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Aspects of (Super)Conformal Field Theories in Even Dimensions

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Life will tell.

Declaration

I, Enrico Andriolo, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any UK law, infringe any third party's copyright or other Intellectual Property Right, or contain any confidential material.

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Details of collaboration and publications:

This thesis describes research carried out with my supervisor Constantinos Papageorgakis, which was published in [1–4]. We collaborated with Neil Lambert in [1]; with Neil Lambert and Tristan Orchard in [2], with Prarit Agarwal and Gergely Kántor in [3]; with Vasilis Niarchos and Elli Pomoni in [4]. It also contains some unpublished material. Where other sources have been used, they are cited in the bibliography.

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The mountains are calling.

Abstract

The thorough study of Conformal-Field-Theories (CFTs) and Super-Conformal Field Theories (SCFTs) lies at the heart of the towering progress that our understanding of Quantum Field Theories (QFTs) has witnessed in the last decades or so. SCFTs are admitted to exist in spacetimes with dimension no greater than six and in this thesis we closely examine some aspects of those Superconformal Field Theories (SCFTs) that live in an even dimensional spacetime.

We start with the 6D (2,0) theory, which describes the low-energy dynamics of M5-branes in M-theory. While in the case of multiple M5-branes, such model is believed to be inherently non-lagrangian, writing a Lagrangian that captures the low-energy dynamics of a single M5-brane is feasible but non-trivial, as one has to overcome the notorious difficulties that arise when formulating a manifestly Lorentz-invariant action for self-dual forms. Building on recent work of A. Sen, we surmount such difficulties and introduce a lagrangian formalism that enables us to elucidate how self-dual forms couple to a curved spacetime background and compute their partition function via a path integral approach. As we show in full detail for the case of the compact chiral boson on a two-dimensional torus, to evaluate their partition function via a path integral approach it is crucial to use a Wick-rotation procedure obtained from a complex deformation of the physical spacetime metric.

We then move down to 4D, where we are mainly interested in the non-perturbative dynamics of SCFTs with $\mathcal{N} = 2$ or $\mathcal{N} = 3$ supersymmetry. In theories with $\mathcal{N} = 2$ supersymmetry, one can find the so-called Coulomb-Branch Operators (CBOs), and we give strong evidence for the fact that the Type-B Weyl anomalies associated to them are covariantly constant along the conformal manifold, in both the conformal and the spontaneously broken phases. In the case of $\mathcal{N} = 3$ theories, we evaluate the Macdonald limit of the superconformal index for some rank one and rank two S-fold SCFTs; we achieve this by computing the vacuum character for the two dimensional $\mathcal{N} = 2$ vertex operator algebras which, via the 4D/2D correspondence, are supposed to correspond to such theories.

Contents

Chapter 1

Introduction and Outline

According to the modern perspective, Quantum Field Theory (QFT) is a powerful and universal language that can be traced across many corners of physics (statistical mechanics, cosmology, condensed matter...) and mathematics (where, for instance, QFT is behind the advancements in knot theory that led to one of the Fields Medals issued in 1990).
QFT keeps playing a prominent role right in the field where it was first conceived – particle physics – as it provides, since its very birth, a framework for constructing quantum mechanical models of the sub-atomic word. Despite many experimentally-verified successes and Nobel prize-awarded discoveries (such as such as quantum-electrodynamics, the Higgs boson, the standard model of electroweak theory...), numerous open questions remain, and in general little is known of QFTs when considered beyond the perturbative regime, where quantum corrections become dominant and standard mathematical techniques break down. For example, more than a half-century has passed since we learnt that the strong interactions are described by Quantum-Chromo-Dynamics (QCD) and, yet, still we lack a complete explanation of how quarks stay confined in hadrons (which is one of the "Millennum Prize Problems").

The most important advancements in the understanding of QFTs have been unlocked by learning how to exploit the symmetries that characterize a system: the larger the set of symmetries that a QFT enjoys, the more constrained its dynamics.
There are two types of symmetries that make a QFT particularly amenable to mathematical treatment: Conformal symmetry and Supersymmetry.

Conformal Field Theories describe phenomena where length or mass scales are irrelevant, which is the case, for example, for statistical systems approaching their critical points.
From a phenomenological perspective, CFTs are important because theories that are

asymptotically free - like QCD, at high energies get close to being conformal. In practice, these are alluring models because with modern techniques they can be studied even without making reference to a Lagrangian, and therefore they constitute the right arena where to develop the machinery needed to then tackle non perturbative phenomena. The conformal bootstrap program [5] is one of these techniques, which can provide precision calculations for the spectrum of a theory and it was successfully used to derive bounds on the scaling dimensions of operators in the 3D Ising model at the critical temperature [6] which are in agreement with the experimental measurements done in the lab.
Moreover, from a theoretical point of view, CFTs sit in a privileged seat amongst the vast landscape of QFTs, as they are the protagonists appearing on one side of the famous and extensively celebrated AdS/CFT duality [7]: a CFT in a particular limit (the planar one) can equivalently be reformulated as a certain theory of gravity in a spacetime of one higher dimension. Remarkably, this is a weak/strong correspondence, as it relates the non-perturbative regime of the CFT with the perturbative one of the gravity theory, and viceversa. One can then apply familiar perturbative techniques on the theory appearing in one side of the correspondence to learn about the complicated non-perturbative physics happening in the dual theory.

Supersymmetric QFTs instead are invariant under a symmetry mapping fermions and bosons into each other, through means of which it is possible to reorganize the whole theory into rigid mathematical structures and thus keep the quantum corrections under control. Although supersymmetry has not yet been detected in particle accelerators, supersymmetric theories represent very interesting models which retain some of the non-trivial phenomenology of experimentally realized QFTs, such as confinement. At the same time, supersymmetric QFTs provide a simplified setting where extremely powerful techniques make computations (more) feasible and progress can be made. Among them, it is mandatory to mention the supersymmetric localization procedure of [8] and the *modus operandi* introduced in the seminal work of Seiberg-Witten [9, 10] which in turn has led to a cascade of new technologies in the field (such as the spectral network machinery of [11]).

Conformal symmetry and supersymmetry get together in the superconformal group, which tightly regulates the dynamics of ***Super Conformal Field Theories*** (SCFTs). These are the very special theories that have played a main role in research on QFT over the last twenty years or so, as their highly constrained nature has enabled us to make sensational progress in the understanding of non-perturbative phenomena and

QFT in general. With this thesis we want to give our modest contribution to this successful and well-established line of research.
According to the pioneering classification done by Nahm in [12], SCFTs are admitted to live in a spacetime that is at most six-dimensional. At that time, physicists and mathematicians used to believe that four was the maximum dimension for non-trivial unitary conformal field theory, so of Nahm's result little note was taken, initially. Indeed, while his classification dates 1977, it was only much later that the first examples of five and six-dimensional SCFTs were actually proven to exist [13–18]. This happened during the second string revolution, in 1995.

One of the most intriguing examples of these theories are the so-called (2,0) theories, which can be identified as the maximally supersymmetric local CFTs in the maximum number of dimensions: they possess 2 chiral supersymmetries (16 supercharges) and superconformal invariance in six dimensions. These are very captivating models. They essentially yield an organizing principle for lower-dimensional supersymmetric dynamics. From them it is possible to derive a multitude of lower-dimensional supersymmetric field theories and understand that many of the latter actually must be connected via a web of dualities ([19–22]) and unexpected correspondences, like AGT, [23]. Despite being at the very heart of some of the most towering successes of contemporary QFT, most of the structures underlying (2,0) SCFTs are still a mystery, which lie beyond the reach of existing techniques[1] as these theories are believed, in their non-abelian realizations, to be inherently non-lagrangian (see [26] for an executive summary).
Even the construction of a lagrangian describing the dynamics of the abelian (2,0) SCFT - locally governed by gauge algebra $\mathfrak{u}(1)$ - is non-trivial, although Lorentz invariant supersymmetric equations of motion have been constructed to all orders in [27]. This is due to the fact that the physical spectrum of this theory contains a chiral 2-form[2] and, as we will soon review in Chapter (2), there are plenty of difficulties in writing down lagrangians for theories involving chiral forms.

The first half of this thesis, consisting of Chapters (2) and (3), is devoted to the study of chiral forms. This is an old topic that we will consider with a fresh pair of eyes, by exploiting the lagrangian for chiral forms that Sen proposed in 2015. In Chapter (2) we will use his action to build an abelian (2,0) lagrangian in 6D and, upon clarifying how the coupling to a curved spacetime background is achieved within this formulation, we will test our construction by exploring various compactifications down to lower dimensional field theories. In Chapter (3), we will move to the quantum aspects, which are

[1] With some few but important exceptions, such as the chiral algebra and the bootstrap approach of [24, 25].
[2] A form is said to be chiral when its field strength is self-dual (with respect to the usual Hodge-star operation).

quite subtle in the presence of a chiral form, and we will compute the path-integral associated to Sen's action, with particular attention to the case of a chiral boson in 2D and a chiral 2-form in 6D. Interestingly enough, to take care of the convergence properties of Sen's action, we will be forced to implement the Wick rotation via a procedure where, in place of the time-coordinate, the curved spacetime metric gets complexified instead, along the lines of [28–30].

In the second half of the thesis we will instead consider SCFTs in a four-dimensional spacetime and we will study two of the tools most commonly employed in the analysis of non-perturbative phenomena in SCFTs – anomalies and indices, which will be the protagonists respectively of Chapters (4) and (5). In short, these are very useful objects as they prove to be robust quantities which change in a controlled way upon certain deformations (like turning up the values of the couplings that regulate the intensity of interactions, for instance) and many times they effectively introduce non-trivial constraints on the dynamics of the system, no matter how complicated this might be.
In Chapter (4), we will focus on a particular kind of Weyl anomalies – the Type-B Weyl anomaly associated to integer scaling dimension operators [31,32]. These anomalies can be easily identified in 4D $\mathcal{N} = 2$ SCFTs, where the so-called Coulomb-Branch operators are available to study. These are operators that are protected by supersymmetry in a way that makes their scaling dimension an integer number. We will study their associated Weyl anomalies which, in general, exhibit a non-trivial dependence on the marginal couplings of the theory, [33]. We will give strong evidence that these anomalies must be covariantly constant across the conformal manifold, even when the theory is considered in a vacuum that spontaneously breaks conformal invariance. As our arguments will be non-perturbative in their nature, when the anomalies in the conformal phase and in the spontaneously broken phase match at a particular point of the conformal manifold, they must keep doing so even away from it and in this way we might be able to constrain some non-perturbative physics via performing a tree-level check. For example, it is worth mentioning that this sort of anomaly matching has found an interesting application in [34] within the context of dimensional deconstruction [35]. According to the latter conjecture, the dynamics of a certain non-abelian (2,0) SCFT on a two-dimensional torus can be understood[3] as a limit-case of a particular 4D $\mathcal{N} = 2$ SCFT in the so-called Higgs phase, where conformal invariance is spontaneously broken; therefore, the fact that the anomaly of the four-dimensional theory in this vacuum agreed, at tree-level, with the one computed in the conformal phase implied that some

[3]For the sake of completeness, we mention that there is also another conjecture expressing the non-abelian (2,0) SCFT in terms of a lower-dimensional theory. In [36,37] indeed it was suggested that the (2,0) theory on a circle could be viewed as the 5D $\mathcal{N} = 2$ super Yang-Mills theory upon including in the latter also all the instanton contributions.

data of the six-dimensional (2,0) SCFT should be captured by doing a supersymmetric localization computation in the conformal phase of the four-dimensional theory.
In Chapter (5), we will investigate another robust observable, the superconformal index [38] which, for any four-dimensional $\mathcal{N} \geq 1$ SCFT, can be understood as the Witten index [39] for the theory in radial quantization. It encodes the information related to the protected spectrum of the theory modulo recombination rules and it is thus invariant under exactly marginal deformations, see [40]. We will be especially interested in the Macdonald limit of this observable, which collects only the contributions coming from a particular set of protected operators, the so-called Schur operators [41]. Remarkably, in [42] it was discovered that the Schur subsector of any 4D $\mathcal{N} \geq 2$ SCFT is captured by a chiral algebra. The latter, roughly speaking, is something analogous to a two-dimensional CFT and it can be used to easily compute quantities of the associated four-dimensional theory. As in two dimensions the conformal group enhances to a infinite-dimensional symmetry, the Schur/chiral correspondence of [42] gives a very good computational handle on the Schur spectrum of four-dimensional theories, to such a point that a minimal amount of info about the four-dimensional theory (central charges, moduli spaces) is sufficient to make an educated guess for the associated chiral algebra. It is in this spirit that in [43] chiral algebras for some non-lagrangian theories were put forward and, by exploiting these findings, we will compute the Macdonald limit of the superconformal index of some $\mathcal{N} = 3$ SCFTs in 4D. The latter are isolated non-lagrangian SCFTs that were discovered to actually exist only in 2015 ([44]) and our results yield the first Macdonald indices for such theories in the literature. As a byproduct of our brute-force computations, we will be able to confirm the proposal of [45], according to which some 4D $\mathcal{N} = 3$ SCFTs could be realized, up to a $\mathcal{N} = 1$ preserving marginal deformation, as the IR description of some lagrangian $\mathcal{N} = 1$ SCFTs.

We will conclude the thesis with some final words in Chapter (6).

Chapter 2

Geometrical Aspects of An Abelian (2,0) Action

In this chapter we first review the problems associated with a lagrangian formulation of chiral forms and we explain how Sen action differs from the ones that had previously appeared in the literature. Then, we focus on the particular case of a chiral 2-form in six dimensions and, by building on his work [46, 47], we construct the full supersymmetric completion to an abelian $(2,0)$ superconformal lagrangian including matter. We explore various geometrical aspects of such action; we elucidate the coupling to general backgrounds and investigate the non-standard diffeomorphism properties of the fields and their relation to the hamiltonian formulation. We also test the action by considering compactifications on a circle, K3 and a Riemann surface. The results are consistent with expectations for an action describing the low-energy physics of a single M5-brane in M-theory.

The contents of this chapter are based on the paper [1]. The abelian (2,0) theory proposed in this paper was later generalized in [48] to an action which fully describes an M5-brane in the eleven-dimensional supergravity background and was reformulated via a superspace approach in the work of [49].

2.1 Introduction

Let M be a $D-$dimensional-spacetime endowed with a Lorentzian metric g and with local coordinates x^μ. The Hodge operator $\star_g$ with respect to the metric g is given by

$$\star_g \omega_p = \sqrt{-\det(g)} \frac{1}{p!} \frac{1}{(\mathrm{D}-p)!} \varepsilon_{\mu_1 \ldots \mu_{\mathrm{D}-p} \nu_1 \ldots \nu_p} g^{\nu_1 \rho_1} \ldots g^{\nu_p \rho_p} \omega_{\rho_1 \ldots \rho_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\nu_{\mathrm{D}-p}} \quad (2.1)$$

for any $p-$form $\omega_p = \frac{1}{p!} \omega_{\nu_1 \ldots \nu_p} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_p}$. Throughout the whole thesis, we will be using the mostly plus signature and $\varepsilon_{012\ldots(\mathrm{D}-1)\mathrm{D}} = -1$, while the rest of the conventions

for any differential structure (forms, wedge product, inner product, differential etc.) are the same of [50] if not otherwise stated.
We say that a $p-$form ω_p is respectively *self-dual* or *antiself-dual* when its field strength $d\omega_p$ is an eigenvector of the Hodge operator with eigenvalue ± 1, i.e.

$$\star_g d\omega_p = \pm d\omega_p \ . \tag{2.2}$$

Of course, such an equation makes sense only when the spacetime is even-dimensional, $D = 2(p+1)$. On the other hand, from (2.1), we can straightforwardly deduce that $(\star_g)^2\omega_p = (-1)^{1+(p+1)(D-p-1)}\omega_p$ hence (anti)self-dual forms can exist only if $p+1$ is odd, that is only if

$$D = 4k + 2 \quad k \in \mathbb{N}_0 \ . \tag{2.3}$$

(Anti)Self-dual fields are ubiquitous in Physics.

> **D=2.** In two dimensions, the (anti)self-duality equation (2.2) for a boson ϕ reads as $\partial_0\phi = \pm\partial_1\phi$, and the propagating degrees of freedom consists of only right/left-moving modes. This is the reason why (anti)self-dual forms are also called *(anti)chiral forms*. Chiral bosons enter the description of the two-dimensional quantum Hall effect [51] and they can be found at the very heart of the world-sheet description of Heterotic String Theory [52–54] - the String Theory most commonly used in String Phenomenology.

The (anti)self-dual boson is the only (anti)self-dual form that figures in Condensed Matter, as Low-Energy physicists like to work in a world that is at maximum four-dimensional. String theorists, instead, believe that our universe is secretly ten dimensional and that otherwise curled-up dimensions should open up at extremely high energies.[4] Remarkably, self-dual forms enter the spectrum of string theory in ten dimensions in an indispensable way.

> **D=10.** Indeed, one of the excitations of Type-IIB superstring is a 4-form with self-dual field strength, which contributes to the miraculous cancellation of the gravitational anomaly [56], making type-IIB supergravity/superstring theories quantum mechanically consistent.

Type-IIB is one of the five different Superstring Theories that, perturbatively, are known to yield a consistent theory of Quantum Gravity. There is an intricate web of dualities connecting them, which makes them appear as if they were just different facets of the same underlying theory. This is called M-theory [57,58], which is populated

[4] We refer to [55] for a complete introduction to the basic concepts underlying String Theory and related ideas (such as M-theory, F-theory, etc).

not by strings but by extended objects (M2- and M5-branes) interacting in an eleven-dimensional spacetime. Little is known about this mysterious M-theory. One way to study it is via the AdS/CFT correspondence which, in its original and simplest inception [7], essentially introduced an equivalence between M-theory on particular backgrounds and the non-abelian field theories that live on a stack of multiple M2- or M5-branes. This is just one of the many reasons why string theorists have extensively worked to arrive at a proper formulation of the low-energy theory on the world-volume of M2/M5-branes. While this has successfully been achieved for the M2-branes [59–61], much is still left to be discovered about the theory of multiple coincident M5-branes.

> **D=6.** The latter is some interacting six-dimensional field theory with $(2,0)$ superconformal symmetry, which is believed to be non-Lagrangian - see [26, 62] and references therein - and where a non-abelian generalization of chiral 2-forms (also known as gerbes) are supposed to play a fundamental role, [63,64]. When the M5-branes are moved far away from each other, this complicated theory simplifies enormously and the dynamics on each M5-brane can be analysed independently. This is what we will do in this chapter, which is devoted to the study of the low-energy theory living on a single M5-brane. This is the theory of a free chiral 2-form and a bunch of free scalars, appropriately supersymmetrized.

The cases mentioned above do not constitute an exhaustive list of all the corners of High Energy Physics where chiral forms appear. For example, we have not mentioned that in six dimensions chiral 2-forms enter the spectrum also of the so-called Little String Theories [65, 66], which are non-gravitational theories having several string-like[5] properties, from whose low-energy limit six-dimensional superconformal field theories (such as the $(2,0)$ theory) can arise too. Moreover, it is enough to change the signature of spacetime to allow for self-dual fields in dimensions different from (2.3).

The bottom-line is that (anti)self-dual fields feature prominently in High-energy Physics and it is therefore desirable to have a proper formulation for them.

2.1.1 Lagrangians for self-dual forms

Chiral forms are peculiar objects. Even though their excitations obey to the Bose statistics, it is fair to say that they are somehow reminiscent of a fermionic behaviour as, in the first place, the (anti)self-duality condition itself is a field equation which is linear in the derivatives. Furthermore, they lead to species doubling [67] when one tries to define them on a lattice [68] and they are sources for gravitational anomalies [56, 69, 70]. This analogy becomes even more striking in two dimensions, if one thinks about the

[5] Relativistic Lorentz-invariant superstring theories necessarily leave in ten dimensions only when the interaction among strings is perturbative.

bosonization procedure, first introduced in [71] and then completed in [72] with an analysis of the underlying global aspects; this essentially is an equivalence between chiral bosons and Weyl fermions in two dimensions, in light of which it is only natural to expect that chiral forms must come with something analogous to a spin structure, in order to be properly defined [73].
As we often describe fermionic theories in terms of their chiral constituents, we would like to arrive at a formulation of bosonic theories where their chiral parts are seen as the true fundamental and independent building blocks. This is easily achieved within the hamiltonian formalism. But things get soon complicated if one wants to describe chiral forms via the lagrangian formalism, where traditional folklore used to require Lorentz invariance to be present in a manifest fashion.
That something must go wrong with an action principle can be appreciated already by noticing that there is no natural lagrangian for a chiral form, as the standard and manifestly-Lorentz invariant kinetic term would identically vanish:

$$d\omega_p \wedge \star_g d\omega_p = d\omega_p \wedge (\pm d\omega_p) = 0\ . \tag{2.4}$$

One could introduce the self-duality condition as a constraint imposed by hand on the equations of motion, after deriving the latter from an action principle for a non-chiral form. This is fine at the classical level, at which stage the self-duality equation (2.2) together with the Bianchi identity for the field strength are indeed enough to describe a chiral form. Nevertheless, we demand an action principle which automatically leads to (2.2) because we will ultimately want to study the quantized theory - via the path integral technique, most notoriously. Furthermore, it is worth struggling for a proper lagrangian of chiral forms, as this would naturally shed light on the global aspects and the geometric nature of self-dual fields and clarify how they couple to curved-backgrounds, external sources, etc, some of which features are not that transparent within an hamiltonian formalism.

In the next section we will briefly scan through some of the methodologies that have been proposed to write a lagrangian for chiral fields. There is an enormous literature on the subject; we will sketch only those ideas that will help us appreciate the novelties of Sen's work [46, 47], which is the approach to chiral fields which we will employ in this and in the following chapter of the thesis.

2.1.2 A little bit of history

It was already pointed out by Marcus and Schwarz [74] in 1982 that a theory written in terms of only the chiral field and the metric could not be described by a local and manifestly Lorentz invariant action functional[6]. All the various formulations that emerged in the subsequent 15 years schematically fall in two classes, as they all worked around Marcus&Schwarz's no-go theorem by either incorporating auxiliary fields into the Lagrangian or by giving up on manifest Lorentz invariance.

Additional auxiliary fields. Let's consider, for concreteness, the case of a chiral boson ϕ in 2D. The naivest idea of introducing a Lagrangian multiplier λ to impose the self-duality constrain on ϕ as in

$$L_{\text{linear}} = -\frac{1}{2} d\phi \wedge \star d\phi - \lambda \wedge (d\phi - \star d\phi) \tag{2.5}$$

does not work, as this system describes not one, but two dynamical chiral forms and there is no natural way to single out just one of them. Indeed, the equation of motion following from (2.5) are

$$d\phi = \star d\phi \qquad \text{and} \qquad d(\lambda + \star\lambda) = d \star d\phi = 0 \ , \tag{2.6}$$

where in the second equation we have used the first one. What we learn from the simple model (2.5) is that to avoid doubling the degrees of freedom of the system, we should introduce the auxiliary field in a more sophisticated way.
One of the first efforts in this direction was made by Siegel [75] who considered the possibility of imposing the self-duality via a lagrangian multiplier λ which enters the lagrangian via a cubic term as in[7]

$$L_{\text{Siegel}} = \frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\partial_1\phi)^2 + \lambda\,(\partial_0\phi + \partial_1\phi)^2 \ , \tag{2.7}$$

so that λ drops out of the field equations and there is indeed hope for it to be gauged away through some non-standard gauge transformations. However, the quantization of (2.7) turned out to be technically very subtle - even controversial [76–80] - and it is possible [81] that, at the end of the day, λ becomes propagating at the quantum level albeit it can be gauged away at the classical level. To fix this problem, one could think of introducing more auxiliary fields that are able to compensate each other's propagating degrees of freedom. It should not surprise then to know that formulations of chiral forms with an infinite number of auxiliary fields soon appeared. This approach was first proposed by McClain, Wu and Yu in [82] (see also [83] and [68]) for the case of the chiral boson in 2D and was later extended to chiral forms in higher dimensions in [84, 85]. Interestingly, this type of lagrangians was derived, independently, from a formulation of type-IIB closed

[6] See also [68].
[7] This action can be recast in a manifestly Lorentz-invariant way.

superstring field theory[8] in [86, 87].

Non-manifest Lorentz invariance. Dealing with an infinite number of auxiliary fields might seem a little bit cumbersome. A more tractable approach is then to write a lagrangian where Lorentz invariance is not apparent [81, 88]. However, the system is Lorentz invariant as the action is invariant under some transformations of the fields which - possibly upon using the equations of motion - reduce to the standard Lorentz ones [89] or satisfy the Lorentz algebra in flat spacetime [90, 91].
The archetype of such mechanism was explored for the chiral boson in 2D by Floreanini and Jackiw in [89]. There are more equivalent ways to recast their action, here we display the most commonly used one in the literature, which reads as

$$L_{\mathrm{FJ}} = \partial_0\phi\partial_1\phi + \partial_1\phi\partial_1\phi \ . \tag{2.8}$$

The equation of motion are of the second order in the derivatives

$$\partial_1(\partial_0\phi + \partial_1\phi) = 0 \tag{2.9}$$

from which the chirality condition follows only up to an undetermined function φ of time, i.e. $(\partial_0 + \partial_1)\phi(x^0, x^1) = \varphi(x^0)$. Just by looking at the particular recipe[9] that Floreanini and Jackiw followed to arrive at the lagrangian (2.8), it is clear that, somehow, φ cannot be propagating; and indeed φ can be gauged away, via a mechanism carefully reviewed in [92].
The work of Floreanini and Jackiw was soon after adapted to chiral forms in dimension greater than two by Henneaux and Teitelboim [81].
Another lagrangian for chiral forms that has been extensively used in the literature which does not exhibit Lorentz invariance manifestly is the one proposed by Perry and Schwarz in [90].

In 1997, these two different philosophies found a meeting point in the work of Pasti Sorokin and Tonin [93, 94], which proved to be very successful [70, 95–103]. By introducing a single auxiliary scalar field a they wrote a manifestly Lorentz invariant action

[8] In principle, it should be possible to derive type-IIB supergravity from a particular limit of type-IIB superstring field theory. Therefore, a formulation for the latter can be used to guess the rough structure of the former. As type-IIB supergravity contains a 4-form with self-dual field strength, the bosonic part of its action can then be generalized to an action for chiral forms in any dimensions.

[9] The starting point is an hamiltonian system for the field χ yielding the correct self-duality constraint as hamiltonian equation of motion: $(\partial_0 + \partial_1)\chi(x^0, x^1) = 0$. Via Legendre transformation they obtained a first-order lagrangian system in terms of χ which can be more nicely rewritten as in (2.8) upon a field redefinition $\chi \mapsto \phi(\chi)$. As the original hamiltonian described a chiral boson (and nothing else), φ must be a collateral damage of the field redefinition.

that, upon fixing the gauge transformations that makes a a Stuckelberg field, reduces to the Floreanini-Jackiw-Henneaux-Teitelboim or Perry-Schwarz formulations. Moreover, the Pasti-Sorokin-Tonin action depends on a in a non-polynomial way and it is therefore intuitively clear that such an action should correspond to a formulation with infinitely many auxiliary fields; indeed, it can be related to the McClain-Wu-Yu action, see [104].

The Pasti-Sorokin-Tonin action heavily relies on a set of gauge transformations. For example, it enjoys a redundancy parametrized by arbitrary variations of a. As the quantity $\sqrt{g^{\mu\nu}\partial_\nu a \partial_\mu a}$ enters the denominator of the lagrangian, not all the configurations of a are permitted. In particular the fact that a can not be gauged to a constant or to a light-like coordinate implies the existence of two distinct branches within the Pasti-Sorokin-Tonin system: the one in which a can be gauged to coincide with a spatial coordinate (in which case the Perry-Schwarz action is recovered) and the one in which the gauge $a = x^0$ is accessible (which makes the Pasti-Sorokin-Tonin collapse to the Floreanini-Jackiw-Henneaux-Teitelboim action).

As it happens within the Floreanini-Jackiw-Henneaux-Teitelboim formalism (see formula (2.9)), also the variation of the Pasti-Sorokin-Tonin action delivers a second order equation of motion. In $D = 2(p+1)$ dimensions, this essentially is $d(\star_g d\omega_p \pm d\omega_p) = 0$, from which the (anti)self-duality constraint follows, locally, only up to exact terms. The latter can be gauged away as the Pasti-Sorokin-Tonin action possesses one further gauge symmetry, which does not act on a and which should not be confused with the standard gauge redundancy characterizing usual $p-$forms theories. But when the p^{th} Betti number of the spacetime does not vanish, the chirality constraint can be deduced only up to closed (but non-exact) contributions which, surprisingly enough [92,105], can be gauged away as well only in the branch of the theory where the Pasti-Sorokin-Tonin can be reduced to the Floreanini-Jackiw-Henneaux-Teitelboim theory. Therefore, the Floreanini-Jackiw-Henneaux-Teitelboim is a better formulation than the Perry-Schwarz one, as the latter does not seem to correctly describe a single chiral boson when this is supposed to move through a topologically non-trivial spacetime.

That global aspects play an important role within the Pasti-Sorokin-Tonin action could already been appreciated by the fact that, having $\sqrt{\partial_\nu a \partial^\nu a}$ at the denominator, their lagrangian is not well-defined in spacetimes which lack of nowhere vanishing vector fields. However, in this case some generalization of the Pasti-Sorokin-Tonin mechanism which employs not just one but $q > 1$ auxiliary scalars $(a_1, ..., a_q)$ could be still defined [106, 107], as in these formulations the requirement of a nowhere vanishing $\partial_\nu a \partial^\nu a$ is relaxed into the requirement of a nowhere vanishing determinant for the $q \times q$ matrix $Y^{rs} := \partial_\nu a^r \partial^\nu a^s$. By playing with the gauge fixing of these q scalars, one can then generate lagrangian descriptions where Lorentz invariance is not manifest anymore as they involve a $D = q + (D - q)$ splitting of spacetime (as we saw above, for $q = 1$ these are the Perry-Schwarz and Floreanini-Jackiw-Henneaux-Teitelboim actions). This type of lagrangians [108, 109] deserve to be explored more, as they might experience some difficulties when formulated on non-trivial spacetime (as it happens for the Perry-Schwarz action).

Broadly speaking, all the lagrangians that we have mentioned throughout this section are based on the same strategy, which consists in: writing a lagrangian in terms of the non-chiral form ω_p, making sure that it is ingenious enough to admit some non-standard local redundancies, using the latter to derive from the equations of motion the chirality constraint (2.2) on the exact field strength $d\omega_p$. The non-standard gauge transformations are essential in this game given that, without them, there is no hope to turn a second order equation (like the ones that typically follow from a bosonic action) into the chirality constraint, which is a first order field equation. At the same time, the presence of additional redundancies in the system requires some subtle analysis, especially when the theory is formulated in spacetimes with non-trivial topology and when one is interested in quantizing the theory.

Sen's action introduces a concrete shift of paradigm, as it manages to describe the dynamics of chiral forms without employing any non-standard redundancies. In his formulation, the auxiliary degrees of freedom that are deployed to evade Marcus&Schwarz's no-go theorem form a dynamical non-unitary sector which simply decouples from the physics of the chiral form and therefore there is no need to introduce fancy local redundancies to gauge them away.
Maybe the most intuitive way to present Sen construction is to look back at the lagrangian (2.5), which is clearly equivalent to

$$L_{\text{linear}} = -\frac{1}{2} d\phi \wedge \star d\phi - 2\lambda \wedge d\phi \tag{2.10}$$

where λ is now assumed to satisfy the self-duality condition $\lambda = \star\lambda$ even off-shell. The equations of motion can be recast as

$$d\phi = \star d\phi \qquad \text{and} \qquad d\left(\lambda - \frac{1+\star}{2} d\phi\right) = 0\,, \tag{2.11}$$

and of course we still have two chiral forms in the spectrum – λ and $d\phi$ – as in (2.6). We hope to arrive at a lagrangian where there is a clear distinction between these degrees of freedom and it seems a good idea then to switch to a formulation where λ is replaced by $\hat{\lambda} := \lambda - \frac{1+\star}{2} d\phi$, as this is a chiral form satisfying $d\hat{\lambda} = 0$ even without using the equation of motion of ϕ. In terms of $\hat{\lambda}$, (2.10) reads as

$$L_{\text{linear}} = +\frac{1}{2} d\phi \wedge \star d\phi - 2\hat{\lambda} \wedge d\phi \quad . \tag{2.12}$$

If we now forget the relation between λ and $\hat{\lambda}$, then (2.12) is essentially Sen action in flat spacetime. Even though it still contains two chiral forms in the spectrum, there is a big difference with respect to (2.10). The fact that the kinetic term of ϕ has flipped sign yields a criterion through which it is possible to isolate a single chiral form out of the two: the physics must come with $\hat{\lambda}$ and we need to make sure that the non-unitary excitations of ϕ do not interact with the $\hat{\lambda}$ sector at any order in perturbation theory, even in the presence of external sources or when the theory is formulated in a curved background.

In this chapter we will show that indeed this is the case and that ϕ decouples from the physical spectrum of the theory. But before initiating an in-depth analysis of Sen's action, we would like to conclude this long excursus by mentioning other interesting works on chiral form lagrangians [110–118].

2.1.3 Sen's action

In 2015, drawing inspiration from the string field theory that he had previously constructed in [119], Sen put forward a new proposal for a lagrangian description of chiral $2k$-forms in $4k+2$ dimensions [46] (see also [47]), where the self-duality condition holds off-shell. This deploys auxiliary degrees of freedom in a polynomial way, while preserving manifest Lorentz invariance. The invariance of the action under general diffeomorphisms is not manifest, because the coupling to gravity is realised in a non-standard fashion. Moreover, the action does not couple the fields to the metric in the usual covariant way and, therefore, there is room to evade the no-go theorems regarding the compactifications of chiral $2k$-forms actions [120]. These attractive properties make this proposal worthy of further study.

We will focus on the action for chiral 2-form in 6D given by [47]

$$S_H = \int \left(\frac{1}{2} dB \wedge \star_\eta dB - 2H \wedge dB + H \wedge \tilde{\mathcal{M}}(H) \right) \quad . \tag{2.13}$$

Here B is a generic "2-form", while H is a chiral "3-form" subject to the self-duality condition $H = \star_\eta H$. This expression has some unconventional features. For instance, the coupling to the background is achieved via the interaction term involving $\tilde{\mathcal{M}}$, which is a function of the metrics g, η only. We stress that, although the background is generically curved ($g \neq \eta$), the Hodge star entering the kinetic term of B is defined with respect to the flat Minkowski metric η. As a result, B and H are not standard differential forms, a fact that is also reflected in their non-standard transformation properties under diffeomorphisms. It turns out that S_H encodes on-shell the degrees of freedom carried by—not one but—two free 2-forms with self-dual field strength: in the hamiltonian formulation, it can be shown that the theory contains an unphysical sector (with a wrong-sign kinetic term) that explicitly decouples from the physical one [47]. Thus one expects the physical sector to correctly describe the physics of free chiral 2-forms on generic manifolds.

The supersymmetric completion of this model to a $(2,0)$ theory for Minkowski space was constructed in [26]. In this chapter of the thesis we further investigate and extend several aspects of this $(2,0)$ lagrangian. In Sec. 2.2 we first elucidate the nature of the coupling of the dynamical degrees of freedom to arbitrary backgrounds, providing an alternative to the perturbative construction of $\tilde{\mathcal{M}}$ given in [47]; we also discuss the introduction of sources. We then revisit the (non-manifest) diffeomorphism invariance of the theory and show that the action reproduces standard results following from diffeomorphism-invariant theories, by *e.g.* evaluating the energy-momentum tensor. This information allows us to identify two particular combinations of the lagrangian fields B and H

$$
\begin{aligned}
H_{(s)} &:= H + \left(\frac{1+\star_\eta}{2}\right) dB \\
H_{(g)} &:= H - \tilde{\mathcal{M}}(H) \,,
\end{aligned}
\tag{2.14}
$$

which respectively correspond (on shell) to a singlet "3-form" and a standard chiral 3-form under diffeomorphisms. We then re-examine the hamiltonian formulation of the theory and make apparent the fact that $H_{(s)}$ and $H_{(g)}$ are, respectively, the unphysical and physical chiral degrees of freedom of the theory. We also determine the hamiltonian in terms of $H_{(s)}$ and familiar geometric quantities such as the energy-momentum tensor of $H_{(g)}$. At the end of Sec. 2.2, we provide an extension to the supersymmetric completion of the action for arbitrary backgrounds, that is for arbitrary $\tilde{\mathcal{M}}$.

Then, in Sec. 2.3, we proceed to consider some applications and consistency checks of the action by dimensionally reducing it on a circle, K3 and a non-compact Riemann surface. The reductions are non-trivial and we use either the lagrangian or hamiltonian formulation on a case-by-case basis. The first example leads to the expected spectra of a five-dimensional Maxwell theory, whose lagrangian scales inversely with the radius R, whereas the second example leads to the heterotic string transverse to $\mathbb{R}^5 \times \mathbb{T}^3$, plus some unphysical, decoupled degrees of freedom. The case of the Riemann surface is more interesting as the reduction depends on the scalars and hence is itself dynamical. We follow the approach of [121, 122] with the aim to reproduce the four-dimensional

$\mathcal{N} = 2$ Seiberg–Witten effective action [9]. We arrive at an action for two—instead of one—sets of real, abelian gauge fields subject to a constraint that relates them via electric-magnetic duality. Furthermore, in this case the unphysical sector does not entirely decouple but rather acts as a background. We conclude with a summary and some open questions in Sec. 2.4.

2.2 Abelian (2,0) Action on a Generic Manifold

We begin our discussion with a recap of the relevant background. In flat six-dimensional Minkowski spacetime one can write down the following action for the fields of the free (2,0) tensor multiplet [26]

$$S = \int \left(\frac{1}{2}dB \wedge \star dB - 2H \wedge dB - \frac{1}{2}\partial_\mu X^I \partial^\mu X^I + \frac{i}{2}\bar{\Psi}\Gamma^\mu\partial_\mu\Psi\right) , \tag{2.15}$$

where $H = \star_\eta H$. This is invariant under the superconformal transformations

$$\begin{aligned}
\delta X^I &= i\bar{\epsilon}\Gamma^I\Psi \\
\delta B_{\mu\nu} &= -i\bar{\epsilon}\Gamma_{\mu\nu}\Psi \\
\delta H_{\mu\nu\lambda} &= \frac{3i}{2}\bar{\epsilon}\Gamma_{[\mu\nu}\partial_{\lambda]}\Psi + \frac{3i}{2\cdot 3!}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}\bar{\epsilon}\Gamma^{\rho\sigma}\partial^\tau\Psi - \frac{i}{2}\partial^\rho\bar{\epsilon}\Gamma_\rho\Gamma_{\mu\nu\lambda}\Psi \\
\delta\Psi &= \Gamma^\mu\Gamma^I\partial_\mu X^I\epsilon + \frac{1}{3!}\Gamma^{\mu\nu\lambda}H_{\mu\nu\lambda}\epsilon - \frac{2}{3}\Gamma^I X^I\Gamma^\rho\partial_\rho\epsilon ,
\end{aligned} \tag{2.16}$$

with

$$\partial_\mu\epsilon = \frac{1}{6}\Gamma_\mu\Gamma^\rho\partial_\rho\epsilon . \tag{2.17}$$

A key point of this system is that

$$H_{(s)} = \frac{1}{2}(dB + \star dB) + H , \tag{2.18}$$

is a supersymmetry singlet and on-shell decouples from the rest of the fields. Of course, the latter statement is rather trivial as all fields are free and decoupled. But one can come up with interacting lagrangians for which $H_{(s)}$ is still decoupled.

It is desirable to extend this action to a general curved spacetime with metric g. In principle, one could easily try to couple it in the usual way:

$$S = \int \left(\frac{1}{2}dB \wedge \star_g dB - 2H \wedge dB - \frac{1}{2}dX^I \wedge \star_g dX^I + \frac{i}{2}\bar{\Psi}\Gamma_\mu dx^\mu \wedge \star_g \nabla\Psi - \frac{1}{5}RX^I X^I\right) , \tag{2.19}$$

where R is the Ricci scalar, $H = \star_g H$ with $\star_g$ the Hodge dual evaluated with respect to the metric g, and ∇ is the corresponding covariant derivative on spinors. Indeed this will still be supersymmetric if all expressions in (2.16) are replaced with covariant ones and by assuming that there exists a spinor satisfying $\nabla_\mu\epsilon = \frac{1}{6}\Gamma_\mu\Gamma^\rho\nabla_\rho\epsilon$. However, this would imply that the spurious degrees of freedom associated with $H_{(s)}$ also couple to the metric.

Rather, to make B truly decoupled Sen [46, 47] considers the following

$$S = S_H + S_{mat} \ , \tag{2.20}$$

where S_{mat} is the usual action for the matter fields and S_H is given by

$$S_H = \int \left(\frac{1}{2} dB \wedge \star_\eta dB - 2H \wedge dB + H \wedge \tilde{\mathcal{M}}(H) \right) \ , \tag{2.21}$$

while still imposing the self-duality condition $H = \star_\eta H$. Here we have introduced a subscript on $\star_\eta$ to emphasise that, although the spacetime metric is nontrivial, the Hodge dual is evaluated with the flat Minkowski metric. This is not the expected behaviour for 3-forms on a nontrivial metric; we will in fact see in due course that this is reflected in their unusual transformation properties under diffeomorphisms. In the last term above, $\tilde{\mathcal{M}}$ is a linear map:

$$\tilde{\mathcal{M}}(H)_{\mu\nu\lambda} = \frac{1}{3!}\tilde{\mathcal{M}}_{\mu\nu\lambda}^{\alpha\beta\gamma} H_{\alpha\beta\gamma} \ . \tag{2.22}$$

Since only the anti-self-dual part of $\tilde{\mathcal{M}}(H)$ appears in the action, and hence equations of motion, it can be assumed that

$$\tilde{\mathcal{M}}(H) = - \star_\eta \tilde{\mathcal{M}}(H) \, . \tag{2.23}$$

Similarly, it can be assumed that $\tilde{\mathcal{M}}$ is also symmetric the sense that

$$H_1 \wedge \tilde{\mathcal{M}}(H_2) = H_2 \wedge \tilde{\mathcal{M}}(H_1) \ , \tag{2.24}$$

holds for any two self-dual three-forms H_1, H_2. We note that in [46, 47] the following notation is employed

$$\mathcal{M}^{\mu\nu\lambda;\alpha\beta\gamma} = \frac{4}{3!}\varepsilon^{\mu\nu\lambda\rho\sigma\tau}\tilde{\mathcal{M}}_{\rho\sigma\tau}^{\alpha\beta\gamma} = -4\eta^{\mu\rho}\eta^{\nu\sigma}\eta^{\lambda\tau}\tilde{\mathcal{M}}_{\rho\sigma\tau}^{\alpha\beta\gamma} \ , \tag{2.25}$$

where in the last step we used (2.23).

The role of the last term in S_H is to change the equations of motion to

$$\begin{aligned} d\left(\frac{1}{2}\star_\eta dB + H\right) &= 0 \\ dB - \tilde{\mathcal{M}}(H) &= \star_\eta \left(dB - \tilde{\mathcal{M}}(H)\right) \ , \end{aligned} \tag{2.26}$$

which can be recast into

$$\begin{aligned} dH_{(s)} &= 0 \\ d\left(H - \tilde{\mathcal{M}}(H)\right) &= 0 \ . \end{aligned} \tag{2.27}$$

2.2.1 A construction for $\tilde{\mathcal{M}}$

We would next like to find $\tilde{\mathcal{M}}$ such that

$$\star_g \left(H - \tilde{\mathcal{M}}(H)\right) = H - \tilde{\mathcal{M}}(H) \ , \tag{2.28}$$

for arbitrary H, self-dual with respect to $\star_\eta$. One can then define

$$H_{(g)} := H - \tilde{\mathcal{M}}(H) \ , \tag{2.29}$$

which satisfies $H_{(g)} = \star_g H_{(g)}$ by construction and $dH_{(g)} = 0$ by the equations of motion.

To achieve (2.28), observe that $\tilde{\mathcal{M}}$ is a linear map from self-dual three-forms to anti-self-dual three forms (with respect to $\star_\eta$). However, it is helpful to extend its action to arbitrary 3-forms. Requiring that the symmetry property (2.24) holds for arbitrary 3-forms implies that $\tilde{\mathcal{M}}$ should vanish on anti-self-dual three-forms (with respect to $\star_\eta$). This property can be made explicit by re-writing

$$\tilde{\mathcal{M}} \to \frac{1}{4}(1 - \star_\eta)\tilde{\mathcal{M}}(1 + \star_\eta) \ . \tag{2.30}$$

Given that $H = \frac{1}{2}(1 + \star_\eta)H$, the condition (2.28) becomes

$$\frac{1}{4}(1 - \star_g)(1 - \star_\eta)\tilde{\mathcal{M}}(1 + \star_\eta) = \frac{1}{2}(1 - \star_g)(1 + \star_\eta) \ , \tag{2.31}$$

and can be viewed as a linear-operator equation acting on arbitrary 3-forms.

To solve this, we consider a basis of 3-forms given by

$$\omega_+^A \, , \omega_{-A} \qquad \text{for} \qquad A = 1, ..., 10 \ , \tag{2.32}$$

where the subscript $\pm$ indicates their eigenvalue under $\star_\eta$. The number of self-dual and anti-self-dual forms are equal so we have used the same index to label them (but one upstairs and one downstairs). When acting on this basis we can write $\tilde{\mathcal{M}}$ in terms of a matrix $\tilde{\mathcal{M}}^{AB}$:

$$\tilde{\mathcal{M}}(\omega_{-A}) = 0 \ , \qquad \tilde{\mathcal{M}}(\omega_+^A) = \tilde{\mathcal{M}}^{AB}\omega_{-B} \ . \tag{2.33}$$

Note that if we choose a basis where

$$\omega_+^A \wedge \omega_{B-} = 2\delta_B^A dx^0 \wedge ... \wedge dx^5 \ , \tag{2.34}$$

then the symmetry condition (2.24) reduces to $\tilde{\mathcal{M}}^{AB} = \tilde{\mathcal{M}}^{BA}$.

Equation (2.31) is trivially satisfied when acting on ω_{-A}. However, acting on ω_+^A gives

$$\tilde{\mathcal{M}}^{AB}(1 - \star_g)\omega_{-B} = (1 - \star_g)\omega_+^A \ , \tag{2.35}$$

which can be re-arranged to

$$(1 - \star_g)\left(\omega_+^A - \tilde{\mathcal{M}}^{AB}\omega_{-B}\right) = 0 \ , \tag{2.36}$$

implying that $\omega_+^A - \tilde{\mathcal{M}}^{AB}\omega_{-B}$ is self-dual with respect to $\star_g$.

Next, we can also construct a basis φ^A of self-dual three-form solutions with respect to $\star_g$. In particular, at any given point we can write:

$$\varphi^A = \mathcal{N}^A{}_B\omega_+^B + \mathcal{K}^{AB}\omega_{-B} \ . \tag{2.37}$$

The condition that $\omega_+^A - \tilde{\mathcal{M}}^{AB}\omega_{-B}$ is self-dual with respect to $\star_g$ implies that we can find a $\Theta^A{}_B$ such that

$$\begin{aligned}\omega_+^A - \tilde{\mathcal{M}}^{AB}\omega_{-B} &= \Theta^A{}_B\varphi^B \\ &= \Theta^A{}_B\mathcal{N}^B{}_C\omega_+^C + \Theta^A{}_B\mathcal{K}^{BC}\omega_{-C} \ .\end{aligned} \tag{2.38}$$

Since the ω_+^A and ω_{A-} form a basis of three-forms, this implies that

$$\Theta^A{}_B = (\mathcal{N}^{-1})^A{}_B \ , \tag{2.39}$$

and also results into an expression for $\tilde{\mathcal{M}}$:

$$\tilde{\mathcal{M}}^{AB} = -(\mathcal{N}^{-1})^A{}_C\mathcal{K}^{CB} \ . \tag{2.40}$$

It is important to note that these are all local considerations which are valid at a generic point in spacetime. There could be global issues as both $\mathcal{N}^A{}_B$ and $\mathcal{K}^{AB}$ are only defined locally and $\mathcal{N}^A{}_B$ may not be invertible everywhere. However if at any point $\mathcal{N}$ is not invertible then there exists a self-dual 3-form with respect to $\star_g$, which is anti-self-dual with respect to $\star_\eta$. However, this is not possible if the spacetime is orientable.

Lastly, let us check that (2.40) is compatible with the symmetry condition $\tilde{\mathcal{M}}^{AB} = \tilde{\mathcal{M}}^{BA}$. To this end we can construct, for any choice of A and B,

$$\begin{aligned}(\mathcal{N}^{-1})^A{}_C\varphi^C &= \omega_+^A - \tilde{\mathcal{M}}^{AC}\omega_{-C}\\ (\mathcal{N}^{-1})^B{}_D\varphi^D &= \omega_+^B - \tilde{\mathcal{M}}^{BD}\omega_{-D}\ .\end{aligned} \tag{2.41}$$

These are both self-dual forms with respect to $\star_g$ and therefore their wedge product vanishes:

$$\begin{aligned}0 &= (\mathcal{N}^{-1})^A{}_C\varphi^C \wedge (\mathcal{N}^{-1})^B{}_D\varphi^D\\ &= -\tilde{\mathcal{M}}^{BD}\omega_+^A \wedge \omega_{-D} - \tilde{\mathcal{M}}^{AC}\omega_{-A}\wedge\omega_+^B\ ,\end{aligned} \tag{2.42}$$

where we have used the fact that the wedge product of two self-dual or two anti-self-dual forms with respect to $\star_\eta$ also vanishes. Using the condition (2.34) we see that

$$0 = 2(\tilde{\mathcal{M}}^{AB} - \tilde{\mathcal{M}}^{BA})dx^0 \wedge \ldots \wedge dx^5\ , \tag{2.43}$$

which ensures that indeed $\tilde{\mathcal{M}}^{AB} = \tilde{\mathcal{M}}^{BA}$.

It is interesting to observe that, although $H_{(g)} = H - \tilde{\mathcal{M}}(H)$ is self-dual with respect to $\star_g$, it is not typically equal to $\frac{1}{2}(H + \star_g H)$. Rather we find

$$H_{(g)} = \frac{1}{2}(H + \star_g H) - \frac{1}{2}(1+\star_g)\tilde{\mathcal{M}}(H)\ . \tag{2.44}$$

In particular if $H = H_A\omega_+^A$ then (see (2.41))

$$H_{(g)} = (\mathcal{N}^{-1})^A{}_B H_A \varphi^B\ . \tag{2.45}$$

We can introduce a more compact notation as follows: for any (not necessarily self-dual) three-form ω we have $\tilde{\mathcal{M}}(\tilde{\mathcal{M}}(\omega)) = 0$ so that if we define the map

$$\mathfrak{m} : \omega \mapsto \omega - \tilde{\mathcal{M}}(\omega)\ , \tag{2.46}$$

then its inverse is

$$\mathfrak{m}^{-1} : \omega \mapsto \omega + \tilde{\mathcal{M}}(\omega)\ . \tag{2.47}$$

The map $\mathfrak{m}$ takes $\star_\eta$-self-dual 3-forms to $\star_g$-self-dual 3-forms but acts as the identity on $\star_\eta$-anti-self-dual 3-forms. It does not make all 3-forms $\star_g$-self-dual.

