{2}, \text{ where } \widetilde{\mathsf{C}}(p,\beta) := \left\{\mathcal{V}_p < \beta\right\}, \tag{7.5.22}$$

where $\mathcal{V}_p := V_p^T\big((L_i(2T+i-2))_{i \in [\![1,p]\!]}, (z_1, \ldots, z_T)\big)$ (see (7.2.6)). Clearly in view of the definition of D_Q from (7.5.20), the above claim shows that r.h.s. of (7.5.21) is at most ε. Thus it suffices to check (7.5.22). Towards this end, we first claim that

$$\mathbf{P}_{\alpha_p}^{\vec{y};\vec{z};p,2T+2}(\widetilde{\mathsf{C}}(p,\beta)) = \frac{\mathbf{E}_{\alpha_p}^{\vec{y};\vec{w};p,2T+2}\left[\mathbf{1}_{\widetilde{\mathsf{C}}(p,\beta)} \cdot \mathcal{R} \cdot \mathcal{V}_p\right]}{\mathbf{E}_{\alpha_p}^{\vec{y};\vec{w};p,2T+2}\left[\mathcal{R} \cdot \mathcal{V}_p\right]}, \tag{7.5.23}$$

where $\vec{w} \in [-\infty,\infty)^{2T+2}$ defined as $w_i = -\infty$ for $i \le T$ and $w_i = z_i$ for $i > T$, and $\mathcal{R} :=$

$\exp(-e^{z_T - L_2(2T+1)}\mathbf{1}_{p=2})$. We postpone the proof of (7.5.23) to the next step.

Assuming (7.5.23), to prove (7.5.22), we provide upper and lower bounds for the numerator and denominator of the r.h.s. of (7.5.23) respectively. Consider the events

$$\mathsf{R}_1 := \left\{L_1(2T-1) \geq 2QN^{1/3}\right\},$$
$$\mathsf{R}_2 := \left\{L_2(2T) \geq 2QN^{1/3}, L_2(2T+1) \geq 2QN^{1/3}, L_1(2T-1) \geq (2Q-1)N^{1/3}\right\}$$

Note that

$$\begin{aligned}\mathbf{E}_{\alpha_p}^{\vec{y},\vec{w};p,2T+2}\left[\mathcal{R}\cdot\mathcal{V}_p\right] &\geq \mathbf{E}_{\alpha_p}^{\vec{y},\vec{w};p,2T+2}\left[\mathbf{1}_{\mathsf{R}_p}\cdot\mathcal{R}\cdot\mathcal{V}_p\right] \\ &\geq \tfrac{1}{2}\exp(-e^{-QN^{1/3}})\cdot\mathbf{P}_{\alpha_p}^{\vec{y},\vec{w};p,2T+2}\big(\mathsf{R}_p\big) \\ &\geq \tfrac{1}{2}\exp(-e^{-QN^{1/3}})\cdot\mathbf{P}_{\alpha_p}^{\vec{x},(-\infty)^{2T+2};p,2T+2}\big(\mathsf{R}_p\big). \end{aligned} \tag{7.5.24}$$

where the penultimate inequality follows from the definition of $\mathcal{R}$ and Corollary 7.5.5 and the final inequality follows via stochastic monotonicity as R_p is an increasing event with respect to the boundary data (recall $y_i \geq x_i$). To lower bound the above expression, we proceed into two cases depending on the value of p.

Case 1. $p = 1$**.** Note that $\mathsf{R}_1 \supset \mathsf{RP}_{1,Q}$ event defined in (7.4.1). By Lemma 7.4.1, we have $\mathbf{P}_{\alpha_1}^{\vec{x},(-\infty)^{2T+2};1,2T+2}(\mathsf{R}_1) \geq \mathbf{P}_{\alpha_1}^{-QN^{1/3},(-\infty)^{2T+2};1,2T+2}(\mathsf{RP}_{1,Q}) \geq \phi_1 > 0$ for some ϕ_1 free of N.

Case 2. $p = 2$**.** Let $\vec{u} := (-QN^{1/3} + \sqrt{2T+2}, -QN^{1/3})$. Let us use the shorthand notation $\mathbf{P}_2^{\gamma_1,\gamma_2}$ for $\mathbf{P}_{\alpha_2}^{(\gamma_1,\gamma_2),(-\infty)^{2T+2};2,2T+2}$. Note that by stochastic monotonicity and union bound we have

$$\mathbf{P}_2^{\vec{x}}(\mathsf{R}_2) \geq \mathbf{P}_2^{\vec{u}}\big(\{L_2(2T) \geq 2QN^{1/3}\} \cap \{L_2(2T+1) \geq 2QN^{1/3}\}\big) - \mathbf{P}_2^{\vec{u}}\big(L_1(2T-1) \leq L_2(2T) - N^{1/3}\big)$$

Note that $\mathsf{RP}_{2,Q} \subset \{L_2(2T) \geq 2QN^{1/3}\} \cap \{L_2(2T+1) \geq 2QN^{1/3}\}$ (with T replaced by $T+1$ in (7.4.1)). Applying stochastic monotonicity and Lemma 7.4.1 with $p \mapsto 2$ and $T \mapsto T+1$, we see that the first term in the above equation can be bounded as

$$\mathbf{P}_2^{\vec{u}}\big(\{L_2(2T) \geq 2QN^{1/3}\} \cap \{L_2(2T+1) \geq 2QN^{1/3}\}\big) \geq \mathbf{P}_2^{(-QN^{1/3}, -(Q+1)N^{1/3})}\big(\mathsf{RP}_{2,Q}\big) \geq \phi_2,$$

for some $\phi_2 > 0$ free of N. As for the second term, by translation invariance we have

$$\begin{aligned}\mathbf{P}_2^{\vec{u}}\big(L_1(2T-1) \leq L_2(2T) - N^{1/3}\big) &= \mathbf{P}_2^{(0,-\sqrt{2T+1})}\big(L_1(2T-1) \leq L_2(2T) - N^{1/3}\big) \\ &= \frac{1}{\mathbf{E}[W_{\mathrm{sc}}]}\mathbf{E}\left[W_{\mathrm{sc}}\mathbf{1}_{S_{T-1}^{(2T+1,1)} \leq S_{T-1}^{(2T+1,2)} - N^{1/3}}\right].\end{aligned}$$

where the last equality follows from Observation 7.4.5. Here $(S_k^{(2T+1,i)})_{i\in\{1,2\},k\in[\![0,2T+1]\!]} \sim \mathbf{P}_{W_{\mathrm{sc}}}^{(0,-\sqrt{2T+1};f,g)}$ is a WPRW defined in Definition 7.4.4 with $f = G_{\theta,1} * G_{\theta-1}$ and $g = G_{\alpha_2}$. Now by Corollary (7.4.12), $\mathbf{E}[W_{\mathrm{sc}}] \geq \mathrm{C}/\sqrt{2T+1}$ for some absolute constant $\mathrm{C} > 0$. However on the event $\{S_{T-1}^{(2T+1,1)} \leq S_{T-1}^{(2T+1,2)} - N^{1/3}\}$, $W_{\mathrm{sc}} \leq \exp(-e^{N^{1/3}})$ (recall W_{sc} from (7.4.9)). Thus,

$$\mathbf{P}_2^{\vec{u}}\big(L_1(2T-1) \leq L_2(2T) - N^{1/3}\big) \to 0$$

as $N \to \infty$. Hence for all large enough N we have $\mathbf{P}_2^{\vec{x}}(\mathsf{R}_2) \geq \frac{1}{2}\phi$.

Summarizing the above two cases, for all large enough N, (7.5.24) is lower bounded by some $\phi > 0$ free of N. For the numerator in r.h.s. of (7.5.23) observe that as $\mathcal{R} \leq 1$, by definition of the event $\widetilde{\mathsf{C}}(p,\beta)$, we have $\mathbf{1}_{\widetilde{\mathsf{C}}(p,\beta)} \cdot \mathcal{R} \cdot \mathcal{V}_p \leq \beta$. Let us now choose $\beta = \phi\varepsilon$. Plugging these bounds back in r.h.s. of (7.5.23) yields (7.5.22). This completes the proof.

Step 5. We shall prove (7.5.23) for $p = 2$. The $p = 1$ case proof is analogous. Assume $(L_1[\![1, 4T+3]\!], L_2[\![1, 4T+4]\!]) \sim \mathbf{P}_{\alpha_2}^{\vec{y},\vec{z};2,2T+2}$. Let $\mathcal{G} := \sigma(L_i[\![2T+i-2, 4T+i+2]\!])_{i\in[\![1,2]\!]}$. Fix any event F which is measurable with respect to $\mathcal{G}$. Set $L_2(4T+1) = \infty$ and recall the function $W(a;b,c) =$

$\exp(-e^{a-b} - e^{a-c})$. We claim that

$$\begin{aligned}
&\mathbf{E}_{\alpha_p}^{\vec{y},(-\infty)^{2T+2};p,2T+1}\left[\mathbf{1}_{\mathsf{F}} \cdot \prod_{j=1}^{2T+2} W(z_j; L_2(2j+1), L_2(2j-1))\right] \\
&= \mathbf{E}_{\alpha_p}^{\vec{y},(-\infty)^{2T+2};2,2T}\left[\mathbf{1}_{\mathsf{F}} \cdot \exp(-e^{z_T - L_2(2T+1)}) \prod_{j=1}^{2T+2} W(w_j; L_2(2j+1), L_2(2j-1)) \cdot \mathcal{V}_2\right]
\end{aligned} \tag{7.5.25}$$

Clearly (7.5.23) follows from the above identity and (7.2.5) by taking $\mathsf{F} = \Omega$ (the full set, i.e., $\mathbf{1}_{\mathsf{F}} = 1$) and $\mathsf{F} = \neg\mathsf{C}(p,\beta)$. To see (7.5.25), notice that

$$\begin{aligned}
&\mathbf{E}_{\alpha_2}^{\vec{y},(-\infty)^{2T+2};2,2T+2}\left[\mathbf{1}_{\mathsf{F}} \cdot \prod_{j=1}^{2T+2} W(z_j; L_2(2j+1), L_2(2j-1))\right] \\
&= \mathbf{E}_{\alpha_2}^{\vec{y},(-\infty)^{2T+2};2,2T+2}\Bigg[\mathbf{1}_{\mathsf{F}} \cdot \exp(-e^{z_T - L_2(2T+1)}) \prod_{j=T+1}^{2T+2} W(z_j; L_2(2j+1), L_2(2j-1)) \cdot \\
&\qquad \mathbf{E}_{\alpha_2}^{\vec{y},(-\infty)^{2T+2};2,2T+2}\left(\exp(-e^{z_T - L_2(2T-1)}) \prod_{j=1}^{T-1} W(z_j; L_2(2j+1), L_2(2j-1)) \mid \mathcal{G}\right)\Bigg].
\end{aligned}$$

Observe that by the Gibbs property the inner expectation when viewed as a random variable is almost surely equals to $V_2^T((L_i(2T-1), L_2(2T)), (z_1, z_2, \ldots, z_T)) = \mathcal{V}_2$ defined in (7.2.6). On the other hand we have

$$\prod_{j=T+1}^{2T+2} W(z_j; L_2(2j+1), L_2(2j-1)) = \prod_{j=1}^{2T+2} W(w_j; L_2(2j+1), L_2(2j-1)).$$

Combining the above two observations, leads to (7.5.25) completing the proof.

7.5.3 Proof of Lemma 7.5.2

As with the proof of Lemma 7.3.8 and Lemma 7.4.1, we divide the proof of Lemma 7.5.2 into two parts depending on $p = 1$ or $p = 2$.

Proof of Lemma 7.5.2 in the case $p = 1$. Fix any $T \in [\![k_1 N^{\frac{2}{3}}, k_2 N^{\frac{2}{3}}]\!]$. Fix any $\delta \leq \gamma/6\kappa$. For simplicity we write $L := L_1$. We recall the representation of bottom-free law in $p = 1$ case from Observation 7.4.3. Consider the Brownian motion $B := B^{(1)}$ obtained via KMT coupling that satisfies (7.4.10) with $S = T$. Define

$$\mathsf{A}_\delta := \left\{ \sup_{\substack{i_1,i_2\in[\![1,T]\!] \\ |i_1-i_2|\le\frac{\delta}{2}N^{2/3}}} |L(2i_1-1)-L(2i_2-1)| \ge \tfrac{1}{6}\gamma N^{\frac{1}{3}} \right\},$$

$$\mathsf{B}(k) := \left\{ |L(2k-1)-L(2k)|, |L(2k+1)-L(2k)| \ge \tfrac{1}{3}\gamma N^{\frac{1}{3}} \right\}.$$

Fix any $x \in \mathbb{R}$ and set $\mathbf{P}_1^T := \mathbf{P}_{\alpha_1}^{x,(-\infty)^{2T};1,T}$. Observe that by union bound we have

$$\mathbf{P}_1^T \left(\omega_\delta^N(L, [\![1, 2T-1]\!]) \ge \gamma N^{1/3} \right) \le \mathbf{P}_1^T(\mathsf{A}_\delta) + \sum_{k=1}^{T-1} \mathbf{P}_1^T(\neg \mathsf{A}_\delta \cap \mathsf{B}(k)). \tag{7.5.26}$$

We now proceed to bound each of the above term separately. For the first term, by (7.4.7) and (7.4.10), in view of the estimate in (7.4.11) we have for all large enough N we have

$$\begin{aligned} \mathbf{P}_1^T(\mathsf{A}_\delta) &\le \mathbf{P}\Bigg(\sup_{\substack{i_1,i_2\in[\![1,T]\!] \\ |i_1-i_2|\le\frac{\delta}{2}N^{2/3}}} \sigma|B_{T-i_1-1} - B_{T-i_2-1}| \ge \tfrac{\gamma}{12}N^{1/3} - 2\mathrm{C}\log T \Bigg) \\ &\le \mathbf{P}\Bigg(\sup_{\substack{i_1,i_2\in[\![1,T]\!] \\ |i_1-i_2|\le\frac{\delta}{2}N^{2/3}}} \sigma|B_{i_2} - B_{i_2}| \ge \tfrac{\gamma}{24}N^{1/3} \Bigg). \end{aligned}$$

By modulus of continuity of Brownian motion, the r.h.s. of the above equation can be made smaller than $\frac{1}{2}\varepsilon$ by choosing δ small enough depending on $\mu, \theta, \gamma, k_1, k_2$. For the second term on the r.h.s. of (7.5.26) we use Lemma 7.6.5 to get

$$\mathbf{P}_1^T(\neg \mathsf{A}_\delta \cap \mathsf{B}(k)) \le \mathrm{C}e^{-\frac{1}{\mathrm{C}}\gamma N^{\frac{1}{3}}}.$$

Plugging the bounds back in (7.5.26) and taking N large enough we get the desired result. This completes the proof. □

Proof of Lemma 7.5.2 in the case $p = 2$. Fix any $(x_1, x_2) \in I_{2,M}$, and $T \in [\![k_1 N^{2/3}, k_2 N^{2/3}]\!]$. Set

$n := T-1$. Let $(S_k^{(n,1)}, S_k^{(n,2)})_{k=0}^n$ be a paired random bridge defined in Definition 7.4.4 with endpoints (x_1, x_2) and $f = G_{\theta,-1} * G_{\theta,+1}$ and $g = G_{\alpha_2}$. We recall from Observation 7.4.5 that bottom-free law is given by appropriate WPRW for the supercritical case. To proceed with our analysis for the weighted case, we first need an estimate for $\mathbf{E}[W_{\mathrm{sc}}]$ where W_{sc} is the weight defined in (7.4.9).

Lemma 7.5.6. *There exist constants* $\mathrm{C}_1, \mathrm{C}_2 > 0$*, depending on* M*, such that for all* $(x_1, x_2) \in I_{2,M}$ *we have*

$$\mathbf{E}[W_{\mathrm{sc}}] \geq \tfrac{1}{\sqrt{n}}\mathrm{C}_1^{-1} \cdot \mathbf{P}^{n/4}_{(x_1,x_2)}(\mathsf{NI}) \geq \mathrm{C}_2^{-1} e^{-\mathrm{C}_2(\log n)^{5/4}}, \tag{7.5.27}$$

where $\mathbf{P}^{n/4}_{(x_1,x_2)}(\mathsf{NI})$ *denotes the non-intersection probability of two independent random walks of length* $n/4$ *starting at* (x_1, x_2).

We postpone the proof of Lemma 7.5.6 and complete the proof of Lemma 7.5.2 in the following two steps.

Step 1. Fix any $S, \gamma > 0$. Set $v = \gamma/\sqrt{k_2}$, $u = S/\sqrt{k_1}$, and $t = 2\log\log n$. Let $\mathcal{F} := \sigma(S_0^{(n,1)}, S_0^{(n,2)})$. Consider the events

$$\mathsf{MC}_\delta := \left\{ |S_0^{(n,1)}| + |S_0^{(n,2)}| \leq u\sqrt{n}, \omega_\delta^N(S_\cdot^{(n,i)}, [\![0, \tfrac{n}{8}]\!]) \geq \tfrac{1}{6}v\sqrt{n}, \text{ for } i = 1,2 \right\}, \text{ for } \delta > 0.$$

We claim that given $\varepsilon > 0$, there exists δ small enough and N large enough such that

$$\mathbf{P}_{W_{\mathrm{sc}}}(\mathsf{MC}_\delta) := \frac{\mathbf{E}[W_{\mathrm{sc}}\mathbf{1}_{\mathsf{MC}_\delta}]}{\mathbf{E}[W_{\mathrm{sc}}]} \leq \varepsilon. \tag{7.5.28}$$

where W_{sc} is defined in (7.4.9). We finish the proof of the lemma assuming (7.5.28). Indeed from Observation 7.4.5 we know that $(L_1(2j+1), L_2(2j+2))_{j=0}^n$ is distributed as WPRW. Observe that by Lemma 7.6.5 and tail estimates for G (defined in (7.2.3)) for all $k \geq 1$ we have

$$
\begin{aligned}
&\mathbf{P}_2\Big(|L_1(2k) - L_1(2k-1)|, |L_1(2k) - L_1(2k+1)| \geq \tfrac{1}{3}\gamma N^{1/3} \\
&\qquad\qquad \mid |L_1(2k-1) - L_1(2k+1)| \leq \tfrac{1}{6}\gamma N^{1/3}\Big) \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}\gamma N^{1/3}), \\
&\mathbf{P}_2\Big(|L_2(2k+1) - L_2(2k)|, |L_2(2k+1) - L_2(2k+2)| \geq \tfrac{1}{3}\gamma N^{1/3} \\
&\qquad\qquad \mid |L_2(2k) - L_2(2k+2)| \leq \tfrac{1}{6}\gamma N^{1/3}\Big) \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}\gamma N^{1/3}), \\
&\mathbf{P}_2\Big(|L_2(1) - L_2(2)| \leq \tfrac{1}{6}\gamma N^{1/3}\Big) \leq \mathrm{C}\exp(-\tfrac{1}{\mathrm{C}}\gamma N^{1/3}),
\end{aligned}
$$

where $\mathbf{P}_2 := \mathbf{P}_{\alpha_2}^{\vec{x},(-\infty)^{2T};2,T}$. Thus, in view of (7.5.28), by union bound

$$
\sum_{j=1}^{2} \mathbf{P}_2\left(|L_1(1)| + |L_2(2)| \leq SN^{\frac{1}{3}}, \omega_\delta^N(\mathcal{L}_j^N, [\![1, T/4 + j - 2]\!]) \geq \gamma N^{\frac{1}{3}}\right) \leq \varepsilon + \mathrm{C}\cdot 2k_2 N^{\frac{2}{3}} \exp(-\tfrac{1}{\mathrm{C}}\gamma N^{\frac{1}{3}})
$$

which can be made arbitrarily small taking N large enough. This completes the proof.

Step 2. In this step we prove (7.5.28). We first define a few more necessary events.

$$
\begin{aligned}
\mathsf{G}_1 &:= \{|S_0^{(n,1)}| + |S_0^{(n,2)}| \leq u\sqrt{n}, |S_0^{(n,1)} - S_0^{(n,2)}| \leq (\log n)^{3/2}\}, \\
\mathsf{G}_2 &:= \{|S_0^{(n,1)}| + |S_0^{(n,2)}| \leq u\sqrt{n}, 1 \leq S_0^{(n,1)} - S_0^{(n,2)} \leq 2\}.
\end{aligned}
$$

Recall the non-intersection event NI_p from (7.4.25). As $W_{\mathrm{sc}} \leq 1$, we write

$$
\mathbf{E}[W_{\mathrm{sc}}\mathbf{1}_{\mathsf{MC}_\delta}] \leq \underbrace{\mathbf{E}[W_{\mathrm{sc}}\mathbf{1}_{\mathsf{MC}_\delta \cap \mathsf{G}_1 \cap \mathsf{NI}_t}]}_{(\mathbf{I})} + \underbrace{\mathbf{E}[\mathbf{1}_{\neg\mathsf{NI}_t}] + \mathbf{E}[\mathbf{1}_{\neg\mathsf{G}_1}]}_{(\mathbf{II})}.
$$

For (**II**), note that on $\neg\mathsf{NI}_t$, we have $W_{\mathrm{sc}} \leq e^{-e^t} = e^{-(\log n)^2}$ and by Lemma 7.4.7, $\mathbf{P}(\neg\mathsf{G}_1) \leq \mathrm{C}e^{-\mathrm{C}^{-1}(\log n)^{3/2}}$. Thus, (**II**) $\leq \mathrm{C}e^{-\mathrm{C}^{-1}(\log n)^{3/2}}$. In view of Lemma 7.5.6, $(\mathbf{E}[W_{\mathrm{sc}}])^{-1} \cdot (\mathbf{II}) \to 0$. For

$(\mathbf{I})$, note that

$$(\mathbf{I}) = \sum_{p=1}^{t} \mathbf{E}[W_{\mathrm{sc}} \mathbf{1}_{\mathsf{MC}_\delta \cap \mathsf{G}_1 \cap \mathsf{NI}_p \cap \neg \mathsf{NI}_{p-1}}] \le \sum_{p=1}^{t} e^{-e^p} \mathbf{E}[\mathbf{1}_{\mathsf{G}_1} \mathbf{E}[\mathbf{1}_{\mathsf{MC}_\delta \cap \mathsf{NI}_p} \mid \mathcal{F}]]$$
$$\le \sum_{p=1}^{t} \mathrm{C} e^{-e^p} \cdot \mathbf{E}[\mathbf{1}_{\mathsf{G}_1} \sup_{p \in [\![0,t]\!]} \widetilde{\mathbf{P}}_p(\mathsf{MC}_\delta \cap \mathsf{NI})],$$

where $\widetilde{\mathbf{P}}_p$ denote the law of $(n; n/4, n/4)$-modified random bridge defined in Definition 7.4.9 starting from $(S_0^{(n,1)} + p, S_0^{(n,2)})$ to $(x_1 + p, x_2)$. The last inequality follows from Lemma 7.4.10. Here the constant C depends on u. By Lemma 7.7.1 and Lemma 7.7.5, on G_1 we have

$$\widetilde{\mathbf{P}}_p(\mathsf{NI}) \le \tfrac{\mathrm{C}}{\sqrt{n}} \cdot e^{\mathrm{C}p} \cdot \max\{S_0^{(n,1)} - S_0^{(n,2)}, 1\} \cdot \mathbf{P}^{n/4}_{(x_1,x_2)}(\mathsf{NI}).$$

Thus setting $\mathrm{C}_3 := \sum_{r=1}^{\infty} 2\mathrm{C}_1\mathrm{C}^3 e^{\mathrm{C}r} e^{-e^r}$ (with C_1 coming from Lemma 7.5.6) we have

$$(\mathbf{E}[W_{\mathrm{sc}}])^{-1} \cdot (\mathbf{I}) \le \mathrm{C}_3 \cdot \mathbf{E}\left[\mathbf{1}_{\mathsf{G}_1} \cdot \max\{S_0^{(n,1)} - S_0^{(n,2)}, 1\} \cdot \sup_{p \in [\![0,t]\!]} \widetilde{\mathbf{P}}_p(\mathsf{MC}_\delta \mid \mathsf{NI})\right]$$

Now we claim that one can choose δ sufficiently small such that

$$\mathbf{E}\left[\mathbf{1}_{\mathsf{G}_1} \cdot \max\{S_0^{(n,1)} - S_0^{(n,2)}, 1\} \cdot \sup_{p \in [\![0,t]\!]} \widetilde{\mathbf{P}}_p(\mathsf{MC}_\delta \mid \mathsf{NI})\right] \le \tfrac{1}{2}\mathrm{C}_1^{-1}\varepsilon.$$

We write $\mathsf{G}_1 = \mathsf{G}_{1,M_2} \cup \widetilde{\mathsf{G}}_{1,M_2}$, where

$$\mathsf{G}_{1,M_2} := \{|S_0^{(n,1)}| + |S_0^{(n,2)}| \le u\sqrt{n}, |S_0^{(n,1)} - S_0^{(n,2)}| \le M_2\}, \quad \widetilde{\mathsf{G}}_{1,M_2} := \mathsf{G}_1 \cap \neg \mathsf{G}_{1,M_2}.$$

Given the tail estimates, one can choose M_2 large enough such that

$$\mathbf{E}\left[\mathbf{1}_{\widetilde{\mathsf{G}}_{1,M_2}} \cdot \max\{S_0^{(n,1)} - S_0^{(n,2)}, 1\}\right] \le \tfrac{1}{4}\mathrm{C}_1^{-1}\varepsilon.$$

This fixes our choice for M_2. Now note that the event MC_δ depends only on the first $n/8$ points of the two $(n; n/4, n/4)$-modified random bridges. By definition, the first $n/4$ points of a $(n; n/4, n/4)$-modified random bridge is just a random walk. Thus, in view of Lemma 7.7.9, one can then choose δ small enough and N large enough such that on uniformly on G_{1,M_2} we have

$$\sup_{p\in[\![0,t]\!]} \widetilde{\mathbf{P}}_p(\mathsf{MC}_\delta \mid \mathsf{NI}) \le \tfrac{1}{4}\mathrm{C}_1^{-1}M_2^{-1}\varepsilon.$$

This completes the proof. □

Proof of Lemma 7.5.6. Recall the definition of (n, p, q)-modified random bridge from Definition 7.4.9. Let $\widetilde{\mathbf{P}}_{(a_1,a_2)}$ denote the law of two independent (n, p, q)-modified random bridge starting at (a_1, a_2) and ending at (x_1, x_2) with increments from $G_{\theta,+1} * G_{\theta,-1}$. We write $(S_k^{(n,i)})_{k\in[\![1,n]\!],i\in[\![1,2]\!]}$ for the corresponding random variable. We also use the notation $\mathbf{P}^m_{(b_1,b_2)}$ to denote the law of two independent random walks of length m starting at (b_1, b_2) with same increment law. We use $(U_k, V_k)_{k=0}^n$ for the corresponding random variable.

Recall the event Gap_β from (7.4.20). Invoking Lemma 7.7.8 we first fix a $\beta = \beta(M) \le \frac{1}{2}$ small enough so that it satisfies

$$\widetilde{\mathbf{P}}_{(a_1,a_2)}(\mathsf{Gap}_\beta \mid \mathsf{NI}) \ge \tfrac{3}{4},$$

for all $|a_i| \le \sqrt{n}$ with $1 \le a_1 - a_2 \le 2$. Next by Lemma 7.7.2, we fix $\xi = \xi(M) > 0$ so that

$$\mathbf{P}^{n/4}_{(b_1,b_2)}(|U_{n/4}|, |V_{n/4}| \le \xi\sqrt{n} \mid \mathsf{NI}) \ge \sqrt{\tfrac{3}{4}}$$

for all $|b_i| \le (M + 1)\sqrt{n}$.

We consider the following events

$$\begin{aligned}
\mathsf{G}_3 &:= \left\{|S_0^{(n,i)}| \le \sqrt{n} \text{ for } i = 1, 2,\ 1 \le S_0^{(n,1)} - S_0^{(n,2)} \le 2\right\},\\
\mathsf{Tgt}_\xi &:= \left\{|S_{n/4}^{(n,i)}|, |S_{3n/4}^{(n,i)}| \le \xi\sqrt{n} \text{ for } i = 1, 2\right\},
\end{aligned}$$

where Tgt stands for tightness. Observe that by Lemma 7.4.8 we have

$$\mathbf{E}[W_{\mathrm{sc}}] \geq \mathbf{E}[W_{\mathrm{sc}}\mathbf{1}_{\mathsf{Gap}_\beta \cap \mathsf{G}_3 \cap \mathsf{Tgt}_\xi}] \geq \tfrac{1}{\mathrm{C}}\mathbf{P}(\mathsf{Gap}_\beta \cap \mathsf{G}_3 \cap \mathsf{Tgt}_\xi) = \tfrac{1}{\mathrm{C}}\mathbf{E}\left[\mathbf{1}_{\mathsf{G}_3}\mathbf{E}[\mathbf{1}_{\mathsf{Gap}_\beta, \mathsf{Tgt}_\xi} \mid \mathcal{F}]\right] \tag{7.5.29}$$

where $\mathcal{F} := \sigma(S_0^{(n,1)}, S_0^{(n,2)})$. Under the event G_3 and Tgt_ξ we may invoke Lemma 7.4.10 to get

$$\mathbf{1}_{\mathsf{G}_3} \cdot \mathbf{E}[\mathbf{1}_{\mathsf{Gap}_\beta \cap \mathsf{Tgt}_\xi} \mid \mathcal{F}] \geq \mathrm{C}^{-1} \cdot \mathbf{1}_{\mathsf{G}_3} \cdot \widetilde{\mathbf{P}}_{(a_1,a_2)}[\mathsf{Gap}_\beta \cap \mathsf{Tgt}_\xi] \tag{7.5.30}$$

almost surely, where $a_i = S_0^{(n,i)}$. By Corollary 7.7.7

$$\begin{aligned}\widetilde{\mathbf{P}}_{(a_1,a_2)}(\mathsf{Gap}_\beta \cap \mathsf{Tgt}_\xi) &= \widetilde{\mathbf{P}}_{(a_1,a_2)}(\mathsf{Gap}_\beta \cap \mathsf{Tgt}_\xi \mid \mathsf{NI})\widetilde{\mathbf{P}}_{(a_1,a_2)}(\mathsf{NI}) \\ &\geq \mathrm{C}^{-1}\widetilde{\mathbf{P}}_{(a_1,a_2)}(\mathsf{Gap}_\beta \cap \mathsf{Tgt}_\xi \mid \mathsf{NI}) \cdot \mathbf{P}^{n/4}_{(a_1,a_2)}(\mathsf{NI})\mathbf{P}^{n/4}_{(x_1,x_2)}(\mathsf{NI}).\end{aligned} \tag{7.5.31}$$

By our choice of β and ξ, we have $\widetilde{\mathbf{P}}_{(a_1,a_2)}(\mathsf{Gap}_\beta, \mathsf{Tgt}_\xi \mid \mathsf{NI}) \geq \frac{1}{2}$ uniformly over the event G_3. By Lemma 7.7.1, we have $\mathbf{P}^{n/4}_{(a_1,a_2)}(\mathsf{NI}) \geq \frac{\mathrm{C}^{-1}}{\sqrt{n}}$ uniformly over the event G_3. Thus combining (7.5.29), (7.5.30), and (7.5.31) we have

$$\mathbf{E}[W_{\mathrm{sc}}] \geq \tfrac{1}{\sqrt{n}}\mathrm{C}^{-1} \cdot \mathbf{P}^{n/4}_{(x_1,x_2)}(\mathsf{NI}) \cdot \mathbf{P}(\mathsf{G}_3)$$

By Lemma 7.4.7 ((7.4.17) in particular), $\mathbf{P}(\mathsf{G}_3) \geq \mathrm{C}^{-1}$. Plugging this back in the above equation we get the first inequality in (7.5.27). For the second inequality, we consider the event:

$$\mathsf{G}_4 := \{|U_1 - x_1| \leq 1, |V_i - \min\{x_1 - 3, x_2\}| \leq 1\}.$$

Observe that

$$\mathbf{P}^{n/4}_{(x_1,x_2)}(\mathsf{NI}) \geq \mathbf{P}^{n/4}_{(x_1,x_2)}\Big(\mathsf{G}_4 \cap \{U_j \geq V_j \text{ for all } j \in [\![2, n/4]\!]\}\Big).$$

By the tail bounds of the increments from Lemma 7.6.4, and given the condition $x_1 - x_2 \geq -(\log N)^{7/6}$, we have $\mathbf{P}(\mathsf{G}_4) \geq \mathrm{C}^{-1}\exp(-\mathrm{C}(\log n)^{7/6})$ (recall $n \geq k_1 N^{2/3} - 1$). Furthermore, on G_4 we must have $U_1 \geq V_1$. By Lemma 7.7.1, we have $\mathbf{P}^{n/4-1}_{(a_1,a_2)}(\mathsf{NI}) \geq \mathrm{C}^{-1}/\sqrt{n}$ for all $a_1 \geq a_2$. Thus we have

$$\mathbf{P}^{(n/4)}_{(x_1,x_2)}\bigg(\mathsf{G}_4 \cap \{U_j \geq V_j \text{ for all } j \in [\![2, n/4]\!]\}\bigg) \geq \mathrm{C}^{-1}\exp(-\mathrm{C}(\log n)^{7/6}) \cdot \tfrac{1}{\sqrt{n}}.$$

Adjusting the constant we get the second inequality in (7.5.27). This completes the proof. □

7.6 Basic properties of log-gamma type random variables

In this section we collect some basic facts about log-gamma type random variables. Towards this end, for each $\theta, \kappa > 0$, and $m \in \mathbb{Z}_{>0}$ we consider the following function:

$$H_{\theta,\kappa,(-1)^m}(y) := \frac{\kappa^\theta}{\Gamma(\theta)}\exp(\theta(-1)^m y - \kappa e^{(-1)^m y}).$$

It is plain to check H is a valid probability density function. Observe that $H_{\theta,1,(-1)^m} \equiv G_{\theta,(-1)^m}$ where G is defined in (7.2.3). The following lemma collects some useful properties of H. Its proof follows via straightforward computations and is hence omitted.

Lemma 7.6.1. *Suppose $X \sim H_{\theta,\kappa,1}$. We have the following.*

(a) $-X \sim H_{\theta,\kappa,-1}$.

(b) For every $\alpha > -\theta$ we have $\mathbf{E}[e^{\alpha X}] = \frac{\Gamma(\alpha+\theta)}{\kappa^\alpha\Gamma(\theta)}$.

