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Research article

Weyl almost anti-periodic solution to a neutral functional semilinear differential equation

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Abstract: In this work, we first propose a concept of Weyl almost anti-periodic functions. Then, we make use of the contraction mapping principle and analysis techniques to research the existence of a unique Weyl almost anti-periodic solution to a neutral functional semilinear abstract differential equation. Finally, we give an example of a neutral functional partial differential equation to show the validity of the obtained results.

Keywords: Weyl almost anti-periodicity; neutral functional differential equation; semilinear abstract differential equation

1. Introduction

In nature, human society, engineering technology and other fields, periodic and anti-periodic fluctuations are widespread. Differential equation is an important mathematical model to describe the phenomena and processes in these fields. Therefore, the problem of periodic and anti-periodic solutions of differential equations has always been a focus in the field of qualitative research of differential equations, whether in terms of theoretical research or practical application [1–5]. Since H. Bohr [6] introduced the almost periodic function as a natural extension of periodic function into mathematics, almost periodic solutions to differential equations [7–11] and mathematical models described by differential equations [12–15] has also become an important problem. Recently, the concepts of Bohr and Stepanov almost anti-periodic functions were proposed in [16]. As we know, Weyl almost periodic function is an extension of Bohr almost periodic function and Stepanov almost periodic function [15, 17, 18], but unlike them, the space formed by Weyl almost periodic function is incomplete under Weyl seminorm [9, 18]. As a result, it is a difficult and interesting problem to investigate Weyl almost periodic solutions to differential equations and has always attracted the interest of many scholars. Therefore, it is also meaningful and challenging to introduce a definition of Weyl almost anti-periodic function and study solutions of such functions to differential equations and dynamical systems.

Inspired by the above analysis and observation, and considering that the semilinear abstract neutral functional differential equation includes many mathematical models as its special cases [19]. The primary purpose of this work is to introduce a definition of Weyl almost anti-periodic function, and then investigate the existence of a unique Weyl almost anti-periodic solution to a semilinear abstract neutral functional differential equation.

The remainder of this work is structured as follows: In Section 2, we propose a notion of Weyl almost anti-periodic function. In Section 3, we discuss the existence of a unique Weyl almost anti-periodic solution for a neutral functional semi-linear differential equation and we provide an example to show the validity of our result. In Section 4, we provide a brief conclusion.

2. Weyl almost anti-periodic function

Let $(\mathbb{B}, \|\cdot\|)$ denote a Banach space. For $p \geq 1, f \in L^p_{loc}(\mathbb{R}, \mathbb{B})$, the Weyl seminorm of f is defined as:

$$\|f\|_{W^p} = \lim_{T\to+\infty} \sup_{a\in\mathbb{R}} \left(\frac{1}{T}\int_a^{a+T} \|f(t)\|^p dt\right)^{\frac{1}{p}}.$$

Definition 2.1. *[9] Function $f \in L^p_{loc}(\mathbb{R}, \mathbb{B})$ is called a p-th Weyl almost periodic function, if for each $\epsilon > 0$, it is possible to find a constant $l = l(\epsilon) > 0$ such that each interval of length $l(\epsilon)$ contains a point $\tau \in \mathbb{R}$ satisfying*

$$\|f(\cdot + \tau) - f(\cdot)\|_{W^p} < \epsilon.$$

We will denote the collection of such functions by $W^pAP(\mathbb{R}, \mathbb{B})$.

Remark 2.1. *According to [9], for $x \in W^pAP(\mathbb{R}, \mathbb{B})$, we have $\|x\|_{W^p} < \infty$.*

Definition 2.2. *A function $f \in L^p_{loc}(\mathbb{R}, \mathbb{B})$ is called a p-th Weyl almost anti-periodic function, if for any $\epsilon > 0$, it is possible to find an $l = l(\epsilon) > 0$ such that each interval with length $l(\epsilon)$ contains at least one $\tau = \tau(\epsilon) \in \mathbb{R}$ such that*

$$\|f(\cdot + \tau) + f(\cdot)\|_{W^p} < \epsilon.$$

We will use $W^pANP(\mathbb{R}, \mathbb{B})$ to stand for the space of all such functions.

