Electronic Research Archive

ERA, 31(2): 1065–1088.
DOI: 10.3934/era.2023053
Received: 03 October 2022
Revised: 24 November 2022
Accepted: 29 November 2022
Published: 13 December 2022

http://www.aimspress.com/journal/ERA

Research article

Congestion behavior and tolling strategies in a bottleneck model with exponential scheduling preference

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Abstract: The bottleneck model has been widely used in the past fifty years to analyze the morning commute. To reduce the complexity of analysis, most previous studies adopted discontinuous scheduling preference (DSP). However, this handling destroys the continuity in departure rate and differentiability in travel time and cumulative departures. This paper considers an exponential scheduling preference (ESP), which supposes the unit schedule delay cost for commuters exponentially changes with time. With this scheduling preference, we analytically derive solutions and economic properties of user equilibrium and social optimum in the bottleneck model. The first-best, time-varying toll and the optimal single-step toll scheme with ESP are also studied. Results indicate that ESP eliminates the discontinuity in departure rate and non-differentiability in travel time and cumulative departures, which makes the process of morning commute smooth. The ignorance of ESP will lead to underestimation in the queueing time and bias in travel behavior analysis and policymaking.

Keywords: the bottleneck model; traffic congestion; exponential scheduling preference; user equilibrium; social optimum

1. Introduction

Traffic congestion in the morning rush hour is a crucial issue for many metropolitan cities and impedes the development of urban society and economy. To alleviate this problem, lots of research has been done from perspectives of congestion behavior in commute [1–4] or tolling strategies [5,6], where the classical bottleneck model is widely used. The bottleneck model is first proposed by Vickrey [7] and provides a convenient and tractable way to describe the queueing during rush hour [8]. Therefore, this model has been extended in vast specific research directions such as travel demand [9–11], variable bottleneck capacity [12–14] and the incentive scheme [15,16] in recent decades.

In the bottleneck model, commuters are assumed to minimize the trip cost by making a trade-off between the schedule delay cost and travel time cost. This trade-off is essentially influenced by commuters' scheduling preferences. To capture commuting behavior, researchers have proposed several scheduling preference forms based on different assumptions. Among all research, the piecewise formulation developed by Vickrey [7] and Small [17] is the most popular one, where the unit costs for travel time and schedule delay time are constant and piecewise constant respectively. This scheduling preference is also referred to as $\alpha - \beta - \gamma$ preference in Knockaert et al. [18] and discontinuous scheduling preference (DSP) in Li and Huang [19]. However, though DSP simplifies the algebra in analyzing travel behavior, results given by Li and Huang [19] have shown that it breaks the continuity of departure rate in the user equilibrium (UE). Furthermore, DSP essentially assumes the marginal utilities of travel time and schedule delay early or late time are both independent of time. This is inconsistent with our practical experience and our intuitions. Empirical studies also have shown the marginal utilities of time (MUT) at origin (H) and destination (W) will change with time [20–22]. Thus, the DSP assumption may not capture the realistic characteristics in morning commute and leads to bias in travel behavior analysis and traffic policymaking.

To solve the above problems, other kinds of scheduling preferences are considered in research. Vickrey [20] has proposed a departure time choice model, where the marginal utilities of time at origin and destination relative to the time in commuting are linear functions of time. The popular DSP assumption exactly is a particular situation of this formulation. Tseng and Verhoef [21] developed it with a scheduling model which allows the marginal utilities to nonlinearly change with time, and Jenelius et al. [22] further extended this formulation to a two-trip chain. Hjorth et al. [23] have considered four different marginal utility formulations and estimated them with real data. Based on this study, Li and Huang [19] proposed a continuous scheduling preference (CSP). Then, they analyzed the travel behavior of commuters and the properties of the UE flow pattern in the single-entry traffic corridor model with CSP. Li and Huang [24] also investigated the UE and social optimum (SO) states in the bathtub model with CSP. Besides CSP, Hendrickson and Plank [25] developed and estimated the quadratic penalty functions for the schedule delay time. Engelson et al. [26] further proposed a constant-exponential formulation, where the marginal utilities of time at origin and destination relative to the time in commuting are constant and exponential functions respectively. Hjorth et al. [23] also adopted this kind of utility formulation and estimated the scheduling model based on real data from Stockholm.

Congestion tolling is a travel demand management policy that aims at alleviating traffic congestion in the rush hour by adjusting traffic demand. Based on the bottleneck model and various kinds of scheduling preferences, congestion tolling strategies are also widely discussed. With an activity-based bottleneck model which considers the linear marginal activity utility (equivalent to the linear-linear utility formulation), Li et al. [27] studied the efficiency of step tolling strategies and their influence on the departure time choice behavior under homogeneous and heterogeneous commuters. Li and Zhang [28] also proposed a bottleneck model with the two-mode setting and time-varying scheduling preferences, and investigated the influence of congestion tolling on travel mode and departure time choice behaviors. From the perspective of changing unit schedule delay cost, Zhu et al. [29] developed a bottleneck model with CSP and analyzed commuters' departure time choice equilibrium. The following parts of their research focused on the congestion tolling schemes, designed the first-best and second-best tolling strategies and obtained the closed-form solutions for time-varying toll schemes, single-step toll scheme and multi-step toll scheme.

However, most above research focus on the perspective of commuters' utility functions. Few studies are from the perspective of the time-varying schedule delay cost function. This ignores the economic analysis of congestion behaviors and tolling strategies in the morning commute, especially for the constant-exponential formulation. With this formulation, the schedule delay cost function will exponentially change with time such that herein we refer to it as the exponential scheduling preference (ESP). As suggested by Hjorth et al. [23] and Engelson et al. [26], the constant-exponential formulation is a more general formulation than the constant-affine formulation (exactly equivalent to the CSP assumption). Even if the most appropriate scheduling specification and corresponding measure is still an empirical matter [23], ESP assumption probably is more flexible to be adapted to data from stated preference surveys and describes the time-varying process of the unit schedule delay cost in more detail [26]. For another, according to the conclusions of Xiao et al. [30], the ESP assumption may be more reasonable than the DSP assumption in economic analysis since this preference yields unbiased benefit estimates of travel time reliability improvements for the rush hour. Therefore, we adopt the ESP assumption in this paper. Li and Huang [19] and Zhu et al. [29] have shown that CSP significantly influences the travel behavior of commuters and the efficiency of traffic policies, but how ESP influences congestion behavior and tolling strategies in a specific traffic model has not been investigated by studies. To further explore this problem, this paper develops an ESP bottleneck model and discusses congestion behaviors and tolling strategies in the morning commute under this model. Also, the ESP bottleneck model is compared with the DSP bottleneck model to show the difference. The main contributions of this paper are as follows.

Firstly, we formulate a bottleneck model with ESP assumption, which perfectly eliminates the non-differentiability and discontinuity in the equilibrium departure rate function of the classical bottleneck model. The smooth departure pattern avoids the abrupt change in the equilibrium departure rate and may be more reasonable to capture the commuting behavior. Secondly, we have investigated the congestion behavior and the economic properties of commuters' travel behavior in morning commute with the proposed ESP bottleneck model. The equilibrium travel time, departure pattern and corresponding scheduling cost of commuters are analytically derived. Also, the critical clock times in the ESP bottleneck model are given. Thirdly, we have studied the congestion toll scheme, including the first-best, time-varying toll scheme and the optimal single-step toll scheme, based on ESP. We have also found ESP significantly influences the congestion toll scheme. The imposition of the proposed toll schemes provides references for better transportation management.

The following parts are organized as follows: Section 2 gives the settings of the basic bottleneck model and proposes the unit cost function for ESP. Section 3 and Section 4 respectively analyze the travel behavior and economic properties of commuters in UE and SO states. Section 5 investigates the optimal single-step toll scheme. Section 6 conducts several numerical studies to illustrate the properties of the ESP bottleneck model and compares it to the DSP bottleneck model. Section 7 gives conclusions and future research directions.

2. Bottleneck model with ESP

2.1. The basic bottleneck model

In the morning rush hour, we consider N commuters traveling from home to the workplace through a bottleneck, as shown in Figure 1. The bottleneck has a fixed capacity s. If the arrival rate at the bottleneck exceeds the capacity, queueing will appear. As the capacity of the bottleneck is limited, not all commuters can arrive at the workplace on time, so they will endure the schedule delay penalty beyond the travel time cost. To minimize the trip cost, commuters trade off between the travel time cost and schedule delay cost by choosing the departure time. The following settings are adopted to simplify the analysis.