If $H_{(g)}$ is $\star_g$-self-dual then the map $\mathfrak{m}$ can be used to write

$$H_{(g)} = \mathfrak{m}\left(\frac{1}{2}(1+\star_\eta)H_{(g)}\right) . \tag{2.48}$$

This is due to $\tilde{\mathcal{M}}$ being anti-self-dual with respect to $\star_\eta$; see (2.23). Indeed, if $H_{(g)}$ is $\star_g$-self-dual, there is always an $\star_\eta$-self-dual H such that $H_{(g)} = \mathfrak{m}(H)$. We get

$$\frac{1}{2}(1+\star_\eta)H_{(g)} = \frac{1}{2}(1+\star_\eta)(H - \tilde{\mathcal{M}}(H)) = H , \tag{2.49}$$

and hence (2.48).

2.2.2 Introducing Sources

To consider sources J, the action we would like to consider is [47]

$$S_H^J = \int \left(\frac{1}{2}dB\wedge\star_\eta dB - 2H\wedge dB + H\wedge\tilde{\mathcal{M}}(H) + 2H\wedge\tilde{\mathcal{M}}(J) + 2H\wedge J\right) . \tag{2.50}$$

As before, $H_{(s)} = \frac{1}{2}dB + \frac{1}{2}\star_\eta dB + H$ is still a free $\star_\eta$-self-dual form: $H_{(s)} = \star_\eta H_{(s)}$ and, on-shell, $dH_{(s)} = 0$. However, if we now define

$$H_{(g)}^J := \mathfrak{m}(H + J_+) = H + J_+ - \tilde{\mathcal{M}}(H + J_+) , \tag{2.51}$$

where $J_\pm = \frac{1}{2}(1\pm\star_\eta)J$, then the equation of motion becomes

$$dH_{(g)}^J = dJ , \tag{2.52}$$

while $H_{(g)}^J = \star_g H_{(g)}^J$ holds by construction.

With the identification $H_{(g)}^J = dA + J$, (2.52) is the same equation of motion one would find from the usual action

$$S_A = -\frac{1}{2}\int(dA+J)\wedge\star_g(dA+J) + \int dA\wedge J , \tag{2.53}$$

where the self-duality condition $dA + J = \star_g(dA+J)$ must be imposed by hand. One could also add to S_H a term

$$S_J = \int J\wedge\tilde{\mathcal{M}}(J) - \frac{1}{2}J\wedge\star_\eta J , \tag{2.54}$$

which does not affect the equations of motion but makes the actions $S_H^J + S_J$ and S_A identical on-shell if we identify $H_{(g)}^J = dA + J$. In this case the complete action can be written as

$$S_H^J = \int \Big(\frac{1}{2}dB\wedge\star_\eta dB - 2H\wedge dB + (H+J_+)\wedge\tilde{\mathcal{M}}(H+J_+) + 2H\wedge J_- - J_-\wedge J_+\Big) \tag{2.55}$$

and under

$$\delta_\Lambda B = \Lambda \tag{2.56}$$

$$\delta_\Lambda J = d\Lambda \tag{2.57}$$

$$\delta_\Lambda H = -\Big(\frac{1+\star_\eta}{2}\Big)d\Lambda \ , \tag{2.58}$$

it transforms as

$$S_H^J \to S_H^J + \int d\Lambda\wedge(dB-J) \ . \tag{2.59}$$

In the next chapter of the thesis, we will see that (2.59) will play an important role in the interpretation of the partition function for the chiral form.

2.2.3 Diffeomorphisms

We now turn to the issue of diffeomorphisms, which are already known to enter the discussion in a novel way from [46,47]. Here we will expand on the latter discussion by utilising the construction of $\tilde{\mathcal{M}}$ from the previous sections.

Let us begin by examining how diffeomorphisms act on the original fields B and H. In particular, consider an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu(x)$. We will denote the transformation on B by $\delta_\xi B$ and assume that

$$\delta_\xi H = -\Big(\frac{1+\star_\eta}{2}\Big)d\delta_\xi B \ , \tag{2.60}$$

so that $H_{(s)}$ is invariant: $\delta_\xi H_{(s)} = 0$, as one expects from a field that does not gravitate (as we will see later, $H_{(s)}$ completely decouples from the physical degrees of freedom). By neglecting the boundary term $\int d(\delta_\xi B\wedge dB)$ we find

$$\delta_\xi S_H = \int -2(H-\tilde{\mathcal{M}}(H))\wedge d\delta_\xi B + H\wedge\delta_\xi\tilde{\mathcal{M}}(H) \ . \tag{2.61}$$

Note that, since $\tilde{\mathcal{M}}(H)$ and $\delta_\xi\tilde{\mathcal{M}}(H)$ are both anti-self-dual with respect to $\star_\eta$, the second term can be written as $(H-\tilde{\mathcal{M}}(H))\wedge\delta_\xi\tilde{\mathcal{M}}(H)$ and therefore we can also write this as

$$\delta_\xi S_H = \int -2H_{(g)}\wedge d\delta_\xi B + H_{(g)}\wedge\delta_\xi\tilde{\mathcal{M}}(H) \ , \tag{2.62}$$

where $H_{(g)} = H - \tilde{\mathcal{M}}(H)$.

We now need to ensure that $H_{(g)}$ remains self-dual with respect to $\star_g$ after the diffeomorphism:

$$0 = \delta_\xi \left[(1-\star_g)H_{(g)}\right] = -\delta_\xi \star_g H_{(g)} + (1-\star_g)\delta_\xi H_{(g)} \ . \tag{2.63}$$

Note that

$$\delta_\xi H_{(g)} = \mathfrak{m}(\delta_\xi H) - \delta_\xi \tilde{\mathcal{M}}(H) \ , \tag{2.64}$$

with $\delta_\xi H = \star_\eta \delta_\xi H$, so $\mathfrak{m}(\delta_\xi H)$ is $\star_g$-self-dual and on the one hand (2.63) simply gives

$$\delta_\xi \star_g H_{(g)} = (1-\star_g)\delta_\xi H_{(g)} = -(1-\star_g)\delta_\xi \tilde{\mathcal{M}}(H) \ . \tag{2.65}$$

On the other hand, a direct computation results in

$$\delta_\xi \star_g H_{(g)} = \nabla_\rho \xi^\rho H_{(g)} - \frac{1}{2}(\nabla_\mu \xi^\rho + \nabla^\rho \xi_\mu) H^{(g)}_{\nu\lambda\rho} dx^\mu \wedge dx^\nu \wedge dx^\lambda \ ,$$

where we used that $\delta_\xi g_{\mu\nu} = -2\nabla_{(\mu}\xi_{\nu)}$. Therefore we obtain

$$\delta_\xi \tilde{\mathcal{M}}(H) = \frac{1}{2}\nabla_\mu \xi^\pi H^{(g)}_{\nu\lambda\pi} dx^\mu \wedge dx^\nu \wedge dx^\lambda + \Xi - \tilde{\mathcal{M}}(\Xi) \ . \tag{2.66}$$

Here Ξ is any 3-form which is self-dual with respect to $\star_\eta$ so that the combination $\Xi - \tilde{\mathcal{M}}(\Xi)$ is self-dual with respect to $\star_g$ and hence does not contribute to (2.65). We will fix Ξ shortly.

To proceed, we observe that

$$\begin{aligned} -2\int H_{(g)} \wedge d(i_\xi H_{(g)}) &= -\frac{1}{2}\int H_{(g)} \wedge \left(\nabla_\mu \xi^\pi H^{(g)}_{\nu\lambda\pi} dx^\mu \wedge dx^\nu \wedge dx^\lambda\right) \\ &= -\int H_{(g)} \wedge \delta_\xi \tilde{\mathcal{M}}(H) \ , \end{aligned} \tag{2.67}$$

using $H_{(g)} = \star_g H_{(g)}$ and up to a total derivative, with i_ξ the standard inner derivative. Note that once again the $\Xi - \tilde{\mathcal{M}}(\Xi)$ term in $\delta_\xi \tilde{\mathcal{M}}(H)$ does not contribute here as both it and $H_{(g)}$ are self-dual with respect to $\star_g$ and hence their wedge product vanishes. Therefore we can define

$$\delta_\xi B = i_\xi H_{(g)} \ , \tag{2.68}$$

so that $\delta_\xi S = 0$, up to a total derivative.

Lastly, we need to fix Ξ to ensure that $\delta_\xi \tilde{\mathcal{M}}(H)$ is anti-self-dual with respect to $\star_\eta$:

$$\begin{aligned} 0 &= (1+\star_\eta)\delta_\xi\big(\tilde{\mathcal{M}}(H)\big) \\ &= \frac{1}{2}(1+\star_\eta)\nabla_\mu \xi^\pi H^{(g)}_{\nu\lambda\pi} dx^\mu \wedge dx^\nu \wedge dx^\lambda + 2\Xi \ , \end{aligned} \tag{2.69}$$

where we have used the facts $(1+\star_\eta)\tilde{\mathcal{M}}(\Xi) = 0$ and $(1+\star_\eta)\Xi = 2\Xi$. Therefore we let

$$\Xi = -\frac{1}{4}(1+\star_\eta)\nabla_\mu \xi^\pi H^{(g)}_{\nu\lambda\pi} dx^\mu \wedge dx^\nu \wedge dx^\lambda \ , \tag{2.70}$$

and hence, if we introduce the notation

$$\xi(\omega) := \frac{1}{(p-1)!}\nabla_\mu \xi^\lambda \omega_{\lambda\mu_1\ldots\mu_{p-1}} dx^\mu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{p-1}} \ , \tag{2.71}$$

for any p-form ω, then

$$\begin{aligned} \delta_\xi \tilde{\mathcal{M}}(H) &= \frac{1}{2}(1-\star_\eta)\left[\xi(H_{(g)}) + \tilde{\mathcal{M}}\left(\xi(H_{(g)})\right)\right] \\ &= \frac{1}{2}(1-\star_\eta)\left[\xi(H) - \xi(\tilde{\mathcal{M}}(H)) + \tilde{\mathcal{M}}(\xi(H)) - \tilde{\mathcal{M}}(\xi(\tilde{\mathcal{M}}(H)))\right] \ . \end{aligned} \tag{2.72}$$

Note that we also can write this as

$$\delta_\xi \tilde{\mathcal{M}}(H) = \left(\frac{1-\star_\eta}{2}\right)\mathfrak{m}^{-1}(\xi(\mathfrak{m}(H))) \ , \tag{2.73}$$

where the map $\mathfrak{m}(\omega) = \omega - \tilde{\mathcal{M}}(\omega)$ was defined in (2.47). This transformation law for $\tilde{\mathcal{M}}$ is analogous to that of a connection. In particular, if $\tilde{\mathcal{M}}$ vanishes in one frame it need not vanish in another and it is not consistent to set it to zero by fiat in (2.21) if one wants to maintain diffeomorphism invariance.

We can use the above result to finally determine the transformation properties of $H_{(g)}$. From its definition, we find that

$$\begin{aligned} \delta_\xi H_{(g)} &= \delta_\xi H - \tilde{\mathcal{M}}(\delta_\xi H) - \delta_\xi \tilde{\mathcal{M}}(H) \\ &= -\xi(H_{(g)}) + \mathfrak{m}\left(\frac{1}{2}(1+\star_\eta)\left(-d(i_\xi H_{(g)}) + \xi(H_{(g)})\right)\right) \ , \end{aligned} \tag{2.74}$$

but since

$$-d(i_\xi H_{(g)}) + \xi(H_{(g)}) = i_\xi(dH_{(g)}) - \xi^\pi \nabla_\pi H_{(g)} \tag{2.75}$$

we have, on-shell *i.e.* using $dH_{(g)} = 0$, that

$$\begin{aligned}\delta_\xi H_{(g)} &= -\xi(H_{(g)}) - m\left(\frac{1}{2}(1+\star_\eta)\xi^\pi\nabla_\pi H_{(g)}\right)\\ &= -\xi(H_{(g)}) - \xi^\pi\nabla_\pi H_{(g)}\\ &= -\pounds_\xi H_{(g)} \ ,\end{aligned} \tag{2.76}$$

where we have used the fact that $\star_g\xi^\pi\nabla_\pi H_{(g)} = \xi^\pi\nabla_\pi H_{(g)}$ along with (2.48), and we denoted the standard Lie derivative with $\pounds_\xi$. Thus we recover on shell the usual tensor transformation law for $H_{(g)}$ under a diffeomorphism.

In the presence of a source J we simply modify (2.68) by considering

$$\delta_\xi B = i_\xi H^J_{(g)} - i_\xi J \ , \tag{2.77}$$

where $H^J_{(g)} = H + J_+ - \tilde{\mathcal{M}}(H + J_+)$. Using the usual expression for the variation of J

$$\delta_\xi J = -\xi(J) - \xi^\pi\nabla_\pi J = -\pounds_\xi J \ , \tag{2.78}$$

we recover the standard tensorial variation $\delta_\xi H^J_{(g)} = -\pounds_\xi H^J_{(g)}$ on-shell.

It is worth emphasising that, although B and H have many properties associated with familiar differential forms, they have non-standard transformations under diffeomorphisms. Therefore, it might be more appropriate to refer to them as "pseudo-forms".

2.2.4 Energy-Momentum Tensor

To further exhibit how the action (2.21) reproduces standard results following from diffeomorphism-invariant theories, we can use the $\tilde{\mathcal{M}}$ term to compute the energy-momentum tensor as the response to the action from a variation of the spacetime metric.[10]

As usual we define

$$\begin{aligned}T_{\mu\nu} &:= -\frac{2}{\sqrt{-g}}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}\\ &= -\frac{2}{\sqrt{-g}}H_A H_B\omega^A_+ \wedge \frac{\partial\tilde{\mathcal{M}}}{\partial g^{\mu\nu}}(\omega^B_+) \ ,\end{aligned} \tag{2.79}$$

where we have expanded $H = H_A\omega^A_+$. To compute this we note that

$$(1-\star_g)(\omega^B_+ - \tilde{\mathcal{M}}(\omega^B_+)) = 0 \ , \tag{2.80}$$

[10] Here we will set the matter fields to zero as their contribution can be computed by regular means.

which, when varied with respect to the metric g, yields

$$(1-\star_g)\delta\tilde{\mathcal{M}}(\omega_+^B) = -\delta\star_g(\omega_+^B - \tilde{\mathcal{M}}(\omega_+^B)) \ . \tag{2.81}$$

Therefore, for any $\varphi^A = \star_g\varphi^A$,

$$2\varphi^A\wedge\delta\tilde{\mathcal{M}}(\omega_+^B) = -\varphi^A\wedge\delta\star_g(\omega^B - \tilde{\mathcal{M}}(\omega_+^B)) \ , \tag{2.82}$$

and hence from (2.45) we find

$$2\delta\tilde{\mathcal{M}}^{BC}\varphi_+^A\wedge\omega_{C-} = -(\mathcal{N}^{-1})^B{}_C\varphi^A\wedge\delta\star_g\varphi^C \ . \tag{2.83}$$

On the other hand from (2.34) we have

$$\varphi^A\wedge\omega_{C-} = \mathcal{N}^A{}_D\omega_+^D\wedge\omega_{C-} \ , \tag{2.84}$$

and hence

$$\begin{aligned}\delta\tilde{\mathcal{M}}^{BC}\omega_+^D\wedge\omega_{C-} &= -\frac{1}{2}(\mathcal{N}^{-1})^D{}_A(\mathcal{N}^{-1})^B{}_C\varphi^A\wedge\delta\star_g\varphi^C\\ &= -\frac{1}{2}\left(\omega_+^D-\tilde{\mathcal{M}}(\omega_+^D)\right)\wedge\delta\star_g\left(\omega_+^B-\tilde{\mathcal{M}}(\omega_+^B)\right) \ .\end{aligned} \tag{2.85}$$

Lastly, we contract this with H_B, H_D to find

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}}\left(H-\tilde{\mathcal{M}}(H)\right)\wedge\frac{\partial\star_g}{\partial g^{\mu\nu}}\left(H-\tilde{\mathcal{M}}(H)\right) \ . \tag{2.86}$$

This has a simple interpretation. We first consider the familiar lagrangian

$$\tilde{\mathcal{L}} = -\frac{1}{2}\tilde{H}\wedge\star_g\tilde{H} \ , \tag{2.87}$$

where $\tilde{H}$ is an arbitrary 3-form and compute its energy-momentum tensor:

$$\begin{aligned}\tilde{T}_{\mu\nu} &= \frac{1}{\sqrt{-g}}\tilde{H}\wedge\frac{\partial\star_g}{\partial g^{\mu\nu}}\tilde{H}\\ &= \frac{1}{2}\tilde{H}_{\mu\lambda\rho}\tilde{H}_\nu{}^{\lambda\rho} - \frac{1}{12}g_{\mu\nu}\tilde{H}_{\lambda\rho\tau}\tilde{H}^{\lambda\rho\tau} \ .\end{aligned} \tag{2.88}$$

Then to find our energy-momentum tensor $T_{\mu\nu}$ we set $\tilde{H} = H - \tilde{\mathcal{M}}(H) = H_{(g)}$ and so

$$T_{\mu\nu} = \frac{1}{2}H^{(g)}_{\mu\lambda\rho}g^{\lambda\sigma}g^{\rho\tau}H^{(g)}_{\nu\sigma\tau} \ . \tag{2.89}$$

As usual, we can recover the conservation of the energy-momentum tensor from the invariance of the theory under general coordinates transformations. Indeed, consider a constant infinitesimal vector ξ^μ and re-write (2.62) as

$$
\begin{aligned}
0 &= \int -2dH_{(g)} \wedge \delta_\xi B + H \wedge \frac{\partial \tilde{\mathcal{M}}(H)}{\partial g^{\mu\nu}} \delta_\xi g^{\mu\nu} \\
&= \int -2dH_{(g)} \wedge \delta_\xi B - \frac{1}{2} T_{\mu\nu} \delta_\xi g^{\mu\nu} \sqrt{-g} d^6 x \ ,
\end{aligned} \tag{2.90}
$$

where we used (2.79). Thus, by using $\delta_\xi g^{\mu\nu} = 2\nabla^{(\mu}\xi^{\nu)}$ and the equation of motion $dH_{(g)} = 0$, we recover $\nabla^\mu T_{\mu\nu} = 0$.

The above discussion can be straightforwardly extended to include sources by performing the replacement $H_{(g)} = H - \tilde{\mathcal{M}}(H) \ \mapsto \ H^J_{(g)} = H + J_+ - \tilde{\mathcal{M}}(H + J_+)$.

2.2.5 Hamiltonian Formulation

It will be useful to also express the theory in the hamiltonian formulation; this is the language that was first employed in [46, 47]. To this end we introduce $i, j = 1, 2, .., 5$. Using self-duality the only independent fields are H_{ijk}, B_{ij} and $A_i := B_{0i}$. However, only B_{ij} has a conjugate momentum:

$$
\Pi^B_{ij} = -\frac{1}{2}\left(\partial_0 B_{ij} - 2\partial_{[i}A_{j]}\right) + \frac{1}{3!}\varepsilon_{ijklm}H_{klm} \ , \tag{2.91}
$$

where $\varepsilon_{ijklm} = -\varepsilon_{0ijklm}$. The associated Poisson bracket is

$$
\{B_{ij}(\vec{x}, t), \Pi^B_{kl}(\vec{y}, t)\} = \delta_{i[k}\delta_{l]j}\delta(\vec{x} - \vec{y}) \ , \tag{2.92}
$$

and as a result we find that A_i and H_{ijk} impose the constraints

$$
\begin{aligned}
\partial_i \Pi^B_{ij} &= 0 \\
\frac{1}{2}\varepsilon_{ijklm}\Pi^B_{lm} &= H_{ijk} - \tilde{\mathcal{M}}_{ijk}(H) + \frac{3}{2}\partial_{[i}B_{jk]} \ .
\end{aligned} \tag{2.93}
$$

Following [46, 47] we introduce

$$
\Pi^\pm_{ij} := \frac{1}{2}\left(\Pi^B_{ij} \pm \frac{1}{4}\varepsilon_{ijklm}\partial_k B_{lm}\right) \ , \tag{2.94}
$$

so that the constraints (2.93) become

$$
\begin{aligned}
\partial_i \Pi^\pm_{ij} &= 0 \\
\Pi^-_{ij} &= \frac{1}{2 \cdot 3!}\varepsilon_{ijklm}(H_{klm} - \tilde{\mathcal{M}}_{klm}(H)) \ .
\end{aligned} \tag{2.95}
$$

In particular, we use the second constraint to determine H_{ijk} as a function of Π^-_{ij}, $H = H(\Pi^-)$. The dynamical variables are then simply $\Pi^\pm_{ij}$ with Poisson brackets:[11]

$$\begin{aligned}\{\Pi^\pm_{ij}(\vec{x},t),\Pi^\pm_{kl}(\vec{y},t)\} &= \pm\frac{1}{8}\varepsilon_{ijklm}\frac{\partial}{\partial x^m}\delta(\vec{x}-\vec{y})\\ \{\Pi^+_{ij}(\vec{x},t),\Pi^-_{kl}(\vec{y},t)\} &= 0\ .\end{aligned} \tag{2.96}$$

Explicit calculation reveals that the hamiltonian density can be written as

$$\mathcal{H} = \Pi_{ij}\partial_0 B_{ij} - \mathcal{L} = \mathcal{H}_+ + \mathcal{H}_-\ , \tag{2.97}$$

with

$$\begin{aligned}\mathcal{H}_+ &= -\ 2\Pi^+_{ij}\Pi^+_{ij} - 4\Pi^+_{ij}\partial_i A^+_j\\ \mathcal{H}_- &= 2\Pi^-_{ij}\Pi^-_{ij} + \frac{1}{3}\varepsilon_{ijklm}\Pi^-_{ij}\tilde{\mathcal{M}}_{klm}(H(\Pi^-)) + 4\Pi^-_{ij}\partial_i A^-_j\ .\end{aligned} \tag{2.98}$$

Note that we have introduced two independent constraints to impose $\partial_i\Pi^+_{ij} = 0$ and $\partial_i\Pi^-_{ij} = 0$, rather than the single combined constraint $\partial_i\Pi^B_{ij} = \partial_i(\Pi^+_{ij} + \Pi^-_{ij}) = 0$ that is obtained directly from the Legendre transform of the lagrangian. The reason is that in the lagrangian formulation the constraint $\partial_i\Pi^B_{ij} = 0$ implies both $\partial_i\Pi^+_{ij} = 0$ and $\partial_i\Pi^-_{ij} = 0$, as the difference vanishes due a Bianchi identity. However, in the hamiltonian formulation there is no Bianchi identity and we need to impose independent constraints to ensure we do not just impose the less-restrictive constraint $\partial_i(\Pi^+_{ij} + \Pi^-_{ij}) = 0$. In other words, $A^+_j + A^-_j$ imposes the constraint $\partial_i\Pi^B_{ij}$ and $A^+_j - A^-_j$ imposes the Bianchi identity on $\partial_i(\Pi^+_{ij} - \Pi^-_{ij})$. Thus we see that Π^+_{ij} degrees of freedom are unphysical, with the wrong sign for their energy, but are decoupled from the physical Π^-_{ij} degrees of freedom.

It is interesting to note that in terms of the original lagrangian variables we have

$$\begin{aligned}\Pi^+_{ij} &= -\frac{1}{2}H^{(s)}_{0ij}\\ \Pi^-_{ij} &= \frac{1}{2\cdot 3!}\varepsilon_{ijklm}H^{(g)}_{klm}\\ &= \frac{1}{2}\sqrt{-g}H^{0ij}_{(g)}\ ,\end{aligned} \tag{2.99}$$

[11] In principle, these should be Dirac brackets but in this particular case they reduce to standard Poisson brackets [46, 47].

where indices are raised using $g^{\mu\nu}$. We also observe that

$$\begin{aligned}
2\Pi^-_{ij}\Pi^-_{ij} + \frac{1}{3}\varepsilon_{ijklm}\Pi^-_{ij}\tilde{\mathcal{M}}_{klm}(H) &= \Pi^-_{ij}\Big(2\Pi^-_{ij} + \frac{1}{3}\varepsilon_{ijklm}\tilde{\mathcal{M}}_{klm}(H)\Big) \\
&= \Pi^-_{ij}\Big(\frac{1}{3!}\varepsilon_{ijklm}H_{klm} + \frac{1}{3!}\varepsilon_{ijklm}\tilde{\mathcal{M}}_{klm}(H)\Big) \\
&= \Pi^-_{ij}\Big(-H_{0ij} + \tilde{\mathcal{M}}_{0ij}(H)\Big) \\
&= -\Pi^-_{ij}H^{(g)}_{0ij} \\
&= -\frac{1}{2}\sqrt{-g}H^{0ij}_{(g)}H^{(g)}_{0ij}\ , \qquad (2.100)
\end{aligned}$$

where we first used (2.95), then the (anti)self-duality properties of H and $\tilde{\mathcal{M}}$ with respect to $\star_\eta$, and finally (2.99). Thus in terms of the lagrangian variables we see that, after imposing the constraints $\partial_i\Pi^\pm_{ij} = 0$, the hamiltonian can be written as

$$\mathcal{H} = \left(-\frac{1}{2}H^{(s)}_{0ij}H^{(s)}_{0ij} - \sqrt{-g}T^0{}_0\right)\ . \qquad (2.101)$$

Here $T^0{}_0 = g^{0\mu}T_{\mu 0}$ where $T_{\mu\nu}$ the energy-momentum tensor found in (2.89). Therefore, we can construct the hamiltonian by first using familiar geometric techniques to compute $T^0{}_0$ and then re-writing it in terms of $\Pi^-_{ij} = \frac{1}{2}\sqrt{-g}H^{0ij}_{(g)}$ (*i.e.* one is required to solve for $H^{(g)}_{0ij}$ in terms of $H^{0ij}_{(g)}$ and hence Π^-_{ij}).

As a specific example, let us consider the case of a static-like spacetime with $g_{0i} = 0$. In that case we simply find

$$H^{(g)}_{0ij} = g_{00}g_{ik}g_{jl}H^{0kl}_{(g)} = \frac{2}{\sqrt{-g}}g_{00}g_{ik}g_{jl}\Pi^-_{kl}\ , \qquad (2.102)$$

and hence

$$\mathcal{H} = -2\Pi^+_{ij}\Pi^+_{ij} - 4\Pi^-_{ij}\partial_i A^+_j - \frac{2}{\sqrt{-g}}g_{00}g_{ik}g_{jl}\Pi^-_{ij}\Pi^-_{kl} + 4\Pi^-_{ij}\partial_i A^-_j\ . \qquad (2.103)$$

External sources can be included by leaving the definitions of $\Pi^\pm_{ij}$ unchanged but modifying the constraint for Π^-_{ij} to

$$\begin{aligned}
\Pi^-_{ij} &= \frac{1}{2\cdot 3!}\varepsilon_{ijklm}(H^{(g)}_{klm} - J_{klm}) \\
&= \frac{1}{2}\sqrt{-g}(H^{0ij}_{(g)} - J^{0ij})\ . \qquad (2.104)
\end{aligned}$$

In this case we find, imposing the constraints $\partial_i\Pi^\pm_{ij} = 0$ and focussing once again on the case of static spacetimes for which $g_{0i} = 0$,

$$\mathcal{H} = -\frac{1}{2}H^{(s)}_{0ij}H^{(s)}_{0ij} - \frac{1}{2}\sqrt{-g}(H^{J\ 0ij}_{(g)} - J^{0ij})(H^{J}_{(g)0ij} - J_{0ij}) + \frac{1}{3!}(\star_\eta J)_{ijk}(J - \star_g J)_{ijk} \ , \quad (2.105)$$

where the indices are raised using $g^{\mu\nu}$. In terms of the hamiltonian variables $\Pi^{\pm}_{ij}$ (2.103) remains unchanged but now includes terms quadratic in the sources arising from the last line in (2.105).

2.2.6 Supersymmetry

Here we will write the (on-shell) supersymmetric completion of the action (2.21), generalising the results of [26] to arbitrary backgrounds. We will not introduce sources although some cases along these lines were considered in [26]. We assume that the six-manifold admits a conformal Killing spinor that satisfies

$$\nabla_\mu \epsilon = \Gamma_\mu \zeta \ , \quad (2.106)$$

for some $\zeta = \frac{1}{6}\Gamma^\rho \nabla_\rho \epsilon$.[12] The matter fields X^I and Ψ can be covariantly coupled to the non-trivial metric as usual

$$S_{mat} = \int \left(-\frac{1}{2} dX^I \wedge \star_g dX^I + \frac{i}{2}\bar{\Psi}\Gamma_\mu dx^\mu \wedge \star_g \nabla \Psi - \frac{1}{5} R X^I X^I \right) \ , \quad (2.107)$$

with the action remaining invariant under the extended supersymmetry variations

$$\delta X^I = i\bar{\epsilon}\Gamma^I \Psi$$
$$\delta \Psi = \Gamma^\mu \Gamma^I \partial_\mu X^I \epsilon - \frac{2}{3}\Gamma^I X^I \Gamma^\rho \nabla_\rho \epsilon + \delta_H \Psi \ , \quad (2.108)$$

where $\delta_H \Psi$ is yet to be determined. Here all geometric quantities are those associated with a curved spacetime and hence $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$. A short calculation shows that the terms in δS_{mat} involving X^I cancel out, leaving

$$\delta S_{mat} = -\int i\sqrt{-g}\nabla_\mu \bar{\Psi}\Gamma^\mu \delta_H \Psi \ . \quad (2.109)$$

[12] From this one can derive that $\nabla^2 \epsilon = -\frac{1}{10} R\epsilon$ with R the Ricci curvature. Throughout this section we use the conventions of [123].

Let us now look at δS_H and take

$$\begin{aligned}\delta B_{\mu\nu} &= -i\bar{\epsilon}\Gamma_{\mu\nu}\Psi \\ \delta H_{\mu\nu\lambda} &= \frac{3i}{2}\partial_{[\lambda}(\bar{\epsilon}\Gamma_{\mu\nu]}\Psi) + \frac{3i}{2\cdot 3!}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}\eta^{\rho\alpha}\eta^{\sigma\beta}\eta^{\tau\gamma}\partial_\gamma(\bar{\epsilon}\Gamma_{\alpha\beta}\Psi)\ .\end{aligned} \tag{2.110}$$

A key observation at this point is that

$$\delta H = -\Big(\frac{1+\star_\eta}{2}\Big)d\delta B\ , \tag{2.111}$$

and hence $\delta H_{(s)} = 0$, *i.e.* we have a reducible representation of $(2,0)$ supersymmetry where $H_{(s)}$ is a singlet.[13] On the other hand, from δS_H we have a non-vanishing contribution from $H \wedge d\delta B$ and an additional term[14] from $\delta(H \wedge \tilde{\mathcal{M}}(H)) = -2d\delta B \wedge \tilde{\mathcal{M}}(H)$ which combine to give

$$\begin{aligned}\delta S_H &= \int \frac{i}{3!}\varepsilon^{\mu\nu\lambda\rho\sigma\tau}\partial_\mu(\bar{\Psi}\Gamma_{\nu\lambda}\epsilon)\Big(H - \tilde{\mathcal{M}}(H)\Big)_{\rho\sigma\tau} \\ &= \int i\sqrt{-g}\nabla_\mu(\bar{\Psi}\Gamma_{\nu\lambda}\epsilon)(H - \tilde{\mathcal{M}}(H))^{\mu\nu\lambda}\ ,\end{aligned} \tag{2.112}$$

where we have used the fact that $H - \tilde{\mathcal{M}}(H)$ is self-dual with respect to $\star_g$ and that the Christoffel terms drop out of a covariant derivative involving anti-symmetrised indices. Everything is now in purely geometric terms.

To continue, we note that if $\nabla_\mu\epsilon = \Gamma_\mu\zeta$ then $\Gamma_{012345}\zeta = -\sqrt{-\det g}\ \zeta$, hence $\bar{\Psi}\Gamma_{\mu\nu\lambda}\zeta$ is self-dual. As a result the $\nabla_\mu\epsilon$ term drops out of δS_H and we find

$$\delta S_H = \int i\sqrt{-g}\nabla_\mu\bar{\Psi}\Gamma_{\nu\lambda}\epsilon(H - \tilde{\mathcal{M}}(H))^{\mu\nu\lambda}. \tag{2.113}$$

It is then easy to check that

$$\delta_H\Psi = \frac{1}{3!}\Gamma_{\mu\nu\lambda}(H - \tilde{\mathcal{M}}(H))^{\mu\nu\lambda}\epsilon\ , \tag{2.114}$$

will lead to a supersymmetric action.

In summary, we have that the action $S = S_H + S_{mat}$ is invariant under the on-shell

[13] One expects the fact that $H_{(s)}$ is a supersymmetry singlet. It is also a singlet under all diffeomorphisms and supersymmetry acts, roughly speaking, as the square root of a translation.

[14] Clearly, $\delta\tilde{\mathcal{M}} = 0$, since $\tilde{\mathcal{M}} = 0$ is a function of the background metric only.

supersymmetry, realised by the transformations

$$\begin{aligned}\delta X^I &= i\bar{\epsilon}\Gamma^I\Psi\\ \delta B_{\mu\nu} &= -i\bar{\epsilon}\Gamma_{\mu\nu}\Psi\\ \delta H_{\mu\nu\lambda} &= \frac{3i}{2}\bar{\epsilon}\Gamma_{[\mu\nu}\nabla_{\lambda]}\Psi + \frac{3i}{2\cdot 3!}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}\eta^{\rho\alpha}\eta^{\sigma\beta}\eta^{\tau\gamma}\bar{\epsilon}\Gamma_{\alpha\beta}\nabla_\gamma\Psi\\ &\quad - \frac{i}{4}\nabla^\rho\bar{\epsilon}\Gamma_\rho\Gamma_{\mu\nu\lambda}\Psi - \frac{i}{4\cdot 3!}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}\eta^{\rho\alpha}\eta^{\sigma\beta}\eta^{\tau\gamma}\nabla^\omega\bar{\epsilon}\Gamma_\omega\Gamma_{\alpha\beta\gamma}\Psi\\ \delta\Psi &= \Gamma^\mu\Gamma^I\partial_\mu X^I\epsilon - \frac{2}{3}\Gamma^I X^I\Gamma^\rho\nabla_\rho\epsilon + \frac{1}{3!}\Gamma_{\mu\nu\lambda}(H - \tilde{\mathcal{M}}(H))^{\mu\nu\lambda}\epsilon\ ,\end{aligned} \tag{2.115}$$

for any spinor that satisfies $\nabla_\mu\epsilon = \frac{1}{6}\Gamma_\mu\Gamma^\rho\nabla_\rho\epsilon$ and $\Gamma_{012345}\epsilon = \sqrt{-g}\,\epsilon$.

2.3 Reductions of the Abelian (2,0) Theory

Having developed this geometric formulation we now turn to its compactification. We will focus on three examples: that of a circle, K3 and a Riemann surface. The first reproduces five-dimensional Maxwell theory, while the second gives a heterotic string transverse to $\mathbb{R}^5 \times \mathbb{T}^3$. The Riemann-surface reduction leads to the Seiberg–Witten effective action for a four-dimensional $\mathcal{N} = 2$ Yang-Mills gauge theory. The first two cases are consistent with expectations whereas the third gives rise to some new features. In this section we set the fermions to zero for simplicity as we do not expect them to provide any novel physics.

2.3.1 Reduction on a Circle

The simplest case to consider is a six-dimensional manifold with a product metric of the form

$$g = \begin{pmatrix} \eta_5 & 0 \\ 0 & R^2 \end{pmatrix}\ , \tag{2.116}$$

where η_5 is the flat five-dimensional Minkowski metric. From the M-theory point of view, reducing a single M5-brane on a circle produces a D4-brane in type IIA string theory, which in turn is described by five-dimensional supersymmetric Maxwell theory.[15]

On the one hand we can express the ω_+^A and ω_{-A} basis of six-dimensional (anti)self-

[15] We will explicitly perform the reduction of S_H only; the matter part can be reduced as usual.

dual three-forms with respect to η_6 as

$$\begin{aligned}\omega_+^A &= \Omega^A \wedge dx^5 + \star_5 \Omega^A \\ \omega_{-A} &= \Omega^A \wedge dx^5 - \star_5 \Omega^A \ ,\end{aligned} \tag{2.117}$$

where Ω^A are a basis of two-forms in five dimensions and $\star_5$ is the Hodge dual constructed from η_5. On the other hand, a basis of self-dual three-forms with respect to g is

$$\begin{aligned}\varphi^A &= \Omega^A \wedge dx^5 + \frac{1}{R} \star_5 \Omega^A \\ &= \frac{R+1}{2R}\omega_+^A + \frac{R-1}{2R}\omega_{-A} \ .\end{aligned} \tag{2.118}$$

Then, using the definition (2.37) and the result (2.40), one can extract that for this case

$$\tilde{\mathcal{M}}^{AB} = -\frac{R-1}{R+1}\delta^{AB} \ , \tag{2.119}$$

which is indeed symmetric because the forms ω_+^A, ω_{-A} defined in (2.117) satisfy the condition (2.34). By expanding the self-dual field H in the above basis, $H = H_A\omega_+^A$, we have

$$\begin{aligned}H_{(g)} &= H - \tilde{\mathcal{M}}(H) \\ &= H_A\omega_+^A + \frac{R-1}{R+1}H_A\omega_{-A} \\ &= \frac{2R}{R+1}H_A\Omega^A \wedge dx^5 + \frac{2}{R+1} \star_5 H_A\Omega^A \ .\end{aligned} \tag{2.120}$$

From Eqs. (2.118)-(2.120) it is clear that R is dimensionless. This is due to the fact that in this theory we are dealing with the $\star_g$-self-duality condition of $H_{(g)}$ which, for the metric chosen in (2.116), reads as

$$\begin{aligned}H_{(g)ijk} &= -\frac{1}{R}\epsilon^{ijkl}H_{(g)0l5} \\ H_{(g)ij5} &= -\frac{R}{2}\epsilon^{ijmn}H_{(g)0mn} \ .\end{aligned} \tag{2.121}$$

Thus, to keep the dimensions of $H^{(g)}_{\mu\nu\rho}$ independent of μ, ν, ρ, we work with a convention where R is dimensionless and x^5 is compact with $x^5 \cong x^5 + l$ for some parameter l with dimensions of length. The resulting physical size of the fifth dimension is lR.

By implementing the above in Eq. (2.103), we immediately find

$$\mathcal{H}_- = \frac{2}{R}\Pi^-_{ab}\Pi^-_{ab} + 4R\Pi^-_{a5}\Pi^-_{a5} + 4\Pi^-_{ab}\partial_a A^-_b + 4\Pi^-_{a5}(\partial_a A^-_5 - \partial_5 A^-_a) \ , \tag{2.122}$$

where $a, b = 1, 2, 3, 4$. If we truncate to the zero-mode sector along the circle then we can solve the A_a^- constraint by writing

$$\Pi_{ab}^- = -\frac{\beta}{4l}\varepsilon_{abcd}\partial_c A_d \ , \tag{2.123}$$

for some A_a with β a unitless normalisation factor that can be fixed *ad libitum*. The hamiltonian density reduces to

$$\mathcal{H}_- = \frac{\beta^2}{8lR}(\partial_a A_b - \partial_b A_a)^2 + 4Rl\Pi_{a5}^-\Pi_{a5}^- + 4l\Pi_{a5}^-\partial_a A_5^- \ , \tag{2.124}$$

while the Poisson bracket (2.96) becomes[16]

$$\{A_a(\vec{x}, t), \Pi_{b5}^-(\vec{y}, t)\} = \frac{1}{2\beta}\delta_{ab}\delta_4(\vec{x} - \vec{y}) \ . \tag{2.125}$$

Thus A_a is canonically conjugate to Π_{a5}^- provided that we fix $\beta = 1/2$. We can use this last expression to compute Hamilton's equations

$$\begin{aligned} \partial_0 A_a &= 8Rl\Pi_{a5}^- + 4l\partial_a A_5^- \ , \\ \partial_0 \Pi_{a5} &= -\frac{1}{8Rl}\partial_b(\partial_a A_b - \partial_b A_a) \ , \end{aligned} \tag{2.126}$$

which then yield Maxwell's equations for a gauge potential given by $\{4lA_5^-, A_a\}$. A standard five-dimensional Maxwell lagrangian is then obtained through an inverse Legendre transform; by using (2.126), we get

$$\begin{aligned} \mathcal{L}_- &= \left(\partial_0 A_a \Pi_{a5}^- - \mathcal{H}_-\right)\Big|_{\Pi_{a5}^- = \frac{1}{8Rl}(\partial_0 A_a - 4l\partial_a A_5^-)} \\ &= \frac{1}{32Rl}\left(2\left(\partial_0 A_a - 4l\partial_a A_5^-\right)^2 - (\partial_a A_b - \partial_b A_a)^2\right) \ , \end{aligned} \tag{2.127}$$

which scales with $1/R$.

Alternatively, we can also perform the reduction within the lagrangian formalism; this is an instructive exercise which makes even more transparent how this $1/R$ dependence in front of the 5D theory is due to the non-standard coupling of the 6D theory to the metric. By dimensionally reducing the action (2.21) on a circle, we get

$$\begin{aligned} S_0 = l\int_{\mathbb{R}^{1,4}} \Big[&-\frac{1}{2}d_5 B \wedge \star_5 d_5 B - \frac{1}{2l^2}d_5 B_5 \wedge \star_5 d_5 B_5 \\ &+ \frac{2}{l}H_5 \wedge d_5 B - \frac{2}{l^2}H_5 \wedge \star_5 d_5 B_5 - \frac{2}{l^2}\frac{R-1}{R+1}H_5 \wedge \star_5 H_5 \Big] \ , \end{aligned} \tag{2.128}$$

[16]Note that upon reduction over x^5 the five-dimensional delta function $\delta(\vec{x} - \vec{y})$ changes to l^{-1} times the four-dimensional delta-function $\delta_4(\vec{x} - \vec{y})$.

where all the fields are to be understood as zero-modes and B_5, H_5 stand for $B_5 := lB_{\mu 5}dx^\mu$, $H_5 := \frac{l}{2}H_{\mu\nu 5}dx^\mu \wedge dx^\nu$. The equations of motion yield

$$\begin{aligned} d_5 F^{(g)} &= \frac{1}{R} d_5 \star_5 F^{(g)} = 0 \\ d_5 F^{(s)} &= d_5 \star_5 F^{(s)} = 0 \ , \end{aligned} \tag{2.129}$$

where $F^{(s)}$ and $F^{(g)}$ are defined by

$$\begin{aligned} F^{(g)} :=& l i_5 H^{(g)} = \frac{2R}{R+1} H_5 \\ F^{(s)} :=& H_5 + \frac{1}{2} d_5 B_5 - \frac{l}{2} \star_5 d_5 B \ . \end{aligned} \tag{2.130}$$

Thus we recover two five-dimensional free Maxwell fields.

If one computes the hamiltonian density arising from the compactified lagrangian (2.128) one finds the same result as compactifying the six-dimensional hamiltonian we considered above (including both Π^+_{ij} and Π^-_{ij} sectors). Therefore $F^{(s)}$ is unphysical.

On the other hand, one would like to identify the physical degrees of freedom already at the level of the compactified lagrangian. This is better done in the "dual frame", where the 2-form B is dualised to a vector A^B. That is, in (2.128) we introduce a Lagrange multiplier A^B, which imposes the Bianchi identity on $Q := d_5 B$ as follows:

$$\begin{aligned} S_0 = l \int_{\mathbb{R}^{1,4}} \Big[&- \frac{1}{2} Q \wedge \star_5 Q + \frac{2}{l} H_5 \wedge Q \\ &- \frac{1}{2l^2} d_5 B_5 \wedge \star_5 d_5 B_5 - \frac{2}{l^2} H_5 \wedge \star_5 d_5 B_5 \\ &- \frac{2}{l^2} \frac{R-1}{R+1} H_5 \wedge \star_5 H_5 + \frac{1}{l} Q \wedge d_5 A^B \Big] \ , \end{aligned} \tag{2.131}$$

so that A^B has mass dimension one. By integrating out Q we get

$$\begin{aligned} S_0 = \frac{1}{l} \int_{\mathbb{R}^{1,4}} \Big[&- \frac{1}{2} d_5 A^B \wedge \star_5 d_5 A^B - 2H_5 \wedge \star_5 d_5 A^B \\ &- \frac{1}{2} d_5 B_5 \wedge \star_5 d_5 B_5 - 2H_5 \wedge \star_5 d_5 B_5 \\ &- \frac{4R}{R+1} H_5 \wedge \star_5 H_5 \Big] \ . \end{aligned} \tag{2.132}$$

It is then natural to also integrate out H_5, the equations of motion for which impose

$$\frac{2R}{R+1} H_5 = -\frac{1}{2} d_5 \left(A^B + B_5 \right) \ . \tag{2.133}$$

The action then becomes

$$S_0 = \frac{1}{Rl}\frac{1-R}{4}\int_{\mathbb{R}^{1,4}} d_5A\wedge\star_5 d_5A - \frac{1}{l}\frac{1}{1-R}\int_{\mathbb{R}^{1,4}} d_5A^B\wedge\star_5 d_5A^B \ , \tag{2.134}$$

where the vector A is defined as

$$A := B_5 + \frac{1+R}{1-R}A^B \quad . \tag{2.135}$$

Here we see two free Maxwell fields with opposite signs for their kinetic terms. When we take $R\to 0$, A has the correct sign and its kinetic term scales with $1/R$. In this limit $A = B_5 + A^B$ and (2.133) then states that d_5A is nothing but $F^{(g)}$, *i.e.* $d_5A = -2F^{(g)}$.

In summary, by performing a circle reduction we have found a five-dimensional lagrangian that scales like $1/R$ rather than R; the latter scaling had been previously noted as a challenge for the construction of an action for the M5-brane [120]. Of course, the discussion here might be somewhat unconvincing as we have a free theory and hence we can rescale the fields by any function of R that we like (recall that R is dimensionless), for example by taking a different choice of β in Eqs. (2.124)-(2.125). However the Poisson bracket we used arose from six-dimensions and its normalisation is fixed. Furthermore the dependence on l is determined by dimensional analysis and only the combination Rl has a physical meaning as the size of the fifth dimension. So there is some hope that this calculation is meaningful.

A more stringent test would be to recover the same R scaling in five-dimensional Super-Yang-Mills by considering the non-abelian action constructed in [26], so we close this subsection by sketching some aspects of the corresponding calculation. The lagrangian of [26] employs a covariantly-constant vector field Y^μ with dimensions of length, first introduced in [123].[17] For a circle reduction it is natural to fix $Y^5 = y$,[18] and hence independent of x^5. However, in the cases where Y is not null, it is straightforward to see by looking at the matter terms in the action that the five-dimensional coupling constant will be

$$g^2 = Rl\left(\frac{|\langle y,y\rangle|}{R^2l^2}\right) \ . \tag{2.136}$$

Thus g^2 can be thought of as proportional to Rl but with an arbitrary coefficient given by the dimensionless combination $\langle y,y\rangle/R^2l^2$. Comparing with string theory requires us to identify $|\langle y,y\rangle| = (2\pi Rl)^2$.

[17] In that construction, Y^μ takes values in a three-algebra.

[18] With y some element of the three-algebra.

2.3.2 Reduction on K3

According to U-duality M-theory on K3 is dual to heterotic string theory on $\mathbb{T}^3$ [57,124]. In particular, an M5-brane wrapped on K3 should give the same dynamics as a heterotic string transverse to $\mathbb{R}^5 \times \mathbb{T}^3$. At the worldvolume level this reduction was performed in [125]. We now investigate whether the action (2.21) is also consistent with this expectation.

The reduction on K3 can be performed in the hamiltonian formulation. We take the K3 to span the dimensions $x^1, ..., x^4$. Since the $\mathcal{H}_+$ component in (2.97) is independent of any geometric information, it is not clear how to reduce it on K3. However this does not pose a problem since, as we did for the circle reduction, one can simply think of $\mathcal{H}_+$ as a six-dimensional hamiltonian that decouples from the physical degrees of freedom, and focus on reducing $\mathcal{H}_-$. To this end, we recall from (2.99) that $(a, b \in \{1, 2, 3, 4\})$

$$\begin{aligned}
\Pi^-_{ab} &= \frac{1}{2}\sqrt{g_{\mathrm{K3}}} H^{0ab}_{(g)} = \frac{1}{4}\varepsilon^{abcd} H^{(g)}_{5cd} \\
\Pi^-_{5a} &= \frac{1}{2}\sqrt{g_{\mathrm{K3}}} H^{05a}_{(g)} = \frac{1}{4}\varepsilon^{abcd} H^{(g)}_{bcd} \ .
\end{aligned} \tag{2.137}$$

Next, we make the following ansatz for the Kaluza–Klein reduction of the 3-form fields

$$\begin{aligned}
H^{(g)}_{0ab} &= (-P_A \varphi^A_+ + Q^{A'} \varphi_{A'-})_{ab} \\
H^{(g)}_{5ab} &= (P_A \varphi^A_+ + Q^{A'} \varphi_{A'-})_{ab} \\
H^{(g)}_{bcd} &= 0 \ ,
\end{aligned} \tag{2.138}$$

where $\star_{\mathrm{K3}} \varphi^A_+ = \varphi^A_+$ and $\star_{\mathrm{K3}} \varphi_{A'-} = -\varphi_{A'-}$, with φ^A_+, $\varphi_{A'-}$ harmonic 2-forms on K3. In particular, here $A = 1, 2, ..., 19$ and $A' = 1, 2, 3$. Note that for such an ansatz the constraint $\partial_i \Pi^-_{ij} = 0$ is automatically satisfied ($\Pi^-_{5a} = 0$ and $\partial_b \Pi^-_{ba} = 0$ because φ^A_+ and $\varphi_{A'-}$ are harmonic on K3, hence closed). Note that we have assumed that the usual Kaluza–Klein ansatz can be applied even though, strictly speaking, $H_{(g)}$ is not a differential form. In particular, we assume that the non-standard transformations arising from diffeomorphisms that we discussed in Sec. 2.2.3 can be absorbed by suitably-modified diffeomorphism transformations of P_A and $Q^{A'}$.

