# Revivals in Time Evolution Problems 

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#### Abstract

Subject to periodic boundary conditions, it is known that the solution to a certain family of linear dispersive partial differential equations, such as the free linear Schrödinger and Airy evolution, exhibits a dichotomy at rational and irrational times. At rational times, the solution is decomposed into a finite number of translated copies of the initial condition. Consequently, when the initial function has a jump discontinuity, then the solution also exhibits finitely many jump discontinuities. On the other hand, at irrational times the solution becomes a continuous, but nowhere differentiable function. These two effects form the revival and fractalisation phenomenon at rational and irrational times, respectively.

The main aim of the thesis is to further investigate the phenomenon of revivals in time evolution problems posed under appropriate boundary conditions on a finite interval. We consider both first-order and second-order in time problems. For the former, we examine the influence of non-periodic boundary conditions on the revival effect. For the latter, we study the revivals under periodic and non-periodic boundary conditions.

In terms of first-order in time evolution problems, we show that the revival phenomenon persists in the free linear Schrödinger equation under pseudo-periodic and Robin-type boundary conditions. Moreover, we prove that under quasi-periodic boundary conditions, the Airy equation does not in general exhibit revivals. With respect to second-order in time equations, we first formulate an abstract setting for the revival phenomenon, which we then apply to establish that the periodic, even-order poly-harmonic wave equation exhibits revivals. Finally, following the lack of revivals in Airy's quasi-periodic problem, we characterise quasi-periodic and periodic problems, either of first-order or second-order in time, for which the revival effect breaks.

In general, our approach relies on identifying the canonical periodic components of the generalised Fourier series representations of solutions, in order to utilise the classical periodic theory of revivals.


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## Notation

$\mathbb{Z} \quad$ The set of integers
$\mathbb{N} \quad$ The set of positive integers
$\mathbb{Q} \quad$ The set of rational numbers
$\mathbb{R} \quad$ The set of real numbers
$\mathbb{C} \quad$ The set of complex numbers
$\operatorname{Re}(\mathrm{z}) \quad$ The real part of $z \in \mathbb{C}$
$\operatorname{Im}(z) \quad$ The imaginary part of $z \in \mathbb{C}$
$\bar{z} \quad$ The complex conjugate of $z \in \mathbb{C}$
$|z| \quad$ The absolute value or modulus of either a real or a complex number $z$
$C^{m}[0,2 \pi] \quad$ The space of all $m$-times continuously differentiable functions on $[0,2 \pi]$
$C_{c}^{\infty}(0,2 \pi)$ The space of smooth functions with compact support in $(0,2 \pi)$
$C_{b}(\mathbb{R}) \quad$ The space of bounded, continuous, complex-valued functions defined on $\mathbb{R}$
$(\cdot)^{\prime} \quad$ Differentiation with respect to the space variable $x$
$\|f\| \quad=\sqrt{\int_{0}^{2 \pi}|f(x)|^{2} d x}$
$L^{2}(0,2 \pi) \quad$ The Lebesgue space of complex-valued functions on $(0,2 \pi)$ with $\|f\|<\infty$
$\langle f, g\rangle \quad=\int_{0}^{2 \pi} f(x) \overline{g(x)} d x$
$e_{j}(x) \quad=e^{i j x} / \sqrt{2 \pi}, j \in \mathbb{Z}$
$\widehat{f}(j) \quad=\left\langle f, e_{j}\right\rangle, j \in \mathbb{Z}$
$\langle f\rangle \quad=\frac{1}{2 \pi} \int_{0}^{2 \pi} f(x) d x$
$H^{m}(0,2 \pi) \quad$ The Sobolev space of integer order $m \geq 1$ over the interval $(0,2 \pi)$
$H_{\text {per }}^{s}(0,2 \pi)$ The $2 \pi$-periodic Sobolev space of order $s \geq 0$
$f^{*} \quad$ The $2 \pi$-periodic extension of a function $f$
$\mathcal{T}_{s} \quad$ The periodic translation operator
$\mathcal{R}_{n}(p, q) \quad$ The periodic revival operator of order $n$ at $(p, q)$
$f^{\natural} \quad$ The reflection of a function $f$
$f^{ \pm} \quad$ The even/odd extension of a function $f$
$f * g \quad$ The periodic convolution of two functions $f$ and $g$
$n_{0}(x) \quad=\frac{1}{\sqrt{\pi}}$
$n_{j}(x) \quad=\sqrt{\frac{2}{\pi}} \cos (j x), j \in \mathbb{N}$
$d_{j}(x) \quad=\sqrt{\frac{2}{\pi}} \sin (j x), j \in \mathbb{N}$
$a_{j} \quad=\int_{0}^{\pi} f(x) n_{j}(x) d x, j \in\{0\} \cup \mathbb{N}$
$b_{j} \quad=\int_{0}^{\pi} f(x) d_{j}(x) d x, j \in \mathbb{N}$
$\langle f, g\rangle_{L^{2}(0, \pi)}=\int_{0}^{\pi} f(x) \overline{g(x)} d x$
$\mathcal{H}$
Complex, separable, infinite-dimensional Hilbert space

## Chapter 1

## Introduction

### 1.1 The Phenomenon of Revivals

The revival effect is one part of a dichotomy that is known to appear in the behaviour of the solution to linear dispersive partial differential equations (PDEs) with integer coefficients and under periodic boundary conditions posed on a finite interval. The other part of the dichotomy is the fractalisation effect. The free linear Schrödinger equation and the Airy PDE are two of the main examples that belong in this class of equations which exhibit the phenomenon of revivals and fractalisation.

The two effects appear when we consider initial conditions with finitely many jump discontinuities. The classical example is a piecewise constant initial function at time zero. For such initial data, the solution at certain times, known as rational times, is given in terms of a finite linear combination of translated copies of the initial condition. This implies that the solution evaluated at rational times is also piecewise constant and essentially revives the form of the initial function. This recurrence of the initial condition in the structure of the solution at rational times is known as the revival phenomenon. In stark contrast to the behaviour at rational times, the solution at generic times, also called irrational times, evolves from an initially discontinuous function to a continuous, nowhere differentiable function displaying a fractal-like profile. This smoothing effect on the regularity of the solution at irrational times is the fractalisation phenomenon.

The revival and fractalisation dichotomy is also known as the Talbot effect, see for instance [1, Section 3.19] by Gbur. The Talbot effect refers to a diffraction
phenomenon first discovered by Talbot [2] in 1836. It describes the reappearance at a certain distance, the Talbot distance, of the image of a periodic diffraction grating after light is incident upon the grading. At rational multiples of the Talbot distance, a pattern of shifted copies of the profile of the grating is revealed and as the denominator of the rational number increases a self-similar, fractal form is constructed, see Figure 1.1.


Figure 1.1: Talbot effect : On the left end of the figure, light is diffracted by a periodic grading. On the right end of the picture, at the Talbot distance, the full image of the grading is reproduced. Half-way through the picture, a vertically shifted image of the grading appears. At regular fractions of the Talbot distance self-images of the grading are observed. (Original source [1].)

Talbot's discovery remained unnoticed for almost 50 years. It was re-examined by Lord Rayleigh [3] in 1881, who calculated the Talbot distance to be $a^{2} / \lambda$, where $a$ is the period of the grading and $\lambda$ the wavelength of the incident light. Further experimental and theoretical developments on the Talbot effect seem to originate again much later in the 1950s and 1960s, see for example the works of Cowley and Moodie [4], Hiedemann and Breazeale [5], Winthrop and Worthington [6].

However, it was the contribution of Berry and Klein [7] in 1996, which popularised the subject and attracted the interested of both the physics and mathematics community. Indeed, Berry an Klein examined the Talbot effect based on the Fourier series representation of the solution to the free linear Schrödinger equation with periodic boundary conditions on a bounded interval. They discovered that at rational multiples of the Talbot distance, the solution reduces to a finite superposition of copies of the grading, while at irrational multiples it has a fractal non-differentiable profile.

The theoretical treatment of the Talbot effect as given by Berry and Klein gave an analytical characterisation of the revival of the periodic grading at regular fractions
of the Talbot distance. As mentioned in [7], it further indicated that the mathematical theory behind the Talbot effect is similar to this of the experimentally observed quantum revival, see for example [8] by Vrakking, Villeneuve and Stolow and [9] by Yeazell and Stroud. Indeed, in certain quantum systems, after a long period of time, the wave function can be fully reconstructed in its original form, resulting in a full revival of the initial wave packet. On the other hand, at fractional multiples of this period, it evolves into a fractional revival which refers to a partial reconstruction of the wave function in terms of specific copies of the initial form. In [10], Berry, Marzoli and Schleich provide a good description of the similarities of the underlying mathematics between the Talbot and the quantum revival effects.

The studies outlined above came from the physics community. However, here we should also mention the independent contributions of three mathematicians. In 1992, before the work of Berry and Klein, Oskolkov in [11] had already explored the discontinuous and continuous nature of the solution to the free linear Schrödinger equation and the Airy PDE. Oskolkov rigorously showed that, for bounded variation initial datum with a finite number of jump discontinuities, the solution is bounded and has at most countably many discontinuities at rational times, and thus it revives the initial discontinuity. At irrational times, Oskolkov proved that the solution is a continuous function in space, addressing in this manner the fractalisation phenomenon to some extent.

Independently of Oskolkov and Berry and Klein, Taylor [12] in 2003 showed that the linear Schrödinger equation with zero potential exhibits revivals. In particular, Taylor showed that the solution at rational times is a finite superposition of translations of the initial condition. Similar to the previous works of Oskolkov and Berry and Klein, Taylor noticed that the coefficients in this finite superposition have the form of Gauss sums which are studied in number theory. However, Taylor further showed how a classical reciprocity identity for Gauss sums follows from the analysis of the revival phenomenon.

Finally, in 2010 Olver [13] independently discovered the Talbot effect in the context of the Airy PDE under periodic boundary conditions. By proving the revival property at rational times, Olver showed that the effect persists in the class of firstorder in time, linear dispersive PDEs with integer coefficients. Furthermore, Olver
named the effect dispersive quantisation capturing both the dispersive nature of the equation and the quantisation (revival) of its solution at rational times.

Olver's result settled the mathematical theory of revivals in the periodic setting for first-order in time linear dispersive equations with integer coefficients. In the last 12 years, various others researchers have contributed to the extension of the revival and fractalisation phenomena in other types of time evolution problems. Our contribution is in this exact direction and focuses on the revival effect.

### 1.2 Contribution

In this thesis we examine the revival phenomenon in various time evolution problems posed on a finite interval under appropriate boundary conditions. The time evolution problems under consideration can be distinguished into two classes. First-order and second-order in time evolution problems. For first-order in time equations, we study the revivals in the case of non-periodic boundary conditions, with the principal examples being the free space linear Schrödinger equation

$$
\partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t)
$$

and the Airy PDE

$$
\partial_{t} u(x, t)=\partial_{x}^{3} u(x, t)
$$

On the other hand, for second-order in time problems, such as the bi-harmonic wave equation

$$
\partial_{t}^{2} u(x, t)=-\partial_{x}^{4} u(x, t),
$$

we consider the phenomenon of revivals under both periodic and non-periodic conditions.

Based on the framework of Chapter 3, we formulate our findings in Chapters 4 to 8. All these chapters include new results about the revival phenomenon and extend in various directions the classical theory from the literature, which is presented in Chapter 2. Specifically, our contribution can be summarised as follows.

1. We introduce the concept of the weak revival effect. This is defined as a pure revival effect perturbed by a continuous function in space. The pure revival
refers to the classical, mathematical, interpretation of the revival phenomenon, which means that for a given time evolution problem the solution at a rational time is expressed as a finite linear combination of translations of the initial datum. Both effects, pure and weak revivals, describe revival phenomena in time evolution problems since, in any case, an initial jump discontinuity reappears in the solution at rational times.
2. We show that the revival effect persists in the form of pure or weak revivals in the context of the free linear Schrödinger equation under pseudo-periodic or Robin-type boundary conditions, respectively.
3. We introduce an abstract setting for the revival phenomenon based on a functional calculus for a non-self-adjoint differential operator. As an application, we extend the revival effect for the even-order poly-harmonic wave equation with periodic boundary conditions. In this case, the revivals manifest due to the weak revival effect.
4. We characterise time evolution problems, with either periodic or quasi-periodic boundary conditions, which do not in general support the phenomenon of revivals (pure or weak). A primary example in this direction is the quasiperiodic problem for the Airy PDE.

The results of Chapters 4, 5 and 6 were published in [14] in collaboration with Boulton and Pelloni. Below, we briefly describe the context of each chapter of the thesis whose main body consists of Chapters 2 to 9 .

### 1.3 Structure of the Thesis

In Chapter 2, we consider both the revival and fractalisation effects in the periodic setting. The material here relies on the existing literature and sets the ground to establish some standard notation and terminology used in the thesis. We begin with an illustration of the two phenomena based on the free linear Schrödinger equation and derive the Fourier series representation of the solution by the Fourier method. Within this setting we also introduce the concept of the generalised solution which offers a rigorous framework for the consideration of the revival phenomenon. Then, we present the classical theory of revivals and fractalisation. The classical
theory refers to periodic problems for first-order in time, linear dispersive PDEs with integer coefficients. At this stage, we introduce the notion of pure revivals in order to distinguish the various revival phenomena encountered later. Plainly, the pure revival effect addresses that at rational times the solution of an evolution problem is a finite linear combination of only translations of the initial condition and does not involve other transformations. Furthermore, we review some additional results on the revivals and fractalisation phenomenona in other time evolution problems under periodic conditions. For instance, we consider non-linear dispersive equations and linear Schrödinger equations with potentials which exhibit revivals but in a weaker sense. Thus, we introduce the weak revival effect defined as a perturbation by a continuous function of a pure revival

To prepare a clear mathematical framework for the revival effect, in Chapter 3 a number of special transformations and their properties are considered. Motivated by the pure revival effect in the periodic setting, we define the periodic revival operator in terms of a finite linear superposition of periodic translation operators. The revival operator yields a compact notation for the description of the revival effect. We illustrate this with a specific example from the classical setting. More importantly, as we explicitly demonstrate in later chapters, the revival operator allows us to characterise the revival phenomenon in more complicated time evolution problems, which do not belong in the classical theory. To complete the list of the necessary transformations, we recall the properties of four well known, albeit crucial for the analysis of revivals, transformations. These are the reflection of a function, the even and odd extensions and the periodic convolution.

One of the fundamental chapters of the thesis is Chapter 4. We examine the revival effect in the time evolution of the free linear Schrödinger equation with pseudo-periodic boundary conditions. In [15], Olver, Sheils and Smith established that the solution to this problem exhibits revivals at rational times. In this chapter, we outline a different proof which indicates a more universal treatment of revivals in boundary value problems beyond the classical periodic setting. After deriving the generalised Fourier expansion of the solution, we show that at any time the solution to the pseudo-periodic problem is given by a combination of the solutions of four purely periodic problems for the free linear Schrödinger equation with appropriate
initial conditions. This new representation has two central implications. First, at rational times, it follows that the revival property for the pseudo-periodic problem can be obtained directly from this of the periodic case. Additionally, at irrational times, the fractalisation phenomenon occurs, again, due to the periodic components. The main results are Theorem 4.8 and Corollary 4.9.

The revival phenomenon in the context of the Airy PDE under quasi-periodic boundary conditions is examined in Chapter 5. The main result is Theorem 5.2 which establishes a correspondence between the quasi-periodic problem for the Airy equation and a specific periodic problem for the free linear Schrödinger equation. It shows that, in this case, the persistence of the revival relies on a real parameter that controls the boundary conditions. In stark contrast to the Schödinger equation, when the parameter takes an irrational value then the revival breaks in Airy's quasi-periodic problem and instead the fractalisation phenomenon appears at rational times. The revivals survive only when the parameter is a rational number, illustrating a strong influence of the boundary conditions for equations with higher than two order derivatives in space. To our knowledge, Theorem 5.2 seems to be the first rigorous result showing the lack of revivals in a first-order in time, linear dispersive PDE with integer coefficients and coupled boundary conditions.

In Chapter 6, we show that the phenomenon of revivals persists in the free linear Schrödinger equation under a specific type of Robin boundary conditions. In contrast to the models of previous chapters, here, the boundary conditions do not couple the end points of the interval. Following a similar line of argument as in Chapters 4 and 5, we first establish the generalised Fourier series representation of the solution by analysing the underlying eigenstructure of the problem. We then prove that at any positive time the solution can be separated into two parts. One part is always a continuous function in the space variable. The other part is a periodic function which corresponds to the solution of a periodic problem for the free linear Schrödinger equation with an even initial condition. By evaluating the solution at rational times, we establish a weak revival formula which indicates the existence of the revival phenomenon in the Robin problem. Theorem 6.7 and Corollary 6.9 are the main contributions of this chapter.

In Chapter 7, we develop an abstract framework for the revival phenomenon
based on a functional calculus approach generated by a non-self-adjoint operator. A weaker version of this revival functional calculus is applied to second-order in time evolution problems. In particular, we establish that the periodic problem for the even-order poly-harmonic wave equation, which includes the wave and the biharmonic wave equations, exhibits weak revivals at rational times. The weak revival effect follows by deriving a solution representation which is decomposed into a periodic component, which incorporates the pure revival effect, and a component which is a continuous function in space at all times. The central results are Lemma 7.5, Lemma 7.6, Proposition 7.8 and Corollary 7.9.

In Chapter 8, we present a different approach to the examination of the revival effect in both first-order and second-order in time evolution problems with self-adjoint quasi-periodic boundary conditions. We formulate our results by establishing a correspondence between a given quasi-periodic problem and a periodic problem. In this way, we first extend the lack of revivals in Airy's quasi-periodic problem to quasi-periodic problems with higher order spatial derivatives. The technique further allows a simple generalisation of the Talbot effect in the cubic nonlinear Schrödinger equation with quasi-periodic boundary conditions. Moreover, we show that the weak revival phenomenon is present in the quasi-periodic problem for the bi-harmonic wave equation, but it breaks down for higher order poly-harmonic wave equations, resembling the situation of the first-order problems. Through the correspondence between quasi-periodic and periodic boundary conditions, we are also able to identify periodic problems for linear dispersive PDEs with real, irrational coefficients for which the revival phenomenon fails. As the main results, we refer to Corollary 8.5 and its implications after it, Corollary 8.6 and the conclusions following Proposition 8.9.

We conclude on the material of the thesis in Chapter 9, in which we also present further directions for future work.

The thesis is supported by a number of appendices. The necessary background for the development of our methods and the statement of the results are included in the Appendices A to D. In the first appendix, we recall the definition of onedimensional linear dispersive partial differential equations which are the main class of equations considered in the thesis. Appendix B contains the standard definitions
of the generalised Fourier series and orthonormal and Riesz bases, together with a number of useful properties. In Appendix C, we summarise some classical concepts from the theory of linear operators, such as eigenvalue problems, eigenfunction expansions, self-adjoint, symmetric and essentially self-adjoint operators. A brief overview of the periodic Sobolev spaces and Sobolev spaces over a bounded interval is given in Appendix D. The existence and uniqueness of the generalised solution to the periodic problem for the free linear Schrödinger equation (Theorem 2.4) is given in Appendix E. For the same problem, the proof that the generalised solution is the weak solution (Proposition 2.6) of the problem is included again in Appendix E. In the final appendix, we verify via numerical examples on the phenomena of revivals and fractalisation the results established in Chapters 4, 5 and 6 .

## Chapter 2

## Revivals and Fractalisation in the Periodic Setting

In this chapter, we describe the phenomena of revivals and fractalisation. Beginning with the existing literature, we present a number of results on the revival and fractalisation effects, in the context of dispersive PDEs under periodic boundary conditions. Moreover, we settle the standard terminology and notation to be used in the thesis.

In the first section, we illustrate the phenomenon of revivals and fractalisation based on the free space linear Schrödinger equation. For a step initial function, we show graphs of the profiles of the real and imaginary parts of the solution. We observe that a dichotomy appears at rational and irrational times, which constitutes the revival and fractalisation effects. At rational times, we notice piecewise constant profiles with a finite number of jump discontinuities. Hence, a form of revival of the initial condition. By contrast, at irrational times any discontinuity of the initial data disappears and the profiles become continuous, nowhere differentiable functions in space, displaying a fractal form.

In Section 2.2, the Fourier method is employed to obtain the solution of the free linear Schrödinger equation with arbitrary initial conditions under periodic boundary conditions. The method leads to the Fourier series representation of the solution. Since it provides the first step for a rigorous treatment of the revival effect, we outline the details of the method. Moreover, we introduce the notion of generalised solution which allows for an accurate interpretation of the solution representation
as obtained from the Fourier method. We highlight that throughout the thesis, all solution representations will be considered as generalised solutions (see Remark 2.5).

In Sections 2.3 and 2.4 we present the classical mathematical theory on the phenomenon of revivals and fractalisation in a more systematic manner. The classical setting refers to the formulation of the revivals and fractalisation effects in linear dispersive PDEs with integer coefficients under periodic boundary conditions. This family of equations includes as special cases the free linear Schrödinger equation and the Airy PDE.

In Section 2.3, we characterise analytically the revival effect based on a representation of the solution in terms of a finite superposition of translations of the initial data at rational times. This is the context of Theorem 2.8 which is crucial to our analysis and sets the central statement on which we are going to improve upon. The statement further motivates the definition of pure revivals which asserts that a revival phenomenon occurs in a time evolution due to the decomposition of the solution in terms of a finite number of only translations of the initial function. In Section 2.4, we state known results that describe the behaviour of the solution at irrational times and address the fractalisation effect. Although, our main subject is the revival effect, the value of the results in Section 2.4 will also become apparent in other parts of the thesis.

In Section 2.5 we consider perturbations of linear dispersive equations. We discuss known results that extend to the periodic setting the phenomena of revivals and fractalisation in the cubic non-linear Schrödinger equation, the Korteweg-de Vries equation and linear Schrödinger equations with potential. We elaborate on the proof of the Talbot effect in the context of the non-linear Schrödinger equation, which indicates that there is another type of revival effect different than the pure revival. We use this to motivate the definition of the weak revival effect as a perturbation of a pure revival effect by a continuous function. The weak revival effect manifests also in the case of the linear Schrödinger equation with non-zero potential. Additionally, as we shall see in later chapters, weak revivals occur also in other time evolution problems that do not belong in the classical setting of Section 2.3. Finally, in Section 2.6 we refer to a few more results on the revivals and fractalisation.

### 2.1 Revivals and Fractalisation

In this section, we introduce the phenomenona of revivals and fractalisation based on the free space linear Schrödinger equation

$$
\begin{equation*}
\partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t) . \tag{FSLS}
\end{equation*}
$$

Throughout, the variable $t$ will denote time and $x$ will denote the 1-D space coordinate. It is well-known that the FSLS equation describes the time evolution of the wave function $u(x, t)$ associated with a free quantum particle, [16]. It further provides a typical example of a linear dispersive PDE, where the dispersion relation is $\omega=k^{2}$ (see Appendix A).

To describe the dichotomy of revivals and fractalisation, we consider the Schrödinger equation (FSLS) on the closed interval $[0,2 \pi]$ with an initial condition $u(x, 0)=$ $u_{0}(x)$, subject to periodic boundary conditions

$$
\begin{equation*}
u(0, t)=u(2 \pi, t), \quad \partial_{x} u(0, t)=\partial_{x} u(2 \pi, t) . \tag{2.1}
\end{equation*}
$$

In order to illustrate the main phenomena in question, we depict the solution when $u_{0}$ has jump discontinuities. Specifically, consider

$$
u_{0}(x)= \begin{cases}0, & 0 \leq x \leq \pi  \tag{2.2}\\ 1, & \pi<x \leq 2 \pi\end{cases}
$$

see Figure 2.1.


Figure 2.1: Real (blue) and imaginary (red) parts of the step initial condition $u_{0}$ in (2.2)

In Figures 2.2 and 2.3, we plot the solution in two different regimes. They demonstrate a dichotomy on the behaviour of the solution occurring at rational and
irrational times which we define as follows.

Definition 2.1. A positive $t$ is called $a$ rational time if $t / 2 \pi \in \mathbb{Q}$. Equivalently, $t$ is a rational time if there exist co-prime, positive integers $p, q \in \mathbb{N}$ such that

$$
t=2 \pi \frac{p}{q} .
$$

If $t / 2 \pi \notin \mathbb{Q}$, then we call $t$ an irrational time.


Figure 2.2: Real (blue) and imaginary (red) parts of the solution to problem (FSLS), (2.1), (2.2) at different rational times $t=2 \pi p / q$.

According to Figure 2.2, we observe a revival of the initial condition in the structure of the solution at rational times. By this we mean that the real and imaginary parts of the solution revive the initial jump discontinuity. Moreover, the profiles are not just piecewise continuous, but in particular piecewise constant. Furthermore, it seems that they are constructed by a finite number of certain copies of the initial datum. This recurrence of the initial condition in the solution profile at a rational time is known as the revival phenomenon. As we shall see later, there is an explicitly characterisation of the revival phenomenon in this case, see Theorem 2.8.

On the other hand and in contrast with the behaviour at rational times, in Figure 2.3 we notice that at irrational times the initial condition has evolved to a continuous function. In particular, both the real and imaginary parts are now continuous and they display a non-differentiable structure resembling a fractal curve. This is referred to as the fractalisation phenomenon which is a smoothing effect, meaning that it improves the regularity of the solution. Precise statements regarding the behaviour at irrational times are given in Theorems 2.12 and 2.13.


Figure 2.3: Real (blue) and imaginary (red) parts of the solution to problem (FSLS), (2.1), (2.2) at irrational times.

As mentioned in the introduction, after the work of Berry and Klein [7], the manifestation of the revival and fractalisation effects in the FSLS equation has been closely related to the Talbot effect from optics, with the terminology being interchangeable. We also refer to the recent exposition of this mathematical interplay which is given by Eceizabarrena in [17]. Furthermore, a rigorous justification between the relation of the Talbot effect and the Schrödinger equation is given in [18] by Eceizabarrena, addressing in this way Oskolkov's question in [19]. In the next sections, further references will be made on specific results and conjectures about the Talbot effect under periodic boundary conditions in first-order in time, linear
dispersive PDEs with integer coefficients.

### 2.2 Solution to the Free Linear Schrödinger Equation

We now derive a solution representation to the FSLS equation with the periodic conditions (2.1). This will allow us to introduce the standard terminology and notation to be used in the thesis. The solution will be obtained through the Fourier method, and although it is standard, it will serve as the main technique to derive solutions representations to all initial boundary value problems (IBVPs) encountered later. As it provides a natural starting point to the analysis of the revival phenomenon, see Remark 4.1, we will describe this well known technique in detail.

The Fourier method is a classical method to solve linear partial differential equations with constant coefficients and it applies to a variety of problems [20]. The primary idea is to find a solution representation as a Fourier expansion in terms of a basis of eigenfunctions of the space operator. The basis of eigenfunctions is found by solving the eigenvalue problem associated to the latter. Definitions on the concepts of bases and generalised Fourier series together with some useful characterisations can be found in Appendix B. Also, the notion of a linear operator and its corresponding eigenvalue problem is included in Appendix C.

The solution procedure begins by rewriting the problem in the following differential equation form

$$
\begin{equation*}
\partial_{t} u(x, t)=-i L u(x, t), \quad u(x, 0)=u_{0}(x), \tag{2.3}
\end{equation*}
$$

where $L$ is the linear differential operator defined by the action $L f(x)=-f^{\prime \prime}(x)$ on the domain

$$
\mathrm{D}(L)=\left\{f \in C^{2}[0,2 \pi] ; f(0)=f(2 \pi), f^{\prime}(0)=f^{\prime}(2 \pi)\right\} .
$$

By $C^{2}[0,2 \pi]$ we denote the space of all twice continuously differentiable functions on the interval $[0,2 \pi]$. The notation $(\cdot)^{\prime}$ will always stand for differentiation in the space variable. Problem (2.3) is equivalent to (FSLS) under the periodic conditions
(2.1) and with $u(x, 0)=u_{0}(x)$.

