A PROOF OF LABOR THEORY OF VALUE BASED ON MARGINALIST PRINCIPLE

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Abstract: Because anti-Marxist's criticism against Fundamental Marxian Theorem (FMT) is based on an assertion that this proof can be understood as a "natural-resources exploitation," if we assume, for example, "natural-resources theory of value," we should prove not only exploitation but also labor theory of value itself. Therefore, this article aims to prove labor theory of value mathematically by focusing on the historically conditional proportionality between labor input and amounts of products which is assumed in labor theory of value. By this proof of the conditional proportionality, we show that marginalist principle does not disturb labor theory of value in capitalism at all. Furthermore, marginalist principle is important to show the labor process as a subjective optimization process which is not by the sun but only by human beings. In this way, we use anti-Marxist's marginal principle to object anti-Marxist criticism against labor theory of value.

Key words: Fundamental Marxian Theorem; labor theory of value; marginalist principle; constant return to labor; machine-based production system

Defense of FMT and Marginalist Principle

Fundamental Marxian Theorem (FMT) is the most important achievement in mathematical Marxian economics. Because this proof by Prof. Okishio is robust and strong in mathematics, many mainstream economists have taken an attitude to refuse it in a non-mathematical way. Their argument is that it is mathematically true not only in the case of labor theory of value but also in any type of value theory. For example, its most representative scholar Yoshihara (2008) said that if we accept natural-resources theory of value, this theorem has become just a proof of "natural-resources exploitation." In this sense, if Marxists want to prove labor exploitation, Marxists have to prove labor theory of value first of all.

Although, of course, many Marxist economists fought against this contradiction, the most important points should be the difference between the human being and the sun. The human being has a will to make life better, while the sun does not have any will. The human being expends labor power for this purpose, while the sun expends sun power without any purpose. Therefore, this must be understood as the problem whether there is a subjective object to be realized by such a subjective action. In my opinion, existence of this subjective object in labor theory of value should be shown clearly in the form of a subjective optimization problem. In other words, labor theory of value should be modeled as a model where the agents maximize a certain target.

However, if we want to formulate a subjective optimization problem, it gives rise to another question: whether or not we should accept the marginalist principle. It is because linear functions do not have any non-corner solutions of the optimization problem and it is not realistic. In this sense, some researches tried to link the labor theory of value with the marginalist principles.

For example, in the 1960s, Johansen (1963) linked linear production functions to the principle that the marginal utility of each good is proportional to each price. In this way, he proved that the influence of marginal utilities on prices is limited. Furthermore, Chinese Marxian economics also has seen a similar study by Baoli Bai and his daughter in Bai and Bai (2014a).[1] A prominent superiority of their research is that a linear production function is led as a long-run horizontal supply curve in a perfectly competitive market based on a short-run upward sloping marginal cost curve in the second subsection of chapter three of Bai and Bai (2014a). This theoretical framework is basically the same with the long-run horizontal supply curve in the textbooks of mainstream microeconomics, but a very important difference between them is that Bai and Bai's marginal cost is the marginal labor expense. In this sense, it is obvious that their intention is to support the labor theory of value,[2] and the proportionality between the marginal utility and the labor expense was explained.

However, what we should note is our purpose is to identify the value determination as a subjective optimization problem, and for this purpose, we should compare the utility of the product not with the labor expense itself but with the disutility of the labor expense. Therefore, in the next section, we will define an optimization problem to maximize net utility by equalizing the marginal utility of the produced goods and the marginal disutility of the input labor to produce this product.