1) Commuters are homogeneous and have identical desired arrival time t^*.

2) Only the travel time in the bottleneck is considered. The free-flow travel time from home to the workplace is ignored.

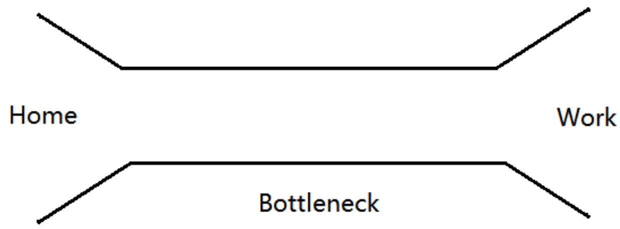

Figure 1. Schematic diagram of morning commute problem with a single-entry bottleneck.

Let $T(t)$ and $D(t)$ be the travel time and length of the queue for commuters departing from home at time t, respectively. Thus, the travel time from home to the workplace can be described as

$$T(t) = T_{free} + T_{var}(t), \tag{1}$$

where T_{free} and $T_{var}(t)$ denote the fixed and variable components of travel time, representing the free-flow travel time and queuing time at the bottleneck, respectively.

Without loss of generality, the free-flow travel time is ignored in following parts, as we state above. Therefore, the travel time is equivalent to the queueing time, which can be expressed as

$$T(t) = T_{var}(t) = \frac{D(t)}{s}. \tag{2}$$

Denote $C(t)$ as the trip cost for the commuter departing at time t. For each commuter, the trip cost consists the travel time cost and the schedule delay cost. The travel time cost is directly proportional to travel time, while the schedule delay early or late cost is relevant to the delay time and scheduling preference function. For commuters, the trip cost equals

$$C(t) = \begin{cases} \alpha T(t) + \int_{t+T(t)}^{t^*} \beta(u)du, & t + T(t) \le t^* \\ \alpha T(t) + \int_{t^*}^{t+T(t)} \beta(u)du, & t + T(t) \ge t^* \end{cases}, \tag{3}$$

where α is the constant unit travel time cost, $\beta(u)$ is the cost for unit schedule delay time, and t^*

denotes the desired arrival time for commuters. In next subsection, we will investigate the unit schedule delay cost for ESP.

2.2. Schedule delay cost for ESP

Let $\beta(u)$ represent the unit schedule delay cost at time u, where $u = t + T(t)$ is the arrival time of commuters departing at time t. In addition, we use $SDC(t)$ to denote the schedule delay cost for commuters departing at time t. The unit schedule delay cost $\beta(u)$ for ESP (as shown in Figure 2(b)) is defined as follows

$$\beta(u) = \begin{cases} -p[e^{\eta(u-t^*)} - 1] , & u < t^* \\ p[e^{\eta(u-t^*)} - 1] , & u \geq t^* \end{cases}, \tag{4a}$$

$$SDC(t) = p\left[\frac{1}{\eta}(e^{\eta(t+T(t)-t^*)} - 1) + t^* - t - T(t)\right], \tag{4b}$$

where $p > 0$ and $\eta > 0$ should hold. The physical meanings of parameters p and η represent commuters' sensitivities to the schedule delay cost and schedule delay time, respectively. Equation (4a) is equivalent to the constant-exponential scheduling formulation in Hjorth et al. [23] and Engelson et al. [26].

For comparison, we plot the general trend of $\beta(u)$ with DSP (a) and ESP (b) respectively in Figure 2, where the light gray area denotes the schedule delay early cost of commuters arriving at t_1 and the dark gray area denotes the schedule delay late cost of commuters arriving at t_2.

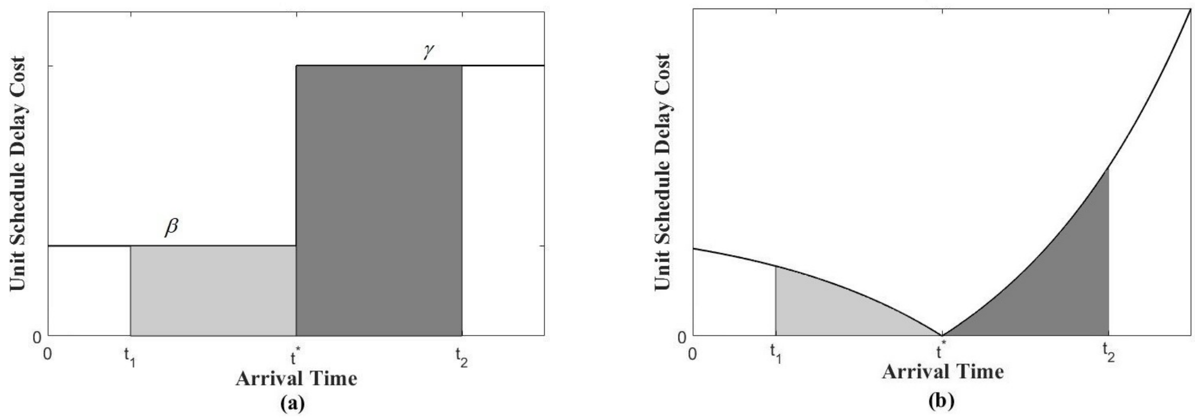

Figure 2. The unit schedule delay time cost, $\beta(u)$, with DSP (a) and ESP (b).

According to the general trend of $\beta(u)$ in Figure 2(b), some properties of ESP can be found:

1) With ESP, the unit schedule delay cost exponentially changes with time. In addition, the change rate of the unit schedule delay cost is also related to time, which describes the changing process of the unit schedule delay cost in more detail.

2) ESP naturally makes the unit schedule delay cost for arriving late change more rapidly than that for arriving early. Thus, commuters will pay more for unit late time than that for unit early time, which is supported by the empirical research in Small [17].

3) The unit schedule delay cost monotonically decreases before the desired arrival time t^* and monotonically increases after that time. In addition, the cost naturally becomes zero at the desired arrival time t^*, which is consistent with reality.

The above properties illustrate ESP maintains the main characteristics of the classical bottleneck model while possessing the merits of realism by considering the continuous change of unit schedule delay cost. Next, we will explore the influence of ESP on the morning commute and investigate commuters' departure time choices based on the ESP bottleneck model.

3. **User equilibrium**

3.1. *Congestion behavior analysis*

Suppose the choices of commuters in the morning commute are rational. The UE condition requires each commuter has identical trip cost so that they have no motivation to alter the departure time. From the above condition, we can have the following inferences:

1) Commuters have identical trip cost in UE state.

2) The bottleneck will be fully utilized during the morning rush hour in UE state, all commuters except the first and the last will experience congestion.

3) The duration of morning rush hour is $\frac{N}{s}$ in UE state.

From Eqs (3) and (4a), we know

$$C(t) = \alpha T(t) + p\left[\frac{1}{\eta}\left(e^{\eta(t+T(t)-t^*)} - 1\right) + t^* - t - T(t)\right]. \tag{5}$$

Let t_s and t_e denote the departure time for the first and the last commuters respectively, and t_{OT} denotes the departure time of the commuter arriving at the workplace punctually. For the three time points, relationships between the departure time, travel time and arrival time follows

$$t_s + T(t_s) = t_s, \tag{6a}$$

$$t_{\mathrm{OT}} + T(t_{\mathrm{OT}}) = t^*, \tag{6b}$$

$$t_e + T(t_e) = t_e. \tag{6c}$$

We normalize time so that the first commuter departs at $t_s = 0$. At the end of this section, we will show the expression of t_s when t^* is given. According to Eq (5) and $T(t_s) = 0$, we know the first commuter's trip cost is

$$C(t_s) = p\left[\frac{1}{\eta}\left(e^{-\eta t^*} - 1\right) + t^*\right]. \tag{7}$$

The UE condition indicates commuters will experience the identical trip cost and no one could lower the trip cost by departing earlier or later if the system reaches equilibrium. We can combine Eqs (5) and (7), and then solve for $T(t)$, which is the travel time for commuters departing at t. It follows

$$T(t) = \frac{pt}{\alpha-p} - \frac{f(t)}{\eta} + \frac{pe^{-\eta t^*}}{\eta(\alpha-p)}, \tag{8}$$

where $f(t) = \text{Lambert}\, W\left\{0, \frac{p}{\alpha-p} \cdot exp\left(\frac{1}{\alpha-p} \cdot (pe^{-\eta t^*} + \alpha\eta t) - \eta t^*\right)\right\}$. The function Lambert $W(k, x)$ represents the solution w for equation $we^w = x$, here actually is the solution for $f(t)e^{f(t)} = \frac{p}{\alpha-p} \cdot exp\left(\frac{1}{\alpha-p} \cdot (pe^{-\eta t^*} + \alpha\eta t) - \eta t^*\right)$. In Figure 3, we have depicted the schematic diagram of travel time in UE state.