Here, $L$ is a linear operator acting on the complex Hilbert space $L^{2}(0,2 \pi)$, the usual space of all square (Lebesgue) integrable complex-valued functions. Here and elsewhere, the inner product and the norm of $L^{2}(0,2 \pi)$ are given by

$$
\begin{equation*}
\langle f, g\rangle=\int_{0}^{2 \pi} f(x) \overline{g(x)} d x, \quad\|f\|=\sqrt{\langle f, f\rangle}, \quad \forall f, g \in L^{2}(0,2 \pi) . \tag{2.4}
\end{equation*}
$$

We seek eigenpairs of the operator. These are found by solving the corresponding eigenvalue problem which in this case has the form of the boundary value problem

$$
\begin{equation*}
-f^{\prime \prime}(x)=\lambda f(x), \quad f(0)=f(2 \pi), f^{\prime}(0)=f^{\prime}(2 \pi) \tag{2.5}
\end{equation*}
$$

on $[0,2 \pi]$. From (2.5), we find that the eigenvalues are all real (as expected since $L$ is symmetric, Definition C.4). In particular, the eigenvalues and the corresponding eigenfunctions are given by

$$
\lambda_{j}=j^{2}, \quad f_{j}(x)=A e^{i j x}, \quad j \in \mathbb{Z}, \quad A \in \mathbb{C} \backslash\{0\} .
$$

Furthermore, by normalising the eigenfunctions $f_{j}(x)$ we see that the operator has an orthonormal basis of eigenfunctions in $L^{2}(0,2 \pi)$ given by

$$
\begin{equation*}
e_{j}(x)=\frac{e^{i j x}}{\sqrt{2 \pi}}, \quad j \in \mathbb{Z} \tag{2.6}
\end{equation*}
$$

Indeed, the family $\left\{e_{j}\right\}_{j \in \mathbb{Z}}$ is the classical orthonormal Fourier basis in $L^{2}(0,2 \pi)$, [20], and we know that any function $f$ in $L^{2}(0,2 \pi)$ admits a complex Fourier series

$$
f(x)=\sum_{j \in \mathbb{Z}} \widehat{f}(j) e_{j}(x), \quad \widehat{f}(j)=\left\langle f, e_{j}\right\rangle,
$$

where the numbers $\widehat{f}(j)$ are the Fourier coefficients of $f$ and the convergence of the series is understood in the norm of $L^{2}(0,2 \pi)$. In essence, the basis property of the eigenfunctions allows the construction of a solution to (2.3) as a Fourier series converging in $L^{2}(0,2 \pi)$.

We should remark that instead of looking for sufficiently smooth solutions that satisfy the problem pointwise, we would like to obtain singular solutions that allow
for jump discontinuities. Additionally, we would prefer to consider initial data that can be piecewise continuous and permit the existence of such singular solutions. As with the example of the step initial condition (2.2), the implications of the revival and fractalisation phenomena on the behaviour of the solution clearly emerge when we consider, in general, initial data of bounded variation with finitely many jumps discontinuities. Such functions belong to the Hilbert space $L^{2}(0,2 \pi)$ which sets a convenient framework for the theoretical development of the revival phenomenon, which is the main subject of our study, both in and outside the classical setting of Section 2.3.

First, we note that $L^{2}(0,2 \pi)$ is the natural space in which the eigenfunction expansion representations of the solutions to our initial boundary value problems will hold. Moreover, the results on the revival phenomenon could then be formulated in terms of (bounded) linear operators on $L^{2}(0,2 \pi)$ based on the form of their Fourier coefficients (see also Remark 2.11). Indeed, in the next chapter, we will set a toolbox of special operators in $L^{2}(0,2 \pi)$, which we will then utilise in the study of the revival phenomenon.

Let us now return to problem (2.3). We will now show that there is a meaningful notion of solution for $u_{0}$ in $L^{2}(0,2 \pi)$. This is the concept of the so called generalised solution, see [21] and [22], which is obtained as the $L^{2}(0,2 \pi)$ limit of a sequence of smooth solutions of (2.3). By a smooth solution to problem (2.3), we mean a bounded function $u(x, t)$ which is smooth in $x \in[0,2 \pi]$ and $t \in[0, \infty)$, with bounded time derivative $\partial_{t} u(x, t)$ and satisfies point-wise the time evolution problem and the boundary conditions. Before we proceed with the definition of the generalised solution, we set the following notation.

Notation 2.2. Let $u:[0,2 \pi] \times[0, \infty] \rightarrow \mathbb{C}$ be a function depending on $x$ and $t$. For every $t \geq 0$ we denote by $u(\cdot, t)$ the function on $[0,2 \pi]$ whose value at almost all $x$, in the sense of Lebesgue measure, is $u(x, t)$.

The notation $u(\cdot, t)$ allows us to separate the roles of the two independent variables $x$ and $t$. With this, we formulate the definition below.

Definition 2.3. A function $u:[0,2 \pi] \times[0, \infty] \rightarrow \mathbb{C}$ is called a generalised solution in $L^{2}(0,2 \pi)$ to the IBVP (2.3) if it defines a continuous map $t \rightarrow u(\cdot, t)$ from $[0, \infty)$
to $L^{2}(0,2 \pi)$ and there exists a family $\left\{u^{n}(x, t)\right\}_{n \in \mathbb{N}}$ of smooth solutions to the IBVP (2.3) such that for every $t \geq 0$

$$
\lim _{n \rightarrow \infty}\left\|u^{n}(\cdot, t)-u(\cdot, t)\right\|=0
$$

The next theorem gives the existence of a unique generalised solution and it is well known. It utilises the ideas of the Fourier method to find a sequence of smooth solutions. The proof is included in Appendix E.

Theorem 2.4. Let $u_{0} \in L^{2}(0,2 \pi)$. Then, there exists a unique generalised solution $u(x, t)$ in $L^{2}(0,2 \pi)$ and for any fixed $t \geq 0$ it is given by the Fourier expansion

$$
\begin{equation*}
u(x, t)=\sum_{j \in \mathbb{Z}} \widehat{u_{0}}(j) e^{-i j^{2} t} e_{j}(x), \tag{2.7}
\end{equation*}
$$

where the convergence is in the norm of $L^{2}(0,2 \pi)$ and $\widehat{u_{0}}(j)=\left\langle u_{0}, e_{j}\right\rangle$ are the Fourier coefficients of $u_{0}$.

Remark 2.5. In what follows, we will always consider the generalised solution of the time evolution problems encountered with initial conditions in $L^{2}(0,2 \pi)$. Sometimes, we will refer to the generalised solution just as the solution of the problem and it will be given as an eigenfunction expansion in $L^{2}(0,2 \pi)$. Motivated by Definition 2.3, each generalised solution will be considered as a continuous map in the time variable with respect to the norm of $L^{2}(0,2 \pi)$. It will be obtained as an $L^{2}(0,2 \pi)$ limit of a sequence of smooth solutions which will always exist due to the Fourier method following the lines of reasoning in the proof of Theorem 2.4.

The concept of the generalised solution is enough for setting a rigorous framework for the revival phenomenon. However, one can proceed a little further and show that the generalised solution obtained in Theorem 2.4, is a solution in the weak sense. In particular, we close this section with the next statement which addresses this direction.

Proposition 2.6. Let $u_{0} \in L^{2}(0,2 \pi)$ and consider the generalised solution $u(x, t)$ of the IBVP (2.3). Then, for arbitrary $\phi \in D(L)$, the $\operatorname{map}\langle u(\cdot, t), \phi\rangle$ is continuous
for every $t \geq 0$ and continuously differentiable for $t>0$ with

$$
\begin{equation*}
\frac{d}{d t}\langle u(\cdot, t), \phi\rangle=-i\langle u(\cdot, t), L \phi\rangle . \tag{2.8}
\end{equation*}
$$

Proof. See Appendix E.
A function $u(x, t)$ that satisfies Definition 2.3 and Proposition 2.6 is called a weak solution to the IBVP (2.3). We note that the definition of a weak solution given here coincides with the definition of a weak solution as found in the context of $C_{0}$ one-parameter semigroups and abstract Cauchy problems in Hilbert spaces, [23]. The equation (2.8) is known as the weak formulation of the problem.

### 2.3 The Classical Theory of the Revival Effect

In the first section we gave a short introduction to the phenomenon of revivals and fractalisation based on the linear Schrödinger equation (FSLS). We now follow a more systematic treatment and present what we call the classical theory, first of the revival effect in this section and of the fractalisation effect in the next section. The classical theory refers to the case of periodic boundary conditions for first-order in time, linear partial differential equations with dispersion relation a polynomial with integer coefficients. Concretely, we are interested in the following initial boundary value problem posed on $[0,2 \pi]$

$$
\begin{align*}
& \partial_{t} u(x, t)=-i P\left(-i \partial_{x}\right) u(x, t), \quad u(x, 0)=u_{0}(x),  \tag{2.9}\\
& \partial_{x}^{m} u(0, t)=\partial_{x}^{m} u(2 \pi, t), \quad m=0,1, \ldots, n-1,
\end{align*}
$$

where $P(\cdot)$ is a polynomial of degree $n \geq 2$ with integer coefficients.
It is readily seen that the linear PDE in (2.9) has dispersion relation $\omega=P(k)$. When $P(x)=x^{2}$, the equation corresponds to (FSLS), and when $P(x)=x^{3}$ we obtain the third-order in space PDE

$$
\begin{equation*}
\partial_{t} u(x, t)=\partial_{x}^{3} u(x, t) . \tag{AI}
\end{equation*}
$$

Equation (AI) is known as the Airy PDE, perhaps because its solution on the real
line can be expressed in terms of the Airy function, [24]. For the same reason, the equation with the negative sign, $\partial_{t} u=-\partial_{x}^{3} u$, is also called Airy PDE, [13], [24] and it can be obtained by (AI) if we change $t$ with $-t$. The negative-sign form is perhaps more popular in the literature, since it is the linear part of the Korteweg-de Vries equation (KdV), which is used to model the unidirectional propagation of surface waves on shallow water, [25], [26]. In this thesis, we will always consider (AI).

In Theorem 2.8 below, we will show that the solution to the periodic problem (2.9) at rational times can be decomposed into a finite number of translated copies of the given initial condition $u_{0}$. Hence, we will obtain a rigorous justification of the revival of the initial condition in the structure of the solution at rational times. This will further explain precisely the numerical example in Figure 2.2 for the FSLS equation. Moreover, as we shall see in the following chapters, Theorem 2.8 will serve as a fundamental ingredient to extend the revival phenomenon beyond periodic boundary conditions and beyond first-order in time evolution problems.

To examine the revival phenomenon, we need to find a solution representation to (2.9). In accordance with Remark 2.5, the Fourier method implies the existence and uniqueness of a generalised solution in $L^{2}(0,2 \pi)$ for any initial function $u_{0} \in$ $L^{2}(0,2 \pi)$. In particular, at any fixed time $t \geq 0$, the (generalised) solution to (2.9) is given by the complex Fourier series

$$
\begin{equation*}
u(x, t)=\sum_{j \in \mathbb{Z}} \widehat{u_{0}}(j) e^{-i P(j) t} e_{j}(x), \tag{2.10}
\end{equation*}
$$

where the convergence is in the sense of $L^{2}(0,2 \pi)$.
The revival effect will follow by showing that at rational times the solution (2.10) admits a representation in $L^{2}(0,2 \pi)$ in terms of a finite linear combination of translations of the initial function. Such a representation was first considered in the work of Berry and Klein [7] for the FSLS equation based on the analysis of the Talbot effect. Independently, Taylor in [12] derived the revival representation for the same equation. Later, Olver in [13] rediscovered the revival and fractalisation effect in the context of the Airy equation (AI) and show that the revival effect extends to the more general class of linear dispersive equations (2.9). Olver also named the effect dispersive quantisation due to the dispersive nature of the model equation and
the property of its solution to quantised into a finite number of copies of the initial datum.

In order to state and prove the revival representation, we first need to give a meaning to the translation of the initial function $u_{0}$ in the periodic setting. Since $u_{0}$ is given originally on $[0,2 \pi]$, we first extend it periodically to $\mathbb{R}$. We will denote by $u_{0}^{*}$ the $2 \pi$-periodic extension of $u_{0}$,

$$
\begin{equation*}
u_{0}^{*}(x)=u_{0}(x-2 \pi m), \quad 2 \pi m \leq x<2 \pi(m+1), \quad m \in \mathbb{Z} \tag{2.11}
\end{equation*}
$$

For any real number $s$, the function $u^{*}(x-s)$ we will be called the translation of $u_{0}$ by $s$.

Since the proof of the revival phenomenon will rely on the uniqueness of the Fourier coefficients of the solution representation (2.10), we are also interested in the form of the Fourier coefficients of the translation. With this in mind, notice that if we restrict $u^{*}(x-s)$ on $[0,2 \pi]$, then $u^{*}(\cdot-s)$ defines a function in $L^{2}(0,2 \pi)$ whenever $u_{0}$ is in $L^{2}(0,2 \pi)$. Indeed,

$$
\begin{equation*}
\left\|u_{0}^{*}(\cdot-s)\right\|^{2}=\int_{0}^{2 \pi}\left|u_{0}^{*}(x-s)\right|^{2} d x=\int_{-s}^{2 \pi-s}\left|u_{0}^{*}(y)\right|^{2} d y=\int_{0}^{2 \pi}\left|u_{0}(y)\right|^{2} d y=\left\|u_{0}\right\|^{2} \tag{2.12}
\end{equation*}
$$

Therefore, $u^{*}(\cdot-s)$ admits a Fourier series representation in $L^{2}(0,2 \pi)$ and its Fourier coefficients have a simple form given by the following lemma.

Lemma 2.7. Let $u_{0} \in L^{2}(0,2 \pi)$ and $s \in \mathbb{R}$. Then, the Fourier coefficients of $u_{0}(\cdot-s)$ are given by

$$
\begin{equation*}
\left\langle u_{0}(\cdot-s), e_{j}\right\rangle=e^{-i j s} \widehat{u_{0}}(j), \quad j \in \mathbb{Z} \tag{2.13}
\end{equation*}
$$

Proof. By definition we have

$$
\left\langle u_{0}(\cdot-s), e_{j}\right\rangle=\frac{1}{\sqrt{2 \pi}} \int_{0}^{2 \pi} u_{0}^{*}(x-s) e^{-i j x} d x
$$

For fixed $s \in \mathbb{R}$ we can find $\ell \in \mathbb{Z}$ such that $2 \pi \ell \leq s<2 \pi(\ell+1)$ or $0 \leq s-2 \pi \ell<2 \pi$,
and thus

$$
\left\langle u_{0}(\cdot-s), e_{j}\right\rangle=\frac{1}{\sqrt{2 \pi}} \int_{0}^{s-2 \pi \ell} u_{0}^{*}(x-s) e^{-i j x} d x+\frac{1}{\sqrt{2 \pi}} \int_{s-2 \pi \ell}^{2 \pi} u_{0}^{*}(x-s) e^{-i j x} d x
$$

If $0 \leq x<s-2 \pi \ell$, then $u_{0}^{*}(x-s)=u_{0}(x-s+2 \pi(\ell+1))$, whereas if $s-2 \pi \ell \leq x<2 \pi$, then $u_{0}^{*}(x-s)=u_{0}(x-s+2 \pi \ell)$. Thus, we have that

$$
\begin{aligned}
\left\langle u_{0}(\cdot-s), e_{j}\right\rangle & =\frac{1}{\sqrt{2 \pi}} \int_{0}^{s-2 \pi \ell} u_{0}(x-s+2 \pi(\ell+1)) e^{-i j x} d x \\
& +\frac{1}{\sqrt{2 \pi}} \int_{s-2 \pi \ell}^{2 \pi} u_{0}(x-s+2 \pi \ell) e^{-i j x} d x
\end{aligned}
$$

Finally, by changing variables in each of the integrals and collecting terms we arrive at (2.13).

We are now ready to show that the solution to the IBVP (2.9) at rational times is constructed by a finite superposition of translations of the initial condition $u_{0}$. The following theorem is the main result of the chapter. The statement can be found in many places, including the monograph [27, Theorem 2.14] by Erdoğan and Tzirakis.

Theorem 2.8 ([13], [27]). Let $u_{0} \in L^{2}(0,2 \pi)$. At a rational time $t=2 \pi \frac{p}{q}$ the solution to the IBVP (2.9) is given by

$$
\begin{equation*}
u\left(x, 2 \pi \frac{p}{q}\right)=\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} G_{p, q}(k) u_{0}^{*}\left(x-2 \pi \frac{k}{q}\right) . \tag{2.14}
\end{equation*}
$$

The equality in (2.14) holds in $L^{2}(0,2 \pi)$ and the coefficients $G_{p, q}(k)$ are given by

$$
\begin{equation*}
G_{p, q}(k)=\sum_{m=0}^{q-1} e^{-2 \pi i P(m) \frac{p}{q}} e_{m}\left(2 \pi \frac{k}{q}\right) . \tag{2.15}
\end{equation*}
$$

Proof. We show that the right-hand side of (2.14) has Fourier coefficients

$$
\widehat{u}_{0}(j) e^{-i P(j) \frac{2 \pi p}{q}} .
$$

Thus, by comparing this with the solution representation (2.10) at a rational time, we conclude that the claim holds due to the uniqueness of the Fourier coefficients.

Denote by $R(x)$ the right-hand side of (2.14). Then, by virtue of (2.13) we have

$$
\begin{aligned}
\widehat{R}(j) & =\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} e^{-2 \pi i P(m) \frac{p}{q}} e_{m}\left(2 \pi \frac{k}{q}\right) e^{-2 \pi i j \frac{k}{q}} \widehat{u}_{0}(j) \\
& =\frac{\widehat{u}_{0}(j)}{q} \sum_{m=0}^{q-1} e^{-2 \pi i P(m)^{\frac{p}{q}}} \sum_{k=0}^{q-1} e^{2 \pi i(m-j) \frac{k}{q}},
\end{aligned}
$$

where we have used the definition of the $e_{m}(x)$ as given by (2.6). We now distinguish two cases based on the integer $m$ running from 0 to $q-1$.

If $m \not \equiv j(\bmod q)$, then there exists $z \in \mathbb{Z}$ not a multiple of $q$, such that $m-j=$ $z_{1} q+z$ for $z_{1} \in \mathbb{Z}$. Hence,

$$
\sum_{k=0}^{q-1} e^{2 \pi i(m-j) \frac{k}{q}}=\sum_{k=0}^{q-1} e^{2 \pi i z_{1} q \frac{k}{q}} e^{2 \pi i z \frac{k}{q}}=\sum_{k=0}^{q-1}\left(e^{2 \pi i \frac{z}{q}}\right)^{k}=\frac{1-\left(e^{2 \pi i \frac{z}{q}}\right)^{q}}{1-e^{2 \pi i \frac{z}{q}}}=0 .
$$

On the other hand, whenever $m \equiv j(\bmod q)$, we have $m-j=z_{2} q$ for $z_{2} \in \mathbb{Z}$ and so

$$
\sum_{k=0}^{q-1} e^{2 \pi i(m-j) \frac{k}{q}}=\sum_{k=0}^{q-1} e^{2 \pi i z_{2} q \frac{k}{q}}=q
$$

Moreover, in this case, we know that $P(m) \equiv P(j)(\bmod q)$, since $P$ is a polynomial of order $n \geq 2$ with integer coefficients. So, $P(m)=P(j)+z_{3} q$ for some other $z_{3} \in \mathbb{Z}$, and this implies that

$$
\begin{equation*}
e^{-2 \pi i P(m) \frac{p}{q}}=e^{-2 \pi i P(j) \frac{p}{q}} e^{-2 \pi i z a s q_{q}^{p}}=e^{-2 \pi i P(j) \frac{p}{q}} . \tag{2.16}
\end{equation*}
$$

Therefore, as $m$ runs from 0 to $q-1$, we find that

$$
\widehat{R}(j)=\widehat{u}_{0}(j) e^{-i P(j) 2 \pi \frac{p}{q}},
$$

as claimed.

The representation (2.14) clearly states that for a given initial condition $u_{0}$, the solution at rational times is constructed by a finite superposition of translations of $u_{0}$. Thus, at rational times the solution revives the structure or the profile of the initial condition. In particular, this implies that for a piecewise constant function
$u_{0}$, the solution is piecewise constant at rational times as a linear combination of piecewise constant functions. In turn, this fully justifies the appearance of the piecewise constant profiles observed at rational times in Figure 2.2, for the real and imaginary part of the solution to the FSLS equation. As mentioned in Section 2.1, this recurrence of the initial condition in the structure of the solution at rational times is known as the revival effect or revival phenomenon. With Theorem 2.8 we achieve a precise meaning to the revival effect for the periodic problem (2.9).

In order to show that a given IBVP exhibits revivals, we look for a revival representation of the type (2.14). But a revival representation, as considered in other parts of the thesis, might involve other transformations of the initial condition apart from translations. For instance, the revival phenomenon for the pseudo-periodic problem associated to the FSLS equation in Chapter 4 follows by a revival representation that includes translations of reflected copies of the initial condition (see Corollary 4.9). On the other hand, the revival representation that we prove in Chapter 6 (see Corollary 6.9) for the FSLS equation with Robin-type boundary conditions implies that the structure of the revival effect is different from both the periodic and the pseudo-periodic case. In general, as it has been noticed in the review paper [28] by Smith, a rigorous definition of the revival property that captures the various revival phenomena remains an open problem.

Similar to the revival representation (2.14), all the analytical results on the revival phenomenon appear to share the following implications. We state them in the context of the periodic problem (2.9) with the following remark.

Remark 2.9. For $u_{0} \in L^{2}(0,2 \pi)$, the revival representation (2.14) implies the following.
(I) If $u_{0}$ is a piecewise continuous function on $[0,2 \pi]$, the solution $u(\cdot, t)$ at rational times $t=2 \pi \frac{p}{q}$ is also a piecewise continuous function of $x$ on $[0,2 \pi]$.
(II) If $u_{0}$ is $2 \pi$-periodic and continuous as a function on $\mathbb{R}$, the solution $u(\cdot, t)$ at rational times $t=2 \pi \frac{p}{q}$ depends on finitely many values of $u_{0}$ and it is also a continuous function of $x$ on $[0,2 \pi]$.

Motivated by the revival formula (2.14), we will establish the following terminology.

Definition 2.10. Let $u(x, t)$ be the solution to an IBVP with initial condition $u_{0}(x)$ at time zero. We say that the IBVP exhibits the phenomenon of pure revivals, if at any rational time $t=2 \pi \frac{p}{q}$ the solution can be expressed as a finite linear combination of translations of the initial function $u_{0}$.

Definition 2.10 serves as a convenient terminology for the exposition of the material in the thesis and we will not regard it as a fully rigorous definition of the revival effect. Its main limitation comes from the fact that it does not necessarily captures the various revival phenomena observed in time evolution problems outside the classical setting.

Nonetheless, it sets a point of reference on the concept of revivals. It will help distinguishing other revival phenomena that we will encountered later and which, sometimes, will incorporate the pure revival effect. Therefore, we stress that essentially the pure revival effect addresses that the revival phenomenon in a time evolution problem occurs due to the reconstruction of the solution in terms of a finite number of only translations of the initial condition and does not involve other transformations.

Remark 2.11. The statement and proof of Theorem 2.8 provides a tool to characterise the revival property in an IBVP from an operational point of view. In particular, the right hand-side of (2.14) can be viewed as an operator in $L^{2}(0,2 \pi)$ constructed by a finite number of translations or in other words translation operators. In Chapter 3, we properly define first the translation operator and then the revival operator. The revival operators will allow a more clear reformulation of Theorem 2.8 in the monomial case $\left(P(x)=x^{n}\right)$ and will help us identify the pure revival effect in time evolution problems which are not in the classical setting of this section.

In the bibliography, the proof of the revival effect usually relies on the fundamental solution of the equation with initial condition a ( $2 \pi$-periodically extended) delta distribution supported on $x=\pi$. By showing that the fundamental solution at a rational time satisfies the revival representation (2.14), then the conclusion for arbitrary initial condition follows by convolution. This was the direction suggested by Olver [13] and a self-contained proof can be found in [27] by Erdoğan and Tzirakis. Taylor, [12], similarly considered the fundamental solution to the FSLS equation
and compared the revival formula (2.14), when $P(x)=x^{2}$, with another revival formula derived by solving the FSLS equation on the real line but with $2 \pi$-periodic initial conditions. This gives an equivalent point of view of the periodic case, which is often considered in the literature. Following this idea, Taylor derived a classical reciprocity identity from number theory involving the coefficients $G_{p, q}(k)$ which are given by

$$
\begin{equation*}
\sum_{m=0}^{q-1} e^{-2 \pi i m^{2} \frac{p}{q}} e^{2 \pi i m \frac{k}{q}}, \tag{2.17}
\end{equation*}
$$

as follows from (2.15) when $P(x)=x^{2}$. For $k=0$, these are known as quadratic Gauss sums, [29]. Note that in general although the coefficients $G_{p, q}(k)$ are $q$-periodic functions and have the form of Gauss sums

$$
\sum_{m=0}^{q-1} \chi(m) e_{m}\left(2 \pi \frac{k}{q}\right) \quad \chi(m)=e^{-2 \pi i P(m) \frac{p}{q}},
$$

they are not Gauss sums since the function $\chi(m)$ is not a Dirichlet character, [29].
The above proof of Theorem 2.8 is partly inspired by Theorem 1 in [13] which address the solution to one of the exercises in [26, Exercise 8.5.8] by Olver. With this theorem, Olver in [13] showed that for a piecewise constant initial function the solution to Airy's equation at rational times corresponds to a piecewise constant function. The underlying ingredient of the argument is the uniqueness of the Fourier coefficients which is combined with the periodicity of the Fourier coefficients of piecewise constant functions. The same characterisation of the revival phenomenon for the Schrödinger equation with piecewise constant initial data is achieved by Theorem 5.1 in Thaller's book [16] by following the same idea and we also refer to [30, Section 9.4 and Exercise 9.46] by Kammler. However, in our case, we were mainly motivated by the argument in [15] by Olver, Sheils and Smith, where they obtained the revival property for the FSLS equation under pseudo-periodic boundary conditions. We will consider these boundary conditions later in Chapter 4. For the moment, we just mention that, roughly speaking, their idea was to carefully construct a linear combination of translations of the initial datum and then show that the (generalised) Fourier coefficients correspond to these of the solution representation at rational times. In the case of Theorem 2.8, the construction was given by Erdoğan and Tzirakis in [27, Thoerem 2.14].

Finally, as noted in [13] by Olver, we should mention that the revival result of Theorem 2.8 can be extended to the case when the coefficients of the polynomial $P$ are integer multiples of a common real number. This follows easily by a suitable rescaling in time and includes the case of rational coefficients.

### 2.4 The Classical Theory of Fractalisation

Recall from Section 2.1 that in contrast to the revival effect in the FSLS equation, the fractalisation effect refers to the improvement of the regularity of the solution at irrational times. In particular, at irrational times the initial discontinuity of the step function (2.2) disappears and the solution profile becomes a continuous, though nowhere differentiable function (see Figure 2.3). In this section we now collect the main results on the fractalisation effect for the time evolution problem (2.9) which includes the free linear Schrödinger equation and the Airy PDE.

Although the main focus of the thesis is on the revival phenomenon, the statements given here on the fractalisation effect, especially for the FSLS equation (see Theorems 2.12 and 2.13), will allow us to draw some interesting conclusions in other cases of boundary conditions. Indeed, in Chapter 4, both the revival and the fractalisation effect will be shown to persist in the FSLS equation with pseudo-periodic boundary conditions. Furthermore, in Chapter 5 we will prove that the revival phenomenon in general breaks in the Airy PDE under quasi-periodic boundary conditions and instead of revivals the fractalisation effect occurs at rational times.

The methods required to analytically study the fractalisation effect are beyond the scope of the thesis. Therefore, in this section, we will refer to the original papers for the proofs and will not include them here. Our approach on the revival effect will rely on the classical theory of linear differential operators and eigenfunction expansions in $L^{2}(0,2 \pi)$. On the other hand, the examination of the fractalisation effect relays on specialised methods in the analysis of periodic linear dispersive partial differential equations. A variety of such methods can be found in [27] by Erdoğan and Tzirakis with applications on the Talbot effect. In this direction, we further refer to the work by Erdoğan and Shakan in [31]. Their study is on the fractal dimension of the graphs of the solutions to a number of periodic dispersive

PDEs. The introduction of [31] offers a concrete overview of the fractalisation effect.
As a smoothing effect, the fractalisation phenomenon cannot be observed in the setting of $L^{2}(0,2 \pi)$ or in general in the setting of the periodic Sobolev space of order $s \geq 0$ (see Appendix D)

$$
H_{\mathrm{per}}^{s}(0,2 \pi)=\left\{f \in L^{2}(0,2 \pi) ; \sum_{j \in \mathbb{Z}}\left(1+j^{2}\right)^{s}|\widehat{f}(j)|^{2}<\infty\right\}
$$

which is often regarded as the standard functional space for the analysis of (2.9). Indeed, notice that for any non-negative reals $s$ and $t$ and integer $j$, we have that

$$
\left(1+j^{2}\right)^{s}\left|e^{-i P(j) t} \widehat{u_{0}}(j)\right|^{2}=\left(1+j^{2}\right)^{s}\left|\widehat{u_{0}}(j)\right|^{2} .
$$

Hence, at any fixed time $t>0$ the solution preserves the Sobolev regularity. That is, whenever the initial data $u_{0}$ belongs in $H_{\mathrm{per}}^{s}(0,2 \pi)$, so does $u(\cdot, t)$.