Furthermore, what we should mention is another important study by Hagendorf (2014) which proved the proportionality between prices and labor values mathematically. Its virtue is that he proved this not by a consumer choice model but by a producer choice model which is just based on the technological characteristics of the capitalistic production process. It is more realistic to express the actual determination process in capitalism, but we cannot neglect the role of utility of the produced goods and disutility of a pain of labor as a sacrifice of human lifetime.[3] A role of this dimension is that it can express the general and essential relation of the labor input and the acquisition of products, because the humans are comparing them through all of history not only in capitalism. Therefore, we introduce the utility–disutility dimension of value determination model which can neglect the price–wage dimension because it is only in the commodity exchange society or capitalist society. By this mode, we will show the special condition of capitalism which realizes the proportionality between amount of products and labor expense, not the proportionality between price and labor expense which was proved by Hagendorf (2014). In this way, we will show the specialty of capitalism.

Utility–Disutility Dimension of the Value-Determination Model

Therefore, now we should introduce the disutility as the cost and the utility as the benefit in this subjective action: labor process. In other words, the subjective purpose is to maximize net utility which is acquired by deducing disutility of labor expense from utility of the products. If so, at the point to maximize this benefit minus cost, the marginal utility of the products produced in this labor process should be equal to the marginal disutility of the labor expense. That is:

$$\frac{\mathrm{d}D}{\mathrm{d}l} = \frac{\mathrm{d}U}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}l}$$

Or in short,

$$\mathrm{d}D(1) = \mathrm{d}U\left(y(1)\right),$$

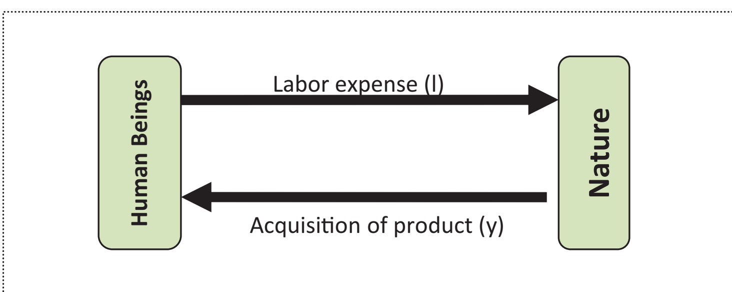

Figure 1 Labor Expense and Acquisition of Product as the Metabolism between Human Beings and Nature

where l, y, D and U express the amount of labor expense and produced goods, the disutility of labor expense and the utility by consuming y produced by labor. Humans make decisions on how much labor should be expended to maximize net utility (=utility minus disutility) as the metabolism between human beings and nature (see Figure 1). This is the determination of labor input, and the determination of value. In this case, the value of one unit of product can be expressed by ($1^*/y^*$), that is, the ratio of optimized input labor (1^*) to produced amount of goods (y^*).

Then, let me introduce more concrete and mathematical formulation of the value determination by unifying the utility of y and disutility of l by the following utility function:

$$\mathrm{U} = \mathrm{U}(y, 1) \qquad \frac{\partial U}{\partial y} > 0, \frac{\partial U}{\partial l} < 0$$

where U is the increasing function of y and the decreasing function of l. However, in order to translate this function into the most popular Cobb-Douglas type of form, we replace labor expense with free time as an explaining variable of this non-linear utility function as follows:

$$\mathrm{U} = \mathrm{C}\mathrm{y}^{\alpha}\left(H - 1\right)^{\beta} \qquad \frac{\partial U}{\partial y} > 0, \ \frac{\partial U}{\partial\left(H - l\right)} > 0$$

where H means the total time that is held by the human beings (e.g., 24 hours in one day) and therefore (H–l) is the free time. Furthermore, we assume positive α and β in order to set this function as an increasing function of consumption (y) and a decreasing function of labor time (l). Furthermore, in order to be consistent to the above explanation, we also assume $0 < \alpha, \beta < 1$ which means diminishing marginal utility to both factors.

However, because the amount of product of the consumption goods is also a function of labor expense, we also specify the following non-linear production function:

$$y = Al^{\gamma}$$

where A and γ express a different kind of labor productivity under the assumption $0 < \gamma < 1$ which is consistent to the above explanation. In this case, the above-mentioned maximization problem becomes the problem to choose the optimal labor input (l) to maximize $U=C(Al^{\gamma})^{\alpha}(H-l)^{\beta}$.

