# SEQUENTIAL PARAMETRIZED MOTION PLANNING AND ITS COMPLEXITY, II

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Abstract. This is a continuation of our recent paper [6] in which we developed the theory of *sequential parametrized* motion planning. A sequential parametrized motion planning algorithm produced a motion of the system which is required to visit a prescribed sequence of states, in a certain order, at specified moments of time. In [6] we analysed the sequential parametrized topological complexity of the Fadell - Neuwirth fibration which is relevant to the problem of moving multiple robots avoiding collisions with other robots and with obstacles in the Euclidean space. In [6] we found the sequential parametrised topological complexity of the Fadell - Neuwirth bundle for the case of the Euclidean space $\mathbb{R}^d$ of odd dimension as well as the case $d = 2$. In the present paper we give the complete answer for an arbitrary $d \geq 2$ even. Moreover, we present an explicit motion planning algorithm for controlling multiple robots in $\mathbb{R}^d$ having the minimal possible topological complexity; this algorithm is applicable to any number $n$ of robots and any number $m \geq 2$ of obstacles.

## 1. Introduction

The topological approach to the motion planning problem of robotics [4] centres around the notion of topological complexity $\mathsf{TC}(X)$, which has several different interpretations, see Theorem 14 in [5]. In particular, $\mathsf{TC}(X)$ is the minimal degree of instability of motion planning algorithms for systems having $X$ as their configuration space. A new *"parametrized"* approach to the motion planning problem was developed recently in [1], [2]. Parametrized algorithms are universal and flexible, they are able to function in a variety of situations involving variable external conditions which are viewed as parameter and are part of the input of the algorithm.

Generalizing this idea, the authors in [6] developed the theory of *sequential parametrized motion planning* in the spirit of Rudyak [9]; it involves an integer parameter $r \geq 2$ with the case $r = 2$ reducing to the model of [1]. In the sequential approach the robot requires to visit a given sequence of $r \geq 2$ states in certain order at prescribed moments of time.

To make this paper readable we add a brief definition referring the reader to [7] and [6] for motivation and further detail. For a Hurewicz fibration $p : E \to B$ with fibre $X$ and

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an integer $r \geq 2$ we denote

$$E^r_B = \{(e_1, \cdots, e_r) \in E^r;\, p(e_1) = \cdots = p(e_r)\}$$

and $E^I_B \subset E^I$ denotes the space of all paths $\alpha : I \to E$ such that $p \circ \alpha : I \to B$ is constant. Fix $r$ points

$$0 \leq t_1 < t_2 < \cdots < t_r \leq 1$$

in $I$ ("the time schedule") and consider the evaluation map

$$\Pi_r : E^I_B \to E^r_B, \quad \Pi_r(\alpha) = (\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_r)). \tag{1}$$

$\Pi_r$ is a Hurewicz fibration, see [2, Appendix], the fibre of $\Pi_r$ is $(\Omega X)^{r-1}$. A section $s : E^r_B \to E^I$ of the fibration $\Pi_r$ can be interpreted as *a parametrized sequential motion planning algorithm*, i.e. a function which assigns to every sequence of points $(e_1, e_2, \ldots, e_r) \in E^r_B$ a continuous path $\alpha : I \to E$ ("the motion of the system") satisfying $\alpha(t_i) = e_i$ for every $i = 1, 2, \ldots, r$ and such that the path $p \circ \alpha : I \to B$ is constant. The latter condition means that the system moves under constant external conditions (such as positions of the obstacles). Typically $\Pi_r$ does not admit continuous sections and hence the motion planning algorithms are necessarily discontinuous. The following definition gives a measure of the complexity of sequential parametrized motion planning algorithms.