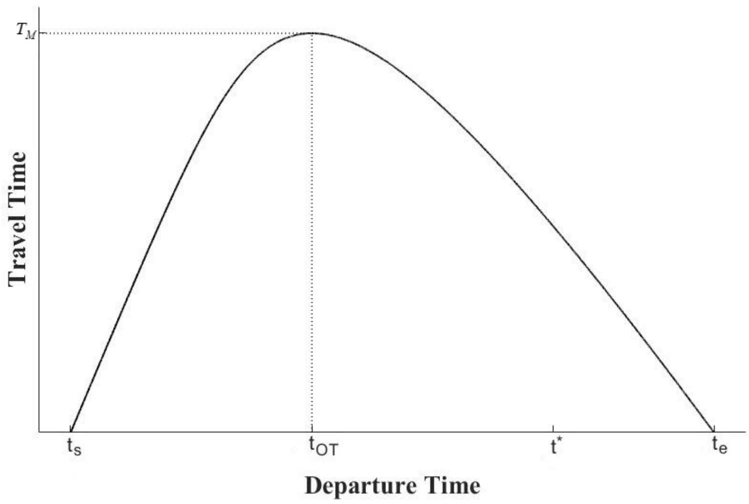

Figure 3. The travel time, $T(t)$, with ESP in UE state.

Let $r(t)$ denote the departure rate of commuters at time t. The length of the queue $D(t)$ in equilibrium state follows

$$D(t) = \int_{t_s}^{t} r(u)du - s(t - t_s). \tag{9}$$

Take the first-order derivative of Eq (9) to time t, we can have the dynamic change process of the queue length. It should be

$$\frac{dD(t)}{dt} = r(t) - s, \quad \text{for } D(t) > 0. \tag{10}$$

From Eq (2), we know $\frac{dT(t)}{dt} = \frac{1}{s} \cdot \frac{dD(t)}{dt}$. Then with Eqs (8) and (10), we can solve for the expression of $r(t)$ in equilibrium state. It follows

$$r(t) = \frac{\alpha s}{(\alpha-p)[f(t)+1]}, \tag{11}$$

where $f(t) = \text{Lambert}\, W\left\{0, \frac{p}{\alpha-p} \cdot exp\left(\frac{1}{\alpha-p} \cdot (pe^{-\eta t^*} + \alpha\eta t) - \eta t^*\right)\right\}$. The schematic diagram of equilibrium departure rate is depicted in Figure 4.

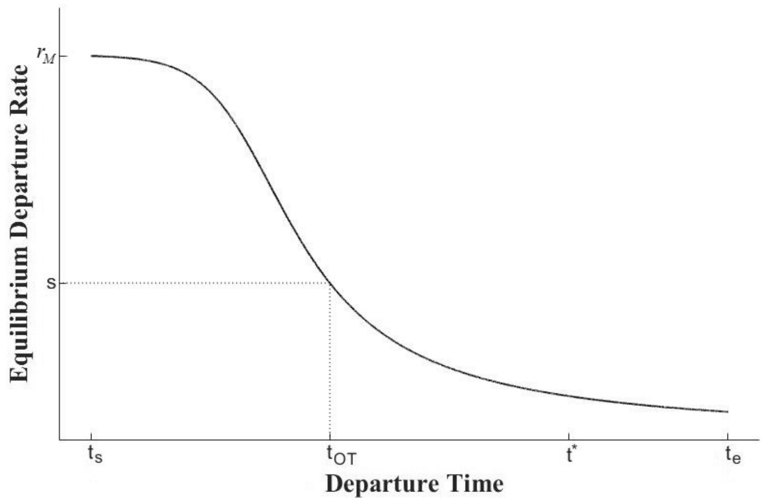

Figure 4. The departure rate, $r(t)$, with ESP in UE state.

Let $Q(t)$ and $A(t)$ denote the cumulative arrivals at the workplace and cumulative departures from home at time t respectively. From above deductions, we can express $Q(t)$ and $A(t)$ as

$$Q(t) = st, \tag{12a}$$

$$A(t) = \int_{t_s}^{t} r(t)dt = s(T(t) + t). \tag{12b}$$

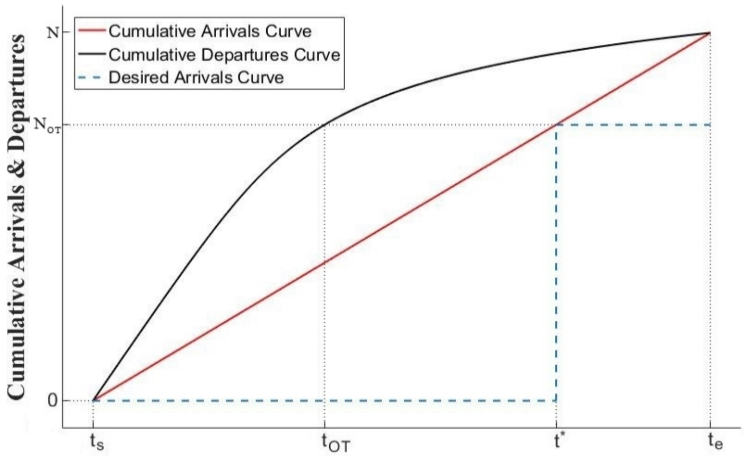

Figure 5. The cumulative arrivals and departures curves, $A(t)$ and $Q(t)$, with ESP in UE state.

The cumulative arrivals and departures curves as functions of time t with ESP are plotted in Figure 5. The queue length and the travel time (queuing time at the bottleneck) are also depicted in Figure 5, which are the vertical and horizontal distances between the two cumulative curves.

3.2. *Properties of the UE solution*

We have known the relationships between the start time t_s and end time t_e of the morning rush hour as

$$t_e - t_s = \frac{N}{s}, \tag{13a}$$

$$C(t_s) = C(t_e). \tag{13b}$$

With the assumption of $t_s = 0$, we can express the desired arrival time t^* from Eqs (13a) and (13b) as

$$t^* = \frac{1}{\eta} ln\left[\frac{s}{\eta N}\left(e^{\frac{\eta N}{s}} - 1\right)\right]. \tag{14}$$

It is easy to find that t^* will be endogenous when t_s is given. However, a more general situation is with a given value of t^*, while t_s needs to be obtained. This situation will not change the qualitative analysis results and the expressions of t_s and t_e when t^* is given can also be deduced as

$$t_s = t^* + \frac{1}{\eta} ln \frac{\eta N}{\left(e^{\frac{\eta N}{s}} - 1\right)s}, \tag{15a}$$

$$t_e = t^* + \frac{N}{s} + \frac{1}{\eta} ln \frac{\eta N}{\left(e^{\frac{\eta N}{s}} - 1\right)s}. \tag{15b}$$

From Eq (15a), we can find that the start time of the morning rush hour t_s merely depends on η while p will not influence t_s. Based on this finding, we can derive the following properties.

Proposition 1. The start time of the morning rush hour t_s monotonically decreases with η and is independent of p.

Proposition 2. The total early arrivals strictly monotonically increase with η while the total late arrivals strictly monotonically decrease with η.

The proofs of **Proposition 1** and **Proposition 2** are given in Appendix. **Proposition 1** and **Proposition 2** indicate the value of the parameter η determines both the start time of the morning rush hour t_s and the percentage of the total early arrivals. This means η is a crucial parameter in ESP, which influences the travel behavior of commuters in the morning rush hour. In ESP, the physical meanings of parameters p and η represent commuters' sensitivities to the schedule delay cost and schedule delay time, respectively. We can find that increasing sensitivity to the schedule delay time for commuters will lead to an earlier start time of the morning rush hour, which is consistent with realism. However, the sensitivity to the schedule delay cost merely influences the trip cost, while the travel behavior pattern of commuters will not be influenced.