Using this input, one finds

$$\begin{aligned}
\mathcal{H}_- &= -\int_{\mathrm{K3}} \frac{1}{2}\sqrt{g_{\mathrm{K3}}} H^{0ab}_{(g)} H^{(g)}_{0ab} \\
&= -\int_{\mathrm{K3}} \frac{1}{4}\varepsilon^{abcd} H^{(g)}_{5cd} H^{(g)}_{0ab} \\
&= \kappa^{AB} P_A P_B + \kappa_{A'B'} Q^{A'} Q^{B'} \ ,
\end{aligned} \tag{2.139}$$

where we defined

$$\kappa^{AB} := \int_{\mathrm{K3}} \varphi_+^A \wedge \varphi_+^B \ , \qquad \kappa_{A'B'} := -\int_{\mathrm{K3}} \varphi_{A'-} \wedge \varphi_{B'-} \ , \tag{2.140}$$

which clearly are invertible matrices.

We also need to reduce the Poisson bracket (here $\vec{x}$ and $\vec{y}$ denote local coordinates on K3 and σ, σ' are coordinates in the remaining x^5 direction):

$$\begin{aligned}-\frac{1}{8}\varepsilon_{abcd}\delta_4(\vec{x}-\vec{y})\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') &= \{\Pi^-_{ab}(\sigma,\vec{x},t), \Pi^-_{cd}(\sigma',\vec{y},t)\} \\ &= \frac{1}{16}\varepsilon^{abef}\varepsilon^{cdgh}\{P_A(\sigma,t)\varphi^A_{+ef}(\vec{x}) + Q^{A'}(\sigma,t)\varphi_{A'-ef}(\vec{x}), \\ &\qquad P_B(\sigma',t)\varphi^B_{+gh}(\vec{y}) + Q^{A'}(\sigma',t)\varphi_{A'-gh}(\vec{y})\} \ ,\end{aligned} \tag{2.141}$$

and hence

$$\begin{aligned}\varepsilon_{abcd}\delta_4(\vec{x}-\vec{y})\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') = -2\det(g_{\mathrm{K3}})\{&P_A(\sigma,t)\varphi_+^{Aab}(\vec{x}) - Q^{A'}(\sigma,t)\varphi^{ab}_{A'-}(\vec{x}), \\ &P_B(\sigma',t)\varphi_+^{Bcd}(\vec{y}) - Q^{A'}(\sigma',t)\varphi^{cd}_{A'-}(\vec{y})\} \ .\end{aligned} \tag{2.142}$$

Multiplying by $\varphi^C_{+ab}(\vec{x})\varphi^D_{+cd}(\vec{y})$, $\varphi^C_{+ab}(\vec{x})\varphi_{D'-cd}(\vec{y})$ and $\varphi_{C'-ab}(\vec{x})\varphi_{D'-cd}(\vec{y})$ and integrating over K3×K3 we respectively find

$$\begin{aligned}\{P_A(\sigma,t), P_B(\sigma',t)\} &= -\frac{1}{2}\kappa^{-1}_{AB}\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') \\ \{P_A(\sigma,t), Q^{B'}(\sigma',t)\} &= 0 \\ \{Q^{A'}(\sigma,t), Q^{B'}(\sigma',t)\} &= \frac{1}{2}(\kappa^{-1})^{A'B'}\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') \ .\end{aligned} \tag{2.143}$$

This returns the same hamiltonian and Poisson-bracket structure as in [47] for (anti)-chiral bosons, albeit without having compactified the x^5 direction. Moreover, Hamilton's equations give

$$\begin{aligned}\frac{\partial P_A}{\partial t} - \frac{\partial P_A}{\partial\sigma} &= 0 \\ \frac{\partial Q^{A'}}{\partial t} + \frac{\partial Q^{A'}}{\partial\sigma} &= 0 \ ,\end{aligned} \tag{2.144}$$

and we have recovered 19 chiral bosons from P_A and 3 anti-chiral bosons from $Q^{A'}$.

The above must be supplemented with the six-dimensional scalar hamiltonian and

Poisson bracket

$$\begin{aligned} \mathcal{H}_{\rm scal} &= \frac{\sqrt{g_{\rm K3}}}{2}\left(\Pi^I\Pi^I + g^{ab}\partial_a X^I\partial_b X^I + \partial_5 X^I\partial_5 X^I\right) \\ \{X^I(\sigma,\vec{x},t),\Pi^J(\sigma',\vec{y},t)\} &= \frac{1}{\sqrt{g_{\rm K3}}}\delta^{IJ}\delta(\sigma-\sigma')\delta_4(\vec{x}-\vec{y}) \ , \end{aligned} \tag{2.145}$$

derived from the scalar part of the action (2.107) (the Ricci curvature vanishes in $\mathbb{R}^{1,1}\times$K3 and this still holds if we compactify the x^5 direction). Reducing $\mathcal{H}_{\rm scal}$ merely requires taking the scalars and their momenta to be independent of K3, and as a result simply introduces a factor, vol(K3),

$$\begin{aligned} \mathcal{H}_{\rm scal} &= \frac{1}{2}{\rm vol(K3)}\left(\Pi^I\Pi^I + \partial_\sigma X^I\partial_\sigma X^I\right) \\ \{X^I(\sigma,t),\Pi^J(\sigma',t)\} &= ({\rm vol(K3)})^{-1}\delta^{IJ}\delta(\sigma-\sigma') \ . \end{aligned} \tag{2.146}$$

If we define

$$P^I := \sqrt{\frac{\rm vol(K3)}{2}}\left(\Pi^I - \partial_\sigma X^I\right) \ , \qquad Q^I := \sqrt{\frac{\rm vol(K3)}{2}}\left(\Pi^I + \partial_\sigma X^I\right) \ . \tag{2.147}$$

we then find

$$\mathcal{H}_{\rm scal} = \frac{1}{2}P^IP^I + \frac{1}{2}Q^IQ^I \ , \tag{2.148}$$

and

$$\begin{aligned} \{P^I(\sigma,t),P^J(\sigma',t)\} &= -\delta^{IJ}\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') \\ \{P^I(\sigma,t),Q^J(\sigma',t)\} &= 0 \\ \{Q^I(\sigma,t),Q^J(\sigma',t)\} &= \delta^{IJ}\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma') \ . \end{aligned} \tag{2.149}$$

This leads to 5 chiral bosons P^I and 5 anti-chiral bosons Q^I. Similarly, the reduction of the fermionic hamiltonian clearly leads to 8 chiral and 8 anti-chiral fermions in two dimensions.

Finally, let us impose a flux-quantisation condition of the form

$$\frac{1}{(2\pi)^3}\int_{C_3} H_{(g)} \in \mathbb{Z} \ , \tag{2.150}$$

over three-cycles C_3 in the full six-dimensional theory. For the purposes of this section, it is enough to consider three-cycles of the form $C_3 = S^1\times C_2$, where S^1 is the compactified x^5 direction with radius $R=1$ (so that $\tilde{\mathcal{M}}=0$) and C_2 is a two-cycle in K3.

The harmonic forms satisfy the quantisation condition

$$\frac{1}{(2\pi)^2}\int_{C_2}\varphi^A_+ \in \mathbb{Z}\ , \qquad \frac{1}{(2\pi)^2}\int_{C_2}\varphi_{A'-} \in \mathbb{Z}\ , \tag{2.151}$$

which implies a quantisation condition

$$\frac{1}{2\pi}\int_{S^1} P_A \in \mathbb{Z}\ , \qquad \frac{1}{2\pi}\int_{S^1} Q^{A'} \in \mathbb{Z}\ . \tag{2.152}$$

This in turn implies an integral constraint on the zero-modes for P_A and $Q^{A'}$. Thus, if we view P_A and $Q^{A'}$ as arising from chiral bosons $P_A = \partial_\sigma \phi_A$, $Q^{A'} = \partial_\sigma \phi^{A'}$, then ϕ_A and $\phi^{A'}$ must be compact with period 2π.

All in all, we find $19 + 5 = 24$ chiral bosons (19 of which are compact) $3 + 5 = 8$ anti-chiral bosons (3 of which are compact), 8 chiral fermions and 8 anti-chiral fermions *i.e.* the physical degrees of freedom of a heterotic string transverse to $\mathbb{R}^5 \times \mathbb{T}^3$.

2.3.3 Reduction on a Riemann Surface

It has been known for some time that the dynamics of a single M5-brane on a non-compact Riemann surface leads at low energies to the Seiberg–Witten effective action [9] of a four-dimensional $\mathcal{N} = 2$ gauge theory [126]. The idea is to wrap the M5-brane worldvolume on a complex curve Σ, whose embedding into spacetime is specified by some holomorphic function $s(z)$. Such a curve is subjected to boundary conditions whose interpretation at infinity is that of intersecting M5-branes. Reducing to type IIA string theory leads to a picture of parallel D4-branes suspended between NS5-branes whose dynamics is given by an $\mathcal{N} = 2$ Yang-Mills gauge theory. One then finds that $s(z)$ depends on various moduli of the Riemann surface u_α, $\alpha = 1, ..., N-1$. To compute the four-dimensional effective action from the M5-brane one is not interested in all of its dynamics, rather just those of its zero-modes: the moduli u_α and their superpartners.

This framework was used to reproduce the scalar sector of the resultant four-dimensional effective action in [121], where a simple kinetic term for the single M5-brane theory can be easily written down. To find the dynamics of the vector fields without an action is more involved. Without scalars, and for a flat torus, the calculation appeared in [127]. For the case of a single M5-brane on a generic Riemann surface the calculation was done in [122] using the equations of motion. This led to interesting integrals over non-holomorphic functions whose evaluation is nevertheless a holomorphic function of the moduli.

But now that we have a proposed action for the self-dual tensors in six-dimensions, this setup provides a natural and non-trivial testing ground for its interpretation as capturing the low-energy dynamics of single M5-brane. The reduction of the action

(2.21) over a rigid compact torus was already performed in [47] and shows the correct $SL(2,\mathbb{Z})$ invariance expected from large diffeomorphisms. Here we will concern ourselves with the case of generic, non-compact Riemann surfaces.

We therefore want to consider an M5-brane where two of its directions (x^4 and x^5 combined into the complex coordinate $z = x^4 + ix^5$), are embedded into spacetime by means of the function $s = X^6 + iX^{10}$. Here X^{10} denotes the M-theory direction and is compact. We label the remaining worldvolume coordinates by x^m, $m = 0, 1, 2, 3$. The embedding of the M5-brane is defined by $X^m = x^m$, $X^7 = X^8 = X^9 = 0$ and in particular is such that $s(z)$ is a holomorphic function [121].[19] The induced metric on the M5 is given by

$$g = \begin{pmatrix} \eta_4 & 0 & 0 \\ 0 & 0 & (1+\partial_z s\bar{\partial}_{\bar{z}}\bar{s})/2 \\ 0 & (1+\partial_z s\partial_{\bar{z}}\bar{s})/2 & 0 \end{pmatrix} . \tag{2.153}$$

Here the coordinates are $0, 1, 2, 3, z, \bar{z}$ so that

$$\eta = \begin{pmatrix} \eta_4 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix} . \tag{2.154}$$

In the usual fashion, the zero-mode dynamics can be determined by working in the Manton approximation [128]: the moduli—and consequently s—are promoted to functions of the remaining four coordinates x^m, $m = 0, 1, 2, 3$ that are slowly varying so that [121]

$$\partial_m s = \sum_\alpha \frac{\partial s}{\partial u_\alpha}\partial_m u_\alpha . \tag{2.155}$$

From this one defines the Seiberg–Witten differential $\lambda_{SW} = s(z)dz$ [129] and the holomorphic 1-forms

$$\lambda_\alpha = \frac{\partial s}{\partial u_\alpha}dz . \tag{2.156}$$

Following [9] one identifies the low-energy scalar fields as

$$a_\alpha = \oint_{A_\alpha} s dz , \tag{2.157}$$

[19] At this stage we neglect terms with $\partial_m s \neq 0$ as these will result into higher-order derivative terms in the Seiberg–Witten effective action.

where A_α, B^α are a basis of cycles of Σ with intersection matrix

$$A_\alpha \cap B^\beta = -B^\beta \cap A_\alpha = \delta_\alpha^\beta \ . \tag{2.158}$$

One also defines the dual variables a_α^D as

$$a_\alpha^D = \oint_{B^\alpha} s dz \ . \tag{2.159}$$

The periods of the holomorphic 1-forms are then

$$\oint_{A_\gamma} \lambda_\alpha = \frac{\partial a_\gamma}{\partial u_\alpha} \ , \qquad \oint_{B_\gamma} \lambda_\alpha = \frac{\partial a_\gamma^D}{\partial u_\alpha} \ , \tag{2.160}$$

while the period matrix can be expressed as

$$\tau_{\alpha\beta} = \frac{\partial a_\alpha^D}{\partial a_\beta} = \tau_{\beta\alpha} \ . \tag{2.161}$$

It is useful to note that

$$\begin{aligned} \int_\Sigma \lambda_\alpha \wedge \bar\lambda_\beta &= \sum_\gamma \left(\oint_{A_\gamma} \lambda_\alpha \oint_{B_\gamma} \bar\lambda_\beta - \oint_{B_\gamma} \lambda_\alpha \oint_{A_\gamma} \bar\lambda_\beta \right) \\ &= \sum_\gamma \left(\frac{\partial a_\gamma}{\partial u_\alpha} \frac{\partial \bar a_\gamma^D}{\partial \bar u_\beta} - \frac{\partial a_\gamma^D}{\partial u_\alpha} \frac{\partial \bar a_\gamma}{\partial \bar u_\beta} \right) \\ &= \sum_\gamma \frac{\partial a_\gamma}{\partial u_\alpha} \frac{\partial \bar a_\delta}{\partial \bar u_\beta} (\bar\tau_{\gamma\delta} - \tau_{\gamma\delta}) \ . \end{aligned} \tag{2.162}$$

We can also consider the holomorphic 1-forms

$$\vartheta_\alpha = \frac{\partial s}{\partial a_\alpha} dz = \sum_\beta \frac{\partial u_\beta}{\partial a_\alpha} \lambda_\beta \ , \tag{2.163}$$

which are normalised to have unit period over the A-cycles:

$$\oint_{A_\gamma} \vartheta_\alpha = \delta_\alpha^\gamma \tag{2.164}$$

and hence

$$\oint_{B^\gamma} \vartheta_\alpha = \tau_{\alpha\gamma} \ , \qquad \int_\Sigma \vartheta_\alpha \wedge \bar\vartheta_\beta = \bar\tau_{\alpha\beta} - \tau_{\alpha\beta} \ . \tag{2.165}$$

This machinery can be applied to the scalar part of the action (2.107). One straightforwardly finds [121]:

$$\begin{aligned}
S_{scal} &= -\frac{1}{2}\int d^4x d^2z \partial_m s \partial^m \bar{s} \\
&= -\frac{1}{2}\sum_{\alpha,\beta}\int d^4x d^2z \,\frac{\partial s}{\partial u_\alpha}\frac{\partial \bar{s}}{\partial \bar{u}_\beta}\partial_m u_\alpha \partial^m \bar{u}_\beta \\
&= -\frac{i}{4}\sum_{\alpha,\beta}\int d^4x \partial_m u_\alpha \partial^m \bar{u}_\beta \int_\Sigma \lambda_\alpha \wedge \bar{\lambda}_\beta \\
&= -\frac{i}{4}\sum_{\alpha,\beta}\int d^4x (\bar{\tau}_{\alpha\beta} - \tau_{\alpha\beta})\partial_m a_\alpha \partial^m \bar{a}_\beta \\
&= -\frac{1}{2}\sum_{\alpha,\beta}\int d^4x \mathrm{Im}\left(\tau_{\alpha\beta}\partial_m a_\alpha \partial^m \bar{a}_\beta\right) \ ,
\end{aligned} \tag{2.166}$$

which is precisely the scalar part of the Seiberg–Witten effective action.

However, our main goal is to use the action (2.21) to reproduce the gauge-field part of the four-dimensional effective action. To proceed, note that if H is of the special form $H = \mathcal{F} \wedge dz$ or $H = \bar{\mathcal{F}} \wedge d\bar{z}$ then one finds $\star_\eta H = H$ provided that $\star_4 \mathcal{F} = -i\mathcal{F}$. The remaining (anti)self-dual forms with respect to η can be expressed in terms of the basis

$$\begin{aligned}
\omega_+ &= h + \frac{i}{2}\star_4 h \wedge dz \wedge d\bar{z} \\
\omega_- &= h - \frac{i}{2}\star_4 h \wedge dz \wedge d\bar{z} \ ,
\end{aligned} \tag{2.167}$$

where $h = \frac{1}{3!}h_{mnl}dx^m \wedge dx^n \wedge dx^l$. Therefore in general we have

$$H = \mathcal{F} \wedge dz + \bar{\mathcal{F}} \wedge d\bar{z} + h + \frac{i}{2}\star_4 h \wedge dz \wedge d\bar{z} \ , \tag{2.168}$$

with $\mathcal{F} = i\star_4 \mathcal{F}$, for which H is real and satisfies $\star_\eta H = H$.

For completeness, let us also determine $H_{(g)}$. When $H = \mathcal{F}\wedge dz$ or $H = \bar{\mathcal{F}}\wedge d\bar{z}$ one has that $\star_g H = H$ and thus $\tilde{\mathcal{M}}(H_{mnz}dx^m \wedge dx^n \wedge dz) = \tilde{\mathcal{M}}(H_{mn\bar{z}}dx^m \wedge dx^n \wedge d\bar{z}) = 0$, whereas the remaining $\star_g$-self-dual forms can be expressed in terms of the basis

$$\begin{aligned}
\varphi &= h + i\frac{1+\partial_z s \partial_{\bar{z}}\bar{s}}{2}\star_4 h \wedge dz \wedge d\bar{z} \\
&= \frac{2+\partial_z s \partial_{\bar{z}}\bar{s}}{2}\omega_+ - \frac{\partial_z s \partial_{\bar{z}}\bar{s}}{2}\omega_- \ ,
\end{aligned} \tag{2.169}$$

from which using (2.33), (2.37) and (2.40) we obtain

$$\tilde{\mathcal{M}}\left(h + \frac{i}{2}(\star_4 h)\wedge dz \wedge d\bar{z}\right) = \frac{\partial_z s \partial_{\bar{z}}\bar{s}}{2+\partial_z s \partial_{\bar{z}}\bar{s}}\left(h - \frac{i}{2}(\star_4 h)\wedge dz \wedge d\bar{z}\right) \ . \tag{2.170}$$

Finally, from (2.45)

$$H_{(g)} = \mathcal{F} \wedge dz + \bar{\mathcal{F}} \wedge d\bar{z} + \frac{2}{2 + \partial_z s \partial_{\bar{z}} \bar{s}} h + i \frac{1 + \partial_z s \partial_{\bar{z}} \bar{s}}{2 + \partial_z s \partial_{\bar{z}} \bar{s}} \star_4 h \wedge dz \wedge d\bar{z} \ . \tag{2.171}$$

To arrive at the desired four-dimensional effective action including gauge fields, one needs to consider a suitable ansatz for H and B by truncating to the lowest Kaluza–Klein modes; this corresponds to restricting to harmonic 1-forms on Σ.[20] We pick the following normalisation:

$$H = \sum_\alpha \mathcal{F}_\alpha \wedge \vartheta_\alpha + \sum_\alpha \bar{\mathcal{F}}_\alpha \wedge \bar{\vartheta}_\alpha \ , \tag{2.172}$$

where $\mathcal{F}_\alpha = i \star_4 \mathcal{F}_\alpha$, while for B we initially set

$$B = \sum_\alpha A_\alpha \wedge \vartheta_\alpha + \sum_\alpha \bar{A}_\alpha \wedge \bar{\vartheta}_\alpha \ , \tag{2.173}$$

where $A_\alpha = A_{\alpha m} dx^m$ are four-dimensional 1-forms.

At this stage recall that the action (2.21) has a gauge symmetry $B \to B + d\Lambda$, where Λ is an arbitrary 1-form. This is expected to descend to a 0-form gauge symmetry for A_α: $A_\alpha \to A_\alpha + d_4\lambda_\alpha$. However, since the ϑ_α are dynamical, under such a transformation

$$B \to B + d\left(\sum_\alpha \lambda_\alpha \vartheta_\alpha + \sum_\alpha \bar{\lambda}_\alpha \bar{\vartheta}_\alpha\right) - \sum_{\alpha,\beta} \lambda_\alpha d_4 a_\beta \wedge \frac{\partial \vartheta_\alpha}{\partial a_\beta} - \sum_{\alpha,\beta} \bar{\lambda}_\alpha d_4 \bar{a}_\beta \wedge \frac{\partial \bar{\vartheta}_\alpha}{\partial \bar{a}_\beta} \ . \tag{2.174}$$

To compensate for this we introduce four-dimensional Stueckelberg-like scalar fields $c_\alpha, \bar{c}_\alpha$ and expand

$$B = \sum_\alpha \left(A_\alpha \wedge \vartheta_\alpha - c_\alpha d_4 a_\beta \wedge \frac{\partial \vartheta_\alpha}{\partial a_\beta}\right) + \sum_\alpha \left(\bar{A}_\alpha \wedge \bar{\vartheta}_\alpha - \bar{c}_\alpha d_4 \bar{a}_\beta \wedge \frac{\partial \bar{\vartheta}_\alpha}{\partial \bar{a}_\beta}\right) \ , \tag{2.175}$$

so that under the combined gauge transformation $A_\alpha \to A_\alpha + d_4\lambda_\alpha$, $c_\alpha \to c_\alpha - \lambda_\alpha$ we recover a one-form gauge transformation

$$B \to B + d\left(\sum_\alpha \lambda_\alpha \vartheta_\alpha + \sum_\alpha \bar{\lambda}_\alpha \bar{\vartheta}_\alpha\right) \ . \tag{2.176}$$

[20] Since Σ is non-compact the zero-form and two-form harmonic forms have divergent integrals and hence do not lead to low-energy modes. Thus $\tilde{\mathcal{M}}$ does not play a role here.

With this in hand, we compute

$$
\begin{aligned}
dB &= \sum_\alpha d_4 A_\alpha \wedge \vartheta_\alpha + \sum_\alpha d_4 \bar{A}_\alpha \wedge \bar{\vartheta}_\alpha \\
&\quad - \sum_\alpha (A_\alpha + d_4 c_\alpha) \wedge d_4 a_\beta \wedge \frac{\partial \vartheta_\alpha}{\partial a_\beta} - \sum_{\alpha,\beta} (\bar{A}_\alpha + d_4 \bar{c}_\alpha) \wedge d_4 \bar{a}_\beta \wedge \frac{\partial \bar{\vartheta}_\alpha}{\partial \bar{a}_\beta} \\
\star_\eta dB &= \sum_\alpha i \star_4 d_4 A_\alpha \wedge \vartheta_\alpha - \sum_\alpha i \star_4 d_4 \bar{A}_\alpha \wedge \bar{\vartheta}_\alpha \\
&\quad - \sum_\alpha i \star_4 ((A_\alpha + d_4 c_\alpha) \wedge d_4 a_\beta) \wedge \frac{\partial \vartheta_\alpha}{\partial a_\beta} + \sum_{\alpha,\beta} i \star_4 ((\bar{A}_\alpha + d_4 \bar{c}_\alpha) \wedge d_4 \bar{a}_\beta) \wedge \frac{\partial \bar{\vartheta}_\alpha}{\partial \bar{a}_\beta} \ ,
\end{aligned}
\tag{2.177}
$$

where d_4 denotes the exterior derivative along x^m. To continue, observe that

$$
\begin{aligned}
\int_\Sigma \vartheta_\alpha \wedge \frac{\partial \bar{\vartheta}_\beta}{\partial \bar{a}_\gamma} &= \frac{\partial}{\partial \bar{a}_\gamma} \int_\Sigma \vartheta_\alpha \wedge \bar{\vartheta}_\beta = \frac{\partial \bar{\tau}_{\alpha\beta}}{\partial \bar{a}_\gamma} \\
\int_\Sigma \frac{\partial \vartheta_\alpha}{\partial a_\gamma} \wedge \bar{\vartheta}_\beta &= \frac{\partial}{\partial a_\gamma} \int_\Sigma \vartheta_\alpha \wedge \bar{\vartheta}_\beta = -\frac{\partial \tau_{\alpha\beta}}{\partial a_\gamma} \ ,
\end{aligned}
\tag{2.178}
$$

and

$$
\int_\Sigma \frac{\partial \vartheta_\alpha}{\partial a_\gamma} \wedge \frac{\partial \bar{\vartheta}_\beta}{\partial \bar{a}_\delta} = \frac{\partial}{\partial \bar{a}_\delta} \int_\Sigma \frac{\partial \vartheta_\alpha}{\partial a_\gamma} \wedge \bar{\vartheta}_\beta = -\frac{\partial}{\partial \bar{a}_\delta} \left(\frac{\partial \tau_{\alpha\beta}}{\partial a_\gamma} \right) = 0 \ . \tag{2.179}
$$

Substituting (2.172) and (2.177) into (2.21) we find[21]

$$
\begin{aligned}
S_H &= \int \Big(-(\tau - \bar{\tau})_{\alpha\beta} \left(-d_4 A_\alpha \wedge i \star_4 d\bar{A}_\beta - 2\mathcal{F}_\alpha \wedge d_4 \bar{A}_\beta + 2\bar{\mathcal{F}}_\alpha \wedge d_4 A_\beta \right) \\
&\quad + \frac{\partial \tau_{\alpha\beta}}{\partial a_\gamma} (-i \star_4 d_4 \bar{A}_\alpha \wedge (A_\beta + d_4 c_\beta) \wedge d_4 a_\gamma + 2\bar{\mathcal{F}}_\alpha \wedge (A_\beta + d_4 c_\beta) \wedge d_4 a_\gamma) \\
&\quad + \frac{\partial \bar{\tau}_{\alpha\beta}}{\partial \bar{a}_\gamma} (i \star_4 d_4 A_\alpha \wedge (\bar{A}_\beta + d_4 \bar{c}_\beta) \wedge d_4 \bar{a}_\gamma + 2\mathcal{F}_\alpha \wedge (\bar{A}_\beta + d_4 \bar{c}_\beta) \wedge d_4 \bar{a}_\gamma) \Big) \\
&= \int \Big((\tau - \bar{\tau})_{\alpha\beta} \left(d_4 A_\alpha \wedge i \star_4 d\bar{A}_\beta + 2\mathcal{F}_\alpha \wedge d_4 \bar{A}_\beta - 2\bar{\mathcal{F}}_\alpha \wedge d_4 A_\beta \right) \\
&\quad + (-i \star_4 d_4 \bar{A}_\alpha \wedge (A_\beta + d_4 c_\beta) \wedge d_4 \tau_{\alpha\beta} + 2\bar{\mathcal{F}}_\alpha \wedge (A_\beta + d_4 c_\beta) \wedge d_4 \tau_{\alpha\beta}) \\
&\quad + (i \star_4 d_4 A_\alpha \wedge (\bar{A}_\beta + d_4 \bar{c}_\beta) \wedge d_4 \bar{\tau}_{\alpha\beta} + 2\mathcal{F}_\alpha \wedge (\bar{A}_\beta + d_4 \bar{c}_\beta) \wedge d_4 \bar{\tau}_{\alpha\beta}) \Big) \ .
\end{aligned}
\tag{2.180}
$$

It is helpful to introduce the two-form

$$
\mathcal{F}^{(s)}_\alpha := \mathcal{F}_\alpha + \frac{1}{2} d_4 A_\alpha + \frac{i}{2} \star_4 d_4 A_\alpha \ , \tag{2.181}
$$

[21] We remind the reader that due to our Kaluza–Klein ansatz $\tilde{\mathcal{M}}$ does not enter this calculation.

and combine the pieces (2.166) and (2.180) to rewrite the action as

$$\begin{aligned} S &= S_{scal} + S_H \\ &= \int \Big(-\frac{1}{4}(\tau-\bar\tau)_{\alpha\beta} d_4 a_\alpha \wedge i \star_4 d_4 \bar a_\beta - (\tau-\bar\tau)_{\alpha\beta} d_4 A_\alpha \wedge i \star_4 d_4 \bar A_\beta + (\tau+\bar\tau)_{\alpha\beta} d_4 A_\alpha \wedge d_4 \bar A_\beta \\ &\quad + 2\mathcal{F}^{(s)}_\alpha \wedge \big((\tau-\bar\tau)_{\alpha\beta} d_4 \bar A_\beta - d_4 \bar\tau_{\alpha\beta} \wedge (\bar A_\beta + d_4 \bar c_\beta)\big) \\ &\quad - 2\bar{\mathcal{F}}^{(s)}_\alpha \wedge \big((\tau-\bar\tau)_{\alpha\beta} d_4 A_\beta + d_4 \tau_{\alpha\beta} \wedge (A_\beta + d_4 c_\beta)\big)\Big) \ . \end{aligned} \tag{2.182}$$

The first line agrees with the Seiberg–Witten effective action [9] but for two sets of $U(1)$ gauge fields, corresponding to the real and imaginary parts of A_α. However, the $\bar{\mathcal{F}}^{(s)}_\alpha$ equation imposes the constraint

$$(\tau-\bar\tau)_{\alpha\beta} d_4 A_\beta + d_4 \tau_{\alpha\beta} \wedge (A_\beta + d_4 c_\beta) = i \star_4 \big((\tau-\bar\tau)_{\alpha\beta} d_4 A_\beta + d_4 \tau_{\alpha\beta} \wedge (A_\beta + d_4 c_\beta)\big) \ . \tag{2.183}$$

This implies that the real and imaginary parts of A_α are related by electric-magnetic duality, *e.g.* if $d_4\tau_{\alpha\beta} = 0$ then this reduces to

$$\mathrm{Im}(d_4 A_\alpha) = \star_4 \mathrm{Re}(d_4 A_\alpha) \ . \tag{2.184}$$

More generally the constraint (2.183) is harder to disentangle. It is worth observing that $A_\alpha + d_4 c_\alpha$ are gauge invariant 1-forms which could provide a restriction on the types of fields that can arise.

Next, we observe that the Stueckelberg fields impose the equation of motion

$$d_4 \mathcal{F}^{(s)}_\alpha \wedge d_4 \bar\tau_{\alpha\beta} = 0 \ , \tag{2.185}$$

which generically implies that $d_4\mathcal{F}^{(s)}_\alpha = 0$. Thus the $\mathcal{F}^{(s)}_\alpha$ decouple in the sense that their equations of motion do not depend on the other fields.

One also finds extra contributions to the scalar and vector equations of motion arising from $\mathcal{F}^{(s)}$. Assuming $d_4\mathcal{F}^{(s)}_\alpha = 0$ we find

$$\begin{aligned} 0 =& (\tau-\bar\tau)_{\alpha\beta} d_4 i \star_4 d_4 a_\beta + \frac{\partial \tau_{\alpha\beta}}{\partial a_\gamma} d_4 a_\gamma \wedge i \star_4 d_4 a_\beta \\ &+ 2\frac{\partial \bar\tau_{\beta\gamma}}{\partial \bar a_\alpha}(d_4 A_\beta + i \star_4 d_4 A_\beta) \wedge (d_4 \bar A_\gamma + i \star_4 d_4 \bar A_\gamma) + 4\frac{\partial \bar\tau_{\beta\gamma}}{\partial \bar a_\alpha} \bar{\mathcal{F}}^{(s)}_\beta \wedge (d_4 A_\gamma - i \star_4 dA_\gamma) \\ 0 =& d_4\left(i \star_4 (\tau-\bar\tau)_{\alpha\beta} dA_\beta - (\tau+\bar\tau)_{\alpha\beta} dA_\beta\right) - 2 d_4 \tau_{\alpha\beta} \wedge \mathcal{F}^{(s)}_\beta \ . \end{aligned} \tag{2.186}$$

One recovers the standard Seiberg–Witten equations [9] in the special case of $\mathcal{F}^{(s)}_\alpha = 0$. More generally, $\mathcal{F}^{(s)}_\alpha$ acts as an non-dynamical background electromagnetic field. Its effects can also be implemented by replacing the last two lines of (2.182) by

$$\mathcal{L}_{background} = 2\tau_{\alpha\beta}\mathcal{F}^{(s)}_{\alpha} \wedge d_4\bar{A}_{\beta} + 2\bar{\tau}_{\alpha\beta}\bar{\mathcal{F}}^{(s)}_{\alpha} \wedge d_4 A_{\beta} \ , \tag{2.187}$$

and imposing the self-duality constraint (2.183) by hand.

Lastly, let us comment on the fact that the equations of motion also admit a sector where $d_4\bar{\tau}_{\alpha\beta} = 0$. On the one hand, for generic $\bar{\tau}_{\alpha\beta}$ the dynamical constraint on $\mathcal{F}^{(s)}$ from (2.185) freezes out the scalars, and hence also the vectors. On the other, if $\tau_{\alpha\beta}$ is constant then we recover a free Seiberg–Witten theory for two gauge fields related by (2.184). Mixed solutions where both $d\mathcal{F}^{(s)}_{\alpha}$ and $d_4\bar{\tau}_{\alpha\beta}$ are non-zero do not seem very likely unless $\tau_{\alpha\beta}$ has some reduced dependence on the moduli.

2.4 Conclusions

In this chapter, we studied the six-dimensional action put forward in [47]—and its (2,0) supersymmetric completion [26]—clarifying many of its unconventional features. This formulation aims to encode the dynamics of a chiral 2-form in 6D into a "2-form" B and an $\star_\eta$-self-dual "3-form" H. Although all of our analysis is performed for chiral 2-forms in six-dimensions, all the techniques we developed can be readily applied to other dimensions.

We elucidated on the coupling of these fields to arbitrary geometries, and in the course of doing so provided a construction of the interaction term $\tilde{\mathcal{M}}$ that goes beyond the perturbative approach of [47]. Moreover, we wrote down how the original fields B, H—which are not conventional differential forms and we dubbed "pseudo-forms"—can be combined into the unphysical $H_{(s)}$ and the physical $H_{(g)}$ fields; the $H_{(s)}$ is a singlet while $H_{(g)}$ has (on-shell) standard transformation properties under diffeomorphisms. We also clarified some aspects of the hamiltonian analysis. First, we showed that $H_{(s)}$ and $H_{(g)}$ correspond precisely to the $\Pi^{\pm}$ variables introduced in [47]. The fact that Π^{+} describes a non-unitary decoupled sector of the theory is consistent with the fact that $H_{(s)}$ is self-dual with respect to η. Indeed, $H_{(s)}$ inherits the non-unitarity of B so it must completely decouple from the physics including gravity. Instead, $H_{(g)}$ carries the physical degrees of freedom and is self-dual with respect to the actual physical metric. Second, we gave a formulation of the hamiltonian in terms of $H_{(s)}$ and the energy-momentum tensor $T^0{}_0$ of $H_{(g)}$. We therefore showed that it is possible to construct the physical hamiltonian by first using familiar geometric techniques to compute $T^0{}_0$ and then re-expressing $H_{(g)}$ in terms of Π^{-}; this leads to particularly simple expressions for static backgrounds (*i.e.* $g_{0i} = 0$).

We then dimensionally reduced the proposed (2,0) action on three backgrounds: a circle, K3 and a Riemann surface. We performed these reductions by implementing the usual Kaluza–Klein ansatz, that is assuming that the only surviving modes at low energies are the zero-modes. While this is standard for theories with physical degrees of freedom, it is not entirely clear that there are no subtleties for the case at hand, where we are dealing with "pseudo-forms"—one of which (B) has the wrong-sign kinetic term. With that disclaimer, we proceeded and found results that are aligned with expectations. For the circle reduction we arrived at a Maxwell theory that scales like $1/R$. Although in a free theory one can always rescale the fields to change the overall coefficient, this $1/R$ scaling is also consistent with the Legendre transform of the 6D hamiltonian reduced on the circle, if one works with canonically-conjugate pairs. A logically straightforward next step in this direction would be to explicitly extend the analysis to the nonabelian, 3-algebra version of the theory constructed in [26]. For the reduction on the Riemann surface, we recovered the expected 4D $\mathcal{N} = 2$ Seiberg–Witten effective action for two sets of abelian gauge fields, subject to a constraint. Perhaps surprisingly, in the special case where the period matrix of the Seiberg–Witten curve $\tau_{\alpha\beta}$ was independent of the Riemann-surface moduli, this constraint related the gauge fields via standard electric-magnetic duality, reminiscent of the work of [130]. Therefore another interesting direction would be to better understand the nature of the constraint and to what extent it encodes information about electric-magnetic duality in general.

Other directions could involve understanding how to couple $H^{J}_{(g)}$ to a self-dual string, or exploiting the ideas introduced here to write down a four dimensional Maxwell theory that is manifestly invariant under both Lorentz transformations and duality symmetry, along the lines of what happens for the PST formalism; *c.f.* [96]. From a more speculative perspective, it would be very interesting if there existed a nice geometric construction that accommodates "pseudo-forms" and explains the properties of $\tilde{\mathcal{M}}$. Moreover, the fact that H and B mix under diffeomorphisms could be due to both originating from the same object in a higher-dimensional theory, after compactification. For example, the idea that the abelian 6D (2,0) theory can be formulated as a 7D Chern–Simons theory has been put forward in [73] and further utilised in [116].

To summarise, the action discussed here is a novel, relatively simple formulation that is consistent with the abelian, low-energy physics of a single M5-brane in M-theory. It has several attractive features: it is Lorentz and diffeomorphism covariant without introducing a scalar field that ultimately requires some non-vanishing preferred direction—as *e.g.* is the case in the PST formalism. Although we require additional modes with the wrong-sign kinetic terms these can be discarded—effectively set to zero—when one examines the physical degrees of freedom.

Chapter 3

Path Integral for Sen's Action

Whereas, in the previous chapter, we mainly consider classical aspects of the action proposed by Sen [46, 47] for chiral forms, we now move to test his approach down to the quantum level. In particular, we evaluate the partition function of the compact chiral boson on a two-dimensional torus using a path integral formulation. Crucially, to ensure the convergence of the path integral, we use a Wick-rotation procedure obtained from a complex deformation of the physical spacetime metric. This will directly reproduce the expected result including general characteristics for the theta functions. We also present results for the chiral 2-form potential in six dimensions which can be readily extended to $4k + 2$ dimensions.

3.1 Introduction

Given the complications of finding action principles for self-dual fields, a popular strategy for obtaining the chiral form partition function *via* a path-integral computation is the so-called *holomorphic factorisation*: one starts with the action for the non-chiral version of the field, evaluates the corresponding path integral in the Wick-rotated theory, observes that the result essentially factorises and reads off the chiral part [72, 131, 132]. For example, the path integral for a compact non-chiral boson on an Euclidean two-dimensional torus with complex structure τ yields this very schematic result

$$Z_{\text{non-chiral}}(J, \tau) \sim \sum_{\alpha,\beta} \left| \frac{\theta\begin{bmatrix}\alpha \\ \beta\end{bmatrix}(J_+|\tau)}{\eta(\tau)} \right|^2 , \tag{3.1}$$

where η is the usual Dedekind's η-function, $J_{\pm}$ are the chiral/anti-chiral parts of the external source J and the sum is over the characteristics α, β of the theta functions $\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$. Therefore, from (3.1) we can deduce that the partition function for a chiral boson should read, again very schematically, as

$$Z_{\text{chiral}}(J_+, \tau) \sim \frac{\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(J_+|\tau)}{\eta(\tau)} \tag{3.2}$$

for some chosen α, β, and that the partition function for the antichiral boson is just the complex conjugate of it. There are two important features about the partition function (3.2) that we can immediately appreciate.

- In defining the quantum theory of a chiral boson, global aspects play an important role, which enter the picture via the α, β characteristics. To get a feeling of which geometric structure α, β actually parametrize, we remind the reader that when the radius of the compact boson is fine-tuned to the so-called free fermion radius, then the chiral boson is dual to a Weyl fermion; in this case, if one restricts α, β to take values in $\{0, \frac{1}{2}\}$ then the characteristics can be identified with the spin structure of the dual fermion (i.e. periodic or anti-periodic boundary conditions). As α, β are real numbers, they actually parametrize an appropriate generalization of the spin structure[22] which, from an heuristic point of view, essentially captures the fact that the winding modes of the chiral boson are not restricted to be just integers but are instead allowed to be any real numbers.

- In general, (3.2) is not invariant under $SL(2, \mathbb{Z})$ modular transformations of τ. A failure of modular invariance implies that the chiral field in $4k + 2$ dimensions is not a genuine $4k + 2$ dimensional system[23]. It is more appropriate to think of it in terms of the dynamics on the boundary of a $4k + 3$ dimensional theory [133]. In [73, 116, 134, 135], the latter was identified with an appropriate generalization of Chern–Simons theory. This is the most rigorous approach to the quantization of a chiral field, which takes into account all the subtle global aspects associated with the quantisation of the self-dual field. For example, in [116] (see also [136] and [137]) it was showed that one is naturally forced to introduce the α, β characteristic when imposing the Gauss law constraint on Chern–Simons theory in one dimension higher.

Since the information about the background spin structure should in principle be

[22] Dubbed Quadratic Refinement of the Intersection Form (QRIF for short) in [116]. With an abuse of language, we will instead keep calling it spin structure.

[23] And the moniker "partition function" could be taken as an abuse of language. In the following sections, with "partition function" we will implicitly mean the path integral evaluated on a spacetime that is an Euclidean torus.

encoded within the action in the form of topological terms [72,131], one could naturally ask whether candidate actions for chiral forms in $4k+2$ dimensions can reproduce the chiral partition function (3.2) *via* a path integral formulation, without resorting to the Chern–Simons description. In this chapter we use the action of [46, 47] as a starting point for precisely such a calculation for the chiral boson on the torus.[24] An immediate obstacle pertains to Wick rotating the action to Euclidean signature, as the standard analytic continuation to imaginary time leads to a path integral that does not converge because of the wrong-sign nature of one of the fields. Moreover, one may wonder how to impose a self-duality constraint when the signature changes and the Hodge star no longer squares to the identity. Here, however, we employ an alternative prescription for Wick rotating *via* a complex deformation of the physical spacetime metric, as suggested by Visser [28]. It is a happy coincidence that the non-standard coupling of the fields in the Sen action to the background metric is precisely such that the resulting path integral is convergent. Furthermore, as this alternative Wick rotation does not modify the reference metric, the self-duality constraint is unaffected and the physical degrees of freedom of the system are explicitly preserved.

This allows us to proceed with the evaluation of the path integral. Note that in our calculation both the physical ($H_{(g)}$) and unphysical modes ($H_{(s)}$) contribute to the partition function. In this way, through a collection of delicate but ordinary manipulations, we recover the standard expression for the chiral-boson partition function in the form of the ratio $\theta(T)/\eta(T)$, where T is related to the complex structure of the torus by an $SL(2,\mathbb{Z})$ transformation. It is interesting to point out that, when the radius-squared of the chiral boson is rational (corresponding to a rational conformal field theory), the resulting partition function is an extended $\widehat{\mathfrak{u}}(1)$ character as expected [139–141]. In our calculation the choice of spin structure corresponds to introducing topological terms (*i.e.* terms that do not affect the equations of motion) to the Sen action along with a change in the boundary conditions for the fields. We make appropriate choices for such terms, leading to more general theta characteristics in the $\theta(T)/\eta(T)$ result, once again as expected for the widely studied chiral boson.

The extension of these results to $4k+2$ dimensional theories is of great interest and we initiate this study by evaluating the path integral of the six-dimensional version of the Sen action on the six-torus, *i.e.* $k=1$, including an additional topological term. One recovers once again appropriate generalisations of the two-dimensional answer, involving higher theta functions with general characteristics. Our results here can be readily extended to more general values of k.

The rest of this chapter, which is based on the paper [2], is organised as follows. We

[24] The path integral approach for the chiral forms actions mentioned in Sec. 2.1.2 can be found in: [82, 84] for the McClain-Wu-Yu action, [138] for the Floreanini-Jackiw action and [70] for the Pasti-Sorokin-Tonin action.

begin in the next section with a summary of the salient features of the Wick-rotation prescription of [28]. We continue in Section 3.2 with the implementation of the Wick rotation and evaluation of the path integral for the chiral boson on the torus. We calculate the oscillator and winding-mode contributions to obtain the $\theta(T)/\eta(T)$ result, with the more general theta characteristics following suit after adding appropriate topological terms to the action and modifying the boundary conditions. In Section 3.3 we sketch the corresponding setup for the chiral 2-form potential in six dimensions and calculate the oscillator and winding-mode contributions to the path integral. We conclude with some a posteriori comments in Section 3.4.

3.1.1 Wick-Rotation via Metric Deformation

When evaluating the path integral, one usually passes to the Euclidean version of the theory to ensure convergence. However, applying the standard Wick-rotation procedure of analytically continuing to imaginary time leads to a non-convergent answer in Sen's action (2.59), because of a wrong sign for the B kinetic term along with the $dB \wedge H$ mixing term. A new prescription is also needed since, due to the self-duality constraint, one would like to Wick rotate the theory without altering the degrees of freedom of the system.

We will employ the proposal of Visser [28], where instead of analytically continuing to imaginary time, one performs a complex deformation of the Lorentzian spacetime metric g, and not the coordinates, *via*

$$(g_\epsilon)_{\mu\nu} := g_{\mu\nu} + i\epsilon \frac{V_\mu V_\nu}{V^\lambda V_\lambda} \ . \tag{3.3}$$

Here, ϵ is the deformation parameter, V^μ is an arbitrary, nowhere-vanishing timelike vector field (which is guaranteed to exist because the spacetime has a global Lorentzian signature) and V_μ is the associated co-vector, *i.e.* $V_\mu := g_{\mu\nu}V^\nu$. For the simple case of flat space and a constant vector $V_\mu = (-1, 0, \ldots, 0)$, (3.3) results in

$$g_\epsilon = \text{diag}(-1 - i\epsilon, +1, \ldots, +1) \ , \tag{3.4}$$

which recovers the standard Minkowski metric for $\epsilon = 0$ and the Euclidean metric for $\epsilon = 2i$. In fact this case is completely equivalent to the standard Wick rotation via analytic continuation to imaginary time; see [28]. Following this, the prescription for analytically continuing to Euclidean signature for arbitrary metrics consists of taking (3.3) and setting $\epsilon = 2i$.[25] Note that the resulting Euclidean metric is in general not unique but depends on the choice of constant timelike vector V^μ.

[25] One cannot make this simply a real deformation by *e.g.* setting $i\epsilon = \lambda$, and taking λ from $0 \to -2$, as the metric would become singular at $\lambda = -1$ or $\epsilon = i$. The complex deformation allows us to go around this point in the ϵ-plane.

Although this recipe was initially put forward to produce Euclidean metrics that are compatible with the existence of a Lorentzian metric [28], it is particularly appropriate in our case where the physical metric dependence of (2.59) comes entirely through $\widetilde{\mathcal{M}}$. We stress that the reference metric η is left untouched by this Wick rotation, bringing in two advantages. First, we can keep the original constraint $H = \star_\eta H$ even in the Wick-rotated theory, and this guarantees that the latter describes the same number of degrees of freedom as the Lorentzian one. Second, the Wick rotation (3.3) makes the path integral convergent. This happens because the wrong-sign term $dB \wedge \star_\eta dB$ remains a purely oscillatory contribution, and this allows us to immediately single out a holomorphic partition function.

It is noteworthy that very recently the idea of Wick rotating a theory via complex deformations of the spacetime metric has re-appeared in an attempt to replace the standard axioms of QFT with the requirement that they be consistently coupled to complex metrics [30]. Applications of this proposal to quantum gravity were considered in [29]. In these works, a complex metric is allowed, under the condition that it leads to a convergent path integral. As will become clear such constraints arise naturally for our metric.

3.2 The Path Integral for the Two-dimensional Chiral Boson

In this section we will write down a well defined path integral for the two-dimensional chiral boson on $\mathbb{T}^2$. Our starting point is the two-dimensional version of the action (2.55) which reads as

$$S = \frac{1}{2\pi}\int \left(\frac{1}{2}d\phi \wedge \star_\eta d\phi - 2H \wedge d\phi + (H + J_+) \wedge \tilde{\mathcal{M}}(H + J_+) + 2H \wedge J_- - J_- \wedge J_+\right) . \tag{3.5}$$

We will first analytically continue it to Euclidean signature by the approach outlined in [28] and we will then evaluate it to directly obtain the chiral boson partition function with particular characteristics for the theta function. More general theta-characteristics will be introduced by including boundary terms in the action, and adjusting the periodicities of the scalar field.

To start we define our path integral on the torus of Figure 3.1 using coordinates (x^0, x^1), subject to the identifications

$$x^0 \cong x^0 + 2\pi l\ , \quad x^1 \cong x^1 + 2\pi l\ , \tag{3.6}$$

where l is an arbitrary length scale. The metric is then dimensionless and in these coordinates reads

$$g_{\mu\nu} = \begin{pmatrix} -L_0^2 + L_0^2 \tan^2\alpha & L_1 L_0 \tan\alpha \\ L_1 L_0 \tan\alpha & L_1^2 \end{pmatrix} . \tag{3.7}$$

Note that the choice of constant timelike vector V^μ needed to implement the Wick rotation in (3.3) is not unique. We find it natural to use the timelike vector $V^\mu := (1/L_0, -\tan\alpha/L_1)$, see Figure 3.1. With this choice, the deformed metric becomes

$$(g_\epsilon)_{\mu\nu} = \begin{pmatrix} -(1+i\epsilon)L_0^2 + L_0^2 \tan^2\alpha & L_1 L_0 \tan\alpha \\ L_1 L_0 \tan\alpha & L_1^2 \end{pmatrix} , \tag{3.8}$$

which leads to the usual flat metric on the flat Euclidean torus for $\epsilon = 2i$. Note that the determinant of g_ϵ is proportional to $(1+i\epsilon)$, so the analytic continuation must be performed by avoiding $\epsilon = +i$.[26] From now on we will set $\epsilon = 2i$.

We next define the 1-forms

$$\begin{aligned} \omega_+ &= dx^0 - dx^1 \\ \omega_- &= dx^0 + dx^1 , \end{aligned} \tag{3.9}$$

satisfying

$$\begin{aligned} \star_\eta \omega_\pm &= \pm\omega_\pm \\ \omega_+ \wedge \omega_- &= 2dx^0 \wedge dx^1 . \end{aligned} \tag{3.10}$$

In two dimensions, the A, B indices appearing in formula (2.40) can take only the value 1. By applying formula (2.40) one arrives at the expression:[27]

$$\widetilde{\mathcal{M}}^{11} = \frac{\frac{L_1}{L_0} + \tan\alpha + i}{\frac{L_1}{L_0} - \tan\alpha - i} . \tag{3.11}$$

By further introducing the complex structure

$$\tau = \frac{L_0}{L_1}(\tan\alpha + i) , \tag{3.12}$$

we can rewrite (3.11) more simply as

$$\widetilde{\mathcal{M}}^{11} = -\frac{\tau+1}{\tau-1} =: \mathcal{M} . \tag{3.13}$$

Remarkably, $\mathcal{M}$ is a meromorphic function of τ, and since $\mathrm{Im}(\tau)$ is strictly positive, so

[26] We also avoid $\epsilon \in i\mathbb{R}_{<0}$, as we have implicitly placed the branch cut of $\sqrt{-g_\epsilon} = L_1 L_0 \sqrt{1+i\epsilon}$ there.