**Definition 1.1.** *The $r$-th sequential parametrized topological complexity of the fibration $p : E \to B$, denoted $\mathsf{TC}_r[p : E \to B]$, is defined as the sectional category of the fibration $\Pi_r$, i.e.*

$$\mathsf{TC}_r[p : E \to B] := \mathsf{secat}(\Pi_r). \tag{2}$$

In more detail, $\mathsf{TC}_r[p : E \to B]$ is the minimal integer $k$ such that there is a open cover $\{U_0, U_1, \ldots, U_k\}$ of $E^r_B$ with the property that each open set $U_i$ admits a continuous section $s_i : U_i \to E^I_B$ of $\Pi_r$. The following Lemma allows using arbitrary partitions instead of open covers:

**Lemma 1.2** (see Proposition 3.6 in [6])**.** *Let $p : E \to B$ be a locally trivial fibration where $E$ and $B$ are metrisable separable ANRs. Then the $r$-th sequential parametrized topological complexity $\mathsf{TC}_r[p : E \to B]$ equals the smallest integer $k \geq 0$ such that the space $E^r_B$ admits a partition*

$$E^r_B = F_0 \sqcup F_1 \sqcup ... \sqcup F_k, \quad F_i \cap F_j = \emptyset \ \ \textit{for } i \neq j,$$

*and on each set $F_i$ there exists a continuous section $s_i : F_i \to E^I_B$ of the fibration $\Pi_r$.*

The aim of this paper is to give an explicit sequential parametrized motion planning algorithm for the problem of moving $n \geq 1$ robots in the presence of $m \geq 2$ obstacles in $\mathbb{R}^d$ where $d \geq 2$ is even. This problem can be expressed through the properties of the Fadell-Neuwirth bundle

$$p : F(\mathbb{R}^d, m + n) \to F(\mathbb{R}^d, m) \tag{3}$$

and its topological complexity $\mathsf{TC}_r[p : F(\mathbb{R}^d, m+n) \to F(\mathbb{R}^d, m)]$ as defined above, see (2). The notation $F(\mathbb{R}^d, m + n)$ stands for the configuration space of $m + n$ pairwise distinct points of $\mathbb{R}^d$ labeled by the integers $1, 2, \ldots, m+n$. The projection (3) associates with a configuration of $m+n$ points the first $m$ points.

The main result of this paper is the following:

**Theorem 1.3.** *For any even $d \geq 2$ and for any integers $n \geq 1$, $m \geq 2$, $r \geq 2$, the sequential parametrized topological complexity of the Fadell-Neuwirth bundle (3) is given by*

$$\mathsf{TC}_r[p : F(\mathbb{R}^d, m+n) \to F(\mathbb{R}^d, m)] = rn + m - 2. \tag{4}$$

In the case $d = 2$ this statement was proven in [6], Theorem 9.2. Besides, Proposition 9.1 from [6] gives in (4) the inequality $\geq$, which is valid for any $d \geq 2$ even. Hence to prove Theorem 1.3 it is enough to establish for $\mathsf{TC}_r[p : F(\mathbb{R}^d, m+n) \to F(\mathbb{R}^d, m)]$ the upper bound $rn + m - 2$. This will follow once we present an explicit sequential parametrized motion planning algorithm of complexity $\leq rn + m - 2$. This task will be accomplished in §3 of this paper.

In the special case of $r = 2$ a motion planning algorithm for the Fadell - Neuwirth bundle for dimension $d$ odd was given in [7]; it has higher by one topological complexity. We refer also to [8] where the case $r = 2$ and $m = 2$ (two obstacles) was considered.

## 2. The obstacle avoiding manoeuvre

In this section we describe an obstacle avoiding manoeuvre which will be used below as a sub-algorithm in the general algorithm. We shall consider the situation of a single robot moving avoiding collisions with $m \geq 2$ obstacles in $\mathbb{R}^d$ where the dimension $d \geq 2$ is assumed to be even.

Let $m \geq 2$ be an integer and let $R$ be an equivalence relation on the set $[m] = \{1, 2, \ldots, m\}$. For $i, j \in [m]$ we shall write $i \sim_R j$ to indicate that $i$ and $j$ are equivalent with respect to $R$. We shall denote by $m(R) \leq m$ the number of equivalence classes of $R$.