Definition 2.3. *Let $f \in L^p_{loc}(\mathbb{R}\times\mathbb{B}, \mathbb{B})$, then it is called p-th Weyl almost anti-periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathbb{B}$, if for any $\epsilon > 0$ and each compact subset $\mathbb{K}$ of $\mathbb{B}$, it is possible to find an $l = l(\epsilon, \mathbb{K}) > 0$ such that each interval with length l contains at least one $\tau = \tau(\epsilon) \in \mathbb{R}$ satisfying*

$$\|f(\cdot + \tau, x) + f(\cdot, x)\|_{W^p} < \epsilon$$

uniformly in $x \in \mathbb{K}$. We will denote the collection of all such functions by $W^pANP(\mathbb{R} \times \mathbb{B}, \mathbb{B})$.

Example 2.1. (1^0) *If $f_1(t) = \cos(\pi t) + \cos(\sqrt{2}\pi t) + \frac{1}{t^2+1}$, then one can easily check that $f_1(t)$ is not periodic, not anti-periodic, nor almost anti-periodic but Weyl almost anti-periodic.*

(2^0) *Let $f_2(t) = \cos(\pi t) + \cos(\sqrt{2}\pi t) + \frac{1}{t^2+1} + 3$, then one can easily show that $f_2 \in W^pAP(\mathbb{R}, \mathbb{B})$, but $f_2 \notin W^pANP(\mathbb{R}, \mathbb{B})$.*

Example 2.2. *Take* $f_3(t) = 2\cos(2t) - \frac{1}{1+t^2}$ *and* $f_4(t) = -\cos(4t) + \frac{1}{1+t^2}$*, then one can easily show that* $f_3(t)$ *and* $f_4(t)$ *are Weyl almost anti-periodic. Since* $f_3(t) + f_4(t) = 1$ *for all* $t \in \mathbb{R}$*,* $\|f_3(\cdot + \tau) + f_4(\cdot + \tau) + f_3(\cdot) + f_4(\cdot)\|_{W^p} = 2 > 0$ *for every* $\tau \in \mathbb{R}$*. Hence,* $f_3(t) + f_4(t)$ *is not Weyl almost anti-periodic.*

Remark 2.2. *Example 2.1 shows that* $W^pANP(\mathbb{R}, \mathbb{B})$ *is a proper subset of* $W^pAP(\mathbb{R}, \mathbb{B})$*. Example 2.2 shows that* $W^pANP(\mathbb{R}, \mathbb{B})$ *does not form a linear space under usual linear operations.*

Remark 2.3. *Based on Definition 2.1 in [16] and Definition 2.2, one can conclude that an anti-periodic function is an almost anti-periodic function and an almost anti-periodic function is also a Weyl almost anti-periodic function, but the converse is not true.*

3. Weyl almost anti-periodic mild solution

The neutral functional semilinear differential equation we are concerned in this paper is as follow:

$$\frac{d}{dt}[u(t) - f(u(t-\theta))] = Au(t) + g(t, u(t-\xi))), \quad t \geq t_0, \tag{3.1}$$

in which, A is the infinitesimal generator of a C_0-semigroup $\{T(t) : t \geq 0\}$ on Banach space $\mathbb{B}$, $f \in C(\mathbb{B}, \mathbb{B})$, $g : \mathbb{R} \times \mathbb{B} \to \mathbb{B}$ is a measurable function, $\theta, \xi > 0$ are constants.

The initial value imposed on system (3.1) is as follow:

$$x(s) = \varphi(s), \quad \varphi \in C([t_0 - \eta, t_0], \mathbb{B}), \quad \eta = \max\{\theta, \xi\}. \tag{3.2}$$

Definition 3.1. *Function* $v : \mathbb{R} \to \mathbb{B}$ *is said to be a mild solution of* (3.1) *and* (3.2) *if it satisfies the initial value condition* (3.2) *and the following equation*

$$\begin{aligned} v(t) =& T(t-t_0)[v(t_0) - f(v(t_0 - \theta))] + f(v(t-\theta)) \\ &+ \int_{t_0}^{t} AT(t-s) f(v(s-\theta))ds + \int_{t_0}^{t} T(t-s) g(s, v(s-\xi))ds \end{aligned}$$

for all $t \in \mathbb{R}$ *with* $t \geq t_0$.