3.3. *Economic properties*

Let TTC denote the total trip cost in the traffic system, $TTTC$ and $TSDC$ denote the total travel time cost and total schedule delay cost respectively. With Eqs (7) and (14), the trip cost for the first commuter and TTC for the system can be rewritten as

$$C(t_s) = p\left[t^* - \frac{1}{\eta} + \frac{N}{s\left(e^{\frac{\eta N}{s}} - 1\right)}\right], \tag{16a}$$

$$TTC_e = Np\left[t^* - \frac{1}{\eta} + \frac{N}{s(e^{\frac{\eta N}{s}} - 1)}\right], \tag{16b}$$

where the subscript e implies the UE state.

We can express the total travel time as the area enclosed by the cumulative departure and arrival curves, as shown in Figure 5. Therefore, $TTTC$ equals $\alpha \int_{t_s}^{t_e} \big(A(t) - Q(t)\big) dt$, and then we have

$$TTTC_e = s(\alpha - p) \cdot \frac{k_2(k_2+2) - k_1(k_1+2)}{2\eta^2} + \frac{N}{s} \cdot \frac{\alpha p(N\eta + 2se^{-\eta t^*})}{2\eta(\alpha - p)}, \tag{17a}$$

$$k_1 = \text{Lambert } W\left\{0, \frac{p}{\alpha - p} \cdot exp\left(\frac{N}{s} \cdot \frac{p\eta\alpha}{(\alpha - p)^2} \cdot e^{-\eta t^*} - \eta t^*\right)\right\}, \tag{17b}$$

$$k_2 = \text{Lambert } W\left\{0, \frac{p}{\alpha - p} \cdot exp\left(\frac{p}{\alpha - p} \cdot e^{-\eta t^*} - \eta t^*\right)\right\}. \tag{17c}$$

With the relationship $TTC = TTTC + TSDC$, we can obtain $TSDC_e$ as

$$TSDC_e = Np\left[t^* - \frac{1}{\eta} + \frac{N}{s(e^{\frac{\eta N}{s}} - 1)}\right] - TTTC_e. \tag{18}$$

In Figure 6, we have depicted TTC, $TTTC$ and $TSDC$ as functions of the total population in UE state to intuitively show the quantitative relationships among the three costs. It can be found from Figure 6 that $TTTC$ is the main component of TTC, while the percentage of $TSDC$ is relatively small.

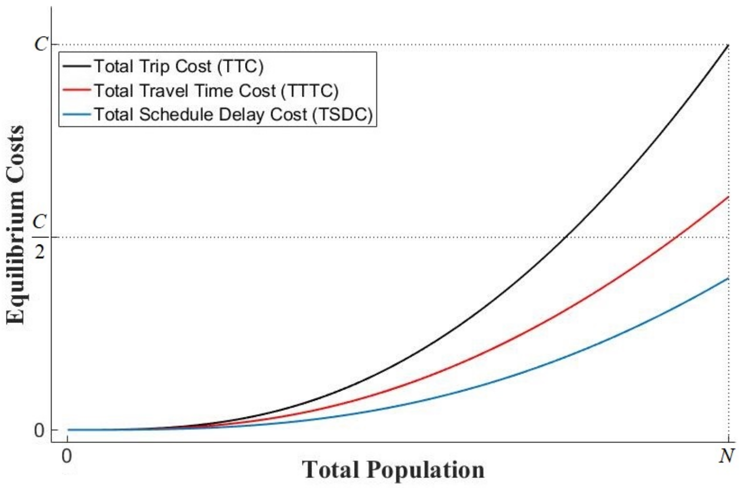

Figure 6. TTC, $TTTC$ and $TSDC$ as functions of total population with ESP in UE state.

From Eqs (5) and (8) we can also plot the trip cost, the travel time cost and the schedule delay cost of commuters who depart from home at time t, as shown in Figure 7. The UE condition requires the trip cost must be identical so that there is a reverse tendency in the travel time cost and the schedule delay cost. This means that the closer the departure time of commuters to t_{OT}, the higher his/her travel time is and the lower his/her schedule delay time is.

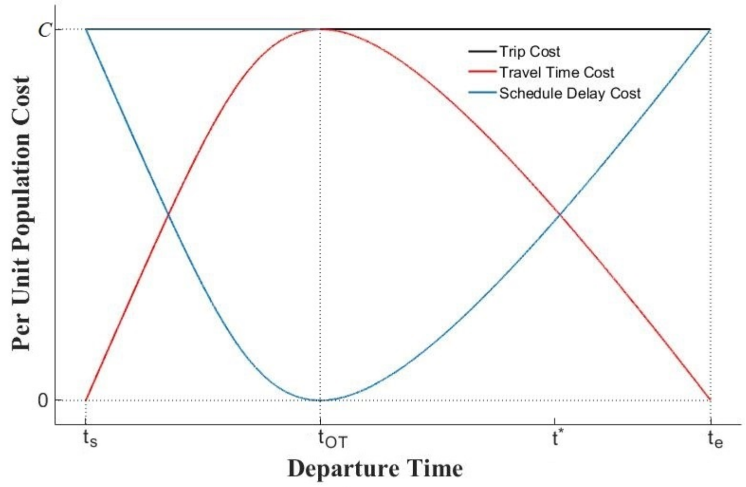

Figure 7. The trip cost, travel time cost and schedule delay cost for commuters with ESP in UE state.

4. The social optimum

4.1. The first-best, time-varying toll

Compared with UE state, SO state entails minimization to TTC. In this model, the travel time is exactly the queuing time at the bottleneck, which is a pure deadweight loss as pointed by Arnott et al. [5]. Therefore, we have the following inferences:

1) $TTTC$ is eliminated in SO state.

2) The schedule delay cost for the first and last commuters must be the same in SO state.

The elimination of queueing at the bottleneck requires the cumulative departure curve to coincide with the cumulative arrival curve. From the second inference, the duration of the morning rush hour and the cumulative arrival curve must be identical in UE and SO states to minimize TTC. The above properties in SO state are shown in Figure 8.

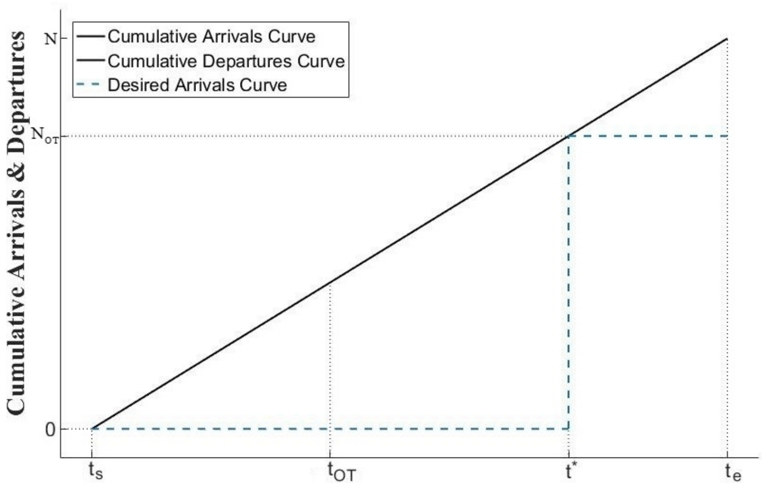

Figure 8. The cumulative arrival and departure curves, $A(t)$ and $Q(t)$, with ESP in SO state.