We next define generalized $\mathcal{HSLG}$ Gibbs measures in the same vein as $\mathcal{HSLG}$ Gibbs measures (see Definition 7.1.2) but by considering the weight function

$$\widetilde{W}_e(x) = \begin{cases} \exp(\theta x - \kappa e^x) & \text{if } e \text{ is } \textbf{blue} \\ \exp(-\gamma e^x) & \text{if } e \text{ is } \textbf{black} \\ \exp(-\alpha x) & \text{if } e \text{ is } \textbf{red}. \end{cases}$$

instead of W defined in (7.1.3). $\kappa = \gamma = 1$ in above weights lead to the usual Gibbs measures. The following result ensures that generalized $\mathcal{HSLG}$ Gibbs measures (and hence the usual ones from Definition 7.1.2) are well defined.

Lemma 7.6.2. *Fix any $\gamma, \kappa, \theta > 0$, and $\alpha > -\theta$. Recall the graph G from Section 7.1.3 used in defining $\mathcal{HSLG}$ Gibbs measures. Given a domain Λ and a boundary condition $\{u_{i,j} : (i,j) \in \partial\Lambda\}$, we have*

$$\int_{\mathbb{R}^{|\Lambda|}} \prod_{e=\{v_1\to v_2\}\in E(\Lambda\cup\partial\Lambda)} \widetilde{W}_e(u_{v_1} - u_{v_2}) \prod_{v\in\Lambda} du_v < \infty.$$

Let us suppose $|u_{i,j}| \le R$ for all $(i,j) \in \Lambda$. Let us assume $\Lambda = \mathcal{K}_{k,T}$ or $\mathcal{K}'_{k,T}$ defined in (7.2.1)*. There exists a constant* C *that depends only on γ, κ, θ, and α such that*

$$\int_{\mathbb{R}^{|\Lambda|}} \prod_{e=\{v_1\to v_2\}\in E(\Lambda\cup\partial\Lambda)} \widetilde{W}_e(u_{v_1} - u_{v_2}) \prod_{v\in\Lambda} du_v \le \mathrm{C}^{kT+R}.$$

Proof. First note that, for red edges $\{v_1 \to v_2\}$, the corresponding weight function $W_e(u_{v_1} - u_{v_2})$ factors out as $e^{-\alpha u_{v_1}} \cdot e^{\alpha u_{v_2}}$. Hence they can be viewed as vertex weight functions. More specifically, at each vertex $(k,1)$ we can associate the vertex weight function $V_k(u) := e^{(-1)^k \alpha u}$. They replace the role of red edge weights. We denote this vertex weights as red circles in Figure 7.20. We now divide our analysis into two cases based on the value of α.

Suppose $\alpha \in (-\theta, \theta)$. As black edge weights are less than 1, we may drop all of them to get a Gibbs measure based on the blue and red edge weights only (see Figure 7.20 B). The integral

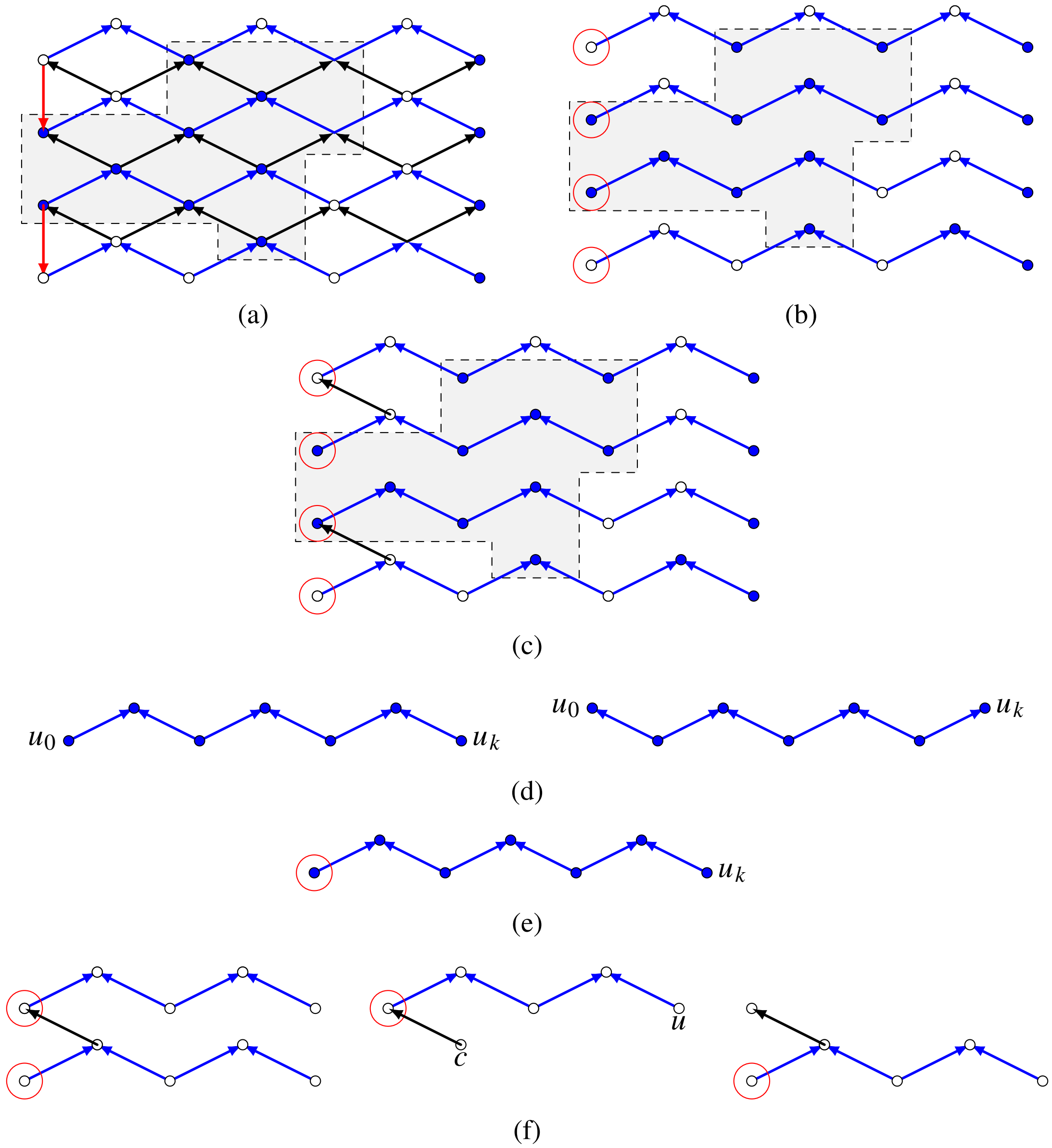


Figure 7.20: (A) A possible domain Λ. (B) Reduction in the case of $\alpha \in (-\theta, \theta)$. (C) Reduction in the case of $\alpha > 0$. (D) Type I Gibbs measures. The figure shows two of them of even length. It may also have odd length with one edge at either of the end removed. (E) Type II Gibbs measures. It may also have odd length with one edge at right end removed. (F) Few examples of Type III Gibbs measures.

of the reduced Gibbs measure can be viewed as a product of integrals of several smaller Gibbs measures that are two types: Type I and Type II (see Figure 7.20 D and E). Type I Gibbs measures are the ones where red vertex weights does not appear. The integral corresponding to Type I takes the following form:

$$\left[\kappa^{\theta}(\Gamma(\theta))^{-1}\right]^{k}\int_{\mathbb{R}^{k-1}}\prod_{i=1}^{k}H_{\theta,\kappa,(-1)^{i+m}}(u_{i-1}-u_i)\prod_{i=1}^{k-1}du_i$$

where u_0 and u_k are in $\partial\Lambda$. In this case, we may use $H_{\theta,\kappa,(-1)^{i+m}}(u_{k-1}-u_k)\leq \mathrm{C}$ and the fact that H is a pdf to get that the integral is bounded by $\mathrm{C}\cdot\left[\kappa^{\theta}(\Gamma(\theta))^{-1}\right]^{k}$. Type II Gibbs measures are the ones where red vertex weights are present. The integral corresponding to the Type II Gibbs measures takes the form

$$\int_{\mathbb{R}^k}\prod_{i=1}^{k}e^{(-1)^m\alpha u_0}\cdot e^{(-1)^{i+m}\theta(u_{i-1}-u_i)-\kappa e^{(-1)^{i+m}(u_{i-1}-u_i)}}\prod_{i=1}^{k}du_i.$$

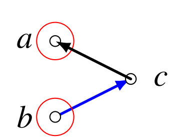
The integrand can be manipulated to show that the above integral is equal to

$$\begin{aligned}&e^{(-1)^{m+k-1}\alpha u_k}\prod_{i=1}^{k-1}(\Gamma(\theta+(-1)^{m+i+1}\alpha))\kappa^{-\theta+(-1)^{m+i}\alpha}\int_{\mathbb{R}^k}\prod_{i=1}^{k}H_{\theta+(-1)^{m+i+1}\alpha,\kappa,(-1)^{i-1}}(x_i)\prod_{i=1}^{k}dx_i\\&=e^{(-1)^{m+k-1}\alpha u_k}\prod_{i=1}^{k-1}(\Gamma(\theta+(-1)^{m+i+1}\alpha))\kappa^{-\theta+(-1)^{m+i}\alpha}.\end{aligned}$$

This verifies the lemma for $\alpha\in(-\theta,\theta)$.

For $\alpha>0$, we remove all the black edges except the ones connecting $(2i-1,1)$ to $(2i,1)$. This leads to a reduced Gibbs measures shown in Figure 7.20 C. The reduced Gibbs measure decomposes into several Type I Gibbs measures and Type III Gibbs measures. Type III Gibbs measures are the ones that has the red vertex weights. Few of the possible Type III Gibbs measures are shown in Figure 7.20.

- If a Type III Gibbs measure has two red vertices in its domain $\cup$ boundary, we may use the fact that the weight of the figure



is $e^{(\theta+\alpha)(b-c)-\kappa e^{b-c}} \cdot e^{\alpha(c-a)-\gamma e^{c-a}} \leq \mathrm{C} e^{(\theta+\alpha)(b-c)-\kappa e^{b-c}}$.

- If a Type III Gibbs measure has only one red vertex in its domain $\cup$ boundary, then it must contain either of the two following figures

with $c \in \partial\Lambda$. The corresponding weights are $e^{\alpha c} \cdot e^{(\theta+\alpha)(b-c)-\kappa e^{b-c}} \leq \mathrm{C} e^{\alpha c}$ and $e^{-\alpha c} \cdot e^{\alpha(c-a)-\gamma e^{c-a}} \leq \mathrm{C} e^{-\alpha c}$ respectively.

Based on the kind of Type III Gibbs measures, we may insert the above obtained bound in the integrand of this type of Gibbs measures. The resulting integral can then be computed explicitly to yield a bound of the form $\mathrm{C}^V e^{|\alpha c|}$ where V is the number of vertices in the Gibbs measures. For example, for the middle figure in Figure 7.20 we have (with $u_4 := u$)

$$\begin{aligned}
&\left[\kappa^{-\theta}\Gamma(\theta)\right]^4 \int_{\mathbb{R}^4} e^{-\alpha u_0} e^{-\gamma e^{c-u_0}} \prod_{i=0}^{3} H_{\theta,\kappa,(-1)^i}(u_i - u_{i+1}) du_i \\
&\leq \left[\kappa^{-\theta}\Gamma(\theta)\right]^4 \cdot \mathrm{C} e^{-\alpha c} \int_{\mathbb{R}^4} \prod_{i=0}^{3} H_{\theta,\kappa,(-1)^i}(u_i - u_{i+1}) du_i \leq \left[\kappa^{-\theta}\Gamma(\theta)\right]^4 \cdot \mathrm{C} e^{|\alpha c|}.
\end{aligned}$$

This establishes the lemma for $\alpha > 0$. $\square$

For the rest of the appendix we fix some $\theta > 0$ and reserve the notation f for the function

$$f(x) := G_{\theta,+1} * G_{\theta,-1}(x) \tag{7.6.1}$$

Note that f is symmetric. We set the variance of f to be $\sigma^2(\theta) > 0$. All the constants appearing in the subsequent lemmas of the appendix may depend on θ. We will not mention this further.

We first state a few properties of f useful for our later analysis. The following lemma concerns with sharp rate of convergence of pdf of $(X_1 + X_2 + \cdots + X_n)/\sqrt{n}$, where $X_i \overset{i.i.d.}{\sim} f$, to normal density with appropriate variance.

Lemma 7.6.3. *Let f^{*n} be the n-fold convolution of f. We have*

$$\sup_{|x|\le(\log n)^2}\left|\frac{\sqrt{n}f^{*n}(x\sqrt{n})}{\phi_\sigma(x)}-1\right|=O(n^{-3/4}).$$

where $\phi_\sigma(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}$.

Proof. The proof below is adapted from Theorem 5 in Chapter XV in [163]. Let ψ denote the Fourier transform (characteristic function) of f. We will prove the lemma for general f satisfying the following two assumptions:

- f is symmetric and
- given any $\delta>0$, $\sup_{t\ge\delta}|\psi(t)|=\eta<1$.

Clearly f in (7.6.1) satisfies the above two assumptions. In what follows, for simplicity we will assume $\sigma^2=1$.

Set $f_n(x):=\int_{\mathbb{R}}e^{itx}\psi^n(t/\sqrt{n})dt$. We have $\sqrt{n}f^{*n}(x/\sqrt{n})=f_n(x)$. Set $\alpha=1/16$. Under the assumption on f, we have

$$\psi(t/\sqrt{n})=1-\tfrac{t^2}{2n}+O(\tfrac{t^4}{n^2}).$$

Thus for $|t|\le n^\alpha$, we have $\psi(t/\sqrt{n})=1-\frac{t^2}{2n}+O(n^{4\alpha-2})=e^{-t^2/2n+O(n^{4\alpha-2})}$. Thus $\psi^n(t/\sqrt{n})=e^{-t^2/2}(1+O(n^{-3/4}))$, where the O term is free of t in that specified range. Thus,

$$\begin{aligned}f_n(x)&=(1+O(n^{-3/4}))\int_{|t|\le n^\alpha}e^{itx}e^{-t^2/2}dt+\int_{|t|\ge n^\alpha}e^{itx}\psi^n(t/\sqrt{n})dt\\&=(1+O(n^{-3/4}))\int_{\mathbb{R}}e^{itx}e^{-t^2/2}dt+\int_{|t|\ge n^\alpha}e^{itx}\psi^n(t/\sqrt{n})dt-(1+O(n^{-3/4}))\int_{|t|\ge n^\alpha}e^{itx}e^{-t^2/2}dt.\end{aligned}$$

We next compute the order of the last two integrals above. Clearly $\int_{|t|\ge n^\alpha}e^{-t^2/2}dt\le Ce^{-cn^{2\alpha}}$. For the second one, we choose $\delta>0$ small enough such that $|\psi(t)|\le e^{-t^2/4}$ for all $|t|\le\delta$. This implies

$$\int_{n^\alpha \le t \le \sqrt{n}\delta} |\psi^n(t/\sqrt{n})| dt \le \mathrm{C}e^{-cn^{2\alpha}}.$$

For $|t| \ge \sqrt{n}\delta$, we know $\sup_{t\ge\delta} |\psi(t)| = \eta < 1$. This forces

$$\int_{|t|\ge\sqrt{n}\delta} |\psi^n(t/\sqrt{n})| dt \le \eta^{n-1}\sqrt{n} \int_{\mathbb{R}} \psi(t) dt.$$

Thus the error integrals are at most $\mathrm{C}\rho^{-n^{1/8}}$ in absolute value uniform in x. Furthermore if we assume $|x| \le (\log n)^2$, $\phi_1(x) \ge \frac{1}{\sqrt{2\pi}} e^{-(\log n)^2/2}$, which dominates the error coming from the integrals. Hence we may divide $\phi_1(x)$ and still obtain that the errors are going to zero. This completes the proof of the lemma. □

The next lemma concerns with the uniform tail behavior f.

Lemma 7.6.4. *For all $x \in \mathbb{R}$ we have*

$$e^{-2e} \le f(x)e^{\theta|x|} \le \Gamma(2\theta).$$

Proof. Since f is symmetric, it suffices to show the lemma for $x > 0$. We have

$$f(x) = \int_{\mathbb{R}} e^{\theta y - e^y + \theta(y-x) - e^{y-x}} dy = e^{-\theta x} \int_{\mathbb{R}} e^{2\theta y - e^y - e^{y-x}} dy$$

Now for the lower bound we observe

$$\int_{\mathbb{R}} e^{2\theta y - e^y - e^{y-x}} dy \ge \int_0^1 e^{2\theta y - e^y - e^{y-x}} dy \ge e^{-2e},$$

whereas for the upper bound we have

$$\int_{\mathbb{R}} e^{2\theta y - e^y - e^{y-x}} dy \le \int_{\mathbb{R}} e^{2\theta y - e^y} dy \ge \Gamma(2\theta).$$

This establishes the lemma. $\square$

We end this section by recording a tail estimate for the ξ-distributions introduced in (7.4.5).

Lemma 7.6.5. *Fix any $a, b \in \mathbb{R}$ and $\theta_0 > 1$. Consider $X \sim \xi^{(a,b)}_{\theta_1,\theta_2;\pm 1}$ where $\xi^{(a,b)}_{\theta_1,\theta_2;\pm 1}$ is defined in* (7.4.5). *There exists a constant* $\mathrm{C} > 0$ *depending on θ_0 such that for all $\theta_1, \theta_2 \in [\theta_0^{-1}, \theta_0]$ and for all $r \geq |a-b|$ we have*

$$\mathbf{P}(X \notin [\min\{a,b\} - 2r, \max\{a,b\} + 2r]) \leq \mathrm{C}e^{-\frac{1}{\mathrm{C}}r}.$$

Proof. Fix any $\theta_1, \theta_2 \in [\theta_0^{-1}, \theta_0]$. We prove the bound for $\xi^{(a,b)}_{\theta_1,\theta_2;1}$. The proof for the case $\xi^{(a,b)}_{\theta_1,\theta_2;-1}$ is analogous. Without loss of generality assume $b \leq a$. Observe that

$$\int_a^{a+1} G_{\theta_1,1}(a-x)G_{\theta_2,1}(b-x) \geq \tfrac{1}{\mathrm{C}} \cdot e^{-\max\{\theta_1,\theta_2\}\cdot(a-b)},$$

where in above we used the fact that $G_{\beta,1}(-y) \geq \mathrm{C}^{-1}e^{-\beta y}$ (recall G from (7.2.3)). Similarly one has

$$\int_{x \leq b-2r} G_{\theta_1,1}(a-x)G_{\theta_2,1}(b-x)\mathrm{d}x + \int_{x \geq a+2r} G_{\theta_1,1}(a-x)G_{\theta_2,1}(b-x)\mathrm{d}x \leq \mathrm{C} \cdot e^{-2(\theta_1+\theta_2)r}.$$

Thus as long as $r \geq a-b$ adjusting the constant C we get the desired result. $\square$

7.7 Estimates for non-intersection probability

In this section, we study non-intersection probability of random walks, random bridges, and modified random bridges defined in Definition 7.4.9.

Let $X_i \sim f$ where f is defined in (7.6.1). Set $S_0^{(n)} := a$ and $S_k^{(n)} := a + \sum_{i=1}^k X_i$ for $k \in [\![1,n]\!]$. We denote the law of $(S_k^{(n)})_{k=0}^n$, the random walk of size n starting at a, to be $\mathbf{P}_a^n$. Given two independent random walks of size n starting at a_1 and a_2, we denote their joint law to be $\mathbf{P}^n_{(a_1,a_2)}$.

Given $(U_k, V_k)_{k=0}^n \sim \mathbf{P}^n_{(a_1,a_2)}$, we define the weak non-intersection event as

$$\mathsf{NI}_p := \{U_k - V_k \geq -p \text{ for all } k \in [\![1, n]\!]\},$$

and $\mathsf{NI} := \mathsf{NI}_0$ (the true non-intersection event). When $a_1 - a_2 = O(1)$, it is well known that $\mathbf{P}^n_{(a_1,a_2)}(\mathsf{NI}) = O(n^{-1/2})$. We record this classical fact in the following lemma.

Lemma 7.7.1. *For all* $(a_1, a_2) \in \mathbb{R}^2$ *we have* $\mathbf{P}^n_{(a_1,a_2)}(\mathsf{NI}) \leq \mathrm{C}\frac{\max\{a_1-a_2,1\}}{\sqrt{n}}$ *for some absolute constant* $\mathrm{C} > 0$. *If in addition* $a_1 \geq a_2$, *we have* $\mathbf{P}^n_{(a_1,a_2)}(\mathsf{NI}) \geq \frac{\mathrm{C}^{-1}}{\sqrt{n}}$.

Proof. The first part is [228, Theorem A] and the second part is [297, Theorem 3.5]. □

Next we study diffusive properties of the random walks under the non-intersecting event.

Lemma 7.7.2. *Given any* $\varepsilon > 0$ *there exists a constant* $\delta(\varepsilon) > 0$ *such that for all* $n \geq 1$ *and* $(a_1, a_2) \in \mathbb{R}^2$ *we have*

$$\mathbf{P}^n_{(a_1,a_2)}\left(U_n - V_n \geq \delta\sqrt{n} \mid \mathsf{NI}\right) \geq 1 - \varepsilon, \tag{7.7.1}$$

$$\mathbf{P}^n_{(a_1,a_2)}\left(\sup_{k\in[\![0,n]\!]} (U_k - V_k) \leq \delta^{-1}\sqrt{n} + \max\{a_1 - a_2, 0\} \mid \mathsf{NI}\right) \geq 1 - \varepsilon, \tag{7.7.2}$$

$$\mathbf{P}^n_{(a_1,a_2)}\left(\inf_{k\in[\![0,n]\!]} U_k - a_1 \geq -\delta^{-1}\sqrt{n} \mid \mathsf{NI}\right) \geq 1 - \varepsilon, \tag{7.7.3}$$

$$\mathbf{P}^n_{(a_1,a_2)}\left(\sup_{k\in[\![0,n]\!]} V_k - a_2 \leq \delta^{-1}\sqrt{n} \mid \mathsf{NI}\right) \geq 1 - \varepsilon, \tag{7.7.4}$$

where $(U_k, V_k)_{k=1}^n \sim \mathbf{P}^n_{(a_1,a_2)}$.

Proof. **Proof of** (7.7.1)**.** Set $S_k := U_k - V_k$. Note that under the event NI we have $U_1 \geq V_1$. Thus it suffices to study

$$\mathbf{P}^{n-1}_{(b_1,b_2)}(S_{n-1} \geq \delta\sqrt{n} \mid \mathsf{NI})$$

with $b_1 \geq b_2$. Note that $(S_k)_{k=1}^{n-1}$ is itself a random walk starting at $b_1 - b_2 \geq 0$ conditioned to stay non-negative. Thus by stochastic monotonicity we have

$$\mathbf{P}^{n-1}_{(b_1,b_2)}(S_{n-1} \geq \delta\sqrt{n} \mid \mathsf{NI}) \geq \mathbf{P}^{n-1}_{(0,0)}(S_{n-1} \geq \delta\sqrt{n} \mid \mathsf{NI}).$$

But under $\mathbf{P}^{n-1}_{(0,0)}$, it is known from [204] that the random walk $(U_k - V_k)_{k=1}^{n-1}$, conditioned to stay non-negative converges weakly to a Brownian meander with appropriate diffusion coefficient under diffusive scaling. Since the endpoint of a Brownian meander is a strictly positive continuous random variable, we thus have (7.7.1).

Proof of (7.7.2)**.** Let $S_k = U_k - V_k$. To obtain (7.7.2), we observe the following string of inequalities

$$\begin{aligned}
&\mathbf{P}^n_{a_1-a_2}\Bigg(\sup_{k\in[\![0,n]\!]} S_k \leq \delta^{-1}\sqrt{n} + \max\{a_1-a_2,0\} \mid \bigcap_{k=1}^{n}\{S_k \geq 0\}\Bigg) \\
&\geq \mathbf{P}^n_{\max\{a_1-a_2,0\}}\Bigg(\sup_{k\in[\![0,n]\!]} S_k \leq \delta^{-1}\sqrt{n} + \max\{a_1-a_2,0\} \mid \bigcap_{k=1}^{n}\{S_k \geq 0\}\Bigg) \\
&\geq \mathbf{P}^n_{\max\{a_1-a_2,0\}}\Bigg(\sup_{k\in[\![0,n]\!]} S_k \leq \delta^{-1}\sqrt{n} + \max\{a_1-a_2,0\} \mid \bigcap_{k=1}^{n}\{S_k \geq \max\{a_1-a_2,0\}\}\Bigg) \\
&= \mathbf{P}^n_0\Bigg(\sup_{k\in[\![0,n]\!]} S_k \leq \delta^{-1}\sqrt{n} \mid \bigcap_{k=1}^{n}\{S_k \geq 0\}\Bigg) \geq 1-\varepsilon.
\end{aligned}$$

Let us briefly explain the above inequalities. The first two inequalities follows from stochastic monotonicity applied to the boundary point and then to the bottom curve. The equality in the last line follows by translating the random walk. The final inequality follows by taking δ small enough due to the tightness of the random walk paths conditioned to stay positive (when scaled by diffusively). This completes the proof.

Proof of (7.7.3) **and** (7.7.4) Note that due to stochastic monotonicity, the non-intersecting condition makes U stochastically larger than a usual random walk. Thus,

$$\mathbf{P}^n_{(a_1,a_2)}\big(\inf_{k\in[\![0,n]\!]} U_k - a_1 \geq -\delta^{-1}\sqrt{n} \mid \mathsf{NI}\big) \geq \mathbf{P}^n_{a_1}\big(\inf_{k\in[\![0,n]\!]} U_k - a_1 \geq -\delta^{-1}\sqrt{n}\big).$$

By diffusive behavior of random walks one can choose δ small enough so that the above quantity is at most $1-\varepsilon$. Similarly one can take δ even smaller so that $\mathbf{P}^n_{(a_1,a_2)}\big(\sup_{k\in[\![0,n]\!]} V_k - a_2 \le \delta^{-1}\sqrt{n}\big) \ge 1-\varepsilon$. $\square$

Corollary 7.7.3. *Fix any $n \ge 1$. a_1, a_2 with $|a_1 - a_2| = o(\sqrt{n})$. Given any $\varepsilon, \gamma > 0$ there exists a constant $\rho(\varepsilon,\gamma) \in (0, \frac{1}{4}]$ such that*

$$\mathbf{P}^n_{(a_1,a_2)}\left(\sup_{k\in[\![0,n\rho]\!],i=1,2} \left|S_k^{(n,i)} - a_i\right| \ge \gamma\sqrt{n} \mid \mathsf{NI}\right) \ge 1-\varepsilon$$

where $(S_{k,i})_{k=1}^{n\rho} \sim \mathbf{P}^n_{(a_1,a_2)}$.

We now study non-intersecting probabilities for random bridges. We use the notation $\mathbf{P}^{n;(b_1,b_2)}_{(a_1,a_2)}$ to denote the law of two independent random bridges of size n starting at (a_1,a_2) and ending at (b_1,b_2) with increments drawn from f. The following lemma shows when the starting points and endpoints are far apart in diffusive scale, non-intersection probability is bounded away from zero.

Lemma 7.7.4. *Fix $\delta > 0$. For each $n \ge 4$, consider the set*

$$R_{n,\delta} := \{(x_1,x_2) : |x_i| \le 2\sqrt{n}(\log n)^{3/2}, x_1 - x_2 \ge \delta\sqrt{n}\} \tag{7.7.5}$$

There exists $\phi = \phi(\delta) > 0$ such that for all $n \ge 4$ and for all $(a_1,a_2),(b_1,b_2) \in R_{n,\delta}$ we have

$$\mathbf{P}^{n;(b_1,b_2)}_{(a_1,a_2)}\left(\inf_{k\in[\![0,n]\!]}\left[S_k^{(n,1)} - S_k^{(n,2)}\right] \ge \tfrac{1}{4}\delta\sqrt{n}\right) \ge \phi.$$

Proof. Fix any $(a_1,a_2),(b_1,b_2) \in R_{n,\delta}$. It suffices to prove the lemma for large enough n. Note that $|b_i - a_i| \le 4\sqrt{n}(\log n)^{3/2}$. By KMT coupling for Brownian bridges (Theorem 2.3 in [153] with $z = b_i - a_i$ and $p = 0$), there exists an absolute constant $\mathrm{C} > 0$ such that for all $n \ge 1$ and $i = 1, 2$ we have

$$\mathbf{P}^n_{a_i\to b_i}(\neg\mathsf{CL}^{(b_1,b_2)}_{(a_1,a_2)}) \le \tfrac{1}{n},\ \mathsf{CL}^{(b_1,b_2)}_{(a_1,a_2)} := \left\{\sup_{0\le k\le n}\left|S_k^{(n,i)} - \sqrt{n}B^{(i)}_{\frac{k}{n}} - a_i - \tfrac{k}{n}(b_i - a_i)\right| \le \mathrm{C}\log^3 n\right\} \tag{7.7.6}$$

where $B^{(1)}, B^{(2)}$ are Brownian bridges with appropriate fixed diffusion coefficient. There exists $\phi = \phi(\delta) > 0$ such that

$$\mathbf{P}\left(\sup_{x\in[0,1]}(|B^{(1)}(x)| + |B^{2}(x)|) \le \tfrac{1}{8}\delta\right) \ge 2\phi.$$

Combining the previous two math displays we see that with probability $2\phi - \frac{1}{n}$ we have

$$\begin{aligned}
S_k^{(n,1)} - S_k^{(n,2)} &\ge a_1 - a_2 + \tfrac{k}{n}(b_1 - a_1 - b_2 + a_2) - 2\mathrm{C}(\log n)^3 - \tfrac{1}{4}\delta\sqrt{n} \\
&= \tfrac{n-k}{n}a_1 - a_2 + \tfrac{k}{n}(b_1 - b_2) - 2\mathrm{C}(\log n)^3 - \tfrac{1}{4}\delta\sqrt{n} \\
&\ge -2\mathrm{C}(\log n)^3 + \tfrac{1}{2}\delta\sqrt{n} > 0
\end{aligned}$$

for all large enough n. This forces non-intersection. Thus $\mathbf{P}_{(a_1,a_2)}^{n;(b_1,b_2)}(\mathsf{NI}) \ge 2\phi - \frac{1}{n} \ge \phi$ for all large enough n. This completes the proof. □

Our next lemma gives a crude bound for the weak non-intersection probability in terms of true non-intersection probability.

Lemma 7.7.5. *There exists a constant* $\mathrm{C} > 0$ *such that for all* $p \ge 0$, $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$, $n \ge 1$

$$\mathbf{P}_{(a_1,a_2)}^{n;(b_1,b_2)}(\mathsf{NI}_p) \le e^{\mathrm{C}p} \cdot \mathbf{P}_{(a_1,a_2)}^{n;(b_1,b_2)}(\mathsf{NI}).$$

Proof. By lifting the first random bridge by p units we see that

$$\mathbf{P}_{(a_1,a_2)}^{n;(b_1,b_2)}(\mathsf{NI}_p) = \mathbf{P}_{(a_1+p,a_2)}^{n;(b_1+p,b_2)}(\mathsf{NI}).$$

Conditioning on the second point and the penultimate point of both the random bridges we get

$$\mathbf{P}_{a_1+p,a_2}^{n;(b_1+p,b_2)}(\mathsf{NI}) = \frac{\int_{x_1\ge x_2, y_1\ge y_2} \Lambda_{x_1,x_2}^{n;(y_1,y_2)}(\mathsf{NI})\Upsilon_p(x_1,x_2;y_1,y_2)dx_1dx_2dy_1dy_2}{f^{*n}(a_1-b_1)f^{*n}(a_2-b_2)}. \tag{7.7.7}$$

where

$$\Upsilon_p(x_1,x_2;y_1,y_2) := f(a_1+p-x_1)f(a_2-x_2)f(y_1-b_1-p)f(y_2-b_2),$$

$$\Lambda_{x_1,x_2}^{n;(y_1,y_2)}(\mathsf{NI}) := \int_{x_{j,1}\geq x_{j,2}, j\in[\![2,n-2]\!]} \prod_{j=1}^{n-2} f(x_{j,1}-x_{j+1,1})f(x_{j,2}-x_{j+1,2}) \prod_{j=2}^{n-2} dx_{j,1}dx_{j,2}.$$

Here in the above integration we set $x_{1,1} := x_1$, $x_{1,2} := x_2$, $x_{n-1,1} := y_1$, $x_{n-1,2} := y_2$. From Lemma 7.6.4, we have that $\Upsilon_p(x_1,x_2;y_1,y_2) \leq e^{\mathrm{C}p}\Upsilon_0(x_1,x_2;y_1,y_2)$, where the $\mathrm{C} > 0$ depends only on θ. Plugging this bound back in (7.7.7) we get the desired result. $\square$

The following technical lemma, which can be thought of as the bridge analog of Lemma 7.7.1, studies the non-intersection probability for random bridges when the starting points are close.

Lemma 7.7.6. *Fix $M > 0$ and $n \geq 1$. Suppose $|a_i| \leq \sqrt{n}(\log n)^{3/2}$, $|b_i| \leq M\sqrt{n}$, with $b_1 \geq b_2$, and $|a_1 - a_2| \leq (\log n)^{3/2}$. There exist a constant $\mathrm{C} = \mathrm{C}(M) > 0$ such that*

$$\mathbf{P}_{(a_1,a_2)}^{n;(b_1,b_2)}(\mathsf{NI}) \leq \mathrm{C}\tfrac{1}{\sqrt{n}}(\max\{a_1-a_2,0\}+1)\cdot\max\left\{\tfrac{1}{\sqrt{n}}|a_1-b_1|,2\right\}^{3/2}.$$

Proof. It suffices to prove the lemma only for large enough n. Set $r = \max\{\frac{1}{\sqrt{n}}|a_1-b_1|,2\}$ and $p = nr^{-3}$. We first claim that there exists $m(M) > 0$ such that

$$\mathbf{P}(\mathsf{NI}) \leq 2\mathbf{P}\left(|S_p^{(n,i)} - a_i| \leq m\sqrt{n}r^{-1} \text{ for } i=1,2,\ \mathsf{NI}\right). \tag{7.7.8}$$

Note that the density of $S_p^{(n,i)}$ at x is given by $\frac{f^{*p}(x-a_i)f^{*(n-p)}(b_i-x)}{f^{*n}(b_i-a_i)}$. By Lemma 7.6.3, we have

$$\begin{aligned}\sup_{|x-a_i|\leq m\sqrt{n}r^{-1}} \frac{f^{*(n-p)}(b_i-x)}{f^{*n}(b_i-a_i)} &= (1+o(1))\exp\left(\tfrac{1}{2\sigma^2}\left[r^2 - \tfrac{(r-mr^{-1})^2}{1-r^{-3}}\right]\right) \\ &= (1+o(1))\exp\left(\tfrac{1}{2\sigma^2(1-r^{-3})}\left[-r^{-1}-m^2r^{-2}+2m\right]\right) \leq 2e^{2m/\sigma^2}.\end{aligned}$$

Thus $\frac{f^{*p}(x-a_i)f^{*(n-p)}(b_i-x)}{f^{*n}(b_i-a_i)} \leq 2e^{2m/\sigma^2}\cdot f^{*p}(x-a_i)$ whenever $|x-a_i| \leq m\sqrt{n}r^{-1}$. This allows us to go from random bridge laws to random walk laws. Indeed, we have

$$
\begin{aligned}
&\mathbf{P}^{n;(b_1,b_2)}_{(a_1,a_2)}\left(|S_p^{(n,i)}-a_i|\le m\sqrt{n}r^{-1} \text{ for } i=1,2,\ \mathsf{NI}\right)\\
&\le \mathbf{P}^{n;(b_1,b_2)}_{(a_1,a_2)}\left(|S_p^{(n,i)}-a_i|\le m\sqrt{n}r^{-1} \text{ for } i=1,2,\ \bigcap_{k=1}^{p}\{S_k^{(n,1)}\ge S_k^{(n,2)}\}\right)\\
&\le 2e^{2m/\sigma^2}\cdot \mathbf{P}^{n}_{(a_1,a_2)}\left(|S_p^{(n,i)}-a_i|\le m\sqrt{n}r^{-1} \text{ for } i=1,2,\ \bigcap_{k=1}^{p}\{S_k^{(n,1)}\ge S_k^{(n,2)}\}\right)\\
&\le \mathbf{P}^{p}_{(a_1,a_2)}(\mathsf{NI})\le \tfrac{\mathrm{C}}{\sqrt{n}}r^{3/2}\cdot(\max\{a_1-a_2,0\}+1).
\end{aligned}
$$

where the last inequality uses Lemma 7.7.1. This completes the proof modulo (7.7.8). The rest of the proof is devoted to showing (7.7.8).