In order to gain our main result, we assume that:

(H_1) There are two positive constants M, ζ satisfying $\|T(t)\| \leq Me^{-\zeta t}$.

(H_2) The function $\varpi \to AT(\varpi)$ defined on $[0, \infty)$ is a strongly measurable one and there exists a nonincreasing function $J : [0, \infty) \to [0, \infty)$ with $e^{-\zeta \cdot} J(\cdot) \in L^1([0, \infty), [0, \infty))$ satisfying

$$\int_0^{+\infty} e^{-\zeta\varpi} J(\varpi) dh \leq K_1, \quad \|AT(\varpi)\| \leq e^{-\zeta\varpi} J(\varpi), \quad \varpi \geq 0.$$

(H_3) Function $f \in C(\mathbb{B}, \mathbb{B}), g \in W^pANP(\mathbb{R} \times \mathbb{B}, \mathbb{B})$ and there are constants $L^f, L^g > 0$ such that for all $x, y \in \mathbb{B}$ and $t \in \mathbb{R}$,

$$\|f(x) \pm f(y)\| \leq L^f \|x \pm y\|, \quad \|g(t, x) \pm g(t, y)\| \leq L^g \|x \pm y\|.$$

Moreover, $f(0) = g(t, 0) = 0$.

(H$_4$) The constant $\widetilde{M} := L^f + K_1 L^f + \frac{ML^g}{\zeta}$ satisfies $\frac{2ML^g}{\zeta} < \widetilde{M} < 1$, where M is mentioned in (H$_1$), K_1 is mentioned in (H$_2$), ζ is mentioned in (H$_1$) and (H$_2$).

Let $L^\infty(\mathbb{R},\mathbb{B})$ denote the space of all essentially bounded and measurable functions from $\mathbb{R}$ to $\mathbb{B}$. Then, it is a Banach space when endowed with the norm $\|x\|_\infty = \operatorname{ess}\sup_{t\in\mathbb{R}} \|x(t)\|$ for $x \in L^\infty(\mathbb{R},\mathbb{B})$.

Lemma 3.1. *If $\Theta \in L^p_{loc}(\mathbb{R},\mathbb{B})$ with $\|\Theta\|_{W^p} < \infty$, then one has*

$$\left\| \int_0^{+\infty} e^{-\zeta\varpi}\Theta(\cdot - \varpi)d\varpi \right\|_{W^p} \le \zeta^{-1}\|\Theta\|_{W^p}$$

and

$$\left\| \int_0^{+\infty} e^{-\zeta\varpi}J(\varpi)\Theta(\cdot - \varpi)d\varpi \right\|_{W^p} \le K_1\|\Theta\|_{W^p}.$$

Proof. By Fubini's theorem, we deduce that

$$\begin{aligned}
&\left\| \int_0^{+\infty} e^{-\zeta\varpi}\Theta(\cdot - \varpi)d\varpi \right\|_{W^p}\\
=&\lim_{T\to+\infty}\sup_{a\in\mathbb{R}}\left[\frac{1}{T}\int_a^{a+T}\left(\int_0^{+\infty} e^{-\frac{\zeta\varpi}{p}}e^{-\frac{(p-1)\zeta\varpi}{p}}\|\Theta(t-\varpi)\|d\varpi\right)^p dt\right]^{\frac{1}{p}}\\
\le&\lim_{T\to+\infty}\sup_{a\in\mathbb{R}}\left[\frac{1}{T}\int_a^{a+T}\int_0^{+\infty} e^{-\zeta\varpi}\|\Theta(t-\varpi)\|^p d\varpi\left(\int_0^{+\infty} e^{-\zeta\varpi}d\varpi\right)^{p-1} dt\right]^{\frac{1}{p}}\\
\le&\zeta^{-\frac{p-1}{p}}\lim_{T\to+\infty}\sup_{a\in\mathbb{R}}\left[\frac{1}{T}\int_a^{a+T}\int_0^{+\infty} e^{-\zeta\varpi}\|\Theta(t-\varpi)\|^p d\varpi dt\right]^{\frac{1}{p}}\\
=&\zeta^{-\frac{p-1}{p}}\lim_{T\to+\infty}\sup_{a\in\mathbb{R}}\left[\frac{1}{T}\int_0^{+\infty} e^{-\zeta\varpi}\int_a^{a+T}\|\Theta(t-\varpi)\|^p dt d\varpi\right]^{\frac{1}{p}}\\
\le&\zeta^{-\frac{p-1}{p}}\lim_{T\to+\infty}\sup_{a+\varpi\in\mathbb{R}}\left[\int_0^{+\infty} e^{-\zeta\varpi}\frac{1}{T}\int_{a+\varpi}^{a+\varpi+T}\|\Theta(s)\|^p ds d\varpi\right]^{\frac{1}{p}}\\
=&\zeta^{-\frac{p-1}{p}}\left[\int_0^{+\infty} e^{-\zeta\varpi}\|\Theta\|^p_{W^p}d\varpi\right]^{\frac{1}{p}}\\
\le&\zeta^{-1}\|\Theta\|_{W^p}
\end{aligned}$$

