Edge proximity conditions for extendability in regular bipartite graphs

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Abstract

Let m and r be positive integers with $r \geq 3$, let G be an r-regular cyclically $((m-1)r+1)$-edge-connected bipartite graph and let M be a matching of size m in G. In [10], Plummer showed that whenever $r \geq m+1$, there is a perfect matching of G containing M. When $r = 3$, Aldred and Jackson [1], extended this result to the case when $m+1 \geq r = 3$ by showing there is a perfect matching in G containing M whenever the edges in M are pairwise at least distance $f(m)$ apart where

$$f(m) = \begin{cases} 1, & m = 2 \\ 3, & 3 \leq m \leq 4 \\ 4, & 5 \leq m \leq 8 \\ 5, & m \geq 9. \end{cases}$$

In this paper we relax the condition that $r = 3$ and the distance restriction introduced by Aldred and Jackson to show that, for $m \geq r \geq 3$ and G an r-regular cyclically-$((m-1)r+1)$-edge-connected bipartite graph, for each matching M in G with $|M| = m$ and such that each pair of edges in M is distance at least 3 apart, there is a perfect matching in G containing M.

1 Introduction

Throughout this paper the graphs considered will be finite simple graphs. A graph G with at least $2m+2$ vertices is said to be *m-extendable* if for each set $M \subset E(G)$ of m independent edges in G, there is a perfect matching F of G with $M \subset F$. We also say that a given set $M \subseteq E(G)$ *extends* to a perfect matching in G if there is a perfect matching F of G with $M \subset F$.

Perfect matchings in r-regular bipartite graphs have been extensively studied. It is well known, for example, (c.f. König [5, 6]) that every r-regular bipartite graph is r-edge-colourable and hence 1-extendable. See, for example, [2] - [11] for other results in this area.

Plummer [12] showed that an m-extendable graph must also be k-extendable for all $0 \leq k \leq m$ and also, if G is an m-extendable graph, then G must be $(m+1)$-connected. Thus an r-regular graph cannot be r-extendable. Indeed, in the case of an r-regular graph G with $|V(G)| \geq 2r+2$, we can select any vertex $v \in V(G)$ and readily find a matching M with $|M| \leq r$ such that $N_G(v) \subseteq V(M)$ and $v \notin V(M)$. Clearly such a matching M cannot extend to a perfect matching of G.

There exist r-connected r-regular bipartite graphs which are not even 2-extendable. To see this we can take two copies of $K_{r,r-1}$, one black dominated and the other white dominated. Form an r-connected r-regular bipartite graph G by joining these two graphs by a matching between r black vertices of the first copy of $K_{r,r-1}$ and the r white vertices of the second copy of $K_{r,r-1}$. Clearly no pair of edges in this matching can be included in a perfect matching of G.

While the connectivity of an r-regular graph is bounded above by r, there is no upper bound on the cyclic edge-connectivity of an r regular graph and we may ask whether the extendability of r-regular bipartite graphs increases with their cyclic edge-connectivity. This was confirmed, for $m \leq r-1$ by the following result of Plummer [10].

Theorem 1.1 *Let G be an r-regular bipartite graph with $r \geq m+1$ for some positive integers m and r. Then G is m-extendable if the cyclic edge-connectivity of G is at least $(m-1)r+1$.*

□

This result is best possible since no r-regular graph can be m-extendable for $m \geq r$. Moreover, Plummer [10] showed that the cyclic edge-connectivity requirement is also sharp.

We will consider which m-tuples of edges in a given r-regular bipartite graph *do* extend to perfect matchings when $m \geq r$. To do this we must rule out the possibility that our m-tuple covers all vertices in the neighbourhood of a single vertex. Clearly, we cannot exclude this possibility with cyclic edge-connectivity alone.

In [1], Aldred and Jackson considered cubic bipartite graphs of high cyclic edge-connectivity with the additional requirement that the edges we want to extend to a perfect matching are pairwise suitably far apart. For edges $e, f \in E(G)$, we define the distance, $dist(e, f)$ to be the length of a shortest path in G joining an end-vertex of e to an end-vertex of f. Using this idea, they established the following theorem.