Similar to the proof of (7.7.3) and (7.7.4), by stochastic monotonicity we have

$$
\mathbf{P}^{n;(b_1,b_2)}_{(a_1,a_2)}(S_p^{(n,1)}-a_1\le -m\sqrt{n}r^{-1}\mid \mathsf{NI})\le \mathbf{P}^{n}_{a_1\to b_1}(S_p^{(n,1)}-a_1\le -m\sqrt{n}r^{-1}) \tag{7.7.9}
$$

Now by KMT coupling for random bridges (see (7.7.6)) with probability $1-\frac{1}{n}$,

$$
S_p^{(n,1)}-a_1\ge \sqrt{n}B_{p/n}+\tfrac{p}{n}(b_1-a_1)-\mathrm{C}\log^3 n=\sqrt{n}B_{p/n}-\sqrt{n}r^{-2}-\mathrm{C}\log^3 n\ge \sqrt{n}B_{p/n}-2\sqrt{n}r^{-1}.
$$

Here $B_{p/n}$ is a standard Brownian bridge on $[0,1]$. By Brownian bridge calculations, there exists an absolute constant $c>0$ such that $\mathbf{P}(B_{p/n}=B_{r^{-3}}\ge -(m-2)r^{-1})\ge 1-e^{-cm^2r}$. For this choice of m we see that r.h.s. of (7.7.9) $\le \frac{1}{8}$ for large enough n.

For the other inequality we use stochastic monotonicity at the starting points to get

$$
\begin{aligned}
\mathbf{P}^{n;(b_1,b_2)}_{(a_1,a_2)}(S_p^{(n,1)}-a_1\ge m\sqrt{n}r^{-1}\mid \mathsf{NI}) &\le \mathbf{P}^{n;(b_1,b_2)}_{(a_1+\sqrt{n}r^{-1},a_2)}(S_p^{(n,1)}-a_1\ge m\sqrt{n}r^{-1}\mid \mathsf{NI})\\
&\le \frac{\mathbf{P}^{n;(b_1,b_2)}_{(a_1+\sqrt{n}r^{-1},a_2)}(S_p^{(n,1)}-a_1\ge m\sqrt{n}r^{-1})}{\mathbf{P}^{n;(b_1,b_2)}_{(a_1+\sqrt{n}r^{-1},a_2)}(\mathsf{NI})}.
\end{aligned} \tag{7.7.10}
$$

We now proceed to show appropriate upper and lower bounds for the numerator and denominator of r.h.s. of (7.7.10) respectively. We apply (7.7.6) with $a_1 \mapsto a_1 + \sqrt{n}r^{-1}$, $a_2 \mapsto a_2$, and $b_i \mapsto b_i$ for $i = 1, 2$. Following the same above Brownian calculations for large enough n and m we have

$$\mathbf{P}^{n;(b_1,b_2)}_{(a_1+\sqrt{n}r^{-1},a_2)}(S_p^{(n,1)} - a_1 \geq m\sqrt{n}r^{-1}) \leq \tfrac{1}{n} + e^{-cm^2 r}. \tag{7.7.11}$$

This gives an upper bound for the numerator. For the denominator, recall the event $\mathsf{CL}^{(b_1,b_2)}_{(a_1,a_2)}$ and the Brownian bridges $B^{(1)}, B^{(2)}$ from (7.7.6). Note that on the event $\mathsf{CL}^{(b_1,b_2)}_{(a_1+\sqrt{n}r^{-1},a_2)} \cap \{\inf_{x\in[0,1]}(B_x^{(1)} - B_x^{(2)}) \geq -\frac{1}{2}r^{-1}\}$, for large enough n we have

$$\begin{aligned} S_k^{(n,1)} &\geq \sqrt{n}B^{(1)}_{k/n} + a_1 + \sqrt{n}r^{-1} + \tfrac{k}{n}(b_1 - a_1) - \mathrm{C}(\log n)^3 \\ &\geq \sqrt{n}B^{(2)}_{k/n} + \tfrac{1}{2}\sqrt{n}r^{-1} + a_2 + \tfrac{k}{n}(b_2 - a_2) - 2\mathrm{C}(\log n)^3 \\ &\geq S_k^{(n,2)} + \tfrac{1}{2}\sqrt{n}r^{-1} - 3\mathrm{C}(\log n)^3 \geq S_k^{(n,2)}. \end{aligned}$$

where in above lines we use the fact $|a_1 - a_2| \leq (\log n)^{3/2}$, $b_1 \geq b_2$, and $r \leq (\log n)^{3/2}$. Thus for large enough n,

$$\begin{aligned} \mathbf{P}^{n;(b_1,b_2)}_{(a_1+\sqrt{n}r^{-1},a_2)}(\mathsf{NI}) &\geq \mathbf{P}\left(\inf_{x\in[0,1]}(B_x^{(1)} - B_x^{(2)}) \geq -\tfrac{1}{2}r^{-1}\right) - \mathbf{P}^{n;(b_1,b_2)}_{(a_1+\sqrt{n}r^{-1},a_2)}\left(\neg\mathsf{CL}^{(b_1,b_2)}_{(a_1+\sqrt{n}r^{-1},a_2)}\right) \\ &\geq \mathrm{C}r^{-2} - \tfrac{1}{n} \geq \tfrac{1}{2}\mathrm{C}r^{-2}, \end{aligned}$$

where the penultimate inequality follows from (7.7.6) and Brownian bridge calculations (see Lemma 2.11 in [110] for example). Combining (7.7.11) and the above lower bound we have

$$\text{r.h.s. of (7.7.10)} \leq \tfrac{2}{\mathrm{C}}\left[\tfrac{r^2}{n} + r^2 e^{-cm^2 r}\right] \leq \tfrac{1}{8},$$

for all large enough n and m. Thus in conclusion we have

$$\mathbf{P}^{n;(b_1,b_2)}_{(a_1,a_2)}(|S_p^{(n,1)} - a_1| \geq m\sqrt{n}r^{-1} \mid \mathsf{NI}) \leq \tfrac{1}{4}$$

for all large enough n and m. Similar for all large enough n and m one has

$$\mathbf{P}^{n;(b_1,b_2)}_{(a_1,a_2)}(|S^{(n,2)}_p - a_2| \geq m\sqrt{n}r^{-1} \mid \mathsf{NI}) \leq \tfrac{1}{4}.$$

The last two math displays imply (7.7.8), thus completing the proof. $\square$

Corollary 7.7.7. *Fix any $M > 0$ and $n \geq 1$. Suppose $|a_i|, |b_i| \leq M\sqrt{n}$ for $i = 1, 2$. We have that*

$$\mathbf{P}^{n,(b_1,b_2)}_{(a_1,a_2)}(\mathsf{NI}) \asymp_M \mathbf{P}^{\lfloor n/4\rfloor}_{(a_1,a_2)}(\mathsf{NI})\mathbf{P}^{\lfloor n/4\rfloor}_{(b_1,b_2)}(\mathsf{NI}).$$

Proof. The upper bound follows by applying (7.4.21) with $\delta = \frac{1}{4}$ and integrating over the non-intersection event. Let us focus on the lower bound. Set $U^{(n,i)}_k \sim \mathbf{P}^{\lfloor n/4\rfloor}_{(a_1,a_2)}$ and $V^{(n,i)}_k \sim \mathbf{P}^{\lfloor n/4\rfloor}_{(b_1,b_2)}$. By Lemma 7.7.2, we choose a constant $\widetilde{M}$ depending only on M such that for all $(x_1, x_2) \in \mathbb{R}^2$ with $|x_i| \leq M\sqrt{n}$, we have

$$\mathbf{P}\Big(|U^{(n,i)}_{\lfloor n/4\rfloor}| \leq \widetilde{M}\sqrt{n}, \text{ for } i = 1, 2 \mid \mathsf{NI}\Big) \geq \tfrac{3}{4}, \quad \mathbf{P}\Big(|V^{(n,i)}_{\lfloor n/4\rfloor}| \leq \widetilde{M}\sqrt{n}, \text{ for } i = 1, 2 \mid \mathsf{NI}\Big) \geq \tfrac{3}{4}. \quad (7.7.12)$$

By Lemma 7.7.2 we next choose a $\delta = \delta(M) > 0$ small enough such that

$$\mathbf{P}\Big((U^{(n,1)}_{\lfloor n/4\rfloor}, U^{(n,2)}_{\lfloor n/4\rfloor}) \in R_{n,\delta} \mid \mathsf{NI}\Big) \geq \tfrac{3}{4}, \quad \mathbf{P}\Big((V^{(n,1)}_{\lfloor n/4\rfloor}, V^{(n,2)}_{\lfloor n/4\rfloor}) \in R_{n,\delta} \mid \mathsf{NI}\Big) \geq \tfrac{3}{4}. \quad (7.7.13)$$

where $R_{n,\delta}$ is defined in (7.7.5). By Lemma 7.7.4 there exists $\phi(\delta) > 0$ such that for all $(x_1, x_2), (y_1, y_2) \in R_{n,\delta}$ we have

$$\mathbf{P}^{n-2\lfloor n/4\rfloor,(y_1,y_2)}_{(x_1,x_2)}(\mathsf{NI}) \geq \phi. \quad (7.7.14)$$

We next consider the events

$$\mathsf{E}_1 := \Big\{|S^{(n,i)}_{\lfloor n/4\rfloor}| \leq \widetilde{M}\sqrt{n}\Big\}, \quad \mathsf{E}_2 := \Big\{|S^{(n,i)}_{\lfloor n/4\rfloor}| \leq \widetilde{M}\sqrt{n}\Big\}.$$

Using (7.4.22) with $\delta = \frac{1}{4}$ we have

$$\mathbf{P}^{n,(b_1,b_2)}_{(a_1,a_2)}(\mathsf{NI}) \geq \mathbf{P}^{n,(b_1,b_2)}_{(a_1,a_2)}(\mathsf{E}_1 \cap \mathsf{E}_2 \cap \mathsf{NI}) \gtrsim_M \widetilde{\mathbf{P}}(\mathsf{E}_1 \cap \mathsf{E}_2 \cap \mathsf{NI}).$$

where $\widetilde{\mathbf{P}}$ denotes the joint law of two independent $(n; n/4, n/4)$-modified random bridges of length n starting at (a_1, a_2) and ending at (b_1, b_2) (see Definition 7.4.9). In view of the of our $\widetilde{M}$ choice, we have $\widetilde{\mathbf{P}}(\mathsf{E}_i) \geq \frac{3}{4}$ for $i = 1, 2$ from (7.7.12) and (7.7.13). Furthermore, in view of (7.7.14), we have

$$\mathbf{P}(\mathsf{NI}) \geq \phi \cdot \mathbf{P}^{\lfloor n/4 \rfloor}_{(a_1,a_2)}(\mathsf{NI})\mathbf{P}^{\lfloor n/4 \rfloor}_{(b_1,b_2)}(\mathsf{NI}).$$

We thus have the desired lower bound. $\square$

Fix any $M > 0$, $n \geq 1$, $\varepsilon > 0$ and $(a_1, a_2), (b_1, b_2) \in \mathbb{R}$. Suppose $|a_i|, |b_i| \leq M\sqrt{n}$ and $a_1 \geq a_2$. Take $p, q \in [\![0, n]\!]$ with $p + q \leq n/2$. Suppose further that there exists $\delta > 0$ such that either $q \geq n\rho$ or $b_1 - b_2 \geq \rho\sqrt{n}$. Consider two independent $(n; p, q)$- modified random bridges $(S_k^{(n,i)})_{k\in[\![0,n]\!],i=1,2}$ starting and ending at (a_1, a_2) and (b_1, b_2) respectively. For $\beta > 0$ recall the event Gap_β defined in (7.4.20). The following lemma asserts Gap_β event is very likely under non-intersection.

Lemma 7.7.8. *Given an $\varepsilon > 0$, there exists $\beta(\varepsilon, \rho, M) > 0$, $N_0(\varepsilon, \rho, M) > 0$, such that for all $N \geq N_0$ we have*

$$\mathbf{P}\left(\mathsf{Gap}_\beta \mid \mathsf{NI}\right) \geq 1 - \varepsilon.$$

Proof. We assume n is large enough throughout the proof. Recall from Section 7.4.3 that Gap_β event is intersection of six smaller 'Gap' events: $\mathsf{Gap}_{i,\beta}$. In what follows, we analyze each 'Gap' event separately.

- $\mathsf{Gap}_{1,\beta}$ **and** $\mathsf{Gap}_{2,\beta}$**.** Note that for $k \in [\![1, p]\!]$, $S_k^{(n,1)} - S_k^{(n,2)}$ is itself a random walk. The NI event corresponds to the event of this random walk being non-negative. By classical result about growth of random walks conditioned to stay non-negative (see [285, Theorem 2]) it follows that one can choose β small enough such that $\mathbf{P}(\mathsf{Gap}_{1,\beta}) \geq 1 - \frac{\varepsilon}{6}$. By same argument one has $\mathbf{P}(\mathsf{Gap}_{1,\beta}) \geq 1 - \frac{\varepsilon}{6}$ for all large enough n by choosing β small enough.

- $\mathsf{Gap}_{3,\beta}$**.** Applying Lemma 7.7.2, one can choose γ small enough such that

$$\mathbf{P}\big((S_p^{(n,1)}, S_p^{(n,2)}), (S_{n-q}^{(n,1)}, S_{n-q}^{(n,2)}) \in \mathcal{P}_{n,\gamma}\big) \geq 1 - \tfrac{\varepsilon}{12}, \tag{7.7.15}$$

where

$$\mathcal{P}_{n,\gamma} := \{(z_1, z_2) \in \mathbb{R}^2 : |z_i| \leq \gamma^{-1}\sqrt{n}, z_1 - z_2 \geq \gamma\sqrt{n}\}.$$

In other words, with probability $1 - \frac{\varepsilon}{12}$, the endpoints of the middle portions of the modified random bridges are in $\mathcal{P}_\gamma$. Thus,

$$\mathbf{P}(\mathsf{Gap}_{3,\beta}) \geq (1 - \tfrac{\varepsilon}{12}) \cdot \inf_{(a_1,a_2),(b_1,b_2)\in\mathcal{P}_{n,\gamma}} \mathbf{P}_{(a_1,a_2)}^{n-p-q;(b_1,b_2)}(\mathsf{Gap}_{3,\beta} \mid \mathsf{NI}) \tag{7.7.16}$$

By continuity with respect to the boundary data, the infimum is attained at some point $(a_1^*, a_2^*), (b_1^*, b_2^*) \in \mathcal{P}_{n,\gamma}$. Take any subsequential limit of $\frac{1}{\sqrt{n}}(a_1^*, a_2^*)$, $\frac{1}{\sqrt{n}}(b_1^*, b_2^*)$ say $(u_1, u_2), (v_1, v_2)$. Then $|u_i|, |v_i| \leq \gamma^{-1}$ and $u_1 - u_2, v_1 - v_2 \geq \gamma$. By Lemma 3.10 from [295], this conditional law under diffusive scaling converges to non-intersecting Brownian bridges (with appropriate variance) (B_1, B_2) starting at (u_1, u_2) ending at (v_1, v_2). We have $\mathbf{P}(\inf_{x\in[0,1]}(B_1(x) - B_2(x)) > 0) = 1$. This implies along this subsequence the limit of $\mathbf{P}_{(a_1^*,a_2^*)}^{n-p-q;(b_1^*,b_2^*)}(\mathsf{Gap}_{3,\beta} \mid \mathsf{NI})$ is 1. Since this holds for all subsequence, we thus see that for all large enough n, the r.h.s. of (7.7.16) can be made at least $1 - \frac{\varepsilon}{6}$.

- $\mathsf{Gap}_{4,\beta}$ **and** $\mathsf{Gap}_{5,\beta}$**.** Given β_0, we consider the event

$$\mathsf{A}_i(\beta_0) := \bigcap_{k=1}^{p} \left\{ S_k^{(n,i)} - S_{k-1}^{(n,i)} \leq \beta_0^{-1} k^{1/8} \right\}$$

We first claim that we have

$$\mathbf{P}(S_k^{(n,2)} - S_{k-1}^{(n,2)} \geq \tau \mid \mathsf{NI}) \leq \mathbf{P}(|S_k^{(n,2)} - S_{k-1}^{(n,2)} \geq \tau). \tag{7.7.17}$$

for all $k \in [\![1, p]\!]$. Recall that this part of the modified random bridges corresponds to random walks. The non-intersection condition forces the increments of the lower random walk to be stochastically smaller than that of a usual random walk. Thus by stochastic monotonicity we have (7.7.17).

Let us suppose $X \sim f$. By Lemma 7.6.4, we may choose β_0 small enough that

$$\sum_{k=1}^{p} \mathbf{P}(X \geq \beta_0^{-1} k^{1/8}) \leq \tfrac{\varepsilon}{12}, \quad \sum_{k=1}^{p} \mathbf{P}(X \geq 2\beta_0^{-1} k^{1/8} \mid X \geq \beta_0^{-1} k^{1/8}) \leq \tfrac{\varepsilon}{12}, \tag{7.7.18}$$

By the above choice of β_0, in view of (7.7.17), we have $\mathbf{P}(\mathsf{A}_2(\beta_0)) \geq 1 - \frac{\varepsilon}{12}$.

We have

$$\mathbf{P}(\neg \mathsf{A}_1(2\beta_0) \mid \mathsf{NI}) \leq \tfrac{\varepsilon}{12} + \sum_{k=1}^{p} \mathbf{P}\left(\mathsf{A}_2(\beta_0) \cap \{S_k^{(n,1)} - S_{k-1}^{(n,1)} \geq 2\beta_0^{-1} k^{1/8}\} \mid \mathsf{NI}\right). \tag{7.7.19}$$

Note that non-intersection forces $S_k^{(n,1)} - S_{k-1}^{(n,1)} \geq S_k^{(n,2)} - S_{k-1}^{(n,2)} + S_{k-1}^{(n,2)} - S_{k-1}^{(n,1)}$. Under $\mathsf{NI} \cap \mathsf{A}_2(\beta)$, this lower bound is at most $\beta_0^{-1} k^{1/8}$. Thus by stochastic monotonicity

$$\mathbf{P}\left(\mathsf{A}_2(\beta_0) \cap \{S_k^{(n,1)} - S_{k-1}^{(n,1)} \geq 2\beta_0^{-1} k^{1/8}\} \mid \mathsf{NI}\right) \leq \mathbf{P}(X \geq 2\beta_0^{-1} k^{1/8} \mid X \geq \beta_0^{-1} k^{1/8}).$$

Inserting this bound in (7.7.19) leads to $\mathbf{P}(\mathsf{A}_1(2\beta_0)) \mid \mathsf{NI}) \geq 1 - \frac{\varepsilon}{6}$. Here we also used the fact that $S_0^{(n,1)} = a_1 > a_2 = S_0^{(n,2)}$. Thus we have $\mathbf{P}(\mathsf{Gap}_{4,\beta} \mid \mathsf{NI}) \geq 1 - \frac{\varepsilon}{6}$ for $\beta = \beta_0/2$.

For $\mathsf{Gap}_{5,\beta}$ we are interested in the increment of end part of the modified random bridge. Since this part is time-reversed random walk, requiring an upper bound for the increments of the modified random bridge corresponds to requiring a lower bound for the increments of the random walk. In the same spirit as (7.7.17) one has

$$\mathbf{P}(S_k^{(n,1)} - S_{k-1}^{(n,1)} \geq \tau \mid \mathsf{NI}) \leq \mathbf{P}(|S_k^{(n,1)} - S_{k-1}^{(n,1)} \geq \tau). \tag{7.7.20}$$

for $k \in [\![n-q+1, n]\!]$. Here we do not need the endpoints b_1 and b_2 to be ordered. In view of (7.7.18), this leads to $\mathbf{P}(\mathsf{Gap}_{5,\beta_0}) \geq 1 - \frac{\varepsilon}{12}$.

- $\mathsf{Gap}_{6,\beta}$. From (7.7.15), we get that the end points are in $\mathcal{P}_{n,\gamma}$ with probability $1-\frac{\varepsilon}{12}$. Under this event, we apply KMT coupling on the middle portion bridge of the modified random bridge to get a Brownian bridge B such that for large enough n

$$\mathbf{P}(|S_k^{(n,1)} - S_{k-1}^{(n,1)}| \geq \beta^{-1}\log n) \leq \mathbf{P}(\sqrt{n}|B_{k/n} - B_{(k-1)/n}| \geq \tfrac{1}{2}\beta^{-1}\log n - \tfrac{\gamma}{\sqrt{n}}) \leq \tfrac{1}{n^2}.$$

By union bound we have the desired result.

This completes the proof. □

We end this section by proving a modulus of continuity estimate for non-intersecting random walks.

Lemma 7.7.9. *Fix $M, \gamma > 0$. Let $S_k^{(n,1)}, S_k^{(n,2)}$ be two independent n-step random walk with increments from $f := G_{\theta,1} * G_{\theta,-1}$ starting at a_1 and a_2. Assume $0 \leq a_1 - a_2 \leq M + 2\log\log n$. There exists $n_0(M,\gamma)$ and $\delta(M,\gamma)$ such that for all $n \geq n_0$ we have*

$$\sum_{i=1}^{2} \mathbf{P}(\omega_\delta(S_\cdot^{(n,i)}, [\![0,n]\!]) \geq \gamma\sqrt{n} \mid \mathsf{NI}) \leq \varepsilon.$$

where $\mathsf{NI} := \{S_k^{(n,1)} \geq S_k^{(n,2)}$ for all $k \geq 0\}$.

Proof. Fix $\gamma > 0$. We first control modulus of continuity near zero. Note that by stochastic monotonicity and modulus of continuity of single random walk, we can choose ρ small enough such that

$$\mathbf{P}\bigg(\sup_{k\le n\rho}[S_0^{(n,1)}-S_k^{(n,1)}]\ge\gamma\sqrt{n}\mid \mathsf{NI}\bigg)\le\mathbf{P}\bigg(\sup_{k\le n\rho}[S_0^{(n,1)}-S_k^{(n,1)}]\ge\gamma\sqrt{n}\bigg)\le\varepsilon.$$

Similarly one has $\mathbf{P}(\sup_{k\le n\rho}[S_k^{(n,2)}-S_0^{(n,2)}]\ge\gamma\sqrt{n}\mid \mathsf{NI})\le\varepsilon$ Note that the difference $S_k^{(n,1)}-S_k^{(n,2)}$ is a random walk conditioned to stay nonnegative. By classical result from [204] we can choose ρ such that $\mathbf{P}(\sup_{k\le n\rho}[S_k^{(n,1)}-S_k^{(n,2)}]\ge\gamma\sqrt{n}\mid \mathsf{NI})\le\varepsilon$ for all large enough n.

We next control modulus of continuity away from zero. Towards this end let $I_v:=\{(x_1,x_2): |x_i|\le v^{-1}\sqrt{n}, x_1-x_2\ge v\sqrt{n}\}$. By Lemma 7.7.2, one can choose v small enough to get $\mathbf{P}(\mathsf{A}_v)\ge 1-\varepsilon$ where $\mathsf{A}_v:=\{(S_{n\rho}^{(n,1)},S_{n\rho}^{(n,2)})\in I_v\}$. Let $\mathcal{F}:=\sigma\{S_{n\rho}^{(n,1)},S_{n\rho}^{(n,2)}\}$. Note that

$$\mathbf{P}\left(\omega_\delta(S_\cdot^{(n,i)},[\![n\rho,n]\!])\ge\gamma\sqrt{n}\mid \mathsf{NI}\right)\le\varepsilon+\mathbf{E}\left[\mathbf{1}_{\mathsf{A}_v}\mathbf{P}\left(\omega_\delta(S_\cdot^{(n,i)},[\![n\rho,n]\!])\ge\gamma\sqrt{n}\mid \mathsf{NI},\mathcal{F}\right)\right] \quad (7.7.21)$$

Under A_v, by Lemma 7.7.4, $\mathbf{P}(S_k^{(n,1)}\ge S_k^{(n,2)}$ for all $k\ge n\rho\mid\mathcal{F})\ge\phi$. By modulus of continuity of random walks [29, Lemma 2.25] we can choose δ small enough such that $\mathbf{P}(\omega_\delta(S_\cdot^{(n,i)},[\![n\rho,n]\!])\ge\gamma\sqrt{n}\mid\mathcal{F})$ is at most $\varepsilon\phi^{-1}$ uniformly over A_v. Thus, the r.h.s. of (7.7.21) is at most 2ε. Hence combining the near zero and away zero modulus of continuity we get the desired result by appropriately changing γ and ε. $\square$

7.8 Stochastic monotonicity

The goal of this section is to prove the stochastic monotonicity of $\mathcal{HSLG}$ Gibbs measure (Proposition 8.2.3). Let $\Lambda=\{(i,j): k_1\le i\le k_2, a_i\le j\le b_i\}$. Let $w_1,\ldots,w_{|\Lambda|}$ be the enumeration of points in Λ in the lexicographic order. Set $\Lambda_r=\{w_1,w_2,\ldots,w_r\}$, so that $\Lambda_{|\Lambda|}=\Lambda$. Let $E_r:=E(\Lambda_r\cup\partial\Lambda_r)$, and, recalling the weights W_e from (7.1.3), let

$$H_r(x;(u_v)_{v\in\partial\Lambda_r}):=\int_{\mathbb{R}^{|\Lambda_{r-1}|}}\prod_{e=\{v_1\to v_2\}\in E_r}W_e(u_{v_1}-u_{v_2})\prod_{v\in\Lambda_{r-1}}du_v, \quad (7.8.1)$$

where $u_{w_r}=x$. The proof of Proposition 8.2.3 relies on the following technical lemma.

Lemma 7.8.1. *Fix $r \in [\![1, |\Lambda|]\!]$. For each $v \in \partial\Lambda_r$, fix any $u_v, u'_v \in \mathbb{R}$ with $u_v \le u'_v$. For all $s \ge t$*

$$H_r\big(s; (u_v)_{v\in\partial\Lambda_r}\big)H_r\big(t; (u'_v)_{v\in\partial\Lambda_r}\big) \le H_r\big(s; (u'_v)_{v\in\partial\Lambda_r}\big)H_r\big(t; (u_v)_{v\in\partial\Lambda_r}\big) \tag{7.8.2}$$

We prove Lemma 7.8.1 at the end of this section and now complete the proof of Proposition 8.2.3.

Proof of Proposition 8.2.3. Fix $r \in [\![1, |\Lambda|]\!]$. We first claim that for all boundary conditions $(u_v)_{v\in\partial\Lambda_r}$ and $(u'_v)_{v\in\partial\Lambda_r}$ with $u_v \le u'_v$ for all $v \in \partial\Lambda_r$, and $s \in \mathbb{R}$,

$$\mathbf{P}\big(L(w_r) \le s \mid L(v) = u_v \text{ for all } v \in \partial\Lambda_r\big) \ge \mathbf{P}\big(L(w_r) \le s \mid L(v) = u'_v \text{ for all } v \in \partial\Lambda_r\big). \tag{7.8.3}$$

To show this, observe that $H_r(x; (u_v)_{v\in\partial\Lambda_r})$ in (7.8.1) is proportional to the conditional density at x of $L(w_r)$ given $\big(L(v)\big)_{v\in\partial\Lambda_r} = (u_v)_{v\in\partial\Lambda_r}$. Thus,

$$\mathbf{P}\big(L(w_r) \le s \mid L(v) = u_v \text{ for all } v \in \partial\Lambda_r\big) = F_r\big(s; (u_v)_{v\in\partial\Lambda_r}\big) := \frac{\int_{-\infty}^{s} H_r(x; (u_v)_{v\in\partial\Lambda_r})dx}{\int_{-\infty}^{\infty} H_r(x; (u_v)_{v\in\partial\Lambda_r})dx} \tag{7.8.4}$$

To prove (7.8.3) observe that owing to Lemma 7.8.1, the derivative of

$$\log \int_{-\infty}^{s} H_r(x; (u_v)_{v\in\partial\Lambda_r})dx - \log \int_{-\infty}^{s} H_r(x; (u'_v)_{v\in\partial\Lambda_r})dx.$$

is non-positive for all s. This implies for $s' \ge s$ we have

$$\frac{\int_{-\infty}^{s} H_r(x; (u_v)_{v\in\partial\Lambda_r})dx}{\int_{-\infty}^{s} H_r(x; (u'_v)_{v\in\partial\Lambda_r})dx} \ge \frac{\int_{-\infty}^{s'} H_r(x; (u_v)_{v\in\partial\Lambda_r})dx}{\int_{-\infty}^{s'} H_r(x; (u'_v)_{v\in\partial\Lambda_r})dx}$$

Taking $s' \to \infty$ and cross-multiplying yields the desired inequality (7.8.3), in light of (7.8.4).

Given $(u_v)_{v\in\partial\Lambda} \in \mathbb{R}^{|\partial\Lambda|}$, we now define a sequence of random variables according to the following algorithm. Note that below, $x \leftarrow y$ means to assign the value y to the variable x.

Algorithm 1 Defining the random vectors

Generate $U_1, \ldots, U_{|\Lambda|}$ i.i.d. random variables from $U[0,1]$

$Y_{|\Lambda|} \leftarrow (u_v)_{v\in\partial\Lambda}$

$r \leftarrow |\Lambda|$

while $r \geq 1$ **do**

$L(w_r; (u_w)_{v\in\partial\Lambda}) \leftarrow F_r^{-1}(U_1; Y_r)$

$\widetilde{u}_v \leftarrow u_v$ for all $v \in \partial\Lambda_{r-1} \cap \partial\Lambda_r$

$\widetilde{u}_{w_r} \leftarrow L(w_r; (u_w)_{v\in\partial\Lambda})$

$Y_{r-1} \leftarrow (\widetilde{u}_v)_{v\in\partial\Lambda_{r-1}}$

$r \leftarrow r - 1$

end while

This defines a collection of random variables $L(w_i; (u_v)_{v\in\partial\Lambda})$ indexed by $i \in [\![1, |\Lambda|]\!]$ and $(u_v)_{v\in\partial\Lambda} \in \mathbb{R}^{|\partial\Lambda|}$, all on the common probability space on which $U_1, \ldots, U_{|\Lambda|}$ are defined. It is clear from the definition that for each $(u_v)_{v\in\partial\Lambda} \in \mathbb{R}^{|\partial\Lambda|}$, the law of $\big(L(w_i; (u_v)_{v\in\partial\Lambda})\big)_{i\in[\![1,|\Lambda|]\!]}$ is given by the $\mathcal{HSLG}$ Gibbs measure on the domain Λ with boundary condition $(u_v)_{v\in\partial\Lambda}$. Take two boundary conditions $(u_v)_{v\in\partial\Lambda}$ and $(u'_v)_{v\in\partial\Lambda}$ with $u_v \leq u'_v$ for all $v \in \partial\Lambda$. As each F_r is stochastically increasing with respect to the boundary condition, i.e., (7.8.3), sequentially we obtain that with probability 1 on our probability space $L(w_r; (u_v)_{v\in\partial\Lambda}) \leq L(w_r; (u'_v)_{v\in\partial\Lambda})$ for all r, thus completing the proof. $\square$

Proof of Lemma 7.8.1. Let us begin with a few pieces of notations. Fix any $1 \leq r \leq |\Lambda|$. Set $e_r := \{w_r \to (w_r + (0,1)), (w_r + (0,1)) \to w_r\} \cap E_r$. In words, this is the directed blue edge (see Figure 7.21 A) with w_r as the left point of e_r.

Define

$$h_r\big(x; (u_v)_{v\in\partial\Lambda_r}\big) := \int_{\mathbb{R}^{|\Lambda_{r-1}|}} \prod_{e=\{v_1\to v_2\}\in E_r\setminus\{e_r\}} W_e(u_{v_1} - u_{v_2}) \prod_{v\in\Lambda_{r-1}} du_v$$

with the convention $u_{w_r} = x$. Observe that the difference between H_r from (7.8.1) and h_r above

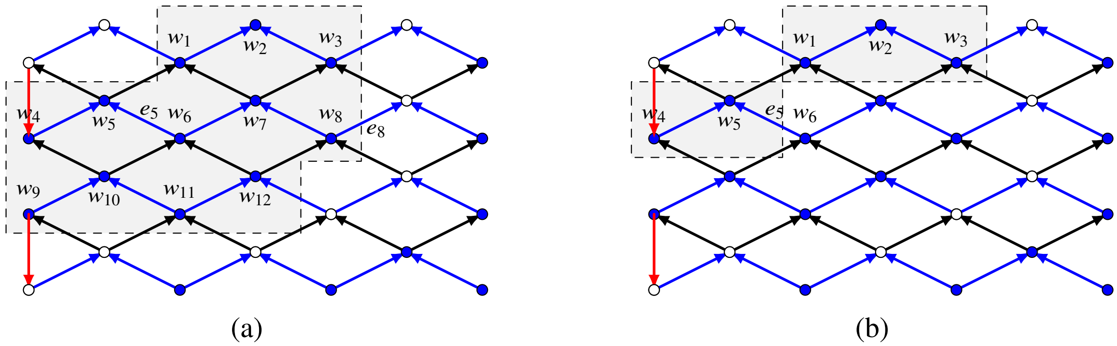


Figure 7.21: (A) A possible domain Λ includes all the vertices in the shaded region. w_i's are the vertices of Λ enumerated in lexicographic order. Directed edges e_r going are shown above for $r = 5$ and $r = 8$. These are the blue edges with w_r as the left point of e_r. (B) The domain Λ_5 includes the vertices in the shaded region. Q_5 is the set of all red and black edges that have one vertex as w_6 and one vertex in $\partial\Lambda_6$. In the above figure, Q_5 is composed of two black edges that points toward w_6.

is that the weight of the directed blue edge e_r is included in the former but not in the latter. Note that the vertices of e_r are not in Λ_{r-1}. Thus in the definition of H_r, the edge weight function corresponding to e_r can be pulled out of the integrand leading to

$$H_r\big(x; (u_v)_{v\in\partial\Lambda_r}\big) = h_r\big(x; (u_v)_{v\in\partial\Lambda_r}\big) \cdot F_r(u_{w_r+(0,1)} - x) \tag{7.8.5}$$

where $F_r(y)$ is the directed blue edge weight corresponding to e_r, i.e., $F_r(y) := e^{\theta y - e^y}$ or $F_r(y) = e^{-\theta y - e^{-y}}$ depending on the direction of the e_r edge between w_r and $w_r + (0, 1)$.