Instead, in [11] Oskolkov focused on the behaviour of the solution (2.10) to the equations (FSLS) and (AI) when the initial condition is of bounded variation over $[0,2 \pi]$ which could possess countably many jump discontinuities. In particular, recall that a function of bounded variation can only have a countable set of points at which it is discontinuous, see [32, Theorem 3.27]. Under this specific assumption and by examining the converge properties of a family of discrete Hilbert transforms, including

$$
H_{N}(x, t)=\sum_{0 \neq j=-N}^{N} \frac{e^{i j^{n} t+i j x}}{j}, \quad n=2 \text { or } 3,
$$

Oskolkov proved that there is a dichotomy at rational and irrational values of $t / 2 \pi$ in the behaviour of the solution to these two equations.

Theorem 2.12 ([11]). Let $P(x)=x^{n}$, with $n=2$ or $n=3$ and consider the solution $u(x, t)$ given by (2.10) to the IBVP (2.9) with initial condition $u_{0}$ of bounded variation over the interval $[0,2 \pi]$. Then, we have the following.
(i) If $t / 2 \pi$ is an irrational number, then $u(x, t)$ is a continuous function of $x$.
(ii) If $t / 2 \pi$ is a rational number and $u_{0}$ has at least one jump discontinuity, then $u(x, t)$ as a function of $x$ is a bounded function with at most countably many discontinuities.
(iii) If $u_{0}$ is also continuous on $[0,2 \pi]$ and such that $u_{0}(0)=u_{0}(2 \pi)$, then $u(x, t)$ is continuous in $x$ and $t$.

A concise proof of Theorem 2.12-(i), and Theorem 2.13 below due to Rodnianski, is also outlined in [33] by Chousionis, Erdoğan and Tzirakis. Oskolkov's Theorem 2.12 explains to some extent the dichotomy of revivals and fractalisation in the case of the Schrödinger and Airy equations. Indeed, for discontinuous initial data of bounded variation, at irrational times the solution becomes continuous, whereas at rational times can exhibit countably many discontinuities. Hence, at irrational times the solution gains regularity whereas at rational times revives the initial discontinuities as we notice in the example of the step function in Section 2.1.

In [7], Berry and Klein observed that for a step initial function the graph of the solution to the free linear Schrödinger equation has a fractal structure at irrational times. In particular, they considered rational approximations to understand the behaviour of the solution at irrational times and they argued that the fractal dimension of the real and imaginary parts of the solution should be $3 / 2$. Moreover, Berry in [34] continued the investigation of the Talbot effect in the context of the FSLS equation confined in a $d$-dimensional box. Berry, [34], conjectured that for almost all times $t$ the graphs of the real and imaginary parts of the solution and its density $|u(x, t)|^{2}$ are fractal sets with dimension $d+1 / 2$. In the one-dimensional case, Rodnianski [35] provided a mathematical proof that justified Berry's conjecture on the fractal dimension of the real and imaginary parts.

Before we state Rodnianski's result, we should mention that with the term fractal dimension we refer to the upper Minkowski dimension (also known as box counting dimension), [36], of a non-empty bounded set $A$ of $\mathbb{R}^{d}$. The fractal dimension of $A$ is given by

$$
\limsup _{\epsilon \rightarrow 0} \frac{\log (N(A, \epsilon))}{\log (1 / \epsilon)},
$$

where $N(A, \epsilon)$ is the smallest number of $\epsilon$-balls needed to cover $A$, see [36] for more details.

Theorem 2.13 ([35]). Let $P(x)=x^{2}$ and consider the solution $u(x, t)$ given by (2.10) to the $I B V P(2.9)$. Let $u_{0}$ be of bounded variation over the interval $[0,2 \pi]$
and such that

$$
\begin{equation*}
u_{0} \notin \bigcup_{s>1 / 2} H_{p e r}^{s}(0,2 \pi) . \tag{2.18}
\end{equation*}
$$

Then, for almost all times $t$, the fractal dimension of the graph of either the function $\operatorname{Re}(u(x, t))$ or the function $\operatorname{Im}(u(x, t))$, or both is $3 / 2$.

Condition (2.18) ensures that the initial function $u_{0}$ of bounded variation does not admit a continuous representative, since Sobolev's embedding does not apply (see Lemma D.5). Thus, any possible discontinuity of $u_{0}$ is a jump discontinuity and not removable. The step function (2.2) is an initial condition of this type. Then, for singular conditions of this form, it follows from Theorem 2.13 that the real and imaginary parts of the solution have fractal dimension $3 / 2$. In turn, this essentially implies that they are continuous, but non-differentiable functions and justifies the fractalisation effect as observed in Figure 2.3 for the FSLS equation.

Rodnianski derived Theorem 2.13 using the results in [37] by Kapitanski and Rodnianski. Kapitanski and Rodnianski, [37], examined the regularity properties of the fundamental solution to the FSLS equation with periodic boundary conditions on the setting of Besov spaces, see [37] and references therein. Measuring the regularity in this framework, they were able to show that, in general, at irrational times the solution to the free linear Schrödinger equation is more regular than at rational times.

In total, Oskolkov's and Rodnianski's theorems completely characterise the fractalisation effect in the case of the FSLS equation. Moroever, Oskolkov's result address the continuity of the solution to the Airy PDE at irrational times. The extension of both Theorems 2.12 and 2.13 in the monomial case, $P(x)=x^{n}$ for any integer $n \geq 3$, of the periodic time evolution problem (2.9) were obtained in [33] by Chousionis, Erdoğan and Tzirakis. The statement of their theorem holds for general polynomials with integer coefficients and can be found in the book [27, Theorem 2.16] with a self-contained proof.

Theorem 2.14 ([27], [33]). Let $P$ be a polynomial of degree $n \geq 2$ with integer coefficients and $P(0)=0$. Consider the solution $u(x, t)$ given by (2.10) to the IBVP (2.9) with initial condition $u_{0}$ of bounded variation over the interval $[0,2 \pi]$. Then, we have the following.
(i) For almost every time $t$, the solution $u(x, t)$ is a continuous function of $x$.
(ii) Assume in addition that

$$
u_{0} \notin \bigcup_{s>1 / 2} H_{p e r}^{s}(0,2 \pi) .
$$

If $P$ is not an odd polynomial, then for almost all times $t$, the fractal dimension of the graph of the functions $\operatorname{Re}(u(x, t))$ and $\operatorname{Im}(u(x, t))$ lies in $\left[1+2^{1-n}, 2-\right.$ $2^{1-n}$, whereas if $P$ is an odd polynomial, the statement holds for the real-valued solutions.

Theorem 2.8 and Theorem 2.14 complement one another and mathematically describe in the periodic setting the dichotomy of revivals and fractalisation for the time evolution problem (2.9). In particular, by Theorem 2.8, for any initial condition $u_{0} \in L^{2}(0,2 \pi)$, at rational times the solution $u(x, t)$ is a finite superposition of translations of the initial function and thus the IBVP exhibits the phenomenon of pure revivals (see Definition 2.10). Moreover, if $u_{0}$ is of bounded variation over the segment $[0,2 \pi]$ and has finitely many jump discontinuities, then at rational times the solution displays finitely many jump-discontinuities. On the other hand, by Theorem 2.14, at irrational times the solution becomes a continuous function of $x$ with its real and imaginary parts being fractal curves.

### 2.5 Perturbations and Weak Revivals

Apart from the family of linear dispersive PDEs (2.9), the phenomenon of revivals and fractalisation has been shown to appear in other type of equations under periodic boundary conditions. In this section, we describe how revivals manifest in non-linear dispersive equations and linear Schrödinger equations with potential.

The results presented here will serve as context of the notion of weak revivals. Plainly, a weak revival effect is understood as a pure revival effect perturbed by a continuous function in space. The term was introduced in [14] to describe the revival phenomenon in the FSLS equation subject to a specific type of separated boundary conditions on $[0, \pi]$. The revival property for this initial boundary value problem will be discussed extensively in Chapter 6.

Starting with the non-linear regime, the numerical studies [38] and [39] by Chen and Olver indicated that, under periodic boundary conditions, the quantisation and fractalisation of the solutions to the free linear Schrödinger and Airy evolution persist into their non-linear counterparts, including both integrable and non-integrable cases. Some of their numerical observations were subsequently proved in the works [40], [41] by Erdoğan and Tzirakis for the cubic non-linear Schrödinger equation

$$
\begin{equation*}
\partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t)+i|u(x, t)|^{2} u(x, t) \tag{NLS}
\end{equation*}
$$

and the Korteweg-de Vries equation

$$
\begin{equation*}
\partial_{t} u(x, t)=-\partial_{x}^{3} u(x, t)-2 u(x, t) \partial_{x} u(x, t), \tag{KdV}
\end{equation*}
$$

respectively.
By virtue of the seminal work of Bourgain, [42] and [43], we know that the $2 \pi$ periodic problems for (NLS) and (KdV) are globally well-posed in $L^{2}(0,2 \pi)$. The results in [41] and [40] showed that for initial data of bounded variation, there is a smoothing effect on the solution at irrational times. This follows from a more general smoothing property that, at any positive time, the difference between the solution and their linear evolution is more regular than the initial data. Thus, in both cases, the revival/fractalisation dichotomy is characterised in terms of change in the regularity of the solution at rational/irrational times. As we will explain shortly, the analysis of the phenomenon for the NLS equation still gives a revival representation, but this time in a weak sense.

Below, we state the main theorem from [41] due to Erdoğan and Tzirakis on the periodic problem for the NLS equation.

Theorem 2.15 ([41]). Consider the equation (NLS) with periodic boundary conditions on $[0,2 \pi]$

$$
u(0, t)=u(2 \pi, t), \quad \partial_{x} u(0, t)=\partial_{x} u(2 \pi, t),
$$

and initial condition $u_{0}$ of bounded variation on $[0,2 \pi]$. Then, we have the following.
(i) If $t / 2 \pi$ is an irrational number, then $u(x, t)$ is a continuous function of $x$.
(ii) For rational values of $t / 2 \pi$, the solution $u(x, t)$ is a bounded function with at
most countably many discontinuities.
(iii) If $u_{0}$ is also continuous on $[0,2 \pi]$ and such that $u_{0}(0)=u_{0}(2 \pi)$, then $u(x, t)$ is continuous in $x$ and $t$.
(iv) Assume in addition that

$$
u_{0} \notin \bigcup_{s>1 / 2} H_{p e r}^{s}(0,2 \pi) .
$$

Then, for almost all times, the fractal dimension of the graph of the functions $\operatorname{Re}(u(x, t))$ and $\operatorname{Im}(u(x, t))$ is $3 / 2$.

From the results of [40], it also follows that parts (i)-(iii) of Theorem 2.15 hold for the solution to (KdV) under periodic boundary conditions on $[0,2 \pi]$. Later, in [33], the graph of the solution was shown to have fractal dimension in $[5 / 4,7 / 4]$. Apart from the original papers, [41], [40], [33], we also refer to the monograph [27, Section 5.3], for the statements and proofs of Theorem 2.15 and the similar result for the Korteweg-de Vries equation.

We would now like to elaborate more on the approach of the proof of Theorem 2.15. This will give a motivation to the weak revival effect. Using Duhamel's principle, Erdoğan and Tzirakis, [41], decomposed the solution into two parts

$$
\begin{equation*}
u(x, t)=u_{F S L S}(x, t)+N(x, t) \tag{2.19}
\end{equation*}
$$

by considering the NLS equation as a non-linear perturbation of the FSLS equation. Here, $u_{F S L S}$ denotes the solution to the periodic problem for the free linear Schrödinger equation with initial condition $u_{0}$ and $N(x, t)$ is the contribution of the non-linearity. When $u_{0}$ is of bounded variation, they proved that the function $N(x, t)$ is continuous in both variables. Thus, by applying Oskolkov's (Theorem 2.12) and Rodnianski's (Theorem 2.13) results to the linear part $u_{F S L S}$, Theorem 2.15 follows.

As a direct implication of (2.19) combined with Theorem 2.8, we notice that the solution at any rational time $t=2 \pi p / q$ is given by

$$
\begin{equation*}
u\left(x, 2 \pi \frac{p}{q}\right)=\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} e^{-2 \pi i m^{2} \frac{p}{q}} e_{m}\left(2 \pi \frac{k}{q}\right) u_{0}^{*}\left(x-2 \pi \frac{k}{q}\right)+N\left(x, 2 \pi \frac{p}{q}\right) . \tag{2.20}
\end{equation*}
$$

From (2.20), it is clear that although the solution is no longer a finite superposition of translations of $u_{0}$, the revival of jump discontinuities in the solution profile still occurs at rational times, whenever the initial condition is a piecewise continuous function in $L^{2}(0,2 \pi)$. The representation (2.20) shows that there exists a weaker type of revivals and motivates the following definition.

Definition 2.16. We say that an initial boundary value problem exhibits weak revivals if its solution evaluated at rational times is given as the sum of a pure revival effect and a continuous function in space.

In accordance with Definition 2.16, the main observation is that the weak revival effect has the same implications as the pure revival effect, outlined in Remark 2.9. Hence, the revival effect in the periodic problem for the NLS equation is viewed as the revival of the initial jump discontinuities and occurs due to a weak revival effect.

Recently, similar results to Theorem 2.15 have been reported on the periodic problem for the linear Schrödinger equation with a potential function $V(x)$,

$$
\begin{equation*}
\partial_{t} u(x, t)=-i\left(-\partial_{x}^{2} u(x, t)+V(x) u(x, t)\right) . \tag{2.21}
\end{equation*}
$$

In [44], Cho, Kim, Kim, Kwon and Seo considered the linear Schrödinger equation (2.21), where the potential $V$ is in $\cup_{s>0} H_{\text {per }}^{s}(0,2 \pi)$. Following, the same line of arguments as in [41], they show that for an initial condition $u_{0}$ of bounded variation on $[0,2 \pi]$, parts (i)-(iii) of Theorem 2.15 hold for (2.21). They also provide lower and upper bounds on the fractal dimension of the real and imaginary parts of $u(x, t)$ and of the density $|u(x, t)|^{2}$. In particular, based again on Duhamel's principle, they show that the solution to (2.21) at any fixed time $t \geq 0$, is given by the sum of the solution to FSLS with initial condition $u_{0}$ and a continuous function in the space variable. Thus, similar to the cubic NLS equation, at rational times the solution to the periodic problem for the Schrödinger equation will exhibit a weak revival effect.

At this point, we should mention that Rodnianski in [45] also studied the Talbot effect for the Schrödinger equation (2.21). In particular, for a potential function in $\cup_{s>1 / 2} H_{\text {per }}^{s}(0,2 \pi)$, it follows from [45] that the fundamental solution to (2.21), resulting from a periodic delta distribution, can be represented by the sum of the fundamental solution to the FSLS equation and a more regular function. Therefore, the
regularity of the fundamental solution to (2.21) is completely determined by that of the fundamental solution to the FSLS which incorporates the revival/fractalisation dichotomy, [37].

We further note that the results on the periodic problems for the NLS equation, [41], and the linear Schrödinger equation (2.21), [44], [45], strongly support that the additional conjecture of Berry in [34] is true. In turn, the Talbot effect should survive under non-linear or smooth perturbations of the free linear Schrödinger equation. Although, as we discussed above, the solution is not given purely by a finite number of translations of $u_{0}$ at rational times, but by a weak revival representation in accordance with Definition 2.16. Surprisingly, the same type of weak revival takes place in the behaviour of the solution to the FSLS equation with separated boundary conditions on $[0, \pi]$ (see Chapter 6). Similarly, for second-order in time evolution problems with periodic boundary conditions the main revival phenomenon seems to be the weak revival effect (see Chapter 7).

### 2.6 Further Results

In [38], Chen and Olver considered also periodic problems for dispersive equations having a non-polynomial dispersion relation. Examples include the linearisations of the Benjamin-Ono and Boussinesq equations from water wave theory. They numerically observed that other types of revival phenomena should exist. Indeed, later in [46], Boulton, Olver, Pelloni and Smith investigated the revivals in three models of integro-differential equations with non-polynomial dispersion relations. They analytically described another type of revival and rigorously confirmed some of the observations of Chen and Olver [38]. For example, in case of the linear BenjaminOno equation, they showed that at rational times the solution is given by a finite linear combination of translates of the initial function and of its periodic Hilbert transform.

An unexpected manifestation of the revival and fractalisation effect was reported by de la Hoz and Vegas in [47] for the vortex filament equation. The equation models the motion of a vortex filament (a vortex curve) in a ideal fluid and there is a correspondence between the equation and the cubic non-linear Schrödinger equation
due to Hasimoto's transformation, [48]. For initial datum a planar regular polygon, de la Hoz and Vegas showed that at rational times the solution becomes a skew polygon. At generic times, their numerical considerations suggest that the solution exhibits a fractal behaviour.

## Chapter 3

## Special Transformations and their Fourier Representation

In this chapter we define a number of isometries and other maps of $L^{2}(0,2 \pi)$. We describe the form of the Fourier coefficients under the action of these operators and summarise their properties. Both the notation and the results presented here will provide a clear mathematical framework of the revival phenomenon beyond the classical setting. This will be used extensively in the upcoming chapters.

Following Remark 2.11, in Section 3.1 we introduce the periodic translation and the revival operators. Based on these two transformations, we reformulate Theorem 2.8 (for $P(x)=x^{n}$ ) in terms of the latter. In this way, we obtain a compact, convenient notation to describe the revival property. Moreover, their consideration allows for further extensions. Indeed, as we shall see in later chapters, combining the transformations described in Sections 3.2 and 3.3 with the translation and revival operators, we will be able to characterise the revival effect in more complicated situations than the periodic problem (2.9) from Chapter 2. Therefore, in Section 3.2 we recall the definitions of the reflection and the even and odd extension of a function, whereas in Section 3.3 we consider the $2 \pi$-periodic convolution of two function. Although the setting of Sections 3.11 and 3.3 is considered well known, we review these transformations due to their fundamental role in the analysis of the revival phenomenon later.

### 3.1 Periodic Translation and Revival Operators

A revival operator is constructed as a finite linear combination of periodic translation operators which in turn are defined from the periodic extension of a function given on $[0,2 \pi)$. We begin by fixing our notation with the following definition.

Definition 3.1. The $2 \pi$-periodic extension to $\mathbb{R}$ of a function $f$ given on $[0,2 \pi)$ is defined by

$$
\begin{equation*}
f^{*}(x)=f(x-2 \pi m), \quad 2 \pi m \leq x<2 \pi(m+1), \quad m \in \mathbb{Z} \tag{3.1}
\end{equation*}
$$

The first important isometry is the translation operator which we properly define as follows.

Definition 3.2. Consider $s \in \mathbb{R}$. The periodic translation operator $\mathcal{T}_{s}: L^{2}(0,2 \pi) \rightarrow$ $L^{2}(0,2 \pi)$ is defined by the formula

$$
\begin{equation*}
\mathcal{T}_{s} f(x)=f^{*}(x-s), \quad x \in[0,2 \pi) \tag{3.2}
\end{equation*}
$$

The periodic translation operator is a linear operator. Moreover, as we see from the next lemma, it is an isometry on $L^{2}(0,2 \pi)$. The first claim below follows from calculation (2.12), whereas the proof of (ii) is exactly the same as the proof of Lemma 2.7.

Lemma 3.3. Let $s \in \mathbb{R}$. Then, we have the following.
(i) $\mathcal{T}_{s}$ is an isometry on $L^{2}(0,2 \pi)$.
(ii) For any $f \in L^{2}(0,2 \pi)$, the Fourier coefficients of $\mathcal{T}_{s} f$ are given by

$$
\begin{equation*}
\widehat{\mathcal{T}_{s} f}(j)=e^{-i j s} \widehat{f}(j) . \tag{3.3}
\end{equation*}
$$

We now introduce the concept of the revival operator which is defined as a finite superposition of translation operators. In particular, the revival operator describes the pure revival effect in the classical periodic setting of the time evolution problem (2.9) in the monomial case $P(x)=x^{n}$, with $n \geq 2$ in $\mathbb{N}$. In accordance with Remark 2.11, the definition of the revival operator is based on the right-hand side of the revival representation (2.14) in Theorem 2.8.

Definition 3.4. Consider $p$ and $q$ co-prime integers and let $n \geq 2$ in $\mathbb{N}$. The periodic revival operator $\mathcal{R}_{n}(p, q): L^{2}(0,2 \pi) \rightarrow L^{2}(0,2 \pi)$ of order $n$ at $(p, q)$ is defined by the formula

$$
\begin{equation*}
\mathcal{R}_{n}(p, q) f=\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} G_{p, q}^{(n)}(k) \mathcal{T}_{2 \pi \frac{k}{q}} f \tag{3.4}
\end{equation*}
$$

where the coefficients $G_{p, q}^{(n)}(k)$ are given by

$$
\begin{equation*}
G_{p, q}^{(n)}(k)=\sum_{m=0}^{q-1} e^{-2 \pi i m^{n} \frac{p}{q}} e_{m}\left(\frac{2 \pi k}{q}\right) . \tag{3.5}
\end{equation*}
$$

Revival operators provide a concise notation for the revival statements. To illustrate this claim, let us consider the IBVP (2.9) for the monomial case $P(x)=x^{n}$, where $n$ is an integer with $n \geq 2$. That is, we consider the time evolution problem

$$
\begin{align*}
& \partial_{t} u(x, t)=-i\left(-i \partial_{x}\right)^{n} u(x, t), \quad u(x, 0)=u_{0}(x),  \tag{3.6}\\
& \partial_{x}^{m} u(0, t)=\partial_{x}^{m} u(2 \pi, t), \quad m=0,1, \ldots, n-1
\end{align*}
$$

on $[0,2 \pi]$. Now, based on the definition of the revival operators, we can reformulate Theorem 2.8 in the case of the periodic problem (3.6) and describe the pure revival effect.

Corollary 3.5. Let $u_{0} \in L^{2}(0,2 \pi)$. Then, at any rational time $t=2 \pi \frac{p}{q}$, the solution to (3.6) admits the representation

$$
\begin{equation*}
u\left(x, 2 \pi \frac{p}{q}\right)=\mathcal{R}_{n}(p, q) u_{0}(x) \tag{3.7}
\end{equation*}
$$

Two main features of the revival operator are given in the next lemma. The first property gives the expression of the Fourier coefficients of $\mathcal{R}(p, q) f$. The second assertion characterises the mathematical invariance of the revival operator. Similar to the translation operator, the revival operator preserves the norm in $L^{2}(0,2 \pi)$.

Lemma 3.6. Let $p$ and $q$ be co-prime integers and $n \geq 2$ in $\mathbb{N}$. Then, we have the following.
(i) For any $f \in L^{2}(0,2 \pi)$, the Fourier coefficients of $\mathcal{R}_{n}(p, q) f$ are given by

$$
\begin{equation*}
\left\langle\mathcal{R}_{n}(p, q) f, e_{j}\right\rangle=e^{-2 \pi i j^{n} \frac{p}{q}} \widehat{f}(j) \tag{3.8}
\end{equation*}
$$

(ii) $\mathcal{R}_{n}(p, q)$ defines an isometry on $L^{2}(0,2 \pi)$.

Proof. From the proof of Theorem 2.8 we directly deduce (i). Moreover, using Parseval's identity (Lemma B.1-(iii)) it follows that $\mathcal{R}_{n}(p, q)$ is an isometry.

Remark 3.7. Revival operators are formed as finite linear combinations of specific translation operators. As such, they are a finite linear combination of isometries of $L^{2}(0,2 \pi)$. In turn, this implies that in the classical periodic setting the revival effect corresponds to the property of the solution being expressed at rational times, in terms of a finite number of isometries (or equivalently in terms of one non-trivial isometry, the revival operator). This observation stays in agreement with the case of pseudo-periodic boundary conditions for the linear Schrödinger equation (FSLS) in Chapter 4. There, we will show that the solution at rational times can be given via a combination of four isometries with each one containing a revival operator (see Corollary 4.9).

Remark 3.8. The expression of the Fourier coefficients of $\mathcal{R}_{n}(p, q) f$ enable us to describe the revival phenomena in more complex cases. In particular, due to the form of the coefficients we can identify the pure revival effect (Definition 2.10) when working with the eigenfunction expansion of the solution to a given IBVP. As we shall see in Chapter 5, using (3.8) an operational approach can be applied on the examination of the revival effect for the Airy PDE (AI) with quasi-periodic boundary conditions (see Proposition 5.2).

### 3.2 Reflections and Even and Odd Extensions

In this section we provide the standard definitions of the reflection with respect to $\pi$ of a function on $[0,2 \pi]$ and the even and odd extension to $[0,2 \pi]$ of a function in $[0, \pi]$. As in the case of the translation and revival operators, we are also interested in their Fourier series representation. Similar to what we mentioned in Remark 3.8, by knowing the form of the Fourier coefficients of these transformations, we can
analyse the revival effect based on an operational manner. More precisely, we will be able to represent the solutions of the time evolution problems of Chapters 4 and 6 through reflections and even and odd extensions of the initial condition (see Theorems 4.8 and 6.7). With this, the study of the revival effect under pseudoperiodic (Chapter 4) or Robin-type boundary conditions (Chapter 6) for the FSLS equation will be based on the classical periodic theory.

We begin by recalling the definition of the reflection of a function $f \in L^{2}(0,2 \pi)$ and set our notation as follows.

Definition 3.9. The reflection with respect to $\pi$ of a function $f \in L^{2}(0,2 \pi)$ is denoted by the symbol $f^{\natural}$ and is defined by

$$
\begin{equation*}
f^{\natural}(x)=f(2 \pi-x) . \tag{3.9}
\end{equation*}
$$

The function $f^{\natural}$ is also in $L^{2}(0,2 \pi)$ and its norm is equal to $f$ since, after the change of variables $y=2 \pi-x$, we have that

$$
\|f(2 \pi-\cdot)\|^{2}=\int_{0}^{2 \pi}|f(2 \pi-x)|^{2} d x=\int_{0}^{2 \pi}|f(y)|^{2} d y=\|f\|^{2} .
$$

The main property of interest is the expression of the Fourier coefficients of $f^{\natural}$. These are given in terms of the Fourier coefficients of $f$ by the following lemma.

Lemma 3.10. Let $f \in L^{2}(0,2 \pi)$ and consider its reflection $f^{\natural}$. Then, for all integers j we have

$$
\begin{equation*}
\widehat{f^{\natural}}(j)=\widehat{f}(-j) . \tag{3.10}
\end{equation*}
$$

Proof. The proof follows by the calculation

$$
\widehat{f^{\natural}}(j)=\frac{1}{\sqrt{2 \pi}} \int_{0}^{2 \pi} f(2 \pi-x) e^{-i j x} d x=\frac{1}{\sqrt{2 \pi}} \int_{0}^{2 \pi} f(y) e^{i j x} d x=\widehat{f}(-j), \quad \forall j \in \mathbb{Z},
$$

where the second equality comes from the change of variables $y=2 \pi-x$.
We now change our point of view and consider a function on $[0, \pi]$ and define its even and odd extension to $[0,2 \pi]$. These extensions provide a convenient way to consider the free linear Schrödinger on $[0, \pi]$ with either zero Neumann or zero Dirichlet boundary conditions as a time evolution problem on $[0,2 \pi]$ with periodic
boundary conditions. We will elaborate on this more in Section 6.1. Furthermore, as stated above, even and odd extensions will also appear in the context of more complicated separated boundary conditions (see Theorem 6.7).

Definition 3.11. Let $f$ be a function defined on $[0, \pi]$. The even ( + ) and odd (-) extension of $f$ to the segment $[0,2 \pi]$ are defined by

$$
f^{ \pm}(x)= \begin{cases}f(x), & 0 \leq x \leq \pi  \tag{3.11}\\ \pm f(2 \pi-x), & \pi<x \leq 2 \pi\end{cases}
$$

Definition 3.11 implies that $f^{ \pm}$is an even/odd function with respect to the middle point $\pi$ of $[0,2 \pi]$. Each extension naturally introduces the cosine or sine Fourier series. Indeed, from [20] recall that the cosine and sine orthonormal Fourier bases in $L^{2}(0, \pi)$ are given accordingly by

$$
\begin{equation*}
n_{0}(x)=\frac{1}{\sqrt{\pi}}, \quad n_{j}(x)=\sqrt{\frac{2}{\pi}} \cos (j x), \quad j \in \mathbb{N}, \tag{3.12}
\end{equation*}
$$

and

$$
\begin{equation*}
d_{j}(x)=\sqrt{\frac{2}{\pi}} \sin (j x), \quad j \in \mathbb{N} . \tag{3.13}
\end{equation*}
$$

Therefore, any element $f \in L^{2}(0, \pi)$ admits, in the $L^{2}(0, \pi)$ sense, a Fourier cosine expansion

$$
\begin{equation*}
f(x)=\sum_{j=0}^{\infty} a_{j} n_{j}(x), \quad a_{j}=\int_{0}^{\pi} f(y) n_{j}(y) d y, \quad j \in\{0\} \cup \mathbb{N}, \tag{3.14}
\end{equation*}
$$

or a Fourier sine expansion

$$
\begin{equation*}
f(x)=\sum_{j=1}^{\infty} b_{j} d_{j}(x), \quad b_{j}=\int_{0}^{\pi} f(y) d_{j}(y) d y, \quad j \in \mathbb{N} . \tag{3.15}
\end{equation*}
$$

The connection of the Fourier coefficients of $f^{ \pm}$with the cosine/sine Fourier coefficients of $f$ is given explicitly in the lemma below. The proof is a direct consequence of Definition 3.11.

