Coincident Rigidity of 2-Dimensional Frameworks

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Abstract

Fekete, Jordán and Kaszanitzky [4] characterised the graphs which can be realised as 2-dimensional, infinitesimally rigid, bar-joint frameworks in which two given vertices are coincident. We formulate a conjecture which would extend their characterisation to an arbitrary set T of vertices and verify our conjecture when $|T| = 3$.

1 Introduction

A d-dimensional (bar-and-joint) *framework* is a pair (G, p) where $G = (V, E)$ is a simple graph and $p : V \to \mathbb{R}^d$ is a map, which we refer to as the *realisation* of the framework. The *length* of each edge of (G, p) is given by the Euclidean distance between its end points. The framework is said to be *rigid* if every continuous motion of the vertices which preserves the lengths of the edges, preserves the distance between all pairs of vertices. It is *infinitesimally rigid* if it satisfies the stronger property that every infinitesimal motion of the vertices which preserves the lengths of the edges is an infinitesimal isometry of $\mathbb{R}^d$.

It is not difficult to see that a 1-dimensional framework is rigid if and only if its underlying graph is connected, but for $d \geq 2$, the decision problem of deciding whether a given d-dimensional framework is rigid is NP-hard by a result of Abbot [1]. This problem becomes more tractable, however, if we restrict our attention to *generic frameworks* i.e. framworks (G, p) for which the set of the coordinates $p(v)$, $v \in V$, is algebraically independent over the rational numbers. Asimow and Roth [2] showed that the properties of rigidty and infinitesimal rigidity are equivalent for such frameworks and depend only on the underlying graph. This allows us to define a graph G as being *rigid* in $\mathbb{R}^d$ if some, or equivalently every, generic realisation of G in $\mathbb{R}^d$ is rigid.

Graphs which are rigid in $\mathbb{R}^d$ have been characterised for $d = 1, 2$: we have already seen that G is rigid in $\mathbb{R}$ if and only if G is connected and a fundamental result of Pollaczek-Geiringer [10], subsequently rediscovered by Laman [7], characterises when G is rigid in $\mathbb{R}^2$. Finding a characterisation when $d \geq 3$ is the main open problem in distance geometry, although characterisations do exist for certian families of graphs. A common technique used to show that a family of graphs G is rigid in $\mathbb{R}^d$ is to reduce G to a smaller graph G' in the family by some operation, apply induction to deduce that G' is rigid, and then show that the inverse operation preserves rigidity. The last step in this proof strategy frequently uses a geometric argument based on a nongeneric realisation of G. More precisely, we extend a generic (and hence infinitesimally rigid) realisation p' of G' to a realisation p of

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G by choosing special positions for the vertices of $V(G) \setminus V(G')$ which make it easy to conclude that (G,p) is also infinitesimally rigid, then use the fact (see Section 2) that if some realsation of (G,p) is infinitesimally rigid then every generic realsation of (G,p) is infinitesimally rigid.

This approach has stimulated interest in such special position frameworks. Jackson and Jordán [6] characterised when a graph G has an infinitesimally rigid realisation in $\mathbb{R}^2$ in which three given vertices are collinear. Another result due to Fekete, Jordán and Kaszanitzky [4], which is closer to our interests in this paper, characterises when a graph G has an infinitesimally rigid realisation in $\mathbb{R}^2$ in which two given vertices are coincident. We need some new notation to describe their theorem. Given a graph $G = (V,E)$ and $T \subseteq V$ we use G/T to denote the graph obtained by contracting the vertices in T to a single vertex. When $T = \{u,v\}$ we often use G/uv instead of G/T. We also use $G - uv$ to denote the graph obtained from G by deleting the edge uv if it exists in E (and putting $G - uv = G$ when $uv \notin E$). We say that a realisation p of G is *T-coincident* if $p(x) = p(y)$ for all $x, y \in T$.

Theorem 1.1. *[4] Let G be a graph and u,v be distinct vertices of G. Then G has an infinitesimally rigid, $\{u,v\}$-coincident realisation in $\mathbb{R}^2$ if and only if $G - uv$ and G/uv are both rigid in $\mathbb{R}^2$.*

We will obtain an analogous characterisation for three coincident vertices.

Theorem 1.2. *Let $G = (V,E)$ be a graph, u,v,w be distinct vertices of G, and $G' = G - uv - uw - vw$. Then G has an infinitesimally rigid, $\{u,v,w\}$-coincident realisation in $\mathbb{R}^2$ if and only if G' and G'/S are rigid in $\mathbb{R}^2$ for all $S \subseteq \{u,v,w\}$ with $|S| \geq 2$.*

We also offer a conjecture which would extend Theorems 1.1 and 1.2 to T-coincident rigidity in $\mathbb{R}^2$ for arbitrary sets T, and provide an example which shows that this conjecture does not extend to $\mathbb{R}^3$ even in the special case when $|T| = 2$.

2 Preliminaries

2.1 Rigidity matrices, infinitesimal rigidity and independent frameworks

The *rigidity matrix* $R(G,p)$ of a d-dimensional framework (G,p) is the matrix of size $|E| \times d|V|$ where, in the row corresponding to an edge $uv \in E$, the entries in the columns corresponding to u and v are $p(u) - p(v)$ and $p(v) - p(u)$, respectively and all other entries are zero. The right kernel of $R(G,p)$ is the space of *infinitesimal motions of* (G,p). This space has dimension at least $\binom{d+1}{2}$ when $|V| \geq d$ since it contains the space of infinitesimal isometries of $\mathbb{R}^d$. This implies that rank $R(G,p) \leq d|V| - \binom{d+1}{2}$ when $|V| \geq d$, and (G,p) is *infinitesimally rigid* if rank $R(G,p)$ acheives this upper bound when $|V| \geq d$. For the case when $|V| < d$, (G,p) is *infinitesimally rigid* if rank $R(G,p) = \binom{|V|}{2}$. Since the rank of $R(G,p)$ is maximised whenever (G,p) is generic, the infinitesimal rigidity of a generic framework (G,p) in $\mathbb{R}^d$ depends only on its underlying graph G.

A framework (G,p) is said to be *independent* if the rows of its rigidity matrix are linearly independent. Maxwell [9] used the upper bound on rank $R(G,p)$ described in the previous paragraph to obtain the following necessary condition for independence.

$$\text{If } (G,p) \text{ is an independent } d\text{-dimensional framework and } G' = (V',E') \text{ is a subgraph of } G \text{ with } |V'| \geq d \text{ then } |E'| \leq d|V'| - \binom{d+1}{2}. \quad (1)$$

2.2 Rigidity matroids

Given a framework (G,p), we can construct a matroid $\mathcal{R}(G,p)$ on $E(G)$ by defining a set $F \subseteq E(G)$ to be independent in $\mathcal{R}(G,p)$ if the corresponding rows of the rigidity matrix $R(G,p)$ are linearly independent. The *d-dimensional rigidity matroid* $\mathcal{R}_d(G)$ of the graph G is the matroid $\mathcal{R}(G,p)$ for any generic d-dimensional framework (G,p). The necessary condition for independence in $\mathcal{R}_d(G)$ given by (1) is also sufficient when $d = 1,2$. When $d = 1$ it is equivalent to saying that G is a forest. When $d = 2$ it is implied by the above mentioned characterisation of generic rigidity in $\mathbb{R}^2$ due to Pollaczek-Geiringer [10]. Lovász and Yemini [8] used the characterisation of independence in $\mathcal{R}_2(G)$ to determine its rank function. We need to introduce some new terminology to describe their result. Given a graph $G = (V,E)$ and $E' \subset E$, a *cover of* E' is a family $\mathcal{X}$ of subsets of V such that each member of $\mathcal{X}$ has cardinality at least two and each edge in E' is induced by some member of $\mathcal{X}$. The cover $\mathcal{X}$ is *1-thin* if $|X_i \cap X_j| \leq 1$ for all distinct $X_i, X_j \in \mathcal{X}$.

Theorem 2.1. *Let $G = (V,E)$ be a graph and $E' \subseteq E$. Then the rank of E' in $\mathcal{R}_2(G)$ is given by $r(E') = \min\{\sum_{X\in\mathcal{X}}(2|X|-3) : \mathcal{X}$ is a 1-thin cover of $E'\}$.*

Coincident rigidity matroids

For $T \subseteq V(G)$, we can define the *T-coincident, d-dimensional rigidity matroid* $\mathcal{R}_{d,T}(G)$ of G in the same way as the d-dimensional rigidity matroid. We first choose a reference vertex $t \in T$. We say that a realisation p of G is *T-coincident* if $p(v) = p(t)$ for all $v \in T$ and is a *generic T-coincident realisation* if $p|_{V(G)\setminus(T-t)}$ is algebraically independent over $\mathbb{Q}$. Then rank $R(G,p)$ will be maximised over all T-coincident realisations in $\mathbb{R}^d$ whenever (G,p) is a generic T-coincident realisation and hence the infinitesimal rigidity of a generic T-coincident framework (G,p) in $\mathbb{R}^d$ depends only on the graph G and the set T. We say that G is *T-coincident rigid in* $\mathbb{R}^d$ if some, or equivalently every, generic T-coincident realisation of G in $\mathbb{R}^d$ is infinitesimally rigid. The *d-dimensional T-coincident rigidity matroid* $\mathcal{R}_{d,T}(G)$ of the pair (G,T) is the matroid $\mathcal{R}(G,p)$ for any generic d-dimensional T-coincident framework (G,p). It is easy to see that $\mathcal{R}_{d,T}(G) = \mathcal{R}_d(G)$ when $|T| = 1$ and that $\mathcal{R}_{1,T}(G) = \mathcal{R}_1(G - E_G(T))$, where $E_G(T)$ is the set of edges of G induced by T. The results of [4] characterise $\mathcal{R}_{d,T}(G)$ when $d = |T| = 2$. We will extend this characterisation to the case when $|T| = 3$ and formulate a conjecture which would characterise $\mathcal{R}_{2,T}(G)$ for all T.