With the above introduced notation, we now turn towards the proof of (7.8.2). Note that given a function $P(x) = e^{-R(x)}$ with R being convex, we have

$$P(\delta - \beta)P(\gamma - \alpha) \geq P(\delta - \alpha)P(\gamma - \beta) \tag{7.8.6}$$

for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha \leq \beta$ and $\gamma \leq \delta$. All our weight functions in (7.1.3) are of this type. In particular, this implies that (7.8.6) holds for $P = F_r$. In view of this and the relation (7.8.5), to show (7.8.2) it suffices to show the same holds for h_r replacing H_r, i.e.,

$$h_r\big(s; (u_v)_{v\in\partial\Lambda_r}\big)h_r\big(t; (u'_v)_{v\in\partial\Lambda_r}\big) \leq h_r\big(s; (u'_v)_{v\in\partial\Lambda_r}\big)h_r\big(t; (u_v)_{v\in\partial\Lambda_r}\big). \tag{7.8.7}$$

We shall prove (7.8.7) via induction. Note that

$$h_1(x; (u_v)_{v\in\partial\{w_1\}}) = \prod_{e=\{v_1\to v_2\}\in E_1\setminus\{e_1\}} W_e(u_{v_1} - u_{v_2})$$

is the product of edge weights without any integration and with the convention $u_{w_1} = x$. Applying (7.8.6) to each such weight function yields (7.8.7) for $r = 1$. Observe the recursion relation for h_r:

$$h_{r+1}\big(x; (u_v)_{v\in\partial\Lambda_{r+1}}\big) = d_r\big(x; (u_v)_{v\in\partial\Lambda_{r+1}}\big) \cdot \int_{\mathbb{R}} h_r\big(y; (u_v)_{v\in\partial\Lambda_r}\big) F_r(x-y)dy$$

where by convention we set $u_{w_{r+1}} = x$ and where we define

$$d_r(x; (u_v)_{v\in\partial\Lambda_{r+1}}) = \prod_{e=\{v_1\to v_2\}\in Q_r} W_e(u_{v_1} - u_{v_2})$$

with Q_r being the set of all red and black edges that have one vertex as w_{r+1} and another vertex in $\partial\Lambda_{r+1}$, see Figure 7.21 (B). Note that the blue edge e_{r+1} between w_{r+1} and $w_{r+1}+(0,1)$ is excluded from Q_r. Appealing to (7.8.6) again, we have

$$d_r(s; (u_v)_{v\in\partial\Lambda_{r+1}})d_r(t; (u'_v)_{v\in\partial\Lambda_{r+1}}) \le d_r(s; (u'_v)_{v\in\partial\Lambda_{r+1}})d_r(t; (u_v)_{v\in\partial\Lambda_{r+1}}) \tag{7.8.8}$$

for all $s \ge t$ and for all $u'_v \ge u_v$ with $v \in \partial\Lambda_{r+1}$. Under same conditions we claim that

$$\begin{aligned}\int_{\mathbb{R}^2} & h_r(y; (u_v)_{v\in\partial\Lambda_r})F_r(s-y)h_r(x; (u'_v)_{v\in\partial\Lambda_r})F_r(t-x)dxdy \\ &\le \int_{\mathbb{R}^2} h_r(y; (u'_v)_{v\in\partial\Lambda_r})F_r(s-y)h_r(x; (u_v)_{v\in\partial\Lambda_r})F_r(t-x)dxdy.\end{aligned} \tag{7.8.9}$$

Combining the above inequality with (7.8.8) we have (7.8.7) completing the proof. To see why (7.8.9) holds, we split the integrals in (7.8.9) over $\{x < y\}$ and $\{y < x\}$ and swap the x, y labels in the region $\{y < x\}$ to get that (7.8.9) is equivalent to

$$\int_{x<y} A(y)Y(y)B(x)X(x)+C(x)Z(x)D(y)W(y) \le \int_{x<y} D(y)Y(y)C(x)X(x)+B(x)Z(x)A(y)Z(y)$$

where we let

$$A(y) = h_r(y; (u_v)_{v\in\partial\Lambda_r}), B(x) = h_r(x; (u'_v)_{v\in\partial\Lambda_r}), C(x) = h_r(x; (u_v)_{v\in\partial\Lambda_r}), D(y) = h_r(y; (u'_v)_{v\in\partial\Lambda_r}),$$

$$X(x) = F_r(t-x), Y(y) = F_r(s-y), W(y) = F_r(t-y), Z(x) = F_r(s-x).$$

The integral above can be rewritten as $\int_{x<y} \big(A(y)B(x) - C(x)D(y)\big)\big(X(x)Y(y) - W(y)Z(x)\big)$ and thus it suffices to show for each $x \leq y$ the integrand is non-positive. By induction hypothesis, $A(y)B(x) \leq C(x)D(y)$ for all $x \leq y$ and since the weight function F_r satisfies (7.8.6) (with $P = F_r$), we also have $X(x)Y(y) \geq W(y)Z(x)$. This proves (7.8.9), completing the proof of the lemma. □

7.9 Supporting calculations

In this section, we provide a detailed verification of the relations (7.2.13), (7.2.14), and (7.2.15). We continue with the same notations as in the proof of Theorem 7.1.3.

Verification of (7.2.13)**.** Note that from the transformation we have

$$e^{u_{j-i+1,2N-2j+1}} = t_{i+2N-2j,i} \quad e^{u_{j-i+1,2N-2j+2}} = t_{i+2N-2j+1,i}.$$

This yields

$$\tau^{\theta}_{2N-2j} = \prod_{i=1}^{j} t^{\theta}_{i+2N-2j,i} = \prod_{i=1}^{j} e^{\theta u_{j-i+1,2N-2j+1}} = \prod_{i=1}^{j} e^{\theta u_{i,2N-2j+1}}.$$

Similarly we have

$$\tau^{\theta}_{2N-2j+2} = e^{-\theta u_{j,2N-2j+3}} \prod_{i=1}^{j} e^{\theta u_{i,2N-2j+3}}, \quad \tau^{\theta}_{2N-2j+1} = \prod_{i=1}^{j} e^{\theta u_{i,2N-2j+2}}.$$

Thus,

$$\begin{aligned}
\prod_{j=1}^{N}\left(\frac{\tau_{2N-2j+2}\tau_{2N-2j}}{\tau_{2N-2j+1}^2}\right)^{\theta} &= \prod_{j=1}^{N}\left[e^{-\theta u_{j,2N-2j+3}}\prod_{i=1}^{j} e^{\theta[u_{i,2N-2j+1}+u_{i,2N-2j+3}-2u_{i,2N-2j+2}]}\right] \\
&= \left[\prod_{i=1}^{N} e^{-\theta u_{i,2N-2i+3}}\right]\cdot\left[\prod_{i=1}^{N}\prod_{j=i}^{N} e^{\theta[u_{i,2N-2j+1}+u_{i,2N-2j+3}-2u_{i,2N-2j+2}]}\right] \\
&= \left[\prod_{i=1}^{N} e^{-\theta u_{i,2N-2i+3}}\right]\cdot\left[\cdot\prod_{i=1}^{N}\prod_{j=1}^{N-i+1} e^{\theta[u_{i,2j-1}+u_{i,2j+1}-2u_{i,2j}]}\right] \quad (j\mapsto N-j+1).
\end{aligned}$$

The last term above is clearly equal to the right-hand side of (7.2.13).

Verification of (7.2.14)**.** Let us write

$$\begin{aligned}
\sum_{i>j}\frac{t_{i-1,j}}{t_{i,j}} = \sum_{j=1}^{N}\sum_{i=j+1}^{2N-j+1}\frac{t_{i-1,j}}{t_{i,j}} &= \sum_{j=1}^{N}\sum_{r=1}^{2N-2j+1}\frac{t_{j+r-1,j}}{t_{j+r,j}} \\
&= \sum_{j=1}^{N}\sum_{r=1}^{N-j}\frac{t_{j+2r-1,j}}{t_{j+2r,j}}+\sum_{j=1}^{N}\sum_{r=1}^{N-j+1}\frac{t_{j+2r-2,j}}{t_{j+2r-1,j}}.
\end{aligned} \tag{7.9.1}$$

Observe that

$$e^{u_{N-r-j+1,2r+1}} = t_{j+2r,j}, \quad e^{u_{N-r-j+2,2r}} = t_{j+2r-1,j}. \tag{7.9.2}$$

Thus we have

$$\begin{aligned}
(7.9.1) &= \sum_{j=1}^{N}\sum_{r=1}^{N-j} e^{u_{N-r-j+2,2r}-u_{N-r-j+1,2r+1}} + \sum_{j=1}^{N}\sum_{r=1}^{N-j+1} e^{u_{N-r-j+2,2r-1}-u_{N-r-j+2,2r}} \\
&= \sum_{j=1}^{N}\sum_{r=1}^{j-1} e^{u_{j-r+1,2r}-u_{j-r,2r+1}} + \sum_{j=1}^{N}\sum_{r=1}^{j} e^{u_{j-r+1,2r-1}-u_{j-r+1,2r}} \quad (j\mapsto N-j+1) \\
&= \sum_{r=1}^{N-1}\sum_{j=r+1}^{N} e^{u_{j-r+1,2r}-u_{j-r,2r+1}} + \sum_{r=1}^{N}\sum_{j=r}^{N} e^{u_{j-r+1,2r-1}-u_{j-r+1,2r}} \\
&= \sum_{r=1}^{N-1}\sum_{i=1}^{N-r} e^{u_{i+1,2r}-u_{i,2r+1}} + \sum_{r=1}^{N}\sum_{i=1}^{N-r+1} e^{u_{i,2r-1}-u_{i,2r}}
\end{aligned}$$

where the last equality follows by setting $j-r \mapsto i$ and $j-r \mapsto i-1$ in the first and second sum respectively. A final interchange of sum in each of the two terms leads to the right hand side of (7.2.14).

Verification of (7.2.15)**.** We follow the same above strategy and write

$$\sum_{i\geq j>1} \frac{t_{i,j-1}}{t_{i,j}} = \sum_{j=2}^{N} \sum_{i=j}^{2N-j+1} \frac{t_{i,j-1}}{t_{i,j}} = \sum_{j=2}^{N} \sum_{r=1}^{2N-2j+2} \frac{t_{j+r-1,j-1}}{t_{j+r-1,j}}$$
$$= \sum_{j=2}^{N} \sum_{r=1}^{N-j+1} \frac{t_{j+2r-1,j-1}}{t_{j+2r-1,j}} + \sum_{j=2}^{N} \sum_{r=1}^{N-j+1} \frac{t_{j+2r-2,j-1}}{t_{j+2r-2,j}} \tag{7.9.3}$$

Due to (7.9.2) we have

$$\begin{aligned}
(7.9.3) &= \sum_{j=2}^{N} \sum_{r=1}^{N-j+1} e^{u_{N-r-j+2,2r+1}-u_{N-r-j+2,2r}} + \sum_{j=2}^{N} \sum_{r=1}^{N-j+1} e^{u_{N-r-j+3,2r}-u_{N-r-j+2,2r-1}} \\
&= \sum_{j=1}^{N-1} \sum_{r=1}^{j} e^{u_{j-r+1,2r+1}-u_{j-r+1,2r}} + \sum_{j=1}^{N-1} \sum_{r=1}^{j} e^{u_{j-r+2,2r}-u_{j-r+1,2r-1}} \quad (j \mapsto N-j+1) \\
&= \sum_{r=1}^{N-1} \sum_{j=r}^{N-1} e^{u_{j-r+1,2r+1}-u_{j-r+1,2r}} + \sum_{r=1}^{N-1} \sum_{j=r}^{N-1} e^{u_{j-r+2,2r}-u_{j-r+1,2r-1}} \\
&= \sum_{r=1}^{N-1} \sum_{i=1}^{N-r} e^{u_{i,2r+1}-u_{i,2r}} + \sum_{r=1}^{N-1} \sum_{i=1}^{N-r} e^{u_{i+1,2r}-u_{i,2r-1}} \quad (j-r \mapsto i-1).
\end{aligned}$$

A final interchange of sum in each of the two terms leads to the right hand side of (7.2.15). This completes the verification of all three equalities.

Chapter 8: The half-space log-gamma polymer in the bound phase

8.1 Introduction

Directed polymers in random environments, first appeared in [203, 206, 61], are a rich class of mathematical physics models that have been extensively studied over the last several decades (see books [303, 181, 143, 99] and the references therein). More recently, a particular variant of the polymer models, the half-space polymers, has garnered considerable attention. The structure of the half-space polymers resembles the behavior of an interface in the presence of an attractive wall and their understanding renders importance to the studies of the wetting phenomena ([1, 269, 76]). Depending on the attraction force of the wall, it was conjectured in [215] that these models exhibit a “depinning" phase transition. When the attraction force exceeds a certain critical threshold (colloquially known as the bound phase), [215] conjectured that the endpoint of the polymer stays within a $O(1)$ window around the wall, i.e., it gets pinned to the wall. In this paper, we focus on the half-space polymers with log-gamma weights which make the model integrable and resolve Kardar's $O(1)$ conjecture in the bound phase. Our work is the first rigorous instance that positively solves Kardar's $O(1)$ conjecture.

Presently, we begin with an introduction to the model and the statements of our main results.

8.1.1 The model and the main results

Fix any $\theta > 0$ and $\alpha > -\theta$ and define the half-space index set: $\mathcal{I}^- = \{(i, j) \in \mathbb{Z}^2_{>0} \mid j \le i\}$. We consider a family of independent variables $(W_{i,j})_{(i,j)\in\mathcal{I}^-}$:

$$W_{i,i} \sim \mathrm{Gamma}^{-1}(\alpha + \theta) \qquad W_{i,j} \sim \mathrm{Gamma}^{-1}(2\theta) \ \text{ for } i < j, \tag{8.1.1}$$

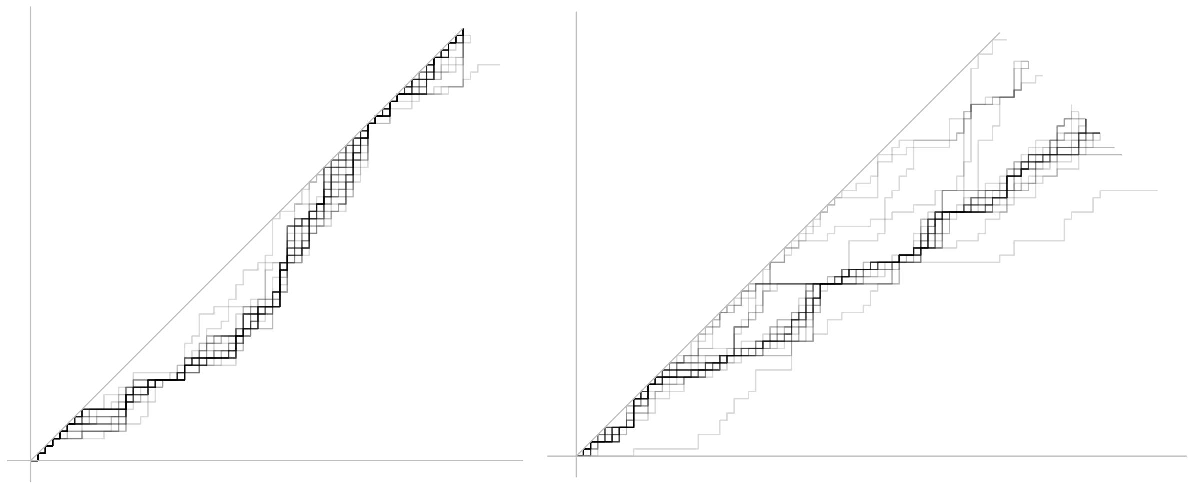


Figure 8.1: The bound and the unbound phase.

where Gamma(β) denotes a random variable with density $\mathbf{1}\{x > 0\}[\Gamma(\beta)]^{-1}x^{\beta-1}e^{-x}$. Let Π_N^{half} be the set of all upright lattice paths of length $2N-2$ starting from $(1,1)$ that are confined to the half-space $\mathcal{I}^-$ (see Figure 8.2). Given the weights in (8.1.1), the half-space log-gamma ($\mathcal{HSLG}$) polymer is a random measure on Π_N^{half} defined as

$$\mathbf{P}^W(\pi) = \frac{1}{Z(N)} \prod_{(i,j)\in\pi} W_{i,j} \cdot \mathbf{1}_{\pi\in\Pi_N^{\text{half}}}, \tag{8.1.2}$$

where $Z(N)$ is the normalizing constant.

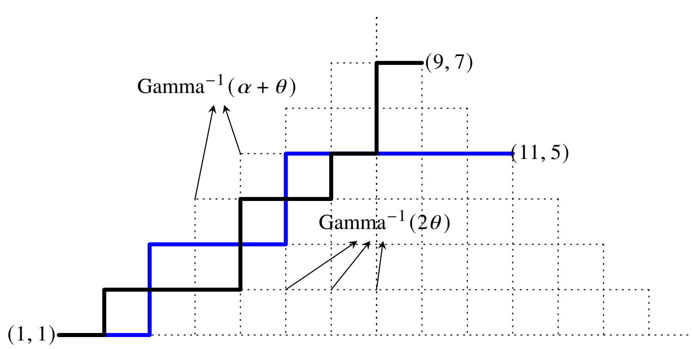


Figure 8.2: Two possible paths of length 14 in Π_8^{half} are shown in the figure.

The parameter α controls the strength of the boundary weights, i.e. the attractiveness of the wall, and a “depinning" phase transition occurs when $\alpha = 0$ (see [215, 275, 25]). When $\alpha \geq 0$, [34, 28] showed that the polymer measure is unpinned and the endpoint lies in a $O(N^{2/3})$ window.

For $\alpha < 0$, the conjecture is that the attraction is strong enough so that the polymer measure is pinned to the diagonal (see Figure 8.1). Indeed, our first main result below confirms that in the bound phase, i.e., when $\alpha \in (-\theta, 0)$, the endpoint of the $\mathcal{HSLG}$ polymer is within $O(1)$ window of the diagonal and is the first such result to capture the "pinning" phenomenon of the half-space polymer measure to the diagonal.

Theorem 8.1.1 (Bounded endpoint)**.** *Fix $\theta > 0$ and $\alpha \in (-\theta, 0)$ and consider the random measure $\mathbf{P}^W$ from* (8.1.2)*. For a path $\pi \in \Pi_N^{\text{half}}$, we denote $\pi(2N-2)$ as the height (i.e., y-coordinate) of the endpoint of the polymer. We have*

$$\limsup_{k\to\infty} \limsup_{N\to\infty} \mathbf{P}^W(\pi(2N-2) \le N-k) = 0, \quad \textit{in probability}. \tag{8.1.3}$$

Theorem 8.1.1 is a quenched result and naturally implies its annealed version. Following the above theorem, our next point of inquiry is the limiting behavior of the quenched distribution of the endpoints around the diagonal. We introduce and clarify a few more notations below before stating our results in this direction. Let $\Pi_{m,n}^{\text{half}}$ is the set of all upright lattice paths starting from $(1,1)$ and ending at (m,n) that reside solely in the half-space $\mathcal{I}^-$. We define the *point-to-point* partition function as

$$Z(m,n) := \sum_{\pi \in \Pi_{m,n}^{\text{half}}} \prod_{(i,j)\in\pi} W_{i,j}. \tag{8.1.4}$$

Under the above definition, the normalizing constant $Z(N)$ in (8.1.2), can also be viewed as the *point-to-line* partition function, i.e.

$$Z(N) = \sum_{p=0}^{N-1} Z(N+p, N-p).$$

The natural logarithm of the partition function is termed as the free energy of the polymer. The aforementioned depinning phase transition can be observed by studying the fluctuations of the free energy of the polymer. In this context, [34] obtained precise one-point fluctuations for the point-to-line free energy $\log Z(N)$ in both the bound and unbound phases and observed the BBP phase transition. A similar fluctuation result and Baik-Rains phase transition were later shown in [205] for the point-to-point free energy $\log Z(N, N)$ on the diagonal. For $\alpha \geq 0$, it was recently proven in [28] that the point-to-point free energy process

$$\left(\log Z(N + pN^{2/3}, N - pN^{2/3})\right)_{p \in [0,r]}$$

after appropriate centering and scaling by $N^{1/3}$ is functionally tight. This result captures the characteristic KPZ $1/3$ fluctuation and $2/3$ transversal scaling exponents. In our present work, we study the point-to-point free energy process under $\alpha < 0$ case. Our second main result below obtains precise fluctuations for the increments of the point-to-point free energy process when $\alpha < 0$. To state the result, we introduce the definition of the *log-gamma random walk*.

Definition 8.1.2. Fix $\theta > 0$ and $\alpha \in (-\theta, 0]$. Let $Y_1 \sim \text{Gamma}(\theta + \alpha)$ and $Y_2 \sim \text{Gamma}(\theta - \alpha)$ be independent random variables. We refer to $X := \log Y_2 - \log Y_1$ as a log-gamma random variable. It has a density given by

$$p(x) := \frac{1}{\Gamma(\theta + \alpha)\Gamma(\theta - \alpha)} \int_{\mathbb{R}} \exp\left((\theta - \alpha)y - e^y + (\theta + \alpha)(y - x) - e^{y-x}\right) dy. \tag{8.1.5}$$

Let $(X_i)_{i \geq 0}$ be a sequence of such iid log-gamma random variables. Set $S_0 = 0$ and $S_k = \sum_{i=1}^{k} X_i$. We refer to $(S_k)_{k \geq 0}$ as a *log-gamma random walk.*

Our next result states that in the bound phase, the above random walk is an *attractor* for the increments of the half-space log-partition function.

Theorem 8.1.3. *Fix $\theta > 0$ and $\alpha \in (-\theta, 0)$. For each $k \geq 1$, as $N \to \infty$, we have the following multi-point convergence in distribution*

$$\left(\frac{Z(N + r, N - r)}{Z(N, N)}\right)_{r \in [\![0,k]\!]} \xrightarrow{d} \left(e^{-S_r}\right)_{r \in [\![0,k]\!]}, \tag{8.1.6}$$

where $(S_r)_{r \geq 0}$ is a log-gamma random walk from Definition 8.1.2.

From the above result, we deduce the following limiting quenched distribution of the endpoint when viewed around the diagonal.

Theorem 8.1.4. *Fix $\theta > 0$ and $\alpha \in (-\theta, 0)$ and consider the random measure $\mathbf{P}^W$ from* (8.1.2)*. Let $(S_k)_{k\geq 0}$ be a log-gamma random walk from Definition 8.1.2. Set $Q := \sum_{p\geq 0} e^{-S_p}$. For a path $\pi \in \Pi_N^{\mathrm{half}}$, we denote $\pi(2N-2)$ as the height (i.e., y-coordinate) of the endpoint of the polymer. Then for each $k \geq 1$, as $N \to \infty$, we have the following multi-point convergence in distribution*

$$\Big(\mathbf{P}^W(\pi(2N-2) = N-r)\Big)_{r\in[\![0,k]\!]} \xrightarrow{d} \Big(Q^{-1}\cdot e^{-S_r}\Big)_{r\in[\![0,k]\!]}. \tag{8.1.7}$$

Beyond proving the $O(1)$ transversal fluctuation around the point (N,N) and pinning down the exact density within this region, our main theorems above also shed light on the attractive properties of half-space log-gamma stationary measures. In [30] a stationary version of the half-space log-gamma polymer was considered for $\alpha \in (-\theta, \theta)$, where the horizontal weights along the first row are assumed to be distributed as $\mathrm{Gamma}^{-1}(\theta-\alpha)$ (i.e., $W_{i,1} \sim \mathrm{Gamma}^{-1}(\theta-\alpha)$). Let us denote $Z^{\mathrm{stat}}(n,m)$ to be the point-to-point $\mathcal{HSLG}$ partition function computed using these weights. It was shown in [30, Proposition 4.5], that this model is stationary in the sense that for all $k \geq 1$, and $N \geq k+1$

$$(\log Z^{\mathrm{stat}}(N,N) - \log Z^{\mathrm{stat}}(N+r, N-r))_{r\in[\![0,k]\!]} \stackrel{d}{=} (S_r)_{r\in[\![0,k]\!]}.$$

where $(S_r)_{r\geq 0}$ is a log-gamma random walk defined in Definition 8.1.2.

Remark 8.1.5. Using the above stationary weights, one can define an associated polymer measure $\mathbf{P}^W_{\mathrm{stat}}$ in the spirit of (8.1.2). We remark that both Theorem 8.1.1 and Theorem 8.1.4 continue to hold under $\mathbf{P}^W_{\mathrm{stat}}$. This is not hard to check from our log-gamma random walk results presented in Appendix 8.6.

Theorem 8.1.3 shows that for $\alpha < 0$ the above log-gamma random walk measure is an *attractor* for the original polymer model in the sense that the increment of the log-partition function of the original model converges to the same log-gamma random walk measure. We believe that our broad techniques should also lead to a similar convergence result for $\alpha \geq 0$. We leave this for future consideration.

We end this section by mentioning a recent work [27] on the stationary measures for the $\mathcal{HSLG}$ polymer. The point-to-point log-gamma polymer partition function $Z(n, m)$ satisfies a recurrence relation

$$Z(n, m) = W_{n,m} \cdot (Z(n-1, m) + Z(n, m-1)) \text{ for } n > m \geq 1,$$
$$Z(n, n) = W_{n,n} \cdot Z(n, n-1) \text{ for } n \geq 1,$$

We refer to a process $(h(k))_{k\geq 0}$ as horizontal-stationary for the $\mathcal{HSLG}$ polymer if the solution to the above recurrence relation with initial data $z(\cdot, 0) = e^{h(\cdot)}$ has stationary horizontal increments. For instance, the distribution of horizontal increments $(\log Z(N+k, N) - \log Z(N, N))_{k\geq 0}$ is same for all $N \geq 0$ (and equal to that of the initial data). Recently, [27] posited a one-parameter family of horizontal-stationary measures for the $\mathcal{HSLG}$ polymer model and conjectured that these stationary measures are attractors for a large class of initial data $(Z(n, 0))_{n\geq 0}$ subject to the condition $\lim_{k\to\infty} \log Z(k, 0)/k = d \in \mathbb{R}$. However, the initial data relevant to our polymer model corresponds to $Z(k, 0) = \mathbf{1}_{k=1}$ and is not covered in [27].

8.1.2 Proof Ideas

In this section we sketch the key ideas behind the proofs of our main results. Our proof relies on inputs from the recently developed $\mathcal{HSLG}$ Gibbsian line ensemble in [28], one-point fluctuation results for point-to-(partial)line half-space log-partition functions from [34] and the localization techniques from [132]. At the heart of our argument lies an innovative combinatorial argument that bridges the aforementioned inputs and enables our proof.

The starting point of our analysis is the $\mathcal{HSLG}$ Gibbsian line ensemble in [28], which allows us to embed the free energy $\log Z(N + r, N - r)$ of the $\mathcal{HSLG}$ polymer as the top curve of a Gibbsian

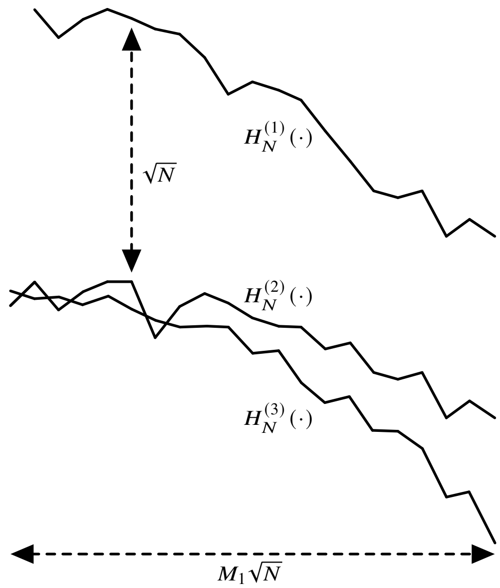


Figure 8.3: First three curves of the $\mathcal{HSLG}$ line ensemble. There is a high probability uniform separation of length $\sqrt{N}$ between the first two curves in the above $M_1\sqrt{N}$ window.

line ensemble $(H_N^{(k)}(\cdot))_{k\in[\![1,N]\!]}$ of log-gamma increment random walks interacting through a soft version of non-intersection (Theorem 8.2.4) conditioning and subject to an energetic interaction at the left boundary (where $r = 0$) depending on the value of α. This fact is due to the geometric RSK correspondence ([121, 263, 260, 59]) and the half-space Whittaker process ([25]). The key idea of our proof is to show that with high probability, the first and the second curves in our line ensemble ensemble (see Figure 8.3) are sufficiently uniformly separated. Then the separation allows us to conclude that the first curve indeed behaves similarly to a log-gamma random walk by a localization analysis.

The existing literature contains some information about the locations of the top two curves. When $\alpha < 0$, one can deduce from the line ensemble description in [28] that the first and the second curves are repulsed from each other at the left boundary. Results in [34] also supply information about the location for the first curve. However, one cannot deduce that the entire second curve lies uniformly much lower than the first curve from the above two inputs and line ensemble techniques alone.

Intuition behind the separation

Before we proceed to further break down our argument about the separation, it is worth dwelling on the mathematical intuition behind the separation between the first and second curves, which originates from the definition of the line ensemble defined in Section 8.2.1. For simplicity, let us focus only on the left boundary. By Definition 8.2.1, we have $H_N^{(1)}(1) = \log Z(N,N)$, and

$$H_N^{(1)}(1) + H_N^{(2)}(1) := \log \left[2 \sum_{\pi_1,\pi_2} \prod_{(i,j)\in\pi_1\cup\pi_2} \widetilde{W}_{i,j} \right], \tag{8.1.8}$$

where the above sum is over all pair of non-intersecting upright paths π_1, π_2 from $(1,1)$ to $(N, N-1)$ and from $(1,2)$ to (N,N) confined in the entire quadrant $\mathbb{Z}_{\geq 1}^2$ (instead of octant). Here $\widetilde{W}_{i,j}$ is the symmetrized version of the weights defined in (8.1.1) on the entire quadrant as:

$$\widetilde{W}_{i,i} = W_{i,i}/2 \ \text{ for } \ i \geq 1, \qquad \widetilde{W}_{i,j} = \widetilde{W}_{j,i} = W_{i,j} \ \text{ for } \ i > j. \tag{8.1.9}$$

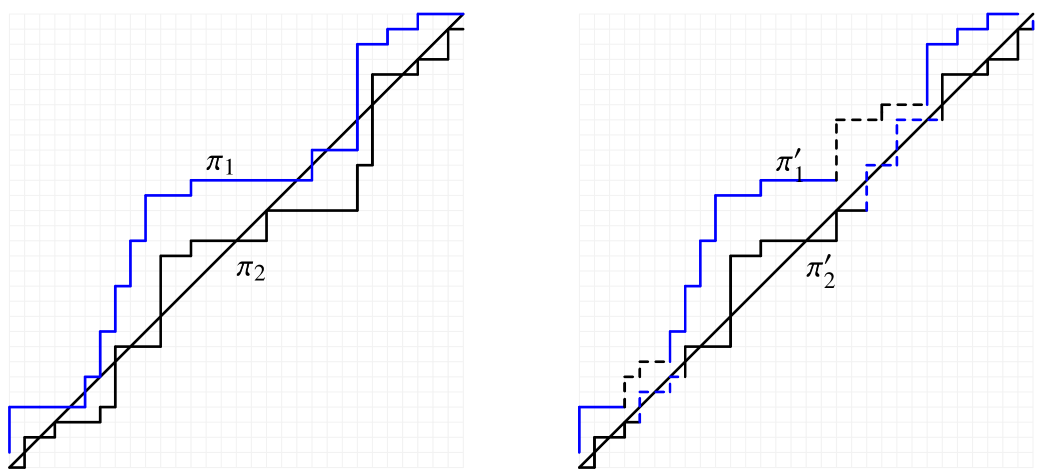
Using point-to-(partial)line log-partition function fluctuation results from [34] and line ensemble techniques, it is not hard to deduce that $\frac{1}{N}H_N^{(1)}(1) \to R := -\Psi(\theta+\alpha) - \Psi(\theta-\alpha)$, where Ψ is the digamma function defined in (8.2.8). However, $H_N^{(2)}(1)$ should follow a different law of large numbers. This can be understood intuitively from (8.1.8) as follows. For α close to $-\theta$, the weights on the diagonal are huge and stochastically dominate all the other weights. The sum in (8.1.8) then concentrates on the pair of paths π_1^*, π_2^* which jointly have the maximal numbers of diagonal points. This occurs when one of the paths carries all the diagonal weights and the other path has no diagonal weights. Thus we expect,

$$\sum_{\pi_1,\pi_2} \prod_{(i,j)\in\pi_1\cup\pi_2} \widetilde{W}_{i,j} \asymp \left[\sum_{\pi_1} \prod_{(i,j)\in\pi_1} \widetilde{W}_{i,j} \right] \cdot \left[\sum_{\pi_2 | \mathrm{diag}(\pi_2)=\varnothing} \prod_{(i,j)\in\pi_2} \widetilde{W}_{i,j} \right] \tag{8.1.10}$$

Upon taking logarithms and dividing by N, the first term goes to R. However, the second term does not feel the effect of the diagonal and hence should follow the law of large numbers corresponding to the unbound phase, i.e., $\alpha > 0$. The unbound phase law of large numbers is given by $-\Psi(\theta)$ noted in [34, 28]. Thus overall, we expect $\frac{1}{N}(H_N^{(1)}(1) + H_N^{(2)}(1)) \to R - \Psi(\theta)$. As Ψ is concave, the above heuristics suggests $H_N^{(2)}(1)$ follow a lower law of large numbers. While our technical arguments to be presented later do not yield exactly (8.1.10), we utilize the above idea to obtain a large enough separation between the two curves, which turns out to be sufficient for proving our main theorems.