[27] Appreciate that, in order to get a convergent path integral, here we are forced to pick the negative branch for the square root on the complex plane ($\sqrt{-1} = -i$).

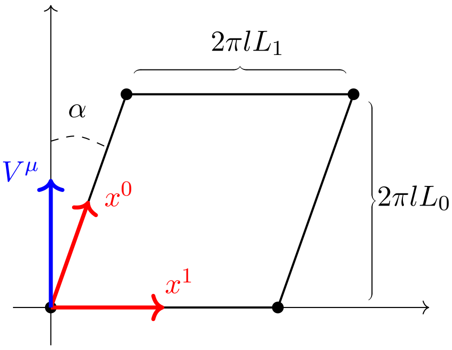


Figure 3.1: The flat torus parametrised by the coordinates x^0, x^1, in red. In blue, the time-like vector that will be used to perform the Wick rotation.

is $\mathrm{Im}(\mathcal{M})$.

We can now expand

$$\begin{aligned} H &= H^+\omega_+ \\ J &= J^+\omega_+ + J^-\omega_- \ , \end{aligned} \tag{3.14}$$

in terms of which the action (3.5) becomes

$$\begin{aligned} S = \frac{1}{2\pi}\int_{\mathbb{T}^2} d^2x \Big(-\frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\partial_1\phi)^2 - 2H^+(\partial_0\phi + \partial_1\phi) \\ + 2\mathcal{M}(H^+ + J^+)^2 + 4H^+J^- + 2J^+J^- \Big) \ . \end{aligned} \tag{3.15}$$

3.2.1 The Dirac Path Integral Prescription

Since we are dealing with a constrained system, we would now like to evaluate the Dirac path integral for the action (3.15). Due to $\star_\eta$-self-duality $H_0 = H^+ = -H_1$ and we can identify ϕ and H_1 as field variables with canonical conjugate momenta

$$\Pi^\phi := \frac{\delta S}{\delta \partial_0 \phi} = \frac{1}{2\pi}\left(-\partial_0\phi + 2H_1\right) \ , \qquad \Pi^{H_1} := \frac{\delta S}{\delta \partial_0 H_1} = 0 \ . \tag{3.16}$$

The Hamiltonian density is thus

$$\mathcal{H} = \frac{1}{2\pi}\left[-2(\pi\Pi^\phi - H_1)^2 - \frac{1}{2}(\partial_1\phi)^2 - 2H_1\partial_1\phi - 2(J^+ - H_1)^2\mathcal{M} + 4H_1J^- - 2J^+J^- \right] \ , \tag{3.17}$$

and the constraint surface is defined by[28]

$$\begin{aligned}\chi_1(x^0,x^1) &:= \Pi^{H_1}(x^0,x^1)\\ \chi_2(x^0,x^1) &:= \left\{\chi_1(x^0,x^1), \int_{S^1} dy^1 \mathcal{H}(x^0,y^1)\right\}_{x^0}\\ &= -\frac{2}{\pi}\left[\pi\Pi^\phi - H_1 - \frac{1}{2}\partial_1\phi + (J^+ - H_1)\mathcal{M} + J^-\right](x^0,x^1)\,, \end{aligned}\tag{3.19}$$

which are primary and secondary constraints respectively. The constraints χ_1 and χ_2 form a pair of second-class constraints,[29] since their Poisson bracket reads

$$\{\chi_1(x^0,x^1),\chi_2(x^0,y^1)\}_{x^0} = -\frac{2}{\pi}(1+\mathcal{M})\delta(x^1-y^1)\,. \tag{3.20}$$

The Dirac path integral is then given by [142, 143]

$$\begin{aligned}Z[J^\pm,\mathcal{M}] = \int [\mathcal{D}\phi\mathcal{D}\Pi^\phi\mathcal{D}H_1\mathcal{D}\Pi^{H_1}]\,\delta\left(\pi\Pi^\phi - H_1 - \frac{1}{2}\partial_1\phi + (J^+ - H_1)\mathcal{M} + J^-\right)\\ \times\,\delta\left(\Pi^{H_1}\right)\sqrt{\det\{\chi_i,\chi_j\}_{x^0}}\times e^{i\int_{\mathbb{T}^2} d^2x\left(\partial_0\phi\Pi^\phi + \partial_0 H_1\Pi^{H_1} - \mathcal{H}\right)}\,.\end{aligned}\tag{3.21}$$

Up to an overall constant factor

$$\begin{aligned}\delta\Big(\pi\Pi^\phi - H_1 - \frac{1}{2}\partial_1\phi + (J^+ - H_1)\mathcal{M} + J^-\Big)\sqrt{\det\{\chi_i,\chi_j\}_{x^0}} \propto\\ \propto\delta\left(H_1 - \frac{1}{1+\mathcal{M}}\left(\pi\Pi^\phi - \frac{1}{2}\partial_1\phi + J^+\mathcal{M} + J^-\right)\right)\,,\end{aligned}\tag{3.22}$$

and the functional integration over Π^{H_1} and H_1 can thus immediately be performed to get

$$Z[J^\pm,\mathcal{M}] = \int [\mathcal{D}\phi\mathcal{D}\Pi^\phi] e^{i\int_{\mathbb{T}^2} d^2x[\partial_0\phi\Pi^\phi - \mathcal{H}]}\Big|_{H_1\to\frac{1}{1+\mathcal{M}}\left(\pi\Pi^\phi - \frac{1}{2}\partial_1\phi + J^+\mathcal{M} + J^-\right)}\,. \tag{3.23}$$

In this framework, the first thing to do is to exploit the translational invariance

[28] Given two functionals F, G of the fields ϕ^i and their conjugate momenta Π^i, we denote with $\{F,G\}_{x^0}$ their Poisson bracket at equal time, *i.e.*

$$\{F,G\}_{x^0} = \int dx^1 \frac{\delta F}{\delta\phi^i(x^0,x^1)}\frac{\delta G}{\delta\Pi^i(x^0,x^1)} - \frac{\delta G}{\delta\phi^i(x^0,x^1)}\frac{\delta F}{\delta\Pi^i(x^0,x^1)}\,. \tag{3.18}$$

[29] There are no first-class constraints for this system. This fact is particular to two dimensions since. When working in $4k+2$ dimensions with $k>0$, the boson ϕ is replaced by a $k-$form in Sen's action, and this comes with its standard gauge redundancy. We will come back to this point in Section 3.3.

over the space of Π^ϕ, to factorise the path integral (3.23) into

$$Z[J^\pm,\mathcal{M}] = \mathcal{W}[J^\pm,\mathcal{M}] \int [\mathcal{D}\Pi^\phi] e^{i\pi\frac{\mathcal{M}}{\mathcal{M}+1}\int_{\mathbb{T}^2} d^2x(\Pi^\phi)^2} \int [\mathcal{D}\phi] e^{iS_{\text{eff}}[\phi,J^\pm]} \,, \tag{3.24}$$

where we have defined the overall field-independent function of the sources

$$\mathcal{W}[J^\pm,\mathcal{M}] := e^{-\frac{i}{\pi}\int_{\mathbb{T}^2} d^2x(J^+J^-+\mathcal{M}^{-1}(J^-)^2)} \,, \tag{3.25}$$

and

$$\begin{aligned} S_{\text{eff}}[\phi,J^\pm] := \frac{1}{2\pi}\int_{\mathbb{T}^2} d^2x\Big(&-\frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\partial_1\phi)^2 - \frac{1}{2\mathcal{M}}(\partial_0\phi+\partial_1\phi)^2 \\ &+ \frac{2}{\mathcal{M}}(\partial_0\phi+\partial_1\phi)(\mathcal{M}J^+ + J^-)\Big) \,. \end{aligned} \tag{3.26}$$

Note that in order to arrive at (3.24), we just used the fact that η is the (reference) Minkowski metric; no assumptions were made about the physical metric g, the data of which is contained inside $\mathcal{M}$. Note also that once we Wick rotate the physical metric as in (3.8), both functional integrations over Π^ϕ and ϕ yield convergent Gaussian integrals, see (3.13).

To proceed with the evaluation, we expand Π^ϕ in terms of an $L^2(\mathbb{T}^2)$ basis, *i.e.*

$$\Pi^\phi(x^0,x^1) = \sum_{n_0,n_1\in\mathbb{Z}^2} \Pi^\phi_{n_0,n_1} e^{-i\frac{x^0}{l}n_0} e^{-i\frac{x^1}{l}n_1} \,, \tag{3.27}$$

with $\Pi^\phi_{-n_0,-n_1} = (\Pi^\phi_{n_0,n_1})^*$ such that

$$\int [\mathcal{D}\Pi^\phi] \sim \prod_{(n_0,n_1)\in\mathbb{Z}^2} \int d\Pi^\phi_{n_0,n_1} \quad . \tag{3.28}$$

Performing a complex integral over all $\Pi^\phi_{n_0,n_1}$ with $(n_0,n_1)\in\mathbb{Z}^2$ would double count the independent fields. Therefore we need to restrict to a domain $U\subset\mathbb{Z}^2$ that has the property that $U\cap(-U)=\emptyset$. Furthermore $\Pi^\phi_{0,0}$ must be treated separately as it is real so $(0,0)\notin U$. But when combined we need $U\cup(-U)\cup(0,0)=\mathbb{Z}^2$. To this end we find it helpful to define

$$U := \{(n_0,n_1)\in\mathbb{Z}^2 | n_0+n_1>0\} \cup \{(n_0,-n_0)|n_0\in\mathbb{N}\} \,. \tag{3.29}$$

Pictorially we can think of this as the set of parallel diagonal lines $(n_0+p,-n_0)$ with $n_0\in\mathbb{Z}$ for a fixed $p\in\mathbb{N}$ along with the half-line $(n_0,-n_0)$ with $n_0\in\mathbb{N}$. Then we can easily rewrite

$$\begin{aligned}\prod_{(n_0,n_1)\in\mathbb{Z}^2}\int d\Pi^\phi_{n_0,n_1} &= \int_{\mathbb{R}} d\Pi^\phi_{0,0}\prod_{(n_0,n_1)\in U}\int_{\mathbb{C}} d\Pi^\phi_{n_0,n_1} d(\Pi^\phi_{n_0,n_1})^* \\ &= \int_{\mathbb{R}} d\Pi^\phi_{0,0}\prod_{(n_0,n_1)\in U}\left(2\int_{\mathbb{R}^2} d\mathrm{Re}[\Pi^\phi_{n_0,n_1}]d\mathrm{Im}[\Pi^\phi_{n_0,n_1}]\right) \;, \end{aligned} \tag{3.30}$$

and hence find for the Π^ϕ functional integral

$$\int [\mathcal{D}\Pi^\phi] e^{i\pi\frac{\mathcal{M}}{\mathcal{M}+1}\int_{\mathbb{T}^2} d^2x(\Pi^\phi)^2} = \sqrt{\frac{i}{(2\pi l)^2}\frac{\mathcal{M}+1}{\mathcal{M}}}\left(2\frac{i}{(2\pi l)^2}\frac{\mathcal{M}+1}{\mathcal{M}}\right)^{|U|} \;, \tag{3.31}$$

where $|U|$ is the cardinality of U.

To compute $|U|$ we note that the half-line $(n_0,-n_0)$, $n_0\in\mathbb{N}$ contributes $\sum_{n_0=1}^{\infty} 1 = \zeta(0) = -1/2$ to $|U|$. On the other hand, each complete line $(n_0+p,-n_0)$ consists of two half-lines plus a point and hence contributes $2\zeta(0)+1=0$ to $|U|$. Thus we simply find

$$|U| = -1/2 \;, \tag{3.32}$$

and so

$$\int [\mathcal{D}\Pi^\phi] e^{i\pi\frac{\mathcal{M}}{\mathcal{M}+1}\int_{\mathbb{T}^2} d^2x(\Pi^\phi)^2} \sim 1 \;. \tag{3.33}$$

From here onwards we will use $\sim$ to denote equality of the partition function up to an irrelevant—although possibly infinite—constant.

All in all, we have reduced the functional integral (3.24) to the evaluation of the Feynman path integral for the effective action (3.26):

$$Z[J^\pm,\mathcal{M}] \sim \mathcal{W}[J^\pm,\mathcal{M}]\int[\mathcal{D}\phi]e^{iS_{eff}[\phi,J^\pm]} \;. \tag{3.34}$$

This result, reached using the Dirac path integral and employing the regularisation (3.32), can in fact be reproduced by considering the Feynman path integral for the original action. To see this we complete the square on H^+ in (3.15) by introducing

$$\hat{H}^+ = H^+ + \left(J^+ + \frac{1}{\mathcal{M}}J^-\right) - \frac{1}{2\mathcal{M}}(\partial_0\phi + \partial_1\phi) \;. \tag{3.35}$$

By construction H is an arbitrary $\star_\eta$-self-dual form and hence $[\mathcal{D}H^+] = [\mathcal{D}\hat{H}^+]$. Thus the Feynman path integral for (3.15) factorises into

$$Z = \mathcal{W}[J^{\pm}, \mathcal{M}] Z_H(\mathcal{M}) \int [\mathcal{D}\phi] e^{iS_{\text{eff}}[\phi, J^{\pm}]} \ , \tag{3.36}$$

where

$$Z_H(\mathcal{M}) = \int [\mathcal{D}\hat{H}^+] e^{\frac{i}{\pi}\int_{\mathbb{T}^2} d^2x \, \mathcal{M}(\hat{H}^+)^2} \ . \tag{3.37}$$

Since the imaginary part of $\mathcal{M}$ is positive, the integration over $\hat{H}^+$ converges and using the same regularisation as for (3.33), one recovers an overall constant

$$Z_H(\mathcal{M}) \sim 1 \ . \tag{3.38}$$

Encouraged by this agreement we will directly employ the Feynman path-integral description in our upcoming discussion, Sections 3.2.4 and 3.3.

After this preliminary work, our task now is to evaluate the functional integral (3.34). We assume that the field ϕ is compact with radius R, *i.e.*

$$\phi \cong \phi + 2\pi R \ , \tag{3.39}$$

and it thus admits the following decomposition on the torus

$$\phi = \phi_{\text{w.m.}} + \phi_{\text{osc}} \ . \tag{3.40}$$

The $\phi_{\text{w.m.}}$ and ϕ_{osc} respectively encode the winding and oscillatory modes of the field:

$$\phi_{\text{w.m.}} = \frac{R}{l}(m_0 x^0 + m_1 x^1) \ , \qquad m_0, m_1 \in \mathbb{Z} \tag{3.41}$$

$$\phi_{\text{osc}} = \sideset{}{'}\sum_{n_0,n_1 \in \mathbb{Z}^2} \phi_{n_0,n_1} e^{-i\frac{x^0}{l}n_0} e^{-i\frac{x^1}{l}n_1} \ , \qquad \phi_{-n_0,-n_1} = (\phi_{n_0,n_1})^* \quad , \tag{3.42}$$

where the prime symbol on top of the sum denotes that the choice $(n_0, n_1) = (0, 0)$ must not be taken into account.

The compact field ϕ can admit topologically non-trivial configurations because it appears in the action only through its derivative: the action is still single-valued on the torus even when ϕ is not. The same is not true for J which thus admits the following expansion on the torus:

$$J = \sum_{n_0,n_1 \in \mathbb{Z}^2} J_{n_0,n_1} e^{-i\frac{x^0}{l}n_0} e^{-i\frac{x^1}{l}n_1} \ , \qquad J_{-n_0,-n_1} = (J_{n_0,n_1})^* \quad . \tag{3.43}$$

Since the effective action (3.26) is quadratic in ϕ, $\phi_{\text{w.m.}}$ and ϕ_{osc} decouple. In the following subsections their contributions to the path integral will be determined separately.

Finally, it is interesting to note that, in the absence of sources, (3.26) closely resembles the Siegel action which is written, as we saw in (2.7), in terms of a the Lagrange multiplier λ:

$$S_{\text{Siegel}}[\phi] = \frac{1}{2\pi}\int d^2x \left(\frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\partial_1\phi)^2 + \lambda\left(\partial_0\phi + \partial_1\phi\right)^2\right) . \tag{3.44}$$

Therefore, much of the algebraic manipulations that we will present in the remainder of this section have already appeared before, see *e.g.* [76,138]. Nonetheless, we will still display the calculations that will lead to the partition function in full detail, not only for clarity but especially because there is one key difference between (3.26) and Siegel's action. Indeed, $\mathcal{M}$ in (3.26), unlike λ in (3.44), is a constant and not a Lagrange multiplier and therefore the effective action (3.26) avoids the potential problems that Siegel's action instead suffers from, see discussion below (2.7).

3.2.2 Oscillator Modes

Evaluating the effective action (3.26) on the oscillator modes we find

$$\begin{aligned} S_{\text{eff}}[\phi_{\text{osc}}, J^{\pm}] = \pi \sum_{n_0,n_1}{}' \Bigg\{ &|\phi_{n_0,n_1}|^2 \left(n_1^2 - n_0^2 - \frac{1}{\mathcal{M}}(n_0+n_1)^2\right) \\ &+4il(n_0+n_1)\phi_{n_0,n_1}\left(J^+_{-n_0,-n_1} + \frac{1}{\mathcal{M}}J^-_{-n_0,-n_1}\right)\Bigg\} . \end{aligned} \tag{3.45}$$

Even though the source J explicitly enters this expression, the dependence on J will be washed away upon integrating over each oscillator, and the contribution Z_{osc} of the oscillatory modes to the partition function will be independent of J. Indeed, let $\phi = \phi_1 + i\phi_2$ represent a generic oscillator mode appearing in (3.45) and let j denote a contribution from the source. For a complex number G with positive imaginary part we then have

$$\begin{aligned} \int_{\mathbb{C}} d\phi d\phi^* e^{iG|\phi|^2 + ij\phi} &= 2\int_{\mathbb{R}^2} d\phi_1 d\phi_2 e^{iG(\phi_1^2+\phi_2^2)+ij(\phi_1+i\phi_2)} \\ &= 2\int_{\mathbb{R}} d\phi_1 e^{iG\left(\phi_1 + \frac{j}{2G}\right)^2} \int_{\mathbb{R}} d\phi_2 e^{iG\left(\phi_2 + i\frac{j}{2G}\right)^2} \\ &= \frac{2\pi i}{G} . \end{aligned} \tag{3.46}$$

Note that the right-moving modes with $n_0 + n_1 = 0$ lead to a vanishing action: since each of them merely contributes to the path integral as an overall infinite constant ($\sim \int d\phi d\phi^* e^{i0}$), we will only include the modes ϕ_{n_0,n_1} with $n_0 + n_1 \neq 0$. Furthermore, due to the reality condition $(\phi_{n_0,n_1})^* = \phi_{-n_0,-n_1}$, we restrict to $n_0 + n_1 > 0$ to avoid double counting. Therefore, Z_{osc} evaluates to

$$\begin{aligned} Z_{\text{osc}} &= \left(\prod_{n_0+n_1>0} \int_{\mathbb{C}} d\phi_{n_0,n_1} d(\phi_{n_0,n_1})^* \right) e^{iS_{\text{eff}}[\phi_{\text{osc}},0]} \\ &= \prod_{n_0+n_1>0} \frac{2i}{(n_0+n_1)(n_1-n_0) - \frac{1}{\mathcal{M}}(n_0+n_1)^2} , \end{aligned} \tag{3.47}$$

which by letting $n := n_0 + n_1$ and $m := n_1$ can be tidied up to

$$Z_{\text{osc}} = \prod_{\substack{m \\ n>0}} \frac{i}{n(m+Tn)} . \tag{3.48}$$

Here we introduced the important parameter

$$T := -\frac{1}{2}(1+\mathcal{M}^{-1}) = -\frac{1}{1+\tau} , \tag{3.49}$$

which in turn can be related to τ *via* an $SL(2,\mathbb{Z})$ transformation. In Appendix A we show that the infinite product appearing in (3.48) can be regularised to

$$\prod_{\substack{m \\ n>0}} \frac{i}{n(m+Tn)} \sim \frac{1}{\eta(T)} , \qquad \eta(T) = e^{\pi i T/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n T}) . \tag{3.50}$$

Thus we arrive at

$$Z_{\text{osc}} \sim \frac{1}{\eta(T)} . \tag{3.51}$$

3.2.3 Winding Modes

Evaluating the effective action (3.26) on the winding modes returns

$$S_{\text{eff}}[\phi_{\text{w.m.}}, J^{\pm}] = \pi \left(-R^2 m_0^2 + R^2 m_1^2 - \frac{R^2}{\mathcal{M}}(m_0+m_1)^2 + 2R(m_0+m_1)\mathcal{J}^{(0)} \right) , \tag{3.52}$$

where we defined a complex structure on the sources by

$$\mathcal{J} = J^+ + \frac{1}{\mathcal{M}} J^- , \tag{3.53}$$

with normalised zero mode defined by

$$\mathcal{J}^{(0)} := \frac{2l}{(2\pi l)^2} \int d^2x \left(J^+ + \frac{1}{\mathcal{M}} J^- \right) . \tag{3.54}$$

Following on from the oscillator discussion we find

$$\begin{aligned} Z_{\text{w.m.}} &= \sum_{m_0,m_1} e^{iS_{\text{eff}}[\phi_{\text{w.m.}},J^{\pm}]} \\ &= \sum_{m_0,m_1} e^{-i\pi R^2\left((m_0+m_1)(m_0-m_1)+(m_0+m_1)^2\mathcal{M}^{-1}\right)+2i\pi R(m_0+m_1)\mathcal{J}^{(0)}} \;. \end{aligned} \tag{3.55}$$

By introducing $n = m_0 + m_1$ and $m = m_1$ we recast this into

$$\begin{aligned} Z_{\text{w.m.}} &= \sum_{m,n} e^{-i\pi R^2\left(n(n-2m)+n^2\mathcal{M}^{-1}\right)+2i\pi Rn\mathcal{J}^{(0)}} \\ &= \sum_{m,n} e^{2\pi i R^2 mn} e^{-i\pi R^2 n^2\left(1+\mathcal{M}^{-1}\right)+2i\pi Rn\mathcal{J}^{(0)}} \;. \end{aligned} \tag{3.56}$$

The sum over m is of the form

$$\sum_{m\in\mathbb{Z}} e^{2\pi i x m} = \sum_{q\in\mathbb{Z}} \delta(x-q) \;, \tag{3.57}$$

which inserted into (3.56) gives

$$Z_{\text{w.m.}} = \sum_n \sum_q \delta(R^2 n - q) e^{2i\pi R^2 n^2 T + 2i\pi Rn\mathcal{J}^{(0)}} \;, \tag{3.58}$$

where T was defined in (3.49).
One then finds that $Z_{\text{w.m.}} = \delta(0)$ for R^2 irrational. From (3.56) it is clear that this $\delta(0)$ divergence arises from the infinite number of degenerate contributions to the partition function that arise from chiral modes in the sum over m at $n = 0$.
Instead, if R^2 is rational—that is when $R^2 = r_1/r_2$ for some coprime integers r_1, r_2—one gets a non-vanishing contribution whenever

$$n = pr_2 \qquad \text{and} \qquad m = pr_1 \;, \tag{3.59}$$

for any $p \in \mathbb{Z}$. Hence for $R^2 = r_1/r_2$ we have

$$Z_{\text{w.m.}} = \delta(0) \sum_p e^{2i\pi r_1 r_2 p^2 T + 2i\pi\sqrt{r_1 r_2} p \mathcal{J}^{(0)}} \sim \theta\begin{bmatrix}0\\0\end{bmatrix}(\sqrt{r_1 r_2}\mathcal{J}^{(0)}|2r_1 r_2 T) \;, \tag{3.60}$$

where we have introduced the theta function which, for complex τ, is defined through

$$\theta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(z|\tau) := \sum_{p\in\mathbb{Z}} e^{i\pi(p+\alpha)^2\tau + 2i\pi(p+\alpha)(z+\beta)} \;, \quad \operatorname{Im}(\tau) > 0 \;, \tag{3.61}$$

with the real parameters α, β referred to as characteristics. In writing the last step of (3.60) we have dropped the $\delta(0)$ as an irrelevant but infinite constant, which is due to the degeneracy originating from shifting a given winding mode (m_0, m_1) by a chiral mode, $(s, -s)$, $s \in \mathbb{Z}$. Indeed, despite not being linear in the winding modes, one sees that upon shifting $\phi_{\text{w.m.}} \to \phi_{\text{w.m.}} + \frac{R}{l}(sx^0 - sx^1)$ the action transforms as (remember that $m_1 + m_0 = n = r_2 p$)

$$S_{\text{eff}}[\phi_{\text{w.m.}}, J^\pm] \to S_{\text{eff}}[\phi_{\text{w.m.}}, J^\pm] - 2\pi r_1 p s \ , \tag{3.62}$$

and hence $e^{iS_{\text{eff}}}$ is invariant. Thus summing over all winding modes induces a divergent contribution arising from an infinite number of states which only differ by a chiral mode of ϕ but which all give the same contribution to the partition function. We accordingly simply discard the infinite constant in Eq. (3.60).

Combining (3.60) with Z_{osc} and the source-dependent prefactor appearing in (3.34), we have

$$\begin{aligned} Z[J^\pm, T] &\sim \mathcal{W}[J^\pm, T] Z_{\text{osc}} Z_{\text{w.m.}} \\ &\sim \mathcal{W}[J^\pm, T] \frac{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(\sqrt{r_1 r_2}\, \mathcal{J}^{(0)} \,|\, 2 r_1 r_2 T \right)}{\eta(T)} \\ &= e^{-\frac{i}{\pi} \int_{\mathbb{T}^2} d^2x (J^+ J^- + \mathcal{M}^{-1} (J^-)^2)} \frac{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(\sqrt{r_1 r_2}\, \mathcal{J}^{(0)} \,|\, 2 r_1 r_2 T \right)}{\eta(T)} \ . \end{aligned} \tag{3.63}$$

In the absence of sources, this can be interpreted as a $\widehat{\mathfrak{u}}(1)_{r_1 r_2}$ character, as expected for a chiral boson on a rational square radius.[30] Moreover, since (3.63) is left invariant by $r_1 \leftrightarrow r_2$, we recognise an underlying duality acting as $R \leftrightarrow 1/R$. This fixes the self-dual radius to be $R^2 = r_1/r_2 = 1$ which, as r_1 and r_2 are coprime, means setting $r_1 = r_2 = 1$ and we find an $\widehat{\mathfrak{su}}(2)_1$ character. All these results are compatible with the formulation of the chiral boson as the edge mode of abelian Chern–Simons theory in one dimension higher [139–141].

To summarise, when R^2 is rational our computation precisely lands—up to the $SL(2, Z)$ twist encoded in $\tau \to T$—on the expected result for the chiral-boson partition function. For R^2 irrational, the theta function collapses to one, and Z is only proportional to $1/\eta(T)$.

[30] For the interested reader a useful resource is [144].

3.2.4 General Theta-Function Characteristics

For the convenience of the presentation, first we remind the reader that under

$$\begin{aligned} \delta_\Lambda \phi &= \Lambda \\ \delta_\Lambda J &= d\Lambda \\ \delta_\Lambda H &= -\Big(\frac{1+\star_\eta}{2}\Big) d\Lambda \ , \end{aligned} \tag{3.64}$$

the two-dimensional action (3.5) tranforms as

$$S \to S + \frac{1}{2\pi}\int d\Lambda \wedge (d\phi - J) \ . \tag{3.65}$$

In what follows, we want to extend the discussion from the previous section so as to introduce general characteristics in the final answer (3.63).
To this end we take

$$\phi_{\text{w.m.}} = \frac{R}{l}(m_0+\alpha_0)x^0 + \frac{R}{l}(m_1+\alpha_1)x^1 \ , \tag{3.66}$$

for some constants α_0, α_1. This will have the effect of shifting the sum over n in (3.56) to $n+\alpha_0+\alpha_1$ and thereby introducing the α-characteristic in (3.61).

As before ϕ is not single-valued over the torus and now satisfies

$$\begin{aligned} \phi(x^0+2\pi l, x^1) &= \phi(x^0,x^1) + 2\pi R(m_0+\alpha_0) \\ \phi(x^0, x^1+2\pi l) &= \phi(x^0,x^1) + 2\pi R(m_1+\alpha_1) \ . \end{aligned} \tag{3.67}$$

Shifting ϕ by a constant is a symmetry of the action, corresponding to a constant choice of Λ in (3.64). Thus we can view this identification as an orbifold whose action is different over the two 1-cycles of $\mathbb{T}^2$. However, we do not want to allow for orbifold actions of the form $\phi \cong \phi + \Lambda$ for any constant Λ as that would completely remove all zero modes. So we restrict to $m_0, m_1 \in \mathbb{Z}$. Another way to say this is that we restrict to orbifold actions for which

$$\frac{1}{2\pi}\oint (d\phi - \mathcal{A}) \in R\,\mathbb{Z} \ , \tag{3.68}$$

where the integral is over any 1-cycle of the torus and

$$\mathcal{A} = \frac{R}{l}\alpha_0 dx^0 + \frac{R}{l}\alpha_1 dx^1 \ , \tag{3.69}$$

is a fixed closed 1-form. Note that we can also think of these boundary conditions in terms of $\Psi = e^{i\phi/R}$:

$$\Psi(x^0 + 2\pi n^0 l, x^1 + 2\pi l n^1) = e^{2\pi i(n^0\alpha_0 + n^1\alpha_1)}\Psi(x^0, x^1) \ . \tag{3.70}$$

Thus if we think of Ψ as a dual fermion then $\mathcal{A}$ encodes the spin structure. It is tempting to interpret $\mathcal{A}$ as a connection 1-form and $d\phi - \mathcal{A}$ as a covariant derivative, as in [73]. However, this interpretation has difficulties in higher dimensions.

Next, we attempt to repeat the winding-mode calculation with the more general modings $m_0 \to m_0 + \alpha_0$, $m_1 \to m_1 + \alpha_1$. One quickly discovers that the sum (3.57) will now involve $x = R^2(n + \alpha_0 + \alpha_1)$ and for generic α_0, α_1 this can never be integer (including zero). To counter this we add the following term to the action (3.15):

$$S_{\mathcal{A}} := S - \frac{1}{2\pi}\int_{\mathbb{T}^2} \mathcal{A} \wedge d\phi \ . \tag{3.71}$$

This term is a total derivative and hence does not affect the equations of motion or any of the symmetries, including infinitesimal diffeomorphisms. As discussed around (3.38), we will bypass the Dirac path integral and work directly with the Feynman path-integral formulation of (3.15), in terms of which one finds that the above term carries through and appears as is in the effective action (3.26). However, it does give the following contribution on the winding modes

$$\begin{aligned} S_{\mathcal{A}\text{w.m.}} &= S_{\text{w.m.}} - 2\pi R^2\alpha_0(m_1 + \alpha_1) + 2\pi R^2\alpha_1(m_0 + \alpha_0) \\ &= S_{\text{w.m.}} - 2\pi R^2(\alpha_0 + \alpha_1)m_1 + 2\pi R^2\alpha_1(m_0 + m_1) \\ &= S_{\text{w.m.}} - 2\pi R^2(\alpha_0 + \alpha_1)m + 2\pi R^2\alpha_1 n \ . \end{aligned} \tag{3.72}$$

Performing the shift $n \to n + \alpha_0 + \alpha_1$, $m \to m + \alpha_1$ in (3.56) and including this extra term we find

$$Z_{\text{w.m.}} = \sum_{n,m} e^{2\pi i R^2(m+\alpha_1)(n+\alpha_0+\alpha_1)} e^{2i\pi R^2(n+\alpha_0+\alpha_1)^2 T + 2i\pi R(n+\alpha_0+\alpha_1)\mathcal{J}^{(0)}} \times \tag{3.73}$$

$$\begin{aligned} &\times e^{-2\pi i R^2(\alpha_0+\alpha_1)m + 2\pi i R^2\alpha_1 n} \\ &= \sum_{n,m} e^{2\pi i R^2 mn} e^{2i\pi R^2(n+\alpha_0+\alpha_1)^2 T + 2\pi i R(n+\alpha_0+\alpha_1)\mathcal{J}^{(0)}} e^{2\pi i R^2\alpha_1(\alpha_0+\alpha_1) + 4\pi i R^2\alpha_1 n} \ . \end{aligned} \tag{3.74}$$

The sum over m reproduces the same δ-function as in (3.58). Thus if $R^2 = r_1/r_2$ we once again find $n = pr_2$ and hence

$$\begin{aligned} Z_{\text{w.m.}} &\sim e^{-\pi i\alpha\beta} \sum_p e^{2i\pi r_1 r_2 (p+\alpha)^2 T + 2i\pi(p+\alpha)(\sqrt{r_1 r_2}\mathcal{J}^{(0)}+\beta)} \\ &= e^{-\pi i\alpha\beta}\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sqrt{r_1 r_2}\mathcal{J}^{(0)}|2r_1 r_2 T) \ , \end{aligned} \tag{3.75}$$

where

$$\alpha = (\alpha_0 + \alpha_1)/r_2 \ , \qquad \beta = 2r_1\alpha_1 \ . \tag{3.76}$$

To sum up, for rational radius ($R^2 = r_1/r_2$), the partition function of the chiral boson computed via Sen action appropriately shifted as in (3.71) with $\mathcal{A}$ given by (3.69) and with ϕ satisfying the boundary condition (3.67) reads as

$$Z[J^\pm, T] \sim \mathcal{W}[J^\pm, T] \frac{\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\sqrt{r_1 r_2}\,\mathcal{J}^{(0)}\,|\,2r_1 r_2 T\right)}{\eta(T)} \ , \tag{3.77}$$

where the α, β-characteristics, the prefactor $\mathcal{W}$ and the parameter T are respectively defined in (3.76), (3.25) and (3.49).

3.2.5 Holomorphic Structure

Now that we have arrived at the partition function (3.77), we can briefly expand on how it depends on the source J. In this way we wish to make contact with the geometric construction of the partition function done in [73, 116], which is based on the fact that the partition function of a chiral form can be interpreted as an almost-holomorphic section of a particular bundle over a complex torus parametrized by the zero modes of the source J.

Let us look at the zero-mode contribution to the partition function:

$$Z \sim \mathcal{W}'[J] e^{\frac{i}{2}\pi\mathcal{J}^{(0)}(\mathcal{J}^{(0)}-\bar{\mathcal{J}}^{(0)})/(T-\bar{T})} \frac{\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\sqrt{r_1 r_2}\mathcal{J}^{(0)}|2r_1 r_2 T)}{\eta(T)} \ , \tag{3.78}$$

where $\mathcal{W}'[J]$ only depends on the non-zero-mode sources. This is almost a holomorphic function of $\mathcal{J}^{(0)}$ which is encoded in the statement that

$$\bar{\mathcal{D}} Z := \left(\frac{\partial}{\partial\bar{\mathcal{J}}^{(0)}} + \frac{i\pi}{2}\frac{\mathcal{J}^{(0)}}{T-\bar{T}}\right) Z = 0 \ . \tag{3.79}$$

It is also helpful to observe that theta functions satisfy, for $m, n \in \mathbb{Z}$,

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z+m+n\tau|\tau) = e^{-in^2\tau+2\pi i(m\alpha-n\beta)-2\pi inz}\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\tau)\ . \tag{3.80}$$

As a result one sees that under a shift

$$\mathcal{J}^{(0)} \to \mathcal{J}^{(0)} + \frac{m}{\sqrt{r_1 r_2}} + 2n\sqrt{r_1 r_2}T\ , \tag{3.81}$$

the theta function in (3.78) changes by:

$$\theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\sqrt{r_1r_2}\mathcal{J}^{(0)}|2r_1r_2T) \to e^{-2r_1r_2Tn^2+2\pi i(m\alpha-n\beta)-2\pi i\sqrt{r_1r_2}\mathcal{J}^{(0)}n}\theta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](\sqrt{r_1r_2}\mathcal{J}^{(0)}|2r_1r_2T)\ . \tag{3.82}$$

This is clearly a function of T for any $m, n \neq 0$ ($\mathcal{J}^{(0)}$ itself has T dependence) and since T is complex this is not a pure phase. To see how the partition function transforms, we must also calculate the change to the anomalous prefactor $\mathcal{W}$. Shifting $\mathcal{J}^{(0)}$ by (3.81) is equivalent to shifting the components of J by

$$\begin{aligned} J^+ &\to J^+ + \frac{1}{2l}\left(\frac{1}{\sqrt{r_1r_2}}m - \sqrt{r_1r_2}n\right) \\ J^- &\to J^- - \frac{1}{2l}\sqrt{r_1r_2}n\ , \end{aligned} \tag{3.83}$$

which can in turn be written as

$$J \to J + d\Lambda\ , \quad \text{with} \quad \Lambda = \frac{1}{2l\sqrt{r_1r_2}}(m-2r_1r_2n)x^0 - \frac{1}{2l}\frac{m}{\sqrt{r_1r_2}}x^1\ . \tag{3.84}$$

This can then be used to find the change in the anomalous term

$$\mathcal{W} \to e^{2\pi ir_1r_2Tn^2+2\pi i\sqrt{r_1r_2}\mathcal{J}^{(0)}n+\frac{i}{2\pi}\int J\wedge d\Lambda+\pi imn}\,\mathcal{W}\ , \tag{3.85}$$

such that the overall change to Z is simply

$$Z \to e^{\pi imn+2\pi i(m\alpha-n\beta))}e^{\frac{i}{2\pi}\int J\wedge d\Lambda}Z\ . \tag{3.86}$$

The change in the partition function is therefore a pure phase for all m, n.

Now we would like to see whether (3.86) could be deduced already at the level of the action. We observe that Eq. (3.84) can be seen as part of the transformation (3.64). However, in this interpretation the winding modes of ϕ are similarly shifted by (3.64):

$$m_0 \to m_0 + \frac{1}{2r_1}(m-2r_1r_2n)\ , \qquad m_1 \to m_1 - \frac{1}{2}\frac{m}{r_1}\ . \tag{3.87}$$

In order for this shift to make sense, *i.e.* for the winding modes to be mapped to winding modes, we see that we must restrict $m = 2r_1 m'$ with $m' \in \mathbb{Z}$. Thus the transformation (3.64) can only account for some of the shift symmetry of the partition function.

The action is not invariant under (3.64), but the change only depends on the sources and theta-characteristics. Explicitly, we find that, when evaluated on a winding mode $\phi_{\text{w.m.}} = R(m_\mu + \alpha_\mu)x^\mu/l$, the action shifts by[31]

$$\begin{aligned}
S &\to S + \frac{1}{2\pi}\int d\Lambda \wedge (d\phi + \mathcal{A} - J) \\
&= S + \frac{\pi}{r_2}((m - 2r_1r_2n)(m_1 + 2\alpha_1) + m(m_0 + 2\alpha_0)) - \frac{2l}{\sqrt{r_1r_2}}(r_1r_2nJ_+^{(0)} - (m - r_1r_2n)J_-^{(0)}) \\
&= S + \frac{2\pi}{r_2}(m(\alpha_0 + \alpha_1) - 2r_1r_2n\alpha_1) - 2l\sqrt{r_1r_2}n(-J_+^{(0)} + J_-^{(0)}) + \frac{2l}{\sqrt{r_1r_2}}mJ_-^{(0)} \\
&\quad + \frac{\pi}{r_2}m(m_1 + m_0) - \pi nm \ .
\end{aligned} \tag{3.88}$$

To remove the dependence on the winding-mode numbers m_0, m_1 we see that in addition to $m = 2r_1m'$, which makes the last term a multiple of 2π, we also require $m' = r_2m''$ so that the second to last term is also a multiple of 2π. Thus for $m = 2r_1r_2m''$ the shift in the action, modulo 2π, is independent of the winding modes and hence we find that

$$Z \to e^{\frac{2\pi}{r_2}(m(\alpha_0+\alpha_1)-2r_1r_2n\alpha_1)-2l\sqrt{r_1r_2}n(-J_+^{(0)}+J_-^{(0)})+\frac{2l}{\sqrt{r_1r_2}}mJ_-^{(0)}} Z \ , \tag{3.89}$$

which agrees with (3.86), as α and β are defined as in (3.76).

This is usually interpreted as saying that the partition function is a section of a line bundle over the space parametrised by $\mathcal{J}^{(0)}$ modulo the identification (3.81), that is an auxiliary complex torus parametrised by the zero-modes of the sources, $\mathcal{J}_{\mathbb{T}^2}$.

3.3 Chiral Two-form Potentials in Six Dimensions

The approach that we used for evaluating the chiral-boson partition function in two dimensions can be extended to $4k+2$-dimensions where one has self-dual $2k+1$ forms. Of particular interest are self-dual three-forms in six dimensions (associated with superconformal field theories such as the (2,0) theory) and self-dual five-forms in ten dimensions (which arise in type IIB string theory). Here, for concreteness, we will detail the computation in six dimensions, *i.e.* $k = 1$. The extension to more general k follows readily, and for $k = 0$ reproduces the chiral boson results.

[31] Note that there is an additional contribution to (3.65) arising from the $\mathcal{A} \wedge d\phi$ term.

3.3.1 Preliminaries

Before explicitly computing the partition function let us first make some general comments. Following on from the discussion of Section 3.2.4 we consider the following version of the action (2.55):

$$\begin{aligned} S_{\mathcal{C}} =& \frac{1}{(2\pi)^5}\int_{\mathbb{T}^6}\left(\frac{1}{2}dB\wedge\star_\eta dB - 2H\wedge dB + (H+J)\wedge\widetilde{\mathcal{M}}(H+J) + 2H\wedge J - \frac{1}{2}J\wedge\star_\eta J\right) \\ & - \frac{1}{(2\pi)^5}\int_{\mathbb{T}^6}\mathcal{C}\wedge dB\ , \end{aligned} \tag{3.90}$$

where, in the second line, we have included a boundary term determined by a closed three-form $\mathcal{C}$, $d\mathcal{C}=0$. Once again, such a term does not contribute to the equations of motion and preserves all the symmetries of the original theory. In particular, the action is invariant under infinitesimal diffeomorphisms $x^\mu \to x^\mu + \xi^\mu(x)$, provided ξ^μ is single valued on $\mathbb{T}^6$. Analogously to what happened in the previous sections, we will see that the role of $\mathcal{C}$ is to allow for more general theta-characteristics.

Next we consider the transformations (2.59). Under such a transformation we find that $S_{\mathcal{C}}$ becomes:

$$S_{\mathcal{C}}[B,H,J] \to S_{\mathcal{C}}[B,H,J] + \frac{1}{(2\pi)^5}\int_{\mathbb{T}^6} d\Lambda\wedge(dB+\mathcal{C}-J)\ . \tag{3.91}$$

These transformations play three roles. First, if $\Lambda = d\lambda$ then we have a familiar abelian gauge transformation $B \to B + d\lambda$ and the action is invariant. These represent redundancies and, for example, we can choose λ to set the timelike components of B to zero. Note that in the case of a two-dimensional chiral boson there are no such gauge symmetries.

Second, if Λ is closed ($d\Lambda = 0$) but not exact then these transformations are genuine symmetries of the action that we can orbifold by. This allows us to introduce the analogues of the winding modes from the chiral boson case. Here B is not single-valued over the torus. Rather we allow for so-called large gauge transformations

$$B \to B + \Lambda\ , \qquad d\Lambda = 0\ , \tag{3.92}$$

where $\Lambda \neq d\lambda$. With this in mind if we go around a loop in the x^μ direction we can take

$$B(x^\mu + 2\pi l) = B(x^\mu) + \Lambda^{(\mu)}\ , \tag{3.93}$$

where $\Lambda^{(\mu)}$ is a closed 2-form. Although this looks like a valid identification for any choice of $\Lambda^{(\mu)}$ one needs to be more careful: we do not want to say that any closed

$\Lambda^{(\mu)}$ is allowed, as this condition is too strong.[32] Rather, we want to impose a flux-quantisation condition

$$\frac{1}{(2\pi)^3}\int_{\Sigma_3} dB \in R^3\,\mathbb{Z}\,, \tag{3.94}$$

for some fixed R and any three-cycle Σ_3. This integral will be non-zero if B is not single valued as in (3.93). By including $\mathcal{C}$ we can be a little more general. We can change the flux-quantisation condition to

$$\frac{1}{(2\pi)^3}\int_{\Sigma_3} (dB-\mathcal{C}) \in R^3\,\mathbb{Z}\,. \tag{3.95}$$

We can also interpret these boundary conditions as acting on the Wilson surface operators of B:

$$W(\Sigma_2) = e^{\frac{i}{R^3}\frac{1}{(2\pi)^2}\int_{\Sigma_2} B}\,, \tag{3.96}$$

where Σ_2 is a 2-cycle. By construction $W(\Sigma_2)$ only depends on the coordinates $x^{\mu'}$ that are normal to Σ_2. Our boundary condition corresponds to

$$W(\Sigma_2)(x^{\mu'}+2\pi l n^{\mu'}) = e^{2\pi i\int_{\Sigma_3}\mathcal{C}}\,W(\Sigma_2)(x^{\mu'})\,, \tag{3.97}$$

where Σ_3 is the 3-cycle obtained by transporting Σ_2 around the closed loop created by $x^{\mu'} \to x^{\mu'} + 2\pi l n^{\mu'}$.

The third application of the transformation (2.59) enables us to think of the source zero-modes as coordinates on an auxiliary complex torus as in Section 3.2.5, $\mathcal{J}_{\mathbb{T}^6}$ [73]. This corresponds to considering transformations where $d\Lambda \neq 0$, so J shifts. The action is no-longer invariant under (2.59) but if we impose another flux-quantisation condition

$$\frac{1}{(2\pi)^3}\int_{\Sigma_3} d\Lambda \in R^{-3}\,\mathbb{Z}\,, \tag{3.98}$$

then the shift in the action is

$$S_{\mathcal{C}}[B,H,J] \to S_{\mathcal{C}}[B,H,J] + \Delta S_{\mathcal{C}}[J] + 2\pi n\,, \tag{3.99}$$

where $n \in \mathbb{Z}$ and

$$\Delta S_{\mathcal{C}}[J] = \frac{1}{(2\pi)^5}\int_{\mathbb{T}^6} d\Lambda \wedge (2\mathcal{C} - J)\,. \tag{3.100}$$

Thus the path integral and partition function is invariant under such shifts, up to a

[32] In the case of a chiral boson such a condition would amount to saying that $\phi \cong \phi + 2\pi R$ for *any* constant R.

phase factor that only depends on $\mathcal{C}$ and J. This is often summarised by saying that the partition function is a section of a line bundle over $\mathcal{J}_{\mathbb{T}^6}$, which consists of sources J modulo $d\Lambda$ subject to (3.98). However, once again one must be a little more careful. Under such a shift we have

$$\begin{aligned} Z[J+d\Lambda] &= \int [\mathcal{D}B'\mathcal{D}H'] e^{iS_{\mathcal{C}}[B',H',J+d\Lambda]} \\ &= \int [\mathcal{D}B'\mathcal{D}H'] e^{iS_{\mathcal{C}}[B+\Lambda,H-(d\Lambda)_+),J+d\Lambda]} \\ &= \int [\mathcal{D}B'\mathcal{D}H'] e^{iS_{\mathcal{C}}[B,H,J]+i\Delta S_{\mathcal{C}}[J]} \\ &= e^{i\Delta S_{\mathcal{C}}[J]} \int [\mathcal{D}B\mathcal{D}H] e^{iS_{\mathcal{C}}[B,H,J]} \\ &= e^{i\Delta S_{\mathcal{C}}[J]} Z[J] \ , \end{aligned} \tag{3.101}$$

where in the second line we chose $B' = B + \Lambda$, $H' = H - (d\Lambda)_+$ and in the fourth line we assumed $[\mathcal{D}B'\mathcal{D}H'] = [\mathcal{D}B\mathcal{D}H]$. When we integrate out B and H we can only perform shifts by Λ if the latter preserves the form of B and H that we integrated over. Since we integrated over all H's this is not a problem for H. However we did not integrate over all B's; rather we restricted to B's that satisfy (3.95). The change $B \to B'$ amounts to just shifting the sum over the winding modes. Thus we can only shift by Λ's such that

$$\frac{1}{(2\pi)^3} \int_{\Sigma_3} d\Lambda \in R^3\mathbb{Z} \ , \tag{3.102}$$

in addition to (3.98). This in turn is only possible if $R^6 = r_1/r_2$ is rational, in which case we take

$$\frac{1}{(2\pi)^3} \int_{\Sigma_3} d\Lambda \in \sqrt{r_1 r_2}\ \mathbb{Z} \ . \tag{3.103}$$

3.3.2 Setup of the Calculation

Having discussed the six-dimensional action in general let us now calculate the partition function. Here we can be more explicit with our discussion. To that end we introduce a basis of (anti)self-dual 3-forms as in (2.32)

$$\begin{aligned} \omega_+^A &= (1+\star_\eta) dx^0 \wedge dx^i \wedge dx^j \\ \omega_{-A} &= (1-\star_\eta) dx^0 \wedge dx^i \wedge dx^j \ , \end{aligned} \tag{3.104}$$

where $i,j = 1,\ldots,5$ and $A = (ij)$ with $i < j$ running over all 10 possibilities. Note that these are chosen such that

$$\omega_+^A \wedge \omega_{-B} = 2\delta^A_B d^6x \ . \tag{3.105}$$

Furthermore we introduce a basis of 2-forms:

$$\omega_2^a = \begin{cases} dx^0 \wedge dx^i \\ dx^i \wedge dx^j \end{cases} , \tag{3.106}$$

where the index a runs over $(0i)$ and (ij) (again with $i < j$). Thus there are $5+10 = 15$ values of a. It is helpful to expand

$$dx^\mu \wedge \omega_2^a = K^{\mu a}{}_B \omega_+^B + L^{\mu a B}\omega_{-B} \ , \tag{3.107}$$

for some $K^{\mu a}{}_B$ and $L^{\mu aB}$. In particular we see that

$$\begin{aligned} K^{\mu a}{}_A &= \frac{1}{2(2\pi l)^6}\int_{\mathbb{T}^6} dx^\mu \wedge \omega_2^a \wedge \omega_{-A} \\ L^{\mu a A} &= -\frac{1}{2(2\pi l)^6}\int_{\mathbb{T}^6} dx^\mu \wedge \omega_2^a \wedge \omega_+^A \ . \end{aligned} \tag{3.108}$$

For future reference we observe that the non-vanishing values of $K^{\mu a}{}_B, L^{\mu aB}$ are $\pm\frac{1}{2}$. However the non-vanishing components of $K^{\mu aB} \pm L^{\mu a}{}_B$ are ± 1. Furthermore we can compute

$$\begin{aligned} (dx^\mu \wedge \omega_2^a) \wedge \star_\eta (dx^\nu \wedge \omega_2^b) &= -2(K^{\mu a}{}_B L^{\nu bB} + K^{\nu b}{}_B L^{\mu aB}) d^6x \\ &=: 2\kappa^{\mu\nu ab} d^6x \ . \end{aligned} \tag{3.109}$$

Next, we need to construct the matrix $\widetilde{\mathcal{M}}^{AB}$ as in (2.40). This is rather cumbersome for a general metric. However, we can make the following important observation. To integrate out H_A from the path integral we need to ensure that $\mathrm{Im}(\widetilde{\mathcal{M}}^{AB}) > 0$. We now prove that this is the case if one chooses the branch $\sqrt{-1} = -i$, as for the 2D chiral boson. With this choice the Lorentzian Hodge operator $\star_{g_{\epsilon=2i}}$ for the Euclidean metric $g_{\epsilon=2i}$ is related to the Euclidean Hodge operator $\star^E_{g_{\epsilon=2i}}$ through $\star_{g_{\epsilon=2i}} = -i\star^E_{g_{\epsilon=2i}}$. Recall that

$$\langle \Omega \mid \Omega' \rangle := \int_{\mathbb{T}^6} \Omega^* \wedge \star^E \Omega' \ , \tag{3.110}$$

defines a positive-definite inner-product over the space of (possibly complex-valued) three-forms in 6D, for any Euclidean metric. Consider now two non-vanishing three-forms $\Omega = \Omega_A \varphi_+^A$ and $\Omega' = \Omega'_A \varphi_+^A$, which are both self-dual with respect to $\star_{g_{\epsilon=2i}}$.