and that

$$\begin{aligned}
&\left\| \int_0^{+\infty} e^{-\zeta\varpi}J(\varpi)\Theta(\cdot - \varpi)d\varpi \right\|_{W^p}\\
=&\lim_{T\to+\infty}\sup_{a\in\mathbb{R}}\left[\frac{1}{T}\int_a^{a+T}\left(\int_0^{+\infty} e^{-\frac{\zeta\varpi}{p}}J(\varpi)^{\frac{1}{p}}e^{-\frac{(p-1)\zeta\varpi}{p}}J(\varpi)^{\frac{p-1}{p}}\|\Theta(t-\varpi)\|d\varpi\right)^p dt\right]^{\frac{1}{p}}\\
\le&\lim_{T\to+\infty}\sup_{a\in\mathbb{R}}\left[\frac{1}{T}\int_a^{a+T}\int_0^{+\infty} e^{-\zeta\varpi}J(\varpi)\|\Theta(t-\varpi)\|^p d\varpi\left(\int_0^{+\infty} e^{-\zeta\varpi}J(\varpi)d\varpi\right)^{p-1} dt\right]^{\frac{1}{p}}\\
\le&K_1^{\frac{p-1}{p}}\lim_{T\to+\infty}\sup_{a\in\mathbb{R}}\left[\frac{1}{T}\int_a^{a+T}\int_0^{+\infty} e^{-\zeta\varpi}J(\varpi)\|\Theta(t-\varpi)\|^p d\varpi dt\right]^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&=K_1^{\frac{p-1}{p}} \lim_{T\to+\infty} \sup_{a\in\mathbb{R}} \left[\frac{1}{T} \int_0^{+\infty} e^{-\zeta\varpi} J(\varpi) \int_a^{a+T} \|\Theta(t-\varpi)\|^p dt d\varpi \right]^{\frac{1}{p}} \\
&\le K_1^{\frac{p-1}{p}} \lim_{T\to+\infty} \sup_{a+\varpi\in\mathbb{R}} \left[\int_0^{+\infty} e^{-\zeta\varpi} J(\varpi) \frac{1}{T} \int_{a+\varpi}^{a+\varpi+T} \|\Theta(s)\|^p ds d\varpi \right]^{\frac{1}{p}} \\
&=K_1^{\frac{p-1}{p}} \left[\int_0^{+\infty} e^{-\zeta\varpi} J(\varpi) \|\Theta\|_{W^p}^p d\varpi \right]^{\frac{1}{p}} \\
&\le K_1 \|\Theta\|_{W^p}.
\end{aligned}
$$

The proof is done.

Theorem 3.1. *Let assumptions* (H_1)–(H_3) *be fulfilled. Then system* (3.1) *admits one and only one Weyl almost anti-periodic solution in* $L^\infty(\mathbb{R}, \mathbb{B})$.

Proof. By Definition 3.1, it is clear that $x : \mathbb{R} \to \mathbb{B}$ is a solution to (3.1) if it meets the equation

$$
x(t) = f(x(t-\theta)) + \int_{-\infty}^{t} AT(t-s) f(x(s-\theta)) ds + \int_{-\infty}^{t} T(t-s) g(s, x(s-\xi)) ds. \tag{3.3}
$$

Define an operator $\Lambda : L^\infty(\mathbb{R}, \mathbb{B}) \to \mathbb{B}$ by

$$
\begin{aligned}
(\Lambda\phi)(t) =& f(\phi(t-\theta)) + \int_{-\infty}^{t} AT(t-s) f(\phi(s-\theta)) ds + \int_{-\infty}^{t} T(t-s) g(s, \phi(s-\xi)) ds \\
=& f(\phi(t-\theta)) + \int_0^{+\infty} AT(\varpi) f(\phi(t-\varpi-\theta)) d\varpi + \int_0^{+\infty} T(\varpi) g(t-\varpi, \phi(t-\varpi-\xi)) d\varpi,
\end{aligned}
$$

where $\phi \in L^\infty(\mathbb{R}, \mathbb{B})$.

First of all, we will confirm that $\Lambda(L^\infty(\mathbb{R}, \mathbb{B})) \subset L^\infty(\mathbb{R}, \mathbb{B})$. Indeed, for every $\phi \in L^\infty(\mathbb{R}, \mathbb{B})$, one gets

$$
\begin{aligned}
\|(\Lambda\phi)\|_\infty \le& \|f(\phi(\cdot-\theta))\|_\infty + \left\| \int_0^{+\infty} AT(\varpi) f(\phi(\cdot-\varpi-\theta)) d\varpi \right\|_\infty \\
&+ \left\| \int_0^{+\infty} T(\varpi) g(\cdot-\varpi, \phi(\cdot-\varpi-\xi)) d\varpi \right\|_\infty \\
\le& L^f \|\phi(\cdot-\theta)\|_\infty + L^f \left\| \int_0^{+\infty} e^{-\zeta\varpi} J(\varpi) \phi(\cdot-\varpi-\theta) d\varpi \right\|_\infty \\
&+ L^g \left\| \int_0^{+\infty} M e^{-\zeta\varpi} \phi(\cdot-\varpi-\xi) d\varpi \right\|_\infty \\
\le& L^f \|\phi\|_\infty + K_1 L^f \|\phi\|_\infty + \frac{ML^g}{\zeta} \|\phi\|_\infty.
\end{aligned}
$$

Hence, Λ is a self-mapping.

Next, we will demonstrate Λ is a contraction mapping. For any $x, y \in L^\infty(\mathbb{R}, \mathbb{B})$, one has

$$
\begin{aligned}
&\|(\Lambda x) - (\Lambda y)\|_\infty \\
\le& \|f(x(\cdot-\theta)) - f(y(\cdot-\theta))\|_\infty + \int_0^{+\infty} AT(\varpi) \|f(x(\cdot-\varpi-\theta)) - f(y(\cdot-\varpi-\theta))\|_\infty d\varpi
\end{aligned}
$$

$$+\int_0^{+\infty} T(\varpi)\|g(\cdot-\varpi,x(\cdot-\varpi-\xi))-g(\cdot-\varpi,y(\cdot-\varpi-\xi))\|_\infty d\varpi$$
$$\le L^f\|x(\cdot-\theta)-y(\cdot-\theta)\|_\infty+L^f\int_0^{+\infty}e^{-\zeta\varpi}J(\varpi)\|x(\cdot-\varpi-\theta)-y(\cdot-\varpi-\theta)\|_\infty d\varpi$$
$$+ML^g\int_0^{+\infty}e^{-\zeta\varpi}\|x(\cdot-\varpi-\xi)-y(\cdot-\varpi-\xi)\|_\infty d\varpi$$
$$\le\Big(L^f+K_1L^f+\frac{ML^g}{\zeta}\Big)\|x-y\|_\infty,$$

which, by (H_4), means that Λ is a contraction mapping. Consequently, we derive that (3.1) admits unique one mild solution $x^*\in L^\infty(\mathbb{R},\mathbb{B})$.

Lastly, we will demonstrate that the x^* is Weyl almost anti-periodic. By (3.3), one can infer that

$$x^*(t)=f(x^*(t-\theta))+\int_0^{+\infty}AT(\varpi)f(x^*(t-\varpi-\theta))d\varpi+\int_0^{+\infty}T(\varpi)g(t-\varpi,x^*(t-\varpi-\xi))d\varpi. \quad (3.4)$$

Since $g\in W^pANP(\mathbb{R}\times\mathbb{B},\mathbb{B})$, for every $\epsilon>0$ and each compact subset $\mathbb{K}$ of $\mathbb{B}$ that contains x^*, there is a constant $l=l(\epsilon,\mathbb{K})>0$ such that in each interval with length l contains at least one $\tau=\tau(\epsilon)\in\mathbb{R}$ satisfying

$$\|g(\cdot+\tau,x)+g(\cdot,x)\|_{W^p}<\epsilon$$

for all $x\in\mathbb{K}$.

On one hand, with the help of (3.4), one can deduce that

$$\|x^*(\cdot+\tau)-x^*(\cdot)\|_{W^p}$$
$$=\Big\|f(x^*(\cdot+\tau-\theta))-f(x^*(\cdot-\theta))+\int_0^{+\infty}AT(\varpi)(f(x^*(\cdot+\tau-\varpi-\theta))-f(x^*(\cdot-\varpi-\theta)))d\varpi$$
$$+\int_0^{+\infty}T(\varpi)(g(\cdot+\tau-\varpi,x^*(\cdot+\tau-\varpi-\xi))+g(\cdot+\tau-\varpi,x^*(\cdot-\varpi-\xi)))d\varpi$$
$$-\int_0^{+\infty}T(\varpi)(g(\cdot+\tau-\varpi,x^*(\cdot-\varpi-\xi))+g(\cdot-\varpi,x^*(\cdot-\varpi-\xi)))d\varpi\Big\|_{W^p}$$
$$\le L^f\|x^*(\cdot+\tau)-x^*(\cdot)\|_{W^p}+L^f\Big\|\int_0^{+\infty}e^{-\zeta\varpi}J(\varpi)(x^*(\cdot+\tau-\varpi-\theta)-x^*(\cdot-\varpi-\theta))d\varpi\Big\|_{W^p}$$
$$+ML^g\Big\|\int_0^{+\infty}e^{-\zeta\varpi}(x^*(\cdot+\tau-\varpi-\xi)+x^*(\cdot-\varpi-\xi))d\varpi\Big\|_{W^p}$$
$$+M\Big\|\int_0^{+\infty}e^{-\zeta\varpi}(g(\cdot+\tau-\varpi,x^*(\cdot-\varpi-\xi))+g(\cdot-\varpi,x^*(\cdot-\varpi-\xi)))d\varpi\Big\|_{W^p}$$
$$:=\Theta_1.$$

Furthermore, by virtue of Lemma 3.1, one can get that

$$\Theta_1<L^f\|x^*(\cdot+\tau)-x^*(\cdot)\|_{W^p}+K_1L^f\|x^*(\cdot+\tau-\varpi-\theta)-x^*(\cdot-\varpi-\theta)\|_{W^p}$$
$$+\frac{ML^g}{\zeta}\|x^*(\cdot+\tau-\varpi-\xi)+x^*(\cdot-\varpi-\xi)\|_{W^p}$$

$$
\begin{aligned}
&+\frac{M}{\zeta}\|g(\cdot+\tau-\varpi,x^*(\cdot-\varpi-\xi))+g(\cdot-\varpi,x^*(\cdot-\varpi-\xi))\|_{W^p}\\
&\leq(L^f+K_1L^f)\|x^*(\cdot+\tau)-x^*(\cdot)\|_{W^p}+\frac{ML^g}{\zeta}\|x^*(\cdot+\tau)+x^*(\cdot)\|_{W^p}+\frac{M}{\zeta}\epsilon. \qquad (3.5)
\end{aligned}
$$