Lemma 3.12. Let $f \in L^{2}(0, \pi)$ and consider the even and odd extension $f^{ \pm}$in $L^{2}(0,2 \pi)$. Then, we have the following.
(i) $\widehat{f^{+}}(j)=a_{j}$ and $\widehat{f^{+}}(-j)=\widehat{f^{+}}(j)$ for all $j \in \mathbb{N}$. For $j=0, \widehat{f^{+}}(0)=\sqrt{2} a_{0}$.
(ii) $\widehat{f^{-}}(j)=-i b_{j}$ and $\widehat{f^{-}}(-j)=-\widehat{f^{-}}(j)$ for all $j \in \mathbb{N}$.

Consequently, by Lemma 3.12, we recover the expansions (3.14) and (3.15) by the Fourier expansion of $f^{ \pm}$. Specifically, we have that

$$
\begin{align*}
& f^{+}(x)=\sum_{j \in \mathbb{Z}} \widehat{f^{+}}(j) e_{j}(x)=\sum_{j=0}^{\infty} a_{j} n_{j}(x),  \tag{3.16}\\
& f^{-}(x)=\sum_{j \in \mathbb{Z}} \widehat{f^{-}}(j) e_{j}(x)=\sum_{j=1}^{\infty} b_{j} d_{j}(x) .
\end{align*}
$$

Remark 3.13. From (3.16) we observe that a cosine Fourier series in $L^{2}(0,2 \pi)$ represents an even function or equivalently the even extension in $[0,2 \pi]$ of a function in $L^{2}(0, \pi)$. Similarly, a sine Fourier series in $L^{2}(0,2 \pi)$ corresponds to an odd $L^{2}(0,2 \pi)$ function or equivalently to the odd extension in $[0,2 \pi]$ of a function in $L^{2}(0, \pi)$.

### 3.3 Periodic Convolution

In this final section, we review two properties of the $2 \pi$-periodic convolution of two functions in $L^{2}(0,2 \pi)$. We present a regularity property of this operation and also recall the form of its Fourier coefficients.

Definition 3.14. Let $f, g$ be in $L^{2}(0,2 \pi)$. The (normalised) $2 \pi$-periodic convolution of $f$ and $g$ is defined by

$$
\begin{equation*}
f * g(x)=\frac{1}{\sqrt{2 \pi}} \int_{0}^{2 \pi} f^{*}(x-y) g^{*}(y) d y, \quad x \in[0,2 \pi] . \tag{3.17}
\end{equation*}
$$

Since the product of two measurable functions is measurable, the integral in (3.17) makes sense for any $x \in[0,2 \pi]$. As it turns out, the convolution is even more regular than both $f$ and $g$. Following the argument in [49, Proposition 3.2] by Stein we prove the following statement.

Lemma 3.15. Let $f$ and $g$ be in $L^{2}(0,2 \pi)$. Then, we have the following.
(i) $f * g$ is a continuous function on $[0,2 \pi]$ and $f * g(0)=f * g(2 \pi)$.
(ii) $f * g$ belongs in $L^{2}(0,2 \pi)$ and its Fourier coefficients are given by

$$
\begin{equation*}
\widehat{f * g}(j)=\widehat{f}(j) \widehat{g}(j) . \tag{3.18}
\end{equation*}
$$

Proof. (i) First we show the implication of the statement for functions $f$ and $g$ in

$$
\{f \in C[0,2 \pi] ; f(0)=f(2 \pi)\} .
$$

The assumption on $f$ implies that the $2 \pi$-periodic extension $f^{*}$ is a $2 \pi$-periodic continuous function on $\mathbb{R}$. Hence, it is uniformly continuous on every closed interval and due to the periodicity, $f^{*}$ is uniformly continuous on $\mathbb{R}$.

Let $\epsilon>0$. Then, there exists $\delta>0$ such that for every $x_{1}, x_{2} \in[0,2 \pi]$ and $y \in \mathbb{R}$, with

$$
\left|x_{1}-x_{2}\right|=\left|\left(x_{1}-y\right)-\left(x_{1}-y\right)\right|<\delta,
$$

we have that

$$
\left|f^{*}\left(x_{1}-y\right)-f^{*}\left(x_{2}-y\right)\right|<\epsilon .
$$

From this and Definition 3.14, we obtain

$$
\left|f * g\left(x_{1}\right)-f * g\left(x_{2}\right)\right| \leq \frac{1}{\sqrt{2 \pi}} \int_{0}^{2 \pi}\left|f^{*}\left(x_{1}-y\right)-f^{*}\left(x_{2}-y\right)\left\|g^{*}(y) \left\lvert\, d y \leq \frac{\epsilon}{\sqrt{2 \pi}}\right.\right\| g \|_{L^{1}(0,2 \pi)} .\right.
$$

Therefore, $f * g$ is also continuous.
For general $f$ and $g$, we use the density of the function space

$$
\{f \in C[0,2 \pi] ; f(0)=f(2 \pi)\}
$$

on $L^{2}(0,2 \pi)$. Let $\left\{f_{n}\right\}_{n=1}^{\infty}$ and $\left\{g_{n}\right\}_{n=1}^{\infty}$ be two sequences in the function space above such that

$$
\left\|f-f_{n}\right\| \text { and }\left\|g-g_{n}\right\| \rightarrow 0
$$

as $n \rightarrow \infty$. Fixing $x \in[0,2 \pi]$,

$$
\begin{aligned}
\left|f * g(x)-f_{n} * g_{n}(x)\right| & \leq\left|\left(f-f_{n}\right) * g(x)\right|+\left|f_{n} *\left(g-g_{n}\right)(x)\right| \\
& \leq \frac{1}{\sqrt{2 \pi}} \int_{0}^{2 \pi}\left|f^{*}(x-y)-f_{n}^{*}(x-y)\right|\left|g^{*}(y)\right| d y \\
& +\frac{1}{\sqrt{2 \pi}} \int_{0}^{2 \pi}\left|f_{n}^{*}(x-y)\right|\left|g^{*}(y)-g_{n}^{*}(y)\right| d y
\end{aligned}
$$

The sequence $\left\{f_{n}\right\}_{n=1}^{\infty}$ is uniformly bounded in $n$ by, say, a positive constant $C$. From this and the Cauchy-Schwarz inequality in $L^{2}(0,2 \pi)$ we obtain

$$
\left|f * g(x)-f_{n} * g_{n}(x)\right| \leq \frac{\|g\|}{\sqrt{2 \pi}}\left\|f-f_{n}\right\|+C\left\|g-g_{n}\right\| .
$$

Thus, for any $x \in[0,2 \pi]$,

$$
\lim _{n \rightarrow \infty} f_{n} * g_{n}(x)=f * g(x)
$$

and we conclude that $f * g$ is continuous on $[0,2 \pi]$ as the uniform limit of continuous functions on $[0,2 \pi]$.

Now, by definition, the periodicity condition is satisfied

$$
f * g(0)=\frac{1}{\sqrt{2 \pi}} \int_{0}^{2 \pi} f^{*}(-y) g^{*}(y) d y=\frac{1}{\sqrt{2 \pi}} \int_{0}^{2 \pi} f(2 \pi-y) g^{*}(y) d y=f * g(2 \pi) .
$$

(ii) Because we have concluded that $f * g$ is continuous on $[0,2 \pi]$, it follows that it also belongs in $L^{2}(0,2 \pi)$. Hence, it admits a Fourier series representation. The proof of (3.18) is based on Fubini's Theorem. We refer to [49] for an approximation argument with continuous $2 \pi$-periodic functions as in (i), or to [50] directly for $L^{1}(0,2 \pi)$ functions (any function in $L^{2}(0,2 \pi)$ belongs in $L^{1}(0,2 \pi)$ ).

We conclude this section with a remark which gives a first indication of the consequences of the continuity of the periodic convolution on the revival phenomenon. We will return to this idea and expand it thoroughly in Chapters 6,7 and 8.

Remark 3.16. As we shall see in Chapter 6, the generalised Fourier series representation of the solution to the FSLS equation with a specific form of separated
boundary conditions on $[0, \pi]$ can be expressed through the solutions of five particular $2 \pi$-periodic problems for the same equation. Four of these problems have initial conditions which are given in terms of $2 \pi$-periodic convolutions, involving the even and odd extensions of the initial condition. The fifth one starts with the even extension of the initial function (see Theorem 6.7). As a consequence of the continuity of the periodic convolution, we will then deduce that the four $2 \pi$-periodic solutions are continuous functions on $[0,2 \pi]$ (and thus on $[0, \pi]$ ) at any rational time. The other periodic solution at rational times will exhibit the phenomenon of pure revivals in accordance with Definition 2.10. Hence, the solution at rational times will be given as a weak revival representation, a pure revival plus a continuous function.

Similar results based on the $2 \pi$-periodic convolution will be derived for the evenorder poly-harmonic wave equation under periodic (Section 7.3) or quasi-periodic (Section 8.4) boundary conditions. Therefore, we will show that the weak revival is possibly the main revival phenomenon in second-order in time, linear dispersive PDEs.

## Chapter 4

## Revivals in the Free Linear Schrödinger Equation with Pseudo-Periodic Boundary Conditions

In this chapter we examine the revival phenomenon for the linear Schrödinger equation with zero potential under pseudo-periodic boundary conditions on $[0,2 \pi]$. For this problem the revival property of the solution at rational times was obtain for the first time in the work of Olver, Sheils and Smith in [15]. Here, we provide a different proof and develop further the results in [15].

In Section 4.1 we outline the basic idea of our approach which comprises two steps. The method is developed in Sections 4.2 and 4.3. In the former, we examine the properties of the underlying spatial differential operator and derive a generalised Fourier series representation of the solution. In the latter, we show that the pseudo-periodic problem exhibits revivals at rational times. The proof follows by manipulating the generalised Fourier series in order to obtain a new representation of the solution. This holds at any positive time. We show that the solution is a combination of the solutions of four purely periodic problems for the FSLS equation. Thus, the revival property follows from that in the periodic case. On the other hand, at irrational times, we can further conclude that the fractalisation effect occurs in the pseudo-periodic problem.

The main results of this chapter are Theorem 4.8 and Corollary 4.9 and were published in [14].

### 4.1 The Time Evolution Problem and a Remark on the Methodology

Let $\beta_{0}$ and $\beta_{1}$ be complex numbers. Consider the initial boundary value problem for the free linear Schrödinger equation on $[0,2 \pi]$

$$
\begin{align*}
& \partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t), \quad u(x, 0)=u_{0}(x)  \tag{4.1}\\
& \beta_{0} u(0, t)=u(2 \pi, t), \quad \beta_{1} \partial_{x} u(0, t)=\partial_{x} u(2 \pi, t)
\end{align*}
$$

The boundary conditions in (4.1) are called pseudo-periodic. When $\beta_{0}=\beta_{1}=1$, these reduce to periodic conditions for which the revival phenomenon holds, as we have seen in Chapter 2. Our goal in this chapter is to show that the revival property at rational times extends to the pseudo-periodic problem (4.1). The method compromises two steps which we describe in the next two sections.

Conditions on $\beta_{1}, \beta_{2}$ that enable a representation of the solution as a generalised Fourier series are given in the next section. From this representation we decompose the solution at each time as the sum of four terms. Each term includes the solution of a periodic problem for the FSLS equation with an initial condition a suitable transformation of the original initial function $u_{0}$. Hence, the revival effect for the pseudo-periodic problem at rational times is recovered from this new representation based on the existence of pure revivals for periodic problems.

This technique provides a new approach to discover revivals beyond periodic boundary conditions. The effectiveness of the method will be further discussed in Chapters 5 and 6 . In the former, we will consider the Airy PDE (AI) on $[0,2 \pi]$ with a particular type of pseudo-periodic boundary conditions called quasi-periodic. In the latter, we discuss the FSLS equation with separated Robin-type boundary conditions on $[0, \pi]$.

Remark 4.1. In general, for a given boundary value problem the strategy can be summarised as follows.

1. We obtain the generalised Fourier series representation of the solution. This is due to the basis property of the eigenfunctions of the underlying spatial differential operator.
2. We identify the canonical periodic components by decomposing the solution representation.

### 4.2 Generalised Fourier Series Representation

Our first task is to derive a representation of the solution. Let us write the initial boundary value problem as

$$
\begin{equation*}
\partial_{t} u(x, t)=-i L u(x, t), \quad u(x, 0)=u_{0}(x), \tag{4.2}
\end{equation*}
$$

where the linear differential operator $L: \mathrm{D}(L) \rightarrow L^{2}(0,2 \pi)$ is given by $L f=-f^{\prime \prime}$ on the domain

$$
\begin{equation*}
\mathrm{D}(L)=\left\{f \in C^{2}[0,2 \pi] ; \beta_{0} f(0)=f(2 \pi), \beta_{1} f^{\prime}(0)=f^{\prime}(2 \pi)\right\} . \tag{4.3}
\end{equation*}
$$

A crucial difference between the pseudo-periodic and periodic boundary conditions comes from the underlying spatial differential operator. The operator is not always symmetric and thus non-self-adjoint in general. This is the context of the following lemma which provides a necessary and sufficient condition on the parameters $\beta_{0}$ and $\beta_{1}$ for the operator to be symmetric or have a self-adjoint extension.

Lemma 4.2. Consider the above linear differential operator $L: D(L) \rightarrow L^{2}(0,2 \pi)$. Then, the following are equivalent.
(i) $\beta_{0} \overline{\beta_{1}}=1$.
(ii) $L$ is symmetric.
(iii) L has a self-adjoint extension.
(iv) $L$ is essentially self-adjoint.

Proof. We first show that $(i) \Rightarrow(i i)$. Let $f$ and $g$ be functions in the domain of $L$. Then, integration parts twice yields

$$
\langle L f, g\rangle=\left(\overline{\beta_{0}} \beta_{1}-1\right) f^{\prime}(0) \overline{g(0)}+\left(\beta_{0} \overline{\beta_{1}}-1\right) f(0) \overline{g^{\prime}(0)}+\langle f, L g\rangle .
$$

Thus, If $\beta_{0} \overline{\beta_{1}}=1$, then $\langle L f, g\rangle=\langle f, L g\rangle$.
To show $(i i) \Rightarrow(i)$, let $f, g \in \mathrm{D}(L)$. Then, since $L$ is symmetric, we have

$$
\left(\overline{\beta_{0}} \beta_{1}-1\right) f^{\prime}(0) \overline{g(0)}+\left(\beta_{0} \overline{\beta_{1}}-1\right) f(0) \overline{g^{\prime}(0)}=0 .
$$

The above holds for any functions $f$ and $g$ that satisfy the boundary conditions and for which $f(0), f^{\prime}(0), g(0)$ and $g^{\prime}(0)$ can be arbitrary values. Set $f$ and $g$ such that $f(0)=1, g^{\prime}(0)=1, f^{\prime}(0)=0$, then $\beta_{0} \overline{\beta_{1}}=1$.

Now, each one of (iii) and (iv) imply (ii) by definition. We finish by showing that (ii) implies (iii) and (iv). So, let $L$ be symmetric, then ( $i$ ) holds. Moreover, from Lemmas 4.5 and 4.6 , which we prove below, we know that when $\beta_{0} \overline{\beta_{1}}=1$, then the eigenfunctions of $L$ form a family of orthonormal basis in $L^{2}(0,2 \pi)$. Hence, $L$ is essentially self-adjoint by Lemma C.6, and thus it has a self-adjoint extension.

The analysis of the pseudo-periodic problem (4.1) will be considered under consistency conditions on $\beta_{0}$ and $\beta_{1}$ as stated below. One of the assumptions depends on the self-adjointness condition $\beta_{0} \overline{\beta_{1}}=1$. However, this does not affect the treatment of the problem and our methods work regardless of the self-adjointness or not of the boundary conditions. Consequently, this further implies that the revival effect for the pseudo-periodic problem does not depend on whether $\beta_{0} \overline{\beta_{1}}=1$.

Assumption 4.3. We pose the following restrictions on the complex parameters $\beta_{0}$ and $\beta_{1}$.
(i)

$$
\beta_{0} \neq 1 \quad \text { and } \quad \beta_{1} \neq 1,
$$

(ii)

$$
\begin{aligned}
& \beta_{0}+\beta_{1} \neq 0 \quad \text { with } \quad \frac{1+\beta_{0} \beta_{1}}{\beta_{0}+\beta_{1}} \in \mathbb{C} \backslash(-\infty,-1) \cup(1,+\infty) \\
& \text { or } \quad \beta_{0}+\beta_{1} \neq 0 \quad \text { for } \beta_{0} \overline{\beta_{1}} \neq 1, \\
& \text { with } \quad \frac{1+\beta_{0} \beta_{1}}{\beta_{0}+\beta_{1}}=\frac{2 \operatorname{Re}\left(\beta_{1}\right)}{1+\left|\beta_{1}\right|^{2}} \in(-1,1) \quad \text { for } \beta_{0} \overline{\beta_{1}}=1,
\end{aligned}
$$

(iii)

$$
\arccos \left(\frac{1+\beta_{0} \beta_{1}}{\beta_{0}+\beta_{1}}\right) \in \mathbb{R}
$$

From the Assumption 4.3, the first condition excludes the periodic case and it will imply that zero is not an eigenvalue of the differential operator $L$. The other two conditions will allow us to obtain a solution in $L^{2}(0,2 \pi)$ to the pseudo-periodic problem (4.1). In particular, the second condition will ensure that the eigenfunctions of the operator $L$ form a Riesz basis in $L^{2}(0,2 \pi)$, whereas the third condition will guarantee that all the eigenvalues are real, and thus avoiding that the equation is ill-posed. Here and in the rest of this chapter, we will assume that the parameters $\beta_{0}$ and $\beta_{1}$ satisfy these three conditions.

To find a solution to the initial boundary value problem (4.1), we follow the Fourier method and express the solution as an eigenfunction expansion. Hence, we consider the eigenvalue problem associated with the linear operator $L$

$$
\begin{equation*}
-\phi^{\prime \prime}(x)=\lambda \phi(x), \quad \beta_{0} \phi(0)=\phi(2 \pi), \quad \beta_{1} \phi^{\prime}(0)=\phi^{\prime}(2 \pi) . \tag{4.4}
\end{equation*}
$$

Set

$$
\begin{equation*}
k_{0}=\frac{1}{2 \pi} \arccos \left(\frac{1+\beta_{0} \beta_{1}}{\beta_{0}+\beta_{1}}\right), \quad \gamma=e^{i 2 \pi k_{0}}=\frac{1+\beta_{0} \beta_{1}}{\beta_{0}+\beta_{1}}+i \sqrt{1-\left(\frac{1+\beta_{0} \beta_{1}}{\beta_{0}+\beta_{1}}\right)^{2}}, \tag{4.5}
\end{equation*}
$$

where the square root in (4.5) is well defined due to Assumption 4.3-(ii).

Lemma 4.4. The eigenvalues of the operator $L$ are all real and are given by

$$
\begin{equation*}
\lambda_{j}=k_{j}^{2}, \quad k_{j}=\left(j+k_{0}\right), \quad j \in \mathbb{Z}, \quad k_{0}=\frac{1}{2 \pi} \arccos \left(\frac{1+\beta_{0} \beta_{1}}{\beta_{0}+\beta_{1}}\right) \neq 0 \tag{4.6}
\end{equation*}
$$

The corresponding eigenfunctions can be written as follows

$$
\begin{equation*}
\phi_{j}(x)=\frac{A}{\sqrt{2 \pi}}\left(e^{i k_{j} x}+\Lambda_{0} e^{-i k_{j} x}\right), \quad j \in \mathbb{Z}, \quad A \in \mathbb{C} \backslash\{0\} \tag{4.7}
\end{equation*}
$$

where $\Lambda_{0}$ is a complex constant given by

$$
\begin{equation*}
\Lambda_{0}=\frac{\gamma-\beta_{0}}{\beta_{0}-\gamma^{-1}}=\frac{\gamma-\beta_{1}}{\gamma^{-1}-\beta_{1}} . \tag{4.8}
\end{equation*}
$$

Proof. Consider the eigenvalue problem (4.4). If $\lambda=0$, then $\phi(x)=A x+B$, with $A$ and $B$ in $\mathbb{C}$. From Assumption 4.3-(i) on $\beta_{0}, \beta_{1}$ and the boundary conditions it follows that $A=B=0$, and so $\phi(x)=0$ for all $x \in[0,2 \pi]$. Hence, $\lambda=0$ is not an eigenvalue.

Let $\lambda \neq 0$. Then, the general solution to (4.4) is

$$
\phi(x)=A e^{i k x}+B e^{-i k x}
$$

where $k$ is taken as the principal branch of the square root of $\lambda \in \mathbb{C}$. From the boundary conditions we obtain a linear system for the two unknown complex constants $A$ and $B$

$$
\begin{aligned}
& A\left(\beta_{0}-e^{i 2 \pi k}\right)+B\left(\beta_{0}-e^{-i 2 \pi k}\right)=0 \\
& A\left(\beta_{1}-e^{i 2 \pi k}\right)+B\left(e^{-i 2 \pi k}-\beta_{1}\right)=0
\end{aligned}
$$

Eigenfunctions do not vanish identically, hence $|A|^{2}+|B|^{2} \neq 0$. Thus,

$$
\left(\beta_{0}+\beta_{1}\right)\left(e^{i 2 \pi k}+e^{-i 2 \pi k}\right)=2\left(1+\beta_{0} \beta_{1}\right) .
$$

So,

$$
\cos (2 \pi k)=\frac{1+\beta_{0} \beta_{1}}{\beta_{0}+\beta_{1}}
$$

and

$$
k_{j}=j+k_{0},
$$

with $k_{0}$ given by (4.5). Hence, the eigenvalues are $\lambda_{j}=k_{j}^{2}$.
Now, the eigenfucntions take the form

$$
\phi_{j}(x)=A e^{i k_{j} x}+B e^{-i k_{j} x} .
$$

Since

$$
e^{-i k_{j} 2 \pi}=e^{-i 2 \pi k_{0}}=\gamma,
$$

the boundary conditions imply that

$$
B=A \frac{\gamma-\beta_{0}}{\beta_{0}-\gamma^{-1}}=A \frac{\gamma-\beta_{1}}{\gamma^{-1}-\beta_{1}} .
$$

Hence,

$$
\phi_{j}(x)=A\left(e^{i k_{j} x}+\Lambda_{0} e^{-i k_{j} x}\right),
$$

with $\Lambda_{0}$ as in (4.8). Rescaling $A^{\prime}=A / \sqrt{2 \pi}$ yields (4.7).
Having found the expression of the eigenfunctions, we now show that they form a basis of $L^{2}(0,2 \pi)$. This will allow us to represent the solution of the pseudoperiodic problem by a generalised Fourier series. In [51], an approach relying on the Cauchy method of residues and the Green's function of the non-self-adjoint problem (4.4) rendered that the family of eigenfunctions $\left\{\phi_{j}\right\}_{j \in \mathbb{Z}}$ is a Riesz basis. On the other hand, in [15], the basis property was obtained from the Unified Transform Method (UTM) applied directly to the evolution problem (4.2). This correspondence between time evolution problems and spectral problems in the context of UTM has been examined in more details in [52]. Nevertheless, noticing that the eigenfunctions (4.7) are a linear combination of two orthonormal bases of $L^{2}(0,2 \pi)$, we can deduce that they form a Riesz basis (see Lemma B.5).

Proposition 4.5. The family of eigenfunctions $\left\{\phi_{j}\right\}_{\in \mathbb{Z}}$ forms a Riesz basis of $L^{2}(0,2 \pi)$.
Proof. Write the eigenfunctions in the form

$$
\phi_{j}(x)=A h_{j}(x), \quad h_{j}(x)=\frac{e^{i k_{j} x}}{\sqrt{2 \pi}}+\Lambda_{0} \frac{e^{-i k_{j} x}}{\sqrt{2 \pi}}, \quad j \in \mathbb{Z}
$$

with $A \in \mathbb{C} \backslash\{0\}$. If $\left\{h_{j}\right\}_{j \in \mathbb{Z}}$ forms a Riesz basis, then the same is true for $\left\{\phi_{j}\right\}_{j \in \mathbb{Z}}$. So, it is enough to check the conditions of Lemma B. 5 for the family $\left\{h_{j}\right\}_{j \in \mathbb{Z}}$.

We know that $e_{j}(x)=e^{i j x} / \sqrt{2 \pi}$ for $j \in \mathbb{Z}$ is an orthonormal basis of $L^{2}(0,2 \pi)$. Moreover, the reflections

$$
e_{-j}(x)=e_{j}(2 \pi-x), \quad j \in \mathbb{Z}
$$

form an orthonormal basis.
Set

$$
\begin{aligned}
& m_{j}(x)=\frac{e^{i k_{j} x}}{\sqrt{2 \pi}}=e^{i k_{0} x} e_{j}(x) \\
& \ell_{j}(x)=\frac{e^{-i k_{j} x}}{\sqrt{2 \pi}}=e^{-i k_{0} x} e_{-j}(x)
\end{aligned}
$$

Then, both $\left\{m_{j}\right\}_{j \in \mathbb{Z}}$ and $\left\{\ell_{j}\right\}_{j \in \mathbb{Z}}$ are orthonormal families in $L^{2}(0,2 \pi)$. By virtue of Lemma B.1-(i), we see that they are actually orthonormal bases. Indeed, let $w(x)=e^{-i k_{0} x}$ for $x \in[0,2 \pi]$ and $f \in L^{2}(0,2 \pi)$. If $\left\langle f, m_{j}\right\rangle=0$, then we equivalently have that $\left\langle w f, e_{j}\right\rangle=0$ or that $w f=0$. Since, $w$ can not be zero, we conclude that $f=0$, which means that $\left\{m_{j}\right\}_{j \in \mathbb{Z}}$ is an orthonormal basis. A similar argument shows that $\left\{\ell_{j}\right\}_{j \in \mathbb{Z}}$ is also an orthonormal basis.

Finally, we show that $\left|\Lambda_{0}\right| \neq 1$. Suppose, the opposite holds true. Then, $\left|\Lambda_{0}\right|^{2}=1$ and using the definition (4.8), we have that

$$
\left(\frac{\gamma-\beta_{0}}{\beta_{0}-\gamma^{-1}}\right)\left(\frac{\gamma^{-1}-\overline{\beta_{0}}}{\overline{\beta_{0}}-\gamma}\right)=1 \Longleftrightarrow\left(\gamma-\beta_{0}\right)\left(\gamma^{-1}-\overline{\beta_{0}}\right)=\left(\beta_{0}-\gamma^{-1}\right)\left(\overline{\beta_{0}}-\gamma\right)
$$

where $\gamma=e^{i k_{0} 2 \pi}$. The above implies that

$$
\gamma=\gamma^{-1}=\bar{\gamma}
$$

Hence, we require $\gamma \in \mathbb{R}$, which gives $\sin \left(2 \pi k_{0}\right)=0$ or $2 \pi k_{0}=n \pi$, for $n \in \mathbb{Z}$. Recall that $k_{0}$ is given by (4.5). Therefore, $\left|\Lambda_{0}\right|=1$ if and only if

$$
\frac{1+\beta_{0} \beta_{1}}{\beta_{0}+\beta_{1}}= \pm 1
$$

But the last condition contradicts Assumption (4.3)-(ii). So, $\left|\Lambda_{0}\right| \neq 1$ and from Lemma B. 5 we conclude that $h_{j}(x)$ forms a Riesz basis.