The U map and its consequences

We now describe the key idea that makes the above intuition work. All the statements mentioned in this subsection should be understood as high probability statements. The above idea of having one path having all diagonal weights is made precise in Section 8.3, where we develop a combinatorial map in Lemma 8.3.1, referred to as the U map.



(a) (b)

Figure 8.4: The U map takes π_1, π_2 from (A) and returns π'_1, π'_2 in (B). The precise description of the map is given in the proof of Lemma 8.3.1

The U map takes every pair of paths π_1, π_2 in the sum in (8.1.8) and returns a pair of non-intersecting paths π'_1, π'_2 while preserving their shared weights up to reflections (see Figure 8.4). Moreover, the diagonal weights collectively carried by the pair will only rest on one of the paths among π'_1, π'_2. The U map is not injective but has at most 2^N many inverses for each pair in its image. When we apply the U map to a single pair of adjacent paths, we get that

$$\tfrac{1}{N}(H_N^{(1)}(1)+H_N^{(2)}(1)) \le \log 2 + R - \Psi(\theta).$$

The $\log 2$ is an entropy factor that comes from overcounting the number of inverses of our U map. To remove its influence, we rely on the definition of the lower curves of the line ensemble. Indeed, similar to (8.1.8), $\sum_{i=1}^{2k} H_N^{(i)}(1)$ admits a representation in terms of $2k$-many non-intersecting paths. When we apply the U map to k pairs of adjacent paths simultaneously, it leads to the following average law of large numbers of the top $2k$ curves:

$$\frac{1}{2kN}\sum_{i=1}^{2k} H_N^{(i)}(1) \le \tfrac{1}{2k}\log 2 - \tfrac{1}{2}\Psi(\theta) - \tfrac{1}{2}\Psi(\theta+\alpha) - \tfrac{1}{2}\Psi(\theta-\alpha).$$

Taking k large enough, one can ensure the right-hand side constant is strictly less than R. In fact, the above argument can be strengthened to conclude that for large enough k

$$\sup_{p\in[\![1,2N-4k+2]\!]} \frac{1}{2kN}\sum_{i=1}^{2k} H_N^{(i)}(p) \le R-\delta,$$

for some $\delta > 0$. This is obtained in Proposition 8.3.4.

As a consequence of this result, using soft non-intersection property of the line ensemble (Theorem 8.2.4), we derive that with high probability, the $(2k+2)$-th curve $H_N^{2k+2}(\cdot)$ is uniformly $\mathrm{Const}\cdot N$ below RN over $[\![1,N]\!]$ in Section 8.4. Employing one-point results from [34], one can ensures the point $H_N^{(1)}(M_1\sqrt{N})$ on the top curve is $(M_2+1)\sqrt{N}$ below RN. Combining the last two results and line ensemble techniques we are able to benchmark the second curve from above:

$$\sup_{p\in[\![1,M_1\sqrt{N}]\!]} H_N^{(2)}(p) \le RN - M_2\sqrt{N} \tag{8.1.11}$$

in Proposition 8.4.2. The details of the argument are presented in Section 8.4. While we are unable to obtain a mismatch in the laws of large numbers for the first two curves following the above procedures, the fact that the second curve is below the diffusive regime of the first curve (since M_2 can be chosen as large as possible) over an interval of length $M_1\sqrt{N}$ is sufficient for our next step of the analysis.

Localization analysis

The remaining piece of our proof of main theorems boils down to a localization analysis of the first curve in Section 8.5. Our proof roughly follows the techniques developed in our paper [132]. First, to prove Theorem 8.1.1 we divide the tail into a deep and a shallow tail depending on the distance away from (N, N), see Figure 8.5.

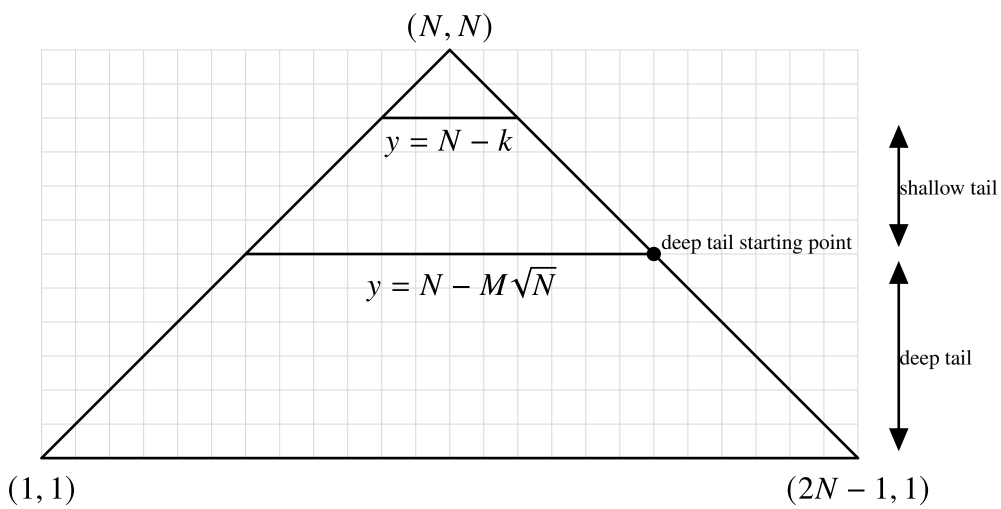


Figure 8.5: If the height of the endpoint of the polymer is less than $N - k$, it either lies in the shallow tail or in the deep tail (illustrated above). Lemma 8.5.1 shows it is exponentially unlikely to lie in the deep tail.

Our argument in Lemma 8.5.1 uses one-point fluctuations results of point-to-(partial)line log-partition function from [34] as input and shows that the probability of the endpoint living in the deep tail region is exponentially small. To show that the shallow tail contribution is also small and to prove our remaining theorems, we establish the following strong convergence result in Proposition 8.5.3:

(a) the law of the top curve within the $[1, M\sqrt{N}]$ window is arbitrarily close to that of a log-gamma random walk for large enough N (Proposition 8.5.3).

In light of (a), the conclusion that the shallow tail contribution is small follows from estimating the probability of the same event under the log-gamma random walk law. Theorem 8.1.3 is immediate from (a) and Theorem 8.1.4 also follows from (a) after some calculations. The details are presented in Section 8.5.2.

Finally, we briefly explain how we establish (a). A detailed discussion appears in the Step 1 of the proof of Proposition 8.5.3. As $H_N^{(1)}(\cdot)$ is a log-gamma random walk subject to soft non-intersecting condition with $H_N^{(2)}(\cdot)$, it suffices to show that there's sufficient distance between the first and the second curves. Indeed, this will imply $H_N^{(1)}$ behaves like a true log-gamma random walk. As we have already benchmarked the second curve in (8.1.11), it remains to determine a suitable lower bound for the first curve. The key idea here is to find a point $p = O(\sqrt{N})$ on the first curve in the deep tail region such that with high probability

$$H_N^{(1)}(p) \geq RN - M'\sqrt{N}$$

for some M'. This is achieved in Lemma 8.5.2 using fluctuation results from [34]. Then using standard random walk tools such as Kolmogorov's maximal inequality, we derive that with high probability $H_N^{(1)}(q) \geq RN - (M'+1)\sqrt{N}$ for all $q \in [\![1, p]\!]$. Choosing $M_2 = M' + 2$ in (8.1.11) implies that with high probability the first curve is at least $\sqrt{N}$ above the second curve, This completes our deduction and consequently establishes (a).

8.1.3 Related works and future directions

Our study of half-space polymers succeeds an extensive history of endeavors that attempt to unravel their full-space variant. These full-space polymer models have rich connections with symmetric functions, random matrices, stochastic PDEs and integrable systems and are believed to belong to the KPZ universality class (see [99, 181, 44]). Yet in spite of intense efforts in the past decade, rigorous results proving either the $1/3$ fluctuation exponent or the $2/3$ transversal exponent for general polymers have been scarce outside a few integrable cases (see [99, 294, 44, 29, 132, 133] and the references therein).

In the half-space geometry, a wealth of literature has focused on the phase diagram for limiting distributions based on the diagonal strength. One of the first mathematical works goes to the series of joint works [15, 17, 16] on the geometric last passage percolation (LPP), i.e. polymers with zero temperature. Their multi-point fluctuations were studied in [289] and similar results were later proved for exponential LPP in [11, 12] using Pfaffian Schur processes. For further recent works on half-space LPP, we refer to [50, 51, 52, 165] and the references therein.

For positive temperature models, i.e., polymers, as they are no longer directly related to the Pfaffian point processes, the first rigorous proof of the depinning transition appeared much later in [34]. Here the authors also included precise fluctuation results such as the BBP phase transition [13] for the point-to-line log-gamma free energy. For the point-to-point log-gamma free energy, the limit theorem as well as the Baik-Rains phase transition were conjectured in [25] based on steepest descent analysis of half-space Macdonald processes. This result has been recently proved in [205] by relating the half- space model to a free boundary version of the Schur process.

Similar to their full-space counterparts, in addition to fluctuations, another dimension of interest to half-space polymers is their localization behaviors, which refer to the concentration of polymers in a very small region given the environment. Figure 8.1 is a simulation of 30 samples of $\mathcal{HSLG}$ polymers of length 120 sampled from the same environment with $\theta = 1$, $\alpha = -0.2$ and $\alpha = +0.2$. The simulation suggests that even in the unbound phase, we expect a localization phenomenon around a favorite site given by the environment. Localization is a unique behavior of the polymer path in the strong disorder regime. In the full space, various levels of localization results have been established for discrete and continuous polymers. The mathematical work began with the *strong* localization result of [84] that confirmed the existence of the favorite sites for the endpoint distributions of point-to-line polymers and has been upgraded to the notion of *atomic* and *geometric* localization for general reference random walks in a series of joint works [42, 44, 20, 19]. An even stronger notion, the “favorite region conjecture", which conjectures the favorite corridor of a polymer to be stochastically bounded, has been proved for two integrable models:

the stationary log-gamma polymer in the discrete case ([101]) and the continuous directed random polymer (CDRP) in the continuous case ([132]). In this direction, building up on [132] work, recently [160] have studied the localization distance of the CDRP.

Investigating the geometry of the half-space CDRP is an interesting question to consider next. Recently, a number of new results have appeared on the half-space KPZ equation, which arises as the free energy of the half-space CDRP [320, 27], in both the mathematics [123, 26, 25, 272, 271, 27, 205] and the physics literature [189, 67, 207, 142, 231, 30, 32, 31]. These results on the free energy render the half-space continuous polymers amenable to analysis. However, the challenge with further studying the geometry of the half-space CDRP remains, due to the lack of an analogous half-space KPZ line ensemble.

Outline

The rest of the paper is organized as follows. Section 8.2 reviews some of the existing results related to $\mathcal{HSLG}$ line ensemble and one-point fluctuations of point-to-(partial)line free energy of $\mathcal{HSLG}$ polymer. In Section 8.3 we prove our key combinatorial lemma and use it to control the average law of large numbers for the top curves of the line ensemble. In Section 8.4, we establish control over the second curve of the line ensemble. Finally, in Section 8.5, we complete the proofs of our main theorems. Appendix 8.6 contains basic properties of log-gamma random walks.

Notation

Throughout this paper, we will assume $\theta > 0$ and $\alpha \in (-\theta, 0)$ are fixed parameters. We write $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$ to denote the set of integers between a and b. We will use serif fonts such as $\mathsf{A}, \mathsf{B}, \ldots$ to denote events. The complement of an event A will be denoted as $\neg\mathsf{A}$.

Acknowledgements

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8.2 Basic framework and tools

In this section, we present the necessary background on the half-space log-gamma ($\mathcal{HSLG}$) line ensemble and point-to-(partial) line partition function. From [28] and [34] we gather a few of the known results on these objects that are crucial in our proofs.

8.2.1 The $\mathcal{HSLG}$ line ensemble and its Gibbs property

We begin with the description of the $\mathcal{HSLG}$ line ensemble and its Gibbs property. The definition of the $\mathcal{HSLG}$ line ensemble is based on the point-to-point symmetrized partition function for multiple paths defined in (8.2.1). These are sum over multiple non-intersecting upright paths on the entire quadrant $\mathbb{Z}_{>0}^2$ of products of the symmetrized version defined in (8.1.9) of the weights defined in (8.1.1). Fix $m, n, r \in \mathbb{Z}_{>0}$ with $n \geq r$, let $\Pi_{m,n}^{(r)}$ be the set of all r-tuples of non-intersecting upright paths in $\mathbb{Z}_{>0}^2$ starting from $(1, r), (1, r-1), \ldots, (1, 1)$ and going to $(m, n), (m, n-1), \ldots, (m, n-r+1)$ respectively. We define the point-to-point symmetrized partition function for r paths as

$$Z_{\mathrm{sym}}^{(r)}(m, n) := \sum_{(\pi_1, \ldots, \pi_r) \in \Pi_{m,n}^{(r)}} \prod_{(i,j) \in \pi_1 \cup \cdots \cup \pi_r} \widetilde{W}_{i,j}. \tag{8.2.1}$$

where $\widetilde{W}_{i,j}$ are defined in (8.1.9). We write $Z_{\mathrm{sym}}(m, n) := Z_{\mathrm{sym}}^{(1)}(m, n)$ and use the convention that $Z_{\mathrm{sym}}^{(0)}(m, n) \equiv 1$. One can recover $\mathcal{HSLG}$ partition function from symmetrized partition function via the following identity. For each $(m, n) \in \mathcal{I}^-$ we have

$$2Z_{\mathrm{sym}}(m, n) = Z(m, n). \tag{8.2.2}$$

The above identity appears in Section 2.1 of [34] and follows easily due to the symmetry of the weights. We stress that the above relation is an exact equality not just in distribution.

Definition 8.2.1 ($\mathcal{HSLG}$ line ensemble)**.** Fix $N > 1$. For each $k \in [\![1, N-1]\!]$ and $p \in [\![1, 2N-2k+2]\!]$ set

$$H_N^{(k)}(p) := \log\left(\frac{2Z_{\mathrm{sym}}^{(k)}(N + \lfloor p/2 \rfloor, N - \lceil p/2 \rceil + 1)}{Z_{\mathrm{sym}}^{(k-1)}(N + \lfloor p/2 \rfloor, N - \lceil p/2 \rceil + 1)}\right) \tag{8.2.3}$$

We view the k-th curve $H_N^{(k)}$ as a random continuous function $H_N^{(k)} : [1, 2(N-k+1)] \to \mathbb{R}$ by linearly interpolating its values on integer points. We call the collection of curves $H_N := (H_N^{(1)}, H_N^{(2)}, \ldots, H_N^{(N)})$ as the $\mathcal{HSLG}$ line ensemble.

We remark that in Definition 2.7 in [28], the authors defined the $\mathcal{HSLG}$ line ensemble by defining $\mathcal{L}_i^N(j) = H_N^{(i)}(j) + \mathrm{Const} \cdot N$ where the 'Const' is explicit and encodes the law of large numbers for the $\mathcal{HSLG}$ free energy process (as well as the entire line ensemble) in the unbound phase. Since the law of large numbers for the first curve and the second curve in the bound phase are possibly different (recall our discussion of the proof idea in the introduction), we choose to not add this constant in our definition of line ensemble. All the results from [28] can be easily translated to results in our setting by adding this appropriate constant.

In view of (8.2.2), for all $p \leq 2N$ we have

$$H_N^{(1)}(p) = \log Z(N + \lfloor p/2 \rfloor, N - \lceil p/2 \rceil + 1). \tag{8.2.4}$$

The $\mathcal{HSLG}$ line ensemble enjoys a property that is known as the $\mathcal{HSLG}$ Gibbs property. To state the$\mathcal{HSLG}$ Gibbs property, we introduce the $\mathcal{HSLG}$ Gibbs measures via graphical representation.

We consider a diamond lattice on the lower-right quadrant with vertices $\{(m, -n), (m + \frac{1}{2}, -n + \frac{1}{2}) \mid m, n \in \mathbb{Z}^2_{>0}\}$ and nearest neighbor edges as shown in Figure 8.6. We label the vertices by setting $\phi((m, n)) = (-\lfloor n \rfloor, 2m - 1)$. We shall always use this labeling to identify a vertex in this lattice and will not mention its actual coordinates further.

On the diamond lattice domain, we add potential *directed-colored edges*. A directed-colored edge $\vec{e} = \{v_1 \to v_2\}$ on this lattice is a directed edge from v_1 to v_2 that has three choices of colors:

blue, **red**, and **black**. Given a directed-colored edge, we associate a weight function based on the color of the edge defined as follows:

$$W_{\vec{e}}(x) = \begin{cases} \exp((\theta-\alpha)x - e^x) & \text{if } \vec{e} \text{ is } \textbf{blue} \\ \exp((\theta+\alpha)x - e^x) & \text{if } \vec{e} \text{ is } \textbf{red} \\ \exp(-e^x) & \text{if } \vec{e} \text{ is } \textbf{black}. \end{cases} \tag{8.2.5}$$

We consider a graph G_N on the diamond lattice with vertex set

$$K_N := \{(i,j) \mid i \in [\![1,N]\!], j \in [\![1, 2N-2i+2]\!]\}.$$

with directed-colored edges described below. For each $(p,q) \in K_N$,

- If q is odd and p is odd (even resp.), we put a **blue** (**red** resp.) edge: $(p,q) \to (p,q+1)$.
- If $q \geq 3$ is odd and p is odd (even resp.), we put a **red** (**blue** resp.) edge: $(p,q) \to (p,q-1)$.
- If q is even, we put two **black** edges: $(p,q) \to (p-1,q)$ and $(p,q) \to (p+1,q)$.

The corresponding graph is shown in Figure 8.6. We write $E(F)$ for the set of edges of any graph $F \subset G_N$.

The following result from [28] shows how the conditional distribution of the $\mathcal{HSLG}$ line ensemble is given by certain measures called $\mathcal{HSLG}$ Gibbs measures.

Theorem 8.2.2 (Gibbs property)**.** *Consider the directed-colored graph G_N described above. Set Λ be a connected subset of the graph G_N on the diamond lattice K_N*

$$\Lambda_N^* = \{(i,j) \mid i \in [\![1,N-1]\!], j \in [\![1,2N-2i+1]\!]\}.$$

Let Λ be a connected subset of Λ_N. Recall the $\mathcal{HSLG}$ line ensemble H^N from Theorem 8.2.1. The law of $\{H_i^N(j) \mid (i,j) \in \Lambda\}$ conditioned on $\{H_i^N(j) \mid (i,j) \in \Lambda^c\}$ is a measure on $\mathbb{R}^{|\Lambda|}$ with

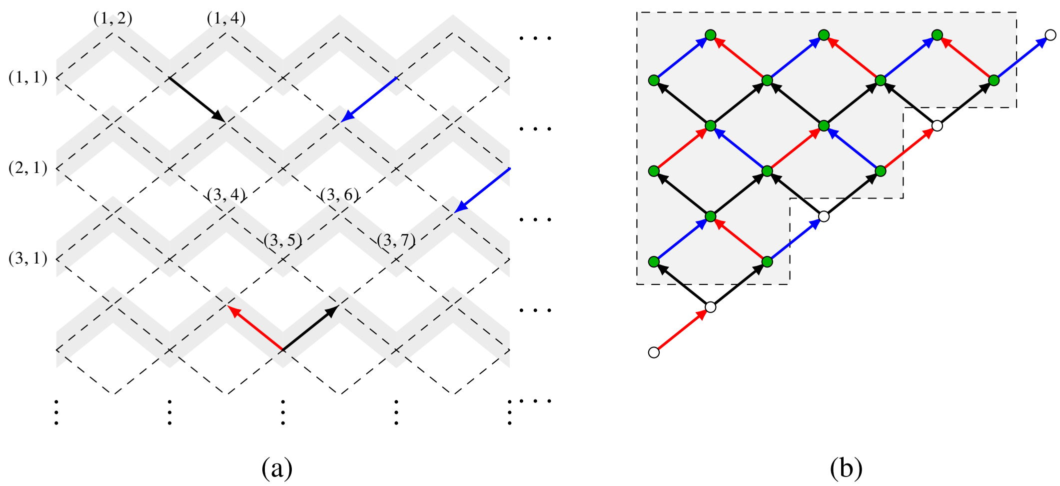


Figure 8.6: (A) Diamond lattice with a few of the labeling of the vertices shown in the figure. The m-th gray-shaded region have vertices with labels of the form $\{(m,n) \mid n \in \mathbb{Z}^2_{>0}\}$. Thus each such region consists of vertices with the same first coordinate labeling. Potential directed-colored edges on the lattice are also drawn above. (B) K_N with $N = 4$. Λ^*_N consists of all vertices in the shaded region.

density at $(u_{i,j})_{(i,j)\in\Lambda}$ *proportional to*

$$\prod_{\vec{e}=\{v_1\to v_2\}\in E(\Lambda\cup\partial\Lambda)} W_{\vec{e}}(u_{v_1} - u_{v_2}), \tag{8.2.6}$$

where $u_{i,j} = H_i^N(j)$ *for* $(i,j) \in \partial\Lambda$.

We call the above conditional law as the $\mathcal{HSLG}$ Gibbs measure with boundary condition $\vec{u} = (u_{i,j})_{(i,j)\in\partial\Lambda}$ and denote this measure as $\mathbf{P}^{\vec{u}}_{\text{gibbs}}(\cdot)$. The above theorem follows directly from the results in [28]). Theorem 1.3 in [28] specifies the Gibbs property for the centered line ensemble $L_i^N(j)$. The same Gibbs property holds for $H_N^{(i)}(j)$ as $\mathcal{HSLG}$ Gibbs measures are translation invariant (Observation 2.1 (b) in [28]. The Gibbs property stated in Theorem 1.3 is different and valid for all $\alpha > -\theta$. When $\alpha \in (-\theta, \theta)$, one can redistribute the edge-weights (see Observation 4.2 in [28]) to obtain the above stated Gibbs property.

The $\mathcal{HSLG}$ Gibbs measures satisfy stochastic monotonicity w.r.t. the boundary data.

Proposition 8.2.3 (Stochastic monotonicity, Proposition 2.6 in [28])**.** *Fix* $k_1 \le k_2$*,* $a_i \le b_i$ *for*

$k_1 \leq i \leq k_2$ and $\alpha > -\theta$. Let

$$\Lambda := \{(i, j) \mid k_1 \leq i \leq k_2, a_i \leq j \leq b_i\}.$$

There exists a probability space consisting of a collection of random variables

$$\{L(v; (u_w)_{w \in \partial\Lambda}) \mid v \in \Lambda, (u_w)_{w \in \partial\Lambda} \in \mathbb{R}^{|\partial\Lambda|}\}$$

such that

1. *For each $(u_w)_{w \in \partial\Lambda} \in \mathbb{R}^{|\partial\Lambda|}$, the law of $\{L(v; (u_w)_{w \in \partial\Lambda}) \mid v \in \Lambda\}$ is given by the $\mathcal{HSLG}$ Gibbs measure for the domain Λ with boundary condition $(u_w)_{w \in \partial\Lambda} \in \mathbb{R}^{|\partial\Lambda|}$.*

2. *With probability 1, for all $v \in \Lambda$ we have*

$$L(v; (u_w)_{w \in \partial\Lambda}) \leq L(v; (u'_w)_{w \in \partial\Lambda}) \text{ whenever } u_w \leq u'_w \text{ for all } w \in \partial\Lambda.$$

As mentioned in the introduction, the $\mathcal{HSLG}$ line ensemble enjoys a certain soft non-intersection property. This property is captured in our next theorem.

Theorem 8.2.4 (Ordering of points, Theorem 3.1 in [28])**.** *Fix any $k \in \mathbb{Z}_{>0}$ and $\rho \in (0, 1)$. There exists $N_0 = N_0(\rho, k) > 0$ such that for all $N \geq N_0$, $i \in [\![1, k]\!]$ and $p \in [\![1, N - i]\!]$ the following inequalities holds:*

$$\mathbf{P}(H_N^{(i)}(2p + 1) \leq H_N^{(i)}(2p) + \log^2 N) \geq 1 - \rho^N,$$
$$\mathbf{P}(H_N^{(i)}(2p - 1) \leq H_N^{(i)}(2p) + \log^2 N) \geq 1 - \rho^N,$$
$$\mathbf{P}(H_N^{(i+1)}(2p) \leq H_N^{(i)}(2p + 1) + \log^2 N) \geq 1 - \rho^N,$$
$$\mathbf{P}(H_N^{(i+1)}(2p) \leq H_N^{(i)}(2p - 1) + \log^2 N) \geq 1 - \rho^N.$$

We remark that the above theorem is true in the unbound phase as well (i.e., for all $\alpha > -\theta$).

We now introduce the *interacting random walks* which are a specialized version of $\mathcal{HSLG}$ Gibbs measures (see Figure 8.7).

Definition 8.2.5 (Interacting random walk)**.** We say $(L_1, L_2) = (L_1[\![1, 2T-2]\!], L_2[\![1, 2T-1]\!])$ is an interacting random walk (IRW) of length T with boundary condition (a, b) if its law is a measure on $\mathbb{R}^{4T-3}$ with density at $(u_{1,j})_{j=1}^{2T-2}, (u_{2,j})_{j=1}^{2T-1}$ proportional to

$$\prod_{j=1}^{T-1} \exp(-e^{u_{2,2j}-u_{1,2j-1}} - e^{u_{2,2j}-u_{1,2j+1}}) \prod_{i=1}^{2} \prod_{j=1}^{2T-1} G_{\theta+(-1)^{i+j}\alpha}\big((-1)^{j+1}(u_{i,j} - u_{i,j+1})\big)$$

where $G_\beta(x) = [\Gamma(\beta)]^{-1} \exp(\beta x - e^x)$, $u_{1,2T-1} = a$, $u_{2,2T} = b$, and $u_{1,2T} = 0$ (which forces $G_{\theta+\alpha}(u_{1,2T-1} - u_{1,2T})$ to be a constant). Figure 8.7 provides the graphical representation of IRW.

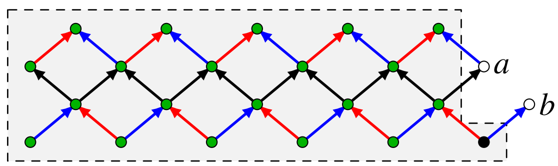


Figure 8.7: IRW of length 6 with boundary condition a and b.

Note that the directed-colored graph associated to IRW can be viewed as a subset of G_N (introduced above). Specifically, for each $i \geq 1$, if we consider the vertex set

$$V_{i,T} := \{(2i, j), (2i+1, j) \mid j \in [\![1, 2T-1]\!]\} \cup \{(2i+1, 2T)\},$$

the subgraph induced by $V_{i,T}, E(V_{i,T})$ corresponds to the graph associated to IRW. Note that the graph associated to IRW can also be viewed as the subgraph induced by $\widehat{V}_T, E(\widehat{V}_T)$ where

$$\widehat{V}_T := \{(1, j), (2, j) \mid j \in [\![1, 2T-1]\!]\} \cup \{(2, 2T)\},$$

provided we switch α to $-\alpha$ in (8.2.5) (i.e., switching red and blue edges). Since we have restricted $\alpha \in (-\theta, 0)$ (bound phase), under this switching IRW can be viewed as certain $\mathcal{HSLG}$ Gibbs measures in the unbound phase. Indeed, after switching α to $-\alpha$, in the language of [28], IRW precisely corresponds to bottom-free measure on the domain $\mathcal{K}_{2,T}$ with boundary condition (a, b) (see Definition 2.3 in [28]). This allows us to use the unbound phase estimates developed in [28].

We end this section by recording one such estimate.

Proposition 8.2.6 (Lemma 5.3 in [28]). *Fix any $T \geq 2$. Let (L_1, L_2) be a* IRW *of length T with boundary condition $(0, -\sqrt{T})$. Fix $\varepsilon \in (0,1)$. There exists $M_0 = M_0(\varepsilon) > 0$ such that*

$$\mathbf{P}\left(\sup_{p\in[\![1,2T-1]\!]} |L_1(p)| + \sup_{q\in[\![1,2T]\!]} |L_2(q)| \geq M_0\sqrt{T}\right) \leq \varepsilon.$$

8.2.2 One-point fluctuations of point-to-(partial)line free energy

In this section, we gather the point-to-(partial)line free energy fluctuation results from [34]. To state the theorem, we introduce a few necessary objects first.

Recall the point-to-point half-space partition function $Z(m,n)$ from (8.1.4). For $k \in [\![0, N-1]\!]$, we define the point-to-(partial)line half-space partition function as

$$\mathcal{Z}_N^{\mathsf{PL}}(m) = \sum_{p=m}^{N-1} Z(N+p, N-p) = \sum_{p=m}^{N-1} e^{H_N^{(1)}(2p+1)}. \tag{8.2.7}$$

For the second equality, note that by (8.2.4) we have $H_N^{(1)}(2p+1) = \log Z(N+p, N-p)$ and thus we can translate the point-to-(partial)line partition function in Definition 1.8 (or equivalently in Definition 1.3) of [34] into sums of $e^{H_N^{(1)}(2p+1)}$ by way of the full-space point-to-point partition function $Z(n+p, n-p)$.

Let $\Psi(\cdot)$ denote the digamma function defined on $\mathbb{R}_{>0}$ by

$$\Psi(z) = \partial_z \log\Gamma(z) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+z}\right), \tag{8.2.8}$$

where γ is the Euler-Mascheroni constant. For any $k \in \mathbb{Z}_{>0}$, we set

$$\begin{aligned} R(\theta, \alpha) &:= -\Psi(\theta+\alpha) - \Psi(\theta-\alpha), \\ \tau(\theta, \alpha) &:= \Psi(\theta-\alpha) - \Psi(\theta+\alpha), \\ \sigma^2(\theta, \alpha) &:= \Psi'(\theta+\alpha) - \Psi'(\theta-\alpha), \\ \Delta_k(\theta, \alpha) &:= \Psi(\theta) - \tfrac{1}{2}[\Psi(\theta+\alpha) + \Psi(\theta-\alpha)] - \tfrac{1}{2k}\log 2. \end{aligned} \tag{8.2.9}$$

For the remainder of the paper, we will make use of the above notation repeatedly. As Ψ is a strictly concave function, for all large enough k (depending on α, θ) we have $\Delta_k > 0$. For the results and proofs in the remainder of the paper, we always choose k large enough such that $\Delta_k > 0$.

We now state the necessary results from [34] about the point-to-(partial)line partition function $\mathcal{Z}_N^{\mathsf{PL}}(m)$ that we need in our subsequent analysis.

Theorem 8.2.7. *Suppose $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. Suppose further that N is an integer that tends to infinity in such a way that $\frac{g(N)}{N} \to 0$. We have*

$$\frac{1}{N^{1/2}\sigma}\left[\log \mathcal{Z}_N^{\mathsf{PL}}(g(N)) - RN + g(N)\tau\right] \xrightarrow{d} \mathcal{N}(0,1). \tag{8.2.10}$$

where R, τ, σ are defined in (8.2.9). *We have the following law of large numbers*

$$\frac{1}{N}\log\left[\sum_{p=1}^{N-1} Z(N+p, N-p)\right] \xrightarrow{p} R \quad \frac{1}{N}\log\left[\sum_{p=1}^{N} Z(N+p, N-p+1)\right] \xrightarrow{p} R. \tag{8.2.11}$$

Furthermore, the above law of large numbers continues to hold when $\alpha = 0$, i.e., the diagonal weights are assumed to be distributed as $\mathrm{Gamma}^{-1}(\theta)$. *In that case $R(\theta, \alpha)$ is interpreted as $R(\theta, 0) = -2\Psi(\theta)$.*

Proof. Theorem 1.10 in [34] discusses several fluctuation results for point-to-(partial)line partition function for the $\mathcal{HSLG}$ polymer, of which Theorem 1.10(3) applies to the bound phase in this paper.

Letting $n = N - g(N)$ and $m = N + g(N)$ in (1.12) of [34] yields

$$\frac{1}{(N-g(N))^{1/2}\sigma_p}\left[\log \mathcal{Z}_N^{\mathsf{PL}}(g(N)) + (N-g(N))\mu_p\right] \xrightarrow{d} \mathcal{N}(0,1).$$

where $\mu_p := \Psi(\theta+\alpha) + p\Psi(\theta-\alpha)$ and $\sigma_p^2 := \Psi'(\theta+\alpha) - p\Psi'(\theta-\alpha)$ with $p = \frac{N+g(N)}{N-g(N)}$. Observe that $(N-g(N))\mu_p = -RN + g(N)\tau$. As $g(N)/N \to 0$, we have that

$$\frac{(N-g(N))^{1/2}\sigma_p}{N^{1/2}\sigma} \to 1.$$

Therefore the above fluctuation result implies (8.2.10). For the law of large numbers, the first one in (8.2.11) follows by taking $g(N) \equiv 1$ and appealing to (8.2.10). The second law of large numbers also follows from Theorem 1.10(3) in [34] as their result also gives fluctuation results for point-to-(partial)line free energy of that form with the same law of large numbers. Finally, the last point of Theorem 8.2.7 follows by from Theorem 1.10(2) in [34] which deals with the $\alpha = 0$ case. □

8.3 Controlling the average law of large numbers of the top curves

In this section, we control the average law of large numbers of the top $2k$ curves for large enough k (Proposition 8.3.4). As explained in the introduction, the key idea behind this proposition is to show that the contribution of diagonal weights in the $2k$ many non-intersecting paths of $Z_{\mathrm{sym}}^{(2k)}(m,n)$ (defined in (8.2.1)) essentially comes from k many paths. The starting point of this idea is Lemma 8.3.1. Given a pair of non-intersecting paths (π_1, π_2) starting and ending at adjacent locations with the same x-coordinate, Lemma 8.3.1 constructs two new non-intersecting paths (π_1', π_2') from (π_1, π_2) such that the new paths collectively carry the same weight variables but the diagonal weights only rest on the lower path. This combinatorial result proceeds to help us decompose the symmetrized multilayer partition function $Z_{\mathrm{sym}}^{(2k)}(m,n)$ into pairs of single-layer ones in Lemmas 8.3.2 and 8.3.3 before culminating into the final result in Proposition 8.3.4.