We will need the following observation which relates the T-coincident rigidity matroid $\mathcal{R}_{d,T}(G)$ to the rigidity matroid $\mathcal{R}_d(G/T)$. Let z_T be the vertex of G/T corresponding to T. Given a framework $(G/T, p_T)$ in $\mathbb{R}^d$, we can obtain a T-coincident realisation (G,p) of G by putting $p(x) = p_T(z_T)$ if $x \in T$ and $p(x) = p_T(x)$ if $x \notin T$. Furthermore, the map $q_T \mapsto q$ given by $q(x) = q_T(z_T)$ if $x \in T$ and $q(x) = q_T(x)$ if $x \notin T$ is an injective linear transformation from $\ker R(G_T, p_T)$ to $\ker R(G,p)$. This gives $\dim\ker R(G,p) \geq \dim\ker R(G/T,p_T)$ and hence

$$\text{rank } R(G,p) \leq \text{rank } R(G/T,p_T) + d(|T|-1). \tag{2}$$

2.3 Graph theoretic notation and terminology

Let $G = (V,E)$ be a graph. For $X \subseteq V$, let $G[X]$ denote the subgraph induced by X. Let $E_G(X)$ be the set and $i_G(X)$ be the number of edges of $G[X]$. For a family $\mathcal{X}$ of subsets of V, we put $E_G(\mathcal{X}) = \bigcup_{X\in\mathcal{X}} E_G(X)$ and $i_G(\mathcal{X}) = |E_G(\mathcal{X})|$. We also define $\text{cov}(\mathcal{X}) = \{xy : \{x,y\} \subseteq X, \text{ for some } X \in \mathcal{X}\}$ and say that $\mathcal{X}$ *covers* a set $E' \subseteq E$

if $E' \subseteq \text{cov}(\mathcal{X})$. The *degree* of a vertex v in G is denoted by $d_G(v)$ and the set of all neighbours of v in G is denoted by $N_G(v)$. We will omit the subscripts referring to G when the graph is clear from the context.

3 A matroid construction

We use a similar strategy to [4] to characterise $\mathcal{R}_{2,T}(G)$ when $|T| = 3$ and prove Theorem 1.2. Suppose $G = (V, E)$ is a graph and $T \subseteq V$. In this section we derive necessary conditions for independence in $\mathcal{R}_{2,T}(G)$, and show that these necessary conditions for independence define a matroid $\mathcal{M}_T(G)$ on $E(G)$ when $|T| \leq 3$. We show in the next section that $\mathcal{M}_T$ is equal to $\mathcal{R}_{2,T}(G)$ and then use our formula for the rank function of $\mathcal{M}_T$ to verify Theorem 1.2.

For a fixed nonempty set $T \subseteq V$, we define the *T-value* of an arbitrary set $X \subseteq V$ by

$$\text{val}_T(X) = \begin{cases} 2|X| - 3 & \text{if } X \not\subseteq T \\ 0 & \text{if } X \subseteq T \end{cases}$$

Note that $\text{val}_T(X) \geq 0$ whenever $|X| \geq 2$.

We say that a non-empty family $\mathcal{H} = \{H_1, \ldots, H_k\}$ of subsets of V is *T-compatible* if T is a proper subset of H_i for all $1 \leq i \leq k$, and define its *T-value* to be

$$\text{val}_T(\mathcal{H}) = \sum_{i=1}^{k} (2|H_i \setminus T| - 1) + 2(|T| - 1).$$

Note that $\text{val}_T(\mathcal{H}) \geq 0$ since $|H_i| > |T| \geq 1$ for all $1 \leq i \leq k$.

The graph G is said to be *T-sparse* if

- $i_G(X) \leq \text{val}_T(X)$ for all $X \subseteq V$ with $|X| \geq 2$ and
- $i_G(\mathcal{H}) \leq \text{val}_T(\mathcal{H})$ for all T-compatible families $\mathcal{H}$.

In particular, if G is T-sparse, then $i_G(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$ so E is independent in $\mathcal{R}_2(G)$ by [10].

We motivate these definitions by showing that T-sparsity is a necessary condition for independence in the 2-dimensional T-coincident rigidity matroid $\mathcal{R}_{2,T}(G)$.

Lemma 3.1. *Let $G = (V, E)$ be a graph and let $T \subseteq V$ with $|T| \geq 1$. Suppose E is independent in $\mathcal{R}_{2,T}(G)$. Then G is T-sparse.*

Proof. Let (G, p) be a generic T-coincident realisation of G in $\mathbb{R}^2$. Then $R(G, p)$ has linearly independent rows. This implies that $R(G', p|_{V'})$ has independent rows for any subgraph $G' = (V', E')$ of G.

Choose $X \subseteq V$ with $|X| \geq 2$ and let $J = G[X]$. Then $(J, p|_X)$ is independent and (1) gives $i_G(X) = |E(J)| \leq 2|X| - 3$. In addition, if $X \subseteq T$ then $i_G(X) = 0$, since any edge $ab \in E(G)$ would give rise to a zero row in $R(J, p|_X)$. Hence $i_G(X) \leq \text{val}_T(X)$.

Now suppose that $|T| \geq 2$ and let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a T-compatible family. Consider the subgraphs of G given by $L_i = G[H_i]$, $1 \leq i \leq k$, and $L = \bigcup_{i=1}^{k} L_i$. Let L_i/T, respectively L/T, be obtained from L_i, respectively L, by contracting T into a single vertex z_T. Let $(L/T, p_T)$ be the realisation of L/T with $p_T(x) = p(x)$ for $x \neq z_T$ and

$p_T(z_T) = p(z)$ for any $z \in T$, and let p_i be the restriction of p_T to the vertices of L_i/T. Then every edge of L/T belongs to one of the subgraphs L_i/T and we can use 1 to obtain

$$\text{rank } R(L/T, p_T) \leq \sum_{i=1}^{k} \text{rank } R(L_i/T, p_i) \leq \sum_{i=1}^{k} (2|V(L_i/T)|) - 3) = \sum_{i=1}^{k} (2|H_i \setminus T| - 1).$$

We can combine this bound with (2) and the fact that $(J, p|_X)$ is independent to obtain

$$i_G(\mathcal{H}) = |E(J)| = \text{rank } R(J, p|_X) \leq \sum_{i=1}^{k} (2|H_i \setminus T| - 1) + 2(|T| - 1) = \text{val}_T(\mathcal{H}).$$

□

The converse of Lemma 3.1 holds for $|T| = 1, 2$. When $|T| = 1$, G is T-sparse if and only if $i_G(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$ and this condition characterises independence in $\mathcal{R}_2(G)$ by [10]. When $|T| = 2$, the condition that G is T-sparse characterises independence in $\mathcal{R}_{2,T}(G)$ by [4]. When $|T| \geq 3$ we we need a stronger condition which follows from the fact that an infinitesimally rigid T-coincident realisation of G is an infinitesimally rigid S-coincident realisation of G for all $S \subseteq T$. Combined with Lemma 3.1, this implies that we need G to be S-sparse for all $S \subseteq T$ with $|S| \geq 2$, for E to be independent in $\mathcal{R}_{2,T}(G)$. (Note that we do not need to include the case when $|S| = 1$ since this follows immediately from the condition that G is T-sparse.) We will show that this strengthened condition characterises independence in $\mathcal{R}_{2,T}(G)$ when $|T| = 3$. We first obtain some preliminary results on S-compatible families.

3.1 S-compatible families

Lemma 3.2. *Let $G = (V, E)$ be a graph, $S \subseteq V$ with $|S| \geq 2$ and $\mathcal{H} = \{H_1, \ldots, H_k\}$ be an S-compatible family in G. Suppose $|H_i \cap H_j| \geq |S| + 1$ for some pair $1 \leq i < j \leq k$. Then there exists an S-compatible family $\overline{\mathcal{H}}$ with $\text{cov}(\mathcal{H}) \subseteq \text{cov}(\overline{\mathcal{H}})$ and $\text{val}_S(\overline{\mathcal{H}}) \leq \text{val}_S(\mathcal{H}) - 1$.*

Proof. We may assume that $i = k - 1$ and $j = k$. Let $\overline{\mathcal{H}} = \{H_1, \ldots, H_{k-2}, \overline{H}_{k-1}\}$ where $\overline{H}_{k-1} = H_{k-1} \cup H_k$. Then we have $\text{cov}(\mathcal{H}) \subseteq \text{cov}(\overline{\mathcal{H}})$ and

$$\begin{aligned}
\text{val}_S(\mathcal{H}) &= \sum_{l=1}^{k} (2|H_l \setminus S| - 1) + 2(|S| - 1) \\
&= \sum_{l=1}^{k-2} (2|H_l \setminus S| - 1) + 2(|S| - 1) + (2|H_{k-1} \setminus S| - 1) + (2|H_k \setminus S| - 1) \\
&= \sum_{l=1}^{k-2} (2|H_l \setminus S| - 1) + 2(|S| - 1) + (2|(H_{k-1} \cup H_k) \setminus S| - 1) \\
&\quad + (2|(H_{k-1} \cap H_k) \setminus S| - 1) \\
&\geq \text{val}_S(\overline{\mathcal{H}}) + 1.
\end{aligned}$$

□

Given an S-sparse graph $G = (V, E)$ with $S \subseteq V$, we define a set $H \subseteq V(G)$ with $|H| \geq 2$ to be *S-tight* if $i_G(H) = \text{val}_S(H)$. Similarly, an S-compatible family $\mathcal{H}$ is said to be *S-tight* if $i_G(\mathcal{H}) = \text{val}_S(\mathcal{H})$.