On the other hand, we can derive that

$$
\begin{aligned}
&\|x^*(\cdot+\tau)+x^*(\cdot)\|_{W^p}\\
=&\Big\|f(x^*(\cdot+\tau-\theta))+f(x^*(\cdot-\theta))+\int_0^{+\infty}AT(\varpi)(f(x^*(\cdot+\tau-\varpi-\theta))+f(x^*(\cdot-\varpi-\theta)))d\varpi\\
&+\int_0^{+\infty}T(\varpi)(g(\cdot+\tau-\varpi,x^*(\cdot+\tau-\varpi-\xi))-g(\cdot+\tau-\varpi,x^*(\cdot-\varpi-\xi)))d\varpi\\
&+\int_0^{+\infty}T(\varpi)(g(\cdot+\tau-\varpi,x^*(\cdot-\varpi-\xi))+g(\cdot-\varpi,x^*(\cdot-\varpi-\xi)))d\varpi\Big\|_{W^p}\\
\leq&L^f\|x^*(\cdot+\tau)+x^*(\cdot)\|_{W^p}+L^f\Big\|\int_0^{+\infty}e^{-\zeta\varpi}J(\varpi)(x^*(\cdot+\tau-\varpi-\theta)+x^*(\cdot-\varpi-\theta))d\varpi\Big\|_{W^p}\\
&+ML^g\Big\|\int_0^{+\infty}e^{-\zeta\varpi}(x^*(\cdot+\tau-\varpi-\xi)-x^*(\cdot-\varpi-\xi))d\varpi\Big\|_{W^p}\\
&+M\Big\|\int_0^{+\infty}e^{-\zeta\varpi}(g(\cdot+\tau-\varpi,x^*(\cdot-\varpi-\xi))+g(\cdot-\varpi,x^*(\cdot-\varpi-\xi)))d\varpi\Big\|_{W^p}\\
:=&\Theta_2,
\end{aligned}
$$

then again by Lemma 3.1, we can get that

$$
\begin{aligned}
\Theta_2<&L^f\|x^*(\cdot+\tau)+x^*(\cdot)\|_{W^p}+K_1L^f\|x^*(\cdot+\tau-\varpi-\theta)+x^*(\cdot-\varpi-\theta)\|_{W^p}\\
&+\frac{ML^g}{\zeta}\|x^*(\cdot+\tau-\varpi-\xi)-x^*(\cdot-\varpi-\xi)\|_{W^p}\\
&+\frac{M}{\zeta}\|g(\cdot+\tau-\varpi,x^*(\cdot-\varpi-\xi))+g(\cdot-\varpi,x^*(\cdot-\varpi-\xi))\|_{W^p}\\
\leq&(L^f+K_1L^f)\|x^*(\cdot+\tau)+x^*(\cdot)\|_{W^p}+\frac{ML^g}{\zeta}\|x^*(\cdot+\tau)-x^*(\cdot)\|_{W^p}+\frac{M}{\zeta}\epsilon. \qquad (3.6)
\end{aligned}
$$

Consequently, from (3.5) and (3.6) it follows that

$$
\begin{aligned}
\|x^*(\cdot+\tau)+x^*(\cdot)\|_{W^p}<&\frac{ML^g}{\zeta}(1-L^f-K_1L^f)^{-1}\|x^*(\cdot+\tau)-x^*(\cdot)\|_{W^p}\\
&+\frac{M}{\zeta}(1-L^f-K_1L^f)^{-1}\epsilon\\
<&\frac{ML^g}{\zeta}(1-L^f-K_1L^f)^{-2}\Big(\frac{ML^g}{\zeta}\|x^*(\cdot+\tau)+x^*(\cdot)\|_{W^p}+\frac{M}{\zeta}\epsilon\Big)\\
&+\frac{M}{\zeta}(1-L^f-K_1L^f)^{-1}\epsilon\\
=&\frac{(M)^2(L^g)^2}{\zeta^2(1-L^f-K_1L^f)^2}\|x^*(\cdot+\tau)+x^*(\cdot)\|_{W^p}
\end{aligned}
$$

$$+\frac{(M)^2L^g}{\zeta^2(1-L^f-K_1L^f)^2}\epsilon+\frac{M}{\zeta(1-L^f-K_1L^f)}\epsilon,$$