In morning rush hour, SO assignment can use a toll scheme, which is referred to as the first-best, time-varying toll, to fully eliminate queueing at the bottleneck. With this time-dependent toll, all commuters will pay for the marginal cost caused by themselves in congestion, which is exactly the travel time cost they would entail in UE state. Let $\tau(u)$ denote the congestion toll for commuters arriving at time $u = t + T(t)$. Then, we can rewrite $C(t)$ with this toll scheme as

$$C(t) = \begin{cases} \tau(u) + \int_{t+T(t)}^{t^*} \beta(v) dv \,, t + T(t) < t^* \\ \tau(u) + \int_{t^*}^{t+T(t)} \beta(v) dv \,, t + T(t) > t^* \end{cases}. \tag{19}$$

Since the first-best, time-varying toll scheme will make commuters have identical trip cost as that in UE state, which is given in Eq (16a). We can combine Eqs (16a) and (19) to have $C(t) = C(t_s)$, and then solve for $\tau(u)$ directly. This toll scheme can be expressed as the following relationship, and we have also depicted it in Figure 9.

$$\tau(u) = p\left[u - \frac{e^{\eta(u-t^*)}}{\eta} + \frac{N}{s(e^{\frac{\eta N}{s}} - 1)}\right]. \tag{20}$$

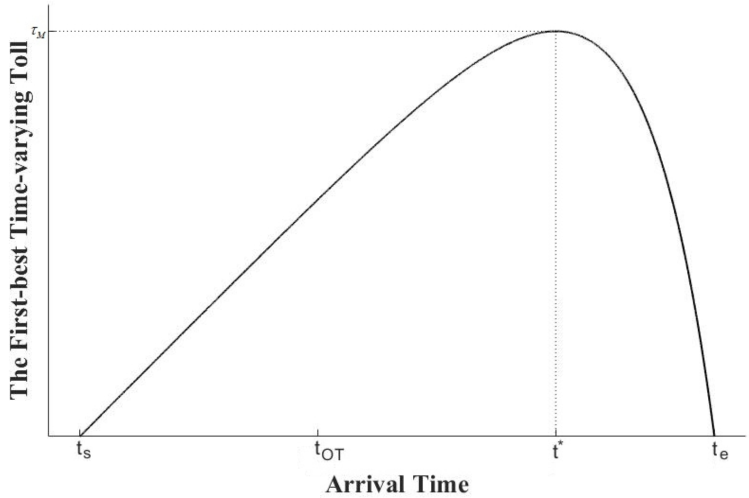

Figure 9. The first-best, time-varying toll, $\tau(u)$, with ESP in SO state.

4.2. *Economic properties*

The above inferences indicate that $TTTC$ is eliminated in SO state. The imposition of the best toll will result in a SO state that coincides with the UE state, which means $TSDC$ in SO state will not change compared to that in UE state. Then we know

$$TTTC_o = 0, \tag{21a}$$

$$TSDC_o = TSDC_e, \tag{21b}$$

where the subscript o implies the SO state.

Thus, according to the relationship $TTC = TTTC + TSDC$, we can obtain TTC for the traffic system in SO state as

$$TTC_o = TSDC_o = Np\left[t^* - \frac{1}{\eta} + \frac{N}{s(e^{\frac{\eta N}{s}} - 1)}\right] - TTTC_e. \tag{22}$$

Although the trip cost for commuters in SO state with the above toll scheme is the same as that in UE state, the SO assignment brings a social saving on $TTTC$. This means the pure deadweight loss caused by queueing at the bottleneck is transformed into the government's toll revenue, which can be employed for improving transportation infrastructures.

5. The optimal single-step toll

The first-best, time-varying toll has an ideal property to eliminate the bottleneck queuing and reach the SO state. However, this time-dependent toll scheme is hard to implement in practice. Therefore, the step-toll strategies, especially the single-step toll, are widely used as an alternative to the time-dependent toll scheme to alleviate traffic congestion. This section aims to propose an optimal single-step toll scheme based on the ESP bottleneck model.

Referring to the step-tolling strategies in Arnott et al. [5], the single-step toll is designed as a scheme that will charge a fixed toll for commuters who arrive at the bottleneck in a fixed period. Compared with the UE state, this scheme can lower TTC to a certain extent and produce economic benefits. Let ρ denote the fixed toll, t^+ and t^- denote the start time and end times for the scheme respectively. Therefore, to obtain the optimal single-step toll, we should determine values of ρ, t^+ and t^- which can minimize TTC.

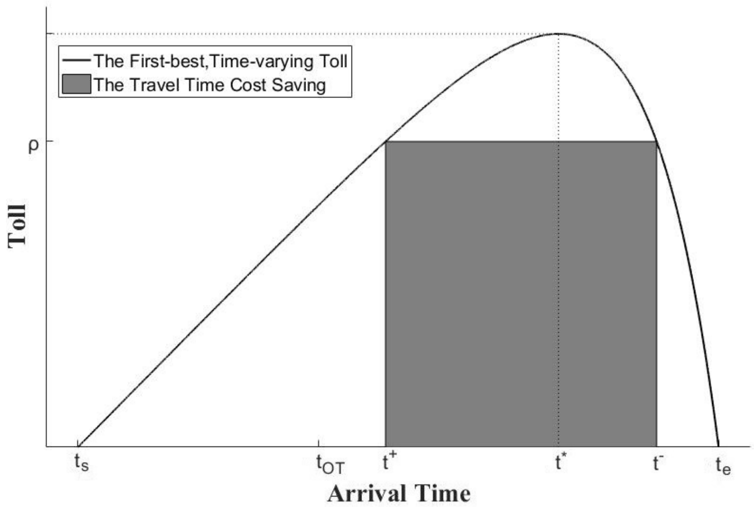

Figure 10. The optimal single-step toll inscribed the first-best, time-varying toll in the ESP bottleneck model.

As shown in Figure 10, the dark rectangular area and the area under the first-best, time-varying toll curve represent the travel time cost saved by the single-step toll scheme and $TTTC$ respectively. Let $F(t^+, t^-)$ denote the travel time cost saving and imposition of the optimal single-step toll should maximize the travel time cost saving, which intuitively is equivalent to the maximization of the dark gray area. Thus, we can express the objective for the optimal single-step toll as maximizing

$$F(t^+, t^-) = \rho s(t^- - t^+). \tag{23}$$

To minimize the total travel time, the bottleneck capacity should not be wasted. Therefore, the fixed toll, toll start and end times must be jointly optimized to eliminate the travel time at t^+ and t^- (i.e., fixed toll should equal the congestion toll of the first-best, time-varying toll at the two times). In this situation, the trip cost for commuters who depart at the two times merely consists the schedule delay cost and congestion toll. We express the above relationships as

$$\rho = \tau(t^+) = \tau(t^-) = p\left[t^+ - \frac{e^{\eta(t^+ - t^*)}}{\eta} + \frac{N}{s(e^{\frac{\eta N}{s}} - 1)}\right], \tag{24a}$$

$$C(t^+) = p\left[\frac{1}{\eta}(e^{\eta(t^+ - t^*)} - 1) - t^+ + t^*\right] + \rho, \tag{24b}$$

$$C(t^-) = p\left[\frac{1}{\eta}(e^{\eta(t^- - t^*)} - 1) - t^- + t^*\right] + \rho. \tag{24c}$$

With the single-step toll scheme, the equilibrium trip cost is same as that in UE state. Combine Eqs (24b) and (24c), we can express the relationship between t^+ and t^- as

$$t^- - t^+ = \frac{N}{s} \cdot \frac{e^{\eta t^-} - e^{\eta t^+}}{e^{\frac{\eta N}{s}} - 1}. \tag{25}$$

Equation (25) indicates the travel time cost saving $F(t^+, t^-)$ can be transformed into a function of the toll start time or the toll end time. Though it is difficult to solve Eq (25) for t^+ or t^- analytically, we can find optimal values of t^+ or t^- via numerical methods. Then from Eqs (24) and (25), the corresponding ρ and other time values of this toll scheme can be obtained.

6. Numerical examples

This section conducts numerical studies to demonstrate the properties of the proposed ESP bottleneck model. Each subsection also numerically compares ESP and DSP bottleneck models to intuitively show the difference between the two models. The results for the DSP bottleneck model can refer to Arnott et al. [5].

6.1. Values of parameters

To better show the influence of ESP, model parameters are set to make the equilibrium trip cost for commuters and the start time of the morning rush hour identical for both ESP and DSP bottleneck models. We adopt parameters in the DSP bottleneck model as Xiao et al. [13]: $\alpha = 6.4$ \$/h, $\beta = 3.0$ \$/h, $\gamma = 8.5$ \$/h and $t^* = 9{:}00$ am, $N = 6000$ veh, $s = 3000$ veh/h. In addition, the desired arrival time, the total number of commuters and the bottleneck capacity are assumed to be identical in ESP and DSP bottleneck models. According to the above conditions, the parameters p and η become endogenous and can be solved. With Eqs (15a) and (16a), we have $p = 3.6134$ \$/h, $\eta = 3.9736\ h^{-1}$.