Lemma 3.3. *Let $G = (V, E)$ be a graph, $S \subseteq V$ with $|S| \geq 2$ and $\mathcal{H} = \{H_1, \dots, H_k\}$ be an S-compatible family such that $H_i \cap H_j = S$ for all $1 \leq i < j \leq k$. Suppose $Y \subseteq V$ with $|Y \cap S| \leq 1$ and $|Y \cap H_i| \geq 2$ for some $1 \leq i \leq k$. Then there exists an S-compatible family $\overline{\mathcal{H}}$ with $\operatorname{cov}(\mathcal{H}) \cup \operatorname{cov}(Y) \subseteq \operatorname{cov}(\overline{\mathcal{H}})$ for which $\operatorname{val}_S(\overline{\mathcal{H}}) \leq \operatorname{val}_S(\mathcal{H}) + \operatorname{val}_S(Y)$. Furthermore, if G is S-sparse and $\mathcal{H}$ and Y are both S-tight, then $\overline{\mathcal{H}}$ is also S-tight.*

Proof. By reordering the elements of $\mathcal{H}$ if necessary, we may assume that $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq j-1$ and $|Y \cap H_i| \geq 2$ for all $j \leq i \leq k$, for some $1 \leq j \leq k$. Let $X = Y \cup \bigcup_{i=j}^{k} H_i$ and $\overline{\mathcal{H}} = \{H_1, \dots, H_{j-1}, X\}$. Then we have $\operatorname{cov}(\mathcal{H}) \cup \operatorname{cov}(Y) \subseteq \operatorname{cov}(\overline{\mathcal{H}})$, and

$$
\begin{aligned}
\operatorname{val}_S(\mathcal{H}) + \operatorname{val}_S(Y) &= \sum_{i=1}^{k} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2|Y| - 3) \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + 2(|S| - 1) + \sum_{i=j}^{k} (2|H_i \setminus S| - 1) + (2|Y| - 3) \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2|X \setminus S| - 1) \\
&\quad + 2|Y \cap S| - (k-j) + 2\sum_{i=j}^{k} |Y \cap (H_i \setminus S)| - 3 \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + (2|X \setminus S| - 1) + 2(|S| - 1) \\
&\quad + 2|Y \cap S| - (k-j) + 2\sum_{i=j}^{k} |Y \cap H_i| - 2\sum_{i=j}^{k} |Y \cap S| - 3 \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + (2|X \setminus S| - 1) + 2(|S| - 1) \\
&\quad - (k-j) + 2\sum_{i=j}^{k} |Y \cap (H_i)| - 2|Y \cap S|(k-j) - 3 \\
&\geq \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + (2|X \setminus S| - 1) + 2(|S| - 1) + 2\sum_{i=j}^{k} |Y \cap H_i| - 3(k-j+1) \\
&= \sum_{i=1}^{j-1} (2|H_i \setminus S| - 1) + (2|X \setminus S| - 1) + 2(|S| - 1) + \sum_{i=j}^{k} (2|Y \cap H_i| - 3) \\
&= \operatorname{val}_S(\overline{\mathcal{H}}) + \sum_{i=j}^{k} \operatorname{val}_S(Y \cap H_i)
\end{aligned}
$$

where for the inequality step we use $|Y \cap S| \leq 1$. Since $\operatorname{val}_S(Y \cap H_i) \geq 0$ for all $j \leq i \leq k$ this gives $\operatorname{val}_S(\mathcal{H}) + \operatorname{val}_S(Y) \geq \operatorname{val}_S(\overline{\mathcal{H}})$.

Now suppose that G is S-sparse, and $\mathcal{H}$ and Y are S-tight. Then we have

$$
i(\overline{\mathcal{H}}) + \sum_{i=j}^{k} i(Y \cap H_i) \geq i(\mathcal{H}) + i(Y) = \operatorname{val}_S(\mathcal{H}) + \operatorname{val}_S(Y)
$$

$$\geq \mathrm{val}_S(\overline{\mathcal{H}}) + \sum_{i=j}^{k} \mathrm{val}_S(Y \cap H_i) \geq i(\overline{\mathcal{H}}) + \sum_{i=j}^{k} i(Y \cap H_i),$$

where the first inequality follows since the edges spanned by $\mathcal{H}$ or Y are spanned by $\overline{\mathcal{H}}$ and if some edge is spanned by both $\mathcal{H}$ and Y, then it is spanned by $Y \cap H_i$ for some i. The first equality holds because $\mathcal{H}$ and Y are S-tight, and the second inequality holds by our calculations above. The last inequality holds because G is S-sparse. Hence equality must hold everywhere, which implies that $\overline{\mathcal{H}}$ is also S-tight. $\square$

Lemma 3.4. *Let $G = (V, E)$ be a graph, $S \subseteq V$ with $|S| \geq 2$ and $\mathcal{H} = \{H_1, \ldots, H_k\}$ be an S-compatible family such that $H_i \cap H_j = S$ for all $1 \leq i < j \leq k$. Suppose that $Y \subseteq V$ with $Y \cap S = \emptyset$, $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq k$ and $|Y \cap H_a| = |Y \cap H_b| = 1$ for some (a, b) with $1 \leq a < b \leq k$. Then there is an S-compatible family $\overline{\mathcal{H}}$ with $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$ for which $\mathrm{val}_S(\overline{\mathcal{H}}) = \mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y)$. Furthermore, if G is S-sparse and $\mathcal{H}$ and Y are both S-tight, then $\overline{\mathcal{H}}$ is also S-tight.*

Proof. We may assume that $a = k-1$ and $b = k$. Let $\overline{\mathcal{H}} = \{H_1, \ldots, H_{k-2}, \overline{H}_{k-1}\}$ where $\overline{H}_{k-1} = H_{k-1} \cup H_k \cup Y$. Then $\mathrm{cov}(\mathcal{H}) \cup \mathrm{cov}(Y) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$ and we have

$$\begin{aligned}
&\mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y) = \sum_{i=1}^{k} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2|Y| - 3) \\
&= \sum_{i=1}^{k-2} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2|H_{k-1} \setminus S| - 1) + (2|H_k \setminus S| - 1) + (2|Y| - 3) \\
&= \sum_{i=1}^{k-2} (2|H_i \setminus S| - 1) + 2(|S| - 1) + (2(|H_{k-1} \setminus S| + |H_k \setminus S| + |Y|) - 1) - 4 \\
&= \sum_{i=1}^{k-2} (2|H_i \setminus S| - 1) + (2|(H_{k-1} \cup H_k \cup Y) \setminus S| - 1) + 2(|S| - 1) \\
&\quad + 2|Y \cap (H_{k-1} \setminus S)| + 2|Y \cap (H_k \setminus S)| - 4 \\
&= \mathrm{val}_S(\overline{\mathcal{H}}).
\end{aligned}$$

Now suppose that G is S-sparse and $\mathcal{H}$ and Y are S-tight. Then we have

$$i(\mathcal{H}) + i(Y) = \mathrm{val}_S(\mathcal{H}) + \mathrm{val}_S(Y) = \mathrm{val}_S(\overline{\mathcal{H}}) \geq i(\overline{\mathcal{H}}) \geq i(\mathcal{H}) + i(Y)$$

where the last inequality follows since $|Y \cap H_i| \leq 1$ for all $1 \leq i \leq k$. Hence equality must hold everywhere which implies that $\overline{\mathcal{H}}$ is also S-tight. $\square$

Lemma 3.5. *Let $G = (V, E)$ be an S-sparse graph for some $S \subseteq V$ with $|S| \geq 2$. Suppose $X, Y \subseteq V$ are S-tight sets in G with $|X \cap Y| \geq 2$ and $X, Y \not\subseteq S$. Then $X \cap Y \not\subseteq S$, and $X \cup Y$ and $X \cap Y$ are both S-tight.*

Proof. First note that, since G is S-sparse, we have

$$\begin{aligned}
2|X| - 3 + 2|Y| - 3 = \mathrm{val}_S(X) + \mathrm{val}_S(Y) &= i(X) + i(Y) \\
&\leq i(X \cap Y) + i(X \cup Y) \\
&\leq \mathrm{val}_S(X \cap Y) + \mathrm{val}_S(X \cup Y) \\
&= \mathrm{val}_S(X \cap Y) + 2|X \cup Y| - 3.
\end{aligned}$$

This implies that $\mathrm{val}_S(X \cap Y) \geq 1$ and hence $X \cap Y \not\subseteq S$. This gives $\mathrm{val}_S(X \cap Y) = 2|X \cap Y| - 3$ and hence equality must hold throughout the above sequence of (in)equalities. In particular, $\mathrm{val}_S(X \cup Y) = i(X \cup Y)$ and $\mathrm{val}_S(X \cap Y) = i(X \cap Y)$, so $X \cup Y$ and $X \cap Y$ are S-tight. $\square$

Given a graph $G = (V, E)$ and $T \subseteq V$ with $|T| \geq 1$, we say that G is *strongly T-sparse* if G is S-sparse for all $\emptyset \neq S \subseteq T$.