Consider the configurations

$$C = (o_1, o_2, \ldots, o_m, z, z'), \tag{5}$$

where $o_i \in \mathbb{R}^d$ denote the positions of the obstacles and the points $z, z' \in \mathbb{R}^d$ denote the current and the desired positions of the robot, correspondingly. Denote by

$$b = (o_1, \ldots, o_m)$$

the configuration of the obstacles. Assuming that $o_1 \neq o_2$ we can consider the unit vector

$$e_b = ||o_2 - o_1||^{-1} \cdot (o_2 - o_1)$$

and the line $L_b \subset \mathbb{R}^d$ through the origin which is parallel to $e_b$. We shall denote by $q_b : \mathbb{R}^d \to L_b$ the orthogonal projection onto the line $L_b$, i.e. $q_b(x) = \langle x, e_b \rangle \cdot e_b$.

Given an equivalence relation $R$ on $[m]$ we define the space

$$\Omega_{m,R}$$

as the set of all configurations (5) satisfying the following conditions:

(a) $o_i \neq o_j$ and $z \neq o_i \neq z'$ for all $i, j \in [m]$, $i \neq j$;
(b) $q_b(o_i) = q_b(o_j)$ if and only if $i \sim_R j$ where $i, j \in [m]$;
(c) $q_b(z) \neq q_b(o_i) \neq q_b(z')$ for $i \in [m]$.

For a configuration $C$ as in (5) we shall denote by $\epsilon(C) > 0$ the minimum of the numbers

$$|o_i - o_j|, \quad i \neq j, \quad i, j \in [m],$$
$$|q_b(z) - q_b(o_i)|, \quad |q_b(z') - q_b(o_i)|, \quad i \in [m],$$
$$|q_b(o_i) - q_b(o_j)|, \quad i \not\sim_R j, \quad i, j \in [m].$$

Note that $\epsilon(C)$ is a continuous function of $C \in \Omega_{m,R}$.

Our goal is to describe an explicit function associating with each configuration $C \in \Omega_{m,R}$ a continuous path $\gamma_C : I = [0,1] \to \mathbb{R}^d$ such that

$$\gamma_C(0) = z, \quad \gamma_C(1) = z', \quad \gamma_C(t) \neq o_i \quad \text{for any} \quad t \in I, \quad i \in [m]. \tag{6}$$

Moreover, we require the function $\gamma_C(t)$ to be continuous function of two variables $(C, t) \in \Omega_{m,R} \times I$.

Clearly $\Omega_{m,R}$ is the disjoint union

$$\Omega_{m,R} = \sqcup_{j=0}^{m(R)} \Omega^j_{m,R}$$

where $\Omega^j_{m,R} \subset \Omega_{m,R}$ denotes the set of configurations (5) such that exactly $j$ distinct projection points $q_b(o_i)$ lie between the projections $q_b(z)$ and $q_b(z')$ onto $L_b$.

Each of the subsets $\Omega^j_{m,R}$ is open and closed in $\Omega_{m,R}$ and hence in it enough to define $\gamma_C(t)$ for $C \in \Omega^j_{m,R}$, where $j = 0, 1, \dots, m(R)$ and $R$ are fixed.

For $C \in \Omega^0_{m,R}$ we can simply define the $\gamma_C(t) = (1-t)z + tz'$, where $t \in [0,1]$.

Next we consider the case when there are $j \geq 1$ projections of the obstacles $q_b(o_i)$ lying between the projections $q_b(z)$ and $q_b(z')$. For simplicity we shall assume that $q_b(z) < q_b(z')$ (recall that the line $L_b$ is oriented by the vector $e_b$) and denote

$$q_b(z) < q_b(o_{i_1}) < \cdots < q_b(o_{i_j}) < q_b(z').$$

Here each $i_s \in [m]$ is the smallest index in its $R$-equivalence class.