Therefore, partially differentiated U with respect to l must be zero. That is:

$$\frac{\partial U}{\partial l} = CA^{\alpha}\alpha l^{\gamma\alpha-1}\left(H - l\right)^{\beta} - CA^{\alpha}\beta l^{\gamma\alpha}\left(H - l\right)^{\beta-1} = 0$$

$$\Leftrightarrow CA^{\alpha} l^{\gamma\alpha-1}\left(H-l\right)^{\beta-1}\{\gamma\alpha\left(H-l\right)-\beta l\}=0.$$

This can be simplified as $\gamma\alpha H=(\gamma\alpha+\beta)l$, and therefore the optimal labor input becomes:

$$l^{*}=\frac{\gamma\alpha}{\gamma\alpha+\beta}H.$$

Furthermore, we can calculate the value for one-unit product as follows:

$$\frac{l^{*}}{y^{*}}=\frac{\dfrac{\gamma\alpha H}{\gamma\alpha+\beta}}{A\left(\dfrac{\gamma\alpha H}{\gamma\alpha+\beta}\right)^{\gamma}}=\frac{1}{A}\left(\frac{\gamma\alpha H}{\gamma\alpha+\beta}\right)^{1-y}$$

which indicates that this value is affected by A、γ、H and $\frac{\gamma\alpha}{\gamma\alpha+\beta}\left(=\dfrac{1}{\left(1+\dfrac{\beta}{\gamma\alpha}\right)}\right)$.

Concretely speaking, (1) increase in labor productivity (A) leads to a decrease in value for the one-unit product. (2) Although γ reflects another kind of labor productivity, γ's effects are not clear.[4] (3) Although H is fixed in nature, if necessary domestic labor can be cut by domestic electrification, actual free time can be lengthened and it is the same with the increase in H. In this case, disutility of labor becomes small and labor supply becomes larger, and finally value for the one-unit product becomes larger. (4) Because the increase of $\frac{\gamma\alpha}{\gamma\alpha+\beta}$ makes leisure time less important, disutility of labor expense becomes smaller and value for the one-unit product becomes larger. It is same with the case of H.[5]

These results show that the value is determined by the technical condition (parameter of production function) such as (1) (2) and the various conditions on the preference such as (3) (4) (the parameters of the utility function). And since Marx did not conduct a detailed analysis on the disutility of labor, he did not explain the effects of (3) and (4), but only the situation of (1) (2) was discussed.

However, when arguing a little more, it turns out that the case of $\gamma = 1$ is the most Marxistic situation in the following sense. In this case, the production function becomes $y = Al$, which is a typical "LTV (labor theory of value) situation" since the input labor per unit product (l/y) is constant as $1/A$. Furthermore, it is important that this situation can be held as a solution of the optimization problem when $\gamma = 1$ independently from the various properties of utility function. The reason why Marx ignored the utility side such as (3) (4) can be explained in this way.

Additionally speaking, because the condition $\gamma = 1$ expresses constant productivity, it means that we do not need the "marginal productivity principle" to explain how input labor (= value) is determined. Only the marginal utility principle is enough to explain, and this was the case in Johansen (1963) and Bai and Bai (2014b). But they just assumed such a condition, while we have proved it.

Means of Labor and "LTV Situation"

Based on the above findings, what we must pay attention to in the next step is much more technological reality after the Industrial Revolution. It is because the Industrial Revolution has changed the form of production functions. Before the Industrial Revolution, the means of labor was not so important, and the productive activities were covered mainly by human powers. However, after the Industrial Revolution, the machines had become the most decisive factor of production.