Lemma 3.6. *Suppose $\mathcal{H}_i$ is an S_i-compatible family in a graph $G = (V, E)$ for $i = 1, 2$ and $S_1 \cap S_2 \neq \emptyset$. Then there exists an $(S_1 \cup S_2)$-compatible family $\mathcal{H}$ in G with $\mathrm{cov}(\mathcal{H}_1) \cup \mathrm{cov}(\mathcal{H}_2) \subseteq \mathrm{cov}(\mathcal{H})$. Furthermore, if G is strongly $(S_1 \cup S_2)$-sparse and $\mathcal{H}_i$ is S_i-tight for $1 \leq i \leq 2$, then $\mathcal{H}$ is $(S_1 \cup S_2)$-tight.*

Proof. Let $\mathcal{H}_1 = \{H_{1,1}, \ldots, H_{1,k}\}$ and $\mathcal{H}_2 = \{H_{2,1}, \ldots, H_{2,l}\}$. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the bipartite graph on $\mathcal{V} = \mathcal{H}_1 \cup \mathcal{H}_2$, with vertex bipartition $(\mathcal{H}_1, \mathcal{H}_2)$, and edge set

$$\mathcal{E} := \{H_{1,i}H_{2,j} : |(H_{1,i} \setminus S_1) \cap (H_{2,j} \setminus S_2)| \geq 1,\ 1 \leq i \leq k,\ 1 \leq j \leq l\}.$$

Let $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{F}_i)$, $1 \leq i \leq r$, be the connected components of $\mathcal{G}$ and put $V_i = \bigcup_{H \in \mathcal{V}_i} H$ for $1 \leq i \leq r$. Let

$$\mathcal{H}_{\text{union}} = \{V_i \cup S_1 \cup S_2 : 1 \leq i \leq r\} \text{ and } \mathcal{H}_{\text{int}} = \{H_{1,i} \cap H_{2,j} : H_{1,i}H_{2,j} \in \mathcal{E}\}.$$

Then $\mathcal{H}_{\text{int}}$ is $(S_1 \cap S_2)$-compatible, $\mathcal{H}_{\text{union}}$ is $(S_1 \cup S_2)$-compatible and we have $\mathrm{cov}(\mathcal{H}_1) \cup \mathrm{cov}(\mathcal{H}_2) \subseteq \mathrm{cov}(\mathcal{H}_{\text{union}})$. So $\mathcal{H}_{\text{union}}$ satisfies the first part of the lemma.

Now suppose that G is strongly $(S_1 \cup S_2)$-sparse and $\mathcal{H}_i$ is S_i-tight for $1 \leq i \leq 2$. We will complete the proof by showing that $\mathcal{H}_{\text{union}}$ is $(S_1 \cup S_2)$-tight. Every edge in E which is covered by either $\mathcal{H}_1$ or $\mathcal{H}_2$ is covered by $\mathcal{H}_{\text{union}}$ and every edge covered by both $\mathcal{H}_1$ and $\mathcal{H}_2$ is covered by $\mathcal{H}_{\text{int}}$. This implies that $i(\mathcal{H}_1) + i(\mathcal{H}_2) \leq i(\mathcal{H}_{\text{union}}) + i(\mathcal{H}_{\text{int}})$. Since $|\mathcal{V}| = k + l$ and r is the number of connected components of $\mathcal{G}$,

$$r + |\mathcal{E}| \geq k + l. \tag{3}$$

We also have

$$\begin{aligned}
&\sum_{i=1}^{r} (|V_i \cup S_1 \cup S_2| - |S_1 \cup S_2|) + \sum_{H_{1,i}H_{2,j} \in \mathcal{E}} (|H_{1,i} \cap H_{2,j}| - |S_1 \cap S_2|) \\
&= \sum_{i=1}^{k} (|H_{1,i}| - |S_1|) + \sum_{i=j}^{l} (|H_{2,j}| - |S_2|)
\end{aligned} \tag{4}$$

as a vertex $x \notin S_1 \cup S_2$ contributes the same amount (one or two) to both sides of (4), and a vertex $x \in S_1 \cup S_2$ contributes zero to both sides of (4).

Since $\mathcal{H}_{\rm int}$ is $(S_1 \cap S_2)$-compatible and $\mathcal{H}_{\rm union}$ is $(S_1 \cup S_2)$-compatible, we have

$$
\begin{aligned}
&\sum_{i=1}^{k}(2|H_{1,i} \setminus S_1|-1)+2(|S_1|-1)+\sum_{j=1}^{l}(2|H_{2,j} \setminus S_2|-1)+2(|S_2|-1)\\
&= \mathrm{val}_{S_1}(\mathcal{H}_1)+\mathrm{val}_{S_2}(\mathcal{H}_2)\\
&= i(\mathcal{H}_1)+i(\mathcal{H}_2)\\
&\le i(\mathcal{H}_{\rm union})+i(\mathcal{H}_{\rm int})\\
&\le \mathrm{val}_{S_1\cup S_2}(\mathcal{H}_{\rm union})+\mathrm{val}_{S_1\cap S_2}(\mathcal{H}_{\rm int})\\
&= \sum_{i=1}^{r}(2|(V_i \cup S_1 \cup S_2)\setminus(S_1 \cup S_2)|-1)+2(|S_1 \cup S_2|-1)\\
&\qquad + \sum_{H_{1,i}H_{2,j}\in\mathcal{E}}(2|(H_{1,i}\cap H_{2,j})\setminus(S_1\cap S_2)|-1)+2(|S_1\cap S_2|-1)\\
&= \sum_{i=1}^{r}2(|V_i \cup S_1 \cup S_2|-|S_1 \cup S_2|)+2(|S_1 \cup S_2|-1)-r\\
&\qquad + \sum_{H_{1,i}H_{2,j}\in\mathcal{E}}2(|H_{1,i}\cap H_{2,j}|-|S_1\cap S_2|)+2(|S_1\cap S_2|-1)-|\mathcal{E}|\\
&\le \sum_{i=1}^{k}2(|H_{1,i}|-|S_1|)+\sum_{j=1}^{l}2(|H_{2,i}|-|S_2|)\\
&\qquad +2(|S_1 \cup S_2|-1)+2(|S_1\cap S_2|-1)-k-l\\
&= \sum_{i=1}^{k}2(|H_{1,i}|-|S_1|)+\sum_{j=1}^{l}2(|H_{2,j}|-|S_2|)+2|S_1|+2|S_2|-2-2-k-l\\
&= \sum_{i=1}^{k}(2|H_{1,i} \setminus S_1|-1)+2(|S_1|-1)+\sum_{j=1}^{l}2|H_{2,j} \setminus S_2|-1+2(|S_2|-1),
\end{aligned}
$$

where the third inequality follows from (3) and (4), and the second last equality follows from the formula $|S_1 \cup S_2|+|S_1 \cap S_2| = |S_1|+|S_2|$. Therefore equality must hold throughout. This implies that $\mathcal{H}_{\rm union}$ is $(S_1 \cap S_2)$-tight (and also that $\mathcal{H}_{\rm int}$ is $(S_1 \cap S_2)$-tight). □

3.2 The matroid $\mathcal{M}_T(G)$

Given a graph $G = (V, E)$ and a set $T \subseteq V$, we will show that the family $\mathcal{I}_T = \{I \subseteq E : G' = (V, I)$ is strongly T-sparse$\}$ is the family of independent sets of a matroid $\mathcal{M}_T(G)$ on E when $1 \le |T| \le 3$.

We need the following concept. For $S \subseteq V$ with $|S| \ge 2$, an *augmented S-compatible family* is a collection $\mathcal{L} = \{\mathcal{H}, X_1, \ldots, X_k\}$ where $\mathcal{H}$ is a (possibly empty) S-compatible family of subsets of V and $X_1, \ldots, X_k$ are subsets of V of size at least two. We say that $\mathcal{L}$ *covers* an edge $e \in E$ if e is induced by either a set $H \in \mathcal{H}$ or by one of the sets X_i for some $1 \le i \le k$ and let $i_G(\mathcal{L})$ be the number of edges of G which are covered by $\mathcal{L}$. The family $\mathcal{L}$ is 1-*thin* if:

(T.1) $|X_i \cap X_j| \le 1$ for all pairs $1 \le i < j \le k$;

(T.2) $H_i \cap H_j = S$ for all distinct $H_i, H_j \in \mathcal{H}$;

(T.3) $|X_i \cap \bigcup_{H \in \mathcal{H}} H| \leq 1$ for all $1 \leq i \leq k$.

We define the S-*value* of $\mathcal{L}$ to be

$$\mathrm{val}_S(\mathcal{L}) = \begin{cases} \mathrm{val}_S(\mathcal{H}) + \sum_{i=1}^k (2|X_i| - 3), & \text{if } \mathcal{H} \neq \emptyset \\ \sum_{i=1}^k (2|X_i| - 3), & \text{if } \mathcal{H} = \emptyset\,. \end{cases}$$

Note that,

$$\text{if } G \text{ is } S\text{-sparse and } \mathcal{L} \text{ is 1-thin, then } i_G(\mathcal{L}) \leq \mathrm{val}_S(\mathcal{L}). \tag{5}$$

Recall that a family $\mathcal{I}$ of subsets of a finite set E is the family of independent sets in a matroid on E if it satisfies the following three axioms:

(I.1) $\emptyset \in \mathcal{I}$;

(I.2) if $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$;

(I.3) for all $E' \subseteq E$, every maximal element of $\{I \in \mathcal{I} : I \subseteq E'\}$ has the same cardinality.

We will use these axioms to verify that $\mathcal{I}_T$ is the independent set system of a matroid and determine its rank function.