We introduce the points

$$o^{\pm}_{i_s} = o_{i_s} \pm \frac{1}{4}\epsilon(C) \cdot e_b \in \mathbb{R}^d, \quad s = 1, \dots, j. \tag{7}$$

These points are small perturbations of the positions of the obstacles in the direction parallel to the line $L_b$ .

Recall that the dimension $d$ of the space is even and hence there exists a continuous function associating with any unit vector $e \in \mathbb{R}^d$ a unit vector $e^\perp \in \mathbb{R}^d$ which is perpendicular to $e$. Such function $e \mapsto e^\perp$ is a tangent vector field of the sphere $S^{d-1}$.

For $s = 1, 2, \dots, j$ the path

$$\alpha_s(t) = o_{i_s} - \frac{\epsilon(C)}{4} \cdot [\cos(\pi t) \cdot e_b + \sin(\pi t) \cdot e_b^\perp], \quad t \in [0,1], \tag{8}$$

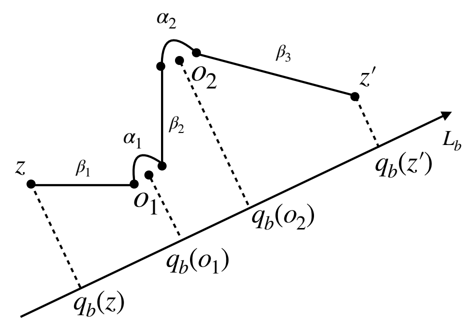


FIGURE 1. Motion of the robot from $z$ to $z'$ avoiding collisions with obstacles.

(semi-circle) connects the point $o_{i_s}^-$ to $o_{i_s}^+$ avoiding the obstacles. The straight line path

$$\beta_s(t) = (1-t)\cdot o_{i_{s-1}}^+ + t\cdot o_{i_s}^-, \quad s = 2, \dots, j$$

connects the point $o_{i_{s-1}}^+$ to $o_{i_s}^-$ avoiding the obstacles. We can similarly define the straight line path $\beta_1(t) = (1-t)\cdot z + t\cdot o_{i_1}^-$ and $\beta_{j+1} = (1-t)\cdot o_{i_j}^+ + t\cdot z'$. The concatenation

$$\gamma_C = \beta_1 * \alpha_1 * \beta_2 * \alpha_2 * \cdots * \alpha_j * \beta_{j+1} \tag{9}$$

(consisting of the straight line segments and semi-circles) is the desired path. Thus we obtain a continuous function $\Omega_{m,R}\times I \to \mathbb{R}^d$, $(C,t)\mapsto \gamma_C(t)$ satisfying (6).

## 3. COLLISION-FREE MOTION PLANNING ALGORITHM FOR MANY ROBOTS AND OBSTACLES IN $\mathbb{R}^d$; THE EVEN-DIMENSIONAL CASE

In this section we describe an explicit parametrized sequential motion planning algorithm for collision free motion of an arbitrary number $n$ of robots in the presence of an arbitrary number of $m$ moving obstacles in $\mathbb{R}^d$, where the dimension $d \geq 2$ is assumed to be even. This algorithm is optimal for $d \geq 2$ even in the sense that its topological complexity is minimal possible and coincides with the cohomological lower bound established in [6].

For the special case $r = 2$ a motion planning algorithm of this kind was described in §6 of [7] which is applicable for any dimension $d \geq 2$, but its topological complexity is minimal only for $d$ odd. The algorithm described in this section is only applicable for even $d \geq 2$, it has a smaller number of local rules and it is *sequential*, i.e. it is applicable for any $r \geq 2$.

We denote by $p : E \to B$ the Fadell - Neuwirth bundle

$$p : F(\mathbb{R}^d, m+n) \to F(\mathbb{R}^d, m) \tag{10}$$

where $F(\mathbb{R}^d, m+n)$ is the configuration space of $m+n$ pairwise distinct points of $\mathbb{R}^d$. A point of $E = F(\mathbb{R}^d, m+n)$ will be denoted by the symbol $(o_1, o_2, ..., o_m, z_1, z_2, ..., z_n)$, where $o_i \in \mathbb{R}^d$ are positions of $m$ obstacles and $z_j$'s are positions of $n$ robots. The projection (10) acts as follows

$$p(o_1, o_2, ..., o_m, z_1, z_2, ..., z_n) = (o_1, o_2, ..., o_m) \in B = F(\mathbb{R}^d, m).$$

The map (10) is a locally trivial bundle, see [3].