Then

$$\begin{aligned}
\langle \Omega \mid \Omega' \rangle &= \int_{\mathbb{T}^6} \Omega^* \wedge \star^E_{g_{\epsilon=2i}} \Omega' \\
&= i \int_{\mathbb{T}^6} \Omega^* \wedge \star_{g_{\epsilon=2i}} \Omega' \\
&= i \int_{\mathbb{T}^6} \Omega^* \wedge \Omega' \\
&= i(2\pi l)^6 \Omega^*_A \Omega'_B (\mathcal{N}^* \mathcal{K}^T - \mathcal{K}^* \mathcal{N}^T)^{AB} \\
&= -2i(2\pi l)^6 \Omega^*_A \Omega'_B (\mathcal{N}^* \widetilde{\mathcal{M}}^T \mathcal{N}^T - \mathcal{N}^* \widetilde{\mathcal{M}}^* \mathcal{N}^T)^{AB} \\
&= -2i(2\pi l)^6 \Omega'_A \Omega^*_B (\mathcal{N}(\widetilde{\mathcal{M}} - \widetilde{\mathcal{M}}^\dagger)\mathcal{N}^\dagger)^{AB} \\
&= -2i(2\pi l)^6 (\mathcal{N}^\dagger \Omega'^*)^\dagger (\widetilde{\mathcal{M}} - \widetilde{\mathcal{M}}^\dagger)(\mathcal{N}^\dagger \Omega^*) \ ,
\end{aligned} \tag{3.111}$$

where we remind the reader that $\varphi^A_+ = \mathcal{N}^A{}_B \omega^B_+ + \mathcal{K}^{AB}\omega_{-B}$ and $\mathcal{K}^{AB} = -\mathcal{N}^A{}_C \widetilde{\mathcal{M}}^{CB}$. Thus we find that choosing $\sqrt{-1} = -i$ leads to $-i(\widetilde{\mathcal{M}} - \widetilde{\mathcal{M}}^\dagger)^{AB}$ being positive definite. This implies that

$$\mathrm{Re}\left(iH_A H_B \widetilde{\mathcal{M}}^{AB} \right) < 0 \ , \tag{3.112}$$

for any real values of H_A, which will be needed to ensure convergence of the functional integrals appearing in the partition function.

To continue we expand the fields, sources and $\mathcal{C}$ as

$$\begin{aligned}
H &= H_A \omega^A_+ \\
B &= B_a \omega^a_2 \\
J &= J^+_A \omega^A_+ + J^{-A}\omega_{-A} \\
\mathcal{C} &= \alpha_{a\mu} dx^\mu \wedge \omega^a_2 \ ,
\end{aligned} \tag{3.113}$$

where H_A and B_a are functions and $\alpha_{\mu a}$ constants. The action can now be written as

$$\begin{aligned}
S_{\mathcal{C}} = \frac{2}{(2\pi)^5} \int_{\mathbb{T}^6} d^6x \Big(&\frac{1}{2}\kappa^{\mu\nu ab}\partial_\mu B_a \partial_\nu B_b - 2L^{\mu a A} H_A \partial_\mu B_a + (H_A + J^+_A)(H_B + J^+_B)\widetilde{\mathcal{M}}^{AB} \\
&+ 2H_A J^{-A} + J^+_A J^{-A} - (K^{\mu a A} L^{\nu b}{}_A - L^{\mu a}{}_A K^{\nu b A})\alpha_{a\mu}\partial_\nu B_b \Big) \\
= \frac{2}{(2\pi)^5} \int_{\mathbb{T}^6} d^6x \Big(&\frac{1}{2} G^{\mu\nu ab} \partial_\mu B_a \partial_\nu B_b + 2L^{\mu a A}\partial_\mu B_a (J^+_A + \widetilde{\mathcal{M}}^{-1}_{AB} J^{-B}) + \hat{H}_A \hat{H}_B \widetilde{\mathcal{M}}^{AB} \\
&- (K^{\mu a A} L^{\nu b}{}_A - L^{\mu a}{}_A K^{\nu b A})\alpha_{a\mu}\partial_\nu B_b - J^+_A J^{-A} - J^{-A} J^{-B} \widetilde{\mathcal{M}}^{-1}_{AB} \Big) \ ,
\end{aligned} \tag{3.114}$$

where

$$\hat{H}_A = H_A + J_A^+ - (L^{\mu a B}\partial_\mu B_a - J^{-B})\widetilde{\mathcal{M}}^{-1}_{AB} \ , \tag{3.115}$$

and

$$\begin{aligned} G^{\mu\nu ab} &= \kappa^{\mu\nu ab} - L^{\mu a C}L^{\nu b D}\widetilde{\mathcal{M}}^{-1}_{CD} \\ &= -K^{\mu a}{}_B L^{\nu b B} - K^{\nu b}{}_B L^{\mu a B} - 2L^{\mu a C}L^{\nu b D}\widetilde{\mathcal{M}}^{-1}_{CD} \ . \end{aligned} \tag{3.116}$$

3.3.3 Explicit Evaluation

At this stage we are ready to evaluate the path integral for (3.114). In principle, one would have to carry this out via the Dirac path-integral procedure, as we implemented for the chiral boson in Section 3.2.1. However, encouraged by the observation in two dimensions that the Dirac and the Feynman path-integral prescriptions led to the same answer (up to multiplicative constants), we will be cavalier and proceed directly with the Feynman path integral

$$Z[J_A^+, J^{-A}, \widetilde{\mathcal{M}}^{AB}] = \int [\mathcal{D}B\mathcal{D}\hat{H}]e^{iS_C} \ . \tag{3.117}$$

We can perform the $\hat{H}_A$ functional integrals since we have seen that $-i(\widetilde{\mathcal{M}}-\widetilde{\mathcal{M}}^\dagger)^{AB}$ is positive definite so that they are convergent. There are ten values of A and each $\hat{H}_A$ has an expansion in terms of a real zero-mode and an infinite tower of complex Fourier modes:

$$\hat{H}_A = h_{0A} + \sum_{n_\mu}{}' h_{n_\mu A}e^{in_\mu x^\mu/l} \ , \tag{3.118}$$

with $(h_{A,n_\mu})^* = h_{A,-n_\mu}$. However, these can be seen to integrate to one after zeta-function regularisation, as we saw above in (3.33).

Thus we are left with evaluating

$$Z[J_A^+, J^{-A}, \widetilde{\mathcal{M}}^{AB}] \sim \int [\mathcal{D}B]e^{iS_{eff}} \ , \tag{3.119}$$

where the effective action left over as the result of the $\hat{H}_A$ integration is given by

$$\begin{aligned} S_{eff} = \frac{2}{(2\pi)^5}\int_{\mathbb{T}^6} d^6x \Big(&\frac{1}{2}G^{\mu\nu ab}\partial_\mu B_a\partial_\nu B_b + 2L^{\mu a A}\partial_\mu B_a\mathcal{J}_A - (K^{\mu a A}L^{\nu b}{}_A - L^{\mu a}{}_A K^{\nu b A})\alpha_{\mu a}\partial_\nu B_b \\ &- \mathcal{J}_A\left(\widetilde{\mathcal{M}}^*(\widetilde{\mathcal{M}}^* - \widetilde{\mathcal{M}})^{-1}\widetilde{\mathcal{M}}\right)^{AB}(\mathcal{J}_B - \bar{\mathcal{J}}_B)\Big) \ , \end{aligned} \tag{3.120}$$

and we have introduced

$$\begin{aligned}\mathcal{J}_A &= J^+_A + \widetilde{\mathcal{M}}^{-1}_{AB} J^{-B} \\ \bar{\mathcal{J}}_A &= J^+_A + J^{-B} (\widetilde{\mathcal{M}}^{-1})^\dagger_{BA} \ .\end{aligned} \tag{3.121}$$

This defines a complex structure on the space of the sources where we view $\mathcal{J}_A$ and $\bar{\mathcal{J}}_A$ as holomorphic and anti-holomorphic coordinates respectively.

It is clear from the simple dependence of S_{eff} on $\bar{\mathcal{J}}_A$ that the partition function is holomorphic in a twisted sense:

$$\bar{\mathcal{D}}^A Z[\mathcal{J}_A, \bar{\mathcal{J}}_B] \ = \left(\frac{\delta}{\delta \bar{\mathcal{J}}_A} - 2\pi i l^6 \left(\widetilde{\mathcal{M}}^* (\widetilde{\mathcal{M}}^* - \widetilde{\mathcal{M}})^{-1} \widetilde{\mathcal{M}} \right)^{AB} \mathcal{J}_B \right) Z[\mathcal{J}_A, \bar{\mathcal{J}}_B] = 0 \ , \tag{3.122}$$

while the holomorphic derivative is

$$\mathcal{D}^B = \frac{\delta}{\delta \mathcal{J}_B} + 2\pi i l^6 \left(\widetilde{\mathcal{M}} (\widetilde{\mathcal{M}} - \widetilde{\mathcal{M}}^*)^{-1} \widetilde{\mathcal{M}}^* \right)_{AB} \bar{\mathcal{J}}_B \ . \tag{3.123}$$

This holomorphic structure captures the simple dependence on $\bar{\mathcal{J}}_B$:

$$Z[\mathcal{J}_A, \bar{\mathcal{J}}_B] = e^{\frac{2i}{(2\pi)^5} \int_{\mathbb{T}^6} d^6x \left(\widetilde{\mathcal{M}}^* (\widetilde{\mathcal{M}}^* - \widetilde{\mathcal{M}})^{-1} \widetilde{\mathcal{M}} \right)^{AB} \mathcal{J}_A \bar{\mathcal{J}}_B} Z_h[\mathcal{J}_A] \ , \tag{3.124}$$

where $Z_h[\mathcal{J}_A]$ can also be further factorised as

$$Z_h[\mathcal{J}_A] = e^{-\frac{2i}{(2\pi)^5} \int_{\mathbb{T}^6} d^6x \left(\widetilde{\mathcal{M}}^* (\widetilde{\mathcal{M}}^* - \widetilde{\mathcal{M}})^{-1} \widetilde{\mathcal{M}} \right)^{AB} \mathcal{J}_A \mathcal{J}_B} Z_h^{(0)}[\mathcal{J}_A] \ . \tag{3.125}$$

So our final task is to compute

$$\begin{aligned} Z_h^{(0)}[\mathcal{J}_A] &= \int [\mathcal{D}B] e^{i S^{(0)}_{eff}[B, \mathcal{J}_A]} \\ S^{(0)}_{eff} &= \frac{2}{(2\pi)^5} \int_{\mathbb{T}^6} d^6x \Big(\frac{1}{2} G^{\mu\nu ab} \partial_\mu B_a \partial_\nu B_b + 2 L^{\mu a A} \partial_\mu B_a \mathcal{J}_A \\ &\qquad - (K^{\mu a A} L^{\nu b}{}_A - L^{\mu a}{}_A K^{\nu b A}) \alpha_{\mu a} \partial_\nu B_b \Big) \ . \end{aligned} \tag{3.126}$$

To this end we expand the fields in Fourier modes

$$B_a = b_a + w_{a\mu} x^\mu + {\sum_{n_{a\mu}}}' b_{a, n_{a\mu}} e^{i n_{a\mu} x^\mu / l} \qquad n_{a\mu} \in \mathbb{Z} \ , \tag{3.127}$$

with $(b_{a,n_{a\mu}})^* = b_{a,-n_{a\mu}}$. We have separated out the zero-modes as they are real. The $w_{a\mu}$ are the analogues of winding modes and the flux-quantisation condition (3.95) implies that

$$w_{a\mu} = \frac{R^3}{l^3}(m_{a\mu} + \alpha_{a\mu}) \qquad m_{a\mu} \in \mathbb{Z} \ . \tag{3.128}$$

Since the action is quadratic the evaluation of the partition function factorises into a contribution arising from a sum over the winding modes $m_{a\mu}$ and an integral over the oscillator modes b_{an_μ}:

$$Z_h^{(0)}[\mathcal{J}_A] = Z_{\text{w.m.}}^{(0)} Z_{\text{osc}}^{(0)} \ . \tag{3.129}$$

Let us first evaluate the action on the oscillator modes. The calculation is similar in form to that of Section 3.2.2. The action evaluates to

$$S_{\text{eff}}^{(0)} = -2\pi l^4 \sum_{a,n_\mu} \Big(G^{\mu\nu ab} n_\mu n_\nu b_{an_\mu} b_{b,-n_\nu} - 4il L^{\mu aA} n_{a\mu} b_{an_\mu} \mathcal{J}_{+A}^{-(n_\mu)} \Big) \ , \tag{3.130}$$

where the integral over $\mathbb{T}^6$ has imposed $n_{a\mu} = -n_{b\mu} = n_\mu$. The integral over the b_{an_μ}'s now produces

$$\begin{aligned} Z_{\text{osc}} &\sim \prod_{n_\mu}{}' \frac{-il^{-4}}{\det(G^{\mu\nu ab} n_\mu n_\nu)} \\ &\sim \prod_{n_\mu}{}' \frac{1}{\det\Big(2K^{\mu a}{}_B L^{\nu bB} n_\mu n_\nu + 2L^{\mu aA} L^{\nu bB} \widetilde{\mathcal{M}}^{-1}_{AB} n_\mu n_\nu\Big)} \\ &\sim \prod_{n_\mu}{}' \frac{1}{\det(2L^{\nu bA}(L^{\mu aA} - K^{\mu a}{}_A) n_\mu n_\nu + 4L^{\mu aA} L^{\nu bB} T_{AB} n_\mu n_\nu)} \ , \end{aligned} \tag{3.131}$$

where the determinant is over the a, b indices and

$$T_{AB} = -\frac{1}{2}(\delta_{AB} + \widetilde{\mathcal{M}}^{-1}_{AB}) \ . \tag{3.132}$$

It is difficult to evaluate the expression (3.131) more precisely in general. We recall that the non-zero values of $(L^{\mu aA} - K^{\mu a}{}_A)$ and $2L^{\mu aA}$ are ± 1 so the determinant is of a matrix which is quadratic in the integers n_μ and linear in T_{AB}, much like (3.48). We will suggestively denote it as:

$$Z_{\text{osc}} := \frac{1}{\eta_{6D}^{10}(T_{AB})} \ . \tag{3.133}$$

Next, we evaluate the action on the winding modes. Again it is helpful to introduce

$$\begin{aligned} w^+_A &:= K^{\mu a}{}_A w_{\mu a} \\ w^{-A} &:= L^{\mu a A} w_{\mu a} \ . \end{aligned} \tag{3.134}$$

We can apply similar maps to $\alpha_{a\mu}$ and $m_{a\mu}$. In this case we find

$$\begin{aligned} S^{(0)}_{\rm eff} &= \frac{2}{(2\pi)^5}\int_{\mathbb{T}^6} d^6x\Big(- w^{-A}w^+_A - w^{-A}w^{-B}\widetilde{\mathcal{M}}^{-1}_{AB} - \frac{R^3}{l^3}\alpha^+_A w^{-A} + \frac{R^3}{l^3}\alpha^{-A}w^+_A + 2w^{-A}\mathcal{J}_A\Big) \\ &= \frac{2}{(2\pi)^5}\int_{\mathbb{T}^6} d^6x\Big(- w^{-A}w^{-B}\widetilde{\mathcal{M}}^{-1}_{AB} + 2w^{-A}(\mathcal{J}_A - \frac{1}{2}(R/l)^3\alpha^+_A) - (w^{-A} - (R/l)^3\alpha^-_A)w^+_A\Big) \\ &= \frac{2}{(2\pi)^5}\int_{\mathbb{T}^6} d^6x\Big(- w^{-A}w^{-B}(\delta_{AB} + \widetilde{\mathcal{M}}^{-1}_{AB}) + 2w^{-A}\left(\mathcal{J}_A + (R/l)^3(\alpha^{-A} - \alpha^+_A)\right) \\ &\qquad + (w^{-A} - (R/l)^3\alpha^{-A})w'_A + (R/l)^6\alpha^{-A}(\alpha^+_A - \alpha^{-A})\Big) \ , \end{aligned} \tag{3.135}$$

where we have introduced

$$w'_A = w^{-A} - w^+_A + \frac{R^3}{l^3}(\alpha^+_A - \alpha^{-A}) \ . \tag{3.136}$$

The point about w'_A is that, given (3.128), then

$$w'_A = w^{-A} - w^+_A = (R/l)^3 m_A \ , \tag{3.137}$$

for some $m_A \in \mathbb{Z}$. Whereas

$$w^{-A} - (R/l)^3\alpha^{-A} = (R/l)^3 m^{-A} \qquad m^{-A} \in \mathbb{Z} \ . \tag{3.138}$$

Therefore we see that the sum over m_A imposes a delta-function constraint

$$\sum_{m_A\in\mathbb{Z}^{10}} e^{4\pi i R^6 m^{-A}m_A} = \sum_{p^A\in\mathbb{Z}^{10}} \delta(2R^6 m^{-A} - p^A) \ . \tag{3.139}$$

For R^6 irrational, the only solution is $m^{-A} = p^A = 0$ and hence $Z^{(0)}_{\rm w.m.} \sim 1$. However, for $R^6 = r_1/r_2$ we find $m^{-A} = r_2 n^A/2$ and $p^A = r_1 n^A$ (recall $m^{-A} \in \frac{1}{2}\mathbb{Z}$). Substituting this back into the action gives

$$\begin{aligned} Z^{(0)}_{\rm w.m.} &= e^{4\pi i R^6\alpha^{-A}(\alpha^+_A - \alpha^{-A})}\sum_{p^A}\sum_{m^{-A}}\delta(2R^6m^{-A} - p^A) \\ &\quad \times e^{-4\pi\pi R^6 i(m^{-A}+\alpha^{-A})(m^{-B}+\alpha^{-B})(\delta_{AB}+\widetilde{\mathcal{M}}^{-1}_{AB}) + 8\pi R^3 i(m^{-A}+\alpha^{-A})(\mathcal{J}^{(0)}_A + R^3(\alpha^{-A}-\alpha^+_A))} \\ &\sim e^{-\pi i\alpha^A\beta_A}\Theta\begin{bmatrix}\alpha^A\\ \beta_A\end{bmatrix}\left(\sqrt{r_1r_2}\mathcal{J}^{(0)}_A \mid 2r_1r_2T_{AB}\right) \ , \end{aligned} \tag{3.140}$$

where we have introduced the higher-dimensional theta function

$$\Theta\begin{bmatrix}\alpha^A\\ \beta_A\end{bmatrix}(z_A \mid \tau_{AB}) := \sum_{n^A\in\mathbb{Z}^{10}} e^{\pi i(n^A+\alpha^A)(n^B+\alpha^B)\tau_{AB}+2\pi i(n^A+\alpha^A)(z_A+\beta_A)} , \tag{3.141}$$

the normalised source zero-mode

$$\mathcal{J}_A^{(0)} = \frac{2l^3}{(2\pi l)^6}\int d^6x \left(J_A^+ + \widetilde{\mathcal{M}}^{-1}_{AB}J^{-B}\right) , \tag{3.142}$$

and the theta-characteristics

$$\begin{aligned}\alpha^A &= 2\alpha^{-A}/r_2\\ \beta_A &= 2r_1(\alpha^{-A}-\alpha_A^+) .\end{aligned} \tag{3.143}$$

In Appendix B, we show that (3.140) factorizes into a product of standard theta functions (3.61) when the spacetime metric is diagonal.

In summary, our final answer for the six-dimensional partition function is

$$Z \sim e^{-\pi i\alpha^A\beta_A} e^{-\frac{2i}{(2\pi)^5}\int_{\mathbb{T}^6} d^6x\left(\widetilde{\mathcal{M}}^*(\widetilde{\mathcal{M}}^*-\widetilde{\mathcal{M}})^{-1}\widetilde{\mathcal{M}}\right)^{AB}\mathcal{J}_{\mathcal{A}}(\mathcal{J}_B-\bar{\mathcal{J}}_B)} \frac{\Theta\begin{bmatrix}\alpha^A\\ \beta_A\end{bmatrix}\left(\sqrt{r_1r_2}\mathcal{J}_A^{(0)} \mid 2r_1r_2T_{AB}\right)}{\eta_{6D}^{10}(\mathcal{T}_{AB})} . \tag{3.144}$$

Note that for vanishing characteristics we find a higher-dimensional analogue of T-duality: $R \leftrightarrow 1/R$.

It is important to make some comments about gauge symmetries. The expressions that we have above are gauge invariant and hence the path integral has over-counted the physical degrees of freedom. As (3.144) is already abstract enough, we did not display the complications that arise from standard gauge-fixing procedures. Maybe the easiest way to take care of the usual redundancy associated to B is to completely fix it via a condition like $B_{0i} = 0$ followed by other 4 constraints (such as $\partial^i B_{ij} = 0$). While it is straightforward to see how the $B_{0i} = 0$ condition would modify (3.144), to incorporate a constraint like $\partial^i B_{ij} = 0$ one would need a more detailed analysis.

3.4 Conclusions

In this chapter we performed a direct calculation of the partition function associated with the Sen action for chiral forms in $4k+2$ dimensions in a path-integral formulation.

As this action contains unphysical fields with the wrong-sign kinetic term, convergence of the path integral was achieved through a non-standard analytic continuation to Euclidean signature *via* a complex deformation of the metric and not time. This procedure had the additional benefit of leaving the self-duality condition of the self-dual form, $H = \star_\eta H$, untouched and directly led to a holomorphic result.

To appreciate this last point, one should take a step back to understand what happens within the holomorphic-factorisation approach to the partition function of the chiral boson. In that framework, one starts with the Lorentzian path integral of a non-chiral boson ϕ, *i.e.* $Z_{\text{n.c.}} \sim \int [\mathcal{D}\phi] \exp\big(i \int d^2x \partial_\mu \phi \partial^\mu \phi\big)$ and by Wick rotating as usual, $x^0 \to -ix^0$, one evaluates the path-integral of a real action

$$\int [\mathcal{D}\phi] \exp\Big(- \int d^2x (\partial_0 \phi)^2 + (\partial_1 \phi)^2\Big) .$$

This leads to a real result of the schematic form

$$Z_{\text{n.c.}} \sim \mathcal{W}(\tau - \bar{\tau}) \sum \frac{\theta}{\eta} \frac{\bar{\theta}}{\bar{\eta}} , \tag{3.145}$$

where the sum is over the characteristics of the theta functions, see [132], and τ is the complex structure of the torus. Then, one would like to conclude that the chiral-boson partition function is indeed the holomorphic theta function, with some undetermined characteristics (which can get fixed, case by case, according to the actual physical system that the chiral boson is meant to describe). In so doing one also needs to ignore the anomalous factor $\mathcal{W}$.

Instead, in the approach taken here, the kinetic term of the non-chiral boson does not get Wick rotated and one computes the path integral of a *complex* action, whose convergence arises from the $\widetilde{\mathcal{M}}$ term in Sen lagrangian. What is more, thanks to implementing the Wick rotation as a metric deformation and to the precise nature of the non-standard coupling of the Sen action to the curved background, $\widetilde{\mathcal{M}}$ is simply related to the torus complex structure τ and enters the computations in a manifestly holomorphic fashion.

In this way, for the chiral boson in two dimensions, we reproduced the classic θ/η result by a calculation of the path integral on $\mathbb{T}^2$. The argument of this expression was an $SL(2, \mathbb{Z})$ transformation away from the usual $\mathbb{T}^2$ complex structure. General theta-function characteristics were incorporated by introducing a topological term to the Sen action and adjusting the periodicities of the scalar on the torus. We then proceeded to repeat the same computation for the significantly more complicated case of the six-dimensional theory on $\mathbb{T}^6$, under certain assumptions about the equivalence of the Dirac and Feynman path-integral prescriptions for the Sen action. The result, which was a generalisation of the two-dimensional one, can be extended to higher k.

It is worth making contact between our calculation and the canonical-quantisation computation of the partition function of the Sen action. As we saw in section (2.2.5), within the hamiltonian formalism one introduces a pair of non-canonically conjugate variables, $\Pi^{\pm}$, in terms of which the Hamiltonian schematically splits into $\mathcal{H} \sim \mathcal{H}_{+} + \mathcal{H}_{-}$. Here $\mathcal{H}_{+}$ is a negative-definite Hamiltonian which completely decouples from the system, while $\mathcal{H}_{-}$ is the physical Hamiltonian which describes the correct spectrum of the chiral form. Therefore, within the canonical approach to quantisation, one can simply recover the partition function of, *e.g.* , the compact chiral boson by computing $\mathrm{Tr} e^{2\pi i \tau R \mathcal{H}_{-}}$; see [47]. Note that, due to the nature of the Legendre transform, the decoupling of the unphysical modes is not straightforward in the Lagrangian formulation of the theory. Nevertheless we found a sensible result by keeping all modes and this is due to the non-standard Wick rotation, which preserved the wrong-sign modes in Lorentzian signature as oscillatory contributions in the Euclidean path-integral. In other words, the Wick rotation (3.13) left the kinetic term of ϕ in (3.15) unaffected and thus it made sense to compute the path-integral of the Wick-rotated theory without removing any contributions.

It is satisfying to see the chiral partition function emerge directly from an honest functional-integral calculation. The computations performed in this chapter hence provide nontrivial evidence that the proposal [46, 47] correctly captures the physics of chiral forms also within the framework of the path-integral approach to quantisation.

Chapter 4

Covariantly Constant Anomalies

Operators with integer scaling dimensions in even-dimensional conformal field theories exhibit well-known type-B Weyl anomalies. In general, these anomalies depend non-trivially on exactly marginal couplings. In this chapter, which is based on the work done in [4], we study the corresponding fully covariantised anomaly functional on conformal manifolds in several examples. We show that a natural consequence of the Wess–Zumino consistency condition is that the anomalies are covariantly constant with respect to the exactly marginal couplings. The argument is general and applies even when the conformal symmetry is spontaneously broken on moduli spaces of vacua.

4.1 Introduction

Sometimes, in a Quantum Field Theory a classical symmetry turns out to be violated by quantum effects. This usually happens because at the quantum level there are divergences which make impossible for the theory to satisfy the Ward identities of a symmetry. Maybe the archetype of such phenomena is the chiral anomaly of theories with fermions in even spacetime dimensions, in which case, the anomaly results as an incompatibility between the Ward identities of vector and axial symmetries, see [145] for a pedagogical review. Chiral anomalies are peculiar and robust observables which can be put to good use in the study of a plethora of phenomena. For instance, they are 1-loop exact and they match in different phases of the theory. Moreover, they are invariant under the renormalization group flow and therefore can be used to constrain strongly coupled dynamics and to test weak-strong coupling dualities between different QFTs, as done in [146].

4.1.1 Weyl Anomalies

Conformal Field Theories can admit another important class of anomalies, the so-called Weyl (or conformal) anomalies, which signal the impossibility of formulating the quantum theory on a curved manifold with simultaneous diffeomorphism and Weyl invariance. We follow the qualitative classification of Weyl-anomalies introduced in [32], according to which a CFT in an even-dimensional spacetime can exhibit Type-A and Type-B Weyl anomalies.

Type-A are the most famous and well-studied Weyl anomalies. They are expressed in terms of topological invariants, which is a feature that makes them akin to chiral anomalies. The c_{2D} anomaly in 2D and the a_{4D} anomaly in 4D belong to this class, which can be isolated by studying the tracelessness condition for the energy-momentum tensor $T_{\mu\nu}$:

$$\langle T^{\mu}_{\mu} \rangle = \begin{cases} -\frac{c_{2D}}{12} E_2 & \text{in 2D} \\ a_{4D} E_4 - c_{4D} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} & \text{in 4D} \end{cases} , \tag{4.1}$$

where E_n is (proportional to) the Euler density in n dimensions, which famously gives rise to a topological quantity upon integration.

In (4.1) we introduced also the coefficient c_{4D} which classifies as the first example of Type-B Weyl anomaly encountered in this chapter, as the square of the Weyl tensor $W_{\mu\nu\rho\sigma}$ is not a topological density.

Like chiral anomalies, also Type-A Weyl anomalies have very peculiar properties, which turn coefficients like a_{4D} and c_{2D} into robust observables, indispensable tools for the analysis of CFTs. For instance, a_{4D} and c_{2D} are guaranteed to match across different phases of the CFT ([147]) and across different points of the conformal manifolds (as a result of the Wess-Zumino consistency condition). Moreover, they obey the $a/c-$theorems of [148, 149], which essentially attribute, to these coefficients, the interpretation of a measure of the amount of degrees of freedom present in the system.

Type-B Weyl anomalies seem to be wilder quantities as, in general, it is not possible to obtain similar results for them. There are various counterexamples for a 4D c-theorem based on the c_{4D} coefficient (see [150] and references therein) and, in absence of supersymmetry, it is hard to figure the fate of c_{4D} under marginal deformation or under the spontaneous breaking of the conformal symmetry; see [151] for a review on these topics.

4.1.2 A subclass of Type-B Weyl anomalies

In this chapter we are interested in a particular subclass of Type-B Weyl anomalies and in probing their behavior under both exactly marginal deformations and spontaneous symmetry breaking of the conformal symmetry.

These are the Weyl anomalies associated to operators $\mathcal{O}_\Delta$ with integer scaling dimensions Δ in a four-dimensional spacetime, which, as we will see below, can be detected by studying the correlator $\langle T^{\mu}_{\mu} \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle$.

Concrete examples of such anomalies can be found in 4D $\mathcal{N} = 2$ SCFTs, meaning superconformal QFTs with eight Poincaré supercharges and R-symmetry group $SU(2)_R \times U(1)_r$. Four of the Poincaré supercharges are left-chiral (which we will denote as $\mathcal{Q}^{\mathcal{I}}_{\alpha}$, where $\alpha = \pm$ are spinor indices and $\mathcal{I} = 1, 2$ labels the doublet of $SU(2)_R$) whereas the other four are right-chiral, $\bar{\mathcal{Q}}_{\mathcal{I}\dot{\alpha}}$. These theories have two kind of $\frac{1}{2}$−BPS scalar superconformal primary operators:

- Coulomb-Branch operators (CBOs), which are charged under the $U(1)_r$ symmetry, but are neural under the $SU(2)_R$. They are either chiral or anti-chiral operators, meaning that they are annihilated by either the four right-chiral supercharges or by the four left-chiral ones. Their scaling dimensions Δ are related to their $U(1)_r$ charge r by the shortening condition[33] $\Delta = |r|$. Coulomb branch operators are naturally endowed with a ring structure, the so-called chiral ring, see [153] for example.
- Higgs-Branch operators (HBOs), which are charged under the $SU(2)_R$ symmetry, but are neural under the $U(1)_r$. They are annihilated by supercharges of both chiralities and obey to the shortening condition $\Delta = 2R$, where R is the charge under the Cartan generator of the $SU(2)_R$. These operators parametrize the Higgs branch (HB) of vacua of the SCFT, where $U(1)_R$ is unbroken and $SU(2)_R$ is spontaneously broken.

Both CBOs and HBOs have their scaling dimensions tied by supersymmetry to the charge of a subgroup of the R-symmetry. As the latter is naturally an integer, these operators have integer scaling dimensions and therefore they lead to a large class of Type-B Weyl anomalies.
As we will be interested in studying how these anomalies change across the conformal manifolds, it is natural to start our investigations with CBOs. Indeed, such operators cannot disappear from the spectrum as we explore the conformal manifold, because they are not allowed to recombine into long multiplets ([152]).
At the same time, we will also want to understand how the anomalies associated to CBOs might vary in the presence of a vacuum $|v\rangle$ that spontaneously breaks conformal invariance. The object that we need to study is $\langle v|T^{\mu}_{\mu}\mathcal{O}_{\Delta}\mathcal{O}_{\Delta}|v\rangle$ and, to make our life simpler, we might want to choose a vacuum $|v\rangle$ parametrized by a vev that is independent of the exactly marginal coupling constants, so that we do not need to worry about how $|v\rangle$ might change as we go around the conformal manifold. Such a choice can be made on the HB, because the connection on the bundle of Higgs-Branch superconformal-primary is flat, [154].

All in all, the main example of anomalies that we will have in mind throughout this chapter are the one associated with CBOs in both the conformal and the HB phase;

[33] Here we use the convention of [152].

nevertheless we will soon indicate which arguments of our discussion can be generalized beyond these specific cases.

The non-perturbative properties of type-B Weyl anomalies associated with CBOs on the Higgs–branch vacuum moduli space of 4D $\mathcal{N}=2$ SCFTs were discussed in [34, 155]. These papers presented examples, where the CBO type-B Weyl anomalies matched across the Higgs branch, and other examples where the matching between the conformally symmetric and spontaneously broken phase does not occur. A complete understanding of the dynamics responsible for these disparate behaviours is still missing, but the existing results have led to a number of non-perturbative conjectures, which were postulated in [155].

In this thesis, we elaborate further on the properties of the CBO type-B Weyl anomalies, and point out that one of the crucial elements in the discussion of Refs [34, 155]—the fact that these anomalies are covariantly constant on conformal manifolds, both in the unbroken and in the HB phase—can be understood in many cases as a natural consequence of the Wess–Zumino consistency conditions of the corresponding anomaly functionals. As explained in Ref. [155], the existence of covariantly constant type-B anomalies in different phases of the theory can lead to non-trivial implications[34].

The main elements of the argument are as follows. For an operator $\mathcal{O}$ with scaling dimension $\Delta = 2+n$ ($n \in \mathbb{N}_0$), the anomaly of interest can be identified (in all phases of the CFT) as a specific contact term in the integrated 3-point function of the trace of the energy-momentum tensor $T \equiv T^{\mu}{}_{\mu}$,

$$\int d^4y \langle T(y)\mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle \propto \Box^n \delta(x) \ . \tag{4.2}$$

In the unbroken conformal phase, the Ward identities of diffeomorphism and Weyl transformations can be used to relate the corresponding anomaly coefficient $G^{(\mathrm{CFT})}_\Delta$, to the 2-point function coefficient of the operator $\mathcal{O}$. In momentum space, the anomaly appears in the logarithmically divergent piece of the 2-point function

$$\langle \mathcal{O}_\Delta(p)\mathcal{O}_\Delta(-p)\rangle = (-1)^{n+1}\frac{\pi^2 G^{(\mathrm{CFT})}_\Delta}{2^{2n}\Gamma(n+1)\Gamma(n+\frac{D}{2})}p^{2n}\log\left(\frac{p^2}{\mu^2}\right) + \dots \ . \tag{4.3}$$

In a phase with spontaneous breaking of the conformal symmetry, the Ward identities do not provide a similar relation between the corresponding type-B conformal anomaly and some datum in the 2-point function of the operator $\mathcal{O}$, see [34]. In that case, the broken-phase anomaly, $G^{(\mathrm{broken})}_\Delta$, must be extracted directly from the 3-point function

[34] Such as potential constraints on the holonomy of superconformal manifolds.

(4.2). In momentum space this reads

$$\lim_{q\to 0}\langle T(q)\mathcal{O}_\Delta(p_1)\mathcal{O}_\Delta(p_2)\rangle = (-1)^n \frac{\pi^2 G_\Delta^{(\text{broken})}}{2^{2n}\Gamma(n+1)\Gamma(n+2)}(p_1^2)^n + \dots \,. \tag{4.4}$$

On the RHS of (4.3) and (4.4) a Dirac-delta imposing momentum conservation is left implicit; in particular, the $q \to 0$ limit in (4.4) is equivalent to taking the $p_2 \to -p_1$ limit.

As an explicit example, let us consider the case of 4D $\mathcal{N} = 2$ SCFTs with a non-trivial chiral ring of CBOs $\mathcal{O}_I$, an anti-chiral ring of conjugate operators $\bar{\mathcal{O}}_J$, and a non-empty conformal manifold $\mathcal{M}$. The latter means that the $\mathcal{N} = 2$ SCFTs of interest possess exactly marginal operators.[35] We will denote the anomaly coefficients in the conformally symmetric or HB phase respectively as $G_{I\bar{J}}^{(\text{CFT})}$ and $G_{I\bar{J}}^{(\text{Higgs})}$. Both quantities are, in general, complicated functions of the exactly marginal couplings, see [33, 34, 155].

A crucial ingredient in the discussion of [34, 155] was the proposal that the anomaly coefficients $G_{I\bar{J}}$ are covariantly constant on the conformal manifold $\mathcal{M}$ in both phases of the theory. Namely, both anomalies obey equations of the form $\nabla G_{I\bar{J}}^{(\text{CFT})} = 0$, $\nabla G_{I\bar{J}}^{(\text{Higgs})} = 0$, where ∇ is a phase-independent connection on the vector bundles of the CBOs. It is straightforward to derive this condition in the conformally symmetric phase as a consequence of superconformal Ward identities. However, as pointed out in [34], a similar argument in the Higgs phase needs to take into account potential contributions from the dilatino. In [34] it was conjectured that such contributions do not affect the contact term that accounts for the anomaly, but it is not straightforward to demonstrate this explicitly. As a result, it would be very useful to have an alternative way to deduce that $\nabla G_{I\bar{J}}^{(\text{Higgs})} = 0$ and in this chapter we will find such an argument. As a bonus, the approach we will present is very general and can be applied to any CFT with a conformal manifold that has operators with integer-valued scaling dimension; it is not restricted to CBOs in $\mathcal{N} = 2$ SCFTs or to Higgs-branch phases.

It is well known (see e.g. [156]) that conformal anomalies can be conveniently packaged into a local anomaly functional that expresses the Weyl variation of the generating functional of correlation functions W[36]

$$\delta_\sigma W \propto \int d^4x \sqrt{\gamma}\,\delta\sigma \mathcal{A}\,. \tag{4.5}$$

W is a non-local functional of the sources of the CFT, but the Weyl anomaly $\mathcal{A}$ is a local term reflecting the above-mentioned fact that in correlation functions it appears as a

[35] These are necessarily supersymmetric descendants of scaling-dimension 2 CBOs.
[36] See also [157, 158].

contact term. The δ_σ variation in (4.5) denotes infinitesimal local Weyl transformations with parameter $\delta\sigma(x)$ that vanish at the boundary of spacetime [147], and $\gamma_{\mu\nu}$ is the background spacetime metric. The locality of $\delta\sigma(x)$ guarantees, among other things, that the Ward identities retain the same form in all phases of the theory, irrespective of whether or not conformal symmetry is spontaneously broken (they are operatorial relations). This fact will be crucial for our upcoming discussion of the structure of the anomaly functional in different phases. In order to encode the CBO type-B anomalies of interest in the anomaly functional one needs to add to the action spacetime-dependent sources for the operators $\mathcal{O}_I, \bar{\mathcal{O}}_J$

$$\delta S = \int d^4x\sqrt{\gamma}\left(\lambda^I(x)\mathcal{O}_I(x) + \bar{\lambda}^J(x)\bar{\mathcal{O}}_J(x)\right) \ . \tag{4.6}$$

The anomaly functional must satisfy certain conditions. It must be invariant under diffeomorphisms or any other unbroken symmetries of the theory. In addition, it must obey the Wess–Zumino (WZ) consistency condition

$$\delta_{[\sigma_1}\delta_{\sigma_2]}W = 0 \ , \tag{4.7}$$

which encodes the fact that the action of the Weyl group is abelian. Finally, terms in $\mathcal{A}$ that are Weyl variations of a local functional express the addition of local counterterms in W [159], which simply correspond to a change in the regularisation scheme. Such terms are considered trivial and can be dropped from $\delta_\sigma W$. This reflects the fact that the anomaly is a scheme-independent quantity.

As emphasised already in Ref. [160], on a conformal manifold $\mathcal{M}$ one should also require that the anomaly functional is suitably invariant under coupling-constant redefinitions. This can be achieved by utilising a connection ∇ on the bundle of operators. For exactly marginal couplings, the WZ consistency condition on the $\mathcal{M}$-covariantised version of the anomaly implies that the connection ∇ is compatible with the Zamolodchikov metric, [160]. In this chapter, we examine whether this argument can be extended beyond the case of the exactly marginal operators.

Since the presence of a contact term like the one in (4.4), in any phase of the theory, has been established independently by the analysis of Ward identities, in all phases the anomaly functional includes a term of the form

$$\delta_\sigma W \propto \int d^4x\sqrt{\gamma}\delta\sigma[G_{I\bar{J}}\lambda^I\Box^n\bar{\lambda}^J + \ldots] \ , \tag{4.8}$$

where G_{IJ} are the corresponding anomaly coefficients. This term should be covariantised on the corresponding vector bundle of operators over $\mathcal{M}$. We perform this covariantisation for operators of scaling dimension $\Delta = 3, 4, 5$ in Sec. 4.2 and show that the WZ consistency condition (4.7) requires that the anomaly is covariantly constant.

In the case of scaling-dimension 4 operators the arguments of Refs [157, 158, 160] are modified to capture the properties of marginal, but not necessarily exactly marginal operators. Our analysis is completely general and does not employ supersymmetry at any stage. We expect that similar arguments can be applied to all higher values of integer scaling dimension Δ, but the anomaly functional becomes significantly more complicated with increasing Δ. Indeed, already at $\Delta = 5$ we present WZ-consistent anomaly functionals, which contain hundreds of terms in the flat-space limit. We notice that in both the cases of $\Delta = 4, 5$ anomalies, new terms in the anomaly functional that involve the curvature of the corresponding operator bundles are crucial in order to satisfy the WZ consistency conditions.

The case of $\Delta = 2$ operators is special and requires a separate discussion: the anomaly functional is automatically WZ-consistent and (4.7) does not lead to further restrictions. To make a non-trivial statement, we need to use the $\mathcal{N} = 2$ supersymmetry to relate the $\Delta = 2$ anomaly to the anomaly of the exactly marginal operators. An argument in favour of this relation is sketched in Sec. 4.3 alongside an explicit tree-level check for $\mathcal{N} = 2$ SCQCD in the conformal and Higgs phases.

4.2 WZ Consistency Conditions in 4D CFTs

We follow closely the discussion and notation of references [161, 162]. $W = \log Z$ is the generating functional of correlation functions. It is a functional of the spacetime-dependent sources (couplings). In this section, we focus on four spacetime dimensions and type-B conformal anomalies of scalar operators. Such anomalies exist when the operators have scaling dimensions $\Delta = 2 + n$ with $n \in \mathbb{N}_0$.

We will denote the operators of interest as $\mathcal{O}_I$ and their corresponding sources as λ^I. Note that although we ultimately have $\mathcal{N} = 2$ applications in mind, we will use a real basis of operators and will not require supersymmetry for any of the arguments presented in this section. When the operators are exactly marginal they will be denoted as Φ_i and their corresponding couplings as λ^i. Clearly, the index i takes values up to the dimension of the conformal manifold $\mathcal{M}$. The more general indices I label conformal primary operators in a sub-bundle of operators of fixed integer dimension and the corresponding conformal anomalies will be denoted G_{IJ}. The background spacetime metric will be denoted $\gamma_{\mu\nu}$ with Greek letters reserved for the spacetime coordinate indices. Vector bundles over the conformal manifold can be equipped with a connection. For a discussion of this connection in the context of conformal perturbation theory see [163, 164]. For a related discussion in radial quantization see [165]. The components of the connection on the sub-bundle of $\mathcal{O}_I$ operators will be denoted as $(A_i)^I_J$, whereas the connection on the tangent space of Φ_i operators as Γ^k_{ij}. The corresponding covariant derivative on the conformal manifold will be denoted as ∇_i.

In this section we follow the general strategy of [157, 158, 160], where the basic ansatz for the Weyl variation of W was covariantised not only in spacetime but also in the tangent bundle of the conformal manifold $T\mathcal{M}$. Accordingly, for the type-B Weyl anomalies of exactly marginal operators [157, 158, 160] proposed the anomaly functional:[37]

$$\delta_\sigma W \propto \int d^4x\sqrt{\gamma}\delta\sigma\, G_{ij}\bigg((\Box\lambda^i + \Gamma^i_{kl}\partial^\mu\lambda^k\partial_\mu\lambda^l)(\Box\lambda^j + \Gamma^j_{mn}\partial^\nu\lambda^m\partial_\nu\lambda^n) - 2\partial_\mu\lambda^i(R^{\mu\nu} - \frac{1}{3}\gamma^{\mu\nu}R)\partial_\nu\lambda^j\bigg)\,, \tag{4.9}$$

with $R_{\mu\nu}$ and R the spacetime Ricci tensor and scalar respectively. Clearly, this functional is sensitive only to the symmetric part $\Gamma^i_{(kl)}$ of the connection. The WZ-condition identifies it to be the Levi–Civita connection on $\mathcal{M}$, and under the further assumption that the connection is torsion-free [163] one obtains that the anomaly is covariantly constant on $\mathcal{M}$,

$$\nabla_i G_{jk} = 0\,. \tag{4.10}$$

This approach can be generalised to generic operators $\mathcal{O}_I$, where covariantisation on the conformal manifold translates into the invariance of $\delta_\sigma W$ under a change of basis in the vector space of $\mathcal{O}_I$s. We discover that imposing the WZ consistency condition will typically lead to $\nabla_i G_{IJ} = 0$.

We emphasise that this result is independent of the phase of the theory. The anomaly functional can be understood as the local Weyl variation of the generating functional W with appropriate boundary conditions for the fields. The infinitesimal local Weyl parameters, $\delta\sigma(x)$, by definition vanish at the boundary of spacetime and parametrise transformations that are valid both in the conformally symmetric and broken phases [147]. Moreover, the asymptotic behaviour of $\delta\sigma(x)$ also guarantees that any boundary terms that involve $\delta\sigma(x)$ (obtained after integration by parts) can be safely ignored in the upcoming discussion.

We will now summarise the key ingredients of the calculation, before specialising to type-B anomalies for operators with $\Delta = 3, 4, 5$.[38] The $\Delta = 2$ case cannot be constrained with a simple analysis of the Wess-Zumino consistency condition and will be treated separately in Sec. 4.3. The expressions $\delta_\sigma W$ for cases with a single source can be found in [162] and form the starting point of our discussion. We study the WZ consistency conditions after we covariantise the expressions in Ref. [162] with respect

[37] For the case of a single coupling, this expression is related to the Fradkin–Tseytlin–Paneitz–Riegert operator [166–169]. A six-dimensional generalisation of this operator was presented in [170].

[38] Here the $\Delta = 4$ case refers exclusively to Weyl anomalies for marginal operators that are not exactly marginal—they can be marginally relevant or irrelevant.

to the conformal manifold. In the process we discover that a fully covariant anomaly functional requires new terms that have not appeared previously in the literature.

4.2.1 Covariantisation on the Conformal Manifold

In what follows we will make an important distinction between the exactly marginal couplings λ^i that parametrise the conformal manifold and the remaining non-exactly marginal sources λ^I. Geometrically, the couplings λ^i are coordinates on the curved conformal manifold, which are allowed to also depend non-trivially on the spacetime coordinates. The couplings λ^I are viewed, instead, as sections of a vector bundle; they can depend both on the spacetime and conformal manifold coordinates.

Under a change of basis on the tangent bundle of the conformal manifold, the spacetime derivatives $\partial_\mu \lambda^i$ transform as

$$\partial_\mu \lambda^i = \frac{\partial \lambda^i}{\partial \lambda'^{j'}} \partial_\mu \lambda'^{j'} \ . \tag{4.11}$$

On the other hand, under a change of basis on each fibre of the λ^I-vector bundle

$$\lambda^I = \frac{\partial \lambda^I}{\partial \lambda'^{I'}} \lambda'^{I'} \ , \tag{4.12}$$

where the transformation matrix $\frac{\partial \lambda^I}{\partial \lambda'^{I'}}$ depends on the $\lambda^i(x^\mu)$ only. As a result, we define covariant derivatives on the conformal manifold in terms of the connection components $(A_i)^I_J$ as

$$\nabla_i \lambda^I = \partial_i \lambda^I + (A_i)^I_J \lambda^J \ . \tag{4.13}$$

The generalised covariant derivative is then naturally given by[39]

$$\widehat{\nabla}_\mu \lambda^I := \nabla_\mu \lambda^i \nabla_i \lambda^I + \nabla_\mu \lambda^I|_{\lambda^i = \text{fixed}} = \partial_\mu \lambda^i \nabla_i \lambda^I + \partial_\mu \lambda^I|_{\lambda^i = \text{fixed}} \ , \tag{4.14}$$

with ∇_μ the standard spacetime-covariant derivative.

Compared to the unhatted differential operators used in [162], commutators of our hatted operators can lead to curvature terms on $\mathcal{M}$. The latter can be easily evaluated by using the definition of the generalised covariant derivative and the fact that $\partial_\mu (A_i)^I_J|_{\lambda^i = \text{fixed}} = 0$, i.e.

$$(F_{\mu\nu})^I_J \lambda^J := [\widehat{\nabla}_\mu, \widehat{\nabla}_\nu] \lambda^I = \partial_\mu \lambda^i \partial_\nu \lambda^j (F_{ij})^I_J \lambda^J \ , \tag{4.15}$$

where $(F_{ij})^I_J = \partial_i (A_j)^I_J - \partial_j (A_i)^I_J + (A_i)^I_K (A_j)^K_J - (A_j)^I_K (A_i)^K_J$.