Theorem 3.7. *Let $G = (V, E)$ be a graph, $T \subseteq V$ with $1 \leq |T| \leq 3$, and E_T be the set of edges of G induced by T. Then*

$$\mathcal{I}_T = \{I \subseteq E : G' = (V, I) \textit{ is strongly } T\textit{-sparse}\} \tag{6}$$

is the family of independent sets in a matroid $\mathcal{M}_T(G)$ on E. In addition, the rank of any $E' \subseteq E$ in $\mathcal{M}_T(G)$ is given by $r(E') = \min\{\mathrm{val}_S(\mathcal{L})\}$ where the minimum is taken over all $S \subseteq T$ with $|S| \geq 2$ and all 1-thin, augmented S-compatible families $\mathcal{L}$ which cover $E' \setminus E_T$.

Proof. We show that $\mathcal{I}_T$ satisfies the independence axioms (I.1), (I.2) and (I.3). Since (I.1) and (I.2) follow immediately from the definition of $\mathcal{I}_T$, we only need to verify that (I.3) holds. Choose $E' \subseteq E$. We first show that

Claim 3.8. *Let F be a maximal element of $\{I \in \mathcal{I}_T : I \subseteq E'\}$. Then $|F| = \mathrm{val}_S(\mathcal{L})$ for some $S \subseteq T$ with $|S| \geq 2$ and some 1-thin, augmented S-compatible family $\mathcal{L}$ which covers $E' \setminus E_T$.*

Proof of Claim. Let $J = (V, F)$ denote the subgraph of G induced by F. Consider the following two cases.

Case 1: J has no tight S-compatible family for all $S \subseteq T$ with $|S| \geq 2$. Let $X_1, X_2, \ldots, X_k$ be the maximal T-tight sets in J and put $\mathcal{L}_1 = \{\emptyset, X_1, X_2, \ldots, X_k\}$. Then $\mathcal{L}_1$ is an augmented T-compatible family. Since $X = \{x, y\}$ is a T-tight set in J for all edges $xy \in F$, $\mathcal{L}_1$ covers F. In addition, Lemma 3.5 and the maximality of the sets $X_1, X_2, \ldots, X_k$ imply that $\mathcal{L}_1$ is 1-thin. Since each X_i is T-tight in J this gives,

$$|F| = \sum_{i=1}^k |E_J(X_i)| = \sum_{i=1}^k (2|X_i| - 3) = \mathrm{val}_T(\mathcal{L}_1).$$

We claim that $\mathcal{L}_1$ covers every edge of $E' \setminus E_T$. To see this consider an edge $ab \in E' \setminus (F \cup E_T)$. Since F is a maximal subset of E' in $\mathcal{I}_T$ we have $F + ab \notin \mathcal{I}_T$. Our assumption that there is no S-tight, S-compatible family in J, now implies that there is a T-tight set X in J with $a, b \in X$. Hence $X \subseteq X_i$ for some $1 \leq i \leq k$. This implies that $\mathcal{L}_1$ covers ab. Hence $\mathcal{L}_1$ covers every edge of $E' \setminus E_T$ and the claim holds in this case.

Case 2: J has an S-tight, S-compatible family $\mathcal{H}$ for some $S \subseteq T$ with $|S| \geq 2$. We may assume by Lemma 3.6 that, for every S'-tight, S'-compatible family $\mathcal{H}'$ in J with $S' \subseteq T$ and $|S'| \geq 2$, we have $S' \subseteq S$ and $\mathrm{cov}(\mathcal{H}') \subseteq \mathrm{cov}(\mathcal{H})$. Let $X_1, X_2, \ldots, X_k$ be the maximal S-tight sets of $J' = (V, F \setminus E_J(\mathcal{H}))$ and put $\mathcal{L}_2 = \{\mathcal{H}, X_1, X_2, \ldots, X_k\}$. Then $\mathcal{L}_2$ is an augmented S-compatible family which covers F and we have

$$i_{J'}(X_i) = 2|X_i| - 3 \text{ for all } 1 \leq i \leq k. \tag{7}$$

We next show that $\mathcal{L}_2$ is 1-thin. Lemma 3.5 and the maximality of the sets $X_1, X_2, \ldots, X_k$ imply that $|X_i \cap X_j| \leq 1$ for all $1 \leq i < j \leq k$, so (T.1) holds. Lemma 3.2 and the fact that $\mathcal{H}$ is S-tight imply that $H_i \cap H_j = S$ for all distinct $H_i, H_j \in \mathcal{H}$ (otherwise we could construct an S-compatible family $\overline{\mathcal{H}}$ in J with $\mathrm{val}_S(\overline{\mathcal{H}}) < \mathrm{val}_S(\mathcal{H}) = i_J(\mathcal{H}) \leq i_J(\overline{\mathcal{H}})$ and this would contradict the hypothesis that $F \in \mathcal{I}_T$). Hence (T.2) holds. Choose a set X_i. We show that (T.3) holds for X_i. If $|X_i \cap S| \geq 2$ then $\mathcal{H}' = \{X_i\}$ would be an $(S \cap X_i)$-tight, $(S \cap X_i)$-compatible family and the maximality of $\mathcal{H}$ would give $E_J(X_i) \subseteq \mathrm{cov}(\mathcal{H}') \subseteq \mathrm{cov}(\mathcal{H})$. This would imply that $E_{J'}(X_i) = \emptyset$ and contradict (7). Hence $|X_i \cap S| \leq 1$.

Suppose $|X_i \cap H_j| \geq 2$ for some $H_j \in \mathcal{H}$. Then Lemma 3.3 gives us an S-tight, S-compatible family $\overline{\mathcal{H}}$ with $\mathrm{cov}(H) \cup \mathrm{cov}(X_i) \subseteq \mathrm{cov}(\overline{\mathcal{H}})$. The maximality of $\mathrm{cov}(\mathcal{H})$ now implies that $\mathrm{cov}(X_i) \subseteq \mathrm{cov}(\mathcal{H})$ and contradicts (7). Hence $|X_i \cap H_j| \leq 1$ for all $H_j \in \mathcal{H}$.

If $|X_i \cap S| = 1$ then the facts that $|X_i \cap H_j| \leq 1$ and $S \subset H_j$ for all $H_j \in \mathcal{H}$ would imply that (T.3) holds for X_i. So we may assume that $X_i \cap S = \emptyset$. We can now use Lemma 3.3 and a similar argument to the previous paragraph to deduce that $|X_i \cap H_j| = 1$ for at most one $H_j \in \mathcal{H}$, so (T.3) holds for X_i. Hence $\mathcal{L}_2$ is 1-thin and we have

$$\begin{aligned} |F| &= \sum_{H_i \in \mathcal{H}} |E_J(H_i)| + \sum_{j=1}^{k} |E_J(X_j)| \\ &= \sum_{H_i \in \mathcal{H}} (2|H_i \setminus S| - 1) + 2(|S| - 1) + \sum_{j=1}^{k} (2|X_j| - 3) = \mathrm{val}_S(\mathcal{L}_2). \end{aligned}$$

We complete the proof of the claim by showing that $\mathcal{L}_2$ is a cover of $E' \setminus E_T$. Choose $ab = e \in E' \setminus (F \cup E_T)$. By the maximality of F we have $F + e \notin \mathcal{I}_T$. Thus J has either an S'-tight set X with $a, b \in X$ or an S'-tight S'-compatible family $\overline{\mathcal{H}}$ with $a, b \in Y_i \in \overline{\mathcal{H}}$ for some $S' \subseteq T$ with $|S'| \geq 2$. In the latter case, the maximality of $\mathrm{cov}(\mathcal{H})$ implies that $\mathrm{cov}(\overline{\mathcal{H}}) \subseteq \mathrm{cov}(\mathcal{H})$ and hence e is covered by $\mathcal{L}_2$. Hence we may assume that $a, b \in X$ for some $X \subseteq V$ with $i_J(X) = 2|X| - 3$. If $|X \cap \bigcup_{H_j \in \mathcal{H}} H_j| \geq 2$, then we can use a similar argument to that used to show that X_i satisfies (T.3) to deduce that $\mathrm{cov}(X) \subseteq \mathrm{cov}(\mathcal{H})$ which implies that $\mathcal{L}_2$ covers e. Hence we may assume that that $|X \cap \bigcup_{H_j \in \mathcal{H}} H_j| \leq 1$. Then $E(X) \subseteq E(J')$ and hence $X \subseteq X_i$ for some $1 \leq i \leq k$. Hence e is covered by $\mathcal{K}_2$. □

We saw in (5) that, if G is S-sparse and $\mathcal{L}$ is a 1-thin, augmented S-compatible family in G, then $i_G(\mathcal{L}) \leq \mathrm{val}_S(\mathcal{L})$. Together with Claim 3.8, this implies that $|F| = \min\{\mathrm{val}_S(\mathcal{L})\}$ where the minimum is taken over all $S \subseteq T$ with $|S| \geq 2$ and all 1-thin, augmented S-compatible families $\mathcal{L}$ which cover $E' \setminus E_T$. Since $\min\{\mathrm{val}_S(\mathcal{L})\}$ is independent of the choice of F, all maximal elements of $\{I \in \mathcal{I}_T : I \subseteq E'\}$ have the same cardinality. Hence (I.3) holds for $\mathcal{I}_T$ and $\mathcal{M}_T(G)$ is a matroid. The assertion that $r(E') = \min\{\mathrm{val}_S(\mathcal{L})\}$ follows immediately since $r(E')$ is equal to the cardinality of any maximal element of $\{I \in \mathcal{I}_T : I \subseteq E'\}$. □

The special cases of Theorem 3.7 when $|T| = 1, 2$ are given in [8] and [4], respectively.