Let us consider the fibration

$$\Pi_r : E_B^I \to E_B^r \tag{11}$$

built out of the Fadell - Neuwirth bundle (10). A point of the space $E_B^r$ can be viewed as a configuration

$$C = (o_1, o_2, ..., o_m, z_1^1, z_2^1, ..., z_n^1, z_1^2, z_2^2, ..., z_n^2, ..., z_1^r, z_2^r, ..., z_n^r), \tag{12}$$

where $o_i, z_j^\ell \in \mathbb{R}^d$ are such that:

(1) $o_i \neq o_{i'}$ for $i \neq i'$,
(2) $o_i \neq z_j^\ell$ for $1 \leq i \leq m$, $1 \leq j \leq n$ and $1 \leq \ell \leq r$,
(3) $z_j^\ell \neq z_{j'}^\ell$ for $j \neq j'$ and $1 \leq \ell \leq r$.

3.1. **Partition of the space $E_B^r$.** For a configuration $C \in E_B^r$ as in (12) let $b$ denote the configuration $b = (o_1, o_2, ..., o_m) \in B = F(\mathbb{R}^d, m)$, and let $L_b \subset \mathbb{R}^d$ denote the line passing through the origin and the vector $o_2 - o_1 \in \mathbb{R}^d$. We shall orient the line $L_b$ such that the vector $o_2 - o_1$ points in the positive direction. Let $e_b \in L_b$ denote the unit vector in the direction of the orientation.

It is well known that for even $d \geq 2$ the unit sphere $S^{d-1} \subset \mathbb{R}^d$ admits a continuous non-vanishing tangent vector field. Fixing such a vector field gives a continuous function $e \mapsto e^\perp$ which assigns to every unit vector $e \in \mathbb{R}^d$ a unit vector $e^\perp \in \mathbb{R}^d$ perpendicular to $e$. The vector $e^\perp$ will help us to specify the obstacle avoiding motions.

For $x \in \mathbb{R}^d$, we denote by $q_b(x)$ the orthogonal projection of $x$ onto $L_b$, i.e. $q_b(x) = \langle x, e_b\rangle \cdot e_b$ where $\langle\, , \rangle$ denotes the scalar product.

For a configuration (12) we denote by

$$q_b(C) = \{q_b(o_i), q_b(z_j^\ell) \mid 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq \ell \leq r\}$$

the set of all projections of the obstacles and robots onto the line $L_b$. Note that the map $q_b$ depends on the positions of the obstacles $o_1$ and $o_2$ of the configuration $C$. Clearly, $q_b(o_1) < q_b(o_2)$; thus, the cardinality of the set $q_b(C)$ satisfies the relation

$$2 \leq |q_b(C)| \leq rn + m.$$

For any $2 \leq c \leq rn + m$ we define the subset $W_c \subset E_B^r$ as follows

$$W_c = \{C \in E_B^r \mid q_b(C) \text{ has cardinality } c\}.$$

Clearly,

$$E_B^r = W_2 \sqcup W_3 \sqcup ... \sqcup W_{rn+m}. \tag{13}$$

Below we construct a continuous section of the fibration $\Pi_r : E_B^I \to E_B^r$ over each set $W_c$, where $c = 2, 3, \ldots, rn + m$. The closure of $W_c$ satisfies

$$\overline{W}_c \subset \bigcup_{c' \leq c} W_{c'}.$$

3.2. **Partitioning the set $W_c$.** Next we partition $W_c$ as follows:

$$W_c = \bigsqcup_{s+t=c} G_{s,t}, \tag{14}$$

where $G_{s,t} \subset W_c$ is defined as the set of all configurations (12) such that the set $\{q_b(o_1), q_b(o_2), ..., q_b(o_m)\}$ has cardinality $s$. Clearly, $s$ takes values $2, 3, \dots, m$. The index $t$ is defined by the relation $s+t=c$; the value of $t$ equals the number of distinct projection values $q_b(z_j^\ell) \in L_b$ which are not projections of the obstacles $q_b(o_i)$. The closure of $G_{s,t}$ satisfies

$$\overline{G}_{s,t} \subset \bigcup_{s'\le s,\ t'\le t} G_{s',t'}. \tag{15}$$

This implies that each set $G_{s,t}$ is open and closed as a subset of $W_c$, where $c = s+t$, and hence continuous sections over the sets $G_{s,t}$ defined below give jointly a continuous section over $W_c$.

3.3. **Partitioning the sets $G_{s,t}$.** Consider the set of $m+rn$ formal symbols

$$S = \{o_1, \dots, o_m, z_1^1, \dots, z_n^1, \dots, z_1^r, \dots, z_n^r\}. \tag{16}$$

A configuration (12) defines *a linear quasi-order* on the set $S$, where for $s, s' \in S$ we say that $s \le s'$ iff $q_b(s) \le q_b(s')$. A quasi-order allows for distinct elements $s, s' \in S$ to satisfy $s \le s'$ and $s' \le s$. In such a case we shall say that the elements $s, s' \in S$ are *equivalent with respect to this quasi-order*.

Consider the set $\mathcal{O}_{s,t}^{m+rn}$ of all possible quasi-orders on the set $S$ having in total $s+t$ equivalence classes and such that the set $\{o_1, \dots, o_m\}$ has $s$ equivalence classes. For $\sigma \in \mathcal{O}_{s,t}^{m+rn}$, let $A_{s,t}^\sigma$ denote the set of all configurations $C \in G_{s,t}$ generating the quasi-order $\sigma$ on the set $S$ of symbols (16). Clearly, one has

$$G_{s,t} = \bigsqcup_{\sigma\in\mathcal{O}_{s,t}^{m+rn}} A_{s,t}^\sigma, \quad \text{where} \quad s \ge 2. \tag{17}$$

Each of the sets $A_{s,t}^\sigma$ is open and closed in $G_{s,t}$ hence a collection of continuous sections over $A_{s,t}^\sigma$ (with various quasi-orders $\sigma$) define together a continuous section over $G_{s,t}$.

3.4. **Motion planning algorithm on $A_{s,rn}^\sigma$.** In this subsection we describe a continuous section of the fibration (11) over $A_{s,rn}^\sigma \subset E_B^r$. For a configuration (12) lying in $A_{s,rn}^\sigma$ all projection points $q_b(z_i^\ell) \in L_b$ are pairwise distinct, where $i = 1, \dots, n$ and $\ell = 1, \dots r$; these points are also distinct from the projections of the obstacles $q_b(o_j)$ where $j = 1, \dots, m$. In general, $s \le m$, and hence the equality $q_b(o_i) = q_b(o_j)$ for $i \ne j$ is not excluded; this happens iff the numbers $i$ and $j$ are equivalent with respect to the quasi-order $\sigma$. A section of (11) over $A_{s,rn}^\sigma$ is determined by $n$ functions

$$\gamma_C^1, \gamma_C^2, \dots, \gamma_C^n : I \to \mathbb{R}^d \tag{18}$$

satisfying the following conditions:

(a) Each $\gamma_C^i(t)$ is continuous as a function of $(C, t) \in A_{s,rn}^\sigma \times I$ for $i = 1, \dots, n$;

(b) $\gamma^i_C(t) \neq \gamma^j_C(t)$, for $i \neq j$ and $t \in I$;
(c) $\gamma^i_C(t) \neq o_j$ for $i = 1, \dots, n$, $j = 1, \dots, m$ and $t \in I$;
(d) $\gamma^i_C(t_\ell) = z^\ell_i$ where $\ell = 1, \dots, r$ and $i = 1, \dots, n$.