[39] Here we are explicitly stressing that ∇_μ and ∂_μ have to be understood at fixed λ^i, but later this will be left implicit.

Under the change of basis (4.11)-(4.12), the connection transforms inhomogeneously as

$$(A_i)^I_J = \frac{\partial\lambda'^{i'}}{\partial\lambda^i}\frac{\partial\lambda'^{J'}}{\partial\lambda^J}\frac{\partial\lambda^I}{\partial\lambda'^{I'}}(A_{i'})^{I'}_{J'} - \frac{\partial\lambda'^{i'}}{\partial\lambda^i}\frac{\partial\lambda'^{J'}}{\partial\lambda^J}\frac{\partial^2\lambda^I}{\partial\lambda'^{i'}\partial\lambda'^{J'}} \tag{4.16}$$

such that

$$\nabla_i\lambda^I = \frac{\partial\lambda'^{i'}}{\partial\lambda^i}\frac{\partial\lambda^I}{\partial\lambda'^{I'}}\nabla_{i'}\lambda'^{I'} , \tag{4.17}$$

which in turn implies

$$\widehat{\nabla}_\mu\lambda^I = \frac{\partial\lambda^I}{\partial\lambda'^{I'}}\widehat{\nabla}_\mu\lambda'^{I'} . \tag{4.18}$$

Therefore, standard differential operators can be covariantised on $\mathcal{M}$ by upgrading the usual spacetime-covariant derivative ∇_μ to $\widehat{\nabla}_\mu$. For example, the $\mathcal{M}-$covariant Laplacian $\widehat{\Box}$, which reads

$$\begin{aligned}\widehat{\Box}\lambda^I &:= \widehat{\nabla}_\mu\widehat{\nabla}^\mu\lambda^I \\ &= \partial^\mu\lambda^i\nabla_i\left(\widehat{\nabla}_\mu\lambda^I\right) + \nabla^\mu\Big|_{\lambda^i\text{fixed}}\widehat{\nabla}_\mu\lambda^I \\ &= \partial^\mu\lambda^i\left(\partial_i\widehat{\nabla}_\mu\lambda^I + (A_i)^I_J\widehat{\nabla}_\mu\lambda^J\right) + \partial^\mu\Big|_{\lambda^i\text{fixed}}\widehat{\nabla}_\mu\lambda^I - g^{\mu\nu}\Gamma^\rho_{\nu\mu}\widehat{\nabla}_\rho\lambda^I ,\end{aligned} \tag{4.19}$$

transforms as

$$\widehat{\Box}\lambda^I = \frac{\partial\lambda^I}{\partial\lambda'^{I'}}\widehat{\Box}\lambda'^{I'} . \tag{4.20}$$

As a result, to get anomaly functionals invariant under a change of basis in the space of $\mathcal{O}$s, one can consider the ones written in [162] and simply replace all spacetime covariant derivatives with their hatted versions. However, because of (4.15) this minimal prescription is, in general, sensitive to ordering choices and does not guarantee WZ consistency.

We conclude this section with some remarks on λ^i and by explicitly stressing how our framework is compatible with the one of [157, 158, 160]. As the exactly marginal coupling λ^i is not a tensor (it is a coordinate on the conformal manifold), the generating functional cannot display an explicit λ^i dependence. Instead, the anomaly can depend on it only through its infinitesimal variation, i.e.

$$\lambda^i_\mu := \partial_\mu\lambda^i = \nabla_\mu\lambda^i . \tag{4.21}$$

This object serves as a pullback from the conformal manifold to spacetime, as it could have been appreciated already at the level of formula (4.14). It has good transformation properties (4.11) and can then be acted upon by the generalised covariant derivative:

$$\widehat{\nabla}_\mu \lambda^i_\nu = \nabla_\mu \lambda^i_\nu + \lambda^j_\mu \Gamma^i_{jk} \lambda^k_\nu \,. \tag{4.22}$$

Thus, within our framework, (4.9) can be more succinctly recast into the form

$$\begin{aligned} \delta_\sigma W \propto & \int d^4x \sqrt{\gamma} \delta\sigma \, G_{ij} \Big(\widehat{\nabla}_\mu \lambda^{i\mu} \widehat{\nabla}_\nu \lambda^{j\nu} - 2\lambda^i_\mu (R^{\mu\nu} - \frac{1}{3}\gamma^{\mu\nu} R) \lambda^j_\nu \Big) \\ = & \int d^4x \sqrt{\gamma} \delta\sigma \left(\widehat{\nabla}_\mu \lambda^{i\mu} \widehat{\nabla}_\nu \lambda^\nu_i - 2\lambda^i_\mu (R^{\mu\nu} - \frac{1}{3}\gamma^{\mu\nu} R) \lambda_{i\nu} \right) \,. \end{aligned} \tag{4.23}$$

In the second line we have implicitly used the fact that Γ^i_{jk} is given by the Christoffel symbol (so that $\widehat{\nabla}_\mu G_{jk} = 0$). The fact that Γ^i_{jk} is symmetric yields many simplifications, e.g.

$$\widehat{\nabla}_{[\mu} \lambda^i_{\nu]} = 0 \,, \qquad \lambda^i_{[\mu} \nabla_i \lambda^j_{\nu]} = 0 \,, \tag{4.24}$$

where the first equation guarantees that the Bianchi identity $\nabla_{[i}(F_{jk]})^I_J = 0$ gets pulled-back onto $\widehat{\nabla}_{[\mu}(F_{\nu\rho]})^I_J = 0$.

4.2.2 Weyl Transformations

In four spacetime dimensions, an infinitesimal local Weyl transformation acts on the spacetime metric $\gamma_{\mu\nu}$ as

$$\delta_\sigma \gamma_{\mu\nu} = 2\delta\sigma \; \gamma_{\mu\nu} \,. \tag{4.25}$$

The Christoffel symbols, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R transform accordingly

$$\begin{aligned} \delta_\sigma \Gamma^\rho_{\mu\nu} &= \gamma^{\rho\sigma} \left(\gamma_{\nu\sigma} \partial_\mu \delta\sigma + \gamma_{\mu\sigma} \partial_\nu \delta\sigma - \gamma_{\mu\nu} \partial_\sigma \delta\sigma \right) \\ \delta_\sigma R_{\mu\nu} &= -2\nabla_\mu \nabla_\nu \delta\sigma - \gamma_{\mu\nu} \Box \delta\sigma \\ \delta_\sigma R &= -2\delta\sigma \; R - 6\Box\delta\sigma \,. \end{aligned} \tag{4.26}$$

For an operator of conformal scaling dimension Δ, one has classically $\delta_\sigma \mathcal{O}_I = -\Delta \mathcal{O}_I \delta\sigma$. Thus,

$$\delta_\sigma \lambda^I = (\Delta - 4)\delta\sigma \; \lambda^I \;, \quad \delta_\sigma \lambda^i = 0 \,. \tag{4.27}$$

Being a number, the anomaly has vanishing classical dimension, so

$$\delta_\sigma G_{IJ} = 0 \,, \tag{4.28}$$

while the uniform Weyl variation of $\nabla_i \lambda^I$ leads to

$$\delta_\sigma (A_i)^I_J = 0 \, . \tag{4.29}$$

One then finds that standard equations such as

$$\delta_\sigma \partial_\mu \lambda^I = (\Delta - 4)\partial_\mu \lambda^I \delta\sigma + (\Delta - 4)\partial_\mu \delta\sigma \, \lambda^I \tag{4.30}$$
$$\delta_\sigma \Box \lambda^I = (\Delta - 6)\delta\sigma \, \Box\lambda^I + 2(\Delta - 3)\partial_\mu \delta\sigma \, \partial^\mu \lambda^I + (\Delta - 4)\lambda^I \Box \delta\sigma \, . \tag{4.31}$$

can be straightforwardly extended to

$$\delta_\sigma \widehat{\nabla}_\mu \lambda^I = (\Delta - 4)\widehat{\nabla}_\mu \lambda^I \delta\sigma + (\Delta - 4)\partial_\mu \delta\sigma \, \lambda^I \tag{4.32}$$
$$\delta_\sigma \widehat{\Box} \lambda^I = (\Delta - 6)\delta\sigma \, \widehat{\Box}\lambda^I + 2(\Delta - 3)\partial_\mu \delta\sigma \, \widehat{\nabla}^\mu \lambda^I + (\Delta - 4)\lambda^I \Box \delta\sigma \, . \tag{4.33}$$

and accordingly for quantities with raised spacetime indices. These expressions will be useful in the calculations that we will be performing below.

4.2.3 $\Delta = 3$ Operators

We begin the construction of fully covariant and WZ-consistent anomaly functionals with the case of $\Delta = 3$. According to the discussion around Eq. (4.8), the ansatz for this case should contain two derivatives. In order to address the Weyl-cohomological problem, we will first characterise terms in the anomaly functional that are cohomologically trivial. We start with the following expression for the generating functional of connected correlation functions

$$W^{\text{exact}} = \int d^4x \sqrt{\gamma} \Big[G_{IJ} \widehat{\nabla}^\mu \lambda^I \widehat{\nabla}_\mu \lambda^J + A_1 \lambda^I \widehat{\Box} \lambda^J G_{IJ} + A_2 G_{IJ} \lambda^I \lambda^J R + A_3 \lambda^I \widehat{\nabla}^\mu \lambda^J \widehat{\nabla}_\mu G_{IJ} + A_4 \lambda^I \lambda^J \widehat{\Box} G_{IJ} \Big] \, . \tag{4.34}$$

By computing its Weyl variation, and after integrating by parts, we find that the most general exact (i.e. cohomologically trivial) anomaly functional is

$$\begin{aligned} \delta_\sigma W^{\text{exact}} \propto \int d^4x \sqrt{\gamma} \delta\sigma \Big[& 2(-1 + A_1 + 6A_2) G_{IJ} \widehat{\nabla}^\mu \lambda^I \widehat{\nabla}_\mu \lambda^J \\ & + 2(-1 + A_1 + 6A_2) \lambda^I \widehat{\Box} \lambda^J G_{IJ} \\ & + 2(-1 + 2A_1 + 12A_2 - A_3 + 2A_4) \lambda^I \widehat{\nabla}^\mu \lambda^J \widehat{\nabla}_\mu G_{IJ} \\ & + (A_1 + 6A_2 - A_3 + 2A_4) \lambda^I \lambda^J \widehat{\Box} G_{IJ} \Big] \, . \end{aligned} \tag{4.35}$$

From the above one can deduce that:

(a) An anomalous Weyl generating functional containing $\delta\sigma G_{IJ}\lambda^I\lambda^J R$ cannot be cohomologically trivial.

(b) $\lambda^I\lambda^J\widehat{\Box}G_{IJ}$ is cohomologically equivalent to $\lambda^I\widehat{\nabla}^\mu\lambda^J\widehat{\nabla}_\mu G_{IJ}$.

(c) $\lambda^I\widehat{\nabla}^\mu\lambda^J\widehat{\nabla}_\mu G_{IJ}$ is cohomologically equivalent to $G_{IJ}\widehat{\nabla}^\mu\lambda^I\widehat{\nabla}_\mu\lambda^J + \lambda^I\widehat{\Box}\lambda^J G_{IJ}$ and by going to momentum space, one sees that the latter does not contribute to the anomaly.

Hence, modulo cohomologically trivial terms and up to integration by parts, the most general Weyl anomalous functional is given by

$$\delta_\sigma W = \int d^4x\sqrt{\gamma}\,\delta\sigma G_{IJ}\left[C_1\widehat{\nabla}^\mu\lambda^I\widehat{\nabla}_\mu\lambda^J + C_2\lambda^I\widehat{\Box}\lambda^J + C_3\lambda^I\lambda^J R\right] \tag{4.36}$$

with $C_1 \neq C_2$. Imposing the WZ consistency condition leads to the following independent solutions for the anomaly functional:

$$\delta_\sigma W^{(1)} = \int d^4x\sqrt{\gamma}\,\delta\sigma G_{IJ}\left[\widehat{\nabla}^\mu\lambda^I\widehat{\nabla}_\mu\lambda^J + \frac{1}{6}\lambda^I\lambda^J R\right], \tag{4.37}$$

$$\delta_\sigma W^{(2)} = \int d^4x\sqrt{\gamma}\,\delta\sigma G_{IJ}\left[\lambda^I\widehat{\Box}\lambda^J - \frac{1}{6}\lambda^I\lambda^J R\right]. \tag{4.38}$$

For $\delta_\sigma W^{(1)}$ one needs to impose $\nabla_i G_{IJ} = 0$, while $\delta_\sigma W^{(2)}$ is automatically WZ consistent.[40] It is interesting to observe that (4.37) and (4.38) are equivalent upon integration by parts when $\nabla_i G_{IJ} = 0$, leading to a self-consistent picture.

4.2.4 $\Delta = 4$ Operators

The classical Weyl variations (4.32) do not distinguish between the exactly-marginal couplings λ^i and the marginally relevant or irrelevant λ^I. However, in our formalism these two sets of couplings are treated differently—the λ^i are non-linear coordinates on the conformal manifold but the λ^I are linear coordinates on a vector bundle. Accordingly, in the conformal phase, we can interpret the anomalies G_{ij} as a Zamolodchikov metric on the conformal manifold, but the anomalies G_{IJ} do not have such an interpretation. This will soon translate to a different type of anomaly functional for the anomalies G_{IJ}, which is sensitive to the curvature of the corresponding operator bundles. Examples of theories with non-exactly marginal $\Delta = 4$ operators, whose curvature is non-trivial, are abundant in 4D $\mathcal{N} = 2$ SCFTs, see e.g. [171].

[40] The WZ consistency condition imposes $C_3 = \frac{1}{6}(C_1 - C_2)$ so the most general anomaly functional is given by $\delta_\sigma W = C_1\delta_\sigma W^{(1)} + C_2\delta_\sigma W^{(2)}$ with $C_1 \neq C_2$. Terms with $C_1 = C_2$ cannot capture the anomaly, see point (c) above.

It is sensible to start with an anomaly functional, which is similar to (4.23) for the exactly marginal operators

$$\delta_\sigma W \propto \int d^4x\sqrt{\gamma}\delta\sigma\, G_{IJ}\left[\widehat{\Box}\lambda^I\widehat{\Box}\lambda^J - 2\widehat{\nabla}_\mu\lambda^I(R^{\mu\nu} - \frac{1}{3}\gamma^{\mu\nu}R)\widehat{\nabla}_\nu\lambda^J\right] . \tag{4.39}$$

For exactly marginal operators $\widehat{\nabla}_{[\mu}\lambda^i_{\nu]} = 0$ from (4.24). Instead, for non-exactly marginal $\Delta = 4$ operators $[\widehat{\nabla}_\mu, \widehat{\nabla}_\nu]\lambda^I = (F_{\mu\nu})^I_J\lambda^J$. As a result, we expect that terms containing either $(F_{\mu\nu})^K_J$ or explicit $(F_{ij})^K_J$ contributions will mark a distinctive difference compared to the exactly-marginal case. Indeed, when checking the WZ-consistency condition for (4.39), one finds that

$$\begin{aligned}\delta_{\sigma_{[2}}\delta_{\sigma_{1]}} W \propto \int d^4x\sqrt{\gamma}\delta\sigma_{[1}\partial^\nu\delta\sigma_{2]}\times \\ \times\widehat{\nabla}^\mu\lambda^I\left[-4\widehat{\nabla}_\mu G_{IJ}\widehat{\nabla}_\nu\lambda^J + 2\widehat{\nabla}_\nu G_{IJ}\widehat{\nabla}_\mu\lambda^J - 4G_{IK}\lambda^J(F_{\mu\nu})^K_J\right] .\end{aligned} \tag{4.40}$$

The expression on the RHS does not vanish automatically even after imposing $\nabla_i G_{IJ} = 0$: extra terms need to be added to (4.39) to cancel the last term in (4.40). One can exhaustively prove that terms constructed out of $(F_{\mu\nu})^I_J$ are closed with respect to the Weyl-cohomology and cannot achieve the desired goal. We are thus forced to use terms where $(F_{ij})^I_J$ appears explicitly and does not combine with pullbacks to give $(F_{\mu\nu})^I_J = (F_{ij})^I_J\lambda^i_\mu\lambda^j_\nu$. We notice, using the first equation in (4.26), that $\delta_\sigma(\widehat{\nabla}_\rho\lambda^i_\nu) \sim \partial_\rho\delta\sigma\lambda^i_\nu$, and as a result

$$\delta_\sigma\left((F_{ij})^J_K\lambda^j_\rho\widehat{\nabla}_\mu\lambda^i_\nu\right) = (F_{ij})^J_K\lambda^j_\rho\delta_\sigma\left(\widehat{\nabla}_\mu\lambda^i_\nu\right) \sim (F_{\rho\nu})^J_K\partial_\mu\delta\sigma \quad . \tag{4.41}$$

We are thus led to consider terms with the schematic structure:

$$G_{IJ}(F_{ij})^I_K\lambda^j_\rho\lambda^K\widehat{\nabla}_\mu\lambda^i_\nu\widehat{\nabla}_\sigma\lambda^J .$$

By taking into account all possible contractions for the spacetime indices, we arrive at the generating functional

$$\begin{aligned}\delta_\sigma W \propto \int d^4x\sqrt{\gamma}\delta\sigma\, G_{IJ}\Big[\widehat{\Box}\lambda^I\widehat{\Box}\lambda^J - 2\widehat{\nabla}_\mu\lambda^I(R^{\mu\nu} - \frac{1}{3}\gamma^{\mu\nu}R)\widehat{\nabla}_\nu\lambda^J + \\ + (F_{ij})^I_K(E_3 g^{\mu\sigma}g^{\nu\rho} + E_2 g^{\mu\rho}g^{\nu\sigma} + E_1 g^{\mu\nu}g^{\rho\sigma})\lambda^j_\rho\widehat{\nabla}_\mu\lambda^i_\nu\lambda^K\widehat{\nabla}_\sigma\lambda^J\Big],\end{aligned} \tag{4.42}$$

where E_1, E_2, E_3 are free constants. The WZ consistency condition can be satisfied by setting $\nabla_i G_{IJ} = 0$ and $E_1 + E_2 + E_3 = -2$. The fact that only the combination $E_1 + E_2 + E_3 = -2$ survives the WZ condition suggests a relation between the three terms in the second line of (4.42). Indeed, the terms parametrised by E_2 and E_3 are identical as a consequence of the identity $\widehat{\nabla}_{[\mu}\lambda^i_{\nu]} = 0$. This leaves a single combination in (4.42)—the difference between the terms parametrised by E_1 and E_2 being closed, but not exact. The resultant anomaly functional (4.42) is the WZ-consistent functional that captures the type-B anomalies G_{IJ} for non-exactly-marginal $\Delta = 4$ operators.

We can draw two lessons from this discussion. First, we verify once again that the condition $\nabla_i G_{IJ} = 0$ is necessary to obtain WZ consistency. Second, and on a more technical level, we notice that in order to cancel $F_{\mu\nu}$-terms in the WZ consistency condition (4.40), one needs to add to the generating functional terms where F_{ij} factors come contracted with (differentiated) pull-backs. The specific terms added in (4.42) contributed to the WZ condition only with $F_{\mu\nu}$ combinations. It turns out that this is a special feature of $\Delta = 4$ operators (for which both $\delta_\sigma \lambda^I$ and $\delta_\sigma \widehat{\nabla} \lambda^I$ vanish). In the next section, we will see that F_{ij}-terms provide contributions to the WZ condition of $\Delta = 5$ anomalies that do not combine to produce $F_{\mu\nu}$. This feature will add to the complexity of the $\Delta = 5$ anomaly functionals.

4.2.5 $\Delta = 5$ Operators

The Osborn equation for type-B anomalies of irrelevant operators in even spacetime dimensions is subtle. Its intricacies were discussed in [162], the main lesson being that in order to ensure the consistency of the anomalous part, one has to introduce a beta function for the spacetime metric. We will generalise the analysis of [162] to the case of multiple irrelevant sources λ^I, starting with the most general ansatz for the spacetime Weyl variation $\delta_\sigma \gamma_{\mu\nu}$ that is quadratic in the sources λ^I. One needs to first impose that $\delta_{[\sigma_2}\delta_{\sigma_1]}\gamma_{\mu\nu} = 0$ and then remove the cohomologically trivial terms from $\delta_\sigma \gamma_{\mu\nu}$.[41] The outcome of this analysis, at quadratic order in the sources, is that the variation of the metric $\delta_\sigma \gamma_{\mu\nu}$ is essentially the covariantised version of the one proposed by [162], i.e.

$$\begin{aligned}\delta_\sigma \gamma_{\mu\nu} = 2\delta\sigma\, \gamma_{\mu\nu} + \alpha\delta\sigma G_{IJ}\left(R_{\mu\nu}\lambda^I\lambda^J + 2\lambda^I\widehat{\nabla}_{(\mu}\widehat{\nabla}_{\nu)}\lambda^J - 3\gamma_{\mu\nu}\widehat{\nabla}^\rho\lambda^I\widehat{\nabla}_\rho\lambda^J + \gamma_{\mu\nu}\lambda^I\widehat{\Box}\lambda^J\right)\\ +\beta\delta\sigma\gamma_{\mu\nu}\left(R\lambda^I\lambda^J + 6\lambda^I\widehat{\Box}\lambda^J - 12\widehat{\nabla}^\rho\lambda^I\widehat{\nabla}_\rho\lambda^J\right) + O(\lambda^4)\,,\end{aligned} \tag{4.43}$$

where α and β are free parameters. Here we have neglected—already at $O(\lambda^2)$—terms that vanish when $\nabla_i G_{IJ} = 0$; one can prove that they sit in a cohomology class different to that of the ones proportional to G_{IJ}, hence their presence would not modify (4.43). Moreover, such terms will not play a role in the computations that we will display below.

As a starting point for the analysis of the $\Delta = 5$ anomaly functional, we consider the covariantised version of the expression derived in [162], which to quadratic order in

[41] The latter are those solutions $(\delta_\sigma\gamma_{\mu\nu})_{\text{trivial}}$ to $\delta_{[\sigma_2}\delta_{\sigma_1]}\gamma_{\mu\nu} = 0$ that can be written as $(\delta_\sigma\gamma_{\mu\nu})_{\text{trivial}} = \delta_\sigma\widehat{\gamma}_{\mu\nu} - 2\delta\sigma\widehat{\gamma}_{\mu\nu}$ for a metric $\widehat{\gamma}_{\mu\nu}$. Therefore the redefined metric $\gamma_{\mu\nu} \mapsto \gamma_{\mu\nu} - \widehat{\gamma}_{\mu\nu}$, continues to transform classically.

the irrelevant sources reads[42]

$$
\begin{aligned}
\mathcal{A} = & \; c\, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \\
&+ \frac{c}{2}\alpha G_{IJ}\Big\{ \widehat{\Box}\lambda^I\widehat{\Box}^2\lambda^J - \frac{13}{8}RR^{\mu\nu}R_{\mu\nu}\lambda^I\lambda^J + \frac{53}{162}R^3\lambda^I\lambda^J + \frac{4}{3}R^{\mu\nu}R^{\lambda\sigma}R_{\mu\lambda\nu\sigma}\lambda^I\lambda^J \\
&- \frac{1}{8}RR_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}\lambda^I\lambda^J + \frac{43}{72}R_{\mu\nu\lambda\sigma}R^{\mu\nu\alpha\beta}R_{\alpha\beta}{}^{\lambda\sigma}\lambda^I\lambda^J - \frac{35}{72}R^2\lambda^I\widehat{\Box}\lambda^J + \frac{25}{24}R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}\lambda^I\widehat{\Box}\lambda^J \\
&- \frac{1}{36}\nabla^\mu R\nabla_\mu R\lambda^I\lambda^J + \frac{167}{12}R^{\mu\nu}R_{\mu\nu}\widehat{\nabla}^\alpha\lambda^I\widehat{\nabla}_\alpha\lambda^J - \frac{101}{24}R^2\widehat{\nabla}^\alpha\lambda^I\widehat{\nabla}_\alpha\lambda^J \\
&- \frac{79}{24}R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}\widehat{\nabla}^\alpha\lambda^I\widehat{\nabla}_\alpha\lambda^J - \frac{1}{3}R\widehat{\Box}\widehat{\nabla}^\mu\lambda^I\widehat{\nabla}_\mu\lambda^J - \frac{10}{9}R^{\mu\nu}\nabla_\mu\nabla_\nu R\lambda^I\lambda^J + \frac{7}{9}R^{\mu\nu}R\lambda^I\widehat{\nabla}_\mu\widehat{\nabla}_\nu\lambda^J \\
&+ \frac{1}{36}\Box R\lambda^I\widehat{\Box}\lambda^J - \frac{16}{9}R\widehat{\Box}\lambda^I\widehat{\Box}\lambda^J + \nabla^\mu R\widehat{\nabla}_\mu\lambda^I\widehat{\Box}\lambda^J + \frac{1}{6}R\lambda^I\widehat{\Box}^2\lambda^J - 4R^{\mu\nu}\widehat{\nabla}_\mu\lambda^I\widehat{\Box}\nabla_\nu\lambda^J \\
&- \frac{37}{18}R_{\mu\nu}\nabla^\mu R\lambda^I\widehat{\nabla}_\nu\lambda^J - 22R^\alpha_\mu R_{\nu\alpha}\widehat{\nabla}^\mu\lambda^I\widehat{\nabla}^\nu\lambda^J + \frac{116}{9}R^{\mu\nu}R\widehat{\nabla}^\mu\lambda^I\widehat{\nabla}^\nu\lambda^J \\
&- 13R^{\alpha\beta}R_{\mu\alpha\nu\beta}\widehat{\nabla}^\mu\lambda^I\widehat{\nabla}^\nu\lambda^J - \frac{5}{18}\nabla^\mu\nabla^\nu R\lambda^I\widehat{\nabla}_\mu\widehat{\nabla}_\nu\lambda^J - \frac{5}{9}R\widehat{\nabla}_\mu\widehat{\nabla}_\nu\lambda^I\widehat{\nabla}^\mu\widehat{\nabla}^\nu\lambda^J \\
&- 5R^{\beta\gamma}\nabla_\gamma R_{\alpha\beta}\lambda^I\widehat{\nabla}^\alpha\lambda^J - \frac{8}{3}R^\gamma_\alpha R^{\alpha\beta}\lambda^I\widehat{\nabla}_\beta\widehat{\nabla}_\gamma\lambda^J + \frac{10}{3}R^{\beta\gamma}\widehat{\nabla}^\alpha\lambda^I\widehat{\nabla}_\gamma\widehat{\nabla}_\beta\widehat{\nabla}_\alpha\lambda^J \\
&+ \frac{5}{6}\Box R^{\mu\nu}\lambda^I\widehat{\nabla}_\mu\widehat{\nabla}_\nu\lambda^J + \frac{22}{3}R^{\mu\nu}\widehat{\nabla}_\mu\widehat{\nabla}_\nu\lambda^I\widehat{\Box}\lambda^J - \frac{5}{3}\nabla^\mu R^{\alpha\beta}\nabla_\mu R_{\alpha\beta}\lambda^I\lambda^I\Big\} + \mathcal{O}\left(\lambda^4\right) \,, \qquad (4.44)
\end{aligned}
$$

where $c = c_{4D}$ is the central charge of the system. From this expression it is apparent that the α parameter entering (4.43) is the normalisation of $\langle T\mathcal{O}\mathcal{O}\rangle$ which, in the unbroken phase, can be related to the normalization of $\langle\mathcal{O}\mathcal{O}\rangle$. However, there is no information about β, since the part of $\delta_\sigma\gamma_{\mu\nu}$ that it parametrises does not contribute to $\delta_\sigma(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma})$.[43] The WZ consistency condition for the anomaly (4.44) is satisfied up to terms that vanish when $\nabla_i G_{IJ} = 0$ and up to bundle-curvature terms (F-terms), since in their absence our expression then reverts to the one of [162].[44] Our next goal will be to introduce new terms $\mathcal{A}^F$ to the anomaly (4.44) that remove the F-terms in the WZ consistency condition.

For the purposes of this chapter, it will be enough to determine the new terms that are needed to make $\mathcal{A} + \mathcal{A}^F$ WZ consistent to leading order around flat spacetime, $\gamma_{\mu\nu} \simeq \delta_{\mu\nu} + \ldots$. We will therefore ignore in $\mathcal{A}$, $\mathcal{A}^F$ terms quadratic (or higher) in the spacetime curvature, like the Weyl-tensor squared. However, terms linear in the spacetime curvature must be taken into account, as the flat spacetime limit of $\delta_\sigma R_{\mu\nu\rho\sigma}$ does not vanish, c.f. (4.26). Accordingly, we will work with the classical Weyl variation of the spacetime metric and up to quadratic order in the λs. In summary, we want to identify the terms $\mathcal{A}^F_{\text{flat}}$ that can remove all F-terms from the flat-spacetime limit of

[42]We thank M. Broccoli for pointing out a missing factor of $\frac{1}{2}$ between the $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ and λ^2 terms in [162]. This factor can also be confirmed by an independent holographic computation [172].

[43]Note that when computing $\delta_\sigma(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma})$, all the ∇_μ operators hitting $\delta_\sigma\gamma_{\mu\nu}$ can be promoted to their hatted versions, i.e. $\nabla_\mu\delta_\sigma\gamma_{\nu\rho} = \widehat{\nabla}_\mu\delta_\sigma\gamma_{\nu\rho}$.

[44]By F-terms we denote contributions that vanish when $F_{ij} = 0$, but do not vanish when $\nabla_i G_{IJ} = 0$.

the WZ condition for $\mathcal{A}_{\text{flat}}$, which reads

$$\mathcal{A}_{\text{flat}} \propto G_{IJ}\Big[\widehat{\Box}\lambda^I\widehat{\Box}^2\lambda^J - \frac{1}{3}R\widehat{\Box}\widehat{\nabla}^\mu\lambda^I\widehat{\nabla}_\mu\lambda^J + \frac{1}{36}\Box R\lambda^I\widehat{\Box}\lambda^J - \frac{16}{9}R\widehat{\Box}\lambda^I\widehat{\Box}\lambda^J + \nabla^\mu R\widehat{\nabla}_\mu\lambda^I\widehat{\Box}\lambda^J + \frac{1}{6}R\lambda^I\widehat{\Box}^2\lambda^J - 4R^{\mu\nu}\widehat{\nabla}_\mu\lambda^I\widehat{\Box}\widehat{\nabla}_\nu\lambda^J - \frac{5}{18}\nabla^\mu\nabla^\nu R\lambda^I\widehat{\nabla}_\mu\widehat{\nabla}_\nu\lambda^J - \frac{5}{9}R\widehat{\nabla}_\mu\widehat{\nabla}_\nu\lambda^I\widehat{\nabla}^\mu\widehat{\nabla}^\nu\lambda^J + \frac{10}{3}R^{\beta\gamma}\widehat{\nabla}^\alpha\lambda^I\widehat{\nabla}_\gamma\widehat{\nabla}_\beta\widehat{\nabla}_\alpha\lambda^J + \frac{5}{6}\Box R^{\mu\nu}\lambda^I\widehat{\nabla}_\mu\widehat{\nabla}_\nu\lambda^J + \frac{22}{3}R^{\mu\nu}\widehat{\nabla}_\mu\widehat{\nabla}_\nu\lambda^I\widehat{\Box}\lambda^J\Big]. \tag{4.45}$$

The F-terms that enter the flat spacetime limit of the WZ consistency condition for $\mathcal{A}_{\text{flat}}$ are[45]

$$\int d^4x\sqrt{\gamma}\delta\sigma_{[1}\nabla^\mu\delta\sigma_{2]}G_{IJ}\times \Big[-8F^J_{\nu\rho L}F^{\nu\rho I}_K\lambda^K\widehat{\nabla}_\mu\lambda^L - \frac{40}{3}F^{\rho I}_{\mu K}F^J_{\nu\rho L}\lambda^K\widehat{\nabla}^\nu\lambda^J - \frac{20}{3}\lambda^K\widehat{\Box}\lambda^L\widehat{\nabla}F^{\rho I}_{\mu K} + \frac{4}{3}\widehat{\nabla}_\nu\lambda^L\widehat{\nabla}^\nu\lambda^J\widehat{\nabla}F^{\rho I}_{\mu L} + \frac{40}{3}\lambda^K\widehat{\nabla}^\nu\widehat{\nabla}_\mu\lambda^J\widehat{\nabla}F^{\rho I}_{\nu K} + \frac{20}{3}F^{\nu I}_{\mu K}\lambda^K\lambda^L\widehat{\nabla}_\rho F^{\rho J}_{\nu L} + 8\lambda^K\widehat{\nabla}^\nu\lambda^J\widehat{\Box}F^I_{\mu\nu K} - 28F^I_{\mu\nu K}\widehat{\nabla}\lambda^K\widehat{\Box}\lambda^J + \frac{4}{3}F^I_{\mu\nu K}\lambda^K\widehat{\Box}\widehat{\nabla}^\nu\lambda^J + 16\widehat{\nabla}\lambda^J\widehat{\nabla}_\rho F^I_{\mu\nu L}\widehat{\nabla}^\rho\lambda^L - \frac{32}{3}F^I_{\nu\rho K}\widehat{\nabla}^\nu\lambda^K\widehat{\nabla}^\rho\widehat{\nabla}_\mu\lambda^J - \frac{8}{3}F^I_{\mu\rho K}\widehat{\nabla}^\nu\lambda^K\widehat{\nabla}^\rho\widehat{\nabla}_\nu\lambda^J\Big]. \tag{4.46}$$

One arrives at this expression by making use of the Bianchi identity for $F_{\mu\nu}$, and rearranging the order of the $\widehat{\nabla}$-operators into terms of the type $\widehat{\nabla}^{(n+2)}G_{IJ}$.[46]

To cancel the terms in (4.46), we start with the most general linear ansatz that is quadratic in the sources λ, that vanishes when $F_{ij} = 0$, and that is at most linear in the spacetime curvature. Moreover, since (4.46) involves only $F_{\mu\nu} = \lambda^i_\mu\lambda^j_\nu F_{ij}$, each F_{ij} must come contracted with corresponding factors of λ^i_μ. Without any additional algebraic simplifications, nor through identifying redundancies due to Weyl-cohomologically trivial terms, we have determined using the xAct Mathematica package [173–178] that such an ansatz comprises ~ 1500 terms. These contribute to the WZ consistency condition with two classes of terms: those that can be rewritten solely in terms of the combination $F_{\mu\nu} = \lambda^i_\mu\lambda^j_\nu F_{ij}$ and those where the curvature components F_{ij} of the $\lambda-$bundle necessarily appear explicitly. We require that the former cancel out the terms in (4.46) and the latter cancel out by themselves. This yields a solution that fixes some of the coefficients of the linear ansatz and leaves the remaining undetermined. By setting the undetermined coefficients to zero the resulting expression has the following 126

[45]To simplify our expressions, we will denote $F^P_{\mu\nu Q} := (F_{\mu\nu})^P{}_Q$, $F^{\mu\nu P}_Q := (F^{\mu\nu})^P{}_Q$ and $F^{\mu P}_{\nu Q} := (F_\nu{}^\mu)^P{}_Q$. Analogous definitions will apply to F_{ij}.

[46]For example, one can rewrite expressions of the type $G_{K(I}\widehat{\nabla}^{(n)}(F_{\mu\nu})^K_{J)}$ solely in terms of $G_{KI}\widehat{\nabla}^{(m)}(F_{\mu\nu})^K_J$ with $m \leq n-2$, and terms that vanish when $\nabla_i G_{IJ} = 0$. In particular, for $n = 0$ we have that $F_{\mu\nu(IJ)} = 0$ when $\nabla_i G_{IJ} = 0$, with $F_{\mu\nu IJ} := G_{KI}(F_{\mu\nu})^K_J$.

terms:[47]

$$
\begin{aligned}
\mathcal{A}^F_{\text{flat}} = G_{IJ}\Big[& -\frac{19}{6}F^I_{\nu\rho L}F^{\nu\rho J}_K\lambda^K\widehat{\Box}\lambda^L + \frac{359}{144}F^I_{ijK}R\lambda^{i\mu}\lambda^J\lambda^K\widehat{\nabla}_\mu\widehat{\nabla}_\nu\lambda^{j\nu} - \frac{23}{3}F^{\nu\rho I}_K\lambda^K\nabla_\rho R_{\mu\nu}\widehat{\nabla}^\mu\lambda^J \\
& - \frac{35}{9}F^{\nu I}_{\mu K}\lambda^J\nabla_\rho R^\rho_\nu\widehat{\nabla}^\mu\lambda^K + \frac{49}{6}F^I_{\nu\rho L}F^{\nu\rho J}_K\widehat{\nabla}_\mu\lambda^L\widehat{\nabla}^\mu\lambda^K + \frac{151}{24}\lambda^{i\mu}\lambda^J\lambda^K\widehat{\nabla}_\mu\widehat{\nabla}^\nu\widehat{\nabla}_\rho\lambda^{j\rho}\widehat{\nabla}_\nu F^I_{ijK} \\
& - \frac{1}{3}F^I_{ijK}R\lambda^{i\mu}\lambda^K\widehat{\nabla}_\mu\lambda^J\widehat{\nabla}_\nu\lambda^{j\nu} + F^I_{ijK}R\lambda^{i\mu}\lambda^J\widehat{\nabla}_\mu\lambda^K\widehat{\nabla}_\nu\lambda^{j\nu} - \frac{359}{144}F^I_{ijK}R\lambda^{i\mu}\lambda^J\lambda^K\widehat{\Box}\lambda^j_\mu \\
& + \frac{89}{24}\lambda^K\widehat{\nabla}^\mu\lambda^J\widehat{\nabla}_\nu\widehat{\Box}F^{\nu I}_{\mu K} + \frac{95}{12}\lambda^J\widehat{\nabla}^\mu\lambda^K\widehat{\nabla}_\nu\widehat{\Box}F^{\nu I}_{\mu K} + \frac{215}{8}\lambda^{i\mu}\lambda^J\lambda^K\widehat{\nabla}_\mu\widehat{\nabla}_\rho F^I_{ijK}\widehat{\nabla}_\nu\widehat{\nabla}^\rho\lambda^{j\nu} \\
& + \frac{255}{4}F^I_{ijK}\lambda^J\lambda^K\widehat{\nabla}_\nu\widehat{\nabla}_\mu\widehat{\nabla}_\rho\lambda^{j\rho}\widehat{\nabla}^\nu\lambda^{i\mu} + \frac{395}{6}\lambda^J\lambda^K\widehat{\nabla}_\mu F^I_{ijK}\widehat{\nabla}_\nu\widehat{\nabla}_\rho\lambda^{j\rho}\widehat{\nabla}^\nu\lambda^{i\mu} \\
& + \frac{2519}{96}F^I_{ijK}\lambda^K\widehat{\nabla}_\mu\lambda^J\widehat{\nabla}_\nu\widehat{\nabla}_\rho\lambda^{j\rho}\widehat{\nabla}^\nu\lambda^{i\mu} + \frac{2509}{96}F^I_{ijK}\lambda^J\widehat{\nabla}_\mu\lambda^K\widehat{\nabla}_\nu\widehat{\nabla}_\rho\lambda^{j\rho}\widehat{\nabla}^\nu\lambda^{i\mu} \\
& - \frac{1139}{24}F^I_{ijK}\lambda^J\lambda^K\widehat{\nabla}_\nu\widehat{\Box}\lambda^j_\mu\widehat{\nabla}^\nu\lambda^{i\mu} + \frac{2}{3}F^I_{ijK}\lambda^{i\mu}\lambda^J\lambda^K\nabla_\rho R^\rho_\nu\widehat{\nabla}^\nu\lambda^j_\mu + \frac{35}{8}F^I_{ijK}\lambda^{i\mu}\lambda^K\widehat{\nabla}_\mu\widehat{\Box}\lambda^j_\nu\widehat{\nabla}^\nu\lambda^J \\
& + \frac{475}{8}\lambda^{i\mu}\lambda^K\widehat{\nabla}_\mu\widehat{\nabla}_\rho\lambda^{j\rho}\widehat{\nabla}_\nu F^I_{ijK}\widehat{\nabla}^\nu\lambda^J + \frac{2519}{96}F^I_{ijK}\lambda^{i\mu}\lambda^K\widehat{\nabla}_\nu\widehat{\nabla}_\mu\widehat{\nabla}_\rho\lambda^{j\rho}\widehat{\nabla}^\nu\lambda^J \\
& - \frac{105}{8}\lambda^{i\mu}\lambda^K\widehat{\nabla}_\mu F^I_{ijK}\widehat{\nabla}_\nu\widehat{\nabla}_\rho\lambda^{j\rho}\widehat{\nabla}^\nu\lambda^J - \frac{2039}{96}F^I_{ijK}\lambda^{i\mu}\lambda^K\widehat{\nabla}_\nu\widehat{\Box}\lambda^j_\mu\widehat{\nabla}^\nu\lambda^J \\
& + \frac{55}{12}F^I_{ijK}\lambda^{i\mu}\lambda^J\widehat{\nabla}_\mu\widehat{\Box}\lambda^j_\nu\widehat{\nabla}^\nu\lambda^K + \frac{2509}{96}F^I_{ijK}\lambda^{i\mu}\lambda^J\widehat{\nabla}_\nu\widehat{\nabla}_\mu\widehat{\nabla}_\rho\lambda^{j\rho}\widehat{\nabla}^\nu\lambda^K \\
& + \frac{9}{4}\lambda^{i\mu}\lambda^J\widehat{\nabla}_\mu F^I_{ijK}\widehat{\nabla}_\nu\widehat{\nabla}_\rho\lambda^{j\rho}\widehat{\nabla}^\nu\lambda^K - \frac{2029}{96}F^I_{ijK}\lambda^{i\mu}\lambda^J\widehat{\nabla}_\nu\widehat{\Box}\lambda^j_\mu\widehat{\nabla}^\nu\lambda^K + 11F^{\rho J}_{\mu L}F^I_{\nu\rho K}\widehat{\nabla}^\mu\lambda^K\widehat{\nabla}^\nu\lambda^L \\
& - F^{\rho J}_{\mu K}F^I_{\nu\rho L}\widehat{\nabla}^\mu\lambda^K\widehat{\nabla}^\nu\lambda^L + \frac{473}{8}\lambda^{i\mu}\lambda^J\lambda^K\widehat{\nabla}_\nu F^I_{ijK}\widehat{\nabla}^\nu\widehat{\nabla}_\mu\widehat{\nabla}_\rho\lambda^{j\rho} - \frac{2}{3}F^I_{\nu\rho K}\widehat{\nabla}_\mu\widehat{\nabla}^\rho\lambda^K\widehat{\nabla}^\nu\widehat{\nabla}^\mu\lambda^J \\
& + \frac{40}{3}F^{\rho J}_{\mu K}F^I_{\nu\rho L}\lambda^K\widehat{\nabla}^\nu\widehat{\nabla}^\mu\lambda^L + \frac{241}{24}\lambda^J\lambda^L\widehat{\nabla}_\mu\lambda^{i\mu}\widehat{\nabla}_\nu F^I_{ijK}\widehat{\nabla}^\nu\widehat{\nabla}_\rho\lambda^{j\rho} - \frac{89}{2}\lambda^{i\mu}\lambda^J\lambda^K\widehat{\nabla}_\nu F^I_{ijK}\widehat{\nabla}^\nu\widehat{\Box}\lambda^j_\mu \\
& + \frac{623}{24}\lambda^J\lambda^K\widehat{\nabla}_\nu\widehat{\nabla}^\rho\lambda^j_\mu\widehat{\nabla}^\nu\lambda^{i\mu}\widehat{\nabla}_\rho F^I_{ijK} + \frac{73}{8}\lambda^{i\mu}\lambda^K\widehat{\nabla}_\mu\widehat{\nabla}^\rho\lambda^j_\nu\widehat{\nabla}^\nu\lambda^J\widehat{\nabla}_\rho F^I_{ijK} \\
& + \frac{175}{12}\lambda^{i\mu}\lambda^K\widehat{\nabla}_\nu\widehat{\nabla}^\rho\lambda^j_\mu\widehat{\nabla}^\nu\lambda^J\widehat{\nabla}_\rho F^I_{ijK} + \frac{22}{3}\lambda^{i\mu}\lambda^J\widehat{\nabla}_\mu\widehat{\nabla}^\rho\lambda^j_\nu\widehat{\nabla}^\nu\lambda^K\widehat{\nabla}_\rho F^I_{ijK} \\
& + \frac{14}{3}R^{\nu\rho}\lambda^K\widehat{\nabla}^\mu\lambda^J\widehat{\nabla}_\rho F^I_{\mu\nu K} + \frac{83}{24}R^\nu_\mu\lambda^K\lambda^K\widehat{\nabla}^\mu\lambda^J\widehat{\nabla}_\rho F^{\rho I}_{\nu K} + 2F^I_{ijK}\lambda^{i\mu}\lambda^K\widehat{\nabla}_\mu\widehat{\Box}\lambda^J\widehat{\nabla}_\rho\lambda^{j\rho} \\
& + 2F^I_{ijK}\lambda^{i\mu}\widehat{\nabla}_\mu\lambda^K\widehat{\Box}\lambda^J\widehat{\nabla}_\rho\lambda^{j\rho} - \frac{85}{6}\lambda^K\widehat{\nabla}_\mu\lambda^J\widehat{\nabla}_\nu F^I_{ijK}\widehat{\nabla}^\nu\lambda^{i\mu}\widehat{\nabla}_\rho\lambda^{j\rho} \\
& + 2\lambda^J\widehat{\nabla}_\mu\lambda^K\widehat{\nabla}_\nu F^I_{ijK}\widehat{\nabla}^\nu\lambda^{i\mu}\widehat{\nabla}_\rho\lambda^{j\rho} + \frac{77}{24}F^I_{ijK}R_{\mu\nu}\lambda^{i\mu}\lambda^K\widehat{\nabla}^\nu\lambda^J\widehat{\nabla}_\rho\lambda^{j\rho} \\
& + 26\lambda^{i\mu}\lambda^K\widehat{\nabla}_\nu\widehat{\nabla}_\mu F^I_{ijK}\widehat{\nabla}^\nu\lambda^J\widehat{\nabla}_\rho\lambda^{j\rho} - 4F^I_{ijK}R_{\mu\nu}\lambda^{i\mu}\lambda^J\widehat{\nabla}^\nu\lambda^K\widehat{\nabla}_\rho\lambda^{j\rho} \\
& - 8F^I_{ijK}\lambda^{i\mu}\widehat{\nabla}_\mu\widehat{\nabla}_\nu\lambda^J\widehat{\nabla}^\nu\lambda^K\widehat{\nabla}_\rho\lambda^{j\rho} - \frac{247}{12}\lambda^{i\mu}\lambda^J\widehat{\nabla}_\nu\widehat{\nabla}_\mu F^I_{ijK}\widehat{\nabla}^\nu\lambda^K\widehat{\nabla}_\rho\lambda^{j\rho} \\
& - \frac{295}{12}\lambda^{i\mu}\lambda^J\lambda^K\widehat{\nabla}_\nu\widehat{\nabla}^\rho\lambda^{j\nu}\widehat{\nabla}_\rho\widehat{\nabla}_\mu F^I_{ijK} + \frac{215}{24}\lambda^{i\mu}\lambda^J\lambda^K\widehat{\nabla}_\nu\lambda^{j\nu}\widehat{\nabla}_\rho\widehat{\nabla}_\mu\widehat{\nabla}^\rho F^I_{ijK} \\
& + \frac{523}{24}\lambda^{i\mu}\lambda^J\lambda^K\widehat{\nabla}_\nu F^I_{ijK}\widehat{\nabla}_\rho\widehat{\nabla}_\mu\widehat{\nabla}^\rho\lambda^{j\nu} + \frac{151}{24}F^I_{ijK}\lambda^{i\mu}\lambda^J\lambda^K\widehat{\nabla}_\rho\widehat{\nabla}_\mu\widehat{\nabla}^\rho\widehat{\nabla}_\nu\lambda^{j\nu} \\
& + \frac{215}{24}\lambda^{i\mu}\lambda^J\lambda^K\widehat{\nabla}_\mu\widehat{\nabla}^\rho\lambda^{j\nu}\widehat{\nabla}_\rho\widehat{\nabla}_\nu F^I_{ijK} - \frac{337}{24}\lambda^{i\mu}\lambda^J\lambda^K\widehat{\nabla}^\nu\lambda^j_\mu\widehat{\nabla}_\rho\widehat{\nabla}_\nu\widehat{\nabla}^\rho F^I_{ijK}
\end{aligned} \tag{4.47}
$$

[47] Our Mathematica notebook with the full solution can be made available upon request.