4 Coincident rigidity

Throughout this section we will only be concerned with 2-dimensional frameworks so will suppress reference to the ambient space $\mathbb{R}^2$. Let $G = (V, E)$ be a graph, $T \subseteq V$ with $|T| = 3$. If G has an independent, T-coincident realisation (G, p) then (G, p) is an independent, S-coincident realisation for all $S \subseteq T$ with $|S| = 2$. We can combine this observation with Lemma 3.1 to deduce that every independent set in the T-coincident rigidity matroid $\mathcal{R}_T(G)$ is independent in the matroid $\mathcal{M}_T(G)$ given by Theorem 3.7. Hence, to show that $\mathcal{R}_T(G) = \mathcal{M}_T(G)$, it only remains to show that every independent set in $\mathcal{M}_T(G)$ is independent in $\mathcal{R}_T(G)$. We will do this by induction on $|V|$: we suppose that E is independent in $\mathcal{M}_T(G)$ and perform a graph theoretic reduction operation to create a smaller graph $G' = (V', E')$ such that E' is independent in $\mathcal{M}_T(G')$; we apply induction to deduce that E' is independent in $\mathcal{R}_T(G')$ then use the fact that the inverse of the reduction operation preserves independence in the T-coincident rigidity matroid to deduce that E is independent in $\mathcal{R}_T(G)$. The last step in this argument uses the following extension operations and geometric lemmas.

Our first two lemmas concern the so called 0- and 1-extension operations. We refer the reader to [12] for their proofs. The 0-*extension operation* on a graph $G = (V, E)$ constructs a new graph by adding a new vertex w and two new edges from w to V. The 1-*extension operation* constructs a new graph from G by deleting an edge uv and then adding a new vertex w and three new edges wu, wv, wx for some $x \in V \setminus \{u, v\}$.

Lemma 4.1. *Suppose that G is a graph and that G' is obtained from G by a 0-extension operation which adds a new vertex w and new edges wu, wv. Suppose further that (G', p) is a realisation of G' and that u, v, w are not colinear in (G, p). Then (G', p) is independent if and only if $(G, p|_{V(G)})$ is independent.*

Lemma 4.2. *Suppose that (G, p) is an independent framework and that G' is obtained from G by a 1-extension operation which adds a new vertex w. Suppose further that neighbours of w in G' are not colinear in (G, p). Then (G', p') is independent for some p' with $p'(x) = p(x)$ for all $x \in V(G)$.*

Our third extension result is a geometric version of a generic vertex splitting lemma of Whiteley which is stated without proof in [12]. Given a graph $G = (V, E)$ and $v \in V$ with neighbour set $N_G(v)$, the *vertex splitting operation* chooses pairwise disjoint sets U_1, U_2, U_3 with $U_1 \cup U_2 \cup U_3 = N_G(v)$ and $|U_2| = 2$, deletes all edges from v to U_3, and then adds a new vertex v' and $|U_3| + 2$ new edges from v' to each vertex in $U_2 \cup U_3$.

Lemma 4.3. *Suppose that (G, p) is an independent framework and that G' is obtained from G by a vertex splitting operation which splits a vertex $z \in V(G)$ into two vertices z, z'. Suppose further that the common neighbours z_1, z_2 of z and z' in G' are not colinear with z in (G, p). Put $p'(z) = p'(z') = p(z)$ and $p'(x) = p(x)$ for all $x \in V(G) - z$. Then (G', p') is independent.*

Proof. Let $N_{G'}(z) = \{z_1, z_2, \ldots, z_k\}$ and $N_{G'}(z') = \{z_1, z_2, z_{k+1}, z_{k+2}, \ldots, z_m\}$. Then the rigidity matrix $R(G',p')$ has the following form.

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \quad z \quad & \quad z' \quad & \quad z_1 \quad & \quad z_2 \quad & \end{array} \\
\begin{array}{c} zz_1 \\ zz_2 \\ z'z_1 \\ z'z_2 \\ zz_3 \\ \vdots \\ zz_k \\ z'z_{k+1} \\ \vdots \\ z'z_m \\ \\ \end{array} &
\left[\begin{array}{ccccc}
p(z)-p(z_1) & (0,0) & p(z_1)-p(z) & (0,0) & \cdots \\
p(z)-p(z_2) & (0,0) & (0,0) & p(z_2)-p(z) & \cdots \\
(0,0) & p(z)-p(z_1) & p(z_1)-p(z) & (0,0) & \cdots \\
(0,0) & p(z)-p(z_2) & (0,0) & p(z_2)-p(z) & \cdots \\
p(z)-p(z_3) & (0,0) & (0,0) & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
p(z)-p(z_k) & (0,0) & (0,0) & (0,0) & \cdots \\
(0,0) & p(z)-p(z_k) & (0,0) & (0,0) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
(0,0) & p(z)-p(z_m) & (0,0) & (0,0) & \cdots \\
\cline{3-5}
0 & 0 & \multicolumn{3}{|c}{R(G-z,p)}
\end{array}\right]
\end{array}
$$

Let M be the matrix obtained from $R(G',p')$ as follows: subtract row 3 from row 1 and row 4 from row 2, then add column 1 to column 2. Then $M = \begin{pmatrix} p(z)-p(z_1) & 0 \\ p(z)-p(z_2) & 0 \\ * & R(G,p) \end{pmatrix}$. The hypotheses that (G,p) is independent and $p(z), p(z_1), p(z_2)$ are not colinear now implies that M has independent rows. Hence (G',p') is independent. $\square$

Our fourth extension lemma gives sufficient conditions for the operation of replacing a rigid subgraph of a graph by a larger rigid subgraph to preserve rigidity.

Lemma 4.4. *Let $G=(V,E)$ be a graph, $Y \subseteq V$ such that $G[Y]$ is rigid and $\{Y_1, Y_2, \ldots, Y_m\}$ be a partition of Y with $m \geq 3$. Let $G'=(V',E')$ be obtained from G by contracting each set Y_i to a single vertex y_i for all $1 \leq i \leq m$ and then adding an edge y_iy_j for all non-adjacent pairs y_i, y_j, $1 \leq i < j \leq m$. Put $Y' = \{y_1, y_2, \ldots, y_m\}$. Suppose that (G',p') is an infintesimally rigid realisation of G'. Then (G,p) is infinitesimally rigid for some $p: V \to \mathbb{R}^2$ with $p|_{V\setminus Y} = p'|_{V'\setminus Y'}$.*

Proof. Let G^* be obtained from G by adding all edges between the vertices of Y. Since $G[Y]$ is rigid, it will suffice to show that (G^*,p) is infinitesimally rigid for some $p: V \to \mathbb{R}^2$ with $p|_{V\setminus Y} = p'|_{V'\setminus Y'}$. Let $p(v) = p'(v)$ for each $v \in V \setminus Y$ and $p(v) = p'(y_i)$ for each $v \in Y_i$, $1 \leq i \leq m$. Then $(G^*[Y], p|_Y)$ is infinitesimally rigid. In addition, each infinitesimal motion of (G^*,p) which fixes Y induces an infinitesimal motion of (G',p') which fixes Y'. Since (G',p') is infintesimally rigid, every such motion fixes $V' \setminus Y'$. Hence (G^*,p) is infinitesimally rigid. $\square$

Our final lemma complements Lemmas 4.1 and 4.2 by showing that the inverse of the 0- and 1-extension operations preserves the property of being strongly T-sparse.

Lemma 4.5. *Let $G=(V,E)$ be a graph and $T \subseteq V$ with $1 \leq |T| \leq 3$. Suppose that G is strongly T-sparse and $z \in V \setminus T$ has at most one neighbour in T.*
(a) If $d(z)=2$ then $G-z$ is strongly T-sparse.
(b) If $d(z)=3$ then $G-z+xy$ is strongly T-sparse for some non-adjacent $x,y \in N_G(z)$.

Proof. Statement (a) follows immediately from the fact if H is a subgraph of a strongly T-sparse graph and $T \subseteq V(H)$ then H is strongly T-sparse.

To verify (b) we let $F = \{ab : a, b \in N(z)\}$, $G_1 = G - z + F$ and $G_2 = G + F$. Let $r(H)$ denote the rank of the edge set of an arbitrary subgraph $H \subseteq G_2$ in the matroid $\mathcal{M}_T(G_2)$ given by Theorem 3.7. Suppose, for a contradiction that (b) is false. Then $r(G_1) = r(G - z)$. Since G is strongly T-sparse, E is independent in $\mathcal{M}_T(G_2)$ and hence $r(G_1) = r(G - z) = r(G) - 3$. Choose a base B_1 of $\mathcal{M}_T(G_1)$ that contains F and extend it to a base B_2 of $\mathcal{M}_T(G_2)$. Since the edges of $G_2[N(G(z) \cup \{z\}] \cong K_4$ is a circuit of $\mathcal{M}_T(G_2)$, we have $r(G_2) = |B_2| \leq |B_1| + 2 = r(G_1) + 2$. Hence $r(G) \leq r(G_2) \leq r(G) - 1$, a contradiction. □

Theorem 4.6. *Let $G = (V, E)$ be a graph and $T \subseteq V$ with $1 \leq |T| \leq 3$. Then E is independent in $\mathcal{R}_T(G)$ if and only if G is strongly T-sparse.*

Proof. If E is independent in $\mathcal{R}_T(G)$ then G is strongly T-sparse by Lemma 3.1. Hence we need only verify the reverse implication. Suppose for a contradiction that this is false and that (G, T) has been chosen to be a counterexample such that: $|V|$ as small as possible; subject to this condition, $|E|$ as large as possible; subject to these two conditions, the number of vertices of degree at most three in G is as large as possible. It is easy to check that E is independent in $\mathcal{R}_T(G)$ when $|V| \leq |T| + 1$ so we may assume that $|V| \geq |T| + 2$. Let K be a complete graph with vertex set V. If E is not a base of $\mathcal{M}_T(K)$ then we could add an edge of K to E to obtain a counterexample with more edges than G. Hence E is a base of $\mathcal{M}_T(K)$. This implies that $|E| = 2|V| - 3$. (We have $|E| \leq 2|V| - 3$ since G is strongly T-sparse. On the other hand, $|E| \geq 2|V| - 3$ since independence in $\mathcal{R}_T(K)$ implies independence in $\mathcal{M}_T(K)$ and we can construct an independent set of size $2|V| - 3$ in $\mathcal{R}_T(K)$ by starting with an edge joining two vertices of $V \setminus T$ and then recursively applying Lemma 4.1.)