6.2. User equilibrium

Figure 11 depicts the departure rate curve in UE state with ESP and DSP bottleneck models. We can find from Figure 11 that the departure rate in the ESP bottleneck model monotonically decreases throughout the morning rush hour. The queue at the bottleneck develops if the departure rate is larger than the bottleneck capacity. This queue begins to dissipate until the departure rate reduces to the bottleneck capacity and will completely dissipate at the end of the morning rush hour. In addition, Figure 11 also shows the departure rate curve changes continuously in the ESP bottleneck model, while this curve has a discontinuity at the time that commuters can arrive at the workplace punctually in the DSP bottleneck model. This is because ESP has made the unit schedule delay cost exponentially change rather than discontinuously change with DSP. It could be more reasonable to introduce ESP to avoid the abrupt change in the equilibrium departure rate.

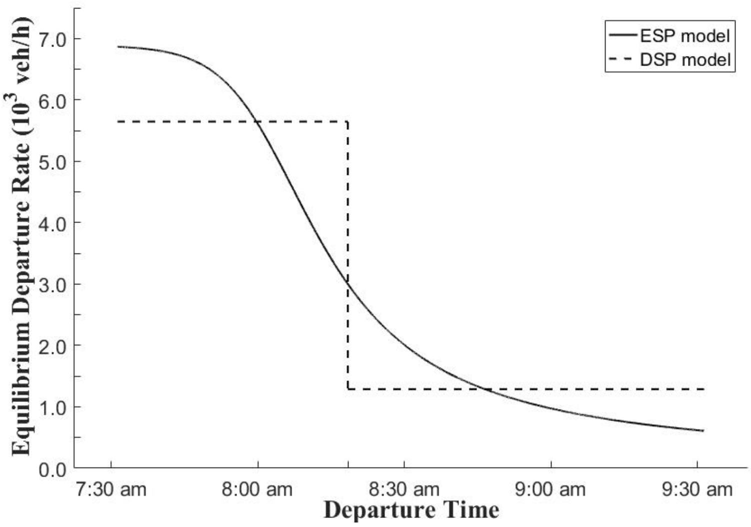

Figure 11. The equilibrium departure rate curve with ESP and DSP bottleneck models.

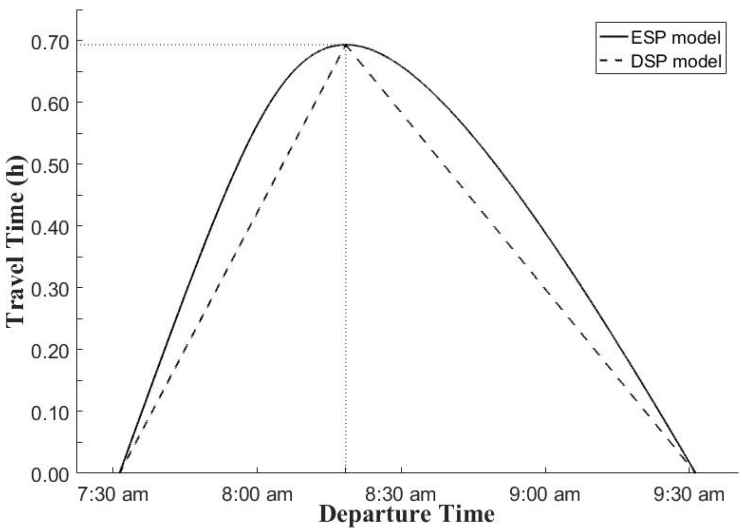

Figure 12. The equilibrium travel time curve with ESP and DSP bottleneck models.

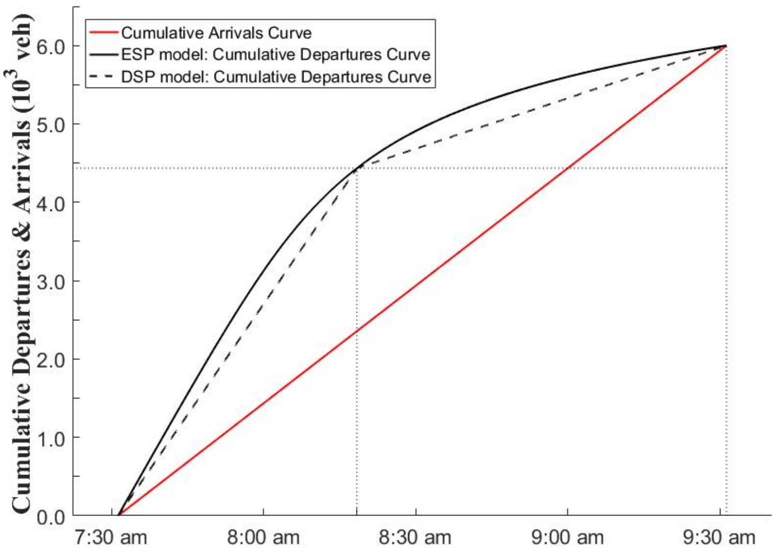

Figure 13. The equilibrium cumulative arrivals & departures curve with ESP and DSP bottleneck models.

Figures 12 and 13 respectively depict the travel time curve and the cumulative arrivals & departures curve in the UE state with ESP and DSP bottleneck models. It can be found from Figures 12 and 13 that non-differential points on the travel time curve and the cumulative departure curve can be perfectly eliminated by ESP, which makes the process of leaving home smoother in the morning commute. Figures 12 and 13 also show that the curves of equilibrium travel time and cumulative departures in the DSP bottleneck model are always below those in the ESP bottleneck model when commuters' trip cost and the morning rush hour are set to be identical in both two models. The above properties imply DSP underestimates the queueing time compared to ESP and most commuters will entail a higher travel time cost than in the DSP bottleneck model. The above two points imply that the travel behavior pattern of commuters in the morning rush hour is significantly influenced by the scheduling preferences, and the difference caused by ESP and other scheduling preferences should not be ignored.

In ESP, the physical meanings of the parameters p and η represent commuters' sensitivities to the schedule delay cost and schedule delay time, respectively. Figure 14 presents the influence of p and η on the travel time of commuters in the ESP bottleneck model. When the values of p and η grow larger, the travel time commuters will experience in the morning commute also becomes larger. In addition, Figure 14(a) shows that p will merely influence the travel time and travel behavior pattern of commuters and will not change the start and end times of the morning rush hour. However, η can influence both the travel time of commuters and the start and end times of the morning rush hour. As shown in Figure 14(b), the larger the value of η, the earlier the morning rush hour start time is. This implies that an increase of the value of η will make commuters choose an earlier departure time to arrive at the workplace early as far as possible for avoiding the large schedule delay late cost. Another set of results when values of p and η become smaller can also be obtained with a similar analytical process.