Theorem 1.2 *Let G be a cyclically $(3m-2)$-edge-connected cubic bipartite graph and let M be a matching in G with $|M| = m \geq 2$. If for each pair of distinct edges $e, f \in M$, $dist(e, f) \geq f(m)$, then M is contained in a perfect matching of G, where*

$$f(m) = \begin{cases} 1, & m = 2 \\ 3, & 3 \leq m \leq 4 \\ 4, & 5 \leq m \leq 8 \\ 5, & m \geq 9. \end{cases}$$

□

The sharpness of Theorem 1.2 with respect to the cyclic edge-connectivity requirement was demonstrated in [1]. Our main result in this paper, Theorem 3.1, extends Theorem 1.2 to r-regular bipartite graphs for all $r \geq 3$ and shows that the distance constraint on the edges of M can be replaced by the much simpler condition that no vertex of G is adjacent to $r-1$ end-vertices of edges in M.

2 Preliminaries

Given a graph G and disjoint proper subsets $U, W \subset V(G)$ we use $E(U, W)$ to denote the set, and $e(U, W)$ the number, of edges of G from U to W. An *edge cut* of G is a set of edges of the form $E(U, \overline{U})$, where $\overline{U} = V(G) \backslash U$. The edge cut $E(U, \overline{U})$ is *cyclic* if both $G[U]$ and $G[\overline{U}]$ contain cycles. The graph G is *cyclically k-edge-connected* if each cyclic edge cut of G has size at least k. We say that an edge cut K *covers* a matching M of G if each edge of M either belongs to K or is adjacent to an edge of K. The matching M is *minimally non-extendable* in G if M is not contained in a perfect matching of G, but $M - e$ is contained in a perfect matching of G for all $e \in M$.

Lemma 2.1 *Let G be an r-regular bipartite graph and let M be a matching in G with $|M| = m$. Suppose M is minimally non-extendable in G. Then there is an edge cut K of G which covers M and is such that*

$$|K| \le \begin{cases} m(r-1) - r + \theta, & r \text{ odd;} \\ mr - r, & r \text{ even.} \end{cases}$$

where $\theta = |M \cap K|$.

Proof. Let (B, W) be the bipartition of G and $M = \{e_1, e_2, \ldots, e_m\}$. Let $e_i = b_i w_i$ for each $e_i \in M$ where $b_i \in B$, and let $M_B = \{b_1, b_2, \ldots, b_m\}$, $M_W = \{w_1, w_2, \ldots, w_m\}$. Since M is not contained in a perfect matching of G, $H = G - M_B - M_W$ contains no perfect matching. By Hall's theorem, there exists a 'Hall set' $X \subseteq V(H) \cap B$ such that $|N_H(X)| < |X|$.

Since $|B| = |W|$ we also have $|N_H(Y)| < |Y|$ for $Y = W \backslash (M_W \cup N_H(X))$. We shall analyze the structure of G based on the Hall sets X and Y for H. To this end we let $R_B = X$, $R_W = N_H(X)$, $L_B = B \backslash (M_B \cup R_B)$ and $L_W = W \backslash (M_W \cup R_W)$, see Figure 1. It follows from the facts that G is r-regular and bipartite and $M - e$ is contained in a perfect matching of G for all $e \in M$, that $|R_B| = |R_W| + 1$, $|L_W| = |L_B| + 1$, $M_W \subseteq N_G(R_B)$ and $M_B \subseteq N_G(L_W)$. Let $e(L_B, R_W) = \rho$, $e(M_B, R_W) = \lambda$, $e(M_B, L_W) = \sigma$, $e(M_W, R_B) = \tau$, $e(M_W, L_B) = \kappa$, and $e(M_W, M_B) = m + \alpha$. Finally let

$$S = \{v \in M_B : e(v, L_W) \ge \lceil \tfrac{r+1}{2} \rceil\},\ s = \sum_{v \in S} e(v, L_W)$$

and

$$T = \{v \in M_W : e(v, R_B) \ge \lceil \tfrac{r+1}{2} \rceil\},\ t = \sum_{v \in T} e(v, R_B).$$