Let $\Pi(v_1 \to v_2, u_1 \to u_2)$ denote the set of pairs of non-intersecting upright paths in $\mathbb{Z}_{>0}^2$

starting from u_1, v_1 and ending at u_2, v_2 respectively. Recall that $\mathcal{I}^- = \left\{(i,j) \in \mathbb{Z}_{>0}^2 | j \le i\right\}$. Define $\mathcal{I}^+ := \left\{(i,j) \in \mathbb{Z}_{>0}^2 | j \ge i\right\}$ which represents the half-space index set that includes points on and above the diagonal. The first lemma constructs the U map.

Lemma 8.3.1 (Construction of U map)**.** *Fix $x \in \mathbb{Z}_{>0}$ and any $(m,n) \in \mathcal{I}^-$ with $n \ge 2$. Then there exists a map $U : \Pi_1 \to \Pi_2$ where*

$$\Pi_1 := \Pi((1, x+1) \to (m,n), (1,x) \to (m, n-1))$$
$$\Pi_2 := \Pi((1, x+1) \to (n-1, m), (1,x) \to (n,m)),$$

such that the following properties hold (let $(\pi_1', \pi_2') := U(\pi_1, \pi_2)$):

(a) π_1' has no diagonal points, i.e., $\{(i,i) \in \mathbb{Z}_{>0}^2\} \cap \pi_1'$ is empty and

$$\{(i,i) \in \mathbb{Z}_{>0}^2\} \cap \pi_2' = \{(i,i) \in \mathbb{Z}_{>0}^2\} \cap \{\pi_1 \cup \pi_2\}.$$

(b) Recall the symmetrized weights $(\widetilde{W}_{i,j})_{(i,j) \in \mathbb{Z}_{>0}^2}$ from (8.1.9)*. We have*

$$\prod_{(i,j) \in \pi_1 \cup \pi_2} \widetilde{W}_{i,j} \stackrel{a.s.}{=} \prod_{(i,j) \in \pi_1' \cup \pi_2'} \widetilde{W}_{i,j}.$$

(c) For each $(\pi_1', \pi_2') \in \Pi_2$ we have

$$\left|U^{-1}(\{(\pi_1', \pi_2')\})\right| \le 2^{|\{(i,i) \in \pi_1 \cup \pi_2\}|}.$$

We remark that Lemma 8.3.1 is entirely combinatorial and does not use any results about the integrability of the model. Lemma 8.3.1 continues to hold for any collection of symmetrized weights that are not necessarily distributed as inverse-Gamma random variables.

Proof. We define a partial order $<$ on the points $\mathbb{Z}_{>0}^2$ by requiring $P_1 = (a_1, b_1) < P_2 = (a_2, b_2)$ whenever $a_1 + b_1 < a_2 + b_2$. Let π_1 denote the path from $(1, x+1)$ to (m,n) and π_2 the path

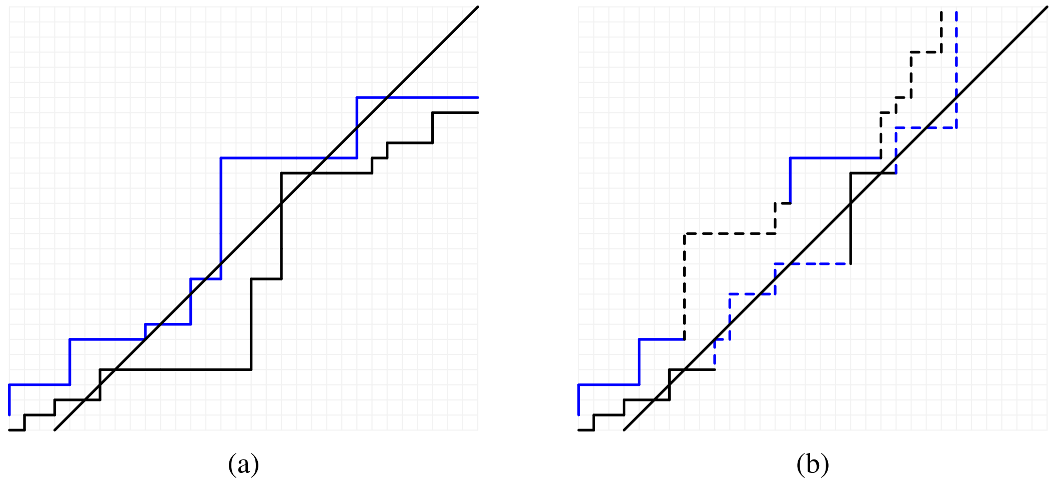


Figure 8.8: The U map takes (A) to (B).

from $(1,x)$ to $(m,n-1)$. We denote $\operatorname{diag}(\pi_i)$ as the set of points on π_i that lie on the diagonal set $D := \{(i,i) \in \mathbb{Z}^2_{>0}\}$. Recall that $\mathcal{I}^+ = \{(i,j) \in \mathbb{Z}^2_{>0} \mid i \leq j\}$ and $\mathcal{I}^- = \{(i,j) \in \mathbb{Z}^2_{>0} \mid j \leq i\}$.

We first define a special collection of points, SPDiag from $\operatorname{diag}(\pi_2)$. Let $D_1 < D_2 < D_3 < \cdots < D_s$ be all the points in $\operatorname{diag}(\pi_1 \cup \pi_2)$ arranged in the increasing order. We put the point $D_j \in \operatorname{diag}(\pi_2)$ in the set SPDiag if $D_{j-1} \in \operatorname{diag}(\pi_1)$ or $D_{j+1} \in \operatorname{diag}(\pi_1)$. In other words, SPDiag consists of the diagonal points in π_2 that bookend contiguous clusters of $\operatorname{diag}(\pi_1)$ in $\operatorname{diag}(\pi_1 \cup \pi_2)$. We enumerate the points in SPDiag as $A_1 < A_2 < \cdots < A_r$. Let B_j be the first point on π_1 that has the same x-coordinate as A_j. Note that by construction, either only π_1 intersects the diagonal or only π_2 intersects the diagonal between A_j and A_{j+1}, $j = 1, \ldots, r$. Let us denote $A_{r+1} := (m, n-1)$ and $B_{r+1} := (m,n)$.

We now construct new paths π_2' and π_1' from π_2 and π_1 by reconstructing each segment between A_j and A_{j+1} (and B_j and B_{j+1} for π_1 respectively), $j = 1, \ldots, r$. We separate the reconstruction procedures for each segment into the following cases: if only π_2 intersects the diagonal and $j \leq r-1$, if only π_1 intersects the diagonal and $j \leq r-1$, or if $j = r$.

1. **When** $1 \leq j \leq r-1$ **and only** π_2 **intersects the diagonal**, we keep the original paths. We set π_1' and π_2' on these segments to be the same as those on π_1 and π_2 respectively.

2. **When** $1 \leq j \leq r-1$ **and only** π_1 **intersects the diagonal between** A_j **and** A_{j+1} **(see Figure**

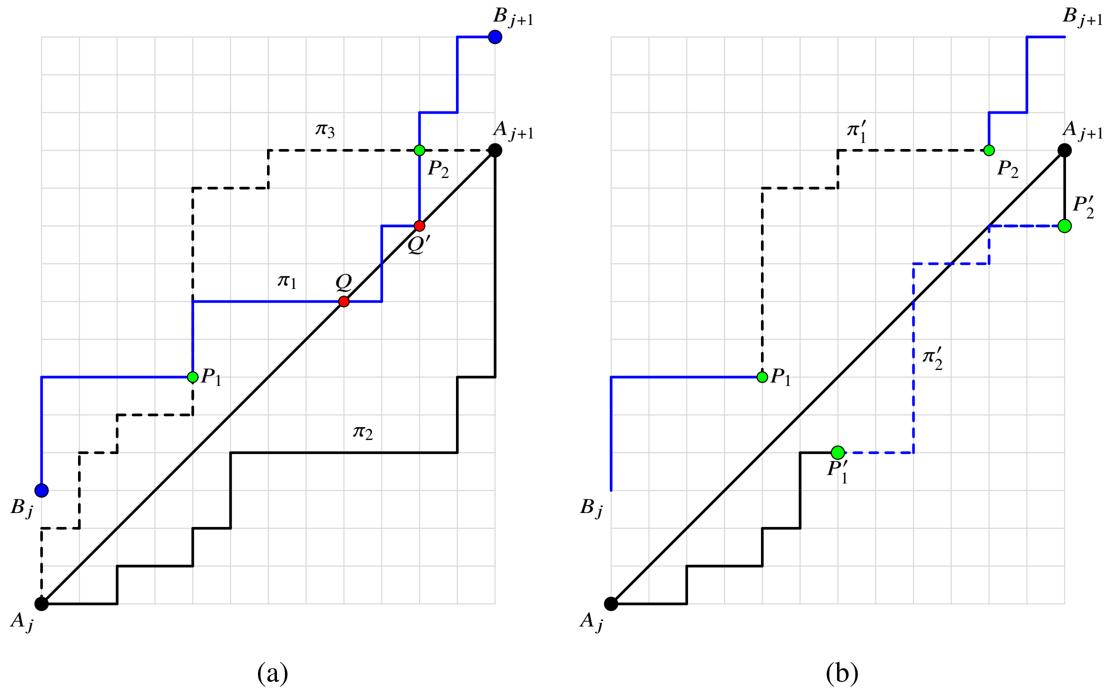


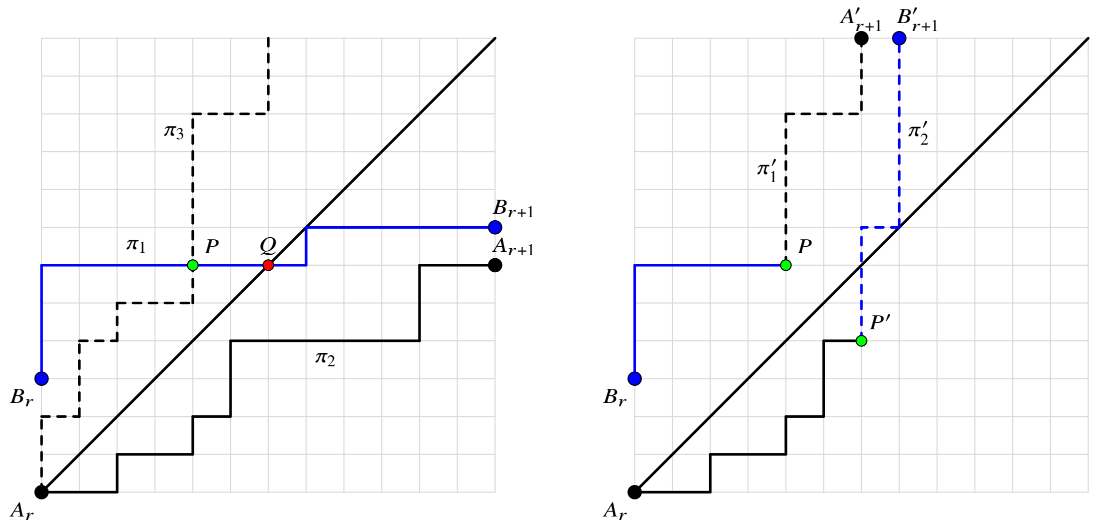
Figure 8.9: The second case when $j \leq r - 1$ and only π_1 intersects with the diagonal. π_1 and π_2 are black and blue paths in Figure (A) respectively. π_3 is the black dashed path in Figure (A). π'_1 is the path in Figure (B) which is formed by the concatenation of solid blue paths and the black dashed path. π'_2 is the path in Figure (B) which is formed by the concatenation of solid black paths and the blue dashed path. The U map takes π_1, π_2 and spits out π'_1, π'_2.

8.9), the portion of the path π_2 from A_j to A_{j+1} (excluding A_j and A_{j+1}) lies in $\mathcal{I}^- \setminus D$. Reflecting the portion of the path π_2 from A_j to A_{j+1} (black path in Figure 8.9) across the diagonal yields a path π_3 (black dashed path in Figure 8.9). Let Q be the first point on diag(π_1) that lies between A_j and A_{j+1} and Q' be the last, which exist by construction of the SPdiag set (Q and Q' may overlap). As the y-coordinate of A_i is strictly smaller than that of B_i and Q, Q' are on the diag(π_1), π_1 and π_3 must intersect on the segments between A_j and Q and Q' and A_{j+1}. Let P_1 be the first point of intersection and P_2 the last point of intersection. Clearly $P_1 \neq P_2$ as the former is between A_j and Q and the latter lies between Q' and A_{j+1}. Replacing the portion of π_1 between P_1 and P_2 with that of π_3 yields a path π'_1 from B_j to B_{j+1}. As the part of π_3 between P_1 and P_2 lies in $\mathcal{I}^+ \setminus D$, π'_1 lies entirely in $\mathcal{I}^+ \setminus D$. We denote the reflections of P_1 and P_2 across the diagonal as P'_1 and P'_2, which must lie on the original π_2 by construction. Similarly replacing the portion of π_2 between P'_1 and P'_2 with the reflection of

π_1 between P_1 and P_2 across the diagonal yields a path π_2' from A_j to A_{j+1}. As π_1 and π_2 are non-intersecting, the reflected paths are also non-intersecting. Thus the new paths π_1' and π_2' are non-intersecting.

3. **When** $j = r$**,** consider the portion of the path π_2 from A_r to A_{r+1} (see Figure 8.10). Note that in this segment, all the diagonal points belong to π_1. Reflecting this portion of π_2 across the diagonal gives us π_3 (black dashed path in Figure 8.10). Let Q be the first point on $\mathrm{diag}(\pi_1)$ that lies between A_j and A_{j+1} and Q exists as π_1 ends at $B_{r+1} := (m, n) \in \mathcal{I}^-$. Note that π_3 lies entirely in $\mathcal{I}^+ \setminus D$, excluding A_r. Thus π_1 and π_3 necessarily intersect in $\mathcal{I}^+ \setminus D$. Again, we locate the first point intersection P and replace the portion of π_1 from P to B_{r+1} with the portion of π_3 from P to $A'_{r+1} := (n-1, m)$. Similarly, reflecting the portion of π_1 from P and B_{r+1} across the diagonal and replacing the portions of π_2 between P' and A_{r+1} with the portion of reflection between P and $B'_{r+1} := (n, m)$ yields a path π_2' from A_r to B'_{r+1}. Clearly, the new path π_1' lies in $\mathcal{I}^+ \setminus D$ and the paths π_1' and π_2' are non-intersecting as the reflected portions are non-intersecting.

As A_j and B_j remain unchanged, connecting all the segments between A_j's (and B_j's respectively) for $j \le r$ and A_r and B'_{r+1} (and B_r and A'_{r+1}) yields the new path π_2' from $(1, x)$ to (n, m) and the new path π_1' from $(1, x+1)$ to $(n-1, m)$ (see Figure 8.8). At each step of the above construction, the paths remain non-intersecting. Thus (π_1', π_2') form a non-intersecting pair. We call this explicitly constructed map U. By construction, π_1' lies entirely in $\mathcal{I}^+ \setminus D$ and has no diagonal points. This proves (a). Since the construction involves only exchanges of reflected portions, due to the symmetry of the weights $\widetilde{W}_{i,j}$ across the diagonal, we have (b). Finally to verify (c), note that there are at most $2^{\mathrm{diag}(\pi_1' \cup \pi_2')}$ possible choices of $\mathrm{diag}(\pi_1)$ and $\mathrm{diag}(\pi_2)$ for a given pair of two paths (π_1', π_2') in the pre-image of U. As $\mathrm{diag}(\pi_1)$ and $\mathrm{diag}(\pi_2)$ uniquely determine SPDiag where reflections are performed, reverting the same operations on π_1' and π_2' between consecutive points in SPDiag leads to original π_1 and π_2. Thus the map has at most $2^{\mathrm{diag}(\pi_1' \cup \pi_2')}$ inverses for (π_1', π_2'), which completes the proof. $\square$



(a) (b)

Figure 8.10: The $j = r$ case. π_1 and π_2 are black and blue paths in Figure (A) respectively. π_3 is the black dashed path in Figure (A). π'_1 is the path in Figure (B) which is formed by the concatenation of the solid blue path and the black dashed path. π'_2 is the path in Figure (B) which is formed by the concatenation of the solid black path and the blue dashed path. The U map takes π_1, π_2 and spits out π'_1, π'_2.

Note that the U map in Lemma 8.3.1 gives us a path that does not contain any diagonal vertex. To capture the contribution of this path, we now introduce *diagonal-avoiding symmetrized* partition function. Let $\widetilde{\Pi}^{(1)}_{m,n}$ be the collection of all upright paths from $(1,1)$ to (m,n) that do not touch the diagonal after $(1,1)$. Set

$$\widetilde{Z}_{\rm sym}(m,n) := \sum_{\pi\in\widetilde{\Pi}^{(1)}_{m,n}} \prod_{(i,j)\in\pi} \widetilde{W}_{i,j}, \quad \widetilde{V}_q := \sum_{(i,j)|i+j=q} \widetilde{Z}_{\rm sym}(i,j) \tag{8.3.1}$$

where $\widetilde{W}_{i,j}$ is defined in (8.1.9). We call $\widetilde{Z}_{\rm sym}(m,n)$ the *diagonal-avoiding symmetrized* partition function. Let us recall $Z_{\rm sym}(m,n)$ from (8.2.1) and we similarly define

$$V_q := \sum_{(i,j)|i+j=q} Z_{\rm sym}(i,j) \tag{8.3.2}$$

The next lemma establishes a relation between $Z^{(2k)}_{\rm sym}(m,n)$, V_{m+n} and $\widetilde{V}_{m+n}$.

Lemma 8.3.2. *For all $(m,n) \in \mathcal{I}^-$, almost surely we have*

$$Z_{\mathrm{sym}}^{(2k)}(m,n) \le 2^n \cdot \prod_{i=2}^{2k} \prod_{j=1}^{i-1} (\widetilde{W}_{1,j})^{-1} \cdot \prod_{i=1}^{k} \left[V_{m+n+2-2i} \widetilde{V}_{m+n+1-2i} \right] \tag{8.3.3}$$

where $Z_{\mathrm{sym}}^{(i)}(m,n)$, $V_{m+n+2-2i}$ and $\widetilde{V}_{m+n+1-2i}$ are defined in (8.2.1), (8.3.2) *and* (8.3.1) *respectively.*

Proof. We extend our definition of U map from Lemma 8.3.1 to the domain $\Pi_{m,n}^{(2k)}$ by defining $U(\pi_1, \dots, \pi_{2k}) := (U(\pi_1, \pi_2), \dots, U(\pi_{2k-1}, \pi_{2k}))$. Let $R_{m,n}^{i,k}$ be the collection of all upright paths from $(1, 2k-2i+1)$ to $(n-2i+2, m)$. Let $\widetilde{R}_{m,n}^{i,k}$ be the collection of all upright paths from $(1, 2k-2i+2)$ to $(n-2i+1, m)$ that avoid the diagonal. Given any $(\pi_1', \dots, \pi_{2k}') \in U(\Pi_{m,n}^{(2k)})$, by (c), there are at most

$$\prod_{i=1}^{k} 2^{|(j,j) \in \pi_{2i-1}' \cup \pi_{2i}'|}$$

many inverses in the pre-image of the U map. The U map preserves the number of diagonal vertices by (a). Furthermore by non-intersection, a $2k$-tuple of paths in $\Pi_{m,n}^{(2k)}$ has at most n many diagonal vertices. Thus there are at most 2^n many inverses. Hence by (a) we have

$$Z_{\mathrm{sym}}^{(2k)}(m,n) \le 2^n \cdot \prod_{i=1}^{k} \left[\sum_{\pi_1 \in \widetilde{R}_{m,n}^{i,k}} \prod_{(i,j) \in \pi_1} \widetilde{W}_{i,j} \right] \cdot \prod_{i=1}^{k} \left[\sum_{\pi_2 \in R_{m,n}^{i,k}} \prod_{(i,j) \in \pi_2} \widetilde{W}_{i,j} \right]. \tag{8.3.4}$$

We may elongate each of the path in $R_{m,n}^{i,k}$ and $\widetilde{R}_{m,n}^{i,k}$ by appending an up-path from $(1,1)$ to $(1, 2k-2i+2)$ and from $(1,1)$ to $(1, 2k-2i+1)$ respectively. This produces elongated paths in $\widetilde{\Pi}_{n-2i+2,m}^{(1)}$ and $\Pi_{n-2i+1,m}$ respectively. In terms of weights, we need to multiply the existing weights in (8.3.4) by $\prod_{j=1}^{2k-2i+1} \widetilde{W}_{1,j}$ and $\prod_{j=1}^{2k-2i} \widetilde{W}_{1,j}$ respectively to get the corresponding weights of elongated paths. After doing precisely the above, we have

$$Z_{\mathrm{sym}}^{(2k)}(m,n) \le 2^n \cdot \prod_{i=2}^{2k} \prod_{j=1}^{i-1} (\widetilde{W}_{1,j})^{-1} \cdot \prod_{i=1}^{k} \left[Z_{\mathrm{sym}}(n-2i+2, m) \widetilde{Z}_{\mathrm{sym}}(n-2i+1, m) \right]. \tag{8.3.5}$$

We get (8.3.3) from the above inequality in (8.3.5) by observing the definition of V_q and $\widetilde{V}_q$ from (8.3.2). This completes the proof. $\square$

The next lemma bounds $\log V_q$ and $\log \widetilde{V}_q$ from above with high probability.

Lemma 8.3.3. *Recall R from* (8.2.9)*. For every $\delta > 0$ and $1 \le p < n$, we have*

$$\lim_{q\to\infty} \mathbf{P}\left(\log V_q \le (R+\delta)\tfrac{q}{2}\right) = 1, \quad \lim_{q\to\infty} \mathbf{P}\left(\log \widetilde{V}_q \le (-2\Psi(\theta)+\delta)\tfrac{q}{2}\right) = 1. \tag{8.3.6}$$

Proof. Fix any $\delta > 0$. By Lemma 8.2.2 we have

$$V_{2N} = \sum_{p=0}^{N-1} Z(N+p, N-p), \quad V_{2N+1} = \sum_{p=1}^{N} Z(N+p, N-p+1).$$

From Theorem 8.2.7 ((8.2.11) in particular) we have that

$$\frac{1}{N}\log\left[\sum_{p=1}^{N} Z(N+p, N-p+1)\right] \xrightarrow{p} R, \quad \frac{1}{N}\log\left[\sum_{p=1}^{N-1} Z(N+p, N-p)\right] \xrightarrow{p} R, \tag{8.3.7}$$

Note that in the above equation, we have excluded $Z(N,N)$ as their result does not contain $Z(N,N)$ in the sum. However, in our case, we may include $Z(N,N)$ by appealing to Theorem 8.2.4. First, in view of the above law of large numbers in (8.3.7), we have

$$\mathbf{P}(\log V_{2N+1} \le (R+\tfrac{1}{2}\delta)N) \to 1. \tag{8.3.8}$$

On the other hand, by (8.2.4) we have $\sum_{p=1}^{N} e^{H_N^{(1)}(2p)} = \sum_{p=1}^{N} Z(N+p, N-p+1)$. Since $H_N^{(1)}(2) \le \log\sum_{p=1}^{N} e^{H_N^{(1)}(2p)} = \log V_{2N+1}$, (8.3.8) implies

$$\mathbf{P}(H_N^{(1)}(2) \le (R+\tfrac{1}{2}\delta)N) \to 1,$$

as $N \to \infty$. In addition, by ordering of points in the line ensemble (Theorem 8.2.4) we know that with probability at least $1 - 2^{-N}$, $H_N^{(1)}(1) \le H_N^{(2)}(2) + \log^2 N$. Thus we have

$$\mathbf{P}(H_N^{(1)}(1) \le (R+\delta)N) \to 1, \tag{8.3.9}$$

as $N \to \infty$. Given that $H_N^{(1)}(1) = Z(N,N)$, combining (8.3.9) and the second convergence in (8.3.7) yields $\mathbf{P}(\log V_{2N} \le (R+\delta)N) \to 1$ and together with (8.3.8) this concludes the proof of the first convergence in (8.3.6).

Next, for the diagonal-avoiding case, let $(W_{i,i}^{\alpha=0})_{i\ge 1}$ be a family of weights distributed as Gamma(θ) independent of $(W_{i,j})$. We set $W_{i,j}^{\alpha=0} := \widetilde{W}_{i,j}$ for $i \neq j$. This gives us a new collection of symmetrized weights. We denote the corresponding symmetrized partition function and the diagonal-avoiding symmetrized partition function as $Z_{\mathrm{sym}}^{\alpha=0}$ and $\widetilde{Z}_{\mathrm{sym}}^{\alpha=0}$ respectively. Observe that

$$\widetilde{Z}_{\mathrm{sym}}(i,j) \le \frac{\widetilde{W}_{1,1}}{W_{1,1}^{\alpha=0}} \cdot \widetilde{Z}_{\mathrm{sym}}^{\alpha=0}(i,j) \le \frac{\widetilde{W}_{1,1}}{W_{1,1}^{\alpha=0}} \cdot Z_{\mathrm{sym}}^{\alpha=0}(i,j). \tag{8.3.10}$$

The first equality in (8.3.10) is due to the fact that the weight corresponding $(1,1)$ is common in all paths and that is the only diagonal weight that appears in the diagonal avoiding symmetrized partition functions. The next inequality is obvious as we have just removed the diagonal avoiding restriction. This leads to

$$\log \widetilde{V}_q \le \log \widetilde{W}_{1,1} - \log W_{1,1}^{\alpha=0} + \log \left[\sum_{(i,j)|i+j=q} Z_{\mathrm{sym}}^{\alpha=0}(i,j) \right].$$

The first two terms on the right-hand side of the above display are tight. An upper bound on the third term can be computed by the exact same analysis as V_q. Indeed, the law of large numbers and Theorem 8.2.4 continue to hold for $\alpha = 0$ when R becomes $-2\Psi(\theta)$ (see the last point in Theorem 8.2.7). This concludes the proof of (8.3.6). □

Finally, with Lemmas 8.3.2 and 8.3.3 in place, we are ready to control the average law of large numbers of the top curves of the $\mathcal{HSLG}$ line ensemble.

Proposition 8.3.4. *Recall Δ_k, R in (8.2.9). Fix any $\varepsilon > 0$ and $k \in \mathbb{Z}_{>0}$ large such that $\Delta_k > 0$. Then there exists $N_0(k,\varepsilon) > 2k+1$ such that for all $N \geq N_0$ we have*

$$\mathbf{P}\left(\sup_{p\in[\![1,2N-4k+2]\!]} \frac{1}{2k}\sum_{i=1}^{2k} H_N^{(i)}(p) \leq (R - \tfrac{1}{2}\Delta_k)N\right) \geq 1-\varepsilon.$$

In plain words, Proposition 8.3.4 claims that when k is taken large enough so that $\Delta_k > 0$, the average law of large numbers of top $2k$ curves is strictly less than R, which is the law of large numbers for point-to-(partial)line free energy process (see Theorem 8.2.7).

Proof. Fix any $\varepsilon > 0$. The definition of the $\mathcal{HSLG}$ line ensemble in (8.2.3) and (8.3.3) collectively yield that, for all $p \in [\![1, N-2k+1]\!]$,

$$\begin{aligned}\sum_{i=1}^{2k} H_N^{(i)}(2p) &= 2k\log 2 + \log Z_{\mathrm{sym}}^{(2k)}(N+p, N-p+1) \\ &\leq 2k\log 2 + N\log 2 - \log\left[\prod_{i=2}^{2k}\prod_{j=1}^{i-1}\widetilde{W}_{1,j}\right] + \log\prod_{i=1}^{k}\left[V_{2N+3-2i}\widetilde{V}_{2N+2-2i}\right],\end{aligned}$$

where the r.h.s. is free of p. Hence we may take supremum over $p \in [\![1, N-2k+1]\!]$ over both sides of the above display to get

$$\begin{aligned}\sup_{p\in[\![1,N-2k+1]\!]}\sum_{i=1}^{2k} H_N^{(i)}(2p) \leq (2k+N)\log 2 - \log\left[\prod_{i=2}^{2k}\prod_{j=1}^{i-1}\widetilde{W}_{1,j}\right] \\ + \log\prod_{i=1}^{k}\left[V_{2N+3-2i}\widetilde{V}_{2N+2-2i}\right].\end{aligned} \tag{8.3.11}$$

We now provide high probability upper bounds for each of the terms on the r.h.s. of (8.3.11). Let us take $\delta := \frac{\Delta_k}{4}$. By Lemma 8.3.3, we may choose $N_0(k,\varepsilon) > 2k+1$ large enough such that for all $N \geq N_0$

$$\mathbf{P}(\log V_N \leq (R+\delta)\tfrac{N}{2}) \geq 1 - \tfrac{\varepsilon}{8k}, \quad \mathbf{P}(\log \widetilde{V}_N \leq (-2\Psi(\theta)+\delta)\tfrac{N}{2}) \geq 1 - \tfrac{\varepsilon}{8k}.$$

Thus applying a union bound we see that for all large enough N, with probability $1 - \frac{\varepsilon}{4}$,

$$\log \prod_{i=1}^{k} \left[V_{N+3-2i} \widetilde{V}_{N+2-2i} \right] \leq RkN - 2\Psi(\theta)kN + 2k\delta N. \tag{8.3.12}$$

Note that the random variable $\log \left[\prod_{i=2}^{2k} \prod_{j=1}^{i-1} \widetilde{W}_{1,j} \right]$ is tight and free of N. Hence with probability $1 - \frac{\varepsilon}{4}$ one can ensure that

$$(2k + N) \log 2 - \log \left[\prod_{i=2}^{2k} \prod_{j=1}^{i-1} \widetilde{W}_{1,j} \right] \leq N \log 2 + 2k\delta N. \tag{8.3.13}$$

holds for all large enough N. Inserting the above two bounds in (8.3.12) and (8.3.13) back in (8.3.11), we have that with probability at least $1 - \frac{\varepsilon}{2}$,

$$\sup_{p \in [\![1, N-2k+1]\!]} \frac{1}{2k} \sum_{i=1}^{2k} H_N^{(i)}(2p) \leq \left[\tfrac{1}{2k} \log 2 + 2\delta + \tfrac{R}{2} - \Psi(\theta) \right] N, \tag{8.3.14}$$

for all large enough N. As $\delta = \frac{\Delta_k}{4}$, the r.h.s. of (8.3.14) is precisely $(R - \frac{1}{2}\Delta_k)N$. By the exact same argument, one can check that with probability at least $1 - \frac{\varepsilon}{2}$ we have

$$\sup_{p \in [\![0, N-2k]\!]} \frac{1}{2k} \sum_{i=1}^{2k} H_N^{(i)}(2p + 1) \leq (R - \tfrac{1}{2}\Delta_k)N, \tag{8.3.15}$$

for all large enough N. Taking another union bound of (8.3.14) and (8.3.15), we get the desired result. $\square$

8.4 Controlling the second curve

In this section, we establish the separation between the first and the second curve of our $\mathcal{HSLG}$ line ensemble. Appealing to Proposition 8.3.4, Lemma 8.4.1 first establishes that for large enough k with high probability the $(2k + 2)$-th curve $H_N^{(2k+2)}(\cdot)$ is uniformly const $\cdot\, N$ below than RN over an interval of $[\![1, N]\!]$, where R defined in (8.2.9) is the law of large numbers for point-to-

(partial)line free energy process (Theorem 8.2.7). This helps us show that with high probability the second curve $H_N^{(2)}(\cdot)$ over an interval of length $O(\sqrt{N})$ is $M\sqrt{N}$ below RN next in Proposition 8.4.2 for any $M > 0$.

Lemma 8.4.1. *Recall R in* (8.2.9). *Fix any $\varepsilon > 0$ and $k \in \mathbb{Z}_{>0}$ large enough such that $\Delta_k > 0$. Then there exists $N_0(k, \varepsilon)$ such that for all $N \geq N_0$ we have*

$$\mathbf{P}\left(\sup_{p \in [\![1,N]\!]} H_N^{(2k+2)}(p) \leq (R - \tfrac{1}{4}\Delta_k)N\right) \geq 1 - \varepsilon. \tag{8.4.1}$$

Proof. Let us consider the following events

$$\begin{aligned}
\mathsf{A} &:= \left\{\sup_{p \in [\![1,N]\!]} H_N^{(2k+2)}(p) \leq (R - \tfrac{1}{4}\Delta_k)N\right\}, \\
\mathsf{B} &:= \left\{H_N^{(i+1)}(p) \leq H_N^{(i)}(p) + 2\log^2 N, \text{ for all } i \in [\![1, 2k+1]\!], p \in [\![1, N]\!]\right\}, \\
\mathsf{C} &:= \left\{\sup_{p \in [\![1,N]\!]} \frac{1}{2k}\sum_{i=1}^{2k} H_N^{(i)}(p) \leq (R - \tfrac{1}{2}\Delta_k)N\right\}.
\end{aligned}$$

We claim that for all large enough N, we have $(\mathsf{B} \cap \neg\mathsf{A}) \subset \neg\mathsf{C}$. To see this, note that on $\mathsf{B} \cap \neg\mathsf{A}$, there exists a point $p^* \in [\![1, N]\!]$ such that $H_N^{(2k+2)}(p^*) > (R - \frac{1}{4}\Delta_k)N)$ and hence (as B holds)

$$H_N^{(i)}(p^*) > (R - \tfrac{1}{4}\Delta_k)N - (4k+4)\log^2 N,$$

for all $i \in [\![1, 2k+1]\!]$. However, the above display also implies that

$$\sup_{p \in [\![1,N]\!]} \frac{1}{2k}\sum_{i=1}^{2k} H_N^{(i)}(p) > (R - \tfrac{1}{4}\Delta_k)N - (4k+4)\log^2 N$$

which is strictly bigger than $(R - \frac{1}{2}\tau)N$ and implies $\neg\mathsf{C}$. Thus by a union bound, we have

$$\mathbf{P}(\neg\mathsf{A}) \leq \mathbf{P}(\neg\mathsf{B}) + \mathbf{P}(\mathsf{B} \cap \neg\mathsf{A}) \leq \mathbf{P}(\neg\mathsf{B}) + \mathbf{P}(\neg\mathsf{C}). \tag{8.4.2}$$

Note that for fixed k, by Theorem 8.2.4 with $\rho = \frac{1}{2}$ and a union bound, we have $\mathbf{P}(\neg\mathsf{B}) \leq N \cdot$

$(2k+1)\cdot 2^{-N} \le \frac{\varepsilon}{2}$ for all $N \ge N_1(k,\varepsilon)$. On the other hand, Proposition 8.3.4 yields that for fixed k and ε, $\mathbf{P}(\neg \mathsf{C}) \le \frac{\varepsilon}{2}$ for all N greater than some $N_2(k, \frac{\varepsilon}{2})$. Letting $N_0(k,\varepsilon) = \max\{N_1, N_2\}$ and inserting these two bounds in (8.4.2) leads to (8.4.1). $\square$

Building on Lemma 8.4.1, the next result demonstrates that on a given interval of length $O(\sqrt{N})$ starting from 1 and any $M_2 > 0$, the second curve $H_N^{(2)}(\cdot)$ is uniformly lower than $RN - M_2\sqrt{N}$ with high probability (see Figure 8.11).