In what follows we choose

$$
\begin{equation*}
A=\frac{1}{\sqrt{\tau}}, \quad \tau=\frac{\left(\gamma^{2}+1\right)\left(\beta_{0} \beta_{1}+1\right)-2 \gamma\left(\beta_{0}+\beta_{1}\right)}{\left(\beta_{0} \gamma-1\right)\left(\beta_{1} \gamma-1\right)}, \tag{4.9}
\end{equation*}
$$

so that the eigenfunctions (4.7) take the form

$$
\begin{equation*}
\phi_{j}(x)=\frac{1}{\sqrt{2 \pi \tau}}\left(e^{i k_{j} x}+\Lambda_{0} e^{-i k_{j} x}\right) . \tag{4.10}
\end{equation*}
$$

The choice of $A$ reflects the biorthogonality property of the basis $\left\{\phi_{j}\right\}_{j \in \mathbb{Z}}$ when paired with the eigenfunctions of the adjoint eigenvalue problem on $[0,2 \pi]$

$$
\begin{equation*}
-\psi^{\prime \prime}(x)=\lambda \psi(x), \quad \psi(0)=\overline{\beta_{1}} \psi(2 \pi), \psi^{\prime}(0)=\overline{\beta_{0}} \psi^{\prime}(2 \pi), \tag{4.11}
\end{equation*}
$$

see [51]. The adjoint eigenvalue problem is associated with the adjoint operator of $L$ which is an extension of the linear differential operator $H f=-f^{\prime \prime}$ with domain

$$
\mathrm{D}(H)=\left\{f \in C^{2}[0,2 \pi] ; f(0)=\overline{\beta_{1}} f(2 \pi), f^{\prime}(0)=\overline{\beta_{0}} f^{\prime}(2 \pi)\right\} .
$$

As with the eigenvalue problem (4.11), we find the family of the adjoint eigenfunctions

$$
\begin{equation*}
\psi_{j}(x)=\frac{1}{\sqrt{2 \pi \tau}}\left(e^{i k_{j} x}+I_{0} e^{-i k_{j} x}\right) \tag{4.12}
\end{equation*}
$$

where $\tau$ is as in (4.9) and

$$
\begin{equation*}
I_{0}=\frac{\gamma-1 / \overline{\beta_{1}}}{1 / \overline{\beta_{1}}-\gamma^{-1}} \tag{4.13}
\end{equation*}
$$

The eigenvalues of $H$ are the same as those of $L$.
We now show the biorthogonality property of the two systems of eigenfunctions. Moreover, under the self-adjointness condition $\beta_{0} \overline{\beta_{1}}=1$, we see that they coincide and become orthonormal.

Lemma 4.6. Consider the eigenfunctions (4.10) and their adjoints (4.12). Then, they form a biorthogonal system

$$
\left\langle\phi_{j}, \psi_{\ell}\right\rangle= \begin{cases}1, & j=k \\ 0, & j \neq \ell\end{cases}
$$

If $\beta_{0} \overline{\beta_{1}}=1$, then $\phi_{j}=\psi_{j}$ for all $j \in \mathbb{Z}$ and $\left\{\phi_{j}\right\}_{j \in \mathbb{Z}}$ becomes an orthonormal family.
Proof. We first prove the biorthogonality condition for general $\beta_{0}$ and $\beta_{1}$. Let $j=$ $\ell \in \mathbb{Z}$, then using the expressions of $\phi_{j}(x)$ and $\psi_{j}(x)$ we have that

$$
\left\langle\phi_{j}, \psi_{j}\right\rangle=\frac{1}{2 \pi \tau} \int_{0}^{2 \pi}\left(1+\Lambda_{0} \overline{I_{0}}\right) d x+\frac{1}{2 \pi \tau} \int_{0}^{2 \pi}\left(\overline{I_{0}} e^{i k_{j} x}+\Lambda_{0} e^{-i k_{j} x}\right) d x .
$$

The second integral on the right-hand side is zero. Indeed, recalling that $\gamma=e^{i 2 \pi k_{0}}$, we have

$$
\overline{I_{0}} \int_{0}^{2 \pi} e^{i k_{j} x} d x+\Lambda_{0} \int_{0}^{2 \pi} e^{-i k_{j} x} d x=\overline{I_{0}} \frac{\gamma^{2}-1}{2 i k_{j}}-\Lambda_{0} \frac{\gamma^{-2}-1}{2 i k_{j}}=\frac{\gamma^{2}-1}{2 i k_{j}}\left(\overline{I_{0}}+\frac{\Lambda_{0}}{\gamma^{2}}\right) .
$$

Then, using the definitions of $I_{0}$ and $\Lambda_{0}$ by (4.8) and (4.13), we find that

$$
\overline{I_{0}}+\frac{\Lambda_{0}}{\gamma}=\frac{\bar{\gamma}-1 / \beta_{1}}{1 / \beta_{1}-\gamma}+\frac{\gamma-\beta_{1}}{\left(\gamma^{-1}-\beta_{1}\right) \gamma^{2}}=0 .
$$

On the other hand,

$$
1+\Lambda \overline{I_{0}}=1+\left(\frac{\gamma-\beta_{0}}{\beta_{0}-\gamma^{-1}}\right)\left(\frac{\bar{\gamma}-1 / \beta_{1}}{1 / \beta_{1}-\gamma}\right)=\tau
$$

thus $\left\langle\phi_{j}, \psi_{j}\right\rangle=1$.
For integers $j$ and $\ell$, such that $j \neq \ell$, we compute

$$
\begin{aligned}
\left\langle\phi_{j}, \psi_{\ell}\right\rangle & =\frac{1}{2 \pi \tau}\left(\int_{0}^{2 \pi} e^{i k_{j} x} e^{-i k_{\ell} x} d x+\Lambda_{0} \overline{I_{0}} \int_{0}^{2 \pi} e^{-i k_{j} x} e^{i k_{\ell} x} d x\right) \\
& +\frac{1}{2 \pi \tau}\left(\overline{I_{0}} \int_{0}^{2 \pi} e^{i k_{j} x} e^{i k_{\ell} x} d x+\Lambda_{0} \int_{0}^{2 \pi} e^{-i k_{j} x} e^{-i k_{\ell} x} d x\right)
\end{aligned}
$$

Since

$$
\int_{0}^{2 \pi} e^{i k_{j} x} e^{-i k_{\ell} x} d x=0 \text { and } \int_{0}^{2 \pi} e^{-i k_{j} x} e^{i k_{\ell} x} d x=0
$$

we find that

$$
\begin{aligned}
\left\langle\phi_{j}, \psi_{\ell}\right\rangle & =\frac{1}{2 \pi \tau}\left(\overline{I_{0}} \int_{0}^{2 \pi} e^{i k_{j} x} e^{i k_{\ell} x} d x+\Lambda_{0} \int_{0}^{2 \pi} e^{-i k_{j} x} e^{-i k_{\ell} x} d x\right) \\
& =\frac{\gamma^{2}-1}{2 \pi \tau\left(2 k_{0}+j+\ell\right)}\left(\overline{I_{0}}+\frac{\Lambda_{0}}{\gamma^{2}}\right) \\
& =0 .
\end{aligned}
$$

Finally, if $\beta_{0} \overline{\beta_{1}}=1$ then by definition $\overline{I_{0}}=\Lambda_{0}$, which implies that $\phi_{j}=\psi_{j}$, for $\tau=1+\left|\Lambda_{0}\right|^{2}$. Then, the orthonormality condition is automatically fulfilled.

From the above analysis of the eigenvalue problem and its adjoint, it follows that every function $f$ in $L^{2}(0,2 \pi)$ admits a unique expansion in $L^{2}(0,2 \pi)$ in terms of $\left\{\phi_{j}\right\}_{j \in \mathbb{Z}}$,

$$
f(x)=\sum_{j \in \mathbb{Z}}\left\langle f, \psi_{j}\right\rangle \phi_{j}(x) .
$$

Therefore, the Fourier method yields a unique solution in $L^{2}(0,2 \pi)$ to the pseudo-
periodic problem (4.2) as an eigenfunction expansion

$$
u(x, t)=\sum_{j \in \mathbb{Z}}\left\langle u(\cdot, t), \psi_{j}\right\rangle \phi_{j}(x) .
$$

Assuming at the moment that $u(x, t)$ is a smooth solution of the problem satisfying the pseudo-periodic boundary conditions, then for each generalised Fourier coefficient we have

$$
\frac{d}{d t}\left\langle u(\cdot, t), \psi_{j}\right\rangle=\left\langle\partial_{t} u(\cdot, t), \psi_{j}\right\rangle=\left\langle i \partial_{x}^{2} u(\cdot, t), \psi_{j}\right\rangle=-i \lambda_{j}\left\langle u(\cdot, t), \psi_{j}\right\rangle,
$$

since each $\psi_{j}$ satisfies the adjoint boundary conditions. Note that, on the lefthand side above, we can exchange differentiation with integration by the Dominated Convergence Theorem since we assume that $u(x, t)$ and $\partial_{t} u(x, t)$ are bounded and continuous functions of $t$ (see Theorem 2.27 in [32]). Now, solving for $\left\langle u(\cdot, t), \psi_{j}\right\rangle$, with initial condition $\left\langle u_{0}, \psi_{j}\right\rangle$ at time zero, we obtain

$$
\left\langle u(\cdot, t), \psi_{j}\right\rangle=\left\langle u_{0}, \psi_{j}\right\rangle e^{-i \lambda_{j} t} .
$$

We finish this section with a proposition which gathers the precise expression of the generalised Fourier series representation of the solution with initial condition in $L^{2}(0,2 \pi)$. This completes the first step of the analysis in Remark 4.1. The proof runs along similar lines as this of Theorem 2.4.

Proposition 4.7. Let $u_{0} \in L^{2}(0,2 \pi)$. Then, there exists a sequence of smooth solutions of the IBVP (4.2) denoted by $\left\{u^{n}(x, t)\right\}_{n \in \mathbb{N}}$ such that for every fixed $t \geq 0$, as $n \rightarrow \infty, u^{n}(x, t)$ converges in $L^{2}(0,2 \pi)$ to

$$
\begin{equation*}
u(x, t)=\sum_{j \in \mathbb{Z}}\left\langle u_{0}, \psi_{j}\right\rangle e^{-i \lambda_{j} t} \phi_{j}(x) . \tag{4.14}
\end{equation*}
$$

Moreover, the map $t \rightarrow u(\cdot, t)$ is continuous in $L^{2}(0,2 \pi)$ with respect to the time variable $t \in[0, \infty)$.

Proof. Let $n \in \mathbb{N}$ and set

$$
u^{n}(x, t)=\sum_{j=-n}^{n}\left\langle u_{0}, \psi_{j}\right\rangle e^{-i k_{j}^{2} t} \phi_{j}(x)
$$

The functions $u^{n}(x, t)$ are smooth in both variables, for $x \in[0,2 \pi]$ and $t \geq 0$. Moreover, they satisfy the free linear Schrödinger equation, the pseudo-periodic boundary conditions in (4.1) and at time zero the initial condition is the partial sum

$$
u^{n}(x, 0)=\sum_{j=-n}^{n}\left\langle u_{0}, \psi_{j}\right\rangle \phi_{j}(x)=u_{0}^{n}(x)
$$

of the initial function $u_{0}$. Hence, the functions $u^{n}(x, t)$ form a sequence of smooth solutions of the pseudo-periodic problem (4.2).

Now for any $t \geq 0,\left|\left\langle u_{0}, \psi_{j}\right\rangle e^{-i j^{2} t}\right|^{2}=\left|\left\langle u_{0}, \psi_{j}\right\rangle\right|^{2}$. Hence

$$
\sum_{j=-n}^{n}\left|\left\langle u_{0}, \psi_{j}\right\rangle e^{-i j^{2} t}\right|^{2}=\sum_{j=-n}^{n}\left|\left\langle u_{0}, \psi_{j}\right\rangle\right|^{2}
$$

Since $u_{0} \in L^{2}(0,2 \pi)$, then as $n \rightarrow \infty$ we see that the series on the right hand side above converges due to the Riesz basis property of $\left\{\phi_{j}\right\}_{j \in \mathbb{Z}}$ (see Lemma B.5-(iii)). Therefore, the sequence of partial sums $u^{n}(\cdot, t)$ converges uniformly in $t$ with respect to the $L^{2}(0,2 \pi)$ norm and defines a map

$$
u(\cdot, t)=\sum_{j \in \mathbb{Z}}\left\langle u_{0}, \psi_{j}\right\rangle e^{-i \lambda_{j} t} \phi_{j},
$$

which takes every $t \in[0, \infty)$ into $L^{2}(0,2 \pi)$, again by Lemma B.5-(iii).
We show continuity in $t$ from the right with respect to the $L^{2}(0,2 \pi)$ norm. A similar argument establishes continuity from left for $t \geq 0$. Fix $t \geq 0$ and let $h>0$. Then, from Lemma B.5-(ii), there exists positive constant $c$ such that

$$
\|u(\cdot, t+h)-u(\cdot, t)\|^{2} \leq \frac{1}{c} \sum_{j \in \mathbb{Z}}\left|e^{-i j^{2} h}-1\right|^{2}\left|\left\langle u_{0}, \psi_{j}\right\rangle\right|^{2}
$$

However, because

$$
\left|e^{-i j^{2} h}-1\right|^{2}\left|\left\langle u_{0}, \psi_{j}\right\rangle\right|^{2} \leq 4\left|\left\langle u_{0}, \psi_{j}\right\rangle\right|^{2}
$$

and $u_{0} \in L^{2}(0,2 \pi)$, the series on the right-hand side above converges absolutely and uniformly with respect to $h$ by the Weierstrass M-test. Moreover, since

$$
\left|e^{-i j^{2} h}-1\right|^{2}=2\left(1-\cos \left(j^{2} h\right)\right)
$$

is continuous as a function of $h$ and vanishes as $h \rightarrow 0$, we see that

$$
\lim _{h \rightarrow 0}\|u(\cdot, t+h)-u(\cdot, t)\|^{2}=0
$$

For each fixed $t \geq 0$, the function $u(x, t)$ given in $L^{2}(0,2 \pi)$ by the generalised Fourier series (4.14) is called the generalised solution of the IBVP (4.2) and from now on we will call it just the solution of the pseudo-periodic problem in accordance with Remark 2.5.

### 4.3 The Revival Effect

In this section we show that the revival effect is exhibited at rational times in the context of the pseudo-periodic problem (4.2) for the free space linear Schrödinger equation. The revival property will be obtained as a corollary of the next more general result which is one of the main contributions of this thesis. Examining the structure of the pseudo-periodic eigenpairs, we will find that the solution $u(x, t)$ at any time (thus at rational times) can be constructed from the solutions of four periodic problems for the FSLS equation with initial data some transformations of the initial condition $u_{0}$.

Recalling from Chapter 3 the definitions of the reflection $f^{\natural}$ of a function and the periodic translation operator $\mathcal{T}_{s}$, we have the following statement.

Theorem 4.8. Let $u_{0} \in L^{2}(0,2 \pi)$ and set

$$
\begin{equation*}
v_{0}(x)=u_{0}(x) e^{-i k_{0} x}, \quad w_{0}(x)=u_{0}(x) e^{i k_{0} x} . \tag{4.15}
\end{equation*}
$$

Consider the FSLS equation with periodic boundary conditions on $[0,2 \pi]$ and denote by

- $v(x, t)$ the solution corresponding to initial condition $v_{0}(x)$,
- $w(x, t)$ the solution corresponding to initial condition $w_{0}(x)$,
- $v^{\natural}(x, t)$ the solution corresponding to initial condition $v_{0}^{\natural}(x)$,
- $w^{\natural}(x, t)$ the solution corresponding to initial condition $w_{0}^{\natural}(x)$.

Then, for every fixed time $t \geq 0$, the solution to the pseudo-periodic problem (4.2) has the $L^{2}(0,2 \pi)$ representation

$$
\begin{align*}
& u(x, t)=\frac{e^{-i k_{0}^{2} t}}{\tau}\left\{e^{i k_{0} x} \mathcal{T}_{2 k_{0} t} v(x, t)+\Lambda_{0} e^{-i k_{0} x} \mathcal{T}_{-2 k_{0} t} v^{\natural}(x, t)\right.  \tag{4.16}\\
&\left.+\bar{I}_{0} e^{i k_{0} x} \mathcal{T}_{2 k_{0} t} w^{\natural}(x, t)+\Lambda_{0} \bar{I}_{0} e^{-i k_{0} x} \mathcal{T}_{-2 k_{0} t} w(x, t)\right\}
\end{align*}
$$

Proof. Let $u_{0} \in L^{2}(0,2 \pi)$. Then for fixed $t \geq 0$, we know from Proposition 4.7 that the solution is given by the eigenfunction expansion

$$
\begin{align*}
u(x, t) & =\sum_{j \in \mathbb{Z}}\left\langle u_{0}, \psi_{j}\right\rangle e^{-i k_{j}^{2} t} \phi_{j}(x) \\
& =\frac{1}{2 \pi \tau} \sum_{j \in \mathbb{Z}}\left(\int_{0}^{2 \pi} u_{0}(y) e^{-i k_{j} y} d y+\bar{I}_{0} \int_{0}^{2 \pi} u_{0}(y) e^{i k_{j} y} d y\right) e^{-i k_{j} t^{2}}\left(e^{i k_{j} x}+\Lambda_{0} e^{-i k_{j} x}\right) . \tag{4.17}
\end{align*}
$$

We consider each of the terms in the series above and recall that $k_{j}=k_{0}+j$. From the definition (4.15) of $v_{0}$ and $w_{0}$, we have

$$
\begin{equation*}
\int_{0}^{2 \pi} u_{0}(y) \frac{e^{-i k_{j} y}}{\sqrt{2 \pi}} d y+\bar{I}_{0} \int_{0}^{2 \pi} u_{0}(y) \frac{e^{i k_{j} y}}{\sqrt{2 \pi}} d y=\widehat{v_{0}}(j)+\overline{I_{0}} \widehat{w_{0}}(-j) . \tag{4.18}
\end{equation*}
$$

We further have the elementary relation,

$$
\begin{equation*}
e^{-i k_{j}^{2} t}=e^{-i k_{0}^{2} t} e^{-2 k_{0} j t} e^{-i j^{2} t}, \tag{4.19}
\end{equation*}
$$

and for the eigenfunctions

$$
\begin{equation*}
\frac{e^{i k_{j} x}}{\sqrt{2 \pi}}+\Lambda_{0} \frac{e^{-i k_{j} x}}{\sqrt{2 \pi}}=e^{i k_{0} x} e_{j}(x)+\Lambda_{0} e^{-i k_{0} x} e_{-j}(x) \tag{4.20}
\end{equation*}
$$

where $e_{j}(x)$ are the periodic eigenfunctions (the classical Fourier basis (2.6)).

By substituting (4.18), (4.19) and (4.20) into (4.17), we obtain

$$
\begin{align*}
u(x, t)=\frac{e^{-i k_{0}^{2} t}}{\tau} & \sum_{j \in \mathbb{Z}} e^{-i 2 k_{0} j t} e^{-i j^{2} t}\left(e^{i k_{0} x} \widehat{v_{0}}(j) e_{j}(x)+\Lambda_{0} e^{-i k_{0} x} \widehat{v_{0}}(j) e_{-j}(x)\right.  \tag{4.21}\\
& \left.+\bar{I}_{0} e^{i k_{0} x} \widehat{w_{0}}(-j) e_{j}(x)+\Lambda_{0} \bar{I}_{0} e^{-i k_{0} x} \widehat{w_{0}}(-j) e_{-j}(x)\right)
\end{align*}
$$

Each term in (4.21) involves the solution of a periodic problem. Indeed, from (3.3) it follows that for $f \in L^{2}(0,2 \pi)$ and $s \in \mathbb{R}$

$$
\begin{equation*}
\mathcal{T}_{s} f(x)=\sum_{j \in \mathbb{Z}} e^{-i j s} \widehat{f}(j) e_{j}(x), \tag{4.22}
\end{equation*}
$$

hence we have for the first term

$$
\begin{equation*}
\sum_{j \in \mathbb{Z}} e^{-i 2 k_{0} j t} e^{-i j^{2} t} e^{i k_{0} x} \widehat{v}_{0}(j) e_{j}(x)=e^{i k_{0} x} \mathcal{T}_{2 k_{0} t}\left(\sum_{j \in \mathbb{Z}} \widehat{v_{0}}(j) e^{-i j^{2} t} e_{j}(x)\right)=e^{i k_{0} x} \mathcal{T}_{2 k_{0} t} v(x, t), \tag{4.23}
\end{equation*}
$$

where $v(x, t)$ solves the FSLS equation with periodic boundary conditions on $[0,2 \pi]$ and with initial condition $v_{0}(x)$.

For the second sum in (4.21), recalling that the Fourier coefficients of the reflection has the form (3.10), we find that

$$
\begin{align*}
\sum_{j \in \mathbb{Z}} e^{-i 2 k_{0} j t} e^{-i j^{2} t} \Lambda_{0} e^{-i k_{0} x} \widehat{v_{0}}(j) e_{-j}(x) & =\Lambda_{0} e^{-i k_{0} x} \sum_{j \in \mathbb{Z}} e^{i 2 k_{0} j t} \widehat{v}_{0}(-j) e^{-i j^{2} t} e_{j}(x) \\
& =\Lambda_{0} e^{-i k_{0} x} \mathcal{T}_{-2 k_{0} t}\left(\sum_{j \in \mathbb{Z}} \widehat{v_{0}}(-j) e^{-i j^{2} t} e_{j}(x)\right) \\
& =\Lambda_{0} e^{-i k_{0} x} \mathcal{T}_{-2 k_{0} t} v^{\natural}(x, t), \tag{4.24}
\end{align*}
$$

where according to the hypothesis $v^{\natural}(x, t)$ solves the FSLS equation with periodic boundary conditions on $[0,2 \pi]$ and with initial condition the reflection of $v_{0}$, that is $v_{0}^{\natural}(x)$.

Similarly, the fourth sum can be written as

$$
\begin{align*}
\sum_{j \in \mathbb{Z}} e^{-i 2 k_{0} j t} e^{-i j^{2} t} \Lambda_{0} \overline{I_{0}} e^{-i k_{0} x} \widehat{w_{0}}(-j) e_{-j}(x) & =\Lambda_{0} \overline{I_{0}} e^{-i k_{0} x} \sum_{j \in \mathbb{Z}} e^{-i 2 k_{0} j t} e^{-i j^{2} t} \widehat{w_{0}}(-j) e_{-j}(x) \\
& =\Lambda_{0} \overline{I_{0}} e^{-i k_{0} x} \sum_{j \in \mathbb{Z}} e^{i 2 k_{0} j t} \widehat{w_{0}}(j) e^{-i j^{2} t} e_{j}(x) \\
& =\Lambda_{0} \overline{I_{0}} e^{-i k_{0} x} \mathcal{T}_{-2 k_{0} t}\left(\sum_{j \in \mathbb{Z}} \widehat{w_{0}}(j) e^{-i j^{2} t} e_{j}(x)\right) \\
& =\Lambda_{0} \bar{I}_{0} e^{-i k_{0} x} \mathcal{T}_{-2 k_{0} t} w(x, t) . \tag{4.25}
\end{align*}
$$

Finally, the third sum gives

$$
\begin{align*}
\sum_{j \in \mathbb{Z}} e^{-i 2 k_{0} j t} e^{-i j^{2} t} \overline{I_{0}} e^{i k_{0} x} \widehat{w_{0}}(-j) e_{j}(x) & =\overline{I_{0}} e^{i k_{0} x} \sum_{j \in \mathbb{Z}} e^{-i 2 k_{0} j t} \widehat{w_{0}}(-j) e^{-i j^{2} t} e_{j}(x) \\
& =\overline{I_{0}} e^{i k_{0} x} \mathcal{T}_{2 k_{0} t}\left(\sum_{j \in \mathbb{Z}} \widehat{w}_{0}(-j) e^{-i j^{2} t} e_{j}(x)\right)  \tag{4.26}\\
& =\bar{I}_{0} e^{i k_{0} x} \mathcal{T}_{2 k_{0} t} w^{\natural}(x, t),
\end{align*}
$$

where $w^{\natural}(x, t)$ solves the FSLS equation with periodic boundary conditions on $[0,2 \pi]$ and with initial condition $w_{0}^{\natural}(x)$, the reflection of $w_{0}$.

Combining (4.23), (4.24), (4.25) and (4.26) with (4.21), yields (4.16)
As we have mentioned earlier, in [15] the authors showed that the FSLS equation with pseudo-periodic boundary conditions exhibits the phenomenon of revivals at rational times $t=2 \pi p / q$. In particular, they showed that at rational times the solution (4.14) is constructed by a finite linear combination not only of translations of the initial condition but also of reflections. This gives a more complex structure of the revival phenomenon than that found in the periodic case. Indeed, motivated by numerical investigation, in [15] the authors carefully constructed a finite superposition of translated and reflected pseudo-periodic extensions of $u_{0}(x)$ of the form

$$
\begin{equation*}
\widetilde{u}_{0}(x)=\gamma^{n} u_{0}(x-2 \pi n), \quad 2 \pi n \leq x<2 \pi(n+1), \quad n \in \mathbb{Z} \tag{4.27}
\end{equation*}
$$

with $\gamma$ given in (4.5). Then, by comparing the generalised Fourier coefficients of
the solution at rational times with the generalised Fourier coefficients of the finite superposition they were able to rigorously confirm the revival effect.

In contrast to the argument in [15], (the proof of) Theorem 4.8 clearly indicates the mathematical reason for the persistence of revivals in the FSLS equation, subject to this general class of non-self-adjoint boundary conditions. It shows that we can solve the pseudo-periodic problem via certain associated periodic problems. This enables us to deduce from the existing results on the periodic case, that at rational times the pseudo-periodic problem (4.2) exhibits the revival effect. On the other hand, due to Oskolkov's (Theorem 2.12) and Rodnianski's (Theorem 2.13) results on the periodic setting, at irrational times the fractalisation effect arises under the pseudo-periodic boundary conditions. To be precise, when the initial profile has a finite number of jump discontinuities, the solution at irrational times becomes a continuous, though nowhere differentiable, function of the space variable.

As a consequence of Theorem 4.8, we can derive a revival representation at rational times. Recall from Lemma 3.5, that the solution at a rational time to a periodic problem for the FSLS equation is given by the second-order revival operator $\mathcal{R}_{2}(p, q)$ as defined in Chapter 3.

Corollary 4.9. Let $u_{0} \in L^{2}(0,2 \pi)$. Then, at any rational time $t=2 \pi \frac{p}{q}$, the solution to the pseudo-periodic problem (4.2) is given in $L^{2}(0,2 \pi)$ by

$$
\begin{align*}
u\left(x, 2 \pi \frac{p}{q}\right)= & \frac{e^{-i \frac{2 \pi k_{0}^{2} p}{q}}}{\tau}\left\{e^{i k_{0} x}\left[\mathcal{T}_{\frac{4 \pi k_{0} p}{q}} \mathcal{R}_{2}(p, q)\right] e^{-i k_{0} x} u_{0}(x)\right. \\
& +e^{-i k_{0} x}\left[\Lambda_{0} \gamma^{-1} \mathcal{T}_{-\frac{4 \pi k_{0} p}{q}} \mathcal{R}_{2}(p, q)\right] e^{i k_{0} x} u_{0}^{\natural}(x)  \tag{4.28}\\
& +e^{i k_{0} x}\left[\bar{I}_{0} \gamma \mathcal{T}_{\frac{4 \pi k_{0} p}{q}} \mathcal{R}_{2}(p, q)\right] e^{-i k_{0} x} u_{0}^{\natural}(x) \\
& \left.+e^{-i k_{0} x}\left[\Lambda_{0} \bar{I}_{0} \mathcal{T}_{-\frac{4 \pi k_{0 p}}{q}} \mathcal{R}_{2}(p, q)\right] e^{i k_{0} x} u_{0}(x)\right\} .
\end{align*}
$$

Similar to the classical periodic case and in accordance with Remark 3.7, the revival effect in the pseudo-periodic setting can be characterised as the property of the solution to be given in terms of a finite linear combination of isometries in
$L^{2}(0,2 \pi)$. In particular, given in terms of four isometries which have the form

$$
e^{ \pm i k_{0} x} \mathcal{T}_{ \pm \frac{4 \pi k_{0} p}{q}} \mathcal{R}_{2}(p, q) e^{\mp i k_{0} x}
$$

acting on the initial condition $u_{0}$ or its reflection $u^{\natural}$.
We further highlight that the revival formula (4.28) holds in the general case of $\beta_{0}$ and $\beta_{1}$ (under Assumption 4.3). Thus, in the pseudo-periodic case the lack of selfadjointness in the boundary conditions does not affect the existence of the revival phenomenon. On the other hand, in stark contrast, the case of the Airy PDE in the next chapter will not exhibit such behaviour. As we shall see, under self-adjoint quasi-periodic boundary conditions, the revival phenomenon in general breaks for the Airy PDE. Hence, in the context of first-order in time linear dispersive PDEs with integer coefficients the FSLS equation seems to be special with regards to the revival phenomenon. An additional confirmation of this will come in Chapter 6, where we show that the FSLS equation exhibits revivals (in the weak sense) when subject to separated Robin-type boundary conditions.