However, what we must know here is that this transformation of the production system made labor expend twice; first to produce means of production, and second to produce final goods as shown in Figure 2, and this duplicity can provide a crucially important and preferable characteristic for labor theory of value. It is because labor theory of value needs a proportionality of labor input or "value"[6] and physical amounts of products, and this proportionality can be led only by the above duplicity of labor input which makes technology constant return to labor. In other words, although the original single production system had a diminishing return to

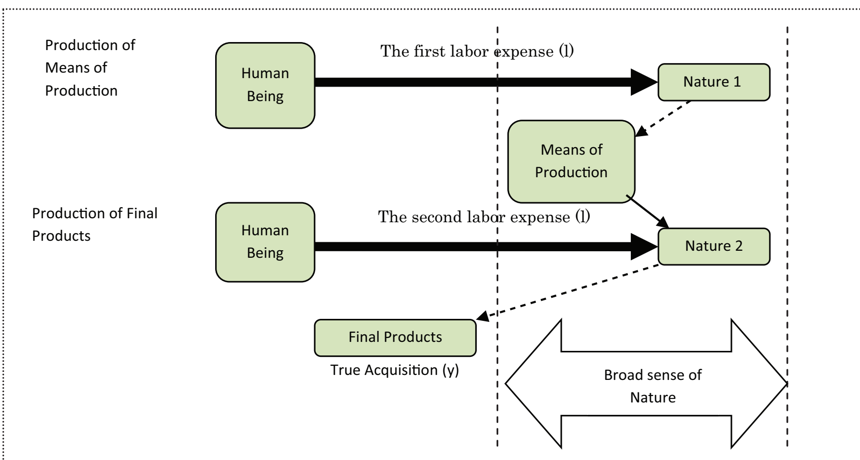

Figure 2 Production of Means of Labor and Final Products

labor shown as the production function $y=Al^{\gamma}$ $(0 < \gamma < 1)$ in the previous section, the above duplicity of labor input changes the ultimate form of production function. Now we will introduce this ultimate form of the new production function.

In this case, the most normal type of the new production function which includes means of production might be:

$$y = Al^{\gamma_1}k^{\gamma_2}.^{7}$$

Here, we assume k is a "flow" variable for simplification, although actual k is a stock variable. This simplification is not essential on this issue. And we also assume $0 < \gamma_1 < 1$, $0 < \gamma_2 < 1$ to express that there is a diminishing return to both factors. Furthermore, we need to introduce one more production function of means of production as follows:

$$k = Bl^{\gamma_3}k^{\gamma_4}.$$

Here, we also assume $0 < \gamma_3 < 1$, $0 < \gamma_4 < 1$ to express that there is a diminishing return to both factors in this function, and it can be transformed into:

$$K = B^{\frac{1}{1-\gamma_4}}l^{\frac{\gamma_3}{1-\gamma_4}}.$$

Furthermore, this equation can be inserted into the above production function y and transform it into:

$$y = Al^{\gamma_1}\left(B^{\frac{1}{1-\gamma_4}}l^{\frac{\gamma_3}{1-\gamma_4}}\right)^{\gamma_2} = AB^{\frac{\gamma_2}{1-\gamma_4}}l^{\gamma_1}l^{\frac{\gamma_2\gamma_3}{1-\gamma_4}} = AB^{\frac{\gamma_2}{1-\gamma_4}}l^{\gamma_1+\frac{\gamma_2\gamma_3}{1-\gamma_4}}.$$

Therefore, our present problem is $\gamma_1 + \frac{\gamma_2\gamma_3}{1-\gamma_4}$ can be 1 or not in the consumption goods sector, and $\frac{\gamma_3}{1-\gamma_4}$ can be 1 or not. If both of them are equal to 1 under the condition that γ_1, γ_2, γ_3 and γ_4 are all under 1, this situation can be regarded as the "LTV situation" under the condition of diminishing returns to individual factors of production, and we can assume such a situation to be very natural.