We first consider the edge cuts $K_1 = E(L_W, M_B) \cup E(L_B, M_W) \cup E(L_B, R_W)$ and $K_2 = E(R_B, M_W) \cup E(R_W, M_B) \cup E(R_W, L_B)$. Then $|K_1| = \sigma + \kappa + \rho$ and $|K_2| = \tau + \lambda + \rho$. We next modify K_1, K_2 as follows. For each vertex $v \in S$ replace $E(v, L_W)$ in K_1 by $E(v, M_W \cup R_W)$ to form K_1' with $|K_1'| = \sigma + \kappa + \rho - 2s + |S|r$. Similarly for each vertex $v \in T$ replace $E(v, R_B)$ in K_2 by $E(v, M_B \cup L_B)$ to form K_2' with $|K_2'| = \tau + \lambda + \rho - 2t + |T|r$. This gives us the following identity for $|K_1'| + |K_2'|$.

$$\begin{aligned} |K_1'| + |K_2'| &= \sigma + \kappa + \tau + \lambda + 2\rho - 2(s+t) + r(|S| + |T|) \\ &= 2(r-1)m - 2\alpha + 2\rho - 2(s+t) + r(|S| + |T|). \qquad (1) \end{aligned}$$

As we are trying to establish the existence of a 'small' edge cut which covers M, we shall bound the sizes of s and t from below, starting with s. To

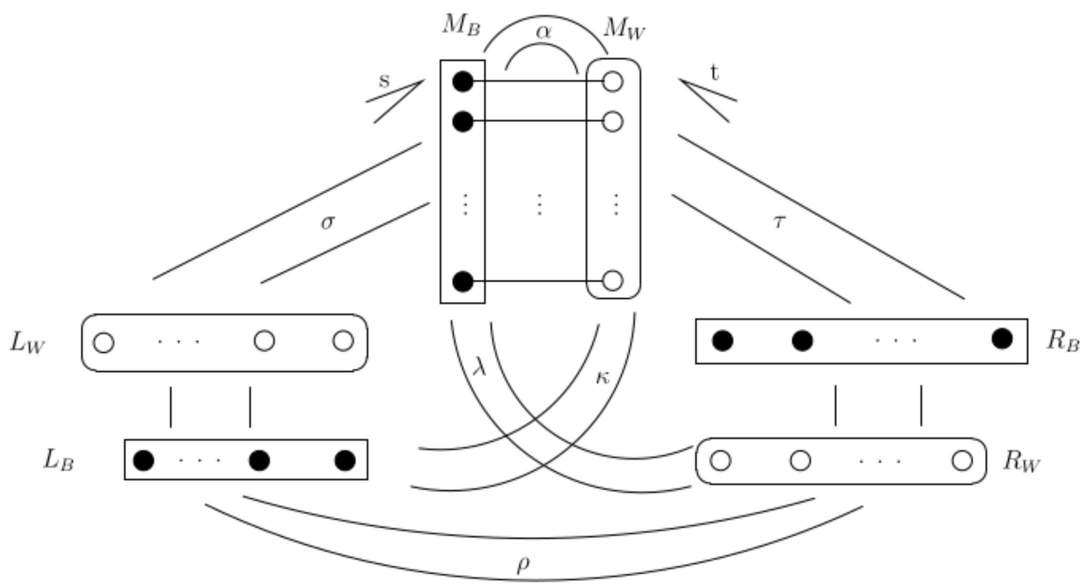

Figure 1: The structure of the graph G in Lemma 2.1.

do this, we consider $e(M_B, W)$.

$$\begin{aligned} e(M_B, W) &= rm \\ &= e(M_B, M_W) + e(M_B, L_W) + e(M_B, R_W) \\ &= m + \alpha + s + e((M_B \backslash S), L_W) + \lambda. \end{aligned}$$