Claim 4.7. *G has minimum degree three.*

Proof of Claim. Since $|E| = 2|V| - 3$, G has minimum degree at most three. Suppose, for a contradiction, that G has a vertex z with $d(z) \leq 2$.

We first consider the case when $z \notin T$. If $|N_G(z) \cap T| \leq 1$ then we can apply Lemma 4.5(a) to deduce that $G - z$ is strongly T-sparse, use the minimality of V to deduce that every generic T-coincident framework $(G - z, p)$ is independent, and then apply Lemma 4.1 to obtain an independent T-coincident realsation of G. This would imply that E is independent in $\mathcal{R}_T(G)$ and contradict the choice of G. Hence $d(z) = 2$ and $N_G(z) \subseteq T$. Let $\mathcal{H} = \{T \cup \{z\}, V - z\}$. Then $\mathcal{H}$ is a T-compatible family with $\mathrm{val}_T(\mathcal{H}) = 2|V| - 4$. This contradicts the fact that G is strongly T-sparse since $i_G(\mathcal{H}) = |E| = 2|V| - 3$.

Now suppose that $z \in T$. Then $N_G(z) \cap T = \emptyset$ since G is strongly T-sparse. Let $T' = T - z$ if $|T| \geq 2$ and otherwise put $T' = \{z'\}$ where z' is an arbitrary vertex in $V - z$. Then $G - z$ is strongly T'-sparse as it is a subgraph of G. The choice of G now implies that every generic T'-coincident framework $(G - z, p)$ is independent. We can now apply Lemma 4.1 to obtain an independent T-coincident realsation of G. This implies that E is independent in $\mathcal{R}_T(G)$ and contradicts the choice of G. □

Let W be the set of all vertices of G having at least two neighbours in T and put $X = T \cup W$.

Claim 4.8. *Every vertex of degree three in G belongs to X.*

Proof of Claim. Suppose for a contradiction that there exists a vertex $z \in V \setminus X$ of degree three in G. Then we can apply Lemma 4.5(b) to deduce that $G - z + xy$ is strongly T-sparse for some non-adjacent $x, y \in N_G(z)$, use the minimality of V to deduce that every generic T-coincident framework $(G-z+xy,p)$ is independent, and then apply Lemma 4.2 to obtain an independent T-coincident realsation of G. This implies that E is independent in $\mathcal{R}_T(G)$ and contradicts the choice of G. $\square$

Let $\mathcal{H}_0 = \{T\cup\{w\} : w \in W\}$. Then $\mathcal{H}_0$ is a T-compatible family and $|W|+2|T|-2 = \mathrm{val}_T(\mathcal{H}_0) \geq i_G(\mathcal{H}_0) \geq 2|W|$ since G is T-sparse. Since $|E| = 2|V|-3$, Claims 4.7 and 4.8 imply that $|X| = |T|+|W| \geq 6$. These two inequalities, combined with the fact that $|T| \leq 3$, give

$$|T| = 3,\ \ 3 \leq |W| \leq 4 \ \text{ and } \ 2|W| \leq i_G(\mathcal{H}_0) \leq |W|+2|T|-2 = |W|+4. \tag{8}$$

Since each vertex in W has at least two neighbours in T, (8) and the hypothesis that G is strongly T-sparse imply that we can label the vertices in T as u,v,w in such a way that:

$$G[X] \text{ contains one of the graphs shown in Figure 1 as a spanning subgraph.} \tag{9}$$

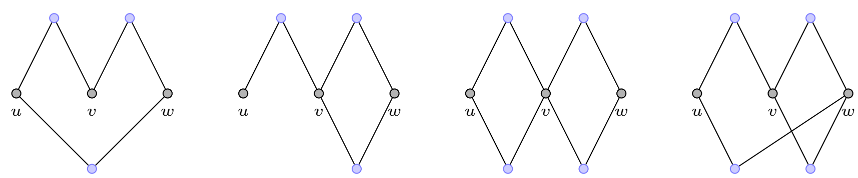

Figure 1: Possible spanning subgraphs of $G[X]$ with $T = \{u,v,w\}$.

We next use Lemma 4.3 to show that the first possibility in Figure 1 must occur.

Claim 4.9. $G[X]$ *does not contain any of the three graphs on the right of Figure 1 as a spanning subgraph.*

Proof of Claim. Suppose, for a contradiction, that the claim is false. Then there exists a 4-cycle C in $G[X]$ with $v,w \in V(C)$ and $E(C) \subseteq E(T,W)$.

Consider the graph $G' = G/vw$. Let z be the vertex of G' coresponding to $\{v,w\}$ and put $T' = \{u,z\}$. If G' is strongly T'-sparse then the choice of G implies that every generic T'-coincident framework (G',p) is independent. We could now apply Lemma 4.3 at z to obtain an independent T-coincident realisation of G and contradict the choice of G. Hence G' is not strongly T'-sparse. Since $|T'| = 2$, G' is not T'-sparse and hence there exists either a set $Z \subseteq V(G')$ such that $i_{G'}(Z) > 2|Z|-3$ or a T'-compatible family $\mathcal{H}$ in G' such that $i_{G'}(\mathcal{H}) > \mathrm{val}_{T'}(\mathcal{H})$.

Suppose the first alternative holds. We may assume that Z has been chosen to be as small as possible. The minimality of Z implies that each vertex in Z has degree at least three in $G'[Z]$, and the fact that G is strongly T-sparse tells us that $z \in Z$. If $u \in Z$ then $\mathcal{H} = \{Z\}$ would be a T'-compatible family with $i_{G'}(\mathcal{H}) > \mathrm{val}_{T'}(\mathcal{H})$. We will consider this possibility when we investigate the second alternative that G' has a T'-compatible family $\mathcal{H}$ with $i_{G'}(\mathcal{H}) > \mathrm{val}_{T'}(\mathcal{H})$, so we now assume that $u \notin Z$. This and the fact that $G[X]$ has one of the three graphs to the right of Figure 1 as a spanning subgraph imply that every vertex of $W \cap Z$ will have lower degree in $G'[Z]$ than in G. Since $G'[Z]$ has minimum degree at least three and W has at most one vertex with degree geater than three in G by (8), we have $|W \cap Z| \leq 1$. We can now obtain a contradiction by examining the three choices for the spanning subgraphs of $G[X]$ given in Figure 1. If $G[X]$ contains the second graph in the figure as a spanning subgraph, we would have $d_{G'}(z) = d_G(v) + d_G(w) - 2 = 4$

and the fact that $|W \cap Z| \leq 1$ now gives $d_{G'[Z]}(z) \leq 2$. On the other hand, if $G[X]$ contains the third or fourth graph in the figure as a spanning subgraph, we would have $d_{G'}(z) = d_G(v) + d_G(w) - 2 \leq 5$ and the fact that $|W \cap Z| \leq 1$ again gives $d_{G'[Z]}(z) \leq 2$. Both possibilities contradict the fact that $G'[Z]$ has minimum degree at least three.

Hence there exists a T'-compatible family $\mathcal{H}$ in G' such that $i_{G'}(\mathcal{H}) > \mathrm{val}_{T'}(\mathcal{H})$. We may assume that $\mathcal{H}$ has been chosen such that $i_{G'}(\mathcal{H})$ is minimal. Let $C = vawbv$. Since at most one vertex in X has degree greater than three, we may also assume that b has degree three in G. Then $d_{G'}(b) = 2$ and $T' \not\subseteq N_{G'}(b)$. We may now use the minimality of $i_{G'}(\mathcal{H})$ to deduce that $b \notin H$ for all $H \in \mathcal{H}$. Consider the two T-compatible families in G given by

$$\mathcal{H}_1 := \{H - z + v + w : H \in \mathcal{H}\} \cup \{\{a, u, v, w\}, \{b, u, v, w\}\}$$

and

$$\mathcal{H}_2 := \{H - z + v + w : H \in \mathcal{H}\} \cup \{\{b, u, v, w\}\}.$$

If $a \notin H$ for all $H \in \mathcal{H}$ we have

$$\mathrm{val}_T(\mathcal{H}_1) = \mathrm{val}_{T'}(\mathcal{H}) + 4 < i_{G'}(\mathcal{H}) + 4 = i_G(\mathcal{H}_1)$$

and, if $a \in H$ for some $H \in \mathcal{H}$, then

$$\mathrm{val}_T(\mathcal{H}_2) = \mathrm{val}_{T'}(\mathcal{H}) + 3 < i_{G'}(\mathcal{H}) + 3 = i_G(\mathcal{H}_2).$$

Both alternatives contradict the the fact that G is T-sparse. □

Our next result extends the previous claim by showing, in particular, that $G[X]$ is equal to the first graph of Figure 1. Let $Y = V \setminus X$.