First, $\frac{\gamma_3}{1-\gamma_4}$ can be 1 when $\gamma_3 + \gamma_4 = 1$. This means a constant return to scale, the most natural and common assumption in microeconomics. Second, $\gamma_1 + \frac{\gamma_2\gamma_3}{1-\gamma_4}$ can be also 1 when $\gamma_1 + \gamma_2=1$ under the condition $\gamma_3 + \gamma_4=1$. This also means a constant return to scale. Therefore, we can say that the "LTV situation" can be introduced in the machine-based production system; that is the technological base of the capitalist system.

Another Explanation by Minimum Necessary Investment

Such a focus on means of production can provide another type of explanation to lead the situation $\gamma = 1$. One of the important characteristics of the machinery system after the Industrial Revolution is that it has a certain minimum amount to be activated. Companies cannot operate below this level but can do only when means of production are accumulated over this level. If we express this level as k_0, this relation can be shown as the following production function of the final goods:

$$y = A\left(k - k_0\right)^{\gamma_4}.$$

Here, we assume the diminishing return to capital as $0 < \gamma_4 < 1$, but at the same time, we want to introduce the same production function of the means of production with the last section as $k = Bl^{\gamma_3}$, and also assume the diminishing return to labor as $0 < \gamma_3 < 1$. In this case, the above production function of the final goods can be translated into:

$$y = A\left(Bl^{\gamma_3} - Bl_0^{\gamma_3}\right)^{\gamma_4},$$ [8]

and labor productivity[9] becomes:

$$\frac{y}{l} = \frac{AB^{\gamma_4}\left(l^{\gamma_3} - l_0^{\gamma_3}\right)^{\gamma_4}}{l}.$$

To check the return to labor, differentiating it with l leads to:

$$\begin{aligned}
\frac{\partial}{\partial 1}\left(\frac{y}{l}\right) &= \frac{AB^{\gamma_4}\gamma_4\left(l^{\gamma_3} - l_0^{\gamma_3}\right)^{\gamma_4 - 1}\cdot\gamma_3 l^{\gamma_3 - 1}\cdot l - AB^{\gamma_4}\left(l^{\gamma_3} - l_0^{\gamma_3}\right)^{\gamma_4}}{l^2} \\
&= \frac{AB^{\gamma_4}\left(l^{\gamma_3} - l_0^{\gamma_3}\right)^{\gamma_4 - 1}\{\gamma_4\gamma_3 l^{\gamma_3} - \left(l^{\gamma_3} - l_0^{\gamma_3}\right)\}}{l^2} \\
&= \frac{AB^{\gamma_4}\left(l^{\gamma_3} - l_0^{\gamma_3}\right)^{\gamma_4 - 1}\{l_0^{\gamma_3} - l^{\gamma_3}\left(1 - \gamma_4\gamma_3\right)\}}{l^2} \\
&> 0 \text{ when } l < l_o\left(1 - \gamma_3\gamma_4\right)^{-\frac{1}{3}} \\
&= 0 \text{ when } l = l_o\left(1 - \gamma_3\gamma_4\right)^{-\frac{1}{3}} \\
&< 0 \text{ when } l > l_o\left(1 - \gamma_3\gamma_4\right)^{-\frac{1}{3}}.
\end{aligned}$$

Here, we assumed $0 < \gamma_3\gamma_4 \leq 1$, and in this case, we could find an area where there is the increasing return and constant return, if we introduce a concept of minimum necessary amount of capital and even if we assume the diminishing return on the production functions. For us, the most important case is the second case where technology becomes a constant return to labor as we saw in the previous section. This is the "LTV situation."

This situation can be realized if human beings choose the maximum efficiency in terms of labor input. It is true that this section does not consider any influence of the utility at all differently from the second section of this article, and in this sense, there is a possibility that l is not chosen as this optimal level. However, if producers of a certain product compete with each other in the market, only the most efficient producers can survive by applying the same technological condition in the long run, and that condition should be the point $l = l_o\left(1-\gamma_3\gamma_4\right)^{-\frac{1}{3}}$. In this case, each producer's size of production is fixed, but total demand for this product is covered by a certain number of producers whose number is determined by $\frac{\textit{total demand determined by this price level}}{\textit{size of production of each producers}}$.[10] This situation makes the long-run supply curve horizontal,[11] and realizes the "LTV situation" in the sense that y/l ratio becomes constant.