Rearranging this gives us

$$e((M_B \backslash S), L_W) = (r-1)m - s - \alpha - \lambda.$$

Since $e((M_B \backslash S), L_W) \leq \lceil \frac{r-1}{2} \rceil (m - |S|)$ by the definition of S, we have

$$\left\lceil \frac{r-1}{2} \right\rceil (m - |S|) \geq (r-1)m - s - \alpha - \lambda.$$

This gives

$$s \geq \left\lfloor \frac{r-1}{2} \right\rfloor m + \left\lceil \frac{r-1}{2} \right\rceil |S| - \lambda - \alpha.$$

Similarly,

$$t \geq \left\lfloor \frac{r-1}{2} \right\rfloor m + \left\lceil \frac{r-1}{2} \right\rceil |T| - \kappa - \alpha.$$

Substituting into (1) above, we have

$$\begin{aligned} |K_1'| + |K_2'| &\leq 2(r-1)m + 2\rho - 4 \left\lfloor \frac{r-1}{2} \right\rfloor m - 2 \left\lceil \frac{r-1}{2} \right\rceil |S| + 2\lambda \\ &\quad -2 \left\lceil \frac{r-1}{2} \right\rceil |T| + 2\kappa + (|S| + |T|)r + 2\alpha \\ &= 2\delta m + 2\rho + (1-\delta)(|S| + |T|) + 2\kappa + 2\lambda + 2\alpha, \\ &\quad \text{where } \delta = \begin{cases} 1, & r \text{ even;} \\ 0, & r \text{ odd.} \end{cases} \end{aligned}$$

We can now use the fact that $\rho = m(r-1) - \kappa - \lambda - r - \alpha$, obtained by equating expressions for $e(L_W, L_B)$ and $e(L_B, L_W)$ and using $|L_W| = |L_B|+1$, to deduce that

$$|K'_1| + |K'_2| \leq 2\delta m + 2m(r-1) - 2r + (1-\delta)(|S| + |T|).$$

Since $|S| = M \cap K'_1$ and $|T| = |M \cap K'_2|$, this implies that either K'_1 or K'_2 will satisfy the conclusion of the lemma. □

3 The Main Result

We will now use Lemma 2.1 to prove our main result.

Theorem 3.1 *Let $m \geq r \geq 3$ be integers and G be an r-regular cyclically-$((m-1)r+1)$-edge-connected bipartite graph. Suppose M is a matching of size $m = |M|$ and no vertex of G is adjacent to $r-1$ end-vertices of edges in M. Then M extends to a perfect matching of G.*

Proof. Suppose, for a contradiction, that M is minimally non-extendable. Then, by Lemma 2.1, there is an edge cut K in G with $|K| \leq (m-1)r$ such that K covers M and G has the general structure indicated in Figure 1.

We may assume, without loss of generality, that K separates $L_B \cup L_W$ from $R_B \cup R_W$. Since, by hypothesis, no vertex v in L_W has more than $(r-2)$ neighbours in M_B, each vertex in L_W has at least 2 neighbours in L_B. Since $|L_W| \geq |L_B|$, this implies that $G[L_W \cup L_B]$ contains a cycle. The same argument tells us that $G[R_B \cup R_W]$ contains a cycle and hence K is a *cyclic* edge cut in G. This contradicts the hypothesis that G is cyclically $((m-1)r+1)$-edge-connected and completes the proof of the theorem. □

This immediately gives:

Corollary 3.2 *Suppose m and r are integers with $m \geq r \geq 3$, and G is an r-regular cyclically-$((m-1)r+1)$-edge-connected bipartite graph. If M is a matching in G of size $m = |M| \geq r$ and each pair of edges in M is distance at least 3 apart, then M extends to a perfect matching of G.*

□

Remark: Clearly, when $r = 3$ and $m \geq 3$, Corollary 3.2 provides a stronger result than Theorem 1.2, requiring only distance 3 for all values of $m \geq 3$. It should be noted however that Corollary 3.2 uses distance 3 to ensure that in the hypotheses of Theorem 3.1, the requirement that no vertex of G is adjacent to $r-1$ end-vertices of edges in M is met. The hypotheses of Theorem 3.1 do not impose any inherent distance restriction on the edges in M per se. Indeed, for Theorem 3.1 it is perfectly fine if pairs of edges in the matching M are distance 1 apart or distance 2 apart, so long as there is no set of $r-1$ edges in M pairwise joined by paths of length 2 all of which pass through a single vertex.