Claim 4.10. *$G[X]$ is a cycle of length six, $|E_G(X, Y)| = 6$, $i_G(Y) = 2|Y| - 3$ and $G[Y]$ is rigid.*

Proof of Claim. Let G_1 be the first graph of Figure 1. Then G_1 is a spanning subgraph of $G[X]$ by (9) and Claim 4.9. If there exists an edge in $E_G(T, W) \setminus E(G_1)$ then $G[X]$ would contain the second graph in Figure 1 as a spanning subgraph, contradicting Claim 4.9. Thus $E_G(T, W) \subseteq E(G_1)$. Claims 4.7, the definition of W and the fact that $E_G(T, W) \subseteq E(G_1)$ imply that each vertex in T is adjacent to a distinct vertex in Y and hence $|Y| \geq 3$. Since G is T-sparse, we have $i_G(Y) \leq 2|Y| - 3$. Let α be the number of edges in $G[X] - E(G_1)$. Then the fact that each vertex of X has degree three in G gives

$$\begin{aligned} 2|V(G)| - 3 = |E(G)| = i_G(X) + |E_G(X, Y)| + i_G(Y) &\leq (6 + \alpha) + (6 - 2\alpha) + 2|Y| - 3 \\ &= 2|V(G)| - 3 - \alpha. \end{aligned}$$

Hence $\alpha = 0$ and equality holds throughout. This gives $G[X] = G_1$, $|E_G(X, Y)| = 6$ and $i_G(Y) = 2|Y| - 3$. Since $G[Y]$ is S-sparse for any $S \subset Y$ with $|S| = 1$, the minimality of G implies that $G[Y]$ is rigid. □

We can now complete the proof of the theorem. We first consider the case when $|Y| = 3$. The seven possibilities for G when $|Y| = 3$ are shown in Figure 2. We can verify that the realisation $p : V \to \mathbb{R}^2$ shown for the first graph gives an infinitesimally rigid, T-coincident realisation for all seven graphs by calculating the rank of the corresponding rigidity matrices. Hence the theorem holds when $|Y| = 3$ so we must have $|Y| \geq 4$.

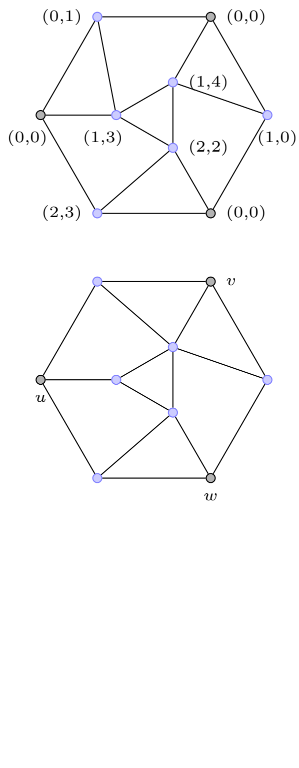

Figure 2: The possible graphs with $G[X] \cong C_6$ and $|Y| = 3$. The vertices in Y are drawn inside the outer six-cycle which corresponds to $G[X]$.

Let y_1, y_2, y_3 be the vertices in Y which are adjacent to T and let $\{Y_1, Y_2, Y_3\}$ be a partition of Y such that $y_i \in Y_i$ for all $1 \leq i \leq 3$. Let $G' = (V', E')$ be obtained from G by contracting each set Y_i to a single vertex y_i for all $1 \leq i \leq 3$ and then adding an edge $y_i y_j$ for all non-adjacent pairs y_i, y_j with $1 \leq i < j \leq 3$. Put $Y' = \{y_1, y_2, y_3\}$. Then G' is one of the graphs in Figure 2 so has an infinitesimally rigid T-coincident realisation (G', p') by the previous paragraph. Then (G, p) is infinitesimally rigid for some $p : V \to \mathbb{R}^2$ with $p|_{V \setminus Y} = p'|_{V' \setminus Y'}$ by Lemma 4.4. This implies that (G, p) is independent since $|E| = 2|V| - 3$. This contradicts the choice of G and completes the proof of the theorem. $\square$

4.1 Proof of Theorem 1.2

Let $G = (V, E)$, $T = \{u, v, w\}$ and E_T be the set of all edges of $G[T]$.

Necessity follows since, if G has an an infinitesimally rigid T-coincident realisation (G, p), then (G', p) is infinitesimally rigid and (2) implies that $(G'/S, p_S)$ is infinitesimally rigid for all $S \subseteq T$ with $|S| \geq 2$.

For sufficiency, we assume that G' and G'/S are rigid for all $S \subseteq T$ with $|S| \geq 2$ and prove that G is T-coincident rigid. Suppose, for a contradiction, that this is not the case. Then Theorems 3.7 and 4.6, imply that, for some $S \subseteq T$ with $|S| \geq 2$, there exists a 1-thin, augmented S-compatible family $\mathcal{L} = \{\mathcal{H}, X_1, X_2, \ldots, X_m\}$ in G which covers $E - E_T$ and has $\mathrm{val}_S(\mathcal{L}) \leq 2|V| - 4$. If $\mathcal{H} = \emptyset$ then we would have $\sum_{i=1}^{m}(2|X_i - 3) \leq 2|V| - 4$ and Theorem 2.1 would imply that G' is not rigid. Hence $\mathcal{H} \neq \emptyset$.

Consider the graph G'/S obtained from G' by contracting the vertices in S into a new vertex z. Then $\mathcal{L}' = \{H'_1, \ldots, H'_k, X_1, \ldots, X_m\}$ is a 1-thin cover of G/S, where $H'_i = (H_i \setminus S) \cup \{z\}$ for each $H_i \in \mathcal{H}$. Then we have

$$\begin{aligned} \sum_{i=1}^{k}(2|H'_i| - 3) + \sum_{i=1}^{m}(2|X_i| - 3) &= \sum_{i=1}^{k}(2|H_i \setminus S| - 1) + \sum_{i=1}^{m}(2|X_i| - 3) \\ &= \mathrm{val}_S(\mathcal{L}) - 2(|S| - 1) \\ &\leq 2|V| - 4 - 2(|S| - 1) \\ &= 2(|V| - (|S| - 1)) - 4. \end{aligned}$$

This contradicts the assumption that G'/S is rigid by Theorem 2.1. $\square$

The graph in Figure 3 shows that we cannot replace the hypothesis of Theorem 1.2 that G'/S is rigid for all $S \subseteq T$ with $|S| \geq 2$ by the weaker hypothesis that G'/T is rigid.

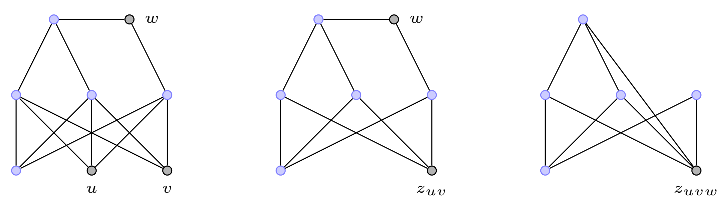

Figure 3: The graph on the left is G, $T = \{u, v, w\}$ and $G' = G$. The graph in the middle is G/uv and the graph on the right is G/T. Both G and G/T are rigid, but G/uv is not. Hence G is not T-coincident rigid by Theorem 1.2.

5 Closing Remarks

5.1 Extension to $|T| \geq 3$

We believe that Theorem 4.6 can be extended to sets T of arbitrary size.

Conjecture 5.1. *Let $G = (V, E)$ be a graph and T be a non-empty subset of V. Then E is independent in $\mathcal{R}_T(G)$ if and only if G is strongly T-sparse.*

The following result from the PhD thesis of the first author [5] gives some evidence in support of this conjecture.

Theorem 5.2. *Let $G = (V, E)$ be a graph, T be a non-empty subset of V and*

$$\mathcal{I} = \{I \subseteq E : G' = (V, I) \text{ is strongly } T\text{-sparse}\}.$$

Then $\mathcal{I}$ is the family of independent sets in a matroid on E.

5.2 Extension to $d \geq 3$

It is natural to ask whether Theorems 1.1 and 1.2 can be extended to $\mathbb{R}^d$ for $d \geq 3$. We will use the following result on flexible realisations of complete bipartite graphs which follows easily from a result of Bolker and Roth [3, Theorem 10]. It is stated explicitly in a paper of Whiteley as an immediate corollary of [11, Theorem 1]. Whiteley makes the simplifying assumption at the beginning of [11] that all frameworks (G, p) have $p(u) \neq p(v)$ whenever $uv \in E(G)$ but this assumption is not used in his proof of [11, Theorem 1].

Lemma 5.3. *Let $(K_{m,n}, p)$ be a realisation of the complete bipartite graph $K_{m,n}$ with all its vertices on a quadric surface in $\mathbb{R}^d$ for some $m, n, d \geq 2$. Then $(K_{m,n}, p)$ is not infinitesimally rigid in $\mathbb{R}^d$.*

Consider a generic $\{u, v\}$-coincident framework $(K_{5,5}, p)$ in $\mathbb{R}^3$ where u, v are vertices on different sides of the bipartition of $K_{5,5}$. Then Lemma 5.3, combined with the fact that any set of nine points lie on a quadric surface in $\mathbb{R}^3$, imply that $(K_{5,5}, p)$ is not infinitesimally rigid. On the other hand, $K_{5,5} - uv$ and $K_{5,5}/uv$ are both rigid in $\mathbb{R}^3$ since both $K_{5,5} - uv$ and the spanning subgraph of $K_{5,5}/uv$ obtained by deleting any three edges incident to the vertex of degree eight can be constructed from K_4 by the 3-dimensional versions of the 0- and 1-extension operations defined at the beginning of Section 4, see [12] for more details.

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