Proposition 8.4.2. *Recall Δ_k, R in* (8.2.9)*. Fix $\varepsilon \in (0,1)$, $M_1, M_2 \ge 1$ and $k \in \mathbb{Z}_{>0}$ such that $\Delta_k > 0$. Then there exists a constant $N_2(\varepsilon, M_1, M_2) > 0$ such that for all $N \ge N_2$ we have*

$$\mathbf{P}\left(\sup_{p\in[1,2\lfloor M_1\sqrt{N}\rfloor+1]} H_N^{(2)}(p) \le RN - M_2\sqrt{N}\right) \ge 1 - \tfrac{1}{2}\epsilon. \tag{8.4.3}$$

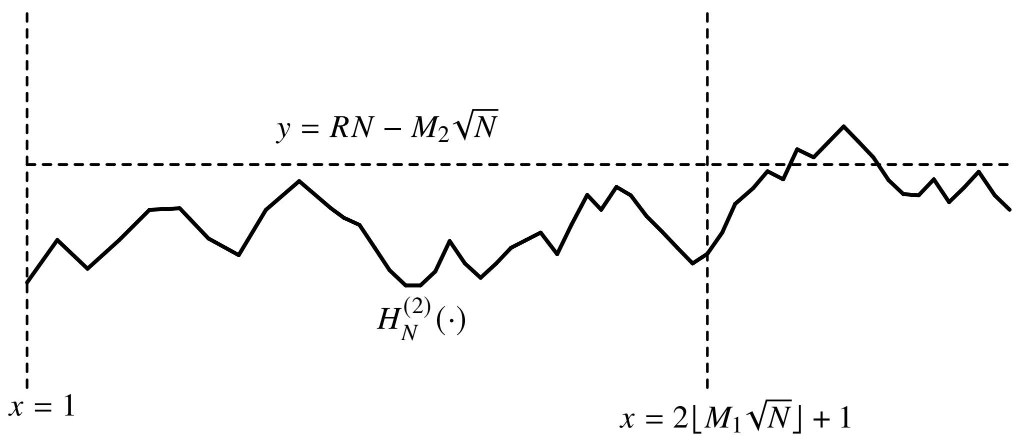


Figure 8.11: The high probability event in Proposition 8.4.2.

Proof. The proof of Proposition 8.4.2 is conducted in the following stages:

- Using Theorem 8.2.7 and Lemma 8.4.1, we determine high probability locations of $H_N^{(1)}(2M\sqrt{N}+1)$ and $H_N^{(2k+2)}(\cdot)$. Using the ordering of points in Theorem 8.2.4, we then bound the end-points $H_N^{(i)}(2M\sqrt{N}+1)$, $i \in [\![1, 2k+1]\!]$ from above based on the high probability locations of $H_N^{(1)}(2M\sqrt{N}+1)$ and the $(2k+2)$-th curve.

- We next consider the conditional law of $(H_N^{(i)}\llbracket 1, 2M\sqrt{N}\rrbracket)_{i\in\llbracket 1,2k+1\rrbracket}$ given the above boundary conditions. By Theorem 8.2.2, this law is given by an appropriate $\mathcal{HSLG}$ Gibbs measure. Applying stochastic monotonicity, we may also assume that the $H_N^{(2i-1)}(2M\sqrt{N}+1)$ and $H_N^{(2i)}(2M\sqrt{N}+1)$ are sufficiently far apart. This will allow us to approximate the Gibbs measure as a product of interacting random walks defined in Definition 8.2.5.

- Lastly, we use the associated estimates of interacting random walks from Proposition 8.2.6 to dissect the Gibbs measure and yield a quantitative bound in our favor.

Let us begin by fleshing out the technical details of the above stages. In the following proof, we assume all the multiples of $\sqrt{N}$ appearing below are integers for convenience in notation. The general case follows verbatim by considering the floor function. For clarity, we split our proof into several steps.

Step 1. In this step, we reduce our proof of (8.4.3) to (8.4.7). Let us consider the $\mathcal{HSLG}$ line ensemble $H_N = (H_N^{(1)}, \dots, H_N^{(N)})$. Fix any $\varepsilon \in (0,1)$, $M_1, M_2 \geq 1$ and any $k \in \mathbb{Z}_{>0}$ such that $\Delta_k > 0$. Let $\Phi(x)$ be the cumulative distribution function of a standard Gaussian random variable. Set $\tau := |\Psi(\theta-\alpha) - \Psi(\theta+\alpha)|$. Let $M \in \mathbb{Z}_{>0}$ whose precise value is to be determined. Taking $g(N) = M\sqrt{N}$ in Theorem 8.2.7 yields

$$\frac{1}{\sigma\sqrt{N}}\left[\log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) - RN + M\tau\sqrt{N}\right] \xrightarrow{d} \mathcal{N}\big(0,1\big). \tag{8.4.4}$$

Note that (8.4.4) implies

$$\mathbf{P}\left(\frac{1}{\sigma\sqrt{N}}\left[\log\left[\mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N})\right] - RN + M\tau\sqrt{N}\right] \leq \Phi(1-\frac{\varepsilon}{2})\right) \to 1 - \frac{\varepsilon}{2}.$$

Thus for N large enough, we have that with probability greater than $1-\varepsilon$,

$$\log\left[\mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N})\right] \leq RN - \big(M\tau - \Phi(1-\tfrac{\varepsilon}{2})\sigma\big)\sqrt{N}. \tag{8.4.5}$$

Set $M \geq \max\{M_1, \frac{1}{\tau}(M_2 + k + 1 + \Phi(1 - \frac{\varepsilon}{2})\sigma)\}$. Note that by definition, $H_N^{(1)}(2M\sqrt{N} + 1) \leq \log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N})$ and as $M\tau - \Phi(1 - \frac{\varepsilon}{2})\sigma > M_2 + k + 1$, (8.4.5) yields that

$$\mathbf{P}(\mathsf{A}) \geq 1 - \varepsilon, \text{ where } \mathsf{A} := \left\{ H_N^{(1)}(2M\sqrt{N} + 1) \leq RN - (M_2 + k + 1)\sqrt{N} \right\} \tag{8.4.6}$$

for all large enough N. Set $T = M\sqrt{N} + 1$. We claim that

$$\mathbf{P}(\neg\mathsf{E}) \leq 3\varepsilon + \frac{k\varepsilon}{(1 - \varepsilon)^{k+1}}, \text{ where } \mathsf{E} := \left\{ \sup_{p \in [\![1, 2T-1]\!]} H_N^{(2)}(p) \leq RN - M_2\sqrt{N} \right\}. \tag{8.4.7}$$

Since $2T - 1 \geq 2M_1\sqrt{N} + 1$, assuming (8.4.7) and adjusting ε yield (8.4.3).

Step 2. In this step we prove (8.4.7). To begin with, we consider several events:

$$\begin{aligned}
\mathsf{B} &:= \bigcap_{i=1}^{2k} \Big\{ H_N^{(i+1)}(2T) \leq H_N^{(i)}(2T - 1) + \log^2 N, \\
&\qquad\qquad\qquad\qquad H_N^{(i+1)}(2T - 1) \leq H_N^{(i+1)}(2T) + \log^2 N \Big\}, \\
\mathsf{C} &:= \left\{ \sup_{p \in [\![1, N]\!]} H_N^{(2k+2)}(p) \leq (R - \tfrac{1}{4}\Delta_k)N \right\}, \\
\mathsf{D} &:= \bigcap_{i=1}^{k} \Big\{ \max\left\{ H_N^{(2i)}(2T - 1), H_N^{(2i)}(2T), H_N^{(2i+1)}(2T) \right\} \\
&\qquad\qquad\qquad\qquad \leq RN - (M_2 + k + 1)\sqrt{N} + 2k \log^2 N \Big\}.
\end{aligned}$$

Let us consider the σ-field

$$\begin{aligned}
\mathcal{F} := \sigma \Big\{ & H_N^{(2i)}[\![2T - 1, 2N - 4i + 2]\!], H_N^{(2i+1)}[\![2T, 2N - 4i]\!],\ i \in [\![1, k]\!], \\
& H_N^{(1)}[\![1, 2N]\!], H_N^{(j)}[\![1, 2N - 2j + 2]\!],\ j \in [\![2k + 2, N]\!] \Big\}.
\end{aligned}$$

By Theorem 8.2.4 with $\rho = \frac{1}{2}$, we have $\mathbf{P}(\neg\mathsf{B}) \leq 4k2^{-N} \leq \varepsilon$ for all large enough N. Observe that $\mathsf{A} \cap \mathsf{B} \subset \mathsf{D}$ and recall that $\mathbf{P}(\neg\mathsf{A}) < \varepsilon$ in (8.4.6). Thus via the union bound, we have $\mathbf{P}(\neg\mathsf{D}) \leq \mathbf{P}(\neg\mathsf{A}) + \mathbf{P}(\neg\mathsf{B}) \leq 2\varepsilon$. Note that $\mathsf{C} \cap \mathsf{D}$ is measurable w.r.t. $\mathcal{F}$. Applying the union bound and tower property of conditional expectation we get

$$\mathbf{P}(\neg \mathsf{E}) \leq \mathbf{P}(\neg \mathsf{C}) + \mathbf{P}(\neg \mathsf{D}) + \mathbf{P}(\mathsf{C} \cap \mathsf{D} \cap \neg \mathsf{E}) \leq 3\varepsilon + \mathbf{E}\left[\mathbf{1}_{\mathsf{C}\cap\mathsf{D}} \cdot \mathbf{E}\left[\mathbf{1}_{\neg\mathsf{E}} \mid \mathcal{F}\right]\right]. \tag{8.4.8}$$

where in the last inequality we have used Lemma 8.4.1 to get that $\mathbf{P}(\neg \mathsf{C}) \leq \varepsilon$ for all large enough N. We claim that

$$\mathbf{E}\left[\mathbf{1}_{\mathsf{C}\cap\mathsf{D}} \cdot \mathbf{E}\left[\mathbf{1}_{\neg\mathsf{E}} \mid \mathcal{F}\right]\right] \leq \frac{k\varepsilon}{(1-\varepsilon)^{k+1}}. \tag{8.4.9}$$

We will demonstrate (8.4.9) in the **Steps 3-4**. Currently, assuming the validity of (8.4.9) and appealing to (8.4.8) prove (8.4.7).

Step 3. In this step we study $\mathbf{1}_{\mathsf{C}\cap\mathsf{D}}\mathbf{E}\left[\mathbf{1}_{\neg\mathsf{E}} \mid \mathcal{F}\right]$ by invoking the Gibbs property (Theorem 8.2.2). Let us consider the domain

$$\Theta_{k,T} := \{(i,j) \mid i \in [\![2, 2k+1]\!], j \in [\![1, 2T-1-\mathbf{1}_{i=\text{even}}]\!]\}.$$

By Theorem 8.2.2, the distribution of the line ensemble conditioned on $\mathcal{F}$ is given by $\mathbf{P}^{\vec{u}}_{\text{gibbs}}$, i.e.

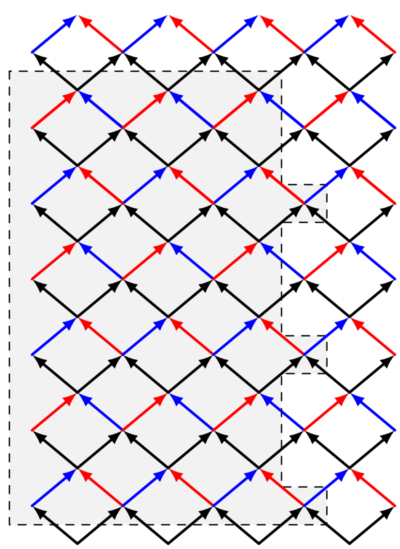


Figure 8.12: $\Theta_{k,T}$ for $k = 3, T = 4$ shown in the shaded region. The $\mathcal{HSLG}$ Gibbs measure on $\Theta_{3,4}$ with boundary condition $(u_{i,j})_{(i,j)\in\partial\Theta_{3,4}}$.

the $\mathcal{HSLG}$ Gibbs measure on the domain $\Theta_{k,T}$ with boundary condition $\vec{u} := (H_N^{(i)}(j))_{(i,j)\in\partial\Theta_{k,T}}$ and the boundary set of $\Theta_{k,T}$ is given by

$$\partial\Theta_{k,T} := \left\{(1,2j-1),(2,2T-1),(3,2T),(2i,2T-1),(2i,2T),(2i+1,2T) \mid i \in [\![2,k]\!], j \in [\![1,T]\!]\right\}.$$

Note that for large enough N, on the event $\mathsf{C}\cap\mathsf{D}$ we have

$$\begin{aligned}
&H_N^{(1)}(2j-1) \le x_{1,2j-1} := \infty, \quad j \in [\![1,T]\!], \\
&H_N^{(2i)}(2T-1) \le x_{2i,2T-1} = RN - (M_2+i)\sqrt{N}, \quad i \in [\![1,k]\!], \\
&H_N^{(2i)}(2T) \le x_{2i,2T} := RN - (M_2+i)\sqrt{N}, \quad i \in [\![2,k]\!], \\
&H_N^{(2i+1)}(2T) \le x_{2i+1,2T} := RN - (M_2+i)\sqrt{N} - \sqrt{T}, \quad i \in [\![1,k]\!], \\
&H_N^{(2k+2)}(2j) \le x_{2k+2,2j} := RN - (M_2+k+1)\sqrt{N}, \quad j \in [\![1,T]\!].
\end{aligned} \tag{8.4.10}$$

where C holds only in the last inequality. Since $\neg\mathsf{E}$ event is increasing with respect to the boundary data, by stochastic monotonicity we have

$$\mathbf{1}_{\mathsf{C}\cap\mathsf{D}} \cdot \mathbf{E}\left[\mathbf{1}_{\neg\mathsf{E}} \mid \mathcal{F}\right] \le \mathbf{1}_{\mathsf{C}\cap\mathsf{D}} \cdot \mathbf{P}^{\vec{u}}_{\text{gibbs}}(\neg\mathsf{E}) \le \mathbf{P}^{\vec{x}}_{\text{gibbs}}(\neg\mathsf{E}). \tag{8.4.11}$$

To bound $\mathbf{P}^{\vec{x}}_{\text{gibbs}}(\neg\mathsf{E})$ we seek for a convenient alternative representation for the $\mathbf{P}^{\vec{x}}_{\text{gibbs}}$ measure. Towards this end, by carefully studying the Gibbs measure, we dissect the $\mathbf{P}^{\vec{x}}_{\text{gibbs}}$ measure into blocks of independent interacting random walks (Definition 8.2.5) and the Radon-Nikodym derivatives interleaved between adjacent blocks (see Figure 8.13). Let us now describe this decomposition.

Recall the interacting random walk (IRW) from Definition 8.2.5. Let $(L_{2i}[\![1,2T-2]\!], L_{2i+1}[\![1,2T-1]\!])_{i=1}^k$ be k independent IRWs of length T with boundary condition $(x_{2i,2T-1}, x_{2i+1,2T})$. Let us denote the joint law and expectation of L as $\mathbf{P}^{\vec{x}}_{\text{block}}$ and $\mathbf{E}^{\vec{x}}_{\text{block}}$ respectively. Set

$$W_{\text{br}} := \exp\left(-\sum_{i=1}^{k}\sum_{j=1}^{T}\left[e^{L_{2i+2}(2j)-L_{2i+1}(2j+1)} + e^{L_{2i+2}(2j)-L_{2i+1}(2j-1)}\right]\right) \tag{8.4.12}$$

with the convention $L_{2i+1}(2T+1)=\infty$ for $i\in[\![1,k]\!]$ and $L_i(j)=x_{i,j}$ for all $(i,j)\in\partial\Theta_{k,T}$. Note that here only $H_N^{(1)}(2j+1)$, $j\in[\![1,T]\!]$ are in the boundary and are set to ∞ in (8.4.10). Thus, their contributions to the Radon-Nikodym derivative W_{br} would be $\prod_{j=1}^{2T-2}\exp(-e^{H_N^{(2)}(j)-\infty})=1$. From the description of the $\mathcal{HSLG}$ Gibbs measure, we have

$$\mathbf{P}_{\mathrm{gibbs}}^{\vec{x}}(\neg\mathsf{E})=\frac{\mathbf{E}_{\mathrm{block}}^{\vec{x}}[W_{\mathrm{br}}\mathbf{1}_{\neg\mathsf{E}}]}{\mathbf{E}_{\mathrm{block}}^{\vec{x}}[W_{\mathrm{br}}]}, \tag{8.4.13}$$

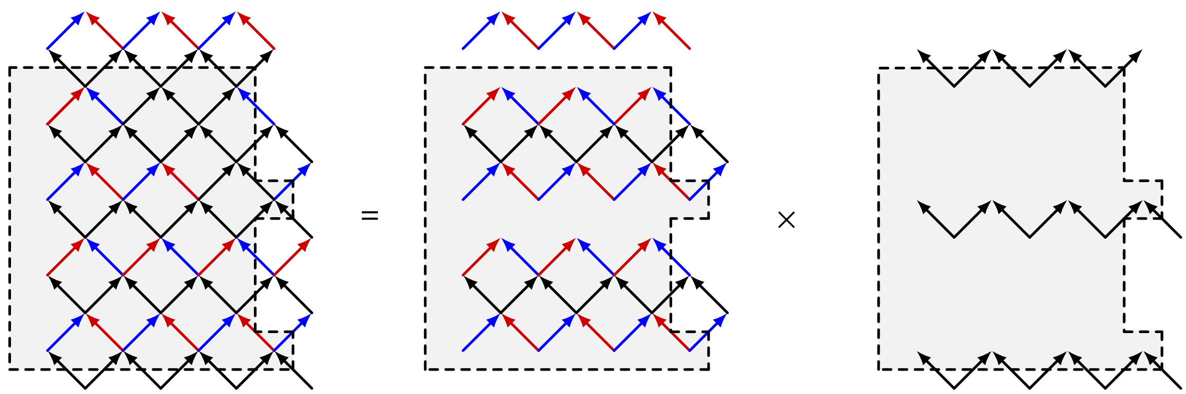


Figure 8.13: Proof Scheme: The Gibbs measure on $\Theta_{2,4}$ domain (left figure) can be decomposed into two parts: One is the combination of the top colored row and 2 IRWs (middle figure) and two are the remaining black weights (right figure) which will be viewed as a Radon-Nikodym derivative. Here note that in the middle figure, the only contribution from the top row comes from the odd points, $H_N^{(1)}(2j-1)$ for $j\in[\![1,T]\!]$, which are set to ∞. Thus, their contribution to (8.4.12) from (8.2.6) would be $\exp(-e^{-\infty})=1$.

Step 4. Finally in this step, we provide an upper bound for the right-hand side of (8.4.13) by bounding its numerator and denominator separately. Let us consider the event:

$$\mathsf{G}:=\bigcap_{i=1}^{k}\left\{\sup_{p\in[\![1,2T-1]\!]}|L_{2i}(p)-x_{2i,2T-1}|+\sup_{q\in[\![1,2T]\!]}|L_{2i+1}(q)-x_{2i,2T-1}|\leq M_0\sqrt{T}\right\}.$$

where M_0 comes from Proposition 8.2.6. From the description of the Gibbs measure, it is clear that if $(L_{2i}(\cdot),L_{2i+1}(\cdot))$ is an IRW with boundary condition $(x_{2i,2T-1},x_{2i,2T-1}-\sqrt{T})$, then $(L_{2i}(\cdot)-x_{2i,2T-1},L_{2i+1}(\cdot)-x_{2i,2T-1})$ is an IRW with boundary condition $(0,-\sqrt{T})$. Thus, appealing to

Proposition 8.2.6, we see that

$$\mathbf{P}^{\vec{x}}_{\text{block}}(\mathsf{G}) \geq (1-\varepsilon)^k \geq 1 - k\varepsilon.$$

Let us assume N is large enough so that $\sqrt{N} - 2M_0\sqrt{T} \geq \frac{1}{2}\sqrt{N}$ (recall $T = O(\sqrt{N})$). Observe that under the event G, we have for all $p \leq 2T-1$

$$L_{2i}(p) \leq x_{2,2T-1} + M_0\sqrt{T} = RN - (M_2+1)\sqrt{N} + M_0\sqrt{T} \leq RN - M_2\sqrt{N}.$$

Thus, E defined in (8.4.7) holds. This implies $\neg\mathsf{E} \subset \neg\mathsf{G}$. Hence

$$\mathbf{E}^{\vec{x}}_{\text{block}}[W_{\text{br}}\mathbf{1}_{\neg\mathsf{E}}] \leq \mathbf{P}^{\vec{x}}_{\text{block}}(\neg\mathsf{E}) \leq \mathbf{P}^{\vec{x}}_{\text{block}}(\neg\mathsf{G}) \leq k\varepsilon. \tag{8.4.14}$$

$$\mathbf{1}_{\mathsf{G}} \cdot W_{\text{br}} \geq \mathbf{1}_{\mathsf{G}} \cdot \exp\left(-k(2T-1)e^{\sqrt{N}-2M_0\sqrt{T}}\right) \geq (1-\varepsilon).$$

where the last one follows by taking N large enough (recall $T = O(\sqrt{N})$). Thus, $\mathbf{E}_{\text{block}}[W_{\text{br}}] \geq (1-\varepsilon)\mathbf{P}_{\text{block}}(\mathsf{G}) \geq (1-\varepsilon)^{k+1}$. Inserting this bound and the bound in (8.4.14) back in (8.4.13) we get that $\mathbf{P}^{\vec{x}}_{\text{gibbs}}(\neg\mathsf{E}) \leq \frac{k\varepsilon}{(1-\varepsilon)^{k+1}}$. Combining this bound with (8.4.11) yields (8.4.9). This completes the proof. $\square$

8.5 Proof of main theorems

In this section, we prove our main theorems, Theorems 8.1.1, 8.1.3, and 8.1.4. This section is structured as follows: In Section 8.5.1 we first present a few supporting technical results. In Section 8.5.2 we complete the proof of our main theorems by assuming a technical proposition (Proposition 8.5.3) which in turn is proved in Section 8.5.3.

8.5.1 Preparatory lemmas

In this section, we prove two preparatory lemmas that will serve as necessary ingredients in proving our main theorems. Recall the polymer measure $\mathbf{P}^W$ from (8.1.2), the partition function $Z(m,n)$ from (8.1.4), and the $\mathcal{HSLG}$ line ensemble H_N from Definition 8.2.1. Note that the quenched distribution of the endpoint of the polymer is related via

$$\mathbf{P}^W(\pi(2N-2) = N-r) = \frac{Z(N+r, N-r)}{\sum_{p=0}^{N-1} Z(N+p, N-p)} = \frac{e^{H_N^{(1)}(2r+1)}}{\sum_{p=0}^{N-1} e^{H_N^{(1)}(2p+1)}}. \tag{8.5.1}$$

where the second equality follows from the relation (8.2.4). Recalling $\mathcal{Z}_N^{\mathsf{PL}}(k) = \sum_{p=k}^{N-1} e^{H_N^{(1)}(2p+1)}$ from (8.2.7), we obtain

$$\mathbf{P}^W(\pi(2N-2) \le N-k) = \frac{\mathcal{Z}_N^{\mathsf{PL}}(k)}{\mathcal{Z}_N^{\mathsf{PL}}(0)}.$$

Theorem 8.1.1 claims that this quenched probability decays as $N \to \infty$ followed by $k \to \infty$. The following lemma settles a weaker version of Theorem 8.1.1 where we take $k = \lfloor M\sqrt{N} \rfloor$. For notational convenience, we assume all the multiples of $\sqrt{N}$ appearing in the proofs in this section are integers. The general case follows verbatim by considering the floor function.

Lemma 8.5.1. *Fix $\varepsilon > 0$ and recall $\mathcal{Z}_N^{\mathsf{PL}}(\cdot)$ from Theorem 8.2.7. There exist constants $M(\varepsilon) > 0, N_1(\varepsilon) > 0$ such that for all $N \ge N_1$,*

$$\mathbf{P}\left(\frac{\mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N})}{\mathcal{Z}_N^{\mathsf{PL}}(1)} \le e^{-\sqrt{N}}\right) \ge 1 - \tfrac{1}{2}\varepsilon. \tag{8.5.2}$$

Proof. Fix $\varepsilon \in (0,1)$. Recall σ from (8.2.9) Taking $g = 1$ and $g = M\sqrt{N}$ in Theorem 8.2.7 yields

$$\frac{1}{\sigma\sqrt{N}}\left[\log \mathcal{Z}_N^{\mathsf{PL}}(1) - RN\right] \xrightarrow{d} \mathcal{N}(0,1),$$

$$\frac{1}{\sigma\sqrt{N}}\left[\log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) - RN + M\tau\sqrt{N}\right] \xrightarrow{d} \mathcal{N}(0,1) \tag{8.5.3}$$

respectively, where R, σ, τ are defined in (8.2.9). Let us set $P := P(\varepsilon) = \Phi^{-1}(1 - \frac{\varepsilon}{8}) + 1$, where $\Phi(\cdot)$ is the cumulative distribution function of $\mathcal{N}(0,1)$. For all large enough N we have

$$\mathbf{P}\left(\log \mathcal{Z}_N^{\mathsf{PL}}(1) \geq RN - P\sigma\sqrt{N}\right) \geq 1 - \tfrac{\varepsilon}{4},$$

$$\mathbf{P}\left(\log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) \leq RN - M\tau\sqrt{N} + P\sigma\sqrt{N}\right) \geq 1 - \tfrac{\varepsilon}{4}.$$

Applying a union bound gives us

$$\mathbf{P}\left(\log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) + (M\tau - 2P\sigma)\sqrt{N} \leq \log \mathcal{Z}_N^{\mathsf{PL}}(1)\right) \geq 1 - \tfrac{\varepsilon}{2},$$

for all large enough N. Taking $M := \frac{1}{\tau}(2P\sigma + 1)$ in above equation leads to (8.5.2). This completes the proof. $\square$

Let us recall our discussion in Section 8.1.2 and Figure 8.5. Let us call the region $[\![N - M\sqrt{N}, N-k]\!]$ and the region $[\![1, N - M\sqrt{N}]\!]$ as shallow tail and deep tail respectively (see Figure 8.5). Lemma 8.5.1 implies that with high probability the quenched probability of $\pi(2N-2)$ living in the deep tail region is exponentially small. Thus the mass accumulates in the window of $M\sqrt{N}$ below the point (N, N). To establish Theorem 8.1.1, we thus have to show the mass in the shallow tail also goes to zero. For convenience, in our proofs below we shall often refer to the point $(N + M\sqrt{N}, N - M\sqrt{N})$ as the deep tail starting point. Given the connection in (8.2.4), the deep tail starting point corresponds to $(2M\sqrt{N}+1)$-th point for the top curve $H_N^{(1)}(\cdot)$ of the $\mathcal{HSLG}$ line ensemble. So, in the coordinates of the $\mathcal{HSLG}$ line ensemble, we shall refer $2M\sqrt{N}+1$ as the deep tail starting point.

Below, we record another important preparatory lemma which claims the existence of a "high point" in $H_N^{(1)}(\cdot)$ not far after the deep tail starting point (see Figure 8.14).

Lemma 8.5.2. *Fix any $\varepsilon > 0$ and recall R, τ from* (8.2.9). *There exists a constant $M_0(\varepsilon) > 0$ such* *that for all $M \geq M_0$, there exists $N_0(\varepsilon, M)$ such that for all $N \geq N_0$,*

$$\mathbf{P}\left(\sup_{p \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]} H_N^{(1)}(2p+1) \geq RN - \tfrac{5}{2}M\tau\sqrt{N}\right) \geq 1 - \tfrac{1}{2}\varepsilon, \tag{8.5.4}$$

where $\tau := \Psi(\theta - \alpha) - \Psi(\theta + \alpha)$.

Proof. Let us set $P := P(\varepsilon) = \Phi^{-1}(1 - \frac{\varepsilon}{6}) + 1$, where $\Phi(\cdot)$ is the cumulative distribution function of $\mathcal{N}(0, 1)$. In view of (8.5.3), for all large enough N we have

$$\mathbf{P}\left(\log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) \geq RN - M\tau\sqrt{N} - P\sigma\sqrt{N}\right) \geq 1 - \tfrac{\varepsilon}{6}, \tag{8.5.5}$$

$$\mathbf{P}\left(\log \mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) \leq RN - 2M\tau\sqrt{N} + P\sigma\sqrt{N}\right) \geq 1 - \tfrac{\varepsilon}{6}.$$

Applying a union bound gives us

$$\mathbf{P}\left(\log \mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) + (M\tau - 2P\sigma)\sqrt{N} \leq \log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N})\right) \geq 1 - \tfrac{\varepsilon}{3}.$$

Thus for any $M \geq \frac{2P\sigma+1}{\tau}$, we have that with probability at least $1 - \frac{\varepsilon}{3}$, $\log \mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) \leq \log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) - \sqrt{N}$, which implies

$$2\mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) \leq \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}).$$

However, by definition of $\mathcal{Z}_N^{\mathsf{PL}}(\cdot)$, the above display implies that with probability at least $1 - \frac{\varepsilon}{3}$,

$$\begin{aligned}\sup_{p \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]} H_N^{(1)}(2p+1) &\geq \log \mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N}) - \log \mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) - \log(2M\sqrt{N}) \\ &\geq \log \mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) - \log(2M\sqrt{N}).\end{aligned}$$

Note that by the first entry in (8.5.5) with M substituted by $2M$, with probability at least $1 - \frac{\varepsilon}{6}$, we have $\log \mathcal{Z}_N^{\mathsf{PL}}(2M\sqrt{N}) \geq RN - 2M\tau\sqrt{N} - P\sigma\sqrt{N}$. Since for all large enough N, we have $RN - (2M\tau + P\sigma)\sqrt{N} - \log(2M\sqrt{N}) \geq RN - \frac{5}{2}M\tau\sqrt{N}$. Thus applying another union bound helps us arrive at (8.5.4) and complete the proof. □

8.5.2 Proof of Theorems 8.1.1, 8.1.3, and 8.1.4

In this section, we prove our main theorems assuming a technical proposition. Let us first begin by describing the proposition. Fix any $M, N \geq 1$ and assume $M\sqrt{N} \in \mathbb{Z}_{>0}$. For any Borel set A of $\mathbb{R}^{M\sqrt{N}}$ we consider the event

$$\mathsf{A} = \left\{ (H_N^{(1)}(1) - H_N^{(1)}(2r+1))_{r=1}^{M\sqrt{N}} \in A \right\}. \tag{8.5.6}$$

for $N > M^2 + 1$. Let $(S_r)_{r=0}^{M\sqrt{N}}$ be the log-gamma random walk defined in Definition 8.1.2. We write

$$\mathbf{P}_{RW}(\mathsf{A}) := \mathbf{P}\left((S_r)_{r=1}^{M\sqrt{N}} \in A \right) \tag{8.5.7}$$

Finally, the last technical proposition below is the main crux of the proof. It claims that $\mathbf{P}$ and $\mathbf{P}_{RW}$ are close to each other when N is large and we postpone its proof to Section 8.5.3.

Proposition 8.5.3. *Fix any $\varepsilon \in (0, \frac{1}{2})$. Set $M(\varepsilon) > 0, N_1(\varepsilon) > 0$ such that Lemma 8.5.1 and Lemma 8.5.2 hold simultaneously for all $N \geq N_1$ for this fixed choice of M. Then there exists $N_0(\varepsilon) > 0$ such that for all $N \geq N_0$,*

$$|\mathbf{P}(\mathsf{A}) - \mathbf{P}_{RW}(\mathsf{A})| \leq 9\varepsilon, \tag{8.5.8}$$

where A and $\mathbf{P}_{RW}(\mathsf{A})$ are defined in (8.5.6) *and* (8.5.7).

In lieu of these results, we are ready to prove our main theorems. Theorems 8.1.3 and 8.1.1 are direct applications of the supporting lemmas. For convenience, we shall assume in the proofs below $M\sqrt{N}$ is an integer. The general case follows verbatim by considering floor functions.

Proof of Theorem 8.1.3. Take the set A as $(-\infty, x_1] \times (-\infty, x_2] \times \cdots \times (-\infty, x_k] \times \mathbb{R}^{M\sqrt{N}-k}$ in (8.5.6). By Proposition 8.5.3,

$$\limsup_{N\to\infty} \left| \mathbf{P}\left(\bigcap_{r=1}^{k} \{H_N^{(1)}(1) - H_N^{(1)}(2r+1) \in (-\infty, x_r]\}\right) - \mathbf{P}_{RW}\left(\bigcap_{r=1}^{k} \{S_r \in (-\infty, x_r]\}\right)\right| \le 9\varepsilon,$$

where $(S_r)_{r=0}^k$ is defined in Definition (8.1.2). As ε is arbitrary, this implies

$$\left(H_N^{(1)}(1) - H_N^{(1)}(2r+1)\right)_{r=0}^{k} \xrightarrow{d} (S_r)_{r=0}^{k}.$$

In conjunction with relation (8.2.4), we get the desired convergence in Theorem 8.1.3. □

Proof of Theorem 8.1.1. Fix any $\varepsilon > 0$. Get $M(\varepsilon), N_1(\varepsilon) > 0$ such that Lemma 8.5.1 and Lemma 8.5.2 hold simultaneously for all $N \ge N_1$ for this fixed choice of M. Using this M we split the probability as follows

$$\begin{aligned}
&\mathbf{P}^W(\pi(2N-2) \le N-k) \\
&\qquad = \mathbf{P}^W\big(\pi(2N-2) \in (N - M\sqrt{N}, N-k]\big) + \mathbf{P}^W\big(\pi(2N-2) \le N - M\sqrt{N}\big).
\end{aligned}$$

For the first term observe that by (8.5.1)

$$\begin{aligned}
\mathbf{P}^W\big(\pi(2N-2) \in (N - M\sqrt{N}, N-k]\big) &= \frac{\sum_{p=k}^{M\sqrt{N}-1} e^{H_N^{(1)}(2p+1)}}{\sum_{p=0}^{N-1} e^{H_N^{(1)}(2p+1)}} \\
&\le \frac{\sum_{p=k}^{M\sqrt{N}} e^{H_N^{(1)}(2p+1)}}{\sum_{p=1}^{M\sqrt{N}} e^{H_N^{(1)}(2p+1)}} = \frac{\sum_{p=k}^{M\sqrt{N}} e^{H_N^{(1)}(2p+1) - H_N^{(1)}(1)}}{\sum_{p=1}^{M\sqrt{N}} e^{H_N^{(1)}(2p+1) - H_N^{(1)}(1)}}.
\end{aligned}$$

Fix any $\delta > 0$ and consider the set

$$\mathsf{A}_\delta := \left\{ \frac{\sum_{p=K}^{M\sqrt{N}} e^{H_N^{(1)}(2p+1) - H_N^{(1)}(1)}}{\sum_{p=1}^{M\sqrt{N}} e^{H_N^{(1)}(2p+1) - H_N^{(1)}(1)}} \ge \delta \right\}.$$

By Proposition 8.5.3, $\mathbf{P}(\mathsf{A}_\delta) \le \mathbf{P}_{RW}(\mathsf{A}_\delta) + 9\varepsilon$ for all large enough N. On the other hand, by

Corollary 8.6.3 we see that $\lim_{k\to\infty}\lim_{N\to\infty}\mathbf{P}_{RW}(\mathsf{A}_\delta)=0$. Thus, as ε is arbitrary,

$$\lim_{k\to\infty}\lim_{N\to\infty}\mathbf{P}^W\big(\pi(2N-2)\in(N-M\sqrt{N},N-k]\big)=0,\ \text{in probability.} \tag{8.5.9}$$

For the second term by Lemma 8.5.1, we see that with probability $1-\frac{\varepsilon}{2}$

$$\mathbf{P}^W\big(\pi(2N-2)\le N-M\sqrt{N}\big)\le\frac{\sum_{p=M\sqrt{N}}^{N-1}e^{H_N^{(1)}(2p+1)}}{\sum_{p=1}^{N-1}e^{H_N^{(1)}(2p+1)}}=\frac{\mathcal{Z}_N^{\mathsf{PL}}(M\sqrt{N})}{\mathcal{Z}_N^{\mathsf{PL}}(1)}\le e^{-\sqrt{N}}.$$

Again, as ε is arbitrary, we have that as $N\to\infty$, $\mathbf{P}^W(\pi(2N-2)\le N-M\sqrt{N})\to 0$ in probability. This completes the proof together with (8.5.9). $\square$

Lastly, with Theorems 8.1.1 and 8.1.3 established, we present the proof of the limiting quenched distribution of the endpoint viewed from around the diagonal.