The appearance of the reflections in the revival formula (4.28) are worth a comment. It is evident that the reflections arise from the solution representation (4.16). They can be viewed as a consequence of the form of the eigenfunctions $\phi_{j}$. We observe from (4.10) that the second component of the eigenfunctions is formed by multiplying the function $e^{-i k_{0} x}$ with the reflection $e^{i j(2 \pi-x)}$ or as a constant multiple of the reflection of the first component of the eigenfunctions

$$
e^{-i k_{j} x}=e^{i k_{0} 2 \pi} e^{i k_{j}(2 \pi-x)}
$$

Remarkably, the presence of reflections does not depend on the self-adjointness of the boundary conditions. If we consider the case when the differential operator $L$ is symmetric, that is when $\beta_{0} \overline{\beta_{1}}=1$, the solution representation (4.16) of the pseudo-periodic problem (4.2) with self-adjoint boundary conditions becomes

$$
\begin{align*}
u(x, t)=\frac{e^{-i k_{0}^{2} t}}{1+\left|\Lambda_{0}\right|^{2}}\{ & e^{i k_{0} x} \mathcal{T}_{2 k_{0} t} v(x, t)+\Lambda_{0} e^{-i k_{0} x} \mathcal{T}_{-2 k_{0} t} v^{\natural}(x, t)  \tag{4.29}\\
& \left.\quad+\bar{\Lambda}_{0} e^{i k_{0} x} \mathcal{T}_{2 k_{0} t} w^{\natural}(x, t)+\left|\Lambda_{0}\right|^{2} e^{-i k_{0} x} \mathcal{T}_{-2 k_{0} t} w(x, t)\right\} .
\end{align*}
$$

Here, at a rational time $t=2 \pi \frac{p}{q}$ a similar revival formula as in Corollary 4.9 holds, and reflected copies of the initial condition are still present as expected from the decomposition (4.29).

However, for $\Lambda_{0}=0$ we should not expect to have reflections. An example of this case is the self-adjoint quasi-periodic boundary conditions, which correspond to the following choice of the boundary parameters

$$
\beta_{0}=\beta_{1}=\beta
$$

The self-adjointness of the boundary conditions requires $|\beta|^{2}=1$. Hence, we can set $\beta=e^{2 \pi i \theta}$ for $\theta \in(0,1)$. Then from (4.5), $k_{0}$ satisfies

$$
\cos \left(2 \pi k_{0}\right)=\frac{1+\beta^{2}}{2 \beta}=\frac{1+e^{4 \pi i \theta}}{2 e^{2 \pi i \theta}}=\cos (2 \pi \theta)
$$

and so we pick $k_{0}=\theta$. From (4.5) and (4.8) we find that

$$
\gamma=e^{2 \pi i \theta}=\beta \text { and } \Lambda_{0}=\frac{\gamma-\beta}{\beta-\gamma^{-1}}=0
$$

Substituting these values into (4.29), yields the reduced expression

$$
\begin{equation*}
u(x, t)=e^{-i \theta^{2} t} e^{i \theta x} \mathcal{T}_{2 \theta t} v(x, t) \tag{4.30}
\end{equation*}
$$

where

$$
v(x, t)=\sum_{j \in \mathbb{Z}} \widehat{v_{0}}(j) e^{-i j^{2} t} e_{j}(x),
$$

is the solution of the periodic problem for the FSLS equation with initial condition $v_{0}(x)=e^{-i \theta x} u_{0}(x)$. Again, because we know that $v(x, t)$ supports the revival effect at rational times, from (4.30) we obtain the revival formula

$$
\begin{equation*}
u\left(x, 2 \pi \frac{p}{q}\right)=e^{-i \theta^{2} 2 \pi \frac{p}{q}} e^{i \theta x} \mathcal{T}_{4 \pi \theta \frac{p}{q}} \mathcal{R}_{2}(p, q) e^{-i \theta x} u_{0}(x), \tag{4.31}
\end{equation*}
$$

with the absence of reflections. Note that (4.31) holds for any any choice of $\theta \in(0,1)$. As mentioned above, in the next chapter we will compare the Schrödinger equation with the Airy PDE under these conditions. We will find that the revival effect in
the quasi-periodic Airy PDE holds only when $\theta$ is a rational number in $(0,1)$.
According to the results above, the revival phenomenon persists under the general class of pseudo-periodic boundary conditions for the free linear Schrödinger equation. Specifically, at all times, the solution of the pseudo-periodic problem is the sum of four components corresponding to solutions of a periodic problem for the same equation with appropriate initial condition. From this decomposition, both the revival and the fractalisation effect follow. In the first section of Appendix F, we provide numerical examples which illustrate and confirm the statements presented here. We close this section with a final remark on the revival representations.

Remark 4.10. Consider the revival representation (4.31). The solution is given explicitly in terms of a finite number of translated copies of $e^{-i \theta x} u_{0}(x)$. Note that the final result is then multiplied by $e^{i \theta x}$, and hence the solution is indeed given in terms of a finite linear combination of translated copies of $u_{0}(x)$. Using the definitions of the revival operator $\mathcal{R}_{2}(p, q)$ and the periodic translation operator $\mathcal{T}_{s}$, then (4.31) can be written as

$$
\begin{gathered}
u\left(x, 2 \pi \frac{p}{q}\right)=e^{i \theta^{2} 2 \pi \frac{p}{q}} \widetilde{F}\left(x-4 \pi \theta \frac{p}{q}\right), \\
F(x)=\frac{1}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} e^{-2 \pi i m^{2} \frac{p}{q}} e^{2 \pi i(m+\theta) \frac{k}{q}} \widetilde{u_{0}}\left(x-2 \pi \frac{k}{q}\right),
\end{gathered}
$$

where the tilde denotes the quasi-periodic extension of a function ( $\gamma=e^{2 \pi i \theta}$ in (4.27)). Similar considerations apply for each one of the four terms in the revival formula (4.28), which reduces to an analogous statement with the main result in [15].

## Chapter 5

## Revivals in Airy's Partial <br> Differential Equation with Self-Adjoint Quasi-Periodic Boundary Conditions

In this chapter we consider the Airy PDE (AI) subject to quasi-periodic boundary conditions of self-adjoint type and examine the revival property. In contrast to the periodic case and even more surprisingly to the quasi-periodic problem of the free space linear Schrödinger equation, we will show that the Airy PDE with quasiperiodic boundary conditions does not in general supports the revival effect. In particular, the revival phenomenon depends explicitly on a real parameter controlling the boundary conditions. Whenever this boundary parameter is an irrational number, then at rational times the revival property breaks and instead, the fractalisation phenomenon takes place. On the other hand, for rational values of the parameter the revival indeed persists.

To confirm our claims, we follow the general method described in the previous chapter. In Section 5.1 we solve the time evolution problem by means of a generalised Fourier series. Then, in Section 5.2 we simplify the solution representation to reveal its periodic components and show that the solution at rational times can be computed via the solution of a periodic problem for the FSLS equation, but at times that depend on the boundary parameter. This is the main result of the chapter,
which corresponds to Theorem 5.2 and is also included in [14]. Theorem 5.2 seems to be the first rigorous result showing the lack of revivals in linear dispersive PDEs of first order in time, with integer coefficients and under coupled boundary conditions on a finite interval.

### 5.1 The Time Evolution Problem and its Solution

Recall from Chapter 2, that under periodic boundary conditions on $[0,2 \pi]$, both the Airy PDE and the free space linear Schrödinger equation exhibit the phenomenon of revivals at rational times (Theorem 2.8). Furthermore, in Chapter 4 we found that the revival property extends to the case of the free linear Schrödinger equation subject to pseudo-periodic boundary conditions, both of self-adjoint and non-selfadjoint type.

Therefore, we now investigate the effect of the boundary conditions on the revivals in the context of the Airy PDE

$$
\begin{equation*}
\partial_{t} u(x, t)=\partial_{x}^{3} u(x, t) . \tag{5.1}
\end{equation*}
$$

We consider an initial state at time zero $u(x, 0)=u_{0}(x)$ and fix quasi-periodic boundary conditions on $[0,2 \pi]$ of the form

$$
\begin{equation*}
e^{i 2 \pi \theta} \partial_{x}^{m} u(0, t)=\partial_{x}^{m} u(2 \pi, t), \quad m=0,1,2, \quad \theta \in(0,1) \tag{5.2}
\end{equation*}
$$

According to Remark 4.1, our first step is to solve the initial boundary value problem and obtain a series representation. Various non-periodic boundary value problems for Airy's PDE and the properties of the underlying eigenvalue problem were thoroughly examined by Pelloni in [53] and an explicit general representation of the solution was given. Here, due to the simplicity of the boundary conditions, we can derive the solution as a generalised Fourier series.

For fixed $\theta \in(0,1)$, we consider the third-order, essentially self-adjoint, linear differential operator $L f=i f^{\prime \prime \prime}$ with dense domain in $L^{2}(0,2 \pi)$ given by

$$
\begin{equation*}
\mathrm{D}(L)=\left\{f \in C^{3}[0,2 \pi]: e^{i 2 \pi \theta} f^{(m)}(0)=f^{(m)}(2 \pi), m=0,1,2\right\} . \tag{5.3}
\end{equation*}
$$

The time evolution problem can be written as follows

$$
\begin{equation*}
\partial_{t} u(x, t)=-i L u(x, t), \quad u(x, t)=u_{0}(x) . \tag{5.4}
\end{equation*}
$$

Integrating by parts three times yields that, for $f$ and $g$ in the domain of the operator, the symmetry condition

$$
\langle L f, g\rangle=\langle f, L g\rangle
$$

is satisfied. Hence, we expect that the eigenvalues of the operator are all real. We confirm this by solving the eigenvalue problem in the lemma below. We also compute the eigenfunctions.

Lemma 5.1. The eigenvalues of $L: D \rightarrow L^{2}(0,2 \pi)$ are all real and are given by

$$
\begin{equation*}
\lambda_{j}=k_{j}^{3}, \quad k_{j}=j+\theta, \quad j \in \mathbb{Z} . \tag{5.5}
\end{equation*}
$$

The corresponding eigenfunctions can be written as follows

$$
\begin{equation*}
\phi_{j}(x)=\frac{e^{i k_{j} x}}{\sqrt{2 \pi}}, \quad j \in \mathbb{Z} \tag{5.6}
\end{equation*}
$$

Proof. We solve the boundary value problem on $[0,2 \pi]$

$$
\begin{equation*}
i \phi^{\prime \prime \prime}(x)=\lambda \phi(x), \quad e^{i 2 \pi \theta} \phi^{(m)}(0)=\phi^{(m)}(2 \pi), \quad m=0,1,2 . \tag{5.7}
\end{equation*}
$$

If $\lambda=0$, then the boundary conditions imply that $\phi(x)=0$ for all $x \in[0,2 \pi]$. Thus, $\lambda=0$ is not an eigenvalue.

Let $\lambda \neq 0$ and write $\lambda=|\lambda| e^{i s}$ with $s \in[0,2 \pi)$. Due to the symmetry of the operator, we know that any eigenvalue should be real, hence $s=0$ or $s=\pi$. However, it is more convenient to treat $\lambda$ as an arbitrary complex number. For this, we set $\lambda=\mu^{3}$ and take $\mu=|\lambda|^{1 / 3} e^{i s / 3}$, the principal branch of the third root of $\lambda$. Then, the general solution to the differential equation $i \phi^{\prime \prime \prime}(x)=\mu^{3} \phi(x)$ is given by

$$
\phi(x)=C_{0} e^{\rho_{0} x}+C_{1} e^{\rho_{1} x}+C_{2} e^{\rho_{2} x},
$$

where

$$
\rho_{0}=i \mu, \quad \rho_{1}=-\frac{\mu}{2}(\sqrt{3}+i) \text { and } \rho_{2}=\frac{\mu}{2}(\sqrt{3}-i) .
$$

Applying the boundary conditions, results to the following linear system

$$
\begin{aligned}
\left(e^{2 \pi i \theta}-e^{2 \pi \rho_{0}}\right) C_{0} & =\left(e^{2 \pi \rho_{1}}-e^{2 \pi i \theta}\right) C_{1}+\left(e^{2 \pi \rho_{2}}-e^{2 \pi i \theta}\right) C_{2} \\
\left(\rho_{0}-\rho_{1}\right)\left(e^{2 \pi \rho_{1}}-e^{2 \pi i \theta}\right) C_{1} & =\left(\rho_{2}-\rho_{0}\right)\left(e^{2 \pi \rho_{2}}-e^{2 \pi i \theta}\right) C_{2} \\
\left(e^{2 \pi \rho_{2}}-e^{2 \pi i \theta}\right) C_{2} & =0
\end{aligned}
$$

We focus on the third equation of the system above and distinguish two cases.
(I) Let $C_{2} \neq 0$. Then, $e^{2 \pi \rho_{2}}=e^{2 \pi i \theta}$ and taking the complex logarithm we find that

$$
\mu_{j}=(j+\theta) \frac{(\sqrt{3}+i) i}{2}, \quad j \in \mathbb{Z}
$$

which gives

$$
\lambda_{j}=(j+\theta)^{3}, \quad j \in \mathbb{Z}
$$

Since $C_{2} \neq 0$ and $\mu=\mu_{j}$ the second equation of the system gives $\left(e^{2 \pi \rho_{1}}-e^{2 \pi i \theta}\right) C_{1}=0$. However, when $\mu=\mu_{j}$, then $e^{2 \pi \rho_{1}} \neq e^{2 \pi i \theta}$, so $C_{1}=0$, and then from the first equation we have $\left(e^{2 \pi i \theta}-e^{2 \pi \rho_{0}}\right) C_{0}=0$. But again for $\mu=\mu_{j}, e^{2 \pi \rho_{2}} \neq e^{2 \pi i \theta}$, and thus $C_{0}=0$. Therefore, we conclude that

$$
\phi_{j}(x)=C_{2} e^{\rho_{2} x}=C_{2} e^{i(j+\theta) x}, \quad \lambda_{j}=(j+\theta)^{3}, \quad j \in \mathbb{Z}
$$

(II) If $C_{2}=0$, then the second equation gives $\left(e^{2 \pi \rho_{1}}-e^{2 \pi i \theta}\right) C_{1}=0$. If moreover, $C_{1} \neq 0$, then we require $e^{2 \pi \rho_{1}}=e^{2 \pi i \theta}$. Thus, in this case, we find that

$$
\mu_{j}=(j+\theta) \frac{(-i)(\sqrt{3}-i)}{2},
$$

which gives

$$
\lambda_{j}=(j+\theta)^{3}, \quad j \in \mathbb{Z}
$$

Now because $C_{2}=0, C_{1} \neq 0$ and $e^{2 \pi \rho_{1}}=e^{2 \pi i \theta}$, the first equation implies that
$C_{0}=0$. Therefore, we see that

$$
\phi_{j}(x)=C_{1} e^{\rho_{1} x}=C_{1} e^{i(j+\theta) x}, \quad \lambda_{j}=(j+\theta)^{3}, \quad j \in \mathbb{Z} .
$$

On the other hand if $C_{1}=0$, we require $C_{0} \neq 0$ and so $e^{2 \pi \rho_{0}}=e^{2 \pi i \theta}$. This implies that $\mu_{j}=(j+\theta)$, for $j \in \mathbb{Z}$ and

$$
\lambda_{j}=(j+\theta)^{3}, \quad \phi_{j}(x)=C_{0} e^{i(j+\theta) x}, \quad j \in \mathbb{Z}
$$

From these two cases above, we conclude that the eigenvalues are given by (5.5) and the eigenfunctions can be written as in (5.6).

It is readily seen from (5.6) that the eigenfuctions form an orthonormal basis in $L^{2}(0,2 \pi)$. In fact,

$$
e^{-i \theta x} \phi_{j}=e_{j}(x)=\frac{e^{i j x}}{\sqrt{2 \pi}},
$$

meaning that they directly satisfy Definition B.2.
The solution, in a generalised sense, to the quasi-periodic problem (5.4) follows as an eigenfunction expansion converging in $L^{2}(0,2 \pi)$. That is, for any initial condition $u_{0} \in L^{2}$, the unique (generalised) solution in $L^{2}(0,2 \pi)$ is given at any fixed time $t \geq 0$, by the generalised Fourier series

$$
\begin{equation*}
u(x, t)=\sum_{j \in \mathbb{Z}}\left\langle u_{0}, \phi_{j}\right\rangle e^{-i k_{m}^{3} t} \phi_{j}(x), \tag{5.8}
\end{equation*}
$$

where the series converges in $L^{2}(0,2 \pi)$.
Using the exact same arguments as for the proof of Theorem 2.4, $u(x, t)$ given by (5.8) is obtained as the $L^{2}(0,2 \pi)$ limit of smooth solutions and defines a map $t \rightarrow u(\cdot, t)$ form $[0, \infty)$ to $L^{2}(0,2 \pi)$, which is continuous in $t$ with respect to the norm of $L^{2}(0,2 \pi)$.

### 5.2 Lack of Revivals at Rational Times

In contrast with the Schrödinger equation, the quasi-periodic problem for Airy's PDE (5.4) does not exhibit any form of revivals at rational times in general. As we
show below, the revival property holds in this case only for rational values of $\theta$ and it breaks otherwise. The manifestation of this dichotomy on $\theta$ is explained through the next theorem which gives a correspondence between the solution to Airy's quasiperiodic problem at a rational time with the solution to the FSLS equation at a time which is a $\theta$ multiple of the given rational time.

Recalling the definitions of the periodic translation and revival operators from Chapter 3, we obtain the main result of this chapter.

Theorem 5.2. Fix $\theta \in(0,1)$ and let $u(x, t)$ be the solution to Airy's quasi-periodic problem with an initial condition $u_{0}$ in $L^{2}(0,2 \pi)$. Let $p$ and $q$ be positive, co-prime integers and set

$$
\begin{equation*}
v_{0}^{(p, q)}(x)=\mathcal{R}_{3}(p, q)\left[u_{0}(x) e^{-i \theta x}\right] . \tag{5.9}
\end{equation*}
$$

Then, the solution at rational time $t_{\mathrm{r}}=2 \pi \frac{p}{q}$ is given by

$$
\begin{equation*}
u\left(x, t_{\mathrm{r}}\right)=e^{-i t_{\mathrm{r}} \theta^{3}} e^{i \theta x} \mathcal{T}_{3 \theta^{2} t_{\mathrm{r}}} v^{(p, q)}\left(x, 3 \theta t_{\mathrm{r}}\right), \tag{5.10}
\end{equation*}
$$

with $v^{(p, q)}(x, t)$ being the solution of the periodic problem for the FSLS equation with initial condition $v_{0}^{(p, q)}$.

Proof. At a fixed rational time $t_{\mathrm{r}}=2 \pi \frac{p}{q}$, the solution to Airy's quasi-periodic problem is given by

$$
\begin{equation*}
u\left(x, t_{\mathrm{r}}\right)=\sum_{j \in \mathbb{Z}}\left\langle u_{0}, \phi_{j}\right\rangle e^{-i k_{j}^{3} t_{\mathrm{r}}} \phi_{j}(x) . \tag{5.11}
\end{equation*}
$$

According to (5.6), $\phi_{j}(x)=e^{i \theta x} e_{j}(x)$, where $\left\{e_{j}\right\}_{j \in \mathbb{Z}}$ is the periodic Fourier basis. Hence,

$$
\begin{equation*}
\left\langle u_{0}, \phi_{j}\right\rangle=\int_{0}^{2 \pi} u_{0}(x) e^{-i \theta x} \overline{e_{j}(x)} d x=\widehat{w_{0}}(j), \quad w_{0}(x)=u_{0}(x) e^{-i \theta x} . \tag{5.12}
\end{equation*}
$$

The exponential term $e^{-i k_{j}^{3} t_{\mathrm{r}}}$ can be written as

$$
\begin{equation*}
e^{-i k_{j}^{3} t_{\mathrm{r}}}=e^{-i(j+\theta)^{3} t_{\mathrm{r}}}=e^{-i \theta^{3} t_{\mathrm{r}}} e^{-i j^{3} t_{\mathrm{r}}} e^{-i j 3 \theta^{2} t_{\mathrm{r}}} e^{-i j^{2} 3 \theta t_{\mathrm{r}}} . \tag{5.13}
\end{equation*}
$$

Substituting all this into (5.11) for the solution of (5.4), we find

$$
\begin{align*}
u\left(x, t_{\mathrm{r}}\right) & =\sum_{j \in \mathbb{Z}}\left\langle u_{0}, \phi_{j}\right\rangle e^{-i k_{j}^{3} t_{\mathrm{r}}} \phi_{j}(x) \\
& =\sum_{j \in \mathbb{Z}} \widehat{w_{0}}(j) e^{-i \theta^{3} t_{\mathrm{r}}} e^{-i j^{3} t_{\mathrm{r}}} e^{-i j 3 \theta^{2} t_{\mathrm{r}}} e^{-i j^{2} 3 \theta t_{\mathrm{r}}} e^{i \theta x} e_{j}(x)  \tag{5.14}\\
& =e^{-i \theta^{3} t_{\mathrm{r}}} e^{i \theta x} \sum_{j \in \mathbb{Z}} \widehat{w_{0}}(j) e^{-i j^{3} t_{\mathrm{r}}} e^{-i j 3 \theta^{2} t_{\mathrm{r}}} e^{-i j^{2} 3 \theta t_{\mathrm{r}}} e_{j}(x) \\
& =e^{-i \theta^{3} t_{\mathrm{r}}} e^{i \theta x} \mathcal{T}_{3 \theta^{2} t_{\mathrm{r}}}\left(\sum_{j \in \mathbb{Z}} \widehat{w_{0}}(j) e^{-i j^{3} t_{\mathrm{r}}} e^{-i j^{2} 3 \theta t_{\mathrm{r}}} e_{j}(x)\right) .
\end{align*}
$$

For the last equality we have used the Fourier representation (3.3) of the translation operator $\mathcal{T}_{s}$.

Now, by virtue of Lemma 3.6,

$$
\widehat{w_{0}}(j) e^{-i j^{3} t_{\mathrm{r}}}=\left\langle\mathcal{R}_{3}(p, q) w_{0}, e_{j}\right\rangle=\left\langle v_{0}^{(p, q)}, e_{j}\right\rangle=\widehat{v_{0}^{(p, q)}}(j),
$$

where the function $v_{0}^{(p, q)}(x)$ is given by (5.9). Substituting this final identity into (5.14), gives
$u\left(x, t_{\mathrm{r}}\right)=e^{-i \theta^{3} t_{\mathrm{r}}} e^{i \theta x} \mathcal{T}_{3 \theta^{2} t_{\mathrm{r}}}\left(\sum_{j \in \mathbb{Z}} \widehat{v_{0}^{(p, q)}}(j) e^{-i j^{2} 3 \theta t_{\mathrm{r}}} e_{j}(x)\right)=e^{-i \theta^{3} t_{\mathrm{r}}} e^{i \theta x} \mathcal{T}_{3 \theta^{2} t_{\mathrm{r}}} v^{(p, q)}\left(x, 3 \theta t_{\mathrm{r}}\right)$.
as claimed.

The surprising statement in Theorem 5.2 can be compared with the case of the free linear Schrödinger equation. As follows, we observe that the fundamental difference between the two equations lies in the fact that the solution of the quasi-periodic problem for the Airy equation corresponds to the solution of a periodic problem for the Schrödinger equation but evaluated at a different time. Indeed, the solution of (5.4) at a rational time $t=t_{\mathrm{r}}$ is obtained via the solution of a periodic problem for the Schrödinger equation evaluated at time $t=3 \theta t_{\mathrm{r}}$. Therefore, if $\theta \notin \mathbb{Q}$, this is an irrational time, for which the fractalisation effect occurs (Theorems 2.12 and 2.13). From this, it follows that the quasi-periodic Airy problem exhibits revivals at rational times if and only if $\theta \in \mathbb{Q}$. More precisely, we have the following dichotomy
on $\theta$.

1. Let $\theta \in \mathbb{Q}$. Then, the time $t=3 \theta t_{\mathrm{r}}$ is a rational time for Schrödinger's periodic problem. Hence, Airy's quasi-periodic problem will exhibit revivals at any rational time $t_{\mathrm{r}}$.
2. Let $\theta \notin \mathbb{Q}$. Then, the time $t=3 \theta t_{\mathrm{r}}$ an is an irrational time for Schrödinger's periodic problem. It follows that for piecewise continuous initial conditions, the solution to Airy's quasi-periodic problem at rational times $t_{\mathrm{r}}$ becomes a continuous but nowhere differentiable function. Hence, there is no revival at rational times in this case.

Remarkably, this additional dichotomy controlled by the parameter $\theta$, does not seem to have been observed in second-order models. We strongly suspect that the revival property carries onto the case of higher-order in space linear dispersive PDEs only under very specific hypotheses, even if the boundary conditions support it for the second-order case. The influence of the boundary conditions on the revival property appears to be crucial for this to hold true. It further suggests that the general pseudo-periodic case for third-order PDEs should not exhibit revivals.

Through the revival operator, we can now establish an explicit representation formula for the solution at rational times, whenever $\theta \in \mathbb{Q}$. A direct consequence of combining Theorem 5.2 with Lemma 3.5 provides the next statement.

Corollary 5.3. Let $(p, q),(c, d)$ be pairs of co-prime positive integers, with $c<d$. Set $\theta_{\mathrm{r}}=c / d$ and let $u_{0} \in L^{2}(0,2 \pi)$. For fixed $\theta=\theta_{\mathrm{r}}$, the solution $u(x, t)$ of Airy's quasi-periodic problem (5.4) at rational time $t_{\mathrm{r}}=2 \pi \frac{p}{q}$ is given by the representation

$$
\begin{equation*}
u\left(x, t_{\mathrm{r}}\right)=e^{i \theta_{\mathrm{r}} x} e^{-i \theta_{\mathrm{r}}^{3} t_{\mathrm{r}}} \mathcal{T}_{3 \theta_{\mathrm{r}}^{2} t_{\mathrm{r}}} \mathcal{R}_{2}(3 c p, d q) \mathcal{R}_{3}(p, q)\left[e^{-i \theta_{\mathrm{r}} x} u_{0}(x)\right] \tag{5.16}
\end{equation*}
$$

Remark 4.10 applies also to the revival identity (5.16). We give below another representation involving the quasi-periodic eigenfunctions $\phi_{j}(x)$. The formula can be verified directly, following the proof of Theorem 2.8 by showing that both sides of the equality have the same generalised Fourier coefficients with respect to the orthonormal basis $\left\{\phi_{j}\right\}_{j \in \mathbb{Z}}$.

Proposition 5.4. Let $p, q, c, d, u_{0}(x)$ and $\theta_{\mathrm{r}}$ be as in Corollary 5.3. Set

$$
\tilde{u}_{0}(x)=e^{2 \pi i \frac{c}{d} n} u_{0}(x-2 \pi n), \quad 2 \pi n \leq x<2 \pi(n+1), \quad n \in \mathbb{Z}
$$

to be the quasi-periodic extension of $u_{0}$. Then, the solution $u(x, t)$ to Airy's quasiperiodic problem (5.4), for $\theta=\theta_{\mathrm{r}}$, is given at a rational time $t_{\mathrm{r}}=2 \pi \frac{p}{q}$ by the representation

$$
\begin{equation*}
u\left(x, t_{\mathrm{r}}\right)=\frac{\sqrt{2 \pi}}{d^{2} q} \sum_{k=0}^{d^{2} q-1} \sum_{m=0}^{d^{2} q-1} e^{-i\left(m+\frac{c}{d}\right)^{3} t_{\mathrm{r}}} \phi_{m}\left(\frac{2 \pi k}{d^{2} q}\right) \tilde{u}_{0}\left(x-\frac{2 \pi k}{d^{2} q}\right) . \tag{5.17}
\end{equation*}
$$

In the second section of Appendix F, we provide numerical examples displaying the implications of Theorem 5.2 on the existence and non-existence of the revival property whenever $\theta \in \mathbb{Q}$ or $\theta \notin \mathbb{Q}$ respectively. Furthermore, we note that in Chapter 8 we will revisit the self-adjoint quasi-periodic problems. By invoking a different method, based on space-time transformations, the results on the Airy and Schrödinger equation will extend to time evolution problems with higher-order spatial differential operators.

Finally, we close this chapter with the remark below. It address the question of considering other times, different than $2 \pi \frac{p}{q}$, for which the revival effect could possibly persist in Airy's quasi-periodic problem when $\theta \notin \mathbb{Q}$. Specifically, times that depend on the parameter $\theta$ and have the form $\frac{2 \pi}{\theta} \frac{p}{q}$, where $p$ and $q$ are positive co-prime integers.