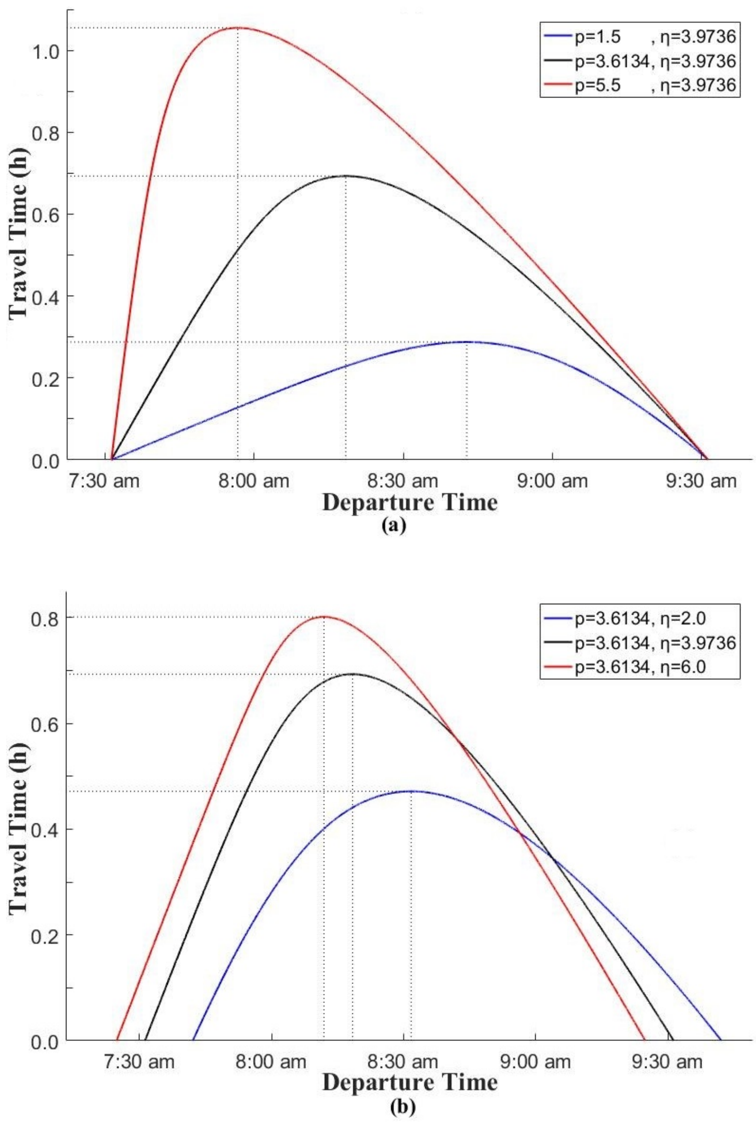

Figure 14. The influence of parameters p (a) and η (b) on the travel time with ESP bottleneck model.

6.3. The social optimum

Figure 15 depicts the time-dependent toll in the SO state with ESP and DSP bottleneck models. The toll for the first and the last commuters departing from home in the morning rush hour equals zero since they impose no external cost on the other commuters. We can find that ESP makes this curve nonlinear, continuous and differentiable. However, at the time that commuters can arrive at the workplace punctually, this toll curve becomes piecewise-linear and non-differentiable with DSP. Furthermore, we can also see that the first-best, time-varying toll curve with ESP is always above that with DSP. This implies that DSP might underestimate the travel time in the morning commute. Thus, the significance of the first-best, time-varying toll scheme is underestimated and the imposition of this toll in practice will bring more benefits than those in previous studies.

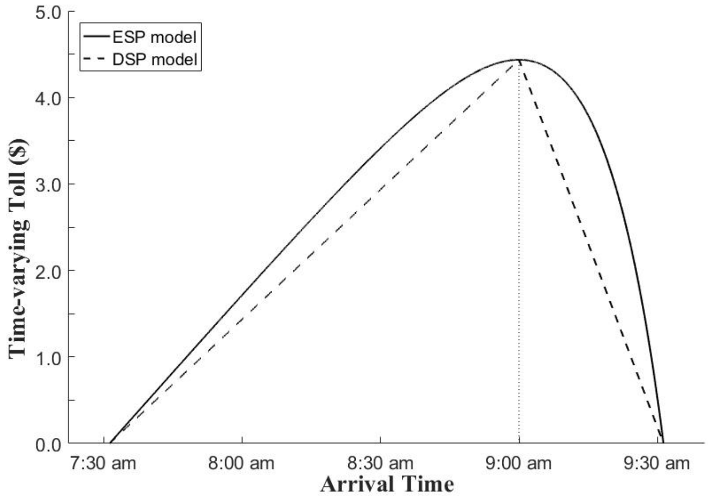

Figure 15. The first-best, time-varying toll curve with ESP and DSP bottleneck models.

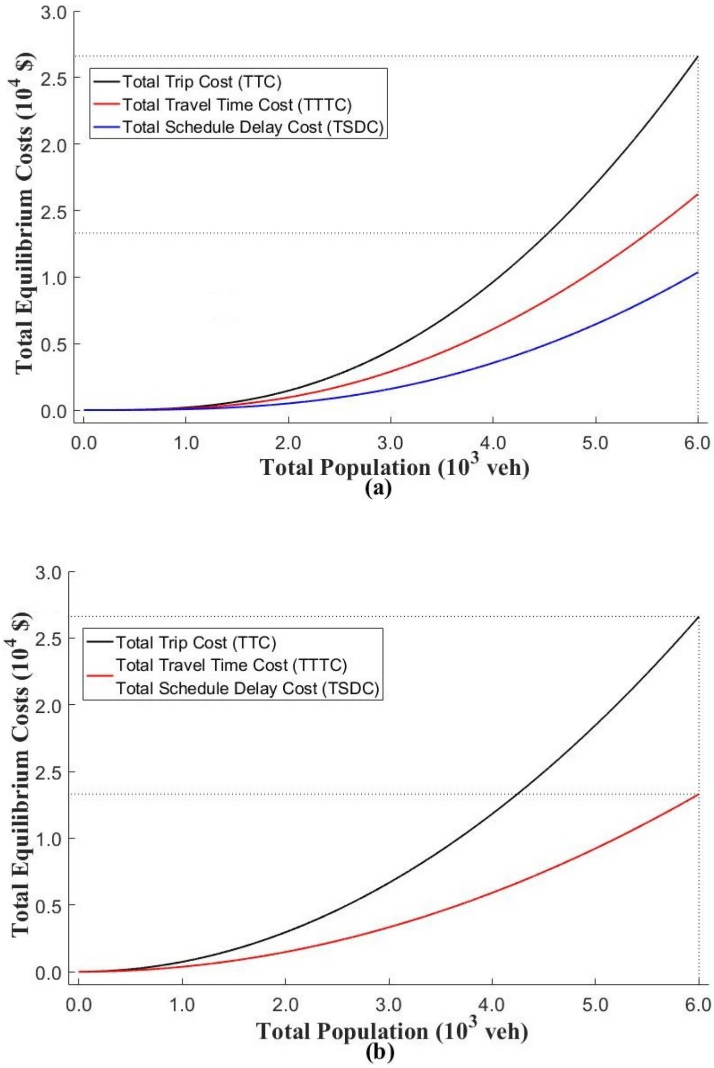

Figure 16. TTC, $TTTC$ and $TSDC$ as functions of the total population with ESP (a) and DSP (b) bottleneck model.

We can describe the effect of the first-best, time-varying toll scheme as the proportion of the travel time cost it eliminates in TTC. Figure 16 depicts the structure of commuters' trip cost in the UE state as a function of the total population with ESP and DSP bottleneck models. In the ESP bottleneck model, we can find $TTTC$ is always larger than $TSDC$ for any choice of total commuters, while those in the classical DSP bottleneck model (as in Arnott et al. [5]) are always equal. The first-best, time-varying toll has a property that can eliminate $TTTC$ in UE assignment and doesn't change $TSDC$. Thus, with the given parameters we know the efficiency of the first-best, time-varying toll in the proposed ESP bottleneck model and classical DSP bottleneck model respectively are 61.91% and 50%. This further indicates the efficiency of the first-best, time-varying toll scheme is underestimated in the DSP bottleneck model and the imposition of this toll in practice will bring more benefits than previous studies show.