Proof of Theorem 8.1.4. Fixed $\theta>0$ and $\alpha\in(-\theta,0)$. Recall from (8.5.1) that

$$\mathbf{P}_{\theta,\alpha;N}(\pi(2N-2)=N-r)=\frac{e^{H_N^{(1)}(2r+1)}}{\sum_{p=0}^{N-1}e^{H_N^{(1)}(2p+1)}}=\frac{e^{H_N^{(1)}(2r+1)-H_N^{(1)}(1)}}{\sum_{p=0}^{N-1}e^{H_N^{(1)}(2p+1)-H_N^{(1)}(1)}} \tag{8.5.10}$$

where the second equality is derived through (8.2.4). Note that by Theorem 8.1.3, a continuous mapping theorem immediately implies that for a positive integer $k<\infty$,

$$\left(\frac{\exp(H_N^{(1)}(2r+1)-H_N^{(1)}(1))}{\sum_{p=0}^{k}\exp(H_N^{(1)}(2p+1)-H_N^{(1)}(1))}\right)_{r\in[\![0,k]\!]}\xrightarrow{d}\left(\frac{e^{-S_r}}{\sum_{p=0}^{k}e^{-S_p}}\right)_{r\in[\![0,k]\!]} \tag{8.5.11}$$

Here $(S_i)_{i\ge0}$ denotes a log-gamma random walk. For simplicity, we denote

$$\Lambda_N(p):=\exp(H_N^{(1)}(2p+1)-H_N^{(1)}(1)).$$

We can then rewrite (8.5.10) as

$$\mathbf{P}_{\theta,\alpha;N}(\pi(2N-2)=N-r)=\frac{\Lambda_N(r)}{\sum_{p=0}^{N-1}\Lambda_N(p)}=\frac{\sum_{p=0}^{k}\Lambda_N(p)}{\sum_{p=0}^{N-1}\Lambda_N(p)}\cdot\frac{\sum_{p=0}^{\infty}e^{-S_p}}{\sum_{p=0}^{k}e^{-S_p}}\cdot\frac{\sum_{p=0}^{k}e^{-S_p}}{\sum_{p=0}^{\infty}e^{-S_p}}\cdot\frac{\Lambda_N(r)}{\sum_{p=0}^{k}\Lambda_N(p)}.$$

Theorem 8.1.1 ensures that

$$\frac{\sum_{p=0}^{k}\Lambda_N(p)}{\sum_{p=0}^{N-1}\Lambda_N(p)}=\mathbf{P}_{\theta,\alpha;N}(\pi(2N-2)\geq N-k)=1-\mathbf{P}_{\theta,\alpha;N}(\pi(2N-2)<N-k)\xrightarrow{p}1$$

as $N\to\infty$ followed by $k\to\infty$. By Lemma 8.6.2 we have

$$\lim_{k\to\infty}\frac{\sum_{p=0}^{\infty}e^{-S_p}}{\sum_{p=0}^{k}e^{-S_p}}\xrightarrow{p}1.$$

Meanwhile, (8.5.11) yields that as $N\to\infty$,

$$\left(\frac{\sum_{p=0}^{k}e^{-S_p}}{\sum_{p=0}^{\infty}e^{-S_p}}\cdot\frac{\Lambda_N(r)}{\sum_{p=0}^{k}\Lambda_N(p)}\right)_{r\in[\![0,k]\!]}\xrightarrow{d}\left(\frac{\sum_{p=0}^{k}e^{-S_p}}{\sum_{p=0}^{\infty}e^{-S_p}}\cdot\frac{\Lambda(r)}{\sum_{p=0}^{k}e^{-S_p}}\right)_{r\in[\![0,k]\!]}=\left(\frac{e^{-S_r}}{\sum_{p=o}^{\infty}e^{-S_p}}\right)_{r\in[\![0,k]\!]}.$$

Thus we establish (8.1.7) and complete the proof of Theorem 8.1.4. □

8.5.3 Proof of Proposition 8.5.3

For clarity, we divide the proof into several steps.

Step 1. In this step we sketch the main ideas behind the proof. At this point, we encourage the readers to consult with Figure 8.14. Recall the event A defined in (8.5.6).

- Let us take M and N_1 as described in the statement of the Proposition 8.5.3. In the language introduced in Figure 8.5 and the text before Lemma 8.5.2, $2M\sqrt{N}+1$ serves as the *deep tail starting point*. As we have assumed Lemma 8.5.2 holds, we thus have a point in $2p^*+1\in[\![2M\sqrt{N}+1,4M\sqrt{N}+1]\!]$ where $H_N^{(1)}(2p^*+1)$ is 'high' enough (see Figure 8.14). This high point event is denoted as event B in **Step 2** which has a probability of at least $1-\frac{1}{2}\varepsilon$ by Lemma 8.5.2.

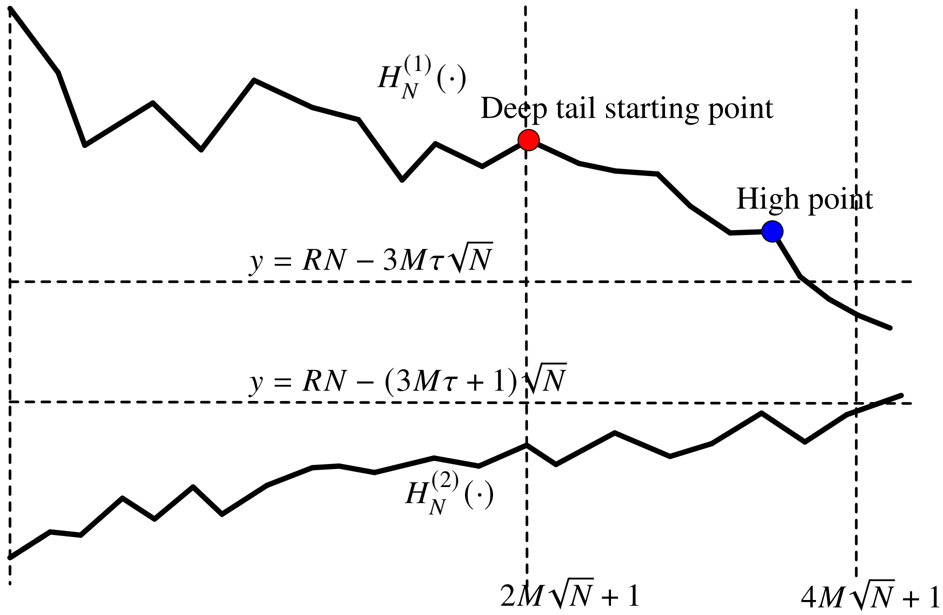


Figure 8.14: Illustration of the proof of Proposition 8.5.3. As claimed by Lemma 8.5.2, there exists a high point in $[\![2M\sqrt{N} + 1, 4M\sqrt{N} + 1]\!]$ such that $H_N^{(1)}(2p^* + 1)$ lies above $RN - \frac{5}{2}M\tau\sqrt{N}$ with high probability. This high point is illustrated as the blue point in the figure. This high point between $[\![2M\sqrt{N}+1, 4M\sqrt{N}+1]\!]$ helps us show that $H_N^{(1)}(\cdot) \geq RN - 3M\tau\sqrt{N}$ between $[\![1, 2p^*+1]\!]$. However, invoking Proposition 8.4.2, we can ensure the second curve stays below the benchmark of $RN - (3M\tau + 1)\sqrt{N}$ on the interval $[\![1, 4M\sqrt{N} + 1]\!]$ with high probability. Thus there is a $\sqrt{N}$ separation (with high probability) between the two curves. By the Gibbs property, this separation ensures that the top curve is close to a log-gamma random walk.

- Invoking Proposition 8.4.2 with high probability we can take the second curve of the line ensemble to be lower than a certain benchmark. More precisely, Proposition 8.4.2 with $M_1 = 2M$ and $M_2 = 3M\tau + 1$ implies that

$$\sup_{p \in [\![1, 4M\sqrt{N}+1]\!]} H_N^{(2)}(p) \leq RN - (3M\tau + 1)\sqrt{N}$$

  with probability at least $1 - \frac{\varepsilon}{2}$. We denote this phenomenon as the Fluc event. As B and Fluc are high probability events, to prove our desired estimate in (8.5.8), it suffices to show that $|\mathbf{P}(\mathsf{A} \cap \mathsf{B} \cap \mathsf{Fluc}) - \mathbf{P}_{RW}(\mathsf{A})|$ is small. This is achieved by considering the measure conditioned on the entire second curve and the first curve beyond $2p^* + 1$. We remark that in reality, this is not exactly how we do it. But for the sketch of the proof, we present it in this way. We refer to the last bullet point for details.

- From the Gibbs property in Theorem 8.2.2, we deduce a key observation regarding the conditional measure in **Step 3**. In colloquial terms, we note that the conditional measure is absolutely continuous w.r.t. a log-gamma random walk $(S_k)_{k\geq 0}$ from Definition 8.1.2 starting at $H_N^{(1)}(2p^*+1)$ and an explicit Radon-Nikodym derivative W_{p^*}. As the free law is precisely the limiting law we are interested in, it suffices to prove that the Radon-Nikodym derivative W_{p^*} over this interval $[1, 2p^*+1]$ is approximately 1.

- Loosely speaking, W_{p^*} is close to 1 whenever there is a wide enough separation between the two curves. Due to the diffusive nature of the random walk (with positive drift), under the free law, the walk does not become too low. This guarantees that under $\mathsf{B} \cap \mathsf{Fluc}$ event we have a uniform separation of $\sqrt{N}$ between the top two curves between $[\![1, 2p^*+1[\![$. Thus, we deduce that $W_{p^*} \approx 1$ when N is large. The details are presented in **Step 5**. This shows that the law of the $H_N^{(1)}(\cdot)$ is close to the free law of a log-gamma random walk starting at $H_N^{(1)}(2p^*+1)$.

- One issue in carrying out the arguments in the last two bullet points is that p^* is *random*. The Gibbs property cannot be applied at p^*, as the property is formulated for *fixed* boundary points. This issue can be circumvented easily by a graining argument. We write B as $\mathsf{B} = \bigsqcup \mathsf{B}_\mathsf{i}$ with B_i being a disjoint collection of events with $\mathsf{B}_i \subset \{H_N^{(1)}(2i+1) \geq RN - \frac{5}{2}M\sqrt{N}\}$ and then apply the Gibbs property for each i.

Step 2. Take $M_1 = 2M$ and $M_2 = 3M\tau + 1$ in Proposition 8.4.2. Taking $N_2(\varepsilon, M_1, M_2) > 0$ (which depends only on ε as M_1, M_2 depends only on ε) from Proposition 8.4.2, we see that

$$\mathbf{P}(\mathsf{Fluc}) \geq 1 - \tfrac{\varepsilon}{2}, \text{ where } \mathsf{Fluc} := \left\{ \sup_{p \in [\![1, 4M\sqrt{N}+1]\!]} H_N^{(2)}(p) \leq RN - (3M\tau+1)\sqrt{N} \right\} \tag{8.5.12}$$

for all $N \geq N_2$. Next we consider the events

$$\mathsf{G}_i := \left\{ H_N^{(1)}(2i+1) \geq RN - \frac{5}{2}M\tau\sqrt{N} \right\} \text{ and } \mathsf{B}_i := \bigcap_{j=i+1}^{2M\sqrt{N}} \mathsf{G}_j^c \cap \mathsf{G}_i.$$

Note that $(\mathsf{B}_i)_{i\in[\![M\sqrt{N},2M\sqrt{N}]\!]}$ forms a disjoint collection of events. Define

$$\begin{aligned}\mathsf{B} &:= \bigsqcup_{i\in[\![M\sqrt{N},2M\sqrt{N}]\!]} \mathsf{B}_i \\ &= \bigcup_{i\in[\![M\sqrt{N},2M\sqrt{N}]\!]} \mathsf{G}_i = \left\{ \sup_{p\in[\![M\sqrt{N},2M\sqrt{N}]\!]} H_N^{(1)}(2p+1) \geq RN - \tfrac{5}{2}M\tau\sqrt{N} \right\},\end{aligned}$$

where we write $\sqcup$ to stress that the events are disjoint in the union. In particular, as Lemma 8.5.2 holds, we have $\mathbf{P}(\mathsf{B}) \geq 1 - \frac{1}{2}\varepsilon$. Thus for all $N \geq N_1 + N_2$, by a union bound we have

$$|\mathbf{P}(\mathsf{A}) - \mathbf{P}(\mathsf{A}\cap\mathsf{B}\cap\mathsf{Fluc})| \leq \mathbf{P}(\neg\mathsf{B}) + \mathbf{P}(\neg\mathsf{Fluc}) \leq \varepsilon.$$

Hence to prove (8.5.8) it suffices to show

$$|\mathbf{P}(\mathsf{A}\cap\mathsf{B}\cap\mathsf{Fluc}) - \mathbf{P}_{RW}(\mathsf{A})| \leq 8\varepsilon. \tag{8.5.13}$$

Define $\mathcal{F}_i$ as the σ-field $\sigma\big(H_N^{(1)}(x)_{x\geq 2i+1}, H_N^{(j)}(x)_{j\geq 2, x\geq 1}\big)$. Note that $\mathsf{B}_i, \mathsf{Fluc}$ are both measurable w.r.t. $\mathcal{F}_i$. Exploiting the fact that B_i's are disjoint yields

$$\mathbf{P}(\mathsf{A}\cap\mathsf{B}\cap\mathsf{Fluc}) = \sum_{i=M\sqrt{N}}^{2M\sqrt{N}} \mathbf{E}\left[\mathbf{1}_{\mathsf{B}_i\cap\mathsf{Fluc}}\mathbf{E}\left[\mathbf{1}_{\mathsf{A}} \mid \mathcal{F}_i\right]\right] \tag{8.5.14}$$

where the last equality is due to the tower property of the conditional expectation. Thus we are left to estimate $\mathbf{E}\left[\mathbf{1}_{\mathsf{A}} \mid \mathcal{F}_i\right]$ for each i.

Step 3. Gibbs law. To analyze $\mathbf{E}\left[\mathbf{1}_{\mathsf{A}} \mid \mathcal{F}_i\right]$, we invoke the Gibbs property (Theorem 8.2.2) for the $\mathcal{HSLG}$ line ensemble. By Theorem 8.2.2, the distribution of $(H_N^{(1)}(j))_{j=1}^{2i}$ conditioned on $\mathcal{F}_i$ has a density at $(u_j)_{j=1}^{2i}$

$$\exp\left(- \sum_{j=1}^{i} \left[e^{H_N^{(2)}(2j)-u_{2j+1}} + e^{H_N^{(2)}(2j)-u_{2j-1}} \right] \right) \tag{8.5.15}$$

$$\cdot \prod_{j=1}^{i} \exp\left((\theta+\alpha)(u_{2j+1}-u_{2j}) - e^{u_{2j+1}-u_{2j}}\right) \tag{8.5.16}$$

$$\cdot \prod_{j=1}^{i} \exp\left((\theta-\alpha)(u_{2j-1}-u_{2j}) - e^{u_{2j-1}-u_{2j}}\right) \tag{8.5.17}$$

with $u_{2i+1} = H_N^{(1)}(2i+1)$. The above explicit expression is obtained from (8.2.6) and (8.2.5). Note that the terms in (8.5.15), (8.5.16), and (8.5.17) correspond to weights of black, red, and blue edges in the graphical representation (see left figure of Figure 8.15) respectively.

Based on the above decomposition, we define a free law $\mathbf{P}_{\text{free},i}$ that depends only on $H_N^{(1)}(2i+1)$. We define that under the law $\mathbf{P}_{\text{free},i}$, the distribution of $(H_N^{(1)}(j))_{j=1}^{2i}$ has a density at $(u_j)_{j=1}^{2i}$ proportional to

$$\prod_{j=1}^{i} \exp\left((\theta+\alpha)(u_{2j+1}-u_{2j}) - e^{u_{2j+1}-u_{2j}}\right) \cdot \prod_{j=1}^{i} \exp\left((\theta-\alpha)(u_{2j-1}-u_{2j}) - e^{u_{2j-1}-u_{2j}}\right)$$

with $u_{2i+1} = H_N^{(1)}(2i+1)$. Note that free law collects all the blue and red edge weights only. A quick comparison of the above formula with (8.1.5) shows that under the free law, $(H_N^{(1)}(1) - H_N^{(1)}(2r+1))_{r=0}^{i}$ is precisely distributed as log-gamma random walk defined in Definition 8.1.2.

In order to obtain the original conditional distribution from the free law, we may introduce the black weights as a Radon-Nikodym derivative (see the decomposition in Figure 8.15). Indeed, we have

$$\mathbf{E}\left[\mathbf{1}_{\mathsf{A}} \mid \mathcal{F}_i\right] = \frac{\mathbf{E}_{\text{free},i}[W_i \mathbf{1}_{\mathsf{A}}]}{\mathbf{E}_{\text{free},i}[W_i]} \tag{8.5.18}$$

where

$$W_i := \exp\left(-\sum_{j=1}^{i}\left[e^{H_N^{(2)}(2j)-H_N^{(1)}(2j+1)} + e^{H_N^{(2)}(2j)-H_N^{(1)}(2j-1)}\right]\right) \tag{8.5.19}$$

We notice that W_i has a trivial upper bound: $W_i \le 1$. For the lower bound, we claim that there

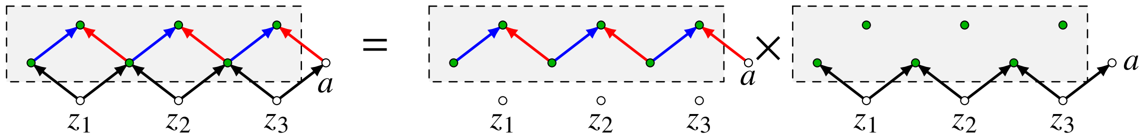


Figure 8.15: Gibbs decomposition. The left figure shows the gibbs measure corresponding to conditioned on $\mathcal{F}_i$ with $i = 3$. Here $a = H_N^{(1)}(2i+1)$, and $z_j := H_N^{(2)}(2j)$ for $j \in [\![1, i]\!]$. The measure has been decomposed into two parts. The free law (middle) and a Radon-Nikodym derivative (right).

exists $N_0(\varepsilon) > 0$ such that for all $N \geq N_0$ we have

$$\mathbf{1}_{\mathsf{Fluc} \cap \mathsf{B}_i} \mathbf{P}_{\text{free},i}(W_i \geq 1 - \varepsilon) \geq \mathbf{1}_{\mathsf{Fluc} \cap \mathsf{B}_i} \cdot (1 - \varepsilon). \tag{8.5.20}$$

Thus, (8.5.20) implies that W_i is close to 1 with high probability under $\mathsf{Fluc} \cap \mathsf{B}_i$. Thus, going back to (8.5.18), we expect $\mathbf{E}\left[\mathbf{1}_{\mathsf{A}} \mid \mathcal{F}_i\right]$ to be close to $\mathbf{P}_{\text{free},i}(\mathsf{A})$. As under the free law $\mathbf{P}_{\text{free},i}(\mathsf{A}) = \mathbf{P}_{RW}(\mathsf{A})$, for all $i \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]$, (8.5.14) eventually leads to (8.5.13), which we make precise in the next step.

Step 4. Assuming (8.5.20), we complete the proof of (8.5.13) in this step. As $W_i \leq 1$, we have

$$\begin{aligned}
\mathbf{1}_{\mathsf{Fluc} \cap \mathsf{B}_i} \frac{\mathbf{E}_{\text{free},i}[W_i \mathbf{1}_{\mathsf{A}}]}{\mathbf{E}_{\text{free},i}[W]} &\geq \mathbf{1}_{\mathsf{Fluc} \cap \mathsf{B}_i} \mathbf{E}_{\text{free},i}[W_i \mathbf{1}_{\mathsf{A}}] \geq (1 - \varepsilon) \cdot \mathbf{1}_{\mathsf{Fluc} \cap \mathsf{B}_i} \mathbf{P}_{\text{free},i}(\mathsf{A} \cap \{W \geq 1 - \varepsilon\}) \\
&\geq (1 - \varepsilon) \cdot \mathbf{1}_{\mathsf{Fluc} \cap \mathsf{B}_i} \left[\mathbf{P}_{\text{free},i}(\mathsf{A}) - \mathbf{P}_{\text{free},i}(W_i < 1 - \varepsilon)\right] \\
&\geq (1 - \varepsilon) \cdot \mathbf{1}_{\mathsf{Fluc} \cap \mathsf{B}_i} \left[\mathbf{P}_{\text{free},i}(\mathsf{A}) - \varepsilon\right]
\end{aligned}$$

where we use (8.5.20) in the last inequality. Recall $\mathbf{P}_{\text{free},i}(\mathsf{A}) = \mathbf{P}_{RW}(\mathsf{A})$. Inserting this bound in (8.5.18) and then going back to (8.5.14) yields

$$\begin{aligned}
\mathbf{P}(\mathsf{A} \cap \mathsf{B} \cap \mathsf{Fluc}) &\geq (1 - \varepsilon) \cdot [\mathbf{P}_{RW}(\mathsf{A}) - \varepsilon] \sum_{i = M\sqrt{N}}^{2M\sqrt{N}} \mathbf{P}(\mathsf{B}_i \cap \mathsf{Fluc}) \\
&= (1 - \varepsilon) \cdot [\mathbf{P}_{RW}(\mathsf{A}) - \varepsilon] \, \mathbf{P}(\mathsf{B} \cap \mathsf{Fluc}) \geq (1 - \varepsilon)^2 \, [\mathbf{P}_{RW}(\mathsf{A}) - \varepsilon] \, .
\end{aligned}$$

for all large enough N. The equality in the above equation follows by recalling that B_i's form a disjoint collection of events and the result implies that $\mathbf{P}(\mathsf{A} \cap \mathsf{B} \cap \mathsf{Fluc}) - \mathbf{P}_{RW}(\mathsf{A}) \geq -3\varepsilon$. This proves the lower bound inequality in (8.5.13). Similarly for the upper bound, as $W_i \leq 1$, we have

$$\mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i} \cdot \frac{\mathbf{E}_{\mathrm{free},i}[W_i \mathbf{1}_{\mathsf{A}}]}{\mathbf{E}_{\mathrm{free},i}[W_i]} \leq \mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i} \cdot \frac{\mathbf{P}_{\mathrm{free},i}(\mathsf{A})}{(1-\varepsilon)\mathbf{P}_{\mathrm{free},i}(W_i \geq 1-\varepsilon)} \leq \mathbf{1}_{\mathsf{Fluc}\cap\mathsf{B}_i} \cdot \frac{\mathbf{P}_{\mathrm{free},i}(\mathsf{A})}{(1-\varepsilon)^2}$$

where the last inequality stems from (8.5.20). Again, Inserting this bound in (8.5.18) and then going back to (8.5.14) gives us

$$\mathbf{P}(\mathsf{A} \cap \mathsf{B} \cap \mathsf{Fluc}) \leq \frac{\mathbf{P}_{RW}(\mathsf{A})}{(1-\varepsilon)^2} \sum_{i=M\sqrt{N}}^{2M\sqrt{N}} \mathbf{P}(\mathsf{B}_i \cap \mathsf{Fluc}) = \frac{\mathbf{P}_{RW}(\mathsf{A})}{(1-\varepsilon)^2} \mathbf{P}(\mathsf{B} \cap \mathsf{Fluc}) \leq \frac{\mathbf{P}_{RW}(\mathsf{A})}{(1-\varepsilon)^2}$$

where again the equality comes from the disjointness of B_i's. As $\varepsilon \leq \frac{1}{2}$, this implies

$$\mathbf{P}(\mathsf{A} \cap \mathsf{B} \cap \mathsf{Fluc}) - \mathbf{P}_{RW}(\mathsf{A}) \leq \frac{1-(1-\varepsilon)^2}{(1-\varepsilon)^2} \leq 8\varepsilon$$

which proves the upper bound in (8.5.13). The proof of Theorem 8.1.3 modulo (8.5.20) is thus complete.

Step 5. Finally in this step we prove (8.5.20). We define the event

$$\mathsf{Sink}(i) := \left\{ \inf_{p \in [\![0,i]\!]} H_N^{(1)}(2p+1) \geq RN - 3M\tau\sqrt{N} \right\}.$$

We claim that there exists $N_0(\varepsilon) > 0$ such that for all $N \geq N_0$, we have

$$\mathbf{1}_{\mathsf{B}_i} \mathbf{P}_{\mathrm{free},i}(\mathsf{Sink}(i)) \geq \mathbf{1}_{\mathsf{B}_i}(1-\varepsilon), \tag{8.5.21}$$

for all $i \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]$.

Recall that the event Fluc in (8.5.12) requires the second curve $H_N^{(2)}(p)$ to lie below certain threshold within the range $p \in [\![1, 4M\sqrt{N}+1]\!]$. Recall the definition of W_j from (8.5.19). Note that on $\mathsf{Sink}(j) \cap \mathsf{Fluc}$ we have

$$W_j \geq \exp(-2je^{-\sqrt{N}}) \geq \exp(-4M\sqrt{N}e^{-\sqrt{N}})$$

as $j \leq 2M\sqrt{N}$. Note that $\exp(-4M\sqrt{N}e^{-\sqrt{N}}) \geq 1 - \varepsilon$ for all large enough N. Therefore, in view of (8.5.21) we have

$$\mathbf{1}_{\mathsf{Fluc} \cap \mathsf{B}_i} \mathbf{P}_{\text{free},i}(W_i \geq 1 - \varepsilon) \geq \mathbf{1}_{\mathsf{Fluc} \cap \mathsf{B}_i} \mathbf{P}_{\text{free},i}(\mathsf{Sink}(i)) \geq \mathbf{1}_{\mathsf{Fluc} \cap \mathsf{B}_i} \cdot (1 - \varepsilon)$$

for all large enough N. This verifies (8.5.20). We are left to show (8.5.21). Towards this end, note that on the event B_i, we have $H_N^{(1)}(2i+1) \geq RN - \frac{5}{2}M\tau\sqrt{N}$. Thus,

$$\mathbf{1}_{\mathsf{B}_i} \mathbf{P}_{\text{free},i}(\mathsf{Sink}(i)) \geq \mathbf{1}_{\mathsf{B}_i} \mathbf{P}_{\text{free},i}\left(\inf_{x \in [\![0,i]\!]} H_N^{(1)}(2x+1) - H_N^{(1)}(2i+1) \geq -\tfrac{1}{2}M\tau\sqrt{N}\right). \tag{8.5.22}$$

Recall from our discussion in **Step 2** that under the law $\mathbf{P}_{\text{free},i}$, $(H_N^{(1)}(1) - H_N^{(1)}(2r+1))_{r=0}^{i}$ is distributed as a log-gamma random walk. Let us use $(S_k)_{k=0}^{i}$ to denote a log-gamma random walk. We have

$$\begin{aligned} \mathbf{P}_{\text{free},i}\left(\inf_{p \in [\![0,i]\!]} H_N^{(1)}(2p+1) - H_N^{(1)}(2i+1) \geq -\tfrac{1}{2}M\tau\sqrt{N}\right) \\ = \mathbf{P}\left(\inf_{p \in [\![0,i]\!]} (S_i - S_p) \geq -\tfrac{1}{2}M\tau\sqrt{N}\right). \end{aligned} \tag{8.5.23}$$

Note that $(S_i - S_p)_{p \geq 0}^{i}$ is again a time-reversed log-gamma random walk. As $i \leq 2M\sqrt{N}$, appealing to Lemma 8.6.1 yields that

$$\mathbf{1}_{\mathsf{B}_i} \mathbf{P}_{\text{free},i}(\mathsf{Sink}(i)) \geq \mathbf{P}\left(\inf_{p \in [\![0,i]\!]} (S_i - S_p) \geq -\tfrac{1}{2}M\tau\sqrt{N}\right) \geq 1 - \frac{8\,\mathrm{Var}(S_1)}{M\tau^2\sqrt{N}} \geq 1 - \varepsilon$$

for all large enough N (uniformly over $i \in [\![M\sqrt{N}, 2M\sqrt{N}]\!]$. Inserting this bound in (8.5.23), in view of the lower bound in (8.5.22), leads to (8.5.21). This completes the proof of Proposition 8.5.3.

8.6 Properties of random walks with positive drift

In this section, we collect some useful properties of random walks with positive drift whose proofs follow by classical analysis. Note that the log-gamma random walk introduced in Definition 8.1.2 is a random walk with positive drift. This is because the density $p(x)$ introduced in (8.1.5) has mean:

$$\int_{\mathbb{R}} xp(x)dx = \Psi(\theta - \alpha) - \Psi(\theta + \alpha),$$

which is positive as the digamma function Ψ is strictly increasing (recall $\alpha < 0$).

Lemma 8.6.1. *Let $(X_i)_{i\geq 0}$ be a sequence of iid random variables with $\mathbf{E}[X_1] = \beta > 0$ and $\mathrm{Var}[X_1] = \gamma < \infty$. Set $S_0 = 0$ and $S_k = \sum_{i=1}^{k} X_i$. For all $M, N, \lambda > 0$ we have*

$$\mathbf{P}\left(\inf_{k\in[\![1,M\sqrt{N}]\!]} S_k \leq -\lambda\right) \leq \frac{M\sqrt{N}\gamma}{\lambda^2}.$$

Proof. As $\beta > 0$, by Kolmogorov's maximal inequality, we have

$$\mathbf{P}\left(\inf_{k\in[\![1,M\sqrt{N}]\!]} S_k \leq -\lambda\right) = \mathbf{P}\left(\sup_{k\in[\![1,M\sqrt{N}]\!]} |S_k - k\beta| \geq \lambda\right) \leq \frac{1}{\lambda^2}\sum_{i=1}^{M\sqrt{N}} \mathrm{Var}(X_i) = \frac{M\sqrt{N}\gamma}{\lambda^2},$$

which is precisely what we want to show. $\square$

Lemma 8.6.2. *Let $(X_i)_{i\geq 0}$ be a sequence of iid random variables with $\mathbf{E}[X_1] = \beta > 0$ and $\mathrm{Var}[X_1] = \gamma < \infty$. Set $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$. We have*

$$\mathbf{P}\left(\sum_{r=0}^{\infty} e^{-S_r} < \infty\right) = 1$$

Proof. By Kolmogorov's maximal inequality

$$\mathbf{P}\big(\sup_{1\leq i\leq n^2} |S_i - i\beta| \geq \tfrac{n^2}{2}\beta\big) \leq \frac{4}{n^4\beta^2}\sum_{i=1}^{n^2} \mathrm{Var}(X_i) = \frac{4\gamma}{n^2\beta^2}.$$

The last bound is summable in n. Thus invoking Borel-Cantelli's lemma we have that there exists a random N with $P(7 \le N < \infty) = 1$ such that

$$S_i \ge i\beta - (N^2/2)\beta \ge -(N^2/2)\beta, \text{ for all } 1 \le i \le N^2,$$

and for all $n \ge N + 1$ we have

$$S_i \ge (n-1)^2\beta - (n^2/2)\beta \ge (n^2/4)\beta, \text{ for all } (n-1)^2 + 1 \le i \le n^2,$$

where above we used the fact that $n \ge N + 1 \ge 8$. Thus with probability 1, we have

$$\begin{aligned}\sum_{r=0}^{\infty} e^{-S_r} &= \sum_{r=0}^{N^2} e^{-S_r} + \sum_{n=N+1}^{\infty} \sum_{i=(n-1)^2+1}^{n^2} e^{-S_i} \\ &\le N^2 e^{(N^2/2)\beta} + \sum_{n=N+1}^{\infty} \sum_{i=(n-1)^2+1}^{n^2} e^{-(n^2/4)\beta} \le N^2 e^{(N^2/2)\beta} + \sum_{n=N+1}^{\infty} n^2 e^{-(n^2/4)\beta} < \infty.\end{aligned}$$

This completes the proof. □

As a corollary, we have the following double-limit result.

Corollary 8.6.3. *Under the setup of Lemma 8.6.2, almost surely we have*

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{\sum_{r=k}^{\infty} e^{-S_r}}{\sum_{r=0}^{n} e^{-S_r}} = 0.$$

Proof. Note that $\sum_{r=0}^{n} e^{S_r}$ is a monotone sequence in n which converges to a random variable that is almost surely finite by Lemma 8.6.2. Thus,

$$\frac{\sum_{r=k}^{\infty} e^{S_r}}{\sum_{r=0}^{n} e^{S_r}} = 1 - \frac{\sum_{r=0}^{k-1} e^{S_r}}{\sum_{r=0}^{n} e^{S_r}} \overset{n\to\infty}{\to} 1 - \frac{\sum_{r=0}^{k-1} e^{S_r}}{\sum_{r=0}^{\infty} e^{S_r}}.$$

Taking $k \to \infty$ yields the desired result. □

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