$$
\begin{aligned}
&-\frac{11}{2}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\mu}F^{I}_{ijK}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}\widehat{\nabla}^{\rho}\lambda^{j\nu}-\frac{151}{24}F^{I}_{ijK}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\mu}\lambda^{i\mu}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}\widehat{\nabla}^{\rho}\lambda^{j\nu}\\
&-\frac{35}{8}F^{I}_{ijK}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}_{\mu}\lambda^{J}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}\widehat{\nabla}^{\rho}\lambda^{j\nu}-\frac{55}{12}F^{I}_{ijK}\lambda^{i\mu}\lambda^{J}\widehat{\nabla}_{\mu}\lambda^{K}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}\widehat{\nabla}^{\rho}\lambda^{j\nu}\\
&+\frac{215}{24}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}\widehat{\nabla}^{\rho}\widehat{\nabla}_{\mu}F^{\mu\nu I}_{K}-\frac{151}{24}F^{I}_{ijK}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}\widehat{\nabla}^{\rho}\widehat{\nabla}^{\nu}\lambda^{j}_{\mu}\\
&-\frac{215}{24}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\nabla}_{\rho}\widehat{\nabla}^{\nu}\widehat{\nabla}^{\rho}\lambda^{j}_{\mu}+\frac{215}{8}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\nu}\lambda^{j\nu}\widehat{\Box}F^{I}_{ijK}\\
&-\frac{47}{2}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}_{\mu}\lambda^{J}\widehat{\nabla}_{\nu}\lambda^{j\nu}\widehat{\Box}F^{I}_{ijK}+\frac{121}{12}\lambda^{i\mu}\lambda^{J}\widehat{\nabla}_{\mu}\lambda^{K}\widehat{\nabla}^{\nu}\lambda^{j\nu}\widehat{\Box}F^{I}_{ijK}\\
&-\frac{215}{8}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\Box}\lambda^{j}_{\mu}\widehat{\Box}F^{I}_{ijK}+\frac{1157}{48}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}_{\nu}\lambda^{J}\widehat{\nabla}^{\nu}\lambda^{j}_{\mu}\widehat{\Box}F^{I}_{ijK}\\
&-\frac{455}{48}\lambda^{i\mu}\lambda^{J}\widehat{\nabla}_{\nu}\lambda^{K}\widehat{\nabla}^{\nu}\lambda^{j}_{\mu}\widehat{\Box}F^{I}_{ijK}+\frac{35}{8}F^{I}_{ijK}\lambda^{K}\widehat{\nabla}^{\mu}\lambda^{J}\widehat{\nabla}_{\nu}\lambda^{i\nu}\widehat{\Box}\lambda^{j}_{\mu}\\
&+\frac{55}{12}F^{I}_{ijK}\lambda^{J}\widehat{\nabla}^{\mu}\lambda^{K}\widehat{\nabla}_{\nu}\lambda^{i\nu}\widehat{\Box}\lambda^{j}_{\mu}-\frac{3073}{48}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\nabla}^{\nu}\lambda^{J}\widehat{\Box}\lambda^{j}_{\mu}\\
&+\frac{431}{16}\lambda^{i\mu}\lambda^{J}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\nabla}^{\nu}\lambda^{K}\widehat{\Box}\lambda^{j}_{\mu}+\frac{215}{8}F^{I}_{ijK}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\mu}\widehat{\nabla}^{\nu}\lambda^{i\mu}\widehat{\Box}\lambda^{j}_{\nu}\\
&-\frac{123}{2}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\mu}F^{I}_{ijK}\widehat{\nabla}^{\nu}\lambda^{i\mu}\widehat{\Box}\lambda^{j}_{\nu}-\frac{3299}{96}F^{I}_{ijK}\lambda^{K}\widehat{\nabla}_{\mu}\lambda^{J}\widehat{\nabla}^{\nu}\lambda^{i\mu}\widehat{\Box}\lambda^{j}_{\nu}\\
&-\frac{3349}{96}F^{I}_{ijK}\lambda^{J}\widehat{\nabla}_{\mu}\lambda^{K}\widehat{\nabla}^{\nu}\lambda^{i\mu}\widehat{\Box}\lambda^{j}_{\nu}+\frac{163}{12}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}_{\mu}F^{I}_{ijK}\widehat{\nabla}^{\nu}\lambda^{J}\widehat{\Box}\lambda^{j}_{\nu}\\
&+\frac{161}{12}\lambda^{i\mu}\lambda^{J}\widehat{\nabla}_{\mu}F^{I}_{ijK}\widehat{\nabla}^{\nu}\lambda^{K}\widehat{\Box}\lambda^{j}_{\nu}-\frac{229}{6}F^{I}_{ijK}\lambda^{J}\lambda^{K}\widehat{\nabla}^{\nu}\widehat{\nabla}_{\mu}\lambda^{i\mu}\widehat{\Box}\lambda^{j}_{\nu}\\
&-\frac{10}{3}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\mu}\lambda^{i\mu}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\Box}\lambda^{j\nu}-\frac{161}{24}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}_{\mu}\lambda^{J}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\Box}\lambda^{j\nu}\\
&-\frac{59}{12}\lambda^{i\mu}\lambda^{J}\widehat{\nabla}_{\mu}\lambda^{K}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\Box}\lambda^{j\nu}+\frac{65}{24}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\nu}\lambda^{j\nu}\widehat{\Box}\widehat{\nabla}_{\mu}F^{I}_{ijK}\\
&+\frac{307}{12}F^{I}_{ijK}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\Box}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\nu}\lambda^{j\nu}+\frac{239}{12}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}^{\nu}\lambda^{j}_{\mu}\widehat{\Box}\widehat{\nabla}_{\nu}F^{I}_{ijK}\\
&-\frac{151}{24}F^{I}_{ijK}\lambda^{J}\lambda^{K}\widehat{\nabla}^{\nu}\lambda^{i\mu}\widehat{\Box}\widehat{\nabla}_{\nu}\lambda^{j}_{\mu}+\frac{57}{8}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\mu}F^{I}_{ijK}\widehat{\Box}\widehat{\nabla}_{\nu}\lambda^{j\nu}\\
&+\frac{151}{24}F^{I}_{ijK}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\mu}\lambda^{i\mu}\widehat{\Box}\widehat{\nabla}_{\nu}\lambda^{j\nu}+\frac{215}{12}\lambda^{J}\lambda^{K}\widehat{\Box}\widehat{\nabla}_{\nu}\widehat{\nabla}_{\mu}F^{\mu\nu I}_{K}\\
&-\frac{247}{12}F^{I}_{ijK}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\Box}\widehat{\Box}\lambda^{j}_{\mu}-\frac{22}{3}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\Box}\widehat{\nabla}^{\nu}\lambda^{j}_{\mu}\\
&+6F^{I}_{\mu\nu K}\widehat{\nabla}^{\mu}\lambda^{K}\widehat{\Box}\widehat{\nabla}^{\nu}\lambda^{J}+\frac{35}{4}F^{I}_{ijK}\lambda^{K}\widehat{\nabla}^{\mu}\lambda^{J}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}\lambda^{j}_{\mu}\widehat{\nabla}^{\rho}\lambda^{i\nu}\\
&+\frac{55}{6}F^{I}_{ijK}\lambda^{J}\widehat{\nabla}^{\mu}\lambda^{K}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}\lambda^{j}_{\mu}\widehat{\nabla}^{\rho}\lambda^{i\nu}+\frac{215}{24}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\nu}\widehat{\nabla}_{\rho}F^{I}_{ijK}\widehat{\nabla}^{\nu}\lambda^{i\mu}\widehat{\nabla}^{\rho}\lambda^{j}_{\mu}\\
&+\frac{19}{24}F^{I}_{ijK}R_{\nu\rho}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}^{\nu}\lambda^{J}\widehat{\nabla}^{\rho}\lambda^{j}_{\mu}-\frac{215}{24}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}_{\nu}\widehat{\nabla}_{\rho}F^{I}_{ijK}\widehat{\nabla}^{\nu}\lambda^{J}\widehat{\nabla}^{\rho}\lambda^{j}_{\mu}\\
&+\frac{449}{12}\lambda^{i\mu}\lambda^{J}\widehat{\nabla}_{\nu}\widehat{\nabla}_{\rho}F^{I}_{ijK}\widehat{\nabla}^{\nu}\lambda^{K}\widehat{\nabla}^{\rho}\lambda^{j}_{\mu}-\frac{471}{8}\lambda^{J}\lambda^{K}\widehat{\nabla}^{\nu}\lambda^{i\mu}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j}_{\mu}\\
&-\frac{875}{48}\lambda^{K}\widehat{\nabla}_{\mu}\lambda^{J}\widehat{\nabla}^{\nu}\lambda^{i\mu}\widehat{\nabla}_{\rho}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j}_{\nu}-\frac{809}{48}\lambda^{J}\widehat{\nabla}_{\mu}\lambda^{K}\widehat{\nabla}^{\nu}\lambda^{i\mu}\widehat{\nabla}_{\rho}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j}_{\nu}\\
&-2F^{I}_{ijK}R_{\nu\rho}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}_{\mu}\lambda^{J}\widehat{\nabla}^{\rho}\lambda^{j\nu}+\frac{215}{24}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j\nu}\\
&+2F^{I}_{ijK}R_{\mu\rho}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}_{\nu}\lambda^{J}\widehat{\nabla}^{\rho}\lambda^{j\nu}+\frac{391}{48}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\rho}F^{I}_{ijK}\widehat{\nabla}_{\nu}\lambda^{J}\widehat{\nabla}^{\rho}\lambda^{j\nu}
\end{aligned}
\tag{4.48}
$$

$$
\begin{aligned}
&+\frac{137}{16}\lambda^{i\mu}\lambda^{I}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\rho}F^{J}_{ijK}\widehat{\nabla}_{\nu}\lambda^{K}\widehat{\nabla}^{\rho}\lambda^{j\nu}+\frac{655}{48}\lambda^{K}\widehat{\nabla}_{\mu}\lambda^{J}\widehat{\nabla}_{\nu}\lambda^{i\mu}\widehat{\nabla}_{\rho}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j\nu}\\
&-\frac{65}{48}\lambda^{J}\widehat{\nabla}_{\mu}\lambda^{K}\widehat{\nabla}_{\nu}\lambda^{i\mu}\widehat{\nabla}_{\rho}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j\nu}-\frac{397}{48}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}_{\nu}\lambda^{J}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\mu}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j\nu}\\
&-\frac{377}{48}\lambda^{i\mu}\lambda^{J}\widehat{\nabla}_{\nu}\lambda^{K}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\mu}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j\nu}+\frac{153}{16}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j\nu}\\
&-\frac{17}{3}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\mu}\lambda^{i\mu}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j\nu}-\frac{29}{24}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}_{\mu}\lambda^{J}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j\nu}\\
&-\frac{29}{24}\lambda^{i\mu}\lambda^{J}\widehat{\nabla}_{\mu}\lambda^{K}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j\nu}-\frac{311}{16}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}\widehat{\nabla}_{\mu}F^{I}_{ijK}\widehat{\nabla}^{\rho}\lambda^{j\nu}\\
&-\frac{215}{24}\lambda^{J}\lambda^{K}\widehat{\nabla}^{\nu}\lambda^{i\mu}\widehat{\nabla}_{\rho}F^{I}_{ijK}\widehat{\nabla}^{\rho}\widehat{\nabla}_{\nu}\lambda^{j}_{\mu}+\frac{31}{8}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\mu}\widehat{\nabla}_{\rho}F^{I}_{ijK}\widehat{\nabla}^{\rho}\widehat{\nabla}_{\nu}\lambda^{j\nu}\\
&+\frac{83}{12}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\mu}F^{I}_{ijK}\widehat{\nabla}^{\rho}\widehat{\nabla}_{\nu}\lambda^{j\nu}-2F^{I}_{\mu\rho K}\widehat{\nabla}^{\mu}\lambda^{K}\widehat{\nabla}^{\rho}\widehat{\Box}\lambda^{J}\\
&+\frac{151}{24}F^{I}_{ijK}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\nu}\widehat{\nabla}_{\rho}\lambda^{j}_{\mu}\widehat{\nabla}^{\rho}\widehat{\nabla}^{\nu}\lambda^{i\mu}-\frac{215}{24}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\nu}\widehat{\nabla}_{\rho}F^{I}_{ijK}\widehat{\nabla}^{\rho}\widehat{\nabla}^{\nu}\lambda^{j}_{\mu}\\
&+\frac{497}{24}\lambda^{i\mu}\lambda^{J}\lambda^{K}\widehat{\nabla}_{\rho}\widehat{\nabla}_{\nu}F^{I}_{ijK}\widehat{\nabla}^{\rho}\widehat{\nabla}^{\nu}\lambda^{j}_{\mu}+12F^{I}_{\nu\rho K}\widehat{\nabla}^{\mu}\lambda^{K}\widehat{\nabla}^{\rho}\widehat{\nabla}^{\nu}\widehat{\nabla}_{\mu}\lambda^{J}\\
&-\frac{71}{16}R_{\mu\sigma\nu\rho}\lambda^{K}\widehat{\nabla}^{\mu}\lambda^{J}\widehat{\nabla}^{\sigma}F^{\nu\rho I}_{K}-\frac{41}{8}F^{I}_{ijK}R_{\mu\rho\sigma\nu}\lambda^{i\mu}\lambda^{K}\widehat{\nabla}^{\nu}\lambda^{J}\widehat{\nabla}^{\sigma}\lambda^{j\rho}\\
&\left.-\frac{22}{3}F^{I}_{ijK}R_{\mu\rho\nu\sigma}\lambda^{i\mu}\lambda^{J}\widehat{\nabla}^{\nu}\lambda^{K}\widehat{\nabla}^{\sigma}\lambda^{j\rho}\right]\ .
\end{aligned}
\tag{4.49}
$$

In particular, one can prove using Mathematica that it is impossible to cancel out the contributions in (4.46) using new terms that contain exclusively the combination $F_{\mu\nu}$.

The flat-spacetime limit is sufficient for the purposes of this chapter. Nevertheless, it is interesting to ask how (4.49) would be modified in the case of arbitrary spacetime curvature. In such a case, one should also add to the ansatz terms that are at least quadratic in the Riemann tensor and linear in the vector-bundle curvature. Dimensional analysis suggests that the only possibility is of the type $G_{RP}F^{R}_{\mu\nu\ Q}R_{abcd}R_{efgh}\lambda^P\lambda^Q$, however by taking into account all possible spacetime contractions all such terms vanish. Consequently, our original ansatz should be sufficient towards determining the anomaly functional for any curved (spacetime and vector-bundle) background. This is a well-defined but computationally challenging problem, to which we hope to return in the future.

4.2.6 $\Delta = 2$ Operators

We have left the case of $\Delta = 2$ operators for last as the associated anomalous functional is trivial. Indeed the anomaly is encoded in

$$
\delta_\sigma W \propto \int d^4x\sqrt{\gamma}\,\delta\sigma\, G_{IJ}\lambda^I\lambda^J\ , \tag{4.50}
$$

which automatically satisfies the WZ consistency condition. Moreover, the above does not involve the connection $(A_i)^I_J$ and, therefore, one cannot infer anything about $\nabla_i G_{IJ}$ from this expression. In the next section, we will analyze the case of $\Delta = 2$ type-B conformal anomalies in the context of 4D $\mathcal{N} = 2$ SCFTs, where supersymmetry will allow us to say more.

4.3 $\Delta = 2$ CBOs in 4D $\mathcal{N} = 2$ SCFTs

In this section we will focus on CBOs $\mathcal{O}_I$ (and their complex conjugates $\bar{\mathcal{O}}_I$) with scaling dimension $\Delta = 2$ in 4D $\mathcal{N} = 2$ SCFTs. We will argue using Poincaré supersymmetry that the $\Delta = 2$ type-B Weyl anomalies are the same as the type-B Weyl anomalies of the exactly marginal $\Delta = 4$ operators. This is obvious in the conformally symmetric phase (see e.g. [34]), but requires a less straightforward argument in phases with spontaneously broken conformal symmetry. We will outline the argument in Sec 4.3.1 and provide tree-level supporting evidence for its validity in Sec 4.3.2. Once the relation with the exactly marginal Weyl anomalies is established, the result $\nabla G = 0$ for $\Delta = 2$ anomalies follows from Eq. (4.10).

4.3.1 Anomalies Related by Poincaré Supersymmetry

The exactly marginal operators of the $\mathcal{N} = 2$ SCFT are of the form

$$\Phi_i \propto Q^4 \cdot \mathcal{O}_I \delta_i^I \ , \quad \bar{\Phi}_i \propto \bar{Q}^4 \cdot \bar{\mathcal{O}}_I \delta_i^I \ , \tag{4.51}$$

where $Q^4\cdot$ stands for four nested commutators, where the $\mathcal{I}$ and α indices of the supercharges $\mathcal{Q}^{\mathcal{I}}_\alpha$ are properly contracted, see Sec. 5.1 of [34] for more details.
In the conformal phase it is straightforward to relate the anomaly of the $\Delta = 2$ operators $\mathcal{O}_I$ to the anomaly of the exactly marginal operators Φ_i by looking at the corresponding 2-point functions (4.3). The Ward identity for Poincaré supercharges

$$\sum_{k=1}^{n} \langle \varphi_1(x_1) \dots Q \cdot \varphi_k(x_k) \dots \varphi_n(x_n) \rangle = 0 \ , \tag{4.52}$$

can be used to move the supercharges around so as to arrive at [164]

$$\Box^2_{x_2} \left\langle \mathcal{O}_I(x_1) \bar{\mathcal{O}}_J(x_2) \right\rangle \propto \left\langle \Phi_i(x_1) \bar{\Phi}_j(x_2) \right\rangle \delta^i_I \delta^j_J \ , \tag{4.53}$$

where the constant of proportionality depends on conventions and will be fixed momentarily.

In a general phase, the type-B anomaly of interest is captured by a particular contact term in the 3-point function (4.4)

$$\langle T(x)\mathcal{O}_I(x_1)\bar{\mathcal{O}}_J(x_2)\rangle \ , \tag{4.54}$$

where $T \equiv T^\mu_\mu$ is the trace of the energy-momentum tensor. The energy-momentum tensor of the $\mathcal{N}=2$ SCFT belongs to a superconformal multiplet with a scalar superconformal primary $\mathcal{T}$ that obeys the shortening conditions $(Q^{\mathcal{I}})^2 \cdot \mathcal{T} = 0$, $(\bar{Q}_{\mathcal{I}})^2 \cdot \mathcal{T} = 0$ (for $\mathcal{I} = 1,2$ the $SU(2)_R$ R-symmetry index), and is of the form (suppressing spacetime indices, spinor indices and sigma-matrices on the RHS)

$$T_{\mu\nu} = Q^1 \cdot Q^2 \cdot \bar{Q}_1 \cdot \bar{Q}_2 \cdot \mathcal{T} + c_1 Q^1 \cdot \bar{Q}_1 \cdot \partial\mathcal{T} + c_2 Q^2 \cdot \bar{Q}_2 \cdot \partial\mathcal{T} + c_3 \partial^2 \mathcal{T} \ . \tag{4.55}$$

In phases with spontaneously broken conformal symmetry it is less straightforward to relate (4.54) to $\langle T(x)\Phi_i(x_1)\bar{\Phi}_j(x_2)\rangle$ by applying Ward identities. In vacua, where Poincaré supersymmetry is unbroken, as e.g. on the Coulomb or Higgs branch of $\mathcal{N}=2$ SCFTs, one can still use the integrated form of the Ward identities (4.52), but their application on 3-point functions of the form (4.54) is complicated. However, since we only care about a contact term in the limit of vanishing momentum for the energy-momentum tensor, it may be natural to anticipate that terms in $T_{\mu\nu}$ with explicit spacetime derivatives (like the c_1, c_2, c_3 terms in (4.55)) will not contribute to the anomaly. Assuming such terms can be dropped, in the 3-point function

$$\langle T(x)(Q^4 \cdot \mathcal{O}_I)(x_1)(\bar{Q}^4 \cdot \bar{\mathcal{O}}_J)(x_2)\rangle \tag{4.56}$$

with two exactly marginal operators, only

$$\langle (Q^1 \cdot Q^2 \cdot \bar{Q}_1 \cdot \bar{Q}_2 \cdot \mathcal{T})(x)(Q^4 \cdot \mathcal{O}_I)(x_1)(\bar{Q}^4 \cdot \bar{\mathcal{O}}_J)(x_2)\rangle \tag{4.57}$$

contributes to the type-B anomaly. Then, as one implements the supersymmetric Ward identity (4.52) and starts moving the supercharges Q around from the x_1-insertion in (4.56), there are terms where the Qs land on the x_2 insertion and terms where the Qs land on the x-insertion. Up to x-derivatives the latter terms vanish. Assuming once again that we can ignore the x-derivatives, we drop all terms where some Qs were moved on the x-insertion of the energy-momentum tensor. This suggests that we can recast the anomalous term of (4.56) as the anomalous term of (4.54), up to a proportionality constant that coincides with the one in the unbroken phase (4.53), i.e.

$$\Box^2_{x_2} \left\langle T(x)\mathcal{O}_I(x_1)\bar{\mathcal{O}}_J(x_2)\right\rangle \propto \left\langle T(x)\Phi_i(x_1)\bar{\Phi}_j(x_2)\right\rangle \delta^i_I \delta^j_J \ . \tag{4.58}$$

To summarise, under the assumption that we can drop terms with x-derivatives, Poincaré supersymmetry guarantees that the anomalies of $\Phi_i \propto Q^4 \cdot \mathcal{O}_I \delta^I_i$ and $\mathcal{O}_I$ are proportional to each other in all phases through a constant of proportionality which is independent of the exactly marginal couplings. Consequently, since G_{ij} is covariantly constant in both the unbroken and broken phases, the same must be true for the G_{IJ} anomaly of the $\Delta = 2$ operators. Notice that the holomorphic part of the tangent bundle (which houses the holomorphic part of the exactly marginal deformations) is a product $\mathcal{L}^4 \otimes \mathcal{V}_2$ of four copies of the bundle of the left-moving supercharges $\mathcal{L}$ and the bundle of $\Delta = 2$ chiral primary operators $\mathcal{V}_2$.[48] Accordingly, the connection on the tangent bundle is a direct sum of the connection on $\mathcal{L}^4$ and $\mathcal{V}_2$, [164,171,179]. However, on the anomalies G_{ij} and G_{IJ} only the part of the connection on $\mathcal{V}_2$ contributes.

4.3.2 Perturbative Checks

As further evidence for the validity of the relation (4.58) in phases with spontaneously broken conformal symmetry, we present an explicit test at leading order in perturbation theory on the Higgs branch of the 4D $\mathcal{N} = 2$ superconformal QCD (SCQCD) theory. We compute at tree-level the anomalies for $\Delta = 2$ CBOs in the CFT and Higgs-branch phases ($G_2^{(\mathrm{CFT})}, G_2^{(\mathrm{Higgs})}$) and relate them to the anomalies of exactly marginal operators ($G_4^{(\mathrm{CFT})}, G_4^{(\mathrm{Higgs})}$) via the series of equalities

$$G_2^{(\mathrm{Higgs})} = G_2^{(\mathrm{CFT})} = \frac{1}{192}\, G_4^{(\mathrm{CFT})} = G_2^{(\mathrm{CFT})} \ . \tag{4.59}$$

The relation $G_2^{(\mathrm{Higgs})} = G_2^{(\mathrm{CFT})}$ is a special case of (4.58).

In 4D SCQCD there is a single $\Delta = 2$ CBO $\mathcal{O}$ and a single exactly marginal operator Φ. In terms of the elementary fields that appear in the SCQCD Lagrangian (see e.g. [34] for a more detailed discussion on notation and conventions)

$$\begin{aligned} \mathcal{O} &= \mathrm{Tr}\varphi^2 \ , \\ \Phi &= 2\,\mathrm{Tr}[\partial_\mu \varphi \partial^\mu \bar{\varphi} + i\lambda\sigma^\mu \partial_\mu \bar{\lambda} + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + O(g)] \, . \end{aligned} \tag{4.60}$$

These operators are related by supersymmetry as in (4.51). In our conventions, the normalisation of the superalgebra is

$$\{Q_\alpha^{\mathcal{I}}, \bar{Q}_{\mathcal{J}\dot{\alpha}}\} = 2\delta_{\mathcal{J}}^{\mathcal{I}} P_{\alpha\dot{\alpha}} \, , \tag{4.61}$$

with $\alpha, \dot{\alpha}$ the 4D Lorentz spinor indices. We will perform a perturbative computation in SCQCD with arbitrary color group G_C.

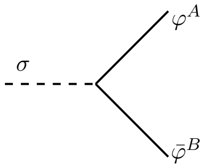
The broken-phase computations that we will present are performed in the Higgs-

[48] Analogous statements apply obviously to anti-holomorphic exactly marginal deformations, right-moving supercharges and $\Delta = 2$ anti-chiral superconformal primaries.

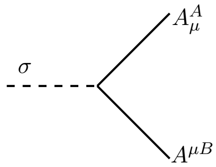
branch vacuum that was analysed in [34], where

$$\langle Q^a_{Ii} \rangle = v \delta_{I1} \delta^a_i \tag{4.62}$$

with $v \in \mathbb{R}$. For $v \neq 0$ the dilatation symmetry is spontaneously broken and a real massless dilaton σ appears in the spectrum. This couples linearly to the energy-momentum tensor of the unbroken phase. By expanding the Lagrangian of $\mathcal{N} = 2$ SCQCD around the vacuum (4.62), one can determine how the dilaton interacts with the elementary fields of the theory. In the following, we will be primarily interested in its couplings with the A_μ and φ fields, which acquire a mass $m = gv$; these are



$$= -i\frac{g^2 v}{k}\delta^{AB}$$



$$= -i\frac{g^2 v}{2k}\delta^{AB} \quad . \tag{4.63}$$

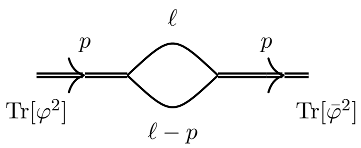
All computations will be performed directly in Euclidean space and the integrals will be evaluated using dimensional regularisation with (μ has the dimensions of a mass and $\epsilon > 0$)

$$\int \frac{d^d l}{(2\pi)^d} \mapsto \mu^{2\epsilon} \int \frac{d^{2(2-\epsilon)} l}{(2\pi)^{2(2-\epsilon)}} \quad . \tag{4.64}$$

$\Delta = 2$ Anomaly in Conformal Phase

The tree level 2-point function of $\mathcal{O}$ in the CFT phase is obtained via simple Wick contraction of the scalar fields φ (which can be carried out in two ways) [34]

$$\langle \mathcal{O}(p)\bar{\mathcal{O}}(-p)\rangle =$$



$$= 2\,\mathcal{C} \times I_1(p) \; . \tag{4.65}$$

Here $\mathcal{C}$ is the color factor

$$\mathcal{C} = \mathrm{Tr}[T^A T^B]\, \mathrm{Tr}[T_A T_B] \tag{4.66}$$

with $A, B = 1, \ldots, \mathrm{rank}(G_c)$, while the integral $I_1(p)$ is the kinematic factor

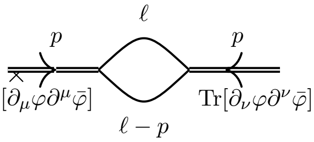
$$I_1(p) = \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{\ell^2} \frac{1}{(\ell - p)^2} = \frac{1}{(4\pi)^2}\left[\frac{1}{\epsilon} - \gamma + 3 - \log\left(\frac{p_1^2}{4\pi\mu^2}\right)\right] \quad . \tag{4.67}$$

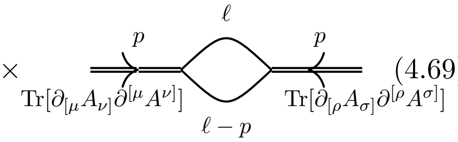
According to (4.3), one then reads off

$$G_2^{(\mathrm{CFT})} = 2\frac{\mathcal{C}}{(2\pi)^4} \quad . \tag{4.68}$$

$\Delta = 4$ Anomaly in Conformal Phase

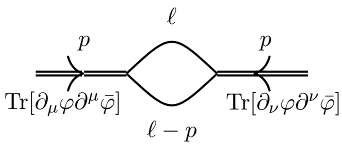
At tree level, the 2-point function of the exactly marginal operators receives only two contributions:

$\langle \Phi(p) \bar{\Phi}(-p) \rangle = 4 \times$ 

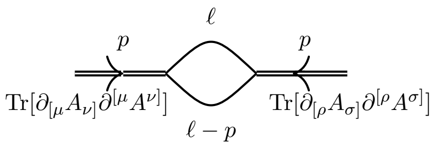
$+ \ 4 \times$ 

(4.69)

The two individual diagrams



$$= \mathcal{C} \times I_2(p) \tag{4.70}$$



$$= 2\,\mathcal{C} \times I_3(p) \tag{4.71}$$

are equally contributing Feynman processes, since the kinematic integrals $I_2(p)$, $I_3(p)$ are given by

$$I_2(p) = \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{\ell^2} \frac{1}{(\ell - p)^2} \times [\ell_\mu \ell^\nu (\ell - p)^\mu (\ell - p)_\nu] = \frac{p^4}{4} I_1(p) \tag{4.72}$$

$$I_3(p) = \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{l^2} \frac{1}{(\ell - p)^2} \times \ell_{[\mu} \ell^{[\rho} \delta^{\sigma]}_{\nu]} \times [(\ell - p)^{[\mu} (\ell - p)_{[\rho} \delta^{\nu]}_{\sigma]}] = \frac{p^4}{8} I_1(p)\,. \tag{4.73}$$

Applying (4.3) one extracts

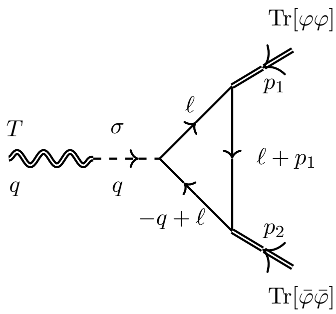
$$G_4^{(\mathrm{CFT})} = 192\, G_2^{(\mathrm{CFT})}\,. \tag{4.74}$$

The factor 192 is part of our conventions. This relation is an explicit tree-level check of the well-known general result (4.53) [164].

$\Delta = 2$ Anomaly in Higgs Phase

Following [34], we compute in the Higgs phase the 3-point function of $\mathcal{O}, \bar{\mathcal{O}}$ with the trace of the energy-momentum tensor $T = T^\mu{}_\mu$. At tree level, this 3-point function receives a contribution due to the dilaton field σ

$$\langle T(q)\mathcal{O}(p_1)\bar{\mathcal{O}}(p_2)\rangle \; = $$



$$= 4\,\mathcal{C} \times I_4(q,p_1,p_2)\,. \qquad (4.75)$$

The combinatorial factor originates from the four possible Wick contractions between the $\varphi\bar{\varphi}$ coming out of the dilaton vertex and the two operators $\mathrm{Tr}[\varphi\varphi]$, $\mathrm{Tr}[\bar{\varphi}\bar{\varphi}]$. The kinematic integral $I_4(q,p_1,p_2)$ is given by

$$I_4(q,p_1,p_2) = v^2 g^2 \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{\ell^2+m^2}\frac{1}{(p_1+\ell)^2+m^2}\frac{1}{(\ell-q)^2+m^2} \xrightarrow{q\to 0} \frac{1}{2}\frac{1}{(4\pi)^2}\,, \quad (4.76)$$

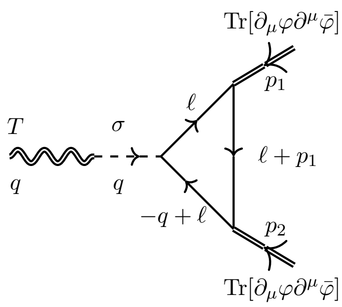
where the mass in the broken phase is proportional to the Higgs vev v, $m^2 = g^2v^2$. From (4.4) one can read off the anomaly in the Higgs phase, as already discussed in [34], which is

$$G_2^{(\mathrm{Higgs})} = G_2^{(\mathrm{CFT})}\,. \qquad (4.77)$$

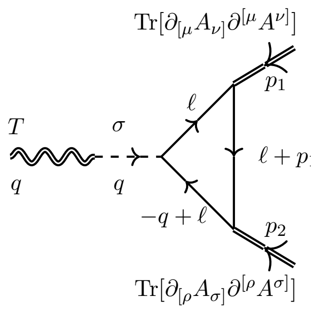
$\Delta = 4$ Anomaly in Higgs Phase

As in the conformal phase, in the Higgs phase the tree-level anomaly also arises from two equally contributing Feynman processes, with a φ and A^μ field running respectively inside the loops,

$$\langle T(q)\Phi(p_1)\bar{\Phi}(p_2)\rangle = 4\times$$

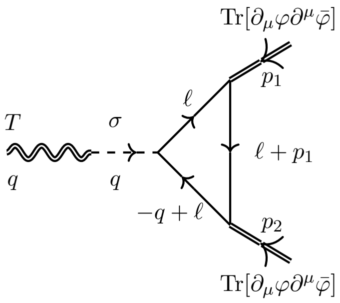


$$+\; 4\times$$

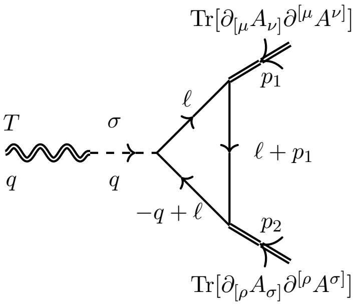


$$(4.78)$$

The two Feynman diagrams above evaluate to



$$= 2\mathcal{C} \times I_5(q, p_1, p_2)\,, \tag{4.79}$$



$$= 8\mathcal{C} \times I_6(q, p_1, p_2)\,, \tag{4.80}$$

with the kinematical integrals $I_5(q, p_1, p_2)$ and $I_6(q, p_1, p_2)$ given by

$$\begin{aligned} I_5(q, p_1, p_2) &= v^2 g^2 \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell_\mu}{\ell^2 + m^2} \frac{(p_1+\ell)^\mu (p_1+\ell)^\nu}{(p_1+\ell)^2 + m^2} \frac{(\ell-q)_\nu}{(\ell-q)^2+m^2} \\ &\xrightarrow{q\to 0} \frac{1}{(4\pi)^2}\left(\frac{1}{8}p_1^4 + \ldots\right)\,, \end{aligned} \tag{4.81}$$

$$\begin{aligned} I_6(q, p_1, p_2) &= \frac{1}{2} g^2 v^2 \int \frac{d^d\ell}{(2\pi)^d} \frac{\ell_{[\mu}\delta_{\nu]\alpha}}{\ell^2 + m^2} \frac{(p_1+\ell)^{[\mu}\delta^{\nu]}_{[\sigma}(p_1+\ell)_{\rho]}}{(p_1+\ell)^2 + m^2} \frac{(\ell-q)^{[\rho}\delta^{\sigma]\alpha}}{(\ell-q)^2+m^2} \\ &\xrightarrow{q\to 0} \frac{1}{(4\pi)^2}\left(\frac{1}{32}p_1^4 + \ldots\right)\,. \end{aligned} \tag{4.82}$$

Therefore, according to (4.4), the anomaly for the marginal operators in the Higgs phase is given by

$$G_4^{(\text{Higgs})} = G_4^{(\text{CFT})}\,. \tag{4.83}$$

Eqs. (4.77), (4.74), (4.83) establish the announced sequence of relations in (4.59).

4.4 Conclusions

In this chapter we investigated the properties of type-B Weyl anomalies of integer-dimension operators on conformal manifolds. We presented evidence that such anomalies are covariantly constant on conformal manifolds in general phases of the theory, where conformal symmetry may be spontaneously broken. By explicitly constructing the corresponding anomaly functionals for operators of dimension $\Delta = 3, 4, 5$, and without relying on supersymmetry, we showed that $\nabla G = 0$ is a condition that guarantees WZ consistency. The anomaly functional for $\Delta = 2$ operators was automatically WZ consistent, but we presented an independent argument in $\mathcal{N} = 2$ SCFTs using Poincaré supersymmetry that also implies $\nabla G = 0$. This argument was explicitly checked to leading order in perturbation theory. It would be useful to examine if there is a more general, supersymmetry-independent argument, that proves $\nabla G = 0$ for $\Delta = 2$ anomalies.

One of the interesting features of the WZ-consistency analysis is that it implies $\nabla G = 0$ in all phases of the theory, even when conformal symmetry is spontaneously broken. The implications of $\nabla G = 0$ in different phases are non-trivial as explained in [155] and reviewed in [151]. We expect the WZ-consistency argument to hold for arbitrary-dimension integer operators, as a consequence of using integration by parts, but, as we showed, the cases of increasing scaling dimension involve increasingly complicated anomaly functionals where the curvature of the bundle of the integer-dimension operators plays a crucial role.

It is important to investigate further the stability of our WZ consistent anomaly functionals under possible deformations, e.g. under turning on nontrivial beta functions for sources/couplings. For instance, one such deformation can arise by having nontrivial beta functions for the exactly marginal couplings.[49] New terms would then enter the computation of the WZ consistency condition through the anomaly functional for exactly marginal operators, (4.9). The simple option of having the standard beta function $\delta_\sigma \lambda^i \propto c^i_{JK} \lambda^J \lambda^K$, where c^i_{JK} is directly related to the 3-point function coefficient of $\langle \mathcal{O}_i \mathcal{O}_J \mathcal{O}_K \rangle$, is not realised in our case, because the operators we consider have (by construction) fixed integer scaling dimensions along the conformal manifold and therefore vanishing coefficients $c^i_{JK} = 0$. However, we cannot rule out the existence of more general beta functions for marginal couplings that may receive contributions from the curvature on the vector bundle of operators.

[49] Using dimensional analysis, one can determine that potential beta functions for the sources λ^I do not affect our construction of anomaly polynomials for $\Delta = 2, 3, 4, 5$ to quadratic order in the λ^I.

Chapter 5

Macdonald Indices for 4D $\mathcal{N}$=3 SCFTs

In this chapter, which is based on the work done in [3], we brute-force evaluate the vacuum character for $\mathcal{N} = 2$ vertex operator algebras labelled by crystallographic complex reflection groups $G(k, 1, 1) = \mathbb{Z}_k$, $k = 3, 4, 6$, and $G(3, 1, 2)$. For $\mathbb{Z}_{3,4}$ and $G(3, 1, 2)$ these vacuum characters have been conjectured (see [43]) to respectively reproduce the Macdonald limit of the superconformal index for rank one and rank two S-fold $\mathcal{N} = 3$ theories in four dimensions. For the $\mathbb{Z}_3$ case, and in the limit where the Macdonald index reduces to the Schur index, we find agreement with predictions from the literature [45].

5.1 Introduction

Before presenting our results, we will first briefly introduce 4D $\mathcal{N} = 3$ theories and the main features of the Schur-chiral 4D/2D correspondence which constitute the theoretical foundations underlying our brute-force computations.

5.1.1 4D $\mathcal{N} = 3$ theories and CCRGs

In recent years there has been intense activity pertaining to the study of superconformal theories (SCFTs) that do not admit a Lagrangian description. Theories with $\mathcal{N} \geq 2$ superconformal symmetry are ideal for such explorations. Despite the lack of perturbative control, one can still extract nontrivial data by exploiting the large amount of symmetry, e.g. by employing the power of dualities [19, 180], implementing the bootstrap programme [181, 182], or evaluating superconformal indices [38].
In this chapter we will apply some of the technology developed to study $\mathcal{N} = 2$ SCFTs to $\mathcal{N} = 3$ SCFTs. These theories are necessarily non-Lagrangian, as representation

theory alone (see [183] for example) is enough to prove that any lagrangian theory possessing $\mathcal{N} = 3$ supersymmetry actually enjoys $\mathcal{N} = 4$ supersymmetry. Also, any "pure" $\mathcal{N} = 3$ SCFTs—$\mathcal{N} = 3$ theories which do not automatically enhance to $\mathcal{N} = 4$—cannot have exactly marginal deformations preserving $\mathcal{N} = 3$ supersymmetry and they are necessarily isolated SCFTs.

Pure $\mathcal{N} = 3$ SCFTs were envisioned in [183,184] and their existence was actually proven only recently, through the S_k-fold constructions of [44], dated 2015, which was immediately generalized in [185] to accommodate different variants of S-folds, distinguished by an analog of discrete torsion. Schematically, S_k-folds can be thought of as generalizations of the F-theory lift of the more familiar type-IIB orientifold planes which includes a $\mathbb{Z}_k$ projection on both the R-symmetry directions as well as the $SL(2, \mathbb{Z})$ S-duality group of type IIB (the torus of F-theory). The theories engineered in [44] admit an F-theory dual on an $\mathrm{AdS}_5 \times (\mathrm{S}^5 \times \mathrm{T}^2)/\mathbb{Z}_k$ background and they are supposed to flow to well-known $\mathcal{N} = 6$ ABJM theories upon reduction on a circle. The gravity description was used in [186, 187] to evaluate the superconformal index in the large-rank limit.

Representation theory of $\mathcal{N} = 3$ superconformal algebra [183, 188, 189] guarantees that the central charges a, c of these theories must be equal $a = c$ and that $\mathcal{N} = 3$ SCFTs cannot have any flavour symmetries, except for its R-symmetry. These features are shared also by all $\mathcal{N} = 4$ theories, while they are not satisfied by generic $\mathcal{N} = 2$ theories. Thus, in a sense, $\mathcal{N} = 3$ SCFTs appear to be more similar to $\mathcal{N} = 4$ theories than to $\mathcal{N} = 2$ theories and one could think of cooking up $\mathcal{N} = 3$ SCFTs via "minor" modifications of $\mathcal{N} = 4$ SCFTs. Gauging a discrete symmetry is the perfect candidate for such a strategy, as it does not introduce any additional interaction and it rather acts simply as a superselection rule on the operator spectrum of the mother theory, without changing its local dynamics (hence it does not modify the values of the central charges either), see [190] and references therein.

Indeed, soon after [44], candidates for additional rank-one and rank-two $\mathcal{N} = 3$ examples were presented in [191, 192], via gaugings of $\mathcal{N} = 4$ theories by a discrete subgroup of the $SU(4)_R$ R-symmetry and $SL(2, \mathbb{Z})$ electromagnetic duality groups. The "Coulomb" limit of the superconformal index [41] and the Higgs-branch Hilbert series for these models were evaluated in [193, 194].

Finally, in [195] a third way to produce four-dimensional $\mathcal{N} = 3$ theories was put forward. Their construction exploits the features of M-theory on non-geometric backgrounds to first engineer (2,0) SCFTs of the exceptional type and to then quotient the latter along with a torus compactification. This strategy has been recently analyzed and further generalized in [196].

In this chapter we will examine some of the S-fold theories of [44, 185]. These are the worldvolume theories of stacks of n D3-branes probing the S_k-fold and, as such, their moduli spaces are given by (see [185])

$$(\mathbb{C}^3)^n/\Gamma_{n,k} \ , \tag{5.1}$$

where $\Gamma_{n,k}$ is a particular *crystallographic-complex-reflection-group (CCRG)*, determined by the values of k and n.

For CCRGs, we use the $G(K,P,N)$-notation of [43], a standard one in the mathematics literature [197], where K,P,N are natural non-zero numbers and P is a divisor of K. The CCRG $G(K,P,N)$ is a discrete group generated by the permutations S_N of the coordinates $\{z_1,...,z_N\}$ of $\mathbb{C}^N$ and the following transformations

$$(z_1,z_2,...,z_N) \mapsto (e^{2\pi a_1 i/K}z_1, e^{2\pi a_2 i/K}z_2..., e^{2\pi a_N i/K}z_N) \tag{5.2}$$

for all possible integers $\{a_1,a_2,...,a_N\}$ satisfying $\sum_{i=1}^N a_i = 0 \mod P$. The CCRG $G(K,P,N)$ can be realized as a discrete subgroup of $U(N)$, $G(K,P,N) \subset U(N)$, and it consists of $\frac{K^N}{P} \times N!$ elements (see last Appendix in [192]). The number N is called rank of the CCRG.

In the presence of an S_k-fold (which exists only for $k = 1,2,3,4,6$), the low-energy theory on n $D3$-branes is characterized by $\mathcal{N} = 3,4$ supersymmetry[50] and a moduli space given by $(\mathbb{C}^3)^n/G(k,p,n)$, where $p|k$ labels some possible variants of the S_k-fold that differ from each other because of some discrete fluxes ([185]) that can be turned on in the background[51].

When $G(k,p,n)$ coincides with a Weyl group[52], the corresponding S-fold construction gives a $\mathcal{N} = 4$ theory with gauge algebra being the one associated to that Weyl group (more details about the enhancement to $\mathcal{N} = 4$ can be found in [185] and [199]). For example, $G(1,1,n)$ is isomorphic to S_n which is the Weyl group of the A_{n-1} Lie algebra: when $k = 1$ the S-fold is trivial, we get a stack of n D3-branes on a flat background and the low energy theory is simply an $\mathcal{N} = 4$ $SU(n)$ gauge theory. Similarly, $G(2,1,n)$ is, depending on the value of n, isomorphic either to the Weyl group of B_n or C_n type Lie algebras, whereas $G(2,2,n)$ is the Weyl group of D_n Lie algebras: for $k = 2$ the S-fold indeed reduces to an orientifold which is well-known to lead to a $\mathcal{N} = 4$ theory.

We want to study pure $\mathcal{N} = 3$ theories and therefore we will consider CCRGs $G(k,p,n)$ with $k = 3,4,6$. In particular, we want to focus only on the so-called "genuine" [53] $\mathcal{N} = 3$ SCFTs, which are the $\mathcal{N} = 3$ theories that cannot be obtained also by discrete gauging

[50] In [198] the S-fold construction was adapted to build four-dimensional $\mathcal{N} = 2$ theories.

[51] Appreciate that the variant with no fluxes (which is the one with $p = k$) always exists, while there might be some restrictions on the possible variants with $p < k$. For example, the analysis of [185] ruled out the possibility of an S-fold leading to a 4D $\mathcal{N} = 3$ with CCRG $G(6,1,1) = \mathbb{Z}_6$.

[52] Weyl groups of familiar Lie algebras can be defined to be those particular CCRGs that are at the same time also Coxeter groups (also known as real reflection groups), [43].

[53] So dubbed in [200].

of another $\mathcal{N}=3$ or $\mathcal{N}=4$ theory; for this reason we restrict our attention to CCRGs $G(k,p,n)$ with $p=1$, see [185]. All in all, we will mainly be concerned with theories for which the $\Gamma_{n,k}$ appearing in (5.1) is the CCRG

$$\Gamma_{n,k} = G(k,1,n) \qquad k=3,4,6\ , \tag{5.3}$$

and which acts on the $\{z_1, z_2, ..., z_n\}$ coordinates of $(\mathbb{C}^3)^n$ as just explained, with the only difference that now each z_i is a triplet of complex numbers. To these genuine theories one can apply the Shapere-Tachikawa method for computing central charges [201] and get ([185]):

$$a_{4D} = c_{4D} = \frac{kn^2+(k-1)n}{4} \quad . \tag{5.4}$$

The size of $G(k,p,n)$ growths extremely fast as n increases and the computations that we will be performing turn out to be out of reach already for rank 3. So we will consider some of the simplest of these models, characterized by n being just 1 or 2; these are

$$G(k,1,1) = \mathbb{Z}_k \quad k=3,4 \tag{5.5}$$

$$G(3,1,2) \qquad . \tag{5.6}$$

We conclude by stressing that different $\mathcal{N}=3$ theories can share the same moduli space and we therefore cannot uniquely label them via the CCRGs associateed to their moduli space. For example, there are two rank-1 (pure) $\mathcal{N}=3$ theories that have $\mathbb{C}^3/\mathbb{Z}_3$ as their moduli spaces: the genuine $\mathcal{N}=3$ theory obtained by the $S_{k=3}$-fold with $n=p=1$ and the $\mathcal{N}=3$ theory constructed via discretely gauging the $\mathbb{Z}_3$ 0-form symmetry that the usual $\mathcal{N}=4$ multiplet with gauge group $U(1)$ exhibits when the gauge coupling is fine tuned to a particular value. Even if they share the same moduli space, they can be distinguished by the values of the central charges; the first theory has $a=c=5/4$ (just apply (5.4)), whereas the second one, being just the discretely gauged version of a $U(1)$ $\mathcal{N}=4$ multiplet, is characterized by $a=c=1/4$.
Nonetheless, we will often use the CCRG $G(k,p,n)$ to denote the particular $\mathcal{N}=3$ theory having $\mathbb{C}^3/G(k,p,n)$ as moduli space and that was constructed via the S-folding procedure. In our cases this should be unambiguous, as we are implicitly interested only in the genuine ones.

In the coming sections, we will discuss an algorithm (discovered in [43]) that to each CCRG $G(k,p,n)$ uniquely associates an observable which allegedly agrees with the Mac-Donald limit of the superconformal index of the four-dimensional $\mathcal{N}=3,4$ theories having moduli space $(\mathbb{C}^3)^n/G(k,p,n)$ and a particular value of the anomalies $a=c$. This conjecture is supposed to work only when the latter is known to actually

exist (via the S-fold construction, for example). To date, in fact, all the methods used to build four-dimensional $\mathcal{N} = 3, 4$ theories yield theories with moduli space $(\mathbb{C}^3)^n/G(k,p,n)$ for some $G(k,p,n)$ and many CCRGs still lack of an interpretation in terms of a four-dimensional theory. For instance, this is the case for $G(6,1,1) = \mathbb{Z}_6$, which we will study anyway just because it will be a natural model to consider after studying $\mathbb{Z}_3$ and $\mathbb{Z}_4$.
As the work of [43] exploits the Schur-chiral 4D-2D correspondence, we briefly review the latter in the next section.

5.1.2 The Schur-chiral 4D-2D correspondence

As $\mathcal{N} = 3$ theories are automatically $\mathcal{N} = 2$, a concrete computational handle can be established through the Schur-chiral 4D-2D correspondence of [42], which states that a certain protected (BPS) subsector of any $\mathcal{N} = 2$ 4D theory is isomorphic to a two-dimensional chiral algebra.

VOAs. Two-dimensional chiral algebras are the holomorphic/left-moving part of a two-dimensional CFT – a meromorphic CFT, which are more commonly referred to as Vertex Operator Algebra (VOA) in the mathematical literature. Through this map it is possible to use the incredible power of meromorphic conformal field theories in two dimensions (where the conformal algebra is enhanced to an infinite dimensional algebra) to learn about the physics of a 4D SCFT. Thanks to meromorphicity, one can compute correlation functions just by knowing its singularities, which in turn are governed by the singular terms appearing in the possible OPE limits between the operators inside the correlation function[54].
OPEs between normal-ordered products of operators can be obtained from the OPEs of the operators making up the normal-ordered product via Wick contractions [144]. Therefore, the singular OPEs between strong generators of the chiral algebra — which are those operators that cannot be written as normal-ordered products, with or without derivatives, of other operators — is all we need to know. When the list of strong generators is finite, we say that the chiral algebras is strongly finitely generated, or that it is a $\mathcal{W}$-algebra [203]. The algebras that we will deal in this chapter are of this kind[55]: once the finite set of the strong generators and (just) the singular OPEs between the latter are known, the chiral algebra is determined and we can study any of its n-point function.
Appreciate that the VOAs employed by the chiral-Schur correspondence is always non-unitary, as its central charge c_{2D} is related to the central charge c_{4D} of the four-dimensional theory via $c_{2D} = -12c_{4D} < 0$. This is not a problem, as the chiral algebra

[54] We refer to [144, 202] for a pedagogical introduction to the world of Conformal Field Theories.
[55] This is true for all the known examples, in fact.

should be thought of as an auxiliary tool that allows us to explore the physics of the four-dimensional theory, which is unitary.

Schur operators. The particular protected subsector of the spectrum that enters the four-dimensional side of the correspondence consists of (and solely of) the so-called "Schur" operators [41], which are all those operators that satisfy these constraints

$$j_1 + j_2 = E - 2R \tag{5.7}$$

$$j_1 - j_2 = -r \; . \tag{5.8}$$

Here, we used the same notations of [42] to denote the charges under the Cartan generators of the bosonic subalgebra of the $\mathcal{N} = 2$ superconformal algebra $SU(2,2|2)$ in 4D: E is the conformal dimension, R, r are the charges under the $SU(2)_R \times U(1)_r$ R-symmetry group and j_1, j_2 are the spin representations of the Lorentz group. We label Schur operators in terms of (E, R, r) and we implicitly consider the remaining Cartan charges (j_1, j_2) to be fixed by (5.8).

Schur operators always belong to shortened representations of the $\mathcal{N} = 2$ superconformal algebra[56] and, within these, Schur operators are always the highest weights of a Lorentz and a $SU(2)_R$ representations.

The operators satisfying (5.8) are called Schur operators because they are precisely the ones that contribute to the Schur limit of the Superconformal Index associated to a four-dimensional $\mathcal{N} = 2$ SCFT $\mathcal{T}$, defined through

$$\mathcal{I}^{\mathcal{T}}_{\text{Schur}}(q, z_i) := \text{Tr}(-1)^F q^{E-R} \prod_{i=1}^{L} z_i^{m_i} \; , \tag{5.9}$$

where F is the fermion number (which in four-dimension is given by $F = j_1 - j_2$), q, z_i are referred to as fugacities [41] and the trace is over the Hilbert space of the radially quantized theory. Here we assumed that $\mathcal{T}$ admits a flavor symmetry with L Cartan generators, so we introduced a fugacity z_i and a charge m_i for each of them.

We can consider a refinement of the Schur Index with an additional fugacity t that enables us to distinguish some Schur operators from each other, which - we stress it - always transform non trivially under $SU(2)_R$. This is achieved by the MacDonald Index

$$\mathcal{I}^{\mathcal{T}}_{\text{MacDonald}}(q, t, z_i) := \text{Tr}(-1)^F q^{E-2R-r} t^{R+r} \prod_{i=1}^{L} z_i^{m_i} \; , \tag{5.10}$$

[56] In the notations of [152], the superconformal multiplets that contain Schur operators are $\hat{\mathcal{B}}_R$, $\mathcal{D}_{R(0,j_2)}$, $\bar{\mathcal{D}}_{R(0,j_2)}$ and $\hat{\mathcal{C}}_{R(j_1,j_2)}$. All of these multiplets possess exactly one Schur operator.

which can be realized as a particular limit of the full superconformal index to which, once again, only Schur operators are allowed to contribute [41].

The correspondence. Let's call χ the map that implements the Schur-chiral 4D-2D corresponce, i.e. that non-trivial map that associates to each Schur operator $\mathcal{O}(0)$ inserted at the origin of the four-dimensional spacetime a chiral operator $\phi(z)$ living in a two-dimensional plane which passes through the origin (this is called the chiral plane). In the following, we will not need the rigorous recipe for χ; so we give a description of χ in simple words and we instead refer the reader to the original work [42] and to the pedagogical introduction [204] for more details.
In short, to build $\phi(z)$, one can first take $\mathcal{O}(0)$ and transport it to the position $(z, \bar{z})$ of the chiral plane via a translation that involves a particular $SU(2)_R$ twist as well; in this way one obtains a field $\phi(z, \bar{z})$ whose dependence on $\bar{z}$ can then be eliminated upon passing to the cohomology of a suitably chosen supercharge.
Given this construction, it is intuitively clear that to the states of the chiral algebra $\chi[\mathcal{T}]$ produced by χ one can associate the quantum numbers related to the global symmetry charges of the 4D theory and so one can consider the graded vacuum character $\mathcal{V}$ of the VOA, which is

$$\mathcal{V}_{\chi[\mathcal{T}]}(q, z_i) = \mathrm{Tr}(-1)^F q^h \prod_{i=1}^{L} z_i^{m_i} \ , \tag{5.11}$$

where h is the (holomorphic) conformal dimension h in 2D. By having a closer look at the construction of χ one can prove that (5.11) precisely agrees with the Schur index (5.9), [42].
A necessary condition for (5.11) to truly correspond to the Schur index is that (5.11) must be independent of any exactly marginal deformations of the four-dimensional SCFT. This is guaranteed by the rigid structure of the chiral algebra. Indeed, by using a non-renormalization theorem proven in [205], it is possible to show that $\chi[\mathcal{T}]$ is left untouched by any such modifications of $\mathcal{T}$.
As it does not depend on marginal deformations[57], $\chi[\mathcal{T}]$ can be easily determined when $\mathcal{T}$ admits a weakly coupled limit: in principle, one just needs to follow the prescription outlined above in the free limit of the four-dimensional theory. Instead, for isolated non-Lagrangian SCFTs, identifying the chiral algebra might be a hard task, as often one does not even know the spectrum of such theories in much detail.
For non-Lagrangian theories all hope is not lost as, remarkably, the Schur sector enjoys some universal properties which essentially fix a (minimal) set of the chiral algebra strong generators and their OPEs.

[57] $\mathcal{N} = 2$ supersymmetry guarantees that any marginal deformation is exactly marginal in 4D, [188].

- For instance, any local SCFT possesses the energy momentum tensor, which sits in the $\hat{\mathcal{C}}_{0,(0,0)}$ multiplet of the 4D $\mathcal{N}=2$ superconformal algebra. This multiplet contains also the supersymmetry and the R-symmetry currents. The heighest weight state of the $SU(2)_R$ current is the Schur operator which is contained in $\hat{\mathcal{C}}_{0,(0,0)}$ and this is the operator that χ maps to the chiral algebra stress tensor $T(z)$, [42].

- A strong generator is obtained when $\mathcal{T}$ has continuous flavor symmetries. The associated Noether current belongs to the $\hat{\mathcal{B}}_1$ multiplet whose superprimary is a Schur operator. Being in the same multiplet of the flavor current, this Schur operator M^A (known as moment map) carries an index A that transforms in the adjoint representation of the flavor symmetry G_F and so we expect to find dimG_F $J^A(z) := \chi[M^A]$ currents in the chiral algebra. A closer look shows that these currents enlarge the Virasoro algebra generated by $T(z)$ (and with central charge $c_{2D}=-12c_{4D}$) to an Affine Kac-Moody algebra, with level $\kappa_{2D}=-\frac{1}{2}\kappa_{4D}$, where κ_{4D} is the central charge associated with the four dimensional flavor symmetry.

- If the four-dimensional $\mathcal{N}=2$ theory $\mathcal{T}$ actually enjoys $\mathcal{N}=3$ or $\mathcal{N}=4$ supersymmetry, some of the extra supercharges survive in the chiral algebra $\chi[\mathcal{T}]$, which enhances respectively to a $\mathcal{N}=2$ or to a small $\mathcal{N}=4$ super Virasoro algebra, see [42]. This is very helpful, as it means that strong generators are grouped together into supersymmetric multiplets of the two-dimensional $\mathcal{N}=2$ or small $\mathcal{N}=4$ algebras.

Then, depending on the information that is available (which typically is retrieved by analyzing the superconformal index or some IR properties of the theory, such as its moduli space) one usually is able to make a more or less educated guess for possible additional generators that should appear in the chiral algebra; in a second step, one writes down the most general ansatz for the singular OPEs between these additional operators which are then fixed by imposing associativity of the OPEs and closure of the algebra. It is in this spirit that one is led to put forward a reasonable proposal for the chiral algebra associated to non-Lagrangian theories.

We conclude this section by stressing that it is completely legitimate to introduce in (5.11) the additional grading by $r+R$ as done in (5.10), simply because each state of the chiral algebra descends from a Schur operator. But as we explained above, the construction of χ employs a $SU(2)_R$ twist and this implies that, depending on the location z on the chiral plane, the operator $\phi(z)=\chi[\mathcal{O}(0)]$ is not built solely in terms of $\mathcal{O}(0)$ but generally also of all its $SU(2)_R$ descendants. This fact has two consequences. First of all it makes very hard to recover, from the chiral algebra perspective, the information about the quantum number R of the Schur operator and therefore there is no natural way, from the two-dimensional perspective, to enhance (5.11) to an observable that can agree with the MacDonald index defined in (5.10). Second, the R charge cannot survive in the chiral algebra as a true (i.e. conserved) quantum number but, if present at all, as a filtration; this means that the two dimensional analogue of R must be a number $\mathcal{R}$ such that: 1) the $\mathcal{R}$ number gets summed when fields get multiplied 2) terms with different values of $\mathcal{R}$ can be summed and the $\mathcal{R}$ of their linear superposition is equal to the maximum value of the $\mathcal{R}$ of the addends 3) the OPE between two operators with numbers $\mathcal{R}_1$ and $\mathcal{R}_2$ gives operators with $\mathcal{R} \leq \mathcal{R}_1 + \mathcal{R}_2$.

5.2 MacDonald Indices for 4D $\mathcal{N} = 3$ theories

VOAs for $\mathcal{N} = 3$ theories were initially constructed in [200, 206] and were then systematically studied in the work of [43].

In that reference, to each complex reflection group G an associated $\mathcal{W}$-algebra ($\mathcal{W}_{\mathsf{G}}$) was put forward. These $\mathcal{W}_{\mathsf{G}}$ algebras are extensions of the $\mathcal{N} = 2$ super Virasoro algebra obtained by introducing additional generators which are in correspondence with the invariants of G. It was conjectured that when $\mathsf{G} = G(k, p, n)$ is a non-Coxeter crystallographic complex reflection group, the associated $\mathcal{W}_{\mathsf{G}}$ encodes the Schur subsector of the known $\mathcal{N} = 3$ S-fold theories with moduli space given by (5.1).

By generalizing the approach of [207], [43] also gave a prescription for an elegant free-field realisation of the VOA associated to G, according to which $\mathcal{W}_{\mathsf{G}}$ is that particular subalgebra of rank(G) copies of the $\beta\gamma bc$ ghost system that is identified with the kernel of a screening operator, $\mathbb{S}_{\mathsf{G}} = \int \mathsf{J}_{\mathsf{G}}$, acting on the $\beta\gamma bc$ systems in a certain way[58]. Remarkably, in these free-field realizations, null states built out of strong generators are identically zero. Moreover, thanks to the free-field realization of $\mathcal{W}_{\mathsf{G}}$, one can naturally introduce an $\mathcal{R}$-filtration of the chiral algebra $\mathcal{W}_{\mathsf{G}}$ and thus recover the Macdonald limit of the superconformal index for $\mathcal{N} = 3$ S-fold theories from the $\mathcal{R}$-filtered version of the graded vacuum character (5.11), which (for $L = 1$) reads as

$$\chi_{\mathcal{W}_{\mathsf{G}}}(q, \xi, z) := \mathrm{Tr}(-1)^F q^h \xi^{\mathcal{R}+r} z^m \,, \tag{5.12}$$

where r, m are the quantum numbers associated with the $\mathsf{gl}(1)$ outer automorphism and $\mathsf{gl}(1)$ subalgebra of the 2D $\mathcal{N} = 2$ SCA respectively. In the free-field realisation the $\beta\gamma bc$ fields carry the quantum numbers presented in Tab. 5.1. Eq. (5.12) was conjectured to correspond to the Macdonald limit of the 4D superconformal index of a theory for which $\mathcal{W}_{\mathsf{G}}$ is the associated VOA, $\mathcal{W}_{\mathsf{G}} = \chi[\mathcal{T}]$; as we have seen above, for $L = 1$ this

[58] The action of $\mathbb{S}_{\mathsf{G}}$ is defined as $\mathbb{S}_{\mathsf{G}} \cdot X = \{\mathsf{J}_{\mathsf{G}} X\}_1$, where $\{\mathsf{J}_{\mathsf{G}} X\}_1$ denotes the coefficient of the order-one pole in the holomorphic OPE of the screening current J_{G} with some operator X.

	h	m	r	$\mathcal{R}$
β_ℓ	$\frac{1}{2}p_\ell$	$\frac{1}{2}p_\ell$	0	$\frac{1}{2}p_\ell$
b_ℓ	$\frac{1}{2}(p_\ell+1)$	$\frac{1}{2}(p_\ell-1)$	$+\frac{1}{2}$	$\frac{1}{2}p_\ell$
c_ℓ	$-\frac{1}{2}(p_\ell-1)$	$-\frac{1}{2}(p_\ell-1)$	$-\frac{1}{2}$	$1-\frac{1}{2}p_\ell$
γ_ℓ	$1-\frac{1}{2}p_\ell$	$-\frac{1}{2}p_\ell$	0	$1-\frac{1}{2}p_\ell$
∂	1	0	0	0

Table 5.1: Quantum numbers for the $\beta_\ell\gamma_\ell b_\ell c_\ell$ ghost systems used in VOA free-field realisations. The $\ell = 1, \ldots, \text{rank}(\mathsf{G})$ labels the ghost-system species and p_ℓ are the degrees of the fundamental invariants of G. These are given for $\mathbb{Z}_k$ by $p_1 = k$ and for $G(3,1,2)$ by $(p_1, p_2) = (3, 6)$.

reads as

$$\mathcal{I}^{\mathcal{T}}_{\text{Macdonald}}(q,t,z) := \text{Tr}(-1)^F q^{E-2R-r} t^{R+r} z^m \,. \tag{5.13}$$

Note that while from the point of view of this 4D $\mathcal{N}=2$ Macdonald index m is a quantum number for a global $U(1)_F$, in the full $\mathcal{N}=3$ description it is part of the $U(3) \supset SU(2)_R \times U(1)_r \times U(1)_F$ R-symmetry group. We should emphasise that in order to connect with (5.12) one needs to redefine $t \to \xi q$ so that

$$\mathcal{I}^{\mathcal{T}}_{\text{Macdonald}}(q,\xi,z) = \text{Tr}(-1)^F q^{E-R} \xi^{R+r} z^m \,. \tag{5.14}$$

Albeit concrete, implementing the findings of [43] in practice quickly becomes computationally intensive. It is difficult to write down the explicit free-field realisation of the relevant VOAs in all but the simplest of cases, and also to evaluate the corresponding vacuum characters in a fugacity expansion for increasing conformal weights. The goal of this chapter is to show how far one can get by implementing a brute-force approach using mathematical software, for the VOAs labelled by the complex reflection groups $G(k,1,1) = \mathbb{Z}_k$, $k = 3,4,6$ ($\mathbb{Z}_{3,4}$ label rank-one S-fold models) and $G(3,1,2)$ (labels a rank-two S-fold model).

Towards that end, we reconstruct the free-field realisations of [43] for the theories of interest. We then provide algorithms for automating the process of finding null states and for evaluating the VOA vacuum characters. Our code, appended to [3], can in principle be executed to obtain the corresponding Macdonald index at arbitrary orders in a fugacity expansion. Note however that the vacuum character computation time increases exponentially as a function of the conformal weight. Our code is also customisable—and we have clearly signposted how to do so—for the reader interested in extending it to the evaluation of vacuum characters for VOAs labelled by other complex reflection groups, once the complete free-field realisation of the VOA has been found.

Our results, all of which have been collected in the ancillary Mathematica notebook in [3] for quick reference, can be used to check the conjecture of [43] against independent calculations of the Macdonald index of 4D $\mathcal{N}=3$ S-fold theories and vice versa. For example, a proposal for the Schur limit of the superconformal index for the rank-one $\mathbb{Z}_3$ S-fold theory was put forward in [45]. In that limit, their and our findings are in complete agreement.

5.3 Implementation and Results

We now describe the strategy behind our code, while detailed results for each case are presented in subsequent subsections. We are interested in the evaluation of the vacuum character for VOAs labelled by crystallographic, non-Coxeter complex reflection groups $\mathbb{Z}_{3,4,6}$ and $G(3,1,2)$, and interpreting them as Macdonald indices for 4D $\mathcal{N}=3$ S-fold theories. To do so one needs to consider (5.12) and trace over all the states created by acting only with normal-ordered products and derivatives of the strong generators of the VOA on the $\mathsf{sl}(2)$-invariant vacuum, up to a given conformal weight, while removing the contributions from null states.

The identification of null states is usually laborious and this is where the free-field realisation becomes helpful. The construction algorithm of [43] starts with a simple free-field prescription for the generators of the 2D $\mathcal{N}=2$ SCA and the chiral strong generators of the $\mathcal{W}_{\mathsf{G}}$-algebra. One then introduces an ansatz for the remaining bosonic strong generators constructed out of all possible super-Virasoro primary, free-field combinations with the requisite quantum numbers and undetermined coefficients, and fixes the latter by closing the VOA under the OPE.

We have diverged slightly from this recipe in the following way. For the VOAs $\chi[\mathcal{T}]$ that have appeared thus far in the literature, the OPE coefficients can be completely fixed by writing down the most general expressions for the expected set of generators and then imposing associativity [200]. The $\mathcal{W}$-algebra presentations of the $\mathcal{W}_{\mathsf{G}}$ VOAs of interest to us were already given in [43] and inserting the free-field ansatze into the existing OPEs straightforwardly fixes the undetermined coefficients for all remaining strong generators (using the OPEdefs [208] and/or SOPEN2defs [209] Mathematica packages); we will expand upon this point on a case-by-case basis when discussing our results.

Equipped with this information we proceed to our main algorithm. In summary, all null states at a given conformal weight can be identified by looking for all possible combinations of states with the same quantum numbers that are identically zero upon using the free-field realisation. This last step requires manipulating normal-ordered products of $\beta\gamma bc$ ghosts, making heavy use of the OPEdefs [208] and ope.math [210] Mathematica packages. The null states that we find contain all those predicted in [43]. It is then straightforward to write down the vacuum character for given values of quantum numbers. Note that by unrefining in the $\mathcal{R}$ fugacity (e.g. if one were interested in the Schur index) the algorithm becomes faster; we have a dedicated section in our code for this special case.

In addition to the above method, we have cross-checked our refined $\mathbb{Z}_{3,4}$ results using a second algorithm that makes no connection to the $\mathcal{W}$-algebra presentation. This procedure constructs the VOA spectrum using all states in rank(G) copies of the $\beta\gamma bc$ ghost system that lie in the kernel of a screening operator, $\mathbb{S}_{\mathsf{G}}$. For $\mathsf{G} = \mathbb{Z}_k$ this is given by [43]

$$\mathsf{J}_{\mathsf{G}} = b\, e^{\left(k^{-1}-1\right)(\chi+\phi)}\ , \tag{5.15}$$

where χ, ϕ are chiral bosons

$$\beta = e^{\chi+\phi}, \quad \gamma = \partial\chi\, e^{-\chi-\phi}\ , \tag{5.16}$$

and all expressions should be considered as normal-ordered. Such an approach is conceptually more straightforward—construct all states using free fields and then keep those in the kernel of $\mathbb{S}_{\mathsf{G}}$—but is computationally more expensive as can be seen from Fig 5.1. E.g. at $h = 9/2$ one already needs to check 941 and 881 terms for $\mathbb{Z}_3$ and $\mathbb{Z}_4$ respectively.

5.3.1 Results: $\mathsf{G} = \mathbb{Z}_3$

This is a rank-one VOA with central charge $c = -15$. Its $\mathcal{W}$-algebra presentation involves the following strong generators: $\mathcal{T}$, $\mathcal{J}$, $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ from the 2D $\mathcal{N} = 2$ SCA, as well as the chiral and anti-chiral generators $\mathcal{W}_3$, $\overline{\mathcal{W}}_3$ and their superpartners $\mathcal{G}_{\mathcal{W}_3}$ and $\widetilde{\mathcal{G}}_{\overline{\mathcal{W}}_3}$. Here $\mathcal{G}_{\mathcal{W}_3} := \{\mathcal{G}\mathcal{W}_3\}_1$ and so on. For the explicit free-field realisation in terms of a single $\beta\gamma bc$ ghost system one starts with a prescription for the $\mathcal{N} = 2$ SCA generators as well as for $\mathcal{W}_3$, $\mathcal{G}_{\mathcal{W}_3}$. The ansatz for $\overline{\mathcal{W}}_3$ contains 8 undetermined coefficients. We always count these before imposing the super-Virasoro primary constraint. Through the OPEs from the $\mathcal{W}$-algebra presentation one can use it to also determine the free-field realisation of $\widetilde{\mathcal{G}}_{\overline{\mathcal{W}}_3}$. We have calculated the fully-refined vacuum character (5.12) up to $O(q^8)$ with the accompanying "null states.nb" and have cross-checked this result using the screening-operator approach in "screening.nb" up to $O(q^4)$. The full expression can be found in "vacuum_characters_summary.nb".

This VOA is expected to encode the Schur sector of a rank one, 4D S-fold $\mathcal{N} = 3$ SCFT, with a Coulomb-branch operator of dimension $\Delta = 3$ and trace-anomaly coefficient $c_{4\mathrm{D}} = \frac{5}{4}$. Through (5.14) the Macdonald index of this S-fold theory—including the global $U(1)_F$ fugacity—can be identified with the refined vacuum character. Below we only present the simpler, Schur limit of these expressions for brevity, where $\xi \to 1$, $z \to 1$. Then:

$$\begin{aligned}\mathcal{I}^{\mathbb{Z}_3}_{\mathrm{Schur}} &= 1 + q + q^2 + 2q^3 - 2q^{7/2} + 3q^4 - 2q^{9/2} + 4q^5 - 4q^{11/2} \\ &\quad + 6q^6 - 6q^{13/2} + 8q^7 - 8q^{15/2} + 11q^8 + O(q^{17/2}).\end{aligned} \tag{5.17}$$

It is interesting to observe that the Schur index for this theory matches the expansion of the following closed-form expression up to $O(q^{10})$, although we currently have neither a derivation for it nor a justification for why it should hold to all orders:

$$\frac{1}{3}\sum_{\epsilon\in\mathbb{Z}_3} \frac{\epsilon}{\sqrt{q}} \mathrm{P.E.}\left[\frac{1}{2} i_{\mathcal{N}=4}(q)\left(\epsilon + \frac{1}{\epsilon}\right)\right] . \tag{5.18}$$

Here $\mathrm{P.E.}[f(z)] := \exp\left[\sum_{n=1}^{+\infty} \frac{1}{n} f(z^n)\right]$ is the plethystic exponential, while $i_{\mathcal{N}=4}(q) = \frac{2q^{\frac{1}{2}}(1-q^{\frac{1}{2}})}{1-q}$ coincides with the single-letter Schur index of $\mathcal{N} = 4$ super-Yang–Mills.

In [45], an independent argument for determining the Schur index of the $\mathsf{G} = \mathbb{Z}_3$ S-fold theory was presented. This entailed starting from an $\mathcal{N} = 1$ 4D UV Lagrangian theory, and flowing to an interacting $\mathcal{N} = 1$ SCFT in the IR, which can also be reached from the $\mathbb{Z}_3$ S-fold theory via an $\mathcal{N} = 1$ preserving marginal deformation. Upon relabelling $q \to p^2$, our result (5.17) agrees with that of [45]—listed to $O(q^7)$—providing a strong consistency check of both calculations.

5.3.2 Results: $\mathsf{G} = \mathbb{Z}_4$

This is a rank-one VOA with central charge $c = -21$. Its $\mathcal{W}$-algebra presentation involves the following strong generators: $\mathcal{T}$, $\mathcal{J}$, $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ from the 2D $\mathcal{N} = 2$ SCA, as well as the chiral and anti-chiral generators $\mathcal{W}_4$, $\overline{\mathcal{W}}_4$ and their superpartners $\mathcal{G}_{\mathcal{W}_4}$ and $\widetilde{\mathcal{G}}_{\overline{\mathcal{W}}_4}$. For the explicit free-field realisation in terms of a single $\beta\gamma bc$ ghost system one starts with a prescription for the $\mathcal{N} = 2$ SCA generators as well as for $\mathcal{W}_4$, $\mathcal{G}_{\mathcal{W}_4}$. The ansatz for $\overline{\mathcal{W}}_4$ contains 19 undetermined coefficients. Through the OPEs in the $\mathcal{W}$-algebra presentation one can use it to also determine the free-field realisation of $\widetilde{\mathcal{G}}_{\overline{\mathcal{W}}_4}$. We have calculated the fully-refined vacuum character (5.12) up to $O(q^8)$ with the accompanying "null states.nb" and exhibited it in "vacuum_characters_summary.nb". We have also cross-checked this result using the screening-operator approach in "screening.nb" up to $O(q^4)$.

This VOA is expected to encode the Schur sector of a rank one 4D S-fold $\mathcal{N} = 3$ SCFT, with a Coulomb-branch operator of dimension $\Delta = 4$ and trace-anomaly coefficient $c_{4\mathrm{D}} = \frac{7}{4}$. Through (5.14) the Macdonald index of this S-fold theory can be identified with the vacuum character. The Schur limit of these expressions yields:

$$\begin{aligned}\mathcal{I}^{\mathbb{Z}_4}_{\text{Schur}} &= 1 + q - 2q^{3/2} + 5q^2 - 6q^{5/2} + 10q^3 - 16q^{7/2} + 27q^4 \\ &- 38q^{9/2} + 56q^5 - 86q^{11/2} + 129q^6 - 178q^{13/2} \\ &+ 251q^7 - 362q^{15/2} + 511q^8 + O(q^{17/2}) \,. \end{aligned} \tag{5.19}$$

In this case the $\mathcal{W}$-algebra construction is such that the bosonic states always appear with integer while the fermionic ones with half-integer conformal weights. Therefore there are no cancellations between bosonic and fermionic states at each level and the chiral algebra partition function reproduces the partition function of Schur operators in the corresponding 4D $\mathcal{N} = 3$ theory.

5.3.3 Results: $\mathsf{G} = \mathbb{Z}_6$

This is a rank-one VOA with central charge $c = -33$. Its $\mathcal{W}$-algebra presentation involves the following strong generators: $\mathcal{T}$, $\mathcal{J}$, $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ from the $\mathcal{N} = 2$ SCA, as well as the chiral and anti-chiral generators $\mathcal{W}_6$, $\overline{\mathcal{W}}_6$ and their superpartners $\mathcal{G}_{\mathcal{W}_6}$ and $\widetilde{\mathcal{G}}_{\overline{\mathcal{W}}_6}$. For the explicit free-field realisation in terms of a single $\beta\gamma bc$ ghost system one starts with a prescription for the 2D $\mathcal{N} = 2$ SCA generators as well as for $\mathcal{W}_6$, $\mathcal{G}_{\mathcal{W}_6}$. The ansatz for $\overline{\mathcal{W}}_6$ contains 87 undetermined coefficients. Through the OPEs in the $\mathcal{W}$-algebra presentation one can use it to also determine the free-field realisation of $\widetilde{\mathcal{G}}_{\overline{\mathcal{W}}_6}$. We have calculated the fully-refined vacuum character (5.12) up to $O(q^8)$ with the accompanying "nullstates.nb". The full result can be found in "vacuum_characters_summary.nb". In the limit $\xi \to 1$, $z \to 1$ this reads:

$$\begin{aligned}\chi_{\mathcal{W}_{\mathbb{Z}_6}} &= 1 + q - 2q^{3/2} + 3q^2 - 4q^{5/2} + 8q^3 - 12q^{7/2} + 19q^4 \\ &- 26q^{9/2} + 38q^5 - 58q^{11/2} + 85q^6 - 116q^{13/2} \\ &+ 165q^7 - 236q^{15/2} + 326q^8 + O(q^{17/2}) \,. \end{aligned} \tag{5.20}$$

No known S-fold theory is associated with this VOA [185].

5.3.4 Results: $\mathsf{G} = G(3, 1, 2)$

This is our only rank 2 example, with central charge $c = -48$. Its $\mathcal{W}$-algebra presentation involves the following strong generators: $\mathcal{T}$, $\mathcal{J}$, $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ from the 2D $\mathcal{N} = 2$ SCA, the chiral and anti-chiral generators $\mathcal{W}_3$, $\mathcal{W}_6$, $\overline{\mathcal{W}}_3$, $\overline{\mathcal{W}}_6$, plus their superpartners $\mathcal{G}_{\mathcal{W}_3}$, $\mathcal{G}_{\mathcal{W}_6}$ and $\widetilde{\mathcal{G}}_{\overline{\mathcal{W}}_3}$, $\widetilde{\mathcal{G}}_{\overline{\mathcal{W}}_6}$, as well as $\mathcal{U}$—which is self conjugate—and its superpartners $\mathcal{G}_{\mathcal{U}}$, $\widetilde{\mathcal{G}}_{\mathcal{U}}$ and $\mathcal{G}_{\widetilde{\mathcal{G}}_{\mathcal{U}}}$. One also needs non-chiral $\mathcal{O}$, $\overline{\mathcal{O}}$ and their superpartners $\mathcal{G}_{\mathcal{O}}$, $\widetilde{\mathcal{G}}_{\mathcal{O}}$, $\mathcal{G}_{\overline{\mathcal{O}}}$, $\widetilde{\mathcal{G}}_{\overline{\mathcal{O}}}$, $\mathcal{G}_{\widetilde{\mathcal{G}}_{\mathcal{O}}}$, $\mathcal{G}_{\widetilde{\mathcal{G}}_{\overline{\mathcal{O}}}}$.

The free-field realisation requires two ghost systems, $\beta_\ell\gamma_\ell b_\ell c_\ell$ with $\ell = 1, 2$. One starts with a prescription for the 2D $\mathcal{N} = 2$ SCA generators as well as for $\mathcal{W}_3$, $\mathcal{W}_6$, $\mathcal{G}_{\mathcal{W}_3}$ and $\mathcal{G}_{\mathcal{W}_6}$. The ansatz for $\overline{\mathcal{W}}_3$ contains 84 undetermined coefficients. It turns out that through the OPEs in the $\mathcal{W}$-algebra presentation, one can fix the coefficients of $\overline{\mathcal{W}}_3$ and by doing so also determine the free-field realisation of all remaining generators. We have calculated the fully-refined vacuum character (5.12) up to $O(q^{9/2})$ with the accompanying "null states.nb". The full result can be found in "vacuum_characters_summary.nb".

This VOA is expected to encode the Schur sector of a rank-two 4D S-fold $\mathcal{N} = 3$ SCFT, with Coulomb-branch operators of dimension $\Delta = 3, 6$ and trace anomaly coefficient $c_{4\text{D}} = 4$. Through (5.14) the Macdonald index of this S-fold theory can be identified with the vacuum character. If for simplicity one considers the limit $z \to 1$:

$$
\begin{aligned}
\mathcal{I}^{G(3,1,2)}_{\text{Macdonald}} &= 1 + q\xi + q^{3/2}\left(-\sqrt{\xi} + \xi^{3/2}\right) + q^2\left(\xi + \xi^2\right) \\
&+ q^{5/2}\left(-\sqrt{\xi} - \xi^{3/2} + 2\xi^{5/2}\right) + q^3\left(\xi - \xi^2 + 2\xi^3\right) \\
&+ q^{7/2}\left(-\sqrt{\xi} - 2\xi^{3/2} + 2\xi^{5/2} + \xi^{7/2}\right) \\
&+ q^4\left(2\xi - \xi^3 + 3\xi^4\right) \\
&+ q^{9/2}\left(-\sqrt{\xi} - 3\xi^{3/2} + \xi^{5/2} + \xi^{7/2} + 2\xi^{9/2}\right) \\
&+ O(q^5)\,.
\end{aligned} \tag{5.21}
$$

5.4 Conclusions

In this chapter we have calculated vacuum characters of rank-one and rank-two VOAs labelled by non-Coxeter, crystallographic complex reflection groups. This involved a brute-force implementation of the algorithms presented in [43] and leads to the Macdonald index of certain 4D $\mathcal{N} = 3$ S-fold SCFTs. Our results were given as an expansion in the fugacity that keeps track of the conformal weight, and were truncated to orders that require short computation times when using a desktop computer; they can be pushed to arbitrary higher orders by allocating appropriate resources. As they stand, they can already be used as new data for $\mathcal{N} = 3$ SCFTs. E.g. the $\mathsf{G} = \mathbb{Z}_3$ result agrees in the Schur limit with [45].

Our code is customisable. We have clearly signposted where changes would need to be made to return vacuum characters of VOAs labelled by different complex reflection groups, for which the free-field realisation is known. In particular, it would be very interesting to extend this approach to the $\mathcal{N} = 3$ S-fold SCFT of rank two associated with $G(4, 1, 2)$ and the rank-three example $G(3, 3, 3)$; a proposal for the Schur index of the latter was also given in [45]. Unfortunately, finding the free-field realisation for both these VOAs—already needed before identifying the null states—is a challenging task: the simplest anti-chiral strong generator ansatze involve 425 and 2265 undetermined coefficients respectively. It would perhaps be more promising to use the screening-operator approach, upon determining $\mathbb{S}_{\mathsf{G}}$. Although our screening-operator code is currently more expensive to run, it could benefit from optimisations that parallelise the computations, hence making it significantly faster on multi-core clusters. It will also be interesting to check these results by directly studying the BPS-states of $\mathcal{N} = 3$ theories. One way to do so would be to study three-string junctions in S-fold backgrounds as in [199].

Finally, it could be relevant to understand whether formula (5.18) actually captures the Schur index of the genuine $\mathcal{N} = 3$ $\mathbb{Z}_3$ S-fold theory at any orders in q. The fact that $\mathcal{I}^{\mathbb{Z}_4}_{\text{Schur}}$ and $\mathcal{I}^{\mathbb{Z}_6}_{\text{Schur}}$ do not seem to admit analogous compact formulae could be a hint that something interesting might affect the $\mathbb{Z}_3$ S-fold theory. What is curious about formula (5.18) is its similarity with the structure of the Schur index of the $\mathcal{N} = 3$ theory obtained by discretely gauging the $\mathbb{Z}_3$ 0-form symmetry that the usual $\mathcal{N} = 4$ multiplet with gauge group $U(1)$ exhibits when the gauge coupling is fine tuned to a particular value. In fact, the Schur index of the latter reads as [193]:

$$\frac{1}{3}\sum_{\epsilon\in\mathbb{Z}_3}\text{P.E.}\left[\frac{1}{2}i_{\mathcal{N}=4}(q)\left(\epsilon+\frac{1}{\epsilon}\right)\right] . \tag{5.22}$$

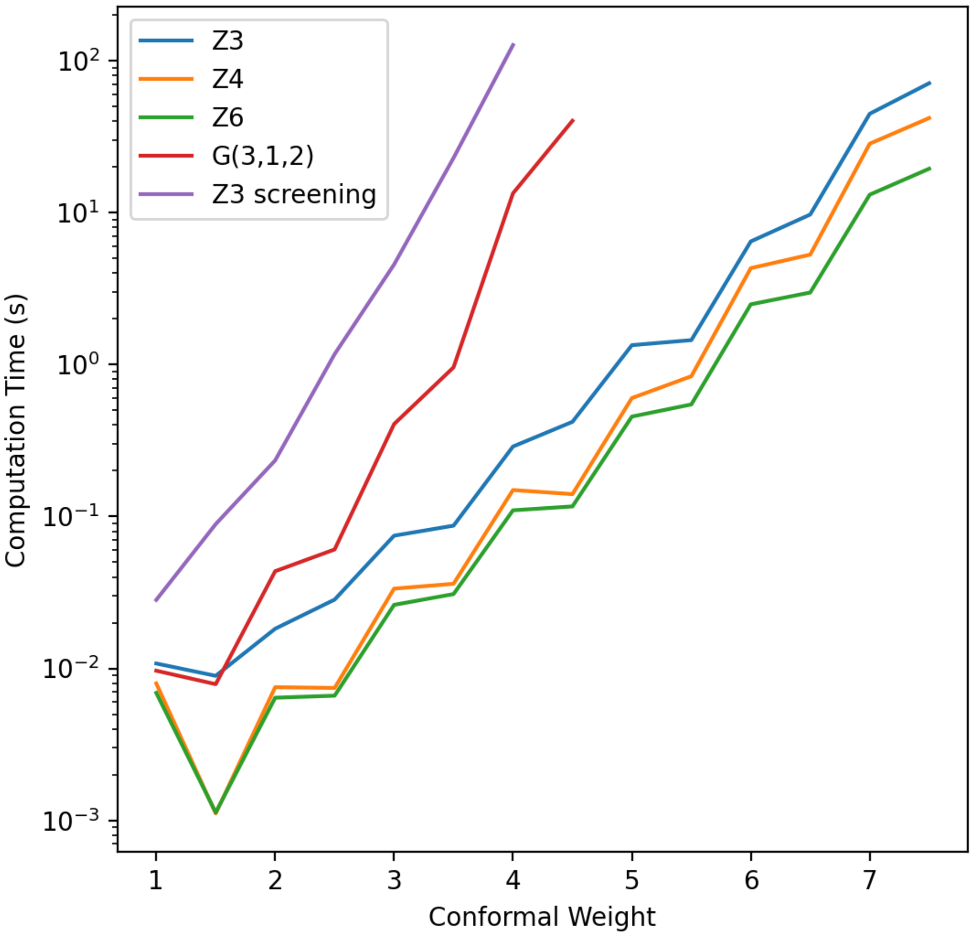


Figure 5.1: Computation times for carrying out the calculation of the vacuum character for VOAs at different conformal weights. Several different VOAs are shown on the graph for comparison, along with the $\mathbb{Z}_3$ theory computed using the kernel of the screening operator. We used a desktop PC with an Intel Core i7-6700K CPU clocked at 4GHz, and 32GB RAM.

Chapter 6

Final Comments

In the first part of this thesis (Chapters (2) and (3)) we have reviewed the problem of writing a Lagrangian formulation for self-dual forms and computing its path-integral. Although this is an old topic that has at least 30 years of academic papers weighing on its shoulders, we hope we have managed to contribute with some new insights to the story, by closely studying the recent proposal of Sen. In a few words, the novelties of his approach lie in achieving manifest Lorentz invariance by deploying an auxiliary field that dynamically decouples from the physics of the chiral form. This mechanism has the advantage of avoiding any non-standard gauge redundancy, which were instead typical in the old literature, and which might lead to complications during the quantization procedure. On the other hand, maybe because of the *conservation of difficulties* principle which is always lurking around in physics, in Sen's Lagrangian one has to give up standard diffeomorphism invariance. Nevertheless, we stress that this is a feature that, in a sense, has to naturally arise from a proper formulation of a chiral form $H_{(g)} = \star_g H_{(g)}$. Indeed, the latter implies $\delta_\xi H_{(g)} = \star_g \delta_\xi H_{(g)}$ as well and, in general, the maximum we can require is that $\delta_\xi H_{(g)}$ agrees with the ordinary Lie derivative only once on-shell. In other words, in a Lagrangian formulation where the chirality condition holds even off-shell, it is just physiological that the coupling to a curved background metric has to be realized in an unconventional fashion. This information is succinctly encoded in the $\mathcal{M}$ term in Sen's action, that we tested first from a classical perspective in Chapter (2) by considering compactifications of the abelian (2,0) Lagrangian down to lower dimensional theories and then from a quantum perspective in Chapter (3) by computing its path integral. The results we found yielded sensible results which are in agreement with expectations.

In the second half of the thesis we made contact with some of the current topics of modern research in SCFTs, where a lot of efforts have been made to understand non-perturbative effects and to study theories which do not necessarily admit a Lagrangian formulation. Two tools that are commonly used in this context are the Superconformal

Index and Anomalies. In Chapter (4) we moved some of the first steps towards a more systematic study of those Type-B Weyl anomalies associated to Coulomb-Branch operators in $\mathcal{N} = 2$ SCFTs in 4D. We gave strong indications that these anomalies are covariantly constant across the conformal manifold, even when the theory is in the Higgs Branch phase, where conformal symmetry is spontaneously broken. This implies that when the CBO Type-B anomalies in the conformal and Higgs phases match at a particular point on the conformal manifold, then they do so also across at least a neighborhood of the latter and, therefore, the matching of the CBO Type-B anomalies in the conformal and Higgs phases can be first examined by a tree-level calculation and then maybe used to constrain some non-perturbative dynamics. In Chapter (5) we instead computed, for the first time in the literature, the Macdonald limit of the superconformal index of some 4D $\mathcal{N} = 3$ SCFTs. As they stand, these indices can already be used as new data for $\mathcal{N} = 3$ SCFTs, which are relatively new non-Lagrangian SCFTs that only recently showed up in town and about which still a lot has to be discovered.

Appendix A

Zeta-regularised product

In this appendix we will detail some aspects of the regularisation that we used in Section 3.2.2. In what follows $T = -\frac{1}{2}(1 + \frac{1}{\mathcal{M}}) = T_1 + iT_2$, with $T_1, T_2 \in \mathbb{R}$ and $T_2 > 0$ as long as $\tau_2 > 0$, see (3.13).

The infinite product

$$P = \prod_{\substack{n>0 \\ m\in\mathbb{Z}}} \frac{1}{n(m+nT)} , \tag{A.1}$$

can be regularised in the following standard fashion. First we will consider the auxiliary sum[59]

$$G(s,T) := \sum_{n=1}^{\infty} \sum_m \frac{1}{n^s (m+nT)^s} , \tag{A.2}$$

which is naturally defined for $\mathrm{Re}(s) > 1$. Then, by freely commuting the infinite sums with each other, and with the integrals that appear, we will analytically continue the latter to $\mathrm{Re}(s) \geq 0$. Finally, we will define (A.1) by

$$P := \exp\left[\frac{d}{ds}\bigg|_{s=0} G(s,T) \right] . \tag{A.3}$$

It is easy to see that

$$G(s, T_1 + 1, T_2) = G(s, T_1, T_2) , \tag{A.4}$$

so we can employ its discrete Fourier transformation $F(s,p,T_2)$ defined by

$$F(s,p,T_2) := \int_0^1 d\chi e^{2\pi i \chi p} G(s,\chi,T_2) , \tag{A.5}$$

[59] From now on, $\sum_m$ will be a shorthand for $\sum_{m\in\mathbb{Z}}$.

to recast $G(s, T_1, T_2)$ as

$$G(s,T_1,T_2) = \sum_p e^{-2\pi i p T_1} F(s,p,T_2)$$
$$= \sum_p e^{-2\pi i p T_1} \int_0^1 d\chi \sum_{n=1}^{\infty} \sum_m \frac{1}{n^s (m + n\chi + inT_2)^s} e^{2\pi i p\chi} \,. \qquad \text{(A.6)}$$

Let $k \in \mathbb{Z}$ and $r \in \{0, \ldots, n-1\}$ so that we can write $m = kn + r$. Then the last line becomes

$$G(s,T_1,T_2) = \sum_p \sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \sum_k \int_0^1 d\chi e^{2\pi i p(\chi - T_1)} \frac{1}{n^s} \frac{1}{(r + n(\chi + k) + inT_2)^s}$$
$$= \sum_p \sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \sum_k \int_k^{k+1} dy e^{2\pi i p(y - T_1)} \frac{1}{n^s} \frac{1}{(r + ny + inT_2)^s}$$
$$= \sum_p \sum_{n=1}^{\infty} \sum_{r=0}^{n-1} \int_{-\infty}^{+\infty} d\chi e^{2\pi i p(\chi - \frac{r}{n} - T_1)} \frac{1}{[n^2(\chi + iT_2)]^s}$$
$$= \sum_k \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} d\chi e^{2\pi i kn(\chi - T_1)} n \frac{1}{[n^2(\chi + iT_2)]^s} \,, \qquad \text{(A.7)}$$

where we performed the change of coordinates $y := \chi + k$ and $\chi := y + \frac{r}{n}$ respectively in the second and third line, whereas in the last step we used

$$\sum_{r=0}^{n-1} e^{-2\pi i \frac{p}{n} r} = \begin{cases} n & \text{if } p = kn \;\; k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \,. \qquad \text{(A.8)}$$

Since $T_2 > 0$ we can now implement the following integral representation of z^{-s}

$$\frac{1}{z^s} = \frac{1}{i^s} \frac{1}{\Gamma(s)} \int_0^{\infty} dt\, t^{s-1} e^{izt} \quad \text{for} \quad \mathrm{Im}(z) > 0 \,, \qquad \text{(A.9)}$$

and, by switching to the $y := n^2 t$ variable, (A.7) becomes

$$G(s,T_1,T_2) = \frac{1}{i^s} \frac{1}{\Gamma(s)} \sum_k \sum_{n=1}^{\infty} n \int_{-\infty}^{+\infty} d\chi e^{2\pi i kn(\chi - T_1)} \int_0^{\infty} dt t^{s-1} e^{in^2(\chi + iT_2)t}$$
$$= \frac{1}{i^s} \frac{1}{\Gamma(s)} \sum_k \sum_{n=1}^{\infty} \frac{1}{n^{2s-1}} \int_0^{\infty} dy y^{s-1} e^{-2\pi i kn T_1 - T_2 y} \int_{-\infty}^{+\infty} d\chi e^{i\chi(2\pi kn + y)}$$
$$= \frac{2\pi}{i^s} \frac{1}{\Gamma(s)} \sum_{k=0} \sum_{n=1}^{\infty} \frac{1}{n^{2s-1}} \int_0^{\infty} dy y^{s-1} e^{-2\pi i kn T_1 - T_2 y} \delta(2\pi kn + y)$$
$$= G_0(s,T_1,T_2) + \frac{1}{i^s} \frac{1}{\Gamma(s)} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{2\pi k}{n}\right)^s \frac{e^{2\pi i kn T}}{k} \,. \qquad \text{(A.10)}$$

Here we have split the $k = 0$ contribution $G_0(s, T_1, T_2)$, which requires additional regularisation, from the $k > 0$ terms which are instead convergent for $\mathrm{Re}(s) \geq 0$.

To compute $G_0(s, T_1, T_2)$ we first observe that it is formally given by

$$\begin{aligned} G_0(s, T_1, T_2) &= \frac{2\pi}{i^s} \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \frac{1}{n^{2s-1}} \int_0^{\infty} dy y^{s-1} e^{-2\pi i k n T_1 - T_2 y} \delta(y) \\ &= \frac{1}{2} \frac{1}{i^s} \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \left(\frac{2\pi k}{n} \right)^s \frac{e^{2\pi i k n T}}{k} \bigg|_{k=0} , \end{aligned} \tag{A.11}$$

where the extra factor of $1/2$ arises since

$$\int_a^b dx \delta(x - c) f(x) = \begin{cases} f(c) & \text{if } c \in (a, b) \\ \frac{1}{2} f(c) & \text{if } c \in \{a, b\} \\ 0 & \text{otherwise} \end{cases} . \tag{A.12}$$

To regularise this we deform $k \to k + \epsilon$ so that

$$G_0(s, T_1, T_2) = \frac{1}{2} \frac{1}{i^s} \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \left(\frac{2\pi \epsilon}{n} \right)^s \frac{e^{2\pi i \epsilon n T}}{\epsilon} \tag{A.13}$$

and therefore, for small s,

$$\begin{aligned} G_0(s, T_2, T_2) &= \frac{1}{2\Gamma(s)} \sum_{n=1}^{\infty} \left(\frac{2\pi \epsilon}{in} \right)^s \frac{e^{2\pi i \epsilon n T}}{\epsilon} \\ &= \frac{s}{2} \sum_{n=1}^{\infty} \frac{e^{2\pi i \epsilon n T}}{\epsilon} + O(s^2) \\ &= \frac{s}{2\epsilon} \zeta(0) + \pi i s T \zeta(-1) + O(s^2, \epsilon) , \end{aligned} \tag{A.14}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function, for which $\zeta(0) = -\frac{1}{2}$ and $\zeta(-1) = -\frac{1}{12}$. Since the divergent part of (A.14) does not depend on T, we regularise the $k = 0$ contribution in (A.10) by neglecting the $\frac{1}{\epsilon}$ divergence. In this case, when $s \to 0^+$, the expression (A.10) looks like[60]

$$\begin{aligned} G(s, T_1, T_2) &= s\pi i T \zeta(-1) + s \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{2\pi k}{n} \right)^s \frac{e^{2\pi i k n T}}{k} + O(s^2) \\ &= s\pi i T \zeta(-1) - s \sum_{n=1}^{\infty} \log\left(1 - e^{2\pi i T n}\right) + O(s^2) , \end{aligned} \tag{A.15}$$

[60] We remind the reader that $\Gamma(s) = \frac{1}{s} - \gamma + O(s)$ where γ is the Euler–Mascheroni constant.

where we identified $\sum_{k=1}^{\infty} \frac{q^k}{k} = -\log(1-q)$. Recalling that $\zeta(-1) = -1/12$ we find

$$P = \prod_{n=1}^{\infty} \prod_{m} \frac{1}{n(m+nT)} = e^{-i\frac{\pi}{12}T} \prod_{n=1}^{\infty} \frac{1}{1-e^{2\pi i T n}} = \frac{1}{\eta(T)} \quad . \tag{A.16}$$

Finally, to make contact with the path-integral computations performed in Section 3.2.2 (see (3.48)) we note that in our regularisation the product (let a be a complex number)

$$P_a = \prod_{\substack{n\neq 0 \\ m}} \frac{a}{n(m+nT)} \ , \tag{A.17}$$

becomes

$$P_a = \exp\left[\frac{d}{ds}\bigg|_{s=0} (a^s G(s,T)) \right] = a^{G(0,T)} \frac{1}{\eta(T)} \ , \tag{A.18}$$

and by using (A.15) we find that $G(0,T) = 0$.

Appendix B

Factorization of Higher-dimensional theta function

Let's consider the case where the physical metric is (in Lorentzian signature)

$$g_{\mu\nu} = \text{diag}\left(-L_0^2, L_1^2, \ldots, L_5^2\right) \ . \tag{B.1}$$

To construct $\widetilde{\mathcal{M}}^{AB}$ we see that a basis of self-dual 3-forms with respect to $\star_g$ are given by

$$\begin{aligned}\varphi_+^A &= (1+\star_g)dx^0 \wedge dx^i \wedge dx^j \\ &= \frac{\omega_+^A + \omega_{A-}}{2} + VL_i^{-2}L_j^{-2}L_0^{-2}\frac{\omega_+^A - \omega_{A-}}{2} \\ &= \frac{1+V/L_i^2L_j^2L_0^2}{2}\omega_+^A + \frac{1-V/L_i^2L_j^2L_0^2}{2}\omega_{A-} \ ,\end{aligned} \tag{B.2}$$

where $V = L_0L_1 \ldots L_5$. From here we can read off

$$\widetilde{\mathcal{M}}^{AB} = -\frac{1-V/A_AL_0^2}{1+V/A_AL_0^2}\delta^{AB} \ , \tag{B.3}$$

where $A_A = L_iL_j$. Finally we want to Wick rotate in the same way as in Section 3.2, which effectively amounts to sending

$$L_0 \to -iL_0 \ . \tag{B.4}$$

Thus we find

$$\begin{aligned}\widetilde{\mathcal{M}}^{AB} &= -\frac{1 - iV/A_A L_0^2}{1 + iV/A_A L_0^2}\delta^{AB} \\ &= -\frac{\tau_A + 1}{\tau_A - 1}\delta^{AB} \ ,\end{aligned} \tag{B.5}$$

where $\tau_A = iL_0 A_A/V$. For such metrics $\widetilde{\mathcal{M}}^{AB}$ becomes diagonal and each component has the form that we saw for the chiral boson but with a purely imaginary complex structure τ_A. Note that in this case

$$\begin{aligned}\mathcal{T}_{AB} &= -\frac{1}{2}\left(\delta_{AB} + \widetilde{\mathcal{M}}^{-1}_{AB}\right) \\ &= -\frac{1}{1+\tau_A}\delta_{AB} \ .\end{aligned} \tag{B.6}$$

As a result, (3.140) factorises into a product of more familiar functions from the two-dimensional chiral boson discussion:

$$Z^{(0)}_{\text{w.m.}} = \prod_{A=1}^{10} \theta \begin{bmatrix} \alpha^A \\ \beta_A \end{bmatrix} \left(\sqrt{r_1 r_2}\mathcal{J}^{(0)}_A \mid -2r_1 r_2/(1+\tau_A)\right) \tag{B.7}$$

Lastly, we note that $-1/(1+\tau_A)$ is simply an S and T modular transformation away from τ_A.

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