Remark 5.5. Fixing $\theta \in(0,1)$ and following a similar approach as in the proof of Theorem 5.2, the solution $u(x, t)$ to Airy's quasi-periodic problem with initial condition $u_{0}$ is given at a fixed time $t_{\theta}=\frac{2 \pi}{\theta} \frac{p}{q}$ by the representation

$$
u\left(x, t_{\theta}\right)=e^{-i t_{\theta} \theta^{3}} e^{i \theta x} \mathcal{T}_{3 \theta^{2} t_{\theta}}\left(\sum_{j \in \mathbb{Z}}\left\langle\mathcal{R}_{2}(6 p, q) v_{0}, e_{j}\right\rangle e^{-i j^{3} t_{\theta}} e_{j}(x)\right),
$$

where $v_{0}$ is as in Theorem 5.10. We observe that the term in the brackets above corresponds to the Fourier series representation of the solution to Airy's periodic problem at time $t_{\theta}$ with initial condition $\mathcal{R}_{2}(6 p, q) v_{0}$. If $\theta \notin \mathbb{Q}$, then $t_{\theta}$ is an irrational time for Airy's periodic problem. Hence, for irrational values of $\theta$, the revival effect
breaks down at times $t_{\theta}$ in the quasi-periodic case.

## Chapter 6

## Weak revivals in the Free Linear <br> Schrödinger Equation with

## Robin-type Boundary Conditions

In this chapter we further explore the revival phenomenon in the context of the free linear Schrödinger equation. The equation is posed on the interval $[0, \pi]$ and we impose separated boundary conditions. This is in contrast to the coupled conditions in the previous chapters. Under a specific type of Robin boundary conditions, which allows an exact analytical treatment of the model, we characterise the revival phenomenon based on the notion of the weak revival effect, see Definition 2.16. Thus, in this case, the dichotomy between the behaviour of the solution at rational and irrational times is still present, resembling the Talbot effect in the context of the cubic non-linear Schrödinger equation or the linear Schröndinger equation with potential described in the $2 \pi$-periodic setting of Section 2.5.

In Section 6.2, by solving the underlying eigenvalue problem, we obtain the solution representation as a generalised Fourier series. Then, in Section 6.3, we show that at all times the solution can be decomposed as the summation of two components. One of them is at all times a continuous function of the space variable, whereas the other part is periodic in space and, at rational times exhibits the pure revival effect, see Definition 2.10. These results were also in [14], however here we provide complete proofs of the properties found. Our main tool described in Section 6.1 involves two auxiliary problems for the free space linear Schrödinger
equation on $[0,2 \pi]$ with periodic boundary conditions.
Note that, since we are going to work on both intervals $[0, \pi]$ and $[0,2 \pi]$, we keep the notation $\langle\cdot, \cdot\rangle$ for the inner-product in $L^{2}(0,2 \pi)$ and we denote the inner-product in $L^{2}(0, \pi)$ by a subscript

$$
\langle f, g\rangle_{L^{2}(0, \pi)}=\int_{0}^{\pi} f(x) \overline{g(x)} d x, \quad \forall f, g \in L^{2}(0, \pi) .
$$

### 6.1 Two auxiliary problems. The Dirichlet and Neumann boundary conditions

We begin by establishing a correspondence between periodic problems on $[0,2 \pi]$ for the free space linear Schrödinger equation with the time evolution problems for the same equation posed on the half interval $[0, \pi]$ with either zero Dirichlet or zero Neumann boundary conditions at the end points. We show that the Dirichlet and Neumann problems can be recast in terms of periodic boundary conditions on $[0,2 \pi]$ by considering odd or even initial conditions with respect to $\pi$. The statements and proofs in the following sections will rely on these two time evolution problems. Furthermore, as we shall see, these correspondences allow periodic revival operators to characterise completely the revivals in these two cases.

We are interested in the following two time evolution problems on $[0, \pi]$. The first is the Dirichlet problem

$$
\begin{equation*}
\partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t), \quad u(x, 0)=u_{0}(x), \quad u(0, t)=u(\pi, t)=0, \tag{6.1}
\end{equation*}
$$

and the second one is the Neumann problem

$$
\begin{equation*}
\partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t), \quad u(x, 0)=u_{0}(x), \quad \partial_{x} u(0, t)=\partial_{x} u(\pi, t)=0 . \tag{6.2}
\end{equation*}
$$

We have chosen the sub-interval $[0, \pi]$, because we want to describe their solutions through periodic boundary conditions on $[0,2 \pi]$. Indeed, on $[0, \pi]$ the eigenvalues of the underlying spectral problems have the same form as the eigenvalues with the periodic problem on $[0,2 \pi]$, although they do not take all the integers values.

Specifically, the eigenvalue problem for the Dirichlet case

$$
-\phi^{\prime \prime}(x)=\lambda \phi(x), \quad \phi(0)=\phi(\pi)=0
$$

has eigenvalues $\lambda_{j}=j^{2}$, with $j \in \mathbb{N}$ and corresponding eigenfunctions the elements $d_{j}(x)$ of the sine Fourier basis, which, recalling from (3.13), are given by

$$
\begin{equation*}
d_{j}(x)=\sqrt{\frac{2}{\pi}} \sin (j x), \quad j \in \mathbb{N} . \tag{6.3}
\end{equation*}
$$

On the other hand, the eigenpairs for the Neumann eigenvalue problem

$$
-\phi^{\prime \prime}(x)=\lambda \phi(x), \quad \phi^{\prime}(0)=\phi^{\prime}(\pi)=0,
$$

are $\left(j^{2}, n_{j}(x)\right)$ with $j$ running on all non-negative integers and $n_{j}(x)$ being the elements of the cosine Fourier basis given by

$$
\begin{equation*}
n_{0}(x)=\frac{1}{\sqrt{\pi}}, \quad n_{j}(x)=\sqrt{\frac{2}{\pi}} \cos (j x), \quad j \in \mathbb{N}, \tag{6.4}
\end{equation*}
$$

as we recall from (3.12).
Hence, for the Dirichlet problem (6.1), it follows that the (generalised) solution representation in $L^{2}(0, \pi)$ is given by the sine Fourier expansion

$$
\begin{equation*}
u(x, t)=\sum_{j=1}^{\infty} b_{j} e^{-i j^{2} t} d_{j}(x), \quad b_{j}=\left\langle u_{0}, d_{j}\right\rangle_{L^{2}(0, \pi)} \tag{6.5}
\end{equation*}
$$

For the Neumann problem (6.2), the (generalised) solution in $L^{2}(0, \pi)$ has the form of the cosine Fourier expansion

$$
\begin{equation*}
u(x, t)=a_{0} n_{0}(x)+\sum_{j=1}^{\infty} a_{j} e^{-j^{2} t} n_{j}(x), \quad a_{j}=\left\langle u_{0}, n_{j}\right\rangle_{L^{2}(0, \pi)} \tag{6.6}
\end{equation*}
$$

Then, as we observe from (6.5) and (6.6), the generalised Fourier coefficients resemble the form of the Fourier coefficients $\widehat{f}(j) e^{-i j^{2} t}$, of a periodic problem on $[0,2 \pi]$ with some initial function $f \in L^{2}(0,2 \pi)$. Therefore, going from $[0, \pi]$ to $[0,2 \pi]$ we match the form of the Fourier coefficients.

Recall further that the revival property is described by a finite number of translations of the initial function. But, to describe the translation of a function defined on $[0, \pi]$ we need to extend its values outside of its interval of definition to the whole real line. The extension follows the behaviour of the eigenfunctions. Thus, since $d_{j}(x)$ are odd in $[0,2 \pi]$ with respect to $\pi$ and $2 \pi$-periodic in $\mathbb{R}$, for Dirichlet boundary conditions, we take odd extensions on $[0,2 \pi]$. Then, the periodic translation operator acts on the odd $2 \pi$-periodic extension of the initial condition. The same idea applies to Neumann boundary conditions, however instead of odd extensions we take even extensions following the behaviour of the eigenfunctions $n_{j}(x)$.

The correspondence between $2 \pi$-periodic and Dirichlet or Neumann problems can be formulated in terms of the following two lemmas. The proofs are elementary and almost the same with the only difference lying on odd and even extensions of the initial condition. So, for the Neumann case we briefly sketch it.

Lemma 6.1. Let $u_{0} \in L^{2}(0, \pi)$ and $u_{0}^{-}$be its odd extension defined by (3.11). Then, $u(x, t)$ solves the periodic problem on $[0,2 \pi]$

$$
\begin{aligned}
& \partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t), \quad u(x, 0)=u_{0}^{-}(x), \\
& u(0, t)=u(x, 2 \pi), \quad \partial_{x} u(0, t)=\partial_{x} u(2 \pi, t),
\end{aligned}
$$

if and only if its restriction on $[0, \pi]$ solves the Dirichlet problem (6.1) on $[0, \pi]$.

Proof. Suppose that $u(x, t)$ is the solution to the periodic problem. Since the initial condition $u_{0}^{-}$is odd with respect to $\pi$, its Fourier coefficients are given by Lemma 3.12-(ii). That is for every $j \in \mathbb{N}$,

$$
\widehat{u_{0}^{-}}(j)=-i \sqrt{\frac{2}{\pi}} \int_{0}^{\pi} u_{0}(x) \sin (j x) d x=-i b_{j} .
$$

For $j=0, \widehat{u_{0}^{-}}(0)=0$. Moreover, we have that

$$
\widehat{u_{0}^{-}}(-j)=-\widehat{u_{0}^{-}}(j), \quad \forall j \in \mathbb{N} .
$$

We know that for every time $t \geq 0$, the solution to the periodic problem has the
$L^{2}(0,2 \pi)$ representation

$$
\begin{equation*}
u(x, t)=\sum_{j \in \mathbb{Z}} \widehat{u_{0}^{-}}(j) e^{-j^{2} t} e_{j}(x) . \tag{6.7}
\end{equation*}
$$

However, the form of the Fourier coefficients $\widehat{u_{0}^{-}}(j)$ imply that

$$
\begin{aligned}
u(x, t) & =\sum_{j \in \mathbb{Z}} \widehat{u_{0}^{-}}(j) e^{-j^{2} t} e_{j}(x) \\
& =\sum_{j=1}^{\infty} \widehat{u_{0}^{-}}(j) e^{-j^{2} t} e_{j}(x)+\sum_{j=1}^{\infty} \widehat{u_{0}^{-}}(-j) e^{-j^{2} t} e_{-j}(x) \\
& =\sum_{j=1}^{\infty} \widehat{u_{0}^{-}}(j) e^{-j^{2} t}\left(e_{j}(x)-e_{-j}(x)\right) \\
& =-i \sum_{j=1}^{\infty} b_{j} e^{-j^{2} t} \frac{2 i}{\sqrt{2 \pi}} \sin (j x), \quad b_{j}=\left\langle u_{0}, d_{j}\right\rangle_{L^{2}(0, \pi)} .
\end{aligned}
$$

Therefore,

$$
\begin{equation*}
u(x, t)=\sum_{j=1}^{\infty} b_{j} e^{-j^{2} t} d_{j}(x) \tag{6.8}
\end{equation*}
$$

Comparing with (6.5), we notice that the right hand-side of (6.8), when restricted on $[0, \pi]$, corresponds to the sine Fourier series representation of the solution to the Dirichlet problem on $[0, \pi]$.

For the reverse direction, starting from (6.5) or (6.8) on $[0, \pi]$, we can equivalently extend to $[0,2 \pi]$ and go back to (6.7) using

$$
d_{j}(x)=\frac{e^{i j x}-e^{-i j x}}{\sqrt{2 \pi} i}
$$

Lemma 6.2. Let $u_{0} \in L^{2}(0, \pi)$ and $u_{0}^{+}$its even extension defined by (3.11). Then, $u(x, t)$ solves the periodic problem on $[0,2 \pi]$

$$
\begin{aligned}
& \partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t), \quad u(x, 0)=u_{0}^{+}(x), \\
& u(0, t)=u(x, 2 \pi), \quad \partial_{x} u(0, t)=\partial_{x} u(2 \pi, t),
\end{aligned}
$$

if and only if its restriction on $[0, \pi]$ solves the Neumann problem (6.2) on $[0, \pi]$.
Proof. Suppose $u(x, t)$ is the solution to the periodic problem. Then for every fixed $t \geq 0$, its solution is given by the Fourier expansion in $L^{2}(0,2 \pi)$

$$
\begin{equation*}
u(x, t)=\sum_{j \in \mathbb{Z}} \widehat{u_{0}^{+}}(j) e^{-j^{2} t} e_{j}(x) . \tag{6.9}
\end{equation*}
$$

Because $u_{0}^{+}$is even with respect to $\pi$, by virtue of Lemma 3.12- $(i)$, its Fourier coefficients are given by

$$
\begin{aligned}
& \widehat{u_{0}^{+}}(0)=\sqrt{\frac{2}{\pi}} \int_{0}^{2 \pi} u_{0}(y) d y=\sqrt{2} a_{0}, \\
& \widehat{u_{0}^{+}}(j)=\sqrt{\frac{2}{\pi}} \int_{0}^{\pi} u_{0}(y) \cos (j y) d y=, \quad j \in \mathbb{N} .
\end{aligned}
$$

Moreover, we have that

$$
\widehat{u_{0}^{+}}(-j)=\widehat{u_{0}^{+}}(j), \quad j \in \mathbb{N} .
$$

Hence, the solution representation (6.9) becomes

$$
\begin{equation*}
u(x, t)=\sum_{j \in \mathbb{Z}} \widehat{u_{0}^{+}}(j) e^{-j^{2} t} e_{j}(x)=a_{0} n_{0}(x)+\sum_{j=1}^{\infty} a_{j} e^{-j^{2} t}\left(e_{j}(x)+e_{-j}(x)\right), \tag{6.10}
\end{equation*}
$$

or

$$
\begin{equation*}
u(x, t)=a_{0} n_{0}(x)+\sum_{j=1}^{\infty} a_{j} e^{-j^{2} t} n_{j}(x), \quad a_{j}=\left\langle u_{0}, n_{j}\right\rangle_{L^{2}(0, \pi)} . \tag{6.11}
\end{equation*}
$$

Finally, comparing with (6.6), we notice that the right hand-side of (6.11), when restricted on $[0, \pi]$, corresponds to the cosine Fourier series representation of the solution to the Neumann problem on $[0, \pi]$. This establishes the correspondence, with the opposite direction obtained by

$$
n_{j}(x)=\frac{e^{i j x}+e^{-i j x}}{\sqrt{2 \pi}}
$$

Remark 6.3. The solution to the Dirichlet problem with initial function $u_{0}$, is the restriction on $[0, \pi]$ of the solution to the periodic problem on $[0,2 \pi]$ with initial condition the odd extension of $u_{0}$. On the other hand, the solution to the Neumann
problem corresponds to the restriction on $[0, \pi]$ of the solution to the periodic problem on $[0,2 \pi]$ with initial condition the even extension of $u_{0}$. Hence, for Dirichlet and Neumann boundary conditions on $[0, \pi]$, we can describe the revival property at rational times as the restriction of the action of the periodic revival operator applied on odd and even extensions.

Therefore, we obtain the following statement addressing the revival effect in the Dirichlet and Neumann problems. In particular, we see that both problems exhibit pure revivals (in accordance with Definition 2.10).

Corollary 6.4. Consider the free space linear Schrödinger equation on $[0, \pi]$ with initial condition $u_{0} \in L^{2}(0, \pi)$. Denote by $u_{0}^{ \pm}$the even and odd extensions of $u_{0}$ to the interval $[0,2 \pi]$. Then, at rational time $t=2 \pi \frac{p}{q}$ the solution under Dirichlet boundary conditions is given in $L^{2}(0, \pi)$ by the restriction on $[0, \pi]$ of the representation

$$
u\left(x, 2 \pi \frac{p}{q}\right)=\mathcal{R}_{2}(p, q) u_{0}^{-}(x),
$$

and under Neumann boundary conditions is given in $L^{2}(0, \pi)$ by the restriction on $[0, \pi]$ of the representation

$$
v\left(x, 2 \pi \frac{p}{q}\right)=\mathcal{R}_{2}(p, q) v_{0}^{+}(x) .
$$

### 6.2 The Time Evolution Problem and its Solution

We turn our attention on the initial boundary value problem on $[0, \pi]$

$$
\begin{align*}
& \partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t), \quad u(x, 0)=u_{0}(x), \\
& b u\left(x_{0}, t\right)=(1-b) \partial_{x} u\left(x_{0}, t\right), x_{0}=0, \pi, \quad b \in(0,1) . \tag{6.12}
\end{align*}
$$

The boundary conditions are a type of Robin boundary conditions which involve the value of the function and its derivative at the each endpoint. These specific form of Robin conditions in (6.12) can be viewed as an interpolation between the Neumann $(b \rightarrow 0)$ and Dirichlet $(b \rightarrow 1)$ boundary conditions. Although from the previous section the revival property at rational times under Neumann or Dirichlet
conditions follows easily from the periodic problem on $[0,2 \pi]$ in terms of the revival operator only, the case when $b \neq 0,1$ turns out to be fundamentally different. The difference lies in the solution representation of problem (6.12), which, at any time $t \geq 0$, consists of the solution to a Neumann problem perturbed by a continuous in space component.

The presence of the Neumann component guarantees that the dichotomy between the persistence versus regularisation of discontinuities at rational versus irrational times still holds. This perturbed revival is exactly the weak revival, see Definition 2.16. The time evolution problem under question provides a benchmark example for which a complete analytical description of the weak revival phenomenon in the context of linear dispersive equations can be conducted.

As in the previous chapters, we begin our analysis with the solution of the underlying eigenvalue problem

$$
\begin{equation*}
-\phi^{\prime \prime}(x)=\lambda \phi(x), \quad b \phi\left(x_{0}\right)=(1-b) \phi^{\prime}\left(x_{0}\right), x_{0}=0, \pi, \quad b \in(0,1) . \tag{6.13}
\end{equation*}
$$

This eigenvalue problem is a regular Sturm-Liouville problem, [20]. The boundary conditions are self-adjoint and the eigenvalue problem (6.13) corresponds to the eigenvalue problem of the essentially self-adjoint differential operator $L f=-f^{\prime \prime}$ defined on the domain

$$
\mathrm{D}(L)=\left\{f \in C^{2}[0, \pi] ; b f\left(x_{0}\right)=(1-b) f^{\prime}\left(x_{0}\right), x_{0}=0, \pi\right\} .
$$

Lemma 6.5. Let $b \in(0,1)$. The eigenvalues and the corresponding normalised eigenfunctions of the eigenvalue problem (6.13) are given by
(i) $\quad \lambda_{b}=-m_{b}^{2}<0, \quad m_{b}=\frac{b}{1-b}, \quad \phi_{b}(x)=A_{b} e^{m_{b} x}, \quad A_{b}=\sqrt{\frac{2 m_{b}}{e^{2 a m_{b}}-1}}$,
(ii) $\lambda_{j}=j^{2}>0, \quad \phi_{j}(x)=\frac{1}{\sqrt{2 \pi}}\left(e^{i j x}-\Lambda_{j} e^{-i j x}\right), \quad j \in \mathbb{N}$,
where

$$
\begin{equation*}
\Lambda_{j}=\frac{b-(1-b) i j}{b+(1-b) i j}, \quad j \in \mathbb{N} . \tag{6.15}
\end{equation*}
$$

Proof. Assume $\lambda=0$. From the boundary conditions follows that $\phi=0$, and so there is no eigenpair. Moreover, since we know that the eigenvalues are all real, we can consider the two cases $\lambda<0$ and $\lambda>0$ separately.

Let $\lambda=-m<0$, with $m>0$. Then, the general solution of the differential equation in (6.13) is

$$
\phi(x)=A e^{\sqrt{m} x}+B e^{-\sqrt{m} x} .
$$

From the boundary conditions we know that $m$ satisfies the equation

$$
\left(b^{2}-(1-b)^{2} m\right)\left(e^{-\sqrt{m} \pi}-e^{\sqrt{m} \pi}\right)=0,
$$

whose solution yields

$$
m=\frac{b^{2}}{(1-b)^{2}} .
$$

Setting $m_{b}=b /(1-b)$, we have $\lambda_{b}=-m_{b}^{2}$. Hence, $\phi(x)=A e^{m_{b} x}+B e^{-m_{b} x}$. However, the boundary conditions imply that $B=0$, and thus $\phi_{b}(x)=A_{b} e^{m_{b} x}$, with the normalising constant $A_{b}$ to be given as in (6.14).

If now $\lambda>0$, then the general solution of the differential equation in (6.13) is

$$
\phi(x)=A e^{i \sqrt{\lambda} x}+B e^{-i \sqrt{\lambda} x} .
$$

Applying the boundary conditions we find that $\lambda$ has to satisfy the equation

$$
\left(b^{2}+(1-b)^{2} \lambda\right)\left(e^{-i \sqrt{\lambda} \pi}-e^{i \sqrt{\lambda} \pi}\right)=0,
$$

which gives solutions of the form $\lambda_{j}=j^{2}$ with $j \in \mathbb{N}$.
Moreover, using the boundary conditions again we can find that $A$ and $B$ satisfy

$$
B=-\Lambda_{j} A, \quad \Lambda_{j}=\frac{b-(1-b) i j}{b+(1-b) i j}, \quad j \in \mathbb{N} .
$$

Consequently, after normalisation, the eigenfunctions $\phi_{j}$ are indeed (6.14).
Remark 6.6. Since $\left\{\phi_{j}\right\}_{j=b, 1}^{\infty}$ is an orthonormal basis in $L^{2}(0, \pi)$, we have that any
$f \in L^{2}(0, \pi)$ has a generalised Fourier expansion in terms of the eigenfunctions

$$
\begin{equation*}
f(x)=\left\langle f, \phi_{b}\right\rangle_{L^{2}(0, \pi)} \phi_{b}(x)+\sum_{j=1}^{\infty}\left\langle f, \phi_{j}\right\rangle_{L^{2}(0, \pi)} \phi_{j}(x) . \tag{6.16}
\end{equation*}
$$

In the limiting cases $b \rightarrow 0$ and $b \rightarrow 1$, this expansion behaves as follows.
(i) When $b \rightarrow 0$, the boundary conditions in (6.13) become of the Neumann type

$$
\phi^{\prime}(0)=\phi^{\prime}(\pi)=0 .
$$

Using the expressions from (6.14) we find that

$$
A_{b} \rightarrow 1 / \sqrt{\pi}, \quad \phi_{b}(x) \rightarrow 1 / \sqrt{\pi}, \quad \Lambda_{j}=-1, \quad \phi_{j}(x)=\sqrt{\frac{2}{\pi}} \cos (j x) .
$$

Hence, (6.16) becomes, as expected, the usual cosine Fourier series of the function $f$ in $L^{2}(0, \pi)$

$$
f(x)=\frac{1}{\pi} \int_{0}^{\pi} f(y) d y+\frac{2}{\pi} \sum_{j=1}^{\infty} \int_{0}^{\pi} f(y) \cos (j y) d y \cos (j x) .
$$

(ii) When $b \rightarrow 1$, then the boundary conditions are of the Dirichlet type

$$
\phi(0)=\phi(\pi)=0 .
$$

Since (6.14) yields $m_{b} \rightarrow \infty$ and $A_{b} \rightarrow \infty$, it follows that $\lambda_{b} \rightarrow-\infty$ and $\phi_{b}(x) \rightarrow \infty$. Hence, $\left(\lambda_{b}, \phi_{b}\right)$ is not an eigenpair. Again from (6.14), we have $\Lambda_{j} \rightarrow 1$ and so

$$
\phi_{j}(x)=i \sqrt{\frac{2}{\pi}} \sin (j x) .
$$

Therefore, as expected, the expansion (6.16) takes the form of the sine Fourier series of $f$

$$
f(x)=\frac{2}{\pi} \sum_{j=1}^{\infty} \int_{0}^{\pi} f(y) \sin (j y) d y \sin (j x) .
$$

Having obtained explicitly the form of the orthonormal basis $\left\{\phi_{j}\right\}_{j=b, 1}^{\infty}$ in (6.14), the solution of the time evolution problem (6.12) can be expressed as a generalised Fourier series. Indeed, for any initial condition $u_{0} \in L^{2}(0, a)$, the Fourier method
provides the generalised solution at any fixed time $t \geq 0$ in terms of the eigenfunction expansion

$$
\begin{equation*}
u(x, t)=\left\langle u_{0}, \phi_{b}\right\rangle_{L^{2}(0, \pi)} e^{i m_{b}^{2} t} \phi_{b}(x)+\sum_{j=1}^{\infty}\left\langle u_{0}, \phi_{j}\right\rangle_{L^{2}(0, \pi)} e^{-i j^{2} t} \phi_{j}(x), \tag{6.17}
\end{equation*}
$$

where equality is understood in the norm of $L^{2}(0, \pi)$.

### 6.3 The Weak Revival

Now, in this section, we follow the idea of decomposing the solution representation (6.17) into $2 \pi$-periodic components in order to derive the revival effect for the Robin problem (6.12). We focus on the infinite sum in (6.17) and break it in terms of specific solutions of the $2 \pi$-periodic problem for the linear Schrödinger equation with zero potential, noticing that the eigenvalues $\lambda_{j}=j^{2}$ have the same form as those of the periodic problem.

Furthermore, to give perhaps an intuition as to why we should expect some form of revivals, let us focus for a moment on the eigenfunctions

$$
\phi_{j}(x)=\frac{1}{\sqrt{2 \pi}}\left(e^{i j x}-\Lambda_{j} e^{-i j x}\right), \quad j \in \mathbb{N} .
$$

On the one hand, the first component $e^{i j x} / \sqrt{2 \pi}$ corresponds to elements of the classical Fourier basis. Thus, if we isolate the terms multiplied by this component, we will obtain a solution representation similar to the periodic problem. Since $j$ runs on $\mathbb{N}$ and not on $\mathbb{Z}$, writing $e^{i j x}=\cos (j x)+i \sin (j x)$, we expect to have Neumann and Dirichlet boundary conditions involved, that is periodic boundary conditions with even and odd extensions. On the other hand, the second component $e^{-i j x} / \sqrt{2 \pi}$ will produce reflections of even and odd extensions. Since we also have multiplication by $\Lambda_{j}$, we expect a new transformation applied to the initial condition. This new transformation will be the $2 \pi$-periodic convolution with a function whose Fourier coefficients will involve the real and imaginary parts of $\Lambda_{j}$.

All the above realises as follows. Let the auxiliary function

$$
\begin{equation*}
f_{1}(x)=\sqrt{\frac{\pi}{2}} \frac{m_{b}}{e^{2 \pi m_{b}}-1} e^{m_{b} x}, \quad x \in[0,2 \pi), \tag{6.18}
\end{equation*}
$$

where $m_{b}$ is the positive constant in (6.14). Note that, following Definition 3.9, its reflection with respect to $\pi$ is given by

$$
\begin{equation*}
f_{1}^{\natural}(x)=f_{1}(2 \pi-x)=\sqrt{\frac{\pi}{2}} \frac{m_{b}}{1-e^{-2 \pi m_{b}}} e^{-m_{b} x}, \quad x \in[0,2 \pi) . \tag{6.19}
\end{equation*}
$$

Then, the Fourier representation of these two functions has

$$
\begin{equation*}
\widehat{f}_{1}(j)=\frac{m_{b}}{2\left(m_{b}-i j\right)} \text { and } \widehat{f_{1}^{\natural}}(j)=\widehat{f}_{1}(-j)=\frac{m_{b}}{2\left(m_{b}+i j\right)}, \quad j \in \mathbb{Z} . \tag{6.20}
\end{equation*}
$$

We now establish the main result of this section. We show that the solution of (6.12) can be expressed at all times in terms of (the restrictions to $[0, \pi]$ of) the solutions to five periodic problems for the free space linear Schödinger equation. The initial condition of each problem is specified by an explicit transformation of $u_{0}$. In particular, four of these initial conditions are obtained as the $2 \pi$-periodic convolution of $f_{1}$ or $f_{1}^{\natural}$ with corresponding odd or even $2 \pi$-periodic extensions of the initial data. The other initial condition is the even extension of $u_{0}(x)$.

Theorem 6.7. Let $u_{0} \in L^{2}(0, \pi)$ and $u_{0}^{ \pm} \in L^{2}(0,2 \pi)$ be its even and odd extension. Consider the following solutions to the $2 \pi$-periodic problem for the free space linear Schrödinger equation

- $n(x, t)$ denotes the solution corresponding to initial condition $n_{0}(x)=u_{0}^{+}(x)$,
- $h(x, t)$ denotes the solution corresponding to initial condition $h_{0}(x)=\left(f_{1}+\right.$ $\left.f_{1}^{\natural}\right) * u_{0}^{+}(x)$,
- $v(x, t)$ denotes the solution corresponding to initial condition $v_{0}(x)=\left(f_{1}^{\natural}-\right.$ $\left.f_{1}\right) * u_{0}^{+}(x)$,
- $z(x, t)$ denotes the solution corresponding to initial condition $z_{0}(x)=\left(f_{1}-\right.$ $\left.f_{1}^{\natural}\right) * u_{0}^{-}(x)$,
- $w(x, t)$ denotes the solution corresponding to initial condition $w_{0}(x)=\left(f_{1}+\right.$ $\left.f_{1}^{\natural}\right) * u_{0}^{-}(x)$,
where $f_{1}(x)$ and $f_{1}^{\natural}(x)$ are defined by (6.18) and (6.19) respectively. Then, for all $t \geq 0$ the solution $u(x, t)$ in $L^{2}(0, \pi)$ to the initial boundary value problem (6.12) is given by

$$
\begin{equation*}
u(x, t)=\left\langle u_{0}, \phi_{b}\right\rangle_{L^{2}(0, \pi)} e^{i m_{b}^{2} t} \phi_{b}(x)+n(x, t)-h(x, t)+v(x, t)+z(x, t)+w(x, t) . \tag{6.21}
\end{equation*}
$$

Here, the pair $\left(m_{b}, \phi_{b}(x)\right)$ is given by (6.14).