hence, we obtain

$$\begin{aligned}\|x^*(\cdot+\tau)+x^*(\cdot)\|_{W^p}<&\epsilon\Big(\frac{(M)^2L^g}{\zeta^2(1-L^f-K_1L^f)^2}+\frac{M}{\zeta(1-L^f-K_1L^f)}\Big)\\ &\times\Big(1-\frac{(M)^2(L^g)^2}{\zeta^2(1-L^f-K_1L^f)^2}\Big)^{-1}\\ =&\epsilon\Big(\frac{(M)^2L^g}{\zeta^2(1-L^f-K_1L^f)^2}+\frac{M}{\zeta(1-L^f-K_1L^f)}\Big)\\ &\times\Big(1+\frac{ML^g}{\zeta(1-L^f-K_1L^f)}\Big)^{-1}\Big(1-\frac{ML^g}{\zeta-L^f\zeta-K_1L^f\zeta}\Big)^{-1}\\ =&\epsilon\Big(\frac{(M)^2L^g}{\zeta^2(1-L^f-K_1L^f)^2}+\frac{M}{\zeta(1-L^f-K_1L^f)}\Big)\\ &\times\Big(1+\frac{ML^g}{\zeta(1-L^f-K_1L^f)}\Big)^{-1}\Big(1-\frac{ML^g}{M\zeta-ML^g}\Big)^{-1}\\ =&\epsilon\Big(\frac{(M)^2L^g}{\zeta^2(1-L^f-K_1L^f)^2}+\frac{M}{\zeta(1-L^f-K_1L^f)}\Big)\\ &\times\Big(1+\frac{ML^g}{\zeta(1-L^f-K_1L^f)}\Big)^{-1}\frac{M\zeta-ML^g}{M\zeta-2ML^g},\end{aligned}$$

which means that x^* is p-th Weyl almost anti-periodic. The proof is finished.

Example 3.1. *Consider the neutral type partial deferential equation*

$$\begin{cases}\frac{\partial}{\partial t}[u(t,x)+f(u(t-\theta,x))]=\frac{\partial^2}{\partial x^2}u(t,x)+g(t,u(t-\xi,x)),\\ u(t,0)=u(t,\pi)=0,\quad t\in(0,+\infty),\\ u(\theta,x)=\phi(\theta,x),\quad \theta\in[-1,0],\quad x\in[0,\pi],\end{cases}\tag{3.7}$$

where

$$f(u(t-\theta,x))=0.25\sin(u(t-1,x))$$

and

$$g(t,u(t-\xi,x))=\cos(\pi t)+\cos(\sqrt{2}\pi t)+\frac{0.25\sin(u(t-0.5,x))}{t^2+1}.$$

Take $\mathbb{B}=L^2([0,\pi])$ with $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ as its norm and inner product, respectively. Let A be an operator defined as $Au=u''$ with its domain

$$D(A):=\{u\in L^2([0,\pi]):u''\in L^2([0,\pi]),u(0)=u(\pi)=0\}.$$

According to the statements in Examples' section of [20], the operator A is the generator of an analytic semigroup $(T(t))_{t\geq0}$ on $\mathbb{B}$. Additionally, one gets, for $\varphi\in\mathbb{B}$,

$$T(t)\varphi=\sum_{n=1}^{\infty}e^{-n^2t}\langle\varphi,z_n\rangle z_n$$

and for $\varphi \in D(A)$,

$$A\varphi = -\sum_{n=1}^{\infty} n^2 \langle \varphi, z_n \rangle z_n,$$

where $z_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n\xi)$.

Hence, $T(t)$ satisfies (H_1) with $M = 1, \zeta = 1$, and $\|T(t)\| \le e^{-t}$, $t \in [0, +\infty)$.

Take

$$J(t) = \begin{cases} 1 & t > 0, \\ 0 & t = 0, \\ -1 & t < 0, \end{cases}$$

then (H_2) holds with $K_1 = 1$.

Consequently, (3.7) can be rewritten in a form as (3.1). It is easy to see that assumption (H_3) holds with $L^f = L^g = 0.25$ and assumption (H_4) holds with $\widetilde{M} = 0.75$ and $\frac{2ML^g}{\zeta} = 0.5 < \widetilde{M} < 1$. Therefore, according to Theorem 3.1, (3.7) admit one unique Weyl almost anti-periodic mild solution.

4. Conclusions

In this work, a definition of Weyl almost anti-periodic function has been introduced and the existence of a unique Weyl almost anti-periodic mild solution to a neutral functional semilinear differential equation has been confirmed by using the compressing mapping theorem and some analytical techniques. The concept and the approach developed in this work may be used to study Weyl almost anti-periodic solutions for other kinds of differential equations and may also be used to discuss Weyl almost anti-periodic solutions for neural networks and population models.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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