In (d) the symbols $t_\ell \in I$ denote the fixed time moments

$$0 = t_1 < t_2 < \dots < t_r = 1$$

used in the definition of the evaluation map (11). The section $s$ determined by the system of curves (18) is given by

$$s(C) = (o_1, \dots, o_m, \gamma^1_C, \dots, \gamma^n_C) \in E^I_B.$$

To construct the curves (18) we shall use the obstacle avoiding manoeuvre of §2. Divide each time interval $[t_\ell, t_{\ell+1}]$, where $\ell = 1, \dots, r-1$, into $n$ subintervals

$$t_\ell = t_{\ell,0} < t_{\ell,1} < \dots < t_{\ell,n} = t_{\ell+1}$$

and add to properties (a) - (d) an additional requirement:

(e) the curve $\gamma^i_C(t)$ is not constant only for $t \in [t_{\ell,i-1}, t_{\ell,i}]$ where $\ell = 1, \dots, r-1$.

Next we describe the system of curves (18) satisfying the conditions (a) - (e). On the first interval $[t_{1,0}, t_{1,1}]$ we apply the algorithm of §2 to move the first robot from $z^1_1$ to $z^2_1$ viewing $o_1, \dots, o_m, z^1_2, z^1_3, \dots, z^1_n$ as obstacles. Note that the assumptions of §2 are satisfied since $C \in A^\sigma_{s,rn}$. This defines the curve $\gamma^1_C|_{[t_{1,0},t_{1,1}]}$. For $i = 2, \dots, n$ the curve $\gamma^i_C|_{[t_{1,0},t_{1,1}]}$ is the constant curve at $z^1_i$.

On the next interval $[t_{1,1}, t_{1,2}]$ we apply the algorithm of §2 to move the second robot from $z^1_2$ to $z^2_2$ viewing $o_1, \dots, o_m, z^2_1, z^1_3, \dots, z^1_n$ as obstacles. This defines the curve $\gamma^2_C|_{[t_{1,1},t_{1,2}]}$. For $i = 1$ the curve $\gamma^i_C|_{[t_{1,1},t_{1,2}]}$ is the constant curve at $z^2_1$ while for $i = 3, \dots, n$ the curve $\gamma^i_C|_{[t_{1,1},t_{1,2}]}$ is the constant curve at $z^1_i$.

Continuing in this fashion we construct the curves $\gamma^i_C|_{[t_{\ell,j-1},t_{\ell,j}]}$ inductively using the algorithm of §2 as follows. On the time interval $[t_{\ell,j-1}, t_{\ell,j}]$ the robot $j$ moves from the position $z^\ell_j$ to $z^{\ell+1}_j$ and this motion is specified by the algorithm of §2 in which we consider the points $o_1, \dots, o_m, z^{\ell+1}_1, \dots, z^{\ell+1}_{j-1}, z^\ell_{j+1}, \dots, z^\ell_n$ as obstacles. This defines the curve $\gamma^j_C|_{[t_{\ell,j-1},t_{\ell,j}]}$. Besides, for $i < j$ the curve $\gamma^i_C|_{[t_{\ell,j-1},t_{\ell,j}]}$ is constant at $z^{\ell+1}_i$ while for $i > j$ the curve $\gamma^i_C|_{[t_{\ell,j-1},t_{\ell,j}]}$ is constant at $z^\ell_i$.