Proof. Write (6.17) as

$$
u(x, t)=\left\langle u_{0}, \phi_{b}\right\rangle_{L^{2}(0, \pi)} e^{i m_{b}^{2} t} \phi_{b}(x)+U(x, t),
$$

where

$$
\begin{equation*}
U(x, t)=\sum_{j=1}^{\infty}\left\langle u_{0}, \phi_{j}\right\rangle_{L^{2}(0, \pi)} e^{-i j^{2} t} \phi_{j}(x) . \tag{6.22}
\end{equation*}
$$

We show that $U(x, t)=n(x, t)-h(x, t)+v(x, t)+z(x, t)+w(x, t)$.
Step 1. Recall that the elements of the sine $d_{j}(x)$ and cosine $n_{j}(x)$ Fourier bases for $j \in \mathbb{N}$ are given by (6.3) and (6.4) respectively. Note that for the cosine basis we have an extra element for $j=0$, which is $n_{0}(x)=1 / \sqrt{\pi}$. Let $j \in \mathbb{N}$. The eigenfunctions can be written as follows

$$
\begin{equation*}
\phi_{j}(x)=\left(\frac{1-\Lambda_{j}}{2}\right) n_{j}(x)+i\left(\frac{1+\Lambda_{j}}{2}\right) d_{j}(x) . \tag{6.23}
\end{equation*}
$$

Moreover,

$$
\begin{aligned}
& \left\langle u_{0}, \phi_{j}\right\rangle_{L^{2}(0, \pi)}=\left(\frac{1-\overline{\Lambda_{j}}}{2}\right) a_{j}+\left(\frac{1+\overline{\Lambda_{j}}}{2 i}\right) b_{j}, \\
& a_{j}=\int_{0}^{\pi} u_{0}(y) n_{j}(y) d y, \quad b_{j}=\int_{0}^{\pi} u_{0}(y) d_{j}(y) d y .
\end{aligned}
$$

Substituting (6.23) and (6.24) into (6.22), yields

$$
\begin{array}{r}
U(x, t)=\sum_{j=1}^{\infty} e^{-i m_{j} t}\left[\frac{\left(1-\overline{\Lambda_{j}}\right)\left(1-\Lambda_{j}\right)}{4} a_{j} n_{j}(x)+\frac{\left(1-\overline{\Lambda_{j}}\right)\left(1+\Lambda_{j}\right)}{4} i a_{j} d_{j}(x)\right.  \tag{6.25}\\
\left.\frac{\left(1+\overline{\Lambda_{j}}\right)\left(1-\Lambda_{j}\right)}{4 i} b_{j} n_{j}(x)+\frac{\left(1+\overline{\Lambda_{j}}\right)\left(1+\Lambda_{j}\right)}{4} b_{j} d_{j}(x)\right] .
\end{array}
$$

However, since $\left|\Lambda_{j}\right|=1$, we have

$$
\begin{aligned}
& \left(1-\overline{\Lambda_{j}}\right)\left(1-\Lambda_{j}\right)=2\left(1-\operatorname{Re}\left(\Lambda_{j}\right)\right), \quad\left(1-\overline{\Lambda_{j}}\right)\left(1+\Lambda_{j}\right)=2 i \operatorname{Im}\left(\Lambda_{j}\right) \\
& \left(1+\overline{\Lambda_{j}}\right)\left(1-\Lambda_{j}\right)=-2 i \operatorname{Im}\left(\Lambda_{j}\right), \quad\left(1+\overline{\Lambda_{j}}\right)\left(1+\Lambda_{j}\right)=2\left(1+\operatorname{Re}\left(\Lambda_{j}\right)\right) .
\end{aligned}
$$

Using the expression of $\Lambda_{j}$ in (6.15), we get

$$
\operatorname{Re}\left(\Lambda_{j}\right)=\frac{m_{b}^{2}-j^{2}}{m_{b}^{2}+j^{2}}, \quad \operatorname{Im}\left(\Lambda_{j}\right)=\frac{-2 m_{b} j}{m_{b}^{2}+j^{2}}
$$

where $m_{b}=b /(1-b)$ as in (6.14). Therefore, the expression (6.25) for $U(x, t)$ takes the form

$$
\begin{equation*}
U(x, t)=S_{1}(x, t)+S_{2}(x, t)+S_{3}(x, t)+S_{4}(x, t), \tag{6.26}
\end{equation*}
$$

where

$$
\begin{array}{ll}
S_{1}(x, t)=\sum_{j=1}^{\infty} \frac{j^{2}}{m_{b}^{2}+j^{2}} a_{j} e^{-i j^{2} t} n_{j}(x), & S_{2}(x, t)=\sum_{j=1}^{\infty} \frac{m_{b} j}{m_{b}^{2}+j^{2}} a_{j} e^{-i j^{2} t} d_{j}(x) \\
S_{3}(x, t)=\sum_{j=1}^{\infty} \frac{m_{b} j}{m_{b}^{2}+j^{2}} b_{j} e^{-i j^{2} t} n_{j}(x), & S_{4}(x, t)=\sum_{j=1}^{\infty} \frac{m_{b}^{2}}{m_{b}^{2}+j^{2}} b_{j} e^{-i j^{2} t} d_{j}(x) .
\end{array}
$$

In the following steps we analyse each of the sums $S_{i}(x, t), i=1,2,3,4$, and show that they give solutions to specific $2 \pi$-periodic problems.

Step 2. Consider $S_{1}(x, t)$. We have

$$
\begin{aligned}
S_{1}(x, t)= & \frac{1}{\pi} \int_{0}^{\pi} u_{0}(y) d y-\frac{1}{\pi} \int_{0}^{\pi} u_{0}(y) d y+\sum_{j=1}^{\infty} \frac{m_{b}^{2}+j^{2}-m_{b}^{2}}{m_{b}^{2}+j^{2}} a_{j} e^{-i j^{2} t} n_{j}(x) \\
= & \left(\frac{1}{\pi} \int_{0}^{\pi} u_{0}(y) d y+\sum_{j=1}^{\infty} a_{j} e^{-i j^{2} t} n_{j}(x)\right) \\
& \quad-\left(\frac{1}{\pi} \int_{0}^{\pi} u_{0}(y) d y+\sum_{j=1}^{\infty} \frac{m_{b}^{2}}{m_{b}^{2}+j^{2}} a_{j} e^{-i j^{2} t} n_{j}(x)\right) \\
= & n(x, t)-h(x, t),
\end{aligned}
$$

where

$$
\begin{aligned}
& n(x, t)=\frac{1}{\pi} \int_{0}^{\pi} u_{0}(y) d y+\sum_{j=1}^{\infty} a_{j} e^{-i j^{2} t} n_{j}(x) \\
& h(x, t)=\frac{1}{\pi} \int_{0}^{\pi} u_{0}(y) d y+\sum_{j=1}^{\infty} \frac{m_{b}^{2}}{m_{b}^{2}+j^{2}} a_{j} e^{-i j^{2} t} n_{j}(x) .
\end{aligned}
$$

By Lemma 6.2 , we know that $n(x, t)$ is a solution to the $2 \pi$-periodic problem with initial condition $n_{0}(x)=u_{0}^{+}(x)$. Furthermore, $h(x, t)$ is the solution to the $2 \pi-$
periodic problem for the free space linear Schrödinger equation with initial condition $h_{0}(x)=\left(f_{1}+f_{1}^{\natural}\right) * u_{0}^{+}(x)$. Indeed,

$$
\sum_{j \in \mathbb{Z}} \widehat{h_{0}}(j) e^{-i j^{2} t} e_{j}(x)=\sum_{j \in \mathbb{Z}}\left(\widehat{f}_{1}(j)+\widehat{f_{1}^{\natural}}(j)\right) \widehat{u_{0}^{+}}(j) e^{-i j^{2} t} e_{j}(x),
$$

where we have used (3.18). Recall that

$$
\widehat{u_{0}^{+}}(0)=\sqrt{\frac{2}{\pi}} \int_{0}^{\pi} u_{0}(y) d y, \quad \widehat{u_{0}^{+}}(j)=\sqrt{\frac{2}{\pi}} \int_{0}^{\pi} u_{0}(y) \cos (j y) d y, \quad j \neq 0 .
$$

Also from (6.20) we have for all integers $j$,

$$
\widehat{f}_{1}(j)+\widehat{f_{1}^{\natural}}(j)=\frac{m_{b}}{2\left(m_{b}-i j\right)}+\frac{m_{b}}{2\left(m_{b}+i j\right)}=\frac{m_{b}^{2}}{m_{b}^{2}+j^{2}}, \quad j \in \mathbb{Z} .
$$

Therefore,

$$
\begin{aligned}
\sum_{j \in \mathbb{Z}}\left(\widehat{f}_{1}(j)+\widehat{f_{1}^{\natural}}(j)\right) \widehat{u_{0}^{+}}(j) e^{-i j^{2} t} e_{j}(x) & =\frac{\widehat{u_{0}^{+}}(0)\left(\widehat{f}_{1}(0)+\widehat{f}_{2}(0)\right)}{\sqrt{2 \pi}} \\
& +\sum_{j=1}^{\infty} \frac{m_{b}^{2}}{m_{b}^{2}+j^{2}} a_{j} e^{-i j^{2} t}\left[\frac{e^{i j x}+e^{-i j x}}{\sqrt{2 \pi}}\right] \\
& =\frac{1}{\pi} \int_{0}^{\pi} u_{0}(y) d y+\sum_{j=1}^{\infty} \frac{m_{b}^{2}}{m_{b}^{2}+j^{2}} a_{j} e^{-i j^{2} t} n_{j}(x) \\
& =h(x, t) .
\end{aligned}
$$

Step 3. Now, we consider $S_{2}(x, t)$. Let $v(x, t)$ be the solution to the $2 \pi$-periodic problem for the free space linear Schrödinger equation with initial condition $v_{0}(x)=$ $\left(f_{1}^{\natural}-f_{1}\right) * u_{0}^{+}(x)$. Then, we know that

$$
v(x, t)=\sum_{j \in \mathbb{Z}} \widehat{v_{0}}(j) e^{-i m_{j} t} e_{j}(x)=\sum_{j \in \mathbb{Z}}\left(\widehat{f_{1}^{\natural}}(j)-\widehat{f_{1}}(j)\right) \widehat{u_{0}^{+}}(j) e^{-i j^{2} t} e_{j}(x) .
$$

Now,

$$
\widehat{f_{1}^{\natural}}(j)-\widehat{f}_{1}(j)=\frac{m_{b}}{2\left(m_{b}+i j\right)}-\frac{m_{b}}{2\left(m_{b}-i j\right)}=-i \frac{m_{b} j}{m_{b}^{2}+j^{2}}, \quad j \in \mathbb{Z} .
$$

Thus, since $\widehat{f_{1}^{\natural}}(0)-\widehat{f}_{1}(0)=0$, we have

$$
\begin{aligned}
v(x, t) & =\sum_{j=1}^{\infty}-i \frac{m_{b} j}{m_{b}^{2}+j^{2}} a_{j} e^{-i j^{2} t}\left[\frac{e^{i j x}-e^{-i j x}}{\sqrt{2 \pi}}\right] \\
& =\sum_{j=1}^{\infty}-i^{2} \frac{m_{b} j}{m_{b}^{2}+j^{2}} a_{j} e^{-i j^{2} t} d_{j}(x) \\
& =S_{2}(x, t)
\end{aligned}
$$

Step 4. For $S_{3}(x, t)$, we consider $z(x, t)$ to be the solution of a $2 \pi$-periodic problem for the free space linear Schrödinger equation with initial condition $z_{0}(x)=$ $\left(f_{1}-f_{1}^{\natural}\right) * u_{0}^{-}(x)$. Then,

$$
z(x, t)=\sum_{j \in \mathbb{Z}} \widehat{z_{0}}(j) e^{-i j^{2} t} e_{j}(x)=\sum_{j \in \mathbb{Z}}\left(\widehat{f}_{1}(j)-\widehat{f_{1}^{\natural}}(j)\right) \widehat{u_{0}^{-}}(j) e^{-i j^{2} t} e_{j}(x) .
$$

Recall that for all $j \in \mathbb{Z}$,

$$
\widehat{u_{0}^{-}}(j)=-i \sqrt{\frac{2}{\pi}} \int_{0}^{\pi} u_{0}(y) \sin (j y) d y .
$$

Hence, $\widehat{u_{0}^{-}}(-j)=-\widehat{u_{0}^{-}}(j)$. Also, from (6.20) we have

$$
\widehat{f}_{1}(j)-\widehat{f_{1}^{\natural}}(j)=\frac{m_{b}}{2\left(m_{b}-i j\right)}-\frac{m_{b}}{2\left(m_{b}+i j\right)}=i \frac{m_{b} j}{m_{b}^{2}+j^{2}}, \quad j \in \mathbb{Z} .
$$

Therefore, since $\widehat{f}_{1}(0)-\widehat{f_{1}^{\natural}}(0)=0$ and $\widehat{u_{0}^{-}}(0)=0$, we have

$$
\begin{aligned}
z(x, t) & =\sum_{j=1}^{\infty}-i^{2} \frac{m_{b} j}{m_{b}^{2}+j^{2}} b_{j} e^{-i j^{2} t}\left[\frac{e^{i j x}+e^{-i j x}}{\sqrt{2 \pi}}\right] \\
& =\sum_{j=1}^{\infty} \frac{m_{b} j}{m_{b}^{2}+j^{2}} b_{j} e^{-i j^{2} t} n_{j}(x) \\
& =S_{3}(x, t) .
\end{aligned}
$$

Step 5. Finally consider $S_{4}(x, t)$. Let $w(x, t)$ be the solution to a $2 \pi$-periodic problem for the free space linear Schrödinger equation with initial condition $w_{0}(x)=$
$\left(f_{1}+f_{1}^{\natural}\right) * u_{0}^{-}(x)$. Then,

$$
w(x, t)=\sum_{j \in \mathbb{Z}} \widehat{w_{0}}(j) e^{-i j^{2} t} e_{j}(x)=\sum_{j \in \mathbb{Z}}\left(\widehat{f_{1}}(j)+\widehat{f_{1}^{\natural}}(j)\right) \widehat{u_{0}^{-}}(j) e^{-i j^{2} t} e_{j}(x) .
$$

Thus, according to Step 2 and Step 4, we have

$$
\begin{aligned}
w(x, t) & =\sum_{j=1}^{\infty}-i \frac{m_{b}^{2}}{m_{b}^{2}+j^{2}} b_{j} e^{-i j^{2} t}\left[\frac{e^{i j x}-e^{-i j x}}{\sqrt{2 \pi}}\right] \\
& =\sum_{j=1}^{\infty}-i^{2} \frac{m_{b}^{2}}{m_{b}^{2}+j^{2}} b_{j} e^{-i j^{2} t} d_{j}(x) \\
& =S_{4}(x, t) .
\end{aligned}
$$

Each of the solutions $n(x, t), h(x, t), v(x, t), z(x, t), w(x, t)$ is an even or odd function with respect to $\pi$, since in each case they come from an even or odd initial function. Consequently, according to Lemmas 6.1 and 6.2 , they also represent solutions to Dirichlet and Neumann problems for the free linear Schrödinger equation with initial conditions the restrictions on $[0, \pi]$ of the initial conditions $n_{0}, h_{0}, v_{0}$, $z_{0}, w_{0}$. Therefore, we could have characterised the solution representation (6.17) in connection with Neumann or Dirichlet problems on $[0, \pi]$. However, our choice on the periodic problems on $[0,2 \pi]$ corresponds to the revival property, which as with Neumann or Dirichlet boundary conditions, can be deduced by applying the revival operator on a specific transformation of the initial condition extended on $[0,2 \pi]$. Thus, making the action of the periodic translation operator possible.

A further point to highlight before we proceed to a revival representation is the following.

Remark 6.8. Due to the linearity of the FSLS equation, the representation (6.22) for $U(x, t)$ corresponds to the solution of the $2 \pi$-periodic problem with initial condition

$$
U_{0}(x)=n_{0}(x)-h_{0}(x)+v_{0}(x)+z_{0}(x)+w_{0}(x) .
$$

In the context of Theorem 6.7 and by the distributive property of the convolution,
we have that

$$
U_{0}(x)=u_{0}^{+}(x)+2 f_{1} *\left(u_{0}^{-}-u_{0}^{+}\right)(x) .
$$

Therefore, at any fixed $t \geq 0, U(x, t)$ admits the $L^{2}(0,2 \pi)$ representation

$$
\begin{equation*}
U(x, t)=\sum_{j \in \mathbb{Z}} \widehat{u_{0}^{+}}(j) e^{-i j^{2} t} e_{j}(x)+\sum_{j \in \mathbb{Z}}\left\langle 2 f_{1} *\left(u_{0}^{-}-u_{0}^{+}\right), e_{j}\right\rangle e^{-i j^{2} t} e_{j}(x) . \tag{6.27}
\end{equation*}
$$

The next corollary characterises the revivals in the Robin problem (6.12). The proof follows from Theorem 6.7 and Remark 6.8

Corollary 6.9. Let $u_{0} \in L^{2}(0, \pi)$ and $u_{0}^{ \pm} \in L^{2}(0,2 \pi)$ be the corresponding even/odd extension. For any fixed rational time $t=2 \pi \frac{p}{q}$, where $p$ and $q$ are co-prime integers, the solution of the time evolution problem (6.12) is given in $L^{2}(0, \pi)$ by the restriction on $[0, \pi]$ of the representation

$$
\begin{align*}
u\left(x, 2 \pi \frac{p}{q}\right)= & \left\langle u_{0}, \phi_{b}\right\rangle_{L^{2}(0, \pi)} e^{2 \pi m_{b}^{2} \frac{p_{q}}{}} \phi_{b}(x)+\mathcal{R}_{2}(p, q) u_{0}^{+}(x)  \tag{6.28}\\
& \left.+\mathcal{R}_{2}(p, q)\left[2 f_{1} *\left(u_{0}^{-}-u_{0}^{+}\right)\right)(x)\right] .
\end{align*}
$$

We distinguish the three components on the right hand side of (6.28).

1. The first and third term are continuous on $x \in[0, \pi]$. For the first term this obvious. The third term is the revival of a $2 \pi$-periodic continuous, and thus continuous on $[0, \pi]$. Indeed, first note that the translation of a continuous $2 \pi$-periodic function on $\mathbb{R}$ would be continuous on $[0,2 \pi]$ as well, and we know that the periodic convolution is a $2 \pi$-periodic continuous function (see Lemma 3.15).
2. The second term is the revival of the (even extension of the) given initial condition.

As a consequence of the representation (6.28), we conclude that (6.12) exhibits the weak form of revivals. The weak revival effect observed in equation (6.12) is manifested as a perturbation by a continuous function of the Neumann revival ( $2 \pi-$ periodic problem with even initial condition), which is a pure revival effect. While the solution is not simply a linear combination of translated copies of the initial condition, and thus not a pure revival, the second term in (6.28) ensures that the
functional class of the initial condition is preserved at rational times. Recall that the implications of the weak revival are the same as those of the pure revival described in Remark 2.9. In particular, whenever $u_{0}$ has a finite number of jump discontinuities, then the same will be true for the solution at rational times. Then, the dichotomy in the behaviour of the solution between rational and irrational times will still be present in the weak revival regime. In fact, if $t / 2 \pi$ is an irrational number, then following Remark 6.8 the solution has the representation

$$
\begin{aligned}
u(x, t)= & \left\langle u_{0}, \phi_{b}\right\rangle_{L^{2}(0, \pi)} e^{i m_{b}^{2} t} \phi_{b}(x)+\sum_{j \in \mathbb{Z}} \widehat{u_{0}^{+}}(j) e^{-i j^{2} t} e_{j}(x) \\
& +\sum_{j \in \mathbb{Z}}\left\langle 2 f_{1} *\left(u_{0}^{-}-u_{0}^{+}\right), e_{j}\right) e^{-i j^{2} t} e_{j}(x)
\end{aligned}
$$

Hence, the first term is again obviously continuous on $[0, \pi]$. The second and third terms will be continuous on $[0,2 \pi]$ due to Oskolkov's result on the periodic problem, see Theorem 2.12. Thus continuous on $[0, \pi]$. Additionally, with regards to the third term, at all times the series represents the Fourier expansion of a $2 \pi$-periodic smooth function in the real line.

In the final section of Appendix F, we provide numerical evidence which illustrates weak revivals and non-revivals in the time evolution problem (6.12). Finally, as we shall see in the next chapter, the weak revival will appear again. It will be the main revival phenomenon in the context of second-order in time evolution problems with periodic boundary conditions.

## Chapter 7

## Functional Calculus for Revivals and Further Applications

We now give an abstract treatment of the revival phenomenon and examine some applications of this scheme. In the first section we introduce a functional calculus on a non-self-adjoint setting and develop a transfer principle which enables a derivation of various properties of the non-self-adjoint case from the periodic self-adjoint model. In Section 7.2, we then generalise the classical revival statement, Theorem 2.8. The actual application, in Section 7.3, is a family of second-order in time evolution problems, including the wave and the bi-harmonic wave equation under periodic boundary conditions. We show that these periodic time evolution problems exhibit weak revivals at rational times. This is in contrast with the pure revival effect in first-order in time evolution problems. The main results, Lemma 7.5, Lemma 7.6, Proposition 7.8 and Corollary 7.9 appear to be new. We are preparing a manuscript to report on these findings.

### 7.1 Functional Calculus of a Non-Self-adjoint Operator

A functional calculus of a linear operator refers to the procedure of constructing functions of this operator, which are again linear operators. In our case, we introduce a functional calculus for the linear differential operator denoted by $L_{h, \theta}$ and defined in $L^{2}(0,2 \pi)$ as follows.

Let $h>0$ and $\theta \in[0,1)$, and consider the linear operator $L_{h, \theta}$ by

$$
\begin{equation*}
L_{h, \theta}=-i \partial_{x}+i \ln h: \mathrm{D}\left(L_{h, \theta}\right) \rightarrow L^{2}(0,2 \pi) \tag{7.1}
\end{equation*}
$$

on the domain

$$
\begin{equation*}
\mathrm{D}\left(L_{h, \theta}\right)=\left\{\phi \in H^{1}(0,2 \pi) ;\left(h e^{i \theta}\right)^{2 \pi} \phi(0)=\phi(2 \pi)\right\}, \tag{7.2}
\end{equation*}
$$

where $H^{1}(0,2 \pi)$ denotes the usual Sobolev space of order one on the interval $(0,2 \pi)$. We refer to Appendix D for the definition of Sobolev spaces and some of their properties.

For specific values of the parameters $h$ and $\theta$, powers of the model operator $L_{h, \theta}$ incorporate (extensions of) some of the differential operators encountered in the time evolution problems from previous chapters. For example, when $h=1$, the time evolution of $L_{1, \theta}^{2}$ or $L_{1, \theta}^{3}$ corresponds to the free space linear Schrödinger equation or the Airy PDE, respectively, under periodic $(\theta=0)$ or quasi-periodic boundary conditions $(\theta \neq 0)$. On the other hand, when $h \neq 1$, then the theory developed below could possibly support the examination of the revival effect in time evolution problems where the underlying spatial differential operator is non-self-adjoint. In the second subsection of Section 9.2, we state an appropriate time evolution problem to be considered in the context of the operator $L_{h, \theta}^{2}$.

With regards to the revival property and within the framework of this and the next section on the differential operator $L_{h, \theta}$, of main interest will be the even-order poly-harmonic wave equation under periodic boundary conditions

$$
\begin{gather*}
\partial_{t}^{2} u(x, t)=-\left(-i \partial_{x}\right)^{2 r} u(x, t), \quad u(x, 0)=f(x), \quad \partial_{t} u(x, 0)=g(x),  \tag{7.3}\\
\partial_{x}^{m} u(0, t)=\partial_{x}^{m} u(2 \pi, t), \quad m=0,1,2, \ldots, 2 r-1,
\end{gather*}
$$

on $[0,2 \pi]$ and with $r \geq 1$ an integer. Notice that when $r=1$, the time evolution problem is posed for the wave equation, whereas for $r=2$ for the bi-harmonic wave equation. As it will be shown, in contrast to first-order in time evolution problems with periodic boundary conditions, the solution to (7.3) at rational times will exhibit weak revivals, enriching the class of linear PDEs for which this perturbed
pure revival effect holds.
Although the applications regarding the revival effect occur when $h=1$, this follows from treating the more general case $h \neq 1$. Therefore, our main focus will be on $L_{h, \theta}$, which, in general, is a non-self-adjoint operator with its adjoint given by the statement below.

Proposition 7.1. The adjoint of the operator $L_{h, \theta}$ is given by

$$
\begin{equation*}
L_{h, \theta}^{*}=-i \partial_{x}-i \ln h: D\left(L_{h, \theta}^{*}\right) \rightarrow L^{2}(0,2 \pi) \tag{7.4}
\end{equation*}
$$

on the domain

$$
\begin{equation*}
D\left(L_{h, \theta}^{*}\right)=\left\{\psi \in H^{1}(0,2 \pi) ; h^{-2 \pi} e^{2 \pi i \theta} \psi(0)=\psi(2 \pi)\right\} . \tag{7.5}
\end{equation*}
$$

Proof. Let $\psi \in \mathrm{D}\left(L_{h, \theta}^{*}\right) \subset L^{2}(0,2 \pi)$. Then, there is $g \in L^{2}(0,2 \pi)$ such that $g=$ $L_{h, \theta}^{*} \psi$ and $\left\langle L_{h, \theta} \phi, \psi\right\rangle=\langle\phi, g\rangle$, for all $\phi \in \mathrm{D}\left(L_{h, \theta}^{*}\right)$. In particular, for any $\phi \in \mathrm{D}\left(L_{h, \theta}^{*}\right)$, using the definition of $L_{h, \theta}$, we have that

$$
\int_{0}^{2 \pi} \overline{\psi(x)} \partial_{x} \phi(x) d x=\int_{0}^{2 \pi} \overline{(-i g(x)+\ln h) \psi(x)} \phi(x) d x,
$$

or equivalently

$$
\int_{0}^{2 \pi} \psi(x) \overline{\partial_{x} \phi(x)} d x=\int_{0}^{2 \pi}(-i g(x)+\ln h) \psi(x) \overline{\phi(x)} d x .
$$

Now, since $C_{c}^{\infty}(0,2 \pi) \subset \mathrm{D}\left(L_{h, \theta}\right)$ and $(-i g+\ln h) \psi \in L^{2}(0,2 \pi)$, it follows that $\psi$ has first weak derivative $\partial_{x} \psi$ in $L^{2}(0,2 \pi)$, and thus $\psi \in H^{1}(0,2 \pi)$. Moreover, we have that

$$
\partial_{x} \psi(x)=-(-i g(x)+\ln h) \psi(x),
$$

which implies that the action of the adjoint $L_{h, \theta}^{*}$ is given by

$$
L_{h, \theta}^{*} \psi(x)=g(x)=\left(-i \partial_{x}-i \ln (h)\right) \psi(x),
$$

for any $\psi \in \mathrm{D}\left(L_{h, \theta}^{*}\right)$.
To specify the domain of the adjoint, let $\psi \in H^{1}(0,2 \pi)$. Then for all $\phi \in \mathrm{D}\left(L_{h, \theta}\right)$,
integration by parts implies that

$$
\left\langle L_{h, \theta} \phi, \psi\right\rangle=-i[\phi(x) \overline{\psi(x)}]_{0}^{2 \pi}+\int_{0}^{2 \pi} \phi(x) \overline{\left(-i \partial_{x}-i \ln h\right) \psi(x)} d x .
$$

Now, if $\psi$ satisfies the boundary condition $h^{-2 \pi} e^{2 \pi i \theta} \psi(0)=\psi(2 \pi)$, then the boundary term above equals zero and thus $\left\langle L_{h, \theta} \phi, \psi\right\rangle=\left\langle\phi, L_{h, \theta}^{*} \psi\right\rangle$. Therefore, $\psi$ belongs in $\mathrