

Abstract 3-Rigidity and Bivariate C_2^1 -Splines II: Combinatorial Characterization

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Abstract: We showed in the first paper of this series that the generic C_2^1 -cofactor matroid is the unique maximal abstract 3-rigidity matroid. In this paper we obtain a combinatorial characterization of independence in this matroid. This solves the cofactor counterpart of the combinatorial characterization problem for the rigidity of generic 3-dimensional bar-joint frameworks. We use our characterization to verify that the counterparts of conjectures of Dress (on the rank function) and Lovász and Yemini (which suggested a sufficient connectivity condition for rigidity) hold for the C_2^1 -cofactor matroid.

Key words and phrases: graph rigidity, rigidity matroid, bivariate spline, cofactor matroid, matroid erection, free elevation

1 Introduction

We will consider a d -dimensional (*bar-joint*) framework, which is a pair consisting of a finite simple graph $G = (V, E)$ and a map $p : V \rightarrow \mathbb{R}^d$. It is *rigid* if every continuous motion of the vertices of G in \mathbb{R}^d which preserves the lengths of the edges, results in a framework which is congruent to (G, p) . The framework has the stronger property of being *infinitesimally rigid* if every length preserving infinitesimal motion of the vertices is induced by an infinitesimal isometry of \mathbb{R}^d . Maxwell [33] gave the following necessary condition for infinitesimal rigidity: if (G, p) is an infinitesimally rigid d -dimensional framework with at least d vertices, then G has a spanning subgraph H such that

- $|E(H)| = d|V(H)| - \binom{d+1}{2}$, and

- $|E(H')| \leq d|V(H')| - \binom{d+1}{2}$ for any subgraph H' of H with $|V(H')| \geq d$.

It is straightforward to check that this condition is also sufficient when $d = 1$. A celebrated result of Pollaczek-Geiringer [38], rediscovered by Laman [28], shows that Maxwell’s condition is also sufficient when $d = 2$ if p is generic, i.e., the set of coordinates in p is algebraically independent over the rational field. When $d \geq 3$, Maxwell’s condition is no longer sufficient to imply infinitesimal rigidity, even for generic frameworks, and finding a combinatorial characterization for generic 3-dimensional rigidity has been the central open problem in graph rigidity for many years, see for example [20, 48]. Maxwell’s condition is known to be sufficient for generic 3-dimensional rigidity of some restricted classes such as triangulations of closed 2-surfaces (with/without holes) [10, 11, 17, 16], graphs with no K_5 -minor [34], and squares of graphs [26]. We refer the reader to [39] for a recent survey on graph rigidity.

Gluck [18] observed that the properties of rigidity and infinitesimal rigidity coincide when p is generic, and are completely determined by the graph G and the dimension d . This fact motivated Asimow and Roth [1] to define a graph G as being *rigid in \mathbb{R}^d* if some, or equivalently every, generic d -dimensional framework (G, p) is infinitesimally rigid. The rigidity of a graph G is determined by the rank of the *generic d -dimensional rigidity matroid $\mathcal{R}_d(G)$* of G and exploring the combinatorial structure of $\mathcal{R}_d(G)$ using machinery from matroid theory is a common approach to attack problems in rigidity, see for example [44, 20, 48]. We will build on this matroidal approach to obtain a combinatorial characterization of a matroid from the theory of bivariate splines which is conjectured to be equal to the generic 3-dimensional rigidity matroid.

Graver [19] defined the class of *abstract d -rigidity matroids* on the edge set of the complete graph K_n using two fundamental properties of rigidity in d -space. The generic d -dimensional rigidity matroid $\mathcal{R}_d(K_n)$ is an example of an abstract d -rigidity matroid, and Graver [19] conjectured that, for all $d \geq 1$, $\mathcal{R}_d(K_n)$ is the unique maximal matroid in the set of all abstract d -rigidity matroids on $E(K_n)$ with respect to the weak order of matroids. Graver showed that his conjecture is true for $d = 1, 2$ but it was subsequently shown to be false for $d \geq 4$. It remains open for $d = 3$. (See Section 4 of this paper or [5] for a more detailed discussion.)

Whiteley [48] found a new candidate for a unique maximal abstract d -rigidity matroid from approximation theory by taking the row matroid of the C_s^{s-1} -cofactor matrix of a(ny) generic 2-dimensional framework (K_n, p) . This is the $|E(K_n)| \times (s + 1)n$ matrix in which sets of consecutive $(s + 1)$ columns are associated with vertices, rows are associated with edges, and the row associated to the edge $e = v_i v_j$ with $i < j$ is

$$e=v_i v_j \left[\begin{array}{cccccc} & & v_i & & v_j & \\ & & & & & \\ 0 & \cdots & 0 & D_{ij} & 0 & \cdots & 0 & -D_{ij} & 0 & \cdots & 0 \end{array} \right],$$

where $D_{i,j} = ((x_i - x_j)^s, (x_i - x_j)^{s-1}(y_i - y_j), \dots, (x_i - x_j)(y_i - y_j)^{s-1}, (y_i - y_j)^s) \in \mathbb{R}^{s+1}$ when $p(v_i) = (x_i, y_i) \in \mathbb{R}^2$ for each $v_i \in V(K_n)$. We refer to the row matroid of this matrix as the C_s^{s-1} -cofactor matroid and denote it by $\mathcal{C}_s^{s-1}(K_n)$. Whiteley [48] showed that $\mathcal{C}_{d-1}^{d-2}(K_n)$ is an abstract d -rigidity matroid, that $\mathcal{C}_{d-1}^{d-2}(K_n) = \mathcal{R}_d(K_n)$ when $d = 1, 2$ and that $\mathcal{C}_{d-1}^{d-2}(K_n) \not\leq \mathcal{R}_d(K_n)$ when $d \geq 4$. He conjectured further that $\mathcal{C}_{d-1}^{d-2}(K_n)$ is the unique maximal abstract d -rigidity matroid on $E(K_n)$ for all $d \geq 1$. Note that Whiteley’s conjecture holds when $d = 1, 2$ by the above mentioned result of Graver. In our first paper [5] in this series, we verified Whiteley’s conjecture for $d = 3$ by showing that $\mathcal{C}_2^1(K_n)$ is the unique maximal abstract 3-rigidity matroid (see Theorem 4.5 below).

Because of the strong similarity between rigidity matroids and cofactor matroids, Whiteley [47, page 55] also remarked that finding a combinatorial characterization of independence in the generic C_2^1 -cofactor matroid may be as challenging as the corresponding problem for the generic 3-dimensional rigidity matroid, and went on to conjecture that these two matroids are equal in [48, Conjecture 10.3.2]. This paper solves the characterization problem for the generic C_2^1 -cofactor matroid (and equivalently for the maximal abstract 3-rigidity matroid) by giving a co-NP type characterization for independence (we will see below that an NP type characterization follows immediately from the Schwartz-Zippel Lemma). We then use our characterization to verify the cofactor counterpart of two long-standing conjectures on the generic 3-dimensional rigidity matroid.

Dress gave two conjectures on the rank of the 3-dimensional generic rigidity matroid in the 1980's. The first conjecture, which appeared in [13], suggested a co-NP type characterization for independence, but was subsequently disproved by Jackson and Jordán [22]. The second conjecture, which was given at a rigidity conference in Montreal in 1987, see [8, 20, 43], describes the rank function in terms of the 'rigid clusters' of the underlying graph and still remains open. It would give a useful structural property of the rigidity matroid but would not obviously give rise to a co-NP type characterization. We prove in Theorem 6.3 below that the corresponding conjecture holds for the generic C_2^1 -cofactor matroid. We also show that the modified versions of Dress's first conjecture given in [23, 21] hold for this matroid.

Lovász and Yemini [31] proved that 6-connectivity is sufficient to imply that graphs are generically rigid in 2-dimensional space, and conjectured that 12-connectivity is sufficient for rigidity in 3-space. We prove in Theorem 7.2 that the corresponding statement is true for the generic C_2^1 -cofactor matroid

The main technical innovation in our work is to find a new kind of rank formula for a matroid. One of the major difficulties in attacking the 3-dimensional rigidity problem is the lack of understanding of how to construct a matroid based on Maxwell's condition when $d \geq 3$. It is well-known that, for $d = 1, 2$, the edge sets satisfying Maxwell's condition form an independent set family of a matroid. Lovász and Yemini [31] showed that the structure of this matroid can be completely understood using the theory of intersecting submodular functions. Currently, this theory does not seem to apply when $d \geq 3$. In particular, it is not clear how to use this theory to obtain a polynomial algorithm to evaluate our formula for the rank function of the C_2^1 -cofactor matroid.

Our rank formula was inspired by a matroid construction due to Crapo [7]. He defined an *erection* of a matroid as an inverse operation to truncation. He showed that the set of all erections of a matroid M forms a lattice under the weak order for matroids, and defined the maximum matroid in this lattice to be the *free erection* of M . A sequence of (free) erections starting from M always terminates after a finite number of steps and we refer to a matroid obtained by such a sequence as a (*free*) *elevation* of M . Examples due to Brylawski [3, Figure 7.9] and the second two authors [25, Theorem 5.4] show that the free elevation may not be the *unique* maximal element in the poset of all elevations of M , but we show in Lemma 3.1 that it is always *a* maximal element. We then obtain an upper bound on the rank function of all elevations of M in terms of the non-spanning circuits of M in Lemma 3.3. Conjecture 3.4 states that this upper bound is tight if and only if the free elevation of M is the unique maximal element in the poset of all elevations of M . Our main result, Theorem 5.7, shows that the upper bound is tight for the generic C_2^1 -cofactor matroid and verifies a special case of this conjecture.

Our results are relevant to another long-standing open problem, the *polynomial identity testing problem for symbolic determinants* (or the *Edmonds problem*). In this problem, we are given a matrix A

with entries in $\mathbb{Q}[x_1, \dots, x_n]$ and we are asked to decide whether the rank of A over $\mathbb{Q}(x_1, \dots, x_n)$ is at least a given number. The Schwartz-Zippel Lemma [40, 49] implies that this problem admits a randomized polynomial time algorithm, but developing a deterministic polynomial time algorithm is a major open problem in theoretical computer science. The same lemma also tells us that the problem is in the class NP, but it is not known whether it is in co-NP. One approach, pioneered by Tutte [46], Edmonds [14] and Lovász [30], is to give a good characterization for the rank of A by showing it is the minimum value for a certain combinatorial optimization problem. Our result offers a new example of this approach. Indeed Lovász [30, Section 5] states that the generic 3-dimensional rigidity problem is an important special case of the polynomial identity testing problem. Our technique solves the closely related problem for generic \mathcal{C}_2^1 -cofactor matrices.

The research direction of this paper was motivated by the fundamental papers of Graver [19] and Whiteley [48], and more recently by a talk given by Meera Sitharam at a BIRS workshop in 2015 on her joint work with Jialong Cheng and Andrew Vince, see [41], in which she described a recursive procedure for constructing the closure operator in a maximal matroid on the edge set of a complete graph in which every K_5 -subgraph is a circuit.

Our paper is structured as follows. In Section 2 we review the theory of matroid erections, using a primal approach instead of the traditional dual approach. We introduce the free elevation of a matroid M in Section 3 and prove two key results: Lemma 3.3 gives an upper bound on the rank of any elevation of M ; Lemma 3.8 shows that if every element of M is contained in a non-spanning circuit of M then every cyclic flat of the free elevation of M is the union of non-spanning circuits of M . We describe the family of abstract d -rigidity matroids and \mathcal{C}_{d-1}^{d-2} -cofactor matroids in Section 4. We obtain our main result, Theorem 5.7, which gives a polynomially verifiable characterization of the rank function of the maximal abstract 3-rigidity matroid \mathcal{C}_2^1 in Section 5. We give two alternative expressions for the rank function of \mathcal{C}_2^1 in Section 6 and use these to obtain sufficient connectivity conditions for the \mathcal{C}_2^1 -matroid of a graph to have maximum possible rank in Section 7. We close with some open problems and remarks in Section 8.

2 Matroid erections

Matroid erection, introduced by Crapo [7], is a key tool in this paper. We give a detailed exposition of matroid erection in this preliminary section for the benefit of readers who are unfamiliar with the topic. For an introduction to the concepts below, see [37].

Given a matroid M , we use E_M to denote its ground set, cl_M to denote its closure operator, and r_M to denote its rank function. We will often suppress the subscript M when it is obvious which matroid we are referring to. For $X \subseteq E$, $M|_X$ denotes the restriction of M to X and $M - X$ denotes $M|_{E \setminus X}$. The contracted matroid M/X is the matroid on $E \setminus X$ in which a set $F \subseteq E \setminus X$ is independent if and only if $F \cup X$ is independent in M . When $X = \{e\}$, we simplify the notation for matroid deletion and contraction to $M - e$ and M/e , respectively.

A set $X \subseteq E$ is said to be *spanning* if $\text{cl}_M(X) = E$ and to be a *flat* if $\text{cl}_M(X) = X$. A *hyperplane* of M is a flat F with $r(F) = r(E) - 1$. The poset of all flats ordered by set inclusion forms a geometric lattice by setting the meet and join of two flats F_1 and F_2 to be $F_1 \cap F_2$, and $\text{cl}_M(F_1 \cup F_2)$, respectively. A pair X, Y of subsets of E is said to be *modular* if $r_M(X) + r_M(Y) = r_M(X \cap Y) + r_M(X \cup Y)$. The *dual* of M is denoted by M^* . The *weak order* on a set of all matroids with the same ground set E is the partial order in

which $M_1 \preceq M_2$ if every independent set of M_1 is independent in M_2 .

One-point extensions and elementary quotients. Given two matroids M and P , we say that P is a *one-point extension* of M if $M = P - p$ for some $p \in E_P$. The structure of one-point extensions can be understood by introducing the concept of modular cuts. A family \mathcal{F} of flats of M is said to be a *modular cut* of M if it is up-closed in the lattice of flats and $X \cap Y \in \mathcal{F}$ for all modular pairs X, Y in \mathcal{F} . Given a modular cut \mathcal{F} of M , we can define a matroid P on $E_M + p$ as follows. For all $X \subseteq E_M$ we put $r_P(X) = r_M(X)$, and

$$r_P(X + p) = \begin{cases} r_M(X) & \text{if } \text{cl}_M(X) \in \mathcal{F}, \\ r_M(X) + 1 & \text{if } \text{cl}_M(X) \notin \mathcal{F}. \end{cases}$$

One can easily check that r_P is indeed a matroid rank function. We will denote the one-point extension of M with respect to the modular cut \mathcal{F} by $M +_{\mathcal{F}} p$. Every one-point extension of M can be uniquely constructed in this manner. More precisely, the map $\mathcal{F} \mapsto M +_{\mathcal{F}} p$ is a bijection between the set of modular cuts of M and the set of one-point extensions of M . (See [37] for more details.)

Given a modular cut \mathcal{F} of M , the matroid $N = (M +_{\mathcal{F}} p)/p$, is called the *elementary quotient* of M with respect to \mathcal{F} . Observe that $E_M = E_N =: E$ and, for all $X \subseteq E$,

$$r_N(X) = \begin{cases} r_M(X) - 1 & \text{if } \text{cl}_M(X) \in \mathcal{F}, \\ r_M(X) & \text{if } \text{cl}_M(X) \notin \mathcal{F}. \end{cases} \quad (1)$$

Elementary lifts. Given two matroids M and N , we say that N is an *elementary lift* of M if M is an elementary quotient of N i.e. $N = P - p$ and $M = P/p$ for some 1-point extension P of N . In this case we can use matroid duality to deduce that $M^* = P^* - p$ and $N^* = P^*/p$. Hence N is an elementary lift of M if and only if N^* is an elementary quotient of M^* . The correspondence between elementary quotients and modular cuts now gives us a bijection between the set of elementary lifts of M and the set of modular cuts of M^* . It will be helpful to describe this bijection in terms of the primal matroid M rather than its dual M^* .

A set $X \subseteq E$ is said to be *cyclic* in M if it is the union of circuits of M . For our purposes, it will make sense to consider the empty set as a cyclic set. Let $\text{cyc}_N(X)$ be the largest cyclic subset of X , i.e., the set obtained from X by removing the coloops in $M|_X$. The poset of all cyclic sets in M ordered by set inclusion forms a lattice by setting the join and the meet of C_1 and C_2 to be $C_1 \cup C_2$ and $\text{cyc}_M(C_1 \cap C_2)$, respectively.

We say that a family \mathcal{C} of cyclic subsets of E is a *modular cyclic family* if \mathcal{C} is down-closed (in the lattice of cyclic sets) and, for every modular pair X, Y in \mathcal{C} , $X \cup Y \in \mathcal{C}$. Note that since modular cyclic families are down-closed and we consider \emptyset to be cyclic, we have $\emptyset \in \mathcal{C}$ for every modular cyclic family \mathcal{C} .

The following result gives a bijection between the modular cuts of M^* and the modular cyclic families in M .

Proposition 2.1. *Let \mathcal{C} be a family of subsets of E . Then \mathcal{C} is a modular cyclic family in M if and only if $\{E \setminus C : C \in \mathcal{C}\}$ is a modular cut in M^* .*

Proof. Recall that X is a circuit in M if and only if $E \setminus X$ is a hyperplane in M^* . Since every flat is the intersection of hyperplanes, X is cyclic in M if and only if $E \setminus X$ is a flat in M^* . In addition, a direct computation shows that X, Y is a modular pair in M if and only if $E \setminus X, E \setminus Y$ is a modular pair in M^* . \square

Proposition 2.1 and the preceding discussion give a bijection between the set of elementary lifts of M and the set of modular cyclic families in M , and allow us to define N to be the *elementary lift of M with respect to the modular cyclic family \mathcal{C} of M* if and only if N^* is the elementary quotient of M^* with respect to the modular cut $\{E \setminus C : C \in \mathcal{C}\}$ of M^* . Moreover, we can use (1), Proposition 2.1, and the fact that $\text{cyc}_M(X) = E \setminus \text{cl}_{M^*}(E \setminus X)$ for $X \subseteq E$ to deduce that the elementary lift N of M with respect to the modular cyclic family \mathcal{C} in M has the following rank function:

$$r_N(X) = \begin{cases} r_M(X) & \text{if } \text{cyc}_M(X) \in \mathcal{C} \\ r_M(X) + 1 & \text{if } \text{cyc}_M(X) \notin \mathcal{C} \end{cases} \quad (X \subseteq E(N)). \quad (2)$$

This formula implies that the above mentioned bijection is an (order reversing) isomorphism between the lattice of elementary lifts of M , ordered by the weak order of matroids, and the lattice of modular cyclic families in M , ordered by inclusion. (The meet and join of two cyclic families in this lattice are given by the intersection of the two families, and the smallest modular cyclic family containing their union, respectively.)

Matroid erections. Let M and N be matroids on the same ground set E with $r_M(E) = k \leq r_N(E)$. We say that M is the *truncation of N to rank k* if $r_M(X) = \min\{r_N(X), k\}$ for all $X \subseteq E(M)$. If $k = r_N(E) - 1$, M is simply called the *truncation* of N . The inverse operation to truncation was used by Crapo [7] and Knuth [27] to recursively generate all the matroids on a given ground set from the rank zero matroid on this set. Following Crapo, we say that N is an *erection of M* if M is the truncation of N . For a technical reason, M is also considered to be an erection of itself, and is referred to as the *trivial* erection.

Crapo [7] showed that the set of all erections of a matroid M forms a lattice under the weak order, where the bottom element corresponds to the trivial erection. The top element in this lattice is called the *free erection* of M . Las Vergnas [29] and Nguyen [35] independently gave a characterization of the free erection of M . Duke [12] subsequently gave a clearer exposition in terms of one-point extensions of the dual matroid M^* . We shall describe Duke’s approach in terms of the primal matroid M .

Observe first that, if M is a truncation of N , then (1) implies that M is the elementary quotient of N with respect to the modular cut $\mathcal{F} = \{E\}$. This in turn implies that N is a special kind of elementary lift of M . Our next result characterizes which elementary lifts are erections.

Theorem 2.2. (A primal version of [12, Lemma 3.1].) *Let \mathcal{C} be a modular cyclic family in a matroid M and N be the elementary lift of M with respect to \mathcal{C} . Then N is an erection of M if and only if \mathcal{C} contains all cyclic flats of M , with the possible exception of E .*

Proof. We first assume that N is an erection of M . Then M is the truncation of N . Let C be a cyclic flat of M with $C \notin \mathcal{C}$. By (2) we have $r_N(C) = r_M(C) + 1$. Since M is the truncation of N this gives $r_M(C) = r_M(E)$. Since C is a flat of M , this implies that $C = E$.

We next assume that \mathcal{C} contains all cyclic flats of M , with the possible exception of E . Suppose F is a flat of M with $r_M(F) < r_M(E)$. Then $\text{cyc}_M(F)$ is a cyclic flat so $\text{cyc}_M(F) \in \mathcal{C}$. By (2), $r_N(F) = r_M(F)$.

It follows that all sets X with $r_N(X) = r_M(X) + 1$ have $r_M(X) = r_M(E)$. Hence M is the truncation of N . \square

We saw above that the lattice of all elementary lifts of M is isomorphic to the lattice of modular cyclic families of M . Theorem 2.2 enables us to determine the restriction of this isomorphism to the lattice of erections of M . Given a family \mathcal{C}_0 of cyclic sets in M , we define its *modular cyclic closure* as:

$$\overline{\mathcal{C}}_0 = \bigcap \{ \mathcal{C} : \mathcal{C} \text{ is a modular cyclic family of } M \text{ with } \mathcal{C}_0 \subseteq \mathcal{C} \}.$$

Let \mathcal{CF}_M be the family of all non-spanning cyclic flats of M , and \mathcal{C}_M be the family of all cyclic sets in M . Then Theorem 2.2 implies that the sublattice $[\overline{\mathcal{CF}}_M, \mathcal{C}_M]$ in the lattice of modular cyclic families corresponds to the lattice of erections of M , where $\overline{\mathcal{CF}}_M$ corresponds to the free erection and \mathcal{C}_M corresponds to the trivial erection. In particular we have:

Corollary 2.3. *A matroid M has no non-trivial erection if and only if $\overline{\mathcal{CF}}_M = \mathcal{C}_M$, i.e., the only modular cyclic family of M containing \mathcal{CF}_M is the family of all cyclic sets in M .*

By (2) we also have the following explicit rank formula for the free erection of M .

Corollary 2.4. *Let N be the free erection of M . Then*

$$r_N(X) = \begin{cases} r_M(X) & \text{if } \text{cyc}_M(X) \in \overline{\mathcal{CF}}_M \\ r_M(X) + 1 & \text{if } \text{cyc}_M(X) \notin \overline{\mathcal{CF}}_M \end{cases} \quad (X \subseteq E). \quad (3)$$

For a set S of elements in a lattice, let S^\downarrow be the lower closure of S . In order to use Corollary 2.4 to determine the rank function for the free erection of M , we need to be able to compute $\overline{\mathcal{CF}}_M$ from \mathcal{CF}_M . An algorithm for constructing the smallest modular cut containing a given set of flats is known (see, e.g., [12, page 367]). Dualizing this algorithm, we may use the following procedure to compute $\overline{\mathcal{CF}}_M$ from \mathcal{CF}_M .

Algorithm 1

- Initialize $\mathcal{S}_0 := \mathcal{CF}_M$.
- Repeatedly construct \mathcal{S}_i from \mathcal{S}_{i-1} by

$$\mathcal{S}_i := \mathcal{S}_{i-1} \cup \{X \cup Y \mid X, Y \in \mathcal{S}_{i-1}^\downarrow : X \text{ and } Y \text{ form a modular pair in } M\}.$$

- If $\mathcal{S}_i = \mathcal{S}_{i-1}$, then \mathcal{S}_i^\downarrow is $\overline{\mathcal{CF}}_M$.

Let M be a matroid on a finite ground set, and let $M = M_0, M_1, M_2, \dots, M_k$ be a sequence of matroids starting from M such that M_i is a non-trivial erection of M_{i-1} for all $1 \leq i \leq k$ and M_k has no non-trivial erection. Since each non-trivial erection increases the rank by one, and the rank is bounded above by $|E|$, the length of any such sequence is bounded. We will refer to the last matroid M_k in such a sequence as an *elevation of M* . The *free elevation of M* is the elevation we obtain from M by taking a maximal sequence of non-trivial free erections.

3 Matroids generated by a set of circuits

Let \mathcal{C} be a family of subsets of a finite set E . A matroid M on E is said to be a \mathcal{C} -matroid if every member of \mathcal{C} is a circuit in M . We will be primarily concerned with matroids constructed by a sequence of free erections from a given matroid. More specifically, let M_0 be a matroid on a finite set E and \mathcal{C}_0 be the family of non-spanning circuits in M_0 . Then every matroid obtained from M_0 by a sequence of erections is a \mathcal{C}_0 -matroid and, in particular, the free elevation of M_0 is a \mathcal{C}_0 -matroid. We will derive some properties of the free elevation of M_0 , which will be crucial tools in the proof of our main theorem.

3.1 Maximality in the weak order

Let \mathcal{C} be a family of subsets of a finite set E . We say that a matroid M is a *maximal* \mathcal{C} -matroid if it is maximal in the weak order on the family of all \mathcal{C} -matroids on E .

Lemma 3.1. *Let M_0 be a matroid on E and \mathcal{C}_0 be the family of all non-spanning circuits in M_0 . Then the free elevation of M_0 is a maximal \mathcal{C}_0 -matroid on E .*

Proof. Let k be the rank of M_0 and M be the free elevation of M_0 . Suppose for a contradiction that $M \prec N$ for some \mathcal{C}_0 -matroid N on E . We first prove:

Claim 3.2. *The truncation of N to rank k is equal to M_0 .*

Proof. Let N_0 be the truncation of N to rank k . The facts that N is a \mathcal{C}_0 -matroid and $M \prec N$ imply that \mathcal{C}_0 is the set of non-spanning circuits of N_0 . This in turn implies that a set $X \subseteq E$ with $|X| = k$ is dependent in M_0 if and only if it is dependent in N_0 , and hence that M_0 and N_0 have the same set of bases. \square

Claim 3.2 implies that M_0 can be obtained from N by a sequence of truncations, and hence that N can be obtained from M_0 by a sequence of erections, say $M_0 = N_0, N_1, \dots, N_\ell = N$. Let $M_0, M_1, \dots, M_m = M$ be the sequence of free erections which construct M from M_0 . Since $N \neq M$, we can choose a smallest possible i such that $N_i \neq M_i$. Then $i \geq 1$, $N_j = M_j$ for all $0 \leq j \leq i - 1$ and N_i is not the free erection of M_{i-1} . Since M_i is the maximum element in the lattice of all erections of M_{i-1} , there is a set $X \subseteq E$ that is dependent in N_i and independent in M_i . The set X remains independent in M but will be dependent in N as $\text{rank}_{N_i}(X) < \text{rank}_{M_i}(X) \leq \text{rank}(M_i) = \text{rank}(N_i)$ and N_i is obtained from N by truncations. This contradicts the hypothesis that $M \prec N$. \square

Since the free erection of a matroid M_0 is the unique maximal element in the lattice of all erections of M_0 , it is tempting to guess that the free elevation of M_0 will be the unique maximal element in the poset of all matroids we can obtain from M_0 by taking sequences of erections. Sadly this is not true in general – counterexamples are given in [3, Figure 7.9] and [25, Theorem 5.4].

3.2 Rank upper bound

We next obtain an upper bound of the rank of any \mathcal{C} -matroid. Given a sequence of circuits (C_1, \dots, C_k) in a matroid M , we put $C_{\leq i} = \bigcup_{j=1}^i C_j$ for all $1 \leq i \leq k$ and $C_{\leq 0} = \emptyset$. The sequence (C_1, \dots, C_k) is said to be *proper* if $C_i \not\subseteq C_{\leq i-1}$ for all $2 \leq i \leq k$, and it is said to be a \mathcal{C} -sequence (for a family of circuits \mathcal{C}) if each

C_i belong to \mathcal{C} . The following lemma is fundamental to our characterization of the rank function of the maximal abstract 3-rigidity matroid.

Lemma 3.3. *Let M be a matroid on a finite set E with rank function r and \mathcal{C} be a family of circuits in M . Then for any $X \subseteq E$ and any proper \mathcal{C} -sequence (C_1, \dots, C_t) ,*

$$r(X) \leq |X \cup C_{\leq t}| - t. \quad (4)$$

Furthermore, if equality holds in (4), then $C_{\leq t} \subseteq \text{cl}(X)$ and each $e \in X \setminus C_{\leq t}$ is a coloop of $M|_X$.

Proof. We first use induction on j to show that, for all $1 \leq j \leq t$,

$$r(C_{\leq j}) \leq \sum_{i=1}^j (|C_i \setminus C_{\leq i-1}| - 1). \quad (5)$$

The base case when $j = 1$ holds since C_1 is a circuit.

Suppose $j > 1$. As the sequence is proper, $C_j \cap C_{\leq j-1}$ is a proper subset of C_j , which is independent. Hence its rank is equal to its cardinality. Thus

$$\begin{aligned} r(C_{\leq j}) &\leq r(C_{\leq j-1}) + r(C_j) - r(C_{\leq j-1} \cap C_j) && \text{(by submodularity)} \\ &= r(C_{\leq j-1}) + |C_j| - 1 - |C_{\leq j-1} \cap C_j| \\ &\leq \sum_{i=1}^j (|C_i \setminus C_{\leq i-1}| - 1) && \text{(by induction)} \end{aligned}$$

and (5) holds.

Putting $j = t$ in (5) gives $r(C_{\leq t}) \leq |C_{\leq t}| - t$. We can now use submodularity and the monotonicity of r to deduce that

$$r(X) \leq r(X \setminus C_{\leq t}) + r(C_{\leq t}) \leq |X \setminus C_{\leq t}| + |C_{\leq t}| - t = |X \cup C_{\leq t}| - t.$$

This completes the proof of the first part of the lemma.

To prove the second part, we assume that $r(X) = |X \cup C_{\leq t}| - t$ for some $X \subseteq E$ and some proper \mathcal{C} -sequence (C_1, \dots, C_t) . If $e \in C_{\leq t}$ then, by the first part of the lemma, $r(X + e) \leq |(X + e) \cup C_{\leq t}| - t = |X \cup C_{\leq t}| - t = r(X)$, and hence $e \in \text{cl}(X)$. Similarly, if $e \in X \setminus C_{\leq t}$ then, by the first part of the lemma, $r(X - e) \leq |(X - e) \cup C_{\leq t}| - t = |X \cup C_{\leq t}| - t - 1 = r(X) - 1$, and hence e is a coloop of $M|_X$. \square

Let M_0 be a matroid on E and \mathcal{C}_0 be the family of non-spanning circuits in M_0 . Then, for any \mathcal{C}_0 -matroid M on E , Lemma 3.3 implies that the function $f_{\mathcal{C}_0} : 2^E \rightarrow \mathbb{Z}$ defined by

$$f_{\mathcal{C}_0}(X) = \min\{|X \cup C_{\leq t}| - t : (C_1, \dots, C_t) \text{ is a proper } \mathcal{C}_0\text{-sequence in } M\} \quad (X \subseteq E)$$

gives an upper-bound for the rank function of M . It follows that, if $f_{\mathcal{C}_0}$ is the rank function of some matroid on E , then this matroid will be the unique maximal \mathcal{C}_0 -matroid and hence will be the free elevation of M_0 by Lemma 3.1. Since the poset of all elevations of M_0 may not have a unique maximal element, $f_{\mathcal{C}_0}$ is not always a matroid rank function. We believe that $f_{\mathcal{C}_0}$ is a matroid rank function whenever there is a unique maximal \mathcal{C}_0 -matroid on E .

Conjecture 3.4. *Let M_0 be a matroid on a finite set E and let \mathcal{C}_0 be the family of non-spanning circuits in M_0 . Suppose that there is a unique maximal \mathcal{C}_0 -matroid on E . Then $f_{\mathcal{C}_0}$ is the rank function of this maximal \mathcal{C}_0 -matroid.*

Our main result verifies Conjecture 3.4 when M_0 is the rank 10 matroid on $E(K_n)$ in which the set of non-spanning circuits \mathcal{C}_0 is the set of copies of K_5 in K_n . (We already showed in [5] that the cofactor matroid $\mathcal{C}_2^1(K_n)$ is the unique maximal K_5 -matroid on $E(K_n)$ and Theorem 5.7 below verifies that $f_{\mathcal{C}_0}$ is its rank function.) The conjecture is verified for several other matroids on $E(K_n)$ in [25].

3.3 A covering lemma

We will use the algorithm for constructing a free elevation given in Section 2 to show that every cyclic flat in the free elevation of a matroid M_0 can be covered by the non-spanning circuits of M_0 (Lemma 3.8 below). This will be a key tool in proving that Conjecture 3.4 holds for the maximal K_5 -matroid on $E(K_n)$.

Recall that \mathcal{CF}_M denotes the family of non-spanning cyclic flats in a matroid M , and that $\overline{\mathcal{CF}}_M$ denotes its modular cyclic closure. We will need the following characterization of the cyclic hyperplanes in the free erection of M .

Lemma 3.5. *Suppose M is a matroid on a finite set E , N is the free erection of M , and $N \neq M$. Let $Z \subseteq E$. Then Z is a cyclic hyperplane in N if and only if Z is a spanning set in M and a maximal element in $\overline{\mathcal{CF}}_M$.*

Proof. Suppose that Z is spanning in M and a maximal element in $\overline{\mathcal{CF}}_M$. Then Z is cyclic in both M and N , and $r_M(Z) = r_N(Z) = r_N(E) - 1$ by (3). It remains to show that Z is a flat in N . To see this, choose a base I of M with $I \subseteq Z$. Then for any $e \notin Z$, $I + e$ contains a circuit C of M with $e \in C$. As $Z \cup C = Z + e$, $Z + e$ is cyclic in M , and the maximality of Z in $\overline{\mathcal{CF}}_M$ now gives $Z + e \notin \overline{\mathcal{CF}}_M$. Hence $r_N(Z + e) > r_N(Z)$ for all $e \notin Z$, which means that Z is a flat in N .

Conversely, suppose that Z is a cyclic hyperplane in N . Then Z is a non-spanning cyclic set in N so $Z \in \overline{\mathcal{CF}}_M$. By the definition of truncation, Z is a spanning set in M . If Z is not maximal in $\overline{\mathcal{CF}}_M$, then there is a maximal set $Y \in \overline{\mathcal{CF}}_M$ with $Z \subsetneq Y$. Then by the first part of the proof, Y would be a hyperplane in N , which is a contradiction since $Z \subsetneq Y$. \square

Our covering lemma for free elevations will follow by recursively applying our next result at each step in the sequence of free erections used to construct a free elevation.

Lemma 3.6. *Let M be a matroid on a finite set E , \mathcal{C} be a family of non-spanning circuits of M , and N be the free erection of M . Suppose that each cyclic flat in M is the union of circuits in \mathcal{C} . Then each cyclic flat in N is the union of circuits in \mathcal{C} .*

Proof. The theorem is trivially true if $M = N$, so we may assume that $M \neq N$. This implies that some spanning circuit of M becomes a base of N . Hence M has no coloops, E is a cyclic flat of M and, by hypothesis, each $e \in E$ belongs to a circuit in \mathcal{C} . We first prove the following.

Claim 3.7. *For each $Z \in \overline{\mathcal{CF}}_M$, there is a $Z' \in \overline{\mathcal{CF}}_M$ such that $Z \subseteq Z'$, $r_M(Z) = r_M(Z')$, and Z' is the union of circuits in \mathcal{C} .*

Proof. Let $\mathcal{S}_0, \dots, \mathcal{S}_k$ be the families of cyclic sets defined in the construction of $\overline{\mathcal{CF}}_M$ from \mathcal{CF}_M in Algorithm 1. Since $Z \in \overline{\mathcal{CF}}_M$, $Z \in \mathcal{S}_j^\downarrow$ for some $0 \leq j \leq k$. We prove that the claim holds for Z by induction on j .

For the base case, we assume that $Z \in \mathcal{S}_0^\downarrow$. If $Z \in \mathcal{S}_0 = \mathcal{CF}_M$, then Z is a cyclic flat of M and hence is the union of circuits in \mathcal{C} by our hypothesis on M . On the other hand, if $Z \in \mathcal{S}_0^\downarrow \setminus \mathcal{S}_0$, then $\text{cl}_M(Z) \in \mathcal{S}_0$ and $Z' = \text{cl}_M(Z)$ is the desired set for Z .

Suppose $Z \in \mathcal{S}_j^\downarrow \setminus \mathcal{S}_{j-1}^\downarrow$ for some $j \geq 1$. We first consider the case when $Z \in \mathcal{S}_j$. Then, by Algorithm 1, $Z = A \cup B$ for some $A, B \in \mathcal{S}_{j-1}^\downarrow$ which form a modular pair in M . By induction on j , there exist $A', B' \in \overline{\mathcal{CF}}_M$ with $A \subseteq A', B \subseteq B', r_M(A) = r_M(A')$ and $r_M(B) = r_M(B')$, such that A' and B' are both the union of sets in \mathcal{C} . We claim that A' and B' form a modular pair in M . This follows since

$$\begin{aligned} r_M(A) + r_M(B) &= r_M(A') + r_M(B') && \text{(by } r_M(A) = r_M(A') \text{ and } r_M(B) = r_M(B')) \\ &\geq r_M(A' \cup B') + r_M(A' \cap B') && \text{(by submodularity)} \\ &\geq r_M(A \cup B) + r_M(A \cap B) && \text{(by } A \subseteq A', B \subseteq B') \\ &= r_M(A) + r_M(B) && \text{(by modularity)} \end{aligned}$$

and hence equality must hold in each inequality. Since $\overline{\mathcal{CF}}_M$ is closed under the union of modular pairs, this gives $A' \cup B' \in \overline{\mathcal{CF}}_M$. The fact that equality holds throughout the above inequality also implies that $r_M(A' \cup B') + r_M(A' \cap B') = r_M(A \cup B) + r_M(A \cap B)$. This and the monotonicity of r_M imply that $r_M(A' \cup B') = r_M(A \cup B) = r_M(Z)$. Thus $Z' = A' \cup B'$ is the desired set for Z .

It remains to consider the case when $Z \in \mathcal{S}_j^\downarrow \setminus (\mathcal{S}_j \cup \mathcal{S}_{j-1}^\downarrow)$. Then $\text{cl}_M(Z) \notin \mathcal{CF}_M$ so Z is a spanning set in M . Choose a maximal element \tilde{Z} of \mathcal{S}_j^\downarrow with $Z \subseteq \tilde{Z}$. Then $\tilde{Z} \in \mathcal{S}_j$, and hence there is a set \tilde{Z}' for \tilde{Z} by the preceding paragraph. Then $r_M(\tilde{Z}') = r_M(\tilde{Z})$ since \tilde{Z} is spanning in M and $Z' = \tilde{Z}'$ is the desired set for Z . This completes the proof of the claim. \square

The claim implies, in particular, that every maximal element in $\overline{\mathcal{CF}}_M$ is the union of circuits in \mathcal{C} .

We can now complete the proof of the lemma. Choose a cyclic flat Z of N . If Z is spanning in N , then $Z = E$ and the lemma follows since each element in E is contained in a circuit in \mathcal{C} . So we may assume that Z is not spanning in N . Then $N|_Z = M|_Z$ follows as M is the truncation of N . As Z is cyclic in N , Z is also cyclic in M . If $r_N(Z) < r_M(E)$, then Z is a non-spanning cyclic flat in M and the claim follows from the hypothesis on M . So we may further assume that $r_N(Z) = r_M(E) = r_N(E) - 1$. Then Z is a hyperplane in N . Lemma 3.5 now implies that Z is a maximal element in $\overline{\mathcal{CF}}_M$, and hence Z is the union of circuits in \mathcal{C} by Claim 3.7. This completes the proof of the lemma. \square

Lemma 3.6 and the fact that non-spanning circuits are preserved when we construct a free erection immediately give:

Lemma 3.8. *Let M be a matroid on a finite set E , \mathcal{C} be a family of non-spanning circuits of M , and N be the free elevation of M . Suppose that each cyclic flat in M is the union of circuits in \mathcal{C} . Then each cyclic flat in N is the union of circuits in \mathcal{C} .*

4 K_{d+2} -matroids and abstract d -rigidity

We will apply the preceding theory to abstract d -rigidity matroids. We will see that abstract d -rigidity matroids are K_{d+2} -matroids, i.e. matroids on $E(K_n)$ in which the edge set of every copy of K_{d+2} is a circuit, and show that they have maximum possible rank over all such matroids. We will then give two important examples of abstract d -rigidity matroids, generic d -dimensional rigidity matroids and C_{d-1}^{d-2} -cofactor matroids, and describe some related results and conjectures.

Let $G = (V, E)$ be a graph. For $X \subseteq V$, let $G[X]$ be the subgraph of G induced by X . For $F \subseteq E$, let $V(F)$ be the set of vertices incident to F , and let $G[F] = (V(F), F)$. For $v \in V$, let $N_G(v)$ be the set of neighbors of v in G , and let $d_G(v) = |N_G(v)|$. For $X = \{v_1, \dots, v_k\} \subseteq V$, let $K(X)$ or $K(v_1, \dots, v_k)$ be the edge set of the complete graph on X .

Given a set F in a matroid M defined on $E(K_n)$, we will often abuse notation and use the same letter for both the set F and the graph $K_n[F]$. It will be clear from the context whether we are referring to an edge set or a graph. In addition, we will refer to bases of $M|_F$ as *bases of F* .

We first use Lemma 3.3 to obtain an upper bound on the rank of any K_{d+2} -matroid. We say that a sequence (C_1, \dots, C_t) of edge sets in K_n is a K_t -sequence if each C_i induces a copy of K_t in K_n .

Lemma 4.1. *Let M be a K_{d+2} -matroid on the edge set of the complete graph K_n with $n \geq d + 2$. Then its rank is at most $dn - \binom{d+1}{2}$.*

Proof. It will suffice to construct a proper K_{d+2} -sequence which covers K_n of length $\binom{n-d}{2}$ since Lemma 3.3 will then give $r(M) \leq \binom{n}{2} - \binom{n-d}{2} = dn - \binom{d+1}{2}$. To this end, we let $K_n = K(v_1, v_2, \dots, v_n)$ and construct a proper K_{d+2} -sequence \mathcal{K}_i of length $\binom{i-d}{2}$ which covers $K(v_1, v_2, \dots, v_i)$ for all $d + 2 \leq i \leq n$, recursively. We first put $\mathcal{K}_{d+2} = (K(v_1, v_2, \dots, v_{d+2}))$. Then, for each $d + 2 \leq i \leq n - 1$, we construct a proper K_{d+2} -sequence \mathcal{K}_{i+1} which covers $K(v_1, \dots, v_{i+1})$ by choosing a set S_i of d vertices in $K(v_1, v_2, \dots, v_i)$, putting $\mathcal{L}_{i+1} = (K(S_i + v_j + v_{i+1}) : 1 \leq j \leq i \text{ and } j \notin S_i)$, and then putting $\mathcal{K}_{i+1} = (\mathcal{K}_i, \mathcal{L}_{i+1})$. It is straightforward to check that \mathcal{K}_{i+1} has length $\binom{i+1-d}{2}$ and is proper (for any ordering of the K_{d+2} 's in each subsequence \mathcal{L}_j). \square

The graphic matroid of K_n and the rank-two uniform matroid on $E(K_n)$ are examples of K_3 -matroids. In addition, the graphic matroid achieves the upper bound on the rank given by Lemma 4.1.

Graver [19] defined an *abstract d -rigidity matroid* to be a matroid on the edge set of the complete graph K_n which satisfies the following two axioms:

- (A1) If $E_1, E_2 \subseteq E(K_n)$ with $|V(E_1) \cap V(E_2)| \leq d - 1$, then $\text{cl}_M(E_1 \cup E_2) \subseteq K(V(E_1)) \cup K(V(E_2))$;
- (A2) If $E_1, E_2 \subseteq E(K_n)$ with $\text{cl}_M(E_1) = K(V(E_1))$, $\text{cl}_M(E_2) = K(V(E_2))$, and $|V(E_1) \cap V(E_2)| \geq d$, then $\text{cl}_M(E_1 \cup E_2) = K(V(E_1 \cup E_2))$.

These two conditions reflect two fundamental rigidity properties of generic d -dimensional bar-joint frameworks. (See, for example, [20] for more details.) Nguyen [36] obtained a simple characterization of abstract d -rigidity matroids which immediately implies that they are special kinds of K_{d+2} -matroids.

Theorem 4.2 (Nguyen[36, Theorem 2.2]). *Let n, d be positive integers with $n \geq d + 2$ and M be a matroid on $E(K_n)$. Then M is an abstract d -rigidity matroid if and only if M is a K_{d+2} -matroid with rank $dn - \binom{d+1}{2}$.*

Lemma 4.1 and Theorem 4.2 imply that abstract d -rigidity matroids are precisely the K_{d+2} -matroids on $E(K_n)$ which attain the maximum possible rank.

4.1 Two fundamental examples

We describe two examples of abstract d -rigidity matroids which have been extensively studied in the literature.

Generic rigidity matroids. A d -dimensional framework is a pair (G, p) consisting of a graph $G = (V, E)$ and a map $p : V \rightarrow \mathbb{R}^d$. An *infinitesimal motion* of (G, p) is a map $q : V \rightarrow \mathbb{R}^d$ such that

$$(p(u) - p(v)) \cdot (q(u) - q(v)) = 0 \quad (uv \in E), \tag{6}$$

where \cdot denotes the Euclidean inner product. The *rigidity matrix* $R(G, p)$ of (G, p) is the matrix representing the linear system of equations (6) in the variables q . Specifically, $R(G, p)$ is a matrix of size $|E| \times d|V|$ in which each row is indexed by an edge, sets of d consecutive columns are indexed by the vertices, and the row indexed by the edge $e = uv$ has the form:

$$e=uv \left[\begin{array}{cccc} 0 \cdots 0 & \overset{u}{p(u) - p(v)} & 0 \cdots 0 & \overset{v}{p(v) - p(u)} & 0 \cdots 0 \end{array} \right].$$

The framework (G, p) is said to be *infinitesimally rigid* if $\text{rank} R(G, p) = dn - \binom{d+1}{2}$.

The *rigidity matroid* of (G, p) is the matroid on E in which a set of edges is independent if the corresponding rows of $R(G, p)$ are linearly independent. The rigidity matroid of (G, p) is the same for any generic $p : V \rightarrow \mathbb{R}^d$ and we refer to this matroid as the *generic d -dimensional rigidity matroid* $\mathcal{R}_d(G)$ of G . The generic d -dimensional rigidity matroid of the complete graph K_n is called the *generic d -dimensional rigidity matroid* (on n vertices) and is denoted by $\mathcal{R}_{d,n}$. A graph $G = (V, E)$ with n vertices is said to be *rigid in \mathbb{R}^d* if E is a spanning set of $\mathcal{R}_{d,n}$ or equivalently, when $n \geq d$, if the rank of E is $dn - \binom{d+1}{2}$. Combinatorial characterizations of graphs which are rigid in \mathbb{R}^d exist for $d = 1, 2$. Obtaining an analogous characterization for $d \geq 3$ is the central open problem in graph rigidity.

Generic cofactor matroids. Let $G = (V, E)$ be a graph and $p : V \rightarrow \mathbb{R}^2$ such that $p(v_i) = (x_i, y_i)$ for all $v_i \in V$. We assume that the vertices are ordered as v_1, v_2, \dots, v_n . As given in the introduction, the C_s^{s-1} -cofactor matrix of (G, p) , denoted by $C_s^{s-1}(G, p)$, is a matrix of size $|E| \times (s+1)|V|$ in which each row is indexed by an edge, sets of $(s+1)$ consecutive columns are indexed by the vertices, and the row associated to the edge $e = v_i v_j$ with $i < j$ is

$$e=v_i v_j \left[\begin{array}{cccc} 0 \cdots 0 & \overset{v_i}{D_{ij}} & 0 \cdots 0 & \overset{v_j}{-D_{ij}} & 0 \cdots 0 \end{array} \right],$$

where $D_{i,j} = ((x_i - x_j)^s, (x_i - x_j)^{s-1}(y_i - y_j), \dots, (x_i - x_j)(y_i - y_j)^{s-1}, (y_i - y_j)^s) \in \mathbb{R}^{s+1}$. (Our definition is slightly different to that given by Whiteley [48], but the two definitions are equivalent up to elementary column operations.) When $s \geq 1$, the space $S_s^{s-1}(\Delta)$ of bivariate C_s^{s-1} -splines over Δ is linearly isomorphic

to the left kernel of $C_s^{s-1}(G, p)$ if (G, p) is the 1-skeleton of a subdivision Δ of a polygonal domain in the plane, see, e.g., [48] for more details. Note that, when $s = 0$, $D_{i,j} \equiv 1$ and $C_s^{s-1}(G, p)$ is just the edge/vertex incidence matrix of G .

The generic C_s^{s-1} -cofactor matroid $\mathcal{C}_{s,n}^{s-1}$ is defined to be the row matroid of $C_s^{s-1}(K_n, p)$ for any generic p . Whiteley [48] showed that $\mathcal{C}_{d-1,n}^{d-2}$ is an abstract d -rigidity matroid for all $d \geq 1$. It follows immediately from the definitions of the rigidity and cofactor matrices that $\mathcal{C}_{d-1,n}^{d-2} = \mathcal{R}_{d,n}$ when $d = 1, 2$. Whiteley [48] showed that that $\mathcal{C}_{d-1,n}^{d-2} \neq \mathcal{R}_{d,n}$ when $d \geq 4$ and $n \geq 2d + 2$ and conjectured that the two matroids are equal when $d = 3$.

4.2 Maximality conjectures

As noted above, abstract d -rigidity matroids are K_{d+2} -matroids which attain the maximum possible rank. Graver [19] conjectured further that the generic d -dimensional rigidity matroid is the unique maximal matroid in the weak order poset of all abstract d -rigidity matroids on $E(K_n)$, and verified his conjecture when $d = 1, 2$. N. J. Thurston (see, [20, page 150]) subsequently showed that Graver’s conjecture is false when $d \geq 4$. The conjecture remains as a long-standing open problem for $d = 3$.

Conjecture 4.3 (Graver’s maximality conjecture [19]). *The generic 3-dimensional rigidity matroid is the unique maximal abstract 3-rigidity matroid on $E(K_n)$.*

Whiteley [48] developed a theory for C_{d-1}^{d-2} -cofactor matroids which is analogous to that for d -dimensional rigidity matroids. In particular, he provided further counterexamples to Graver’s maximality conjecture by noting that the edge set of the complete bipartite graph $K_{d+2,d+2}$ is dependent in $\mathcal{R}_{d,n}$ for all $d \geq 1$ but independent in $\mathcal{C}_{d-1,n}^{d-2}$ when $d \geq 4$. This led him to make the following modified conjecture.

Conjecture 4.4 (Whiteley’s maximality conjecture [48, Conjecture 11.5.1]). *The generic C_{d-1}^{d-2} -cofactor matroid $\mathcal{C}_{d-1,n}^{d-2}$ is the unique maximal abstract d -rigidity matroid on $E(K_n)$ for all $d \geq 1$.*

The conjecture holds for $d = 1, 2$, since Graver’s conjecture holds for $d = 1, 2$ and the generic cofactor and rigidity matroids are the same for these values of d . The main result of our first paper in this series [5] verifies Whiteley’s maximality conjecture when $d = 3$.

Theorem 4.5. *The generic C_2^1 -cofactor matroid $\mathcal{C}_{2,n}^1$ is the unique maximal K_5 -matroid on $E(K_n)$.*

Theorem 4.5 is stronger than Whiteley’s conjecture since it holds for the larger class of K_5 -matroids. We can use a similar proof technique to that given by Graver to show that analogous results hold when $d = 1, 2$: the generic 1-dimensional rigidity matroid is the unique maximal K_3 -matroid and the generic 2-dimensional rigidity matroid is the unique maximal K_4 -matroid on $E(K_n)$. Whiteley’s conjecture, and the corresponding strengthening to the larger class of K_{d+2} -matroids, remain open for all $d \geq 4$.

4.3 Inductive constructions

Inductive constructions are frequently used to solve problems in rigidity. The most common operations in this context are k -extensions, which are defined as follows. Given integers k and d with $0 \leq k \leq d$,

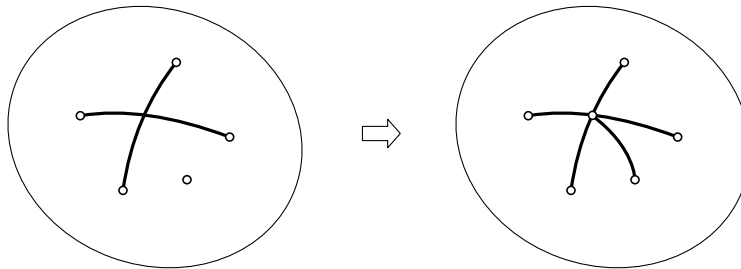


Figure 1: An example of 3-dimensional 2-extension (X-replacement).

a d -dimensional k -extension of a graph G constructs a new graph by deleting k edges from G and then adding a new vertex v_0 and $d + k$ new edges incident to v_0 , with the proviso that each end-vertex of a deleted edge is a neighbor of v_0 . See Figure 1 for an example.

It is well-known that, if G is rigid in \mathbb{R}^d and $k = 0, 1$, then any graph obtained from G by a k -extension is rigid in \mathbb{R}^d , see for example [48]. This fact was used by Pollaczek-Geiringer [38] to show that Maxwell’s necessary condition for rigidity (given in the introduction) is also sufficient in dimension two.

The situation becomes more complicated when $k \geq 2$. In particular, we can distinguish two types of 2-extensions, depending on whether or not the two deleted edges are adjacent: a 2-extension is called a V -replacement if the deleted edges are adjacent, and otherwise is called an X -replacement. See Figure 1. It is conjectured that 3-dimensional X -replacement preserves rigidity in \mathbb{R}^3 .

Conjecture 4.6 (X -replacement conjecture [45]). *Suppose that G and H are graphs and that H can be obtained from G by a 3-dimensional X -replacement. If G is rigid in \mathbb{R}^3 , then H is rigid in \mathbb{R}^3 .*

See [9] for more details about this conjecture. Maehara [32] pointed out that the analogous statement for 4-dimensional X -replacement does not hold in general.

The d -dimensional k -extension operation can be applied to any graph and hence can be used to investigate independence properties of any matroid M defined on the edge set of K_n . We say that M has the d -dimensional k -extension property if every edge set obtained from an independent set by a d -dimensional k -extension operation remains independent.

A basic fact about abstract d -rigidity matroids is that any abstract d -rigidity matroid has the d -dimensional 0-extension property [20]. As noted above, the generic d -dimensional rigidity matroid also has the d -dimensional 1-extension property. For C_{d-1}^{d-2} -cofactor matroids, Whiteley [48, Theorem 11.4.1] proved the following:

Theorem 4.7. *The generic C_{d-1}^{d-2} -cofactor matroid has the d -dimensional 1-extension property for all $d \geq 1$, and the d -dimensional X -replacement property for all $d \geq 2$.*

Theorem 4.7 is an important ingredient in our proof of Theorem 4.5. The fact that the generic 3-dimensional rigidity matroid is not known to have the 3-dimensional X -replacement property is a major barrier to applying the same proof technique to solve Graver’s Maximality Conjecture.

5 Combinatorial characterization

We saw in the last section that $\mathcal{C}_{2,n}^1$ is the unique maximal abstract 3-rigidity matroid on $E(K_n)$. In this section, we obtain a co-NP type characterization of its rank function. Since $\mathcal{C}_{2,n}^1$ is the free matroid on $E(K_n)$ when $n \leq 4$, we assume throughout this section that $n \geq 5$.

Theorem 4.2 implies that the rank 10 truncation of any abstract 3-rigidity matroid on $E(K_n)$ is the matroid R_n on $E(K_n)$ of rank 10 in which the non-spanning circuits are the edge sets of the copies of K_5 in K_n . We can use this observation, together with other results from Sections 3 and 4 to derive two fundamental properties of $\mathcal{C}_{2,n}^1$.

Theorem 5.1. *The generic C_2^1 -cofactor matroid $\mathcal{C}_{2,n}^1$ is the free elevation of R_n .*

Proof. Theorem 4.5 tells us that the generic cofactor matroid $\mathcal{C}_{2,n}^1$ is the unique maximal K_5 -matroid on $E(K_n)$. Lemma 3.1 now implies that $\mathcal{C}_{2,n}^1$ is the free elevation of R_n . \square

Corollary 5.2. *Every cyclic flat of $\mathcal{C}_{2,n}^1$ is the union of copies of K_5 .*

Proof. This follows from Theorem 5.1, Lemma 3.8, and the facts that each non-spanning cyclic flat of R_n is a copy of K_5 and $E(K_n)$ can be covered by copies of K_5 . \square

5.1 Matroids on the edge set of a graph

Our derivation of the rank function of $\mathcal{C}_{2,n}^1$ uses the property that each of its cyclic flats has a base which induces a subgraph of K_n of minimum degree at most four. We will deduce this property from the more general results given in this subsection. We first need to introduce the matroidal concepts of connectivity and ear decomposition.

A matroid M with ground set E is said to be *connected* if, for every pair $e, f \in E$ with $e \neq f$, M has a circuit containing both e and f . A set $X \subseteq E$ is said to be *connected* if $M|_X$ is connected. We can define an equivalence relation on X by saying that $e_1, e_2 \in X$ are related if either $e_1 = e_2$ or there is a circuit of M in X which contains both e_1 and e_2 . The corresponding equivalence classes X_1, X_2, \dots, X_k are the maximal connected subsets of X . We will refer to these equivalence classes as the *connected components* of X in M . We have $X_i \cap X_j = \emptyset$ for all $1 \leq i < j \leq k$ and $r(X) = \sum_{i=1}^k r(X_i)$.

A *partial ear decomposition* of M is a sequence (C_1, C_2, \dots, C_t) of circuits of M such that, for all $2 \leq i \leq t$:

- $C_i \cap C_{\leq i-1} \neq \emptyset$;
- $C_i \setminus C_{\leq i-1} \neq \emptyset$;
- if some circuit C of M satisfies $C \cap C_{\leq i-1} \neq \emptyset \neq C \setminus C_{\leq i-1}$ then $C \setminus C_{\leq i-1}$ is not a proper subset of $C_i \setminus C_{\leq i-1}$.

An *ear decomposition* of M is a partial ear decomposition with the additional property that $C_{\leq t} = E$. More generally, an *ear decomposition* of a set $X \subseteq E$ is a partial ear decomposition with the additional property that $C_{\leq t} = X$. Ear decompositions of matroids were introduced by Coullard and Hellerstein [6]. They showed:

Theorem 5.3. *A matroid M with at least two elements is connected if and only if it has an ear decomposition. Furthermore, if M is connected, then any partial ear decomposition of M can be extended to an ear decomposition of M .*

We will use an ear decomposition of a connected set X in a matroid M defined on the edge set of a graph G to show that, subject to a mild condition on the circuits of M , some base of X induces a subgraph of G of low minimum degree. More precisely, we use induction on the number of circuits in the ear decomposition to show that the average minimum degree over all bases of X is at most $2 \frac{r(X)+1}{|V(X)|} - 1$. For the inductive step we will need the following result which follows easily from the definition of an ear decomposition.

Lemma 5.4. *Let (C_1, C_2, \dots, C_t) be a partial ear decomposition in a matroid M . Then $r(C_{\leq i}) = r(C_{\leq i-1}) + |C_i \setminus C_{\leq i-1}| - 1$ for all $2 \leq i \leq t$. Furthermore, if B_{i-1} is any base of $C_{\leq i-1}$ and Y is any subset of $C_i \setminus C_{\leq i-1}$ of size $|C_i \setminus C_{\leq i-1}| - 1$ then $B_i = B_{i-1} \cup Y$ is a base of $C_{\leq i}$.*

Lemma 5.5. *Let M be a matroid defined on the edge set of a graph $G = (V, E)$ and $X \subseteq E$ be a connected set in M . Suppose that every circuit of M induces a 2-connected subgraph of G . Then*

$$f(X) := \sum_{v \in V(X)} \min\{d_B(v) : B \text{ is a base of } X\} \leq 2(r(X) + 1) - |V(X)|.$$

Proof. We use induction on $|X|$. The lemma holds when X is a coloop since we have $r(X) = |X| = 1$ and $|V(X)| \leq 2$. Hence we may suppose that $|X| \geq 2$ so X is a cyclic set in M . Let $n = |V(X)|$.

We first consider the case when X is a circuit of M . Then $r(X) = |X| - 1$. For each $v \in V(X)$, we can construct a base of X which contains $d_X(v) - 1$ edges incident to v , so the contribution of v to $f(X)$ is $d_X(v) - 1$. Hence

$$f(X) = \sum_{v \in V(X)} (d_X(v) - 1) = 2|X| - n = 2(r(X) + 1) - n$$

as required. Thus we may assume that X is not a circuit.

Let (C_1, C_2, \dots, C_m) be an ear decomposition of X and put $Z = C_m \setminus C_{\leq m-1}$. By Lemma 5.4, $r(X) = r(C_{\leq m-1}) + |Z| - 1$ holds and, for every base B' of $C_{\leq m-1}$ and every subset Y of Z of cardinality $|Z| - 1$, $B = B' \cup Y$ is a base of X .

We will obtain an upper bound on $f(X)$ by considering the separate contributions of the vertices in $V_1 = V(C_{\leq m-1})$ and $V_2 = V(X) \setminus V_1$. To this end, we let $n_1 = |V_1|$, $n_2 = |V_2|$, Z_1 and Z_2 be the sets of edges in Z which belong to the subgraphs of X induced by V_1 and V_2 , respectively, and put $Z_{1,2} = Z \setminus (Z_1 \cup Z_2)$.

We first consider the contribution of the vertices in V_1 to $f(X)$. By Theorem 5.3 and Lemma 5.4, $C_{\leq m-1}$ is connected in M and $r(C_{\leq m-1}) = r(X) - |Z| + 1$. Hence, by induction,

$$f(C_{\leq m-1}) \leq 2(r(X) - |Z| + 2) - n_1. \tag{7}$$

For each $v \in V_1$ which is incident to an edge of Z and each base B' of $C_{\leq m-1}$, we can construct a base B of X with $d_B(v) = d_{B'}(v) + d_Z(v) - 1$. This implies that the contribution of the vertices of V_1 to $f(X)$ is at most

$$f(C_{\leq m-1}) + 2|Z_1| + |Z_{1,2}| - |V(Z) \cap V_1|. \tag{8}$$

We next consider the contribution of the vertices of V_2 to $f(X)$. Choose $v \in V_2$. Then v is only incident to edges of Z in X . Since we can construct a base of X which contains $d_X(v) - 1$ edges incident to v , the contribution of v to $f(X)$ is $d_X(v) - 1$. So the total contribution of the vertices of V_2 to $f(X)$ is

$$\sum_{v \in V_2} (d_X(v) - 1) = 2|Z_2| + |Z_{1,2}| - n_2. \tag{9}$$

Note that the previous sentence remains valid when $V_2 = \emptyset$ since the contribution given by (9) is zero in this case.

We may now combine (7), (8) and (9) with $|Z| = |Z_1| + |Z_2| + |Z_{12}|$ and $n = n_1 + n_2$ to obtain

$$\begin{aligned} f(X) &\leq 2(r(X) - |Z| + 2) - n_1 + 2|Z_1| + |Z_{1,2}| - |V(Z) \cap V_1| + 2|Z_2| + |Z_{1,2}| - n_2 \\ &= 2r(X) - |V(Z) \cap V_1| - n + 4. \end{aligned}$$

Finally we use the hypothesis that the subgraph of G induced by a circuit is 2-connected. This and $C_m \cap C_{\leq m-1} \neq \emptyset$ imply that $|V(Z) \cap V_1| \geq 2$, and we obtain $f(X) \leq 2(r(X) + 1) - n$ as required. \square

Suppose M is a K_{d+2} -matroid with $d \geq 2$. Then Lemma 4.1 implies that $r_M(X) \leq d|V(X)| - \binom{d+1}{2}$ for any connected set X in M and we can now use Lemma 5.5 to deduce that X has a base of minimum degree at most $2d - 2$. We next extend this observation to the case when M is an abstract d -rigidity matroid and X is cyclic but not necessarily connected.

Lemma 5.6. *Suppose M is an abstract d -rigidity matroid defined on $E(K_n)$ for some $d \geq 2$ and X is a nonempty cyclic set in M . Then every base of X has minimum degree at least d , and some base of X has minimum degree at most $2d - 2$.*

Proof. Since M is an abstract d -rigidity matroid, the 0-extension property holds for M and hence each circuit of M induces a subgraph of minimum degree at least $d + 1$. Since X is cyclic, this implies that each vertex of $K_n[X]$ has degree at least $d + 1$.

Suppose $d_B(v) \leq d - 1$ for some base B of X and some $v \in V(X)$. Then we can use the 0-extension property to extend $B - v$ to an independent set of size $|B| + 1$ in X by adding d edges incident to v , contradicting the fact that B is a base of X . Hence $d_B(v) \geq d$ for all bases B of X and all $v \in V(X)$.

It remains to show that some base of X has minimum degree at most $2d - 2$. Let X_1, X_2, \dots, X_q be the connected components of X in M . Then each X_i is a cyclic set in M , $X_i \cap X_j = \emptyset$ for $i \neq j$ and, for any set of bases B_i of X_i , $B = \bigcup_{i=1}^q B_i$ is a base of X . For $v \in V(X)$, let $a(v)$ be the number of components X_i for which $v \in V(X_i)$. Let $U = \{v \in V(X) : a(v) \geq 2\}$ and let $b(X_i) = |U \cap V(X_i)|$ for all $1 \leq i \leq q$. Let $n = |V(X)|$ and $n_i = |V(X_i)|$ for all $1 \leq i \leq q$. Then $\sum_{i=1}^q n_i = n + \sum_{u \in U} (a(u) - 1)$ and $\sum_{u \in U} a(u) = \sum_{i=1}^q b(X_i)$. We may apply Lemmas 4.1 and 5.5 to each X_i to deduce that

$$f(X_i) \leq (2d - 1)n_i - (d - 1)(d + 2). \tag{10}$$

Suppose that $b(X_i) \leq d + 1$ for some $1 \leq i \leq q$. We can use the argument in the first paragraph to deduce that $d_{B_i}(v) \geq d$ for all bases B_i of X_i and all $v \in V(X_i)$. Together with (10) and the hypothesis that $d \geq 2$, this implies that there are at least $d + 2$ vertices $v \in V(X_i)$ such that v has degree at most $2d - 2$ in some base of X_i . Since $b(X_i) \leq d + 1$, we can choose $v \in V(X_i) \setminus U$ such that v has degree at most $2d - 2$

in some base B_i of X_i . We can now extend B_i to a base B of X . Since $v \notin U$, v will have degree at most $2d - 2$ in B .

Hence we may assume that $b(X_i) \geq d + 2$ for all $1 \leq i \leq q$. Since the union of any collection of bases of the X_i gives a base of X , $f(X) = \sum_{i=1}^q f(X_i)$. We may now use Lemma 5.5, (10) and the fact that $\sum_{i=1}^q n_i = n + \sum_{u \in U} (a(u) - 1)$ to deduce that

$$\begin{aligned} (2d - 1)n - f(X) &= (2d - 1)n - \sum_{i=1}^q f(X_i) \\ &= \sum_{i=1}^q ((2d - 1)n_i - f(X_i)) - (2d - 1) \sum_{u \in U} (a(u) - 1) \\ &\geq 2 \sum_{i=1}^q (dn_i - r(X_i) - 1) - (2d - 1) \sum_{u \in U} (a(u) - 1) \\ &= 2(dn - r(X) - q) + \sum_{u \in U} (a(u) - 1) \\ &= 2(dn - r(X)) - 2q - |U| + \sum_{u \in U} a(u). \end{aligned}$$

Since $a(u) \geq 2$ for all $u \in U$ and $b(X_i) \geq d + 2 \geq 5$ for all $1 \leq i \leq q$, we have $\sum_{u \in U} a(u) \geq 2|U|$ and $\sum_{u \in U} a(u) = \sum_{i=1}^q b(X_i) \geq 5q$. Since $r(X) < dn$ by Lemma 4.1, we have $(2d - 1)n - f(X) > 0$ and $f(X) < (2d - 1)n$. Hence some base of X has minimum degree at most $2d - 2$. \square

The bound on the minimum degree of a base in Lemma 5.6 is best possible since the generic d -dimensional rigidity matroid contains circuits of minimum degree $2d - 1$ when $d \geq 2$. We will see in the next subsection, however, that a stronger bound holds for cyclic flats in $\mathcal{C}_{2,n}^1$.

5.2 The rank function of $\mathcal{C}_{2,n}^1$

Theorem 5.1 and Lemma 3.3 tell us that we can bound the rank of any set X in $\mathcal{C}_{2,n}^1$ by using proper K_5 -sequences. More precisely, Theorem 5.1 tells us that $\mathcal{C}_{2,n}^1$ is the free-elevation of the rank 10 matroid on $E(K_n)$ whose non-spanning circuits are the edge sets of all copies of K_5 , and Lemma 3.3 now implies that the rank of a set $X \subseteq E(K_n)$ in $\mathcal{C}_{2,n}^1$ is bounded above by $|X \cup C_{\leq t}| - t$ for any proper K_5 -sequence (C_1, \dots, C_t) . Our next result shows that this bound is tight for some proper K_5 -sequence in K_n . It also establishes an important structural property of cyclic flats in $\mathcal{C}_{2,n}^1$, that they always contain a *simplicial vertex* i.e. a vertex whose neighbor set induces a complete subgraph. Its proof uses a simultaneous induction on both statements.

To simplify notation we put

$$\text{val}(F, \mathcal{C}) := |F \cup C_{\leq t}| - t$$

for all $F \subseteq E(K_n)$ and all proper K_5 -sequences $\mathcal{C} = (C_1, \dots, C_t)$ contained in K_n .

Theorem 5.7. *Suppose $X \subseteq E(K_n)$.*

(a) *The rank of X in $\mathcal{C}_{2,n}^1$ is given by*

$$r(X) = \min\{|X \cup C_{\leq t}| - t : (C_1, \dots, C_t) \text{ is a proper } K_5\text{-sequence in } K_n\}.$$

(b) If X is a cyclic flat in $\mathcal{C}_{2,n}^1$, then there exists a vertex $v \in V(X)$ such that $K(N_X(v)) \subseteq X$, and v has degree three in some base of X .

Proof. We proceed by contradiction. Suppose that the theorem does not hold for some $X \subseteq E(K_n)$. We may suppose that X has been chosen such that $r(X)$ is as small as possible and, subject to this condition, that $|X|$ is as large as possible.

We first show that (b) holds for X when X is a cyclic flat. Choose a vertex $v \in V(X)$ and a base B of X such that $d_B(v)$ is as small as possible. Then $d_B(v) \in \{3, 4\}$ by Lemma 5.6. Let $N_X(v) = \{v_1, \dots, v_k\}$. Note that $k \geq 4$ since X is cyclic and all circuits of $\mathcal{C}_{2,n}^1$ have minimum degree at least four.

Suppose that $d_B(v) = 3$. If $K(N_X(v)) \not\subseteq X$, then relabelling if necessary, we have $e = v_1v_2 \in K(N_X(v)) \setminus X$, and $B - v + e$ is independent since X is a flat. Then $B' = (B - v) \cup \{vv_1, vv_2, vv_3, vv_4\}$ is also independent since it can be obtained from $B - v + e$ by a 1-extension. This contradicts the choice of B , since $|B'| > |B|$ and $B' \subseteq X$. Hence $K(N_X(v)) \subseteq X$ and (b) holds for X .

It remains to consider the case when $d_B(v) = 4$. Then v has degree at least four in all bases of X . We will refer to any base B with the property that $d_B(v) = 4$ as a v -admissible base of X . We will show that this case cannot occur by first showing that v is a simplicial vertex in X and then deducing that v has degree three in some base of X .

Claim 5.8. $r(X - v) = r(X) - 4$.

Proof. The fact that X has a v -admissible base implies that $r(X - v) \geq r(X) - 4$. Let B' be a base of $X - v$. If $|B'| > r(X) - 4$, then B' would extend to a base of X in which v has degree three and would contradict our assumption that v has degree at least four in all bases of X . Hence $|B'| = r(X - v) = r(X) - 4$. \square

As X is a cyclic flat, Corollary 5.2 tells us that X is the union of copies of K_5 . Hence we may choose $S_1 \subseteq X$ such that S_1 induces a copy of K_5 which contains v . We may assume that $V(S_1) = \{v, v_1, v_2, v_3, v_4\}$.

Claim 5.9. *There exists $Y \subseteq X$ such that $d_Y(v) = 5$, $S_1 \subseteq Y$ and $Y - vv_i$ is a v -admissible base of X for all $1 \leq i \leq 4$.*

Proof. Choose a v -admissible base B of X . We will use the term *base exchange on B* to mean the operation which constructs a new base $B + e - f$ by first adding an edge $e \in X \setminus B$ to B and then deleting an edge f from the fundamental circuit of $B + e$. If $vv_i \in B$ for all $1 \leq i \leq 4$ then, since $d_B(v) = 4$, we could perform a sequence of base exchanges using the edges of S_1 to construct a base of X in which v has degree less than four. Hence we may assume that $vv_4 \notin B$. Since $d_B(v) = 4$ and every circuit of $\mathcal{C}_{2,n}^1$ has minimum degree at least four, we can also use a sequence of base exchanges to ensure that $vv_i \in B$ for all $1 \leq i \leq 3$. We can now perform a further sequence of base exchanges using the edges of S_1 to ensure that $S_1 - vv_4 \subseteq B$. Let $Y = B + vv_4$. Then $S_1 \subseteq Y$ and $Y - vv_4 = B$ is a v -admissible base of X . The fact that S_1 is the fundamental circuit of $B + vv_4$ now implies that $Y - vv_i$ is a v -admissible base of X for all $1 \leq i \leq 4$. \square

Since $d_Y(v) = 5$, Y has an edge vv_5 for some $v_5 \notin V(S_1)$. In addition, $Y - v + v_5v_5$ is independent in $\mathcal{C}_{2,n}^1$ for some $1 \leq i \leq 4$, since otherwise $Y - vv_1$ would be dependent (as the closure of $X - v$ would contain $K(v_1, v_2, \dots, v_5)$ and v is joined to $\{v_1, v_2, \dots, v_5\}$ by four edges in $Y - vv_1$). Hence, we may

assume without loss of generality that $Y - v + v_4v_5$ is independent. Since $Y - v$ is a base of $X - v$ by Claims 5.8 and 5.9, this gives

$$r(X - v + v_4v_5) = r(X - v) + 1 = r(X) - 3. \tag{11}$$

Claim 5.10. $K(N_X(v)) \subseteq \text{cl}(X - v + v_4v_5)$.

Proof. We first show that, for all $u \in N_X(v) - v_5$,

$$K(v_1, v_2, v_3, u) \subseteq \text{cl}(X - v + v_4v_5). \tag{12}$$

Suppose to the contrary that $uv_i \notin \text{cl}(X - v + v_4v_5)$ for some $u \in N_X(v) - v_5$ and some $1 \leq i \leq 3$. Then $Y - v + v_4v_5 + uv_i$ is independent in $\mathcal{C}_{2,n}^1$. Since $K(v_1, v_2, v_3, v_4) \subseteq S_1 \subseteq X$, $u \neq v_4$. Hence uv_i and v_4v_5 are disjoint, and we can perform an X -replacement to deduce that $B' = (Y - v) \cup \{vu, vv_i, vv_4, vv_5, vv_j\}$ is independent for any $1 \leq j \leq 3$ with $j \neq i$. Since $|B'| = |Y|$ and $B' \subseteq X$, this would contradict the fact that $Y - vv_1$ is a base of X .

We can now use (12) and axiom (A2) for abstract 3-rigidity from Section 4 to deduce that $\text{cl}(X - v + v_4v_5)$ contains $K(N_X(v) - v_5)$. The same axiom implies that we can complete the proof of the claim by showing that X contains at least three edges between v_5 and $N_X(v) \setminus \{v_5\}$. To see this, recall that X is the union of copies of K_5 . In particular we have $vv_5 \in S_2$ for some copy S_2 of K_5 in X . Since every vertex of $V(S_2) \setminus \{v\}$ is in $N_X(v)$, we have $|(V(S_2) \setminus \{v_5, v\}) \cap N_X(v)| = 3$, and X contains the three edges from v_5 to the vertices of $V(S_2) \setminus \{v_5, v\}$. \square

Since X is a counterexample with minimum rank and $r(X - v + v_4v_5) < r(X)$ by (11), we can apply part (a) of the theorem to $\text{cl}(X - v + v_4v_5)$ to obtain a proper K_5 -sequence $\mathcal{C} = (C_1, \dots, C_t)$ with

$$r(X - v + v_4v_5) = \text{val}(\text{cl}(X - v + v_4v_5), \mathcal{C}). \tag{13}$$

Since $K(N_X(v)) \subseteq \text{cl}(X - v + v_4v_5)$ by Claim 5.10, no $e \in K(N_X(v))$ can be a coloop in the matroid induced by $\text{cl}(X - v + v_4v_5)$ and hence $K(N_X(v)) \subseteq C_{\leq t}$ by Lemma 3.3.

Recall that $N_X(v) = \{v_1, \dots, v_k\}$. Let C_{t+i} be a copy of K_5 on $\{v, v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ for $i = 1, \dots, k-3$, and let $\mathcal{C}'' = (C_1, C_2, \dots, C_t, C_{t+1}, \dots, C_{t+k-3})$ be the K_5 -sequence obtained by appending $(C_{t+1}, \dots, C_{t+k-3})$ to \mathcal{C} . Since $K(N_X(v)) \subseteq C_{\leq t}$, we have

$$\text{val}(X, \mathcal{C}'') = \text{val}(X - v, \mathcal{C}) + k - (k-3) = \text{val}(X - v, \mathcal{C}) + 3 = \text{val}(X - v + v_4v_5, \mathcal{C}) + 3.$$

This gives

$$\begin{aligned} \text{val}(X, \mathcal{C}'') &= \text{val}(X - v + v_4v_5, \mathcal{C}) + 3 \\ &\leq \text{val}(\text{cl}(X - v + v_4v_5), \mathcal{C}) + 3 \\ &= r(X - v + v_4v_5) + 3 && \text{(by (13))} \\ &= r(X) && \text{(by (11)).} \end{aligned}$$

We may now use Lemma 3.3 to deduce that equality must hold throughout and that $C_{\leq t+k-3} \subseteq \text{cl}(X)$. Since X is a flat and $K(N_X(v)) \subseteq C_{\leq t}$, we have $K(N_X(v)) \subseteq X$ and hence v is a simplicial vertex of $K_n[X]$.

We complete the proof that (b) holds for X by showing that v has degree three in some base of X . Choose a base B' of $X - v$ such that B' contains a base of $K(N_X(v))$. Then B' can be extended to a base B of X . The facts that B' contains a base of $K(N_X(v))$ and $|N_X(v)| \geq 4$ (since X is cyclic), imply that B will contain exactly three edges incident to v . Hence (b) holds for X .

By Lemma 3.3, $r(X) \leq |X \cup C_{\leq t}| - t$ holds for any proper K_5 -sequence (C_1, \dots, C_t) . Since X is a counterexample to the theorem, (a) does not hold for X and hence

$$r(X) < \text{val}(X, \mathcal{C}) \text{ for all proper } K_5\text{-sequences } \mathcal{C} \text{ of } K_n. \tag{14}$$

Since $r(X) = \text{val}(X, \emptyset)$ when $|X| \leq 1$, (14) implies that $|X| \geq 2$.

Claim 5.11. X is a cyclic flat in $\mathcal{C}_{2,n}^1$ and $d_X(v) \geq 4$ for all $v \in V(X)$.

Proof. Suppose X is not a flat. Then $r(X + e) = r(X)$ for some $e \in E(K_n) \setminus X$. This implies that $V(X + e) = V(X)$, since the 0-extension property holds for $\mathcal{C}_{2,n}^1$, and the maximality of $|X|$ now gives $r(X) = r(X + e) = \text{val}(X + e, \mathcal{C}) \geq \text{val}(X, \mathcal{C})$ for some proper K_5 -sequence \mathcal{C} of K_n . This contradicts (14). Hence X is a flat.

Suppose X is not cyclic. Then $r(X - e) = r(X) - 1$ for some $e \in X$. By the minimality of $r(X)$, there is a proper K_5 -sequence $\mathcal{C} = (C_1, \dots, C_t)$ such that $r(X - e) = \text{val}(X - e, \mathcal{C})$. Since $e \notin \text{cl}(X - e)$, $e \notin C_{\leq t}$ by Lemma 3.3, and hence $r(X) = r(X - e) + 1 = \text{val}(X - e, \mathcal{C}) + 1 = \text{val}(X, \mathcal{C})$. This again contradicts (14). Hence X is cyclic.

The assertion that $d_X(v) \geq 4$ for all $v \in V(X)$ now follows since all circuits in $\mathcal{C}_{2,n}^1$ have minimum degree at least four. □

Since X is a cyclic flat by Claim 5.11, (b) holds for X and hence there exists a vertex $v \in V(X)$ such that $K(N_X(v)) \subseteq X$, and v has degree three in some base of X . Then $r(X - v) = r(X) - 3$ and we may apply (a) to $X - v$ to obtain a proper K_5 -sequence $\mathcal{C} = (C_1, \dots, C_t)$ with $r(X - v) = \text{val}(X - v, \mathcal{C})$.

Let $N_X(v) = \{v_1, \dots, v_k\}$ and let C_{t+i} be a copy of K_5 on $\{v, v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ for $i = 1, \dots, k - 3$. Let $\mathcal{C}'' = (C_1, C_2, \dots, C_t, C_{t+1}, \dots, C_{t+k-3})$ be the K_5 -sequence obtained by appending C_{t+i} to \mathcal{C} for $i = 1, \dots, k - 3$. Since $K(N_X(v)) \subseteq X - v$, we have

$$\text{val}(X, \mathcal{C}'') = \text{val}(X - v, \mathcal{C}) + k - (k - 3) = \text{val}(X - v, \mathcal{C}) + 3 = r(X - v) + 3 = r(X).$$

This contradicts (14) and completes the proof of the theorem. □

Corollary 5.12. *The problem of deciding whether a given set $X \subseteq E(K_n)$ is independent in $\mathcal{C}_{2,n}^1$ is in $\text{NP} \cap \text{coNP}$.*

Proof. Theorem 5.7 immediately implies that the problem belongs to coNP. The fact that it also belongs to NP follows by applying either the Schwartz-Zippel Lemma [40, 49], or the more elementary coordinate fixing argument of Fekete and Jordán [15] to the C_2^1 -cofactor matrix. □

6 Shellable covers

We use Theorem 5.7 to obtain an alternative formula for the rank function of $\mathcal{C}_{2,n}^1$ which is closely related to existing conjectures on the rank functions of the generic 3-dimensional rigidity matroid [13, 23, 21] and the generic 2-dimensional matrix completion matroid [24].

Let \mathcal{X} be a family of subsets of $V(K_n)$. A *hinge* of \mathcal{X} is a pair of vertices $\{x, y\}$ with $X_i \cap X_j = \{x, y\}$ for two distinct $X_i, X_j \in \mathcal{X}$. Let $H(\mathcal{X})$ be the set of all hinges of \mathcal{X} . The *degree* $\deg_{\mathcal{X}}(h)$ of a hinge h of \mathcal{X} is the number of sets in \mathcal{X} which contain h . The family \mathcal{X} is said to be *t-thin* if $|X_i \cap X_j| \leq t$ for all distinct $X_i, X_j \in \mathcal{X}$.

For $F \subseteq E(K_n)$, we say that \mathcal{X} is a *cover* of F if each set in \mathcal{X} has cardinality at least five and each edge of F is induced by at least one set in \mathcal{X} . Dress et al [13] conjectured that 2-thin covers could be used to characterize the rank function of $\mathcal{R}_{3,n}$. More specifically they defined the *value* of a family \mathcal{X} as

$$\text{val}_{\mathbb{D}}(\mathcal{X}) = \sum_{X \in \mathcal{X}} (3|X| - 6) - \sum_{h \in H(\mathcal{X})} (\deg_{\mathcal{X}}(h) - 1)$$

and conjectured that the rank of any $F \subseteq E(K_n)$ in $\mathcal{R}_{3,n}$ is given by $\min\{|F_0| + \text{val}_{\mathbb{D}}(\mathcal{X})\}$ where the minimum is taken over all $F_0 \subseteq F$ and all 2-thin covers \mathcal{X} of $F \setminus F_0$ with sets of size at least three. This conjecture was shown to be false in [22] by giving an example of a 2-thin family \mathcal{X} with a negative value. Modified conjectures were given in [23, 21] which placed further restrictions on the type of cover used to obtain the minimum. We will show that these restricted 2-thin covers can be used to characterize the rank function of the maximal abstract 3-rigidity matroid $\mathcal{C}_{2,n}^1$.

A family \mathcal{X} of subsets of $V(K_n)$ is said to be *k-shellable* if its elements can be ordered as a sequence (X_1, X_2, \dots, X_m) so that, for all $2 \leq i \leq m$, $|X_i \cap \bigcup_{j=1}^{i-1} X_j| \leq k$. Similarly, \mathcal{X} is said to be *k-degenerate* if its elements can be ordered as a sequence (X_1, X_2, \dots, X_m) so that, for all $2 \leq i \leq m$, the set of hinges of the subfamily $\{X_1, X_2, \dots, X_i\}$ which are contained in X_i has size at most k . The two concepts are related since every *k-shellable* family is $\binom{k}{2}$ -degenerate.

Theorem 6.1. *Suppose r is the rank function of $\mathcal{C}_{2,n}^1$ and $F \subseteq E(K_n)$. Then*

$$r(F) = \min\{|F_0| + \text{val}_{\mathbb{D}}(\mathcal{X})\}, \tag{15}$$

where the minimum is taken over all $F_0 \subseteq F$ and all 4-shellable, 2-thin covers \mathcal{X} of $F \setminus F_0$ with sets of size at least five.

It was conjectured in [21] that the expression on the right hand side of (15) determines the rank function of $\mathcal{R}_{3,n}$ when the minimum is taken over all 9-degenerate 2-thin covers of $F \setminus F_0$ with sets of size at least three. Theorem 6.1 solves the C_2^1 -rigidity counterpart of this conjecture (since 4-shellable covers are 6-degenerate).

The proof of Theorem 6.1 will take up the remainder of this section. Our first result (Lemma 6.2) implies that both 4-shellable covers and 9-degenerate covers give an upper bound on the rank function of any K_5 -matroid.

Given a matroid M on $E(K_n)$ we say that a family \mathcal{X} of subsets of $V(K_n)$ is *M-degenerate* if its elements can be ordered as a sequence (X_1, X_2, \dots, X_m) so that, for all $2 \leq i \leq m$, the set of hinges of the subfamily $\{X_1, X_2, \dots, X_i\}$ which are contained in X_i is independent in M (when viewed as a set of edges of K_n).

Lemma 6.2. *Let M be a K_5 -matroid defined on $E(K_n)$ and r be its rank function. Suppose $F \subseteq E(K_n)$ and \mathcal{X} is an M -degenerate cover of F with sets of size at least five. Then $r(F) \leq \text{val}_D(\mathcal{X})$.*

Proof. We use induction on $|\mathcal{X}|$. If $\mathcal{X} = \{X\}$ then $H(\mathcal{X}) = \emptyset$ and Lemma 4.1 gives

$$r(F) \leq \max\{3|V(F)| - 6, 1\} \leq 3|X| - 6 = \text{val}_D(\mathcal{X})$$

since $V(F) \subseteq X$ and $|X| \geq 5$. Hence suppose that $|\mathcal{X}| \geq 2$.

Let H be the set of all edges of K_n which are induced by a hinge of \mathcal{X} . We may assume that $H \subseteq F$ since adding edges of H to F will not change $\text{val}_D(\mathcal{X})$ or the fact that \mathcal{X} is an M -degenerate cover of F , and can only increase $r(F)$. Since \mathcal{X} is M -degenerate, we can choose an ordering (X_1, X_2, \dots, X_m) of \mathcal{X} so that, for all $2 \leq i \leq m$, the set of hinges of the subfamily $\{X_1, X_2, \dots, X_i\}$ which are contained in X_i are independent in M (when viewed as a set of edges of K_n). Let F_m be the set of edges in F covered by X_m , $H_m = F_m \cap H$ and $F' = F \setminus (F_m \setminus H_m)$. Since H_m is independent in M , we may choose a maximum independent subset B' of F' with $H_m \subseteq B'$. Since $\mathcal{X}' = \mathcal{X} \setminus \{X_m\}$ is an M -degenerate cover of F' , we may use induction to deduce that $|B'| \leq \text{val}_D(\mathcal{X}')$. Let B be a maximum independent subset of F which contains B' . Then $|B \cap F_m| \leq 3|X_m| - 6$ and hence $|B \setminus B'| \leq 3|X_m| - 6 - |H_m|$. This gives

$$r(F) = |B| = |B'| + |B \setminus B'| \leq \text{val}_D(\mathcal{X}') + 3|X_m| - 6 - |H_m| = \text{val}_D(\mathcal{X}).$$

□

Lemma 6.2 and the facts that 4-shellable covers are M -degenerate in any K_5 -matroid on $E(K_n)$ and $r(F) \leq |F_0| + r(F \setminus F_0)$ for all $F_0 \subseteq F \subseteq E(K_n)$, imply that the right hand side of (15) gives an upper bound on $r(F)$. It follows that we may complete the proof of Theorem 6.1 by exhibiting a set $F_0 \subseteq F$ and a 4-shellable, 2-thin cover \mathcal{X} of $F \setminus F_0$ with sets of size at least five, such that $r(F) = |F_0| + \text{val}_D(\mathcal{X})$. To do this, we will choose sets F_0 and \mathcal{X} suggested by a second conjecture of Dress, see [8, 20, 43].

Given a graph $G = (V, E)$, a *maximal clique* of G is a maximal subset $X \subseteq V$ such that $G[X]$ is a complete graph. Suppose that $F \subseteq E(K_n)$ is a flat in $\mathcal{R}_{3,n}$. Let \mathcal{X}^* be the set of all maximal cliques in $K_n[F]$ of size at least three, and F_0 be the set of edges in F which are not covered by \mathcal{X}^* . Dress's second conjecture is that the rank of F in $\mathcal{R}_{3,n}$ is equal to $|F_0| + \text{val}_D(\mathcal{X}^*)$. We will show that a slightly modified form of this conjecture holds for $\mathcal{C}_{2,n}^1$.

Theorem 6.3. *Suppose that $F \subseteq E(K_n)$ is a flat in $\mathcal{C}_{2,n}^1$. Let \mathcal{X}^* be the set of all maximal cliques in $K_n[F]$ of size at least five, and F_0 be the set of edges in F which are not covered by \mathcal{X}^* . Then \mathcal{X}^* is a 4-shellable, 2-thin cover of $F \setminus F_0$ and*

$$r(F) = |F_0| + \text{val}_D(\mathcal{X}^*). \tag{16}$$

Proof. We proceed by contradiction. Suppose that the theorem does not hold for some $F \subseteq E(K_n)$ and that F has been chosen to be as small as possible. The hypothesis that F is a flat and the axiom (A2) of abstract 3-rigidity imply that \mathcal{X}^* is 2-thin. We will obtain a contradiction by showing that \mathcal{X}^* is 4-shellable and $r(F) = |F_0| + \text{val}_D(\mathcal{X}^*)$.

We first show that F is a cyclic set in $\mathcal{C}_{2,n}^1$. Suppose to the contrary that some $e \in F$ is a coloop in $\mathcal{C}_{2,n}^1|_F$. Then $e \in F_0$. Since F is a flat, $F - e$ is a flat and \mathcal{X}^* is the set of all maximal cliques in $K_n[F - e]$ of size at least five. By the minimality of F , \mathcal{X}^* is 4-shellable and we have

$$r(F) = r(F - e) + 1 = |F_0 - e| + \text{val}_D(\mathcal{X}^*) + 1 = |F_0| + \text{val}_D(\mathcal{X}^*).$$

This contradicts the choice of F . Hence F is cyclic. Corollary 5.2 now implies that $F_0 = \emptyset$.

By Theorem 5.7(b), there exists a base B of F and a vertex $v \in V(F)$ such that $d_B(v) = 3$ and $K(N_F(v)) \subseteq F$. Then $r(F - v) = r(F) - 3$ and $\text{cl}(F - v) = \text{cl}(F) - v = F - v$ so $F - v$ is a flat in $\mathcal{C}_{2,n}^1$. Since $K(N_F(v)) \subseteq F$ and $d_F(v) \geq 4$, v is contained in a unique maximal clique $X_v \in \mathcal{X}^*$.

Suppose that $d_F(v) \geq 5$ and let $X'_v = X_v - v$. Then $|X'_v| \geq 6$, so $|X'_v| \geq 5$, $\mathcal{X}' = \mathcal{X}^* - X_v + X'_v$ is the set of maximal cliques of $K_n[F - v]$, and $H(\mathcal{X}') = H(\mathcal{X}^*)$. We can now use the minimality of F to deduce that \mathcal{X}' and \mathcal{X}^* are 4-shellable and

$$r(F) = r(F - v) + 3 = \text{val}_D(\mathcal{X}') + 3 = \text{val}_D(\mathcal{X}^*).$$

This contradicts the choice of F .

It remains to consider the case when $d_F(v) = 4$. Then $|X_v| = 5$ and $\mathcal{X}' = \mathcal{X}^* - X_v$ is the set of maximal cliques of $K_n[F - v]$ of size at least five. Furthermore, the set F'_0 of edges of $F - v$ which do not belong to a clique in \mathcal{X}' is given by $K(N_X(v)) \setminus H(\mathcal{X}^*)$. We can now use induction to deduce that \mathcal{X}' is 4-shellable and $r(F - v) = |F'_0| + \text{val}_D(\mathcal{X}')$. Let (X_1, X_2, \dots, X_t) be a 4-shellable ordering of the cliques in \mathcal{X}' . Since $|X_v \cap V(F - v)| = 4$, $(X_1, X_2, \dots, X_t, X_v)$ will be a 4-shellable ordering of the cliques in \mathcal{X}^* and hence \mathcal{X}^* is 4-shellable. In addition

$$r(F) = r(F - v) + 3 = |F'_0| + \text{val}_D(\mathcal{X}') + 3 = \text{val}_D(\mathcal{X}^*).$$

This contradicts the choice of F and completes the proof of the theorem. \square

As noted above, Theorem 6.1 follows from Lemma 6.2 and Theorem 6.3: Lemma 6.2 implies that the minimum on the right hand side of (15) is an upper bound on $r(F)$, and we may apply Theorem 6.3 to $\text{cl}(F)$ to deduce that equality is attained.

Jackson and Jordán introduced M -degenerate 2-thin covers as ‘iterated 2-thin covers’ in [23] and conjectured that they determine the rank function of $\mathcal{R}_{3,n}$. Theorem 6.1, Lemma 6.2 and the fact that 4-shellable covers are M -degenerate for any K_5 -matroid imply that the conjectured rank formulae in [23, Conjectures 3.2, 3.3] hold for the C_2^1 -cofactor matroid.

Cheng and Sitharam introduced k -degenerate 2-thin covers in [4] as ‘generalized partial k -trees’. They used 3-degenerate 2-thin covers to show that the number of edges in any maximal ‘(3,6)-sparse subgraph’ of a graph gives an upper bound on its rank in $\mathcal{R}_{3,n}$.

7 Sufficient connectivity conditions for C_2^1 -rigidity of graphs

We say that a graph $G = (V, E)$ with n vertices is C_2^1 -rigid if its edge set spans $\mathcal{C}_{2,n}^1$. It is k -connected if $n \geq k + 1$ and $G - U$ is connected for all $U \subset V$ with $|U| \leq k - 1$. We use the results of the preceding section to obtain sufficient connectivity conditions for a graph to be C_2^1 -rigid.

Jackson and Jordán conjectured in [23, Example 2] that, if $G = (V, F)$ is 5-connected and F is a circuit in $\mathcal{R}_{3,n}$, then G is rigid in \mathbb{R}^3 . Our first result implies that the C_2^1 -rigidity counterpart of this conjecture is true.

Theorem 7.1. *Suppose $G = (V, F)$ is a 5-connected graph on n vertices and F is a cyclic set in $\mathcal{C}_{2,n}^1$. Then G is C_2^1 -rigid.*

Proof. We may assume without loss of generality that F is a flat since replacing F by its closure will not affect the hypotheses or conclusion of the theorem. Corollary 5.2 now implies that every edge of G belongs to a copy of K_5 . Let \mathcal{X} be the set of all maximal cliques of size at least five in H . By Theorem 6.3, \mathcal{X} is a 4-shellable cover of F . The hypothesis that G is 5-connected now implies that $|\mathcal{X}| = 1$. Hence G is a clique with at least six vertices and the theorem holds for G . \square

Tay [42] gave examples of 4-connected graphs $G = (V, F)$ on n vertices which are not rigid in \mathbb{R}^3 and are such that F is a circuit in $\mathcal{R}_{3,n}$. The same graphs also show that the connectivity condition of Theorem 7.1 is best possible since they are not $\mathcal{C}_{2,n}^1$ -rigid and F is a circuit in $\mathcal{C}_{2,n}^1$.

Lovász and Yemini [31] proved that every 6-connected graph is rigid in \mathbb{R}^2 and conjectured that every 12-connected graph is rigid in \mathbb{R}^3 . Our second result solves the \mathcal{C}_2^1 -rigidity counterpart of their conjecture.

Theorem 7.2. *Let $G = (V, F)$ be a 12-connected graph on n vertices and $S \subset E$ with $|S| \leq 6$. Then $G - S$ is \mathcal{C}_2^1 -rigid.*

Proof. Suppose, for a contradiction, that $G - S$ is not \mathcal{C}_2^1 -rigid. We may assume that G and S have been chosen to be a counterexample such that $|V|$ is as small as possible and, subject to this condition, $|F \setminus S|$ is as large as possible. Then $F \setminus S$ is a flat in $\mathcal{C}_{2,n}^1$ since if $e \in \text{cl}(F \setminus S) \setminus (F \setminus S)$, then we may apply induction to $(F + e, S - e)$ to deduce that $G - S + e$ is \mathcal{C}_2^1 -rigid and hence, since $e \in \text{cl}(F \setminus S)$, $G - S$ is \mathcal{C}_2^1 -rigid.

Let \mathcal{X} be the set of all maximal cliques of size at least five in $G - S$, \hat{F}_0 be the set of edges in $G - S$ not covered by any member of \mathcal{X} and put $F_0 = \hat{F}_0 \cup S$. By Theorem 6.3, \mathcal{X} is a 4-shellable 2-thin cover of $F \setminus F_0$ and $|\hat{F}_0| + \text{val}_D(\mathcal{X}) = r(F \setminus S) < 3|V(G)| - 6$. Since $|S| \leq 6$ this gives

$$|F_0| + \text{val}_D(\mathcal{X}) < 3|V(G)|. \tag{17}$$

For each vertex $v \in V(F)$, let F_0^v be the set of edges in F_0 incident to v , k_v be the number and $\mathcal{X}^v = \{X_1^v, \dots, X_{k_v}^v\}$ be the set of cliques in \mathcal{X} which contain v , and H_v be the set of hinges of \mathcal{X}^v that contain v . Note that $H_v \subseteq H(\mathcal{X})$ and $\deg_{\mathcal{X}^v}(h) = \deg_{\mathcal{X}}(h)$ for each $h \in H_v$.

Claim 7.3. *For each $v \in V(F)$, either $F_0^v \neq \emptyset$ or $k_v \geq 2$.*

Proof. Suppose that $F_0^v = \emptyset$ and $k_v = 1$, i.e., v is not incident to an edge of F_0 and is contained in exactly one maximal clique $X_1^v \in \mathcal{X}$. Since \mathcal{X} covers $F \setminus F_0$, we have $K(N_G(v)) \subset G$ and either $G - v$ is 12-connected or $G = K_{13}$. Since K_{13} is \mathcal{C}_2^1 -rigid, the first alternative must occur, and we may apply induction to deduce that $G - v - S$ is \mathcal{C}_2^1 -rigid. This and $d_G(v) \geq 12$ in turn imply that G is \mathcal{C}_2^1 -rigid, which is a contradiction. \square

The \mathcal{C}_2^1 -rigidity of $G - S$ will follow easily from our next claim.

Claim 7.4. *For all $v \in V(G)$,*

$$\frac{|F_0^v|}{2} + \sum_{i=1}^{k_v} \left(3 - \frac{6}{|X_i^v|} \right) - \sum_{h \in H_v} \left(\frac{\deg_{\mathcal{X}}(h) - 1}{2} \right) \geq 3. \tag{18}$$

Proof. If $k_v = 0$, then, by 12-connectivity, $|F_0^v| \geq 12$ holds, which implies (18). Hence we may suppose that $k_v \geq 1$.

Suppose that $k_v = 1$, i.e., $\mathcal{X}^v = \{X_1^v\}$. Then $H_v = \emptyset$. By Claim 7.3, $|F_0^v| \geq 1$. Hence, if $|X_1^v| \geq 12$, then (18) holds. It remains to consider the subcase when $5 \leq |X_1^v| \leq 11$. As v has degree at least 12 in G , we have $(|X_1^v| - 1) + |F_0^v| \geq 12$. This implies that

$$\frac{|F_0^v|}{2} + \left(3 - \frac{6}{|X_1^v|}\right) \geq 9.5 - \frac{|X_1^v|}{2} - \frac{6}{|X_1^v|}.$$

When $5 \leq |X_1^v| \leq 11$, the right side of this inequality is at least 3, and we get (18).

Hence we may assume that $k_v \geq 2$. Put $c_1 = 0$ and let $c_i = \left| \left(\bigcup_{j=1}^{i-1} X_j^v \right) \cap X_i^v \right| - 1$ for $2 \leq i \leq k_v$. Then c_i represents the contribution of X_i^v to the ‘hinge count’ at v in $\text{val}_D(X_1^v, \dots, X_{k_v}^v)$ and we have

$$\sum_{h \in H_v} (\deg_{\mathcal{X}^v}(h) - 1) = \sum_{i=1}^{k_v} c_i.$$

Since \mathcal{X} is 4-shellable, \mathcal{X}^v is 4-shellable. Hence, by reordering the sequence $(X_1^v, \dots, X_{k_v}^v)$ if necessary, we can suppose that $c_i \leq 3$ for all $2 \leq i \leq k_v$. Also, since \mathcal{X}^v is 2-thin, $c_2 \leq 1$. Since $|X_i^v| \geq 5$ for all $1 \leq i \leq k_v$, we may deduce that

$$\sum_{i=1}^2 \left(3 - \frac{6}{|X_i^v|} - \frac{c_i}{2}\right) \geq \left(3 - \frac{6}{5}\right) + \left(3 - \frac{6}{5} - \frac{1}{2}\right) > 3,$$

and, for all $3 \leq i \leq k_v$,

$$3 - \frac{6}{|X_i^v|} - \frac{c_i}{2} \geq 3 - \frac{6}{5} - \frac{3}{2} > 0.$$

Combining these inequalities, we obtain

$$\frac{|F_0^v|}{2} + \sum_{i=1}^{k_v} \left(3 - \frac{6}{|X_i^v|}\right) - \sum_{h \in H_v} \left(\frac{\deg_{\mathcal{X}^v}(h) - 1}{2}\right) = \frac{|F_0^v|}{2} + \sum_{i=1}^{k_v} \left(3 - \frac{6}{|X_i^v|} - \frac{c_i}{2}\right) > 3.$$

This completes the proof of the claim. □

Taking the sum of (18) over all vertices of G , we obtain

$$|F_0| + \text{val}_D(\mathcal{X}) = |F_0| + \sum_{X \in \mathcal{X}} (3|X| - 6) - \sum_{h \in H(\mathcal{X})} (\deg_{\mathcal{X}}(h) - 1) \geq 3|V(G)|.$$

This contradicts (17) and completes the proof of the theorem. □

Lovász and Yemini [31] gave examples of 11-connected graphs $G = (V, F)$ which are not rigid in \mathbb{R}^3 . These graphs also show that the connectivity condition of Theorem 7.2 is best possible since they are not $\mathcal{C}_{2,n}^1$ -rigid. The 12-connected graph G consisting of two large complete graphs joined by a set T of 12 disjoint edges fails to be $\mathcal{C}_{2,n}^1$ -rigid if we delete 7 edges from T . This shows that the bound on $|S|$ in Theorem 7.2 is best possible.

8 Problems and Remarks

Many interesting problems remain.

1. Developing a deterministic polynomial time algorithm to solve the combinatorial optimization problem of evaluating the rank formulae in Theorems 5.7 and 6.1 is our most obvious open problem. These formulae imply that the decision problem of determining whether the rank takes any given value is in $\text{NP} \cap \text{co-NP}$, so it is likely that such an algorithm exists. But it is not clear how to use the existing theory of submodular functions to design such an algorithm since our rank formulae are significantly different to other known matroid rank formulae.
2. Let M_0 be the rank $\binom{d+2}{2}$ truncation of $\mathcal{R}_{d,n}$ (or equivalently, of any other K_{d+2} -matroid on $E(K_n)$) for $n \geq d+2$.

Conjecture 8.1. *The rank function of the free elevation of M_0 is given by*

$$r(F) = \min\{|F \cup C_{\leq t}| - t : (C_1, \dots, C_t) \text{ is a proper } K_{d+2}\text{-sequence in } K_n\}$$

for all $F \subseteq E(K_n)$.

This would imply that the free elevation of M_0 is the unique maximal K_{d+2} -matroid on K_n , and hence verify Graver's conjecture that there exists a unique maximal abstract d -rigidity matroid. Conjecture 8.1 holds for $d = 3$ by Theorem 5.7. It is not difficult to see that it also holds for $d = 1, 2$, see for example [25].

Conjecture 8.1 would imply, in particular, that the free elevation of M_0 is an abstract d -rigidity matroid. This would follow from Theorem 4.2 if we could show that the free elevation of M_0 has the 0-extension property.

3. Testing generic 4-dimensional rigidity of graphs is recognized as being an even more difficult problem than generic 3-dimensional rigidity. One reason for this is Whiteley's result that $\mathcal{R}_{4,n}$ is not the unique maximal abstract 4-rigidity matroid since $K_{6,6}$ is independent in $\mathcal{C}_{3,n}^2$ but not in $\mathcal{R}_{4,n}$. Even so, Conjecture 3.4 may still be robust enough to deal with such bad examples.

Conjecture 8.2. *The rank function of $\mathcal{R}_{4,n}$ is given by*

$$r(F) = \min\{|F \cup C_{\leq t}| - t : (C_1, \dots, C_t) \text{ is a proper } \{K_6, K_{6,6}\}\text{-sequence in } K_n\}$$

for all $F \subseteq E(K_n)$.

This would imply that $\mathcal{R}_{4,n}$ is the free elevation of its rank 36 truncation, and also that $\mathcal{R}_{4,n}$ is the unique maximal $\{K_6, K_{6,6}\}$ -matroid on $E(K_n)$ i.e. the unique maximal matroid in the poset of all matroids on $E(K_n)$ in which every copy of K_6 and $K_{6,6}$ is a circuit.

4. A simpler example of a matroid in which all small circuits are copies of complete or complete bipartite graphs arises in the context of the rank 2 completion of partially filled skew-symmetric matrices, see Bernstein [2]. All copies of K_4 and $K_{3,3}$ are circuits in this matroid and we conjecture

that it is the unique maximal $\{K_4, K_{3,3}\}$ -matroid on $E(K_n)$. Combined with Conjecture 3.4, this would imply that its rank function is given by

$$r(F) = \min\{|F \cup C_{\leq t}| - t : (C_1, \dots, C_t) \text{ is a proper } \{K_4, K_{3,3}\}\text{-sequence in } K_n\}$$

for all $F \subseteq E(K_n)$.

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