6.4. The optimal single-step toll

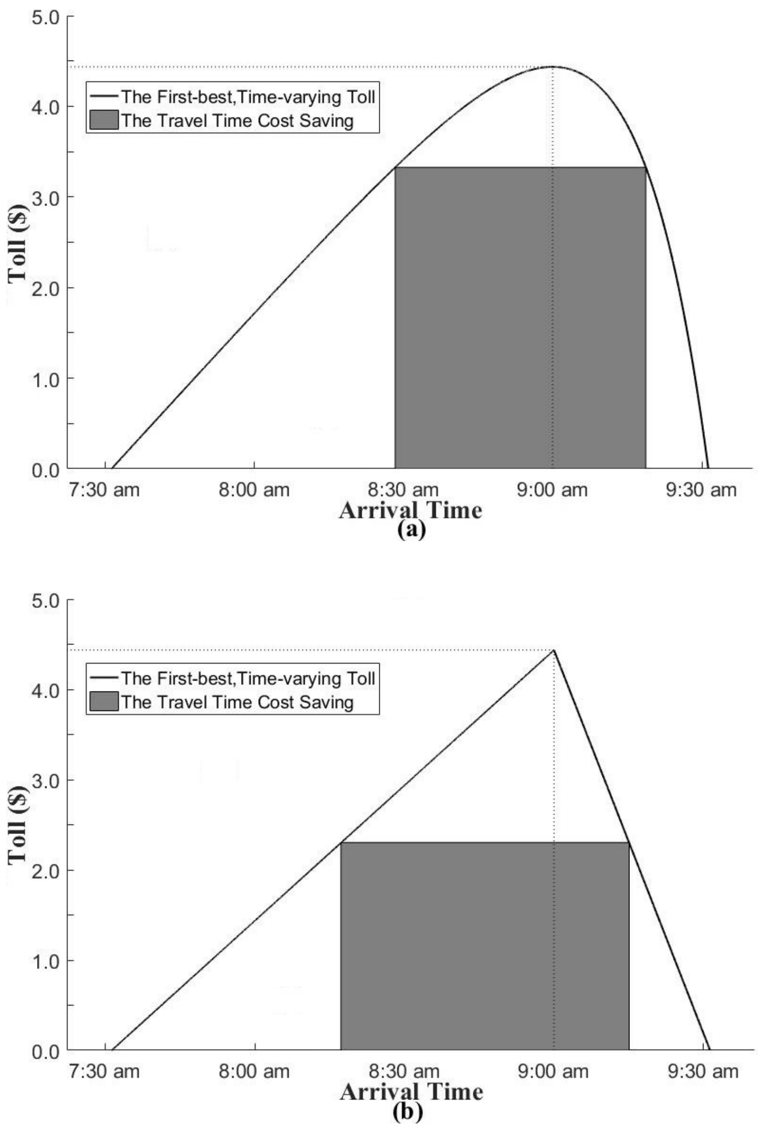

Figure 17. The optimal single-step toll scheme in ESP (a) and DSP (b) bottleneck model.

In the ESP bottleneck model, we can numerically obtain the optimal single-step toll scheme with the given parameters and the method in Section 5. Figure 17 depicts the optimal single-step toll strategy with ESP and DSP bottleneck models. We can find that ESP makes the start and end times of the optimal single-step toll later than those with DSP. In the ESP bottleneck model, the optimal single-step toll scheme begins at 8:28 am and finishes at 9:19 am while this scheme begins at 8:17 am and finishes at 9:15 am in the DSP bottleneck model. This means the scheme lasts for 51 minutes with ESP, which is 7 minutes shorter than that with DSP. However, the optimal toll is 3.326$ in ESP bottleneck model, which is much larger than the optimal toll of 2.303$ in the DSP bottleneck model. These results indicate DSP may overestimate the duration of the optimal toll and underestimate the value of the optimal toll.

Besides, the total toll revenue in the ESP bottleneck model is 8382.52$ and in the DSP bottleneck model is 6655.44$. This result means DSP may underestimate the total toll revenue of the government. However, the efficiency of the optimal single-step toll scheme in the ESP bottleneck model is 51.62%, which is close to the efficiency of 50% in the DSP bottleneck model as the equilibrium costs for the two models under optimal single-step toll are different.

7. Conclusions

This paper formulates a bottleneck model with ESP assumption and fully investigates the influence of ESP on the travel behavior of commuters and congestion tolling schemes. ESP assumes that the unit schedule delay cost for early or late exponentially changes with time. With this scheduling preference, we analytically derive solutions and corresponding economic properties of UE and SO states based on the bottleneck model. The equilibrium travel time, departure pattern and corresponding scheduling cost of commuters are obtained. Also, the critical clock times in the ESP bottleneck model are given. Besides, the first-best, time-varying toll and the optimal single-step toll scheme with ESP are also studied to alleviate the traffic congestion in the morning rush hour. Finally, to show the properties of ESP, several numerical examples of the ESP bottleneck model are presented and compared with the DSP bottleneck model.

In conclusion, ESP can transform commuters' equilibrium travel time and cumulative departures functions from linear to nonlinear, and perfectly eliminate the non-differentiability in these two functions and discontinuity in the equilibrium departure rate function given by the bottleneck model. This makes the equilibrium travel behavior pattern of commuters smooth and indicates that the scheduling preference has a significant influence on commuters' travel behavior pattern in the morning rush hour. In addition, the DSP bottleneck model also underestimated $TTTC$ and efficiency of the first-best, time-varying toll scheme, which means the imposition of this toll in practice will bring more benefits than previous studies imply. However, the efficiency of the optimal single-step toll scheme is with little difference in both DSP and ESP bottleneck models. This paper assumes commuters in the morning commute are homogeneous, which means the unit costs of travel time and schedule delay are identical for all commuters. Thus, the introduction of heterogeneity in the bottleneck model is a significant future research direction to extend the ESP bottleneck model. Another valuable future research is to consider ESP with flow congestion models in a traffic corridor, which can better reflect the features of traffic congestion in practice.

Acknowledgments

This research was supported by grants from the National Natural Science Foundation of China [71801227, 72271248, 72201285], the National Key R&D Program of China [2020YFB1600400], the Higher-end Think-Tank Project of Central South University [2022znzk07], the Fundamental Research Funds for the Central Universities of Central South University [2022ZZTS0721] and the Fundamental Research Funds for the Central Universities, Sun Yat-sen University [22qntd1701].

Conflicts of interest

The authors declare that they have no competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix

A.1. Proof of Proposition 1

Proof. According to the expression of t^* and the condition that parameters p and η satisfy $p > 0$ and $\eta > 0$, the first-order derivatives of t^* to η and p including their properties can be expressed as follows.

$$\frac{\partial t^*}{\partial \eta} = \frac{1}{\eta^2} ln \frac{\eta N}{s(e^{\frac{\eta N}{s}} - 1)} + \frac{s - se^{\frac{\eta N}{s}} + \eta N e^{\frac{\eta N}{s}}}{\eta^2 s(e^{\frac{\eta N}{s}} - 1)} > 0, \tag{A1}$$

$$\frac{\partial t^*}{\partial p} = 0. \tag{A2}$$

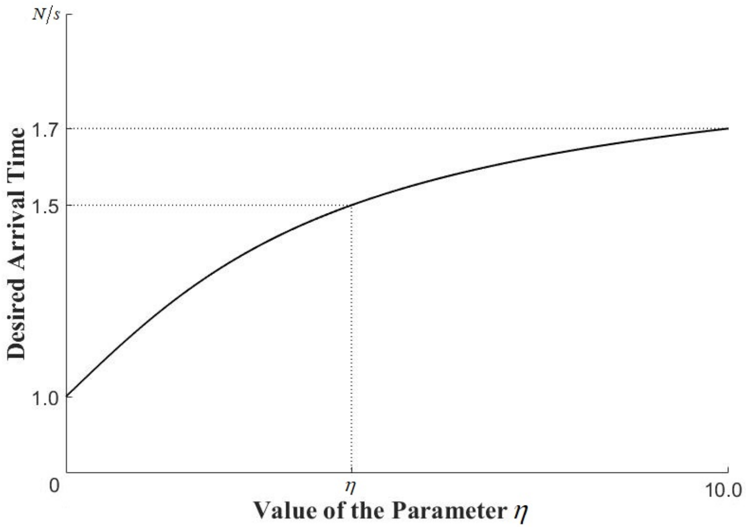

Figure A1. The relationship between desired arrival time t^* and the value of the parameter η.

From $\frac{\partial t^*}{\partial \eta} > 0$ and $\frac{\partial t^*}{\partial p} = 0$, we thus can easily obtain that the desired arrival time t^* monotonically increases with η and is independent of p. The relationship between t^* and η is also intuitively shown in Figure A1. Combine with Eq (15), we then can find that the increase of the desired arrival time t^* will lead to a decrease for the start time of the morning rush hour t_s. This completes the proof.

A.2. Proof of Proposition 2

Proof. In UE state, the bottleneck will be fully utilized and the arrival rates are s throughout the morning rush hour. Thus, the total early arrivals and late arrivals can be expressed as follows.

$$\text{Total early arrivals} = st^*. \tag{A3}$$

$$\text{Total late arrivals} = N - st^*. \tag{A4}$$

The above relationships indicate that the total early and late arrivals are merely depended on the desired arrival time t^*, which is monotonically increasing with respect to η and independent of p. Thus, the total early and late arrivals will strictly monotonically increase and decrease concerning η respectively. This completes the proof.