3.5. **Desingularization: motion planning algorithms on the sets $A^\sigma_{s,t}$ with $t < rn$.** For a configuration $C \in A^\sigma_{s,t}$ define the number $\delta(C) > 0$ as the minimum of the numbers

$$|q_b(z^\ell_i) - q_b(z^{\ell'}_k)|, \quad \text{where} \quad q_b(z^\ell_i) \neq q_b(z^{\ell'}_k),$$

$$|q_b(z^\ell_i) - q_b(o_j)|, \quad \text{where} \quad q_b(z^\ell_i) \neq q_b(o_j),$$

where $i, k \in \{1, \dots, n\}$, $\ell, \ell' \in \{1, \dots, r\}$ and $j \in \{1, \dots, m\}$. Note that $\delta(C)$ is a continuous function of $C \in A^\sigma_{s,t}$. Next we define a homotopy $H : A^\sigma_{s,t} \times I \to E^r_B$ as follows. For $C \in A^\sigma_{s,t}$ we set

$$H(C, \nu) = (o_1, \dots, o_m, \mu^1_1(\nu), \dots, \mu^1_n(\nu), \dots, \mu^r_1(\nu), \dots, \mu^r_n(\nu)) \in E^r_B$$

where

$$\mu_i^\ell(\nu) = z_i^\ell + \frac{((\ell-1)n+i)\cdot\nu\cdot\delta(C)}{rn}\cdot e_b, \quad \nu\in[0,1].$$

Note that for $\nu>0$ the configuration $H(C,\nu)$ lies in $A^\tau_{s,rn}$. Here $\tau$ is the following quasi-ordering on the set (16):

(a) if for two symbols $s, s'$ of (16) one has $s\le_\sigma s'$ then $s\le_\tau s'$;
(b) if two distinct symbols $z_i^\ell$ and $z_{i'}^{\ell'}$ of (16) are equivalent with respect to $\sigma$ then $z_i^\ell <_\tau z_{i'}^{\ell'}$ assuming that $(\ell-1)n+i<(\ell'-1)n+i'$.
(c) if two symbols $o_j$ and $z_i^\ell$ of (16) are equivalent with respect to the quasi-order $\sigma$ then $o_j <_\tau z_i^\ell$.

Denoting $z_i^\ell(1) = {z'}_i^\ell$ we have the configuration

$$C' = (o_1,\dots,o_m,{z'}_1^1,\dots,{z'}_n^1,\dots,{z'}_1^r,\dots {z'}_n^r)\in A^\tau_{s,rn}$$

to which we may apply the algorithm of section 3.4. The path $\mu_i^\ell(\nu)$ connects the points $z_i^\ell$ and ${z'}_i^\ell$. We obtain a path connecting $z_i^\ell$ to $z_i^{\ell+1}$ as a concatenation of $\mu_i^\ell$ with the path $\gamma_{C'}$ from ${z'}_i^\ell$ to ${z'}_i^{\ell+1}$ given the algorithm of §3.4 followed by the inverse of the path $\mu_i^{\ell+1}$, see Figure 2.

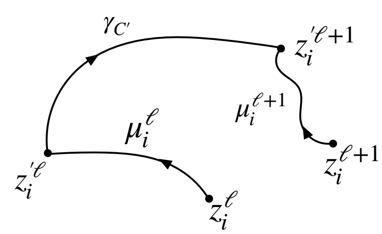


FIGURE 2. Motion of the robot from $z_i^\ell$ to $z_i^{\ell+1}$.

## 4. Proof of Theorem 1.3

As we mentioned above, the inequality $\mathsf{TC}_r[p: F(\mathbb{R}^d, m+n)\to F(\mathbb{R}^d,m)]\ge rn+m-2$ is given by Proposition 9.1 from [6]. To obtain the opposite inequality we shall use Lemma 1.2 applied to the partition (13). The construction of §3 defines a continuous section of the bundle (11) over each of the sets $W_c$, where $c=2,3,\dots,rn+m$. Indeed, we described in §3 a continuous section over each of the sets $A^\sigma_{s,t}$. The sets $A^\sigma_{s,t}$ are open and closed in $G_{s,t}$, see (17), and the sets $G_{s,t}$ are open and closed in $W_c$, where $c=s+t$, see (14). Thus, the continuous sections over the sets $A^\sigma_{s,t}$ with $s+t=c$ jointly define a continuous section over the set $W_c$. This completes the proof.

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