In conclusion, as shown in these ways, the proportionality between input (embodied) labor or "value" and physical amount of products could be realistic conditionally in capitalism where the means of labor have played a more important role in the production process. As we know that there had not been a concept of "commodity value" in the pre-market societies before capitalism, generalization of the market system was the crucial condition for the modern concept of value. But now we also need to know another important condition of the proportionality between input (embodied) labor or "value" and physical products which has been introduced by the machine-based production system after the Industrial Revolution.[12] In this way, labor theory of value can be proved.

Notes

1. They also provided another similar study in Bai and Bai (2014b), but this research has strong assumptions such as a linear and homogeneous utility function and constant ratios of consumed goods. Therefore, this article focuses only on their research in Bai and Bai (2014a).
2. In their understanding, marginal productivity theory is a theory which explains the "contribution" of capital on the production process and justifies capitalists' profit taking. In this sense, they criticized marginal productivity theory, but accepted the short-run upward sloping marginal cost curve.
3. This point is also mentioned by Hagendorf (2014), but he did not introduce any utility functions to show these utility and disutility. Disutility of labor expense was just expressed in the level of the labor time.
4. Differentiating $\left(\frac{\gamma\alpha H}{\gamma\alpha+\beta}\right)^{1-\gamma}$ with respect to γ becomes $\left(\frac{\gamma\alpha H}{\gamma\alpha+\beta}\right)^{1-\gamma}\left[-log\frac{\gamma\alpha H}{\gamma\alpha+\beta}+(1-\gamma)\frac{\beta}{\gamma\alpha+\beta}\right]$. It shows, by a complex calculation, that larger γ increases l^*/y^* if γ is close to zero, and decreases l^*/y^* if γ is close to one in the range $0<\gamma\leq 1$.

5. Strictly speaking, all these explanations are just the determination of the "individual value." However, we can assume that all the agents in the society behave in the same way. Therefore, even if there are differences among "individual values," social value can be calculated just by averaging them.
6. Strictly speaking, because substance of value is input labor in Marxian economics, there is no difference between value and input labor. In this sense, the word "value" here just means a certain substance imagined as "value."
7. Different from the Leontief type of production function, the Cobb-Douglas type of production function is better to accept substitutability of factors of production as a kind of marginalist analysis. Marginal substitution and marginal productivity of each factor are closely related. Bai and Bai (2014a, 116) pointed out Sraffa's limitation to assume unsubstitutability of factors of production.
8. Here, l_0 is the necessary input labor to produce the minimum necessary investment.
9. Take note that the labor productivity is the inverse of the value for the one-unit product which we discussed in the second section of this article.
10. This explanation is completely the same with the ordinary explanation of the long-run horizontal supply curve in the textbooks of mainstream microeconomics. In this case, all the companies, in the long run, choose the amount of production to minimize the unit cost, and then the number of companies for this product is determined by $\frac{\textit{total demand determined by this price level}}{\textit{amount of production of each companies}}$.
11. In the same way, Bai and Bai (2014b) introduced this horizontal supply curve.
12. In fact, the machine-based production system which was decisive for capitalism provided two other ways to establish the concept of commodity value. Its first way is that it destroyed workers' skill, standardized the human labor into simple labor, and substantiated the concept of "abstract human labor." It has been the materialistic base of the concept of commodity value. Second, the machine-based production system had realized the full market system by its "scale-merit" technology which accelerated the social division of labor. Because the concept of commodity value needs a situation where all of the wealth can be exchanged and therefore can be imagined including a certain substance, society should be marketized by its "scale-merit" technology. Consequently, both of these characteristics show that the concept of "value" is historical.

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