The results expressed in Theorem 3.1 and Corollary 3.2 are best possible in the following sense. Given integers m and r with $m \geq r \geq 3$, we can construct a cyclically $((m-1)r)$-edge-connected r-regular bipartite graph G in which there is a set M of m edges, pairwise distance arbitrarily far apart (and hence no $r-1$ of them have a vertex as a common neighbour if this distance is at least 3) such that no perfect matching of G contains M. One such construction is as follows. Let k, m and r be integers such that $k \geq 1$ and $m \geq r \geq 3$. Let H be a cyclically $((r-2)(k+1)(m-1)(r-1)+1)$-edge connected r-regular bipartite graph, with vertices coloured black and white according to the bipartition of H. (That such graphs exist see, for example, Theorem 6 of [9].) Then H has girth at least $(k+1)(m-1)(r-1)+1$. Let H^1, H^2 be two copies of H and choose a set of edges $E^i = \{e^i_j = x^i_j y^i_j \in E(H^i) \;:\; 1 \leq j \leq (m-1)(r-1)-1,\, 1 \leq i \leq 2\}$ and a vertex $z^i \in V(H^i)$ such that x^i_j and z^2 are black, y^i_j and z^1 are white and the elements of $E^i \cup \{z^i\}$ are pairwise distance at least k apart in H^i. (For example by choosing them to be close to equally spaced around a shortest cycle in H^i.)

Let $x^1_{(r-1)(m-1)}, x^1_{(r-1)(m-1)+1}, x^1_{(r-1)(m-1)+2}, \ldots x^1_{(r-1)(m-1)+r-2}, x^1_{(r-1)m}$ be the r neighbours of z^1 in H^1 and $y^2_{(r-1)(m-1)}, y^2_{(r-1)(m-1)+1}, y^2_{(r-1)(m-1)+2}, \ldots, y^2_{(r-1)(m-1)+r-2}, y^2_{(r-1)m}$ be the r neighbours of z^2 in H^2.

Let G be the cyclically $((m-1)r)$-edge-connected r-regular bipartite graph with vertex set $V(G) = (V(H^1) - z^1) \cup (V(H^2) - z^2) \cup \{x_{(r-1)m+i}, y_{(r-1)m+i} : 1 \leq i \leq m\}$ and edge set $E(G) = (E(H^1 - z^1) - E_1) \cup (E(H^2 - z^2) - E_2) \cup \{x_{(r-1)m+i} y_{(r-1)m+i}, x_{(r-1)m+i} y^2_{i+jm}, y_{(r-1)m+i} x^1_{i+jm} : 1 \leq i \leq m, 0 \leq j \leq r-2\} \cup \{x^2_j y^1_j : 1 \leq j \leq (m-1)(r-1)-1\}$.

In G the m edges $\{x_{(r-1)m+i} y_{(r-1)m+i} : 1 \leq i \leq m\}$ are pairwise distance at least k apart and cannot all be contained in the same perfect matching of G since the black vertices in $(V(H^1) - z^1)$ form a Hall set in $G - \{x_{(r-1)m+i}, y_{(r-1)m+i} : 1 \leq i \leq m\}$. (Note that the edges $\{x_{(r-1)m+i} y_{(r-1)m+i} : 1 \leq i \leq m\} \cup \{x^2_j y^1_j : 1 \leq j \leq (m-1)(r-1)-1\}$ form a minimal cyclic edge cut in G of size $(m-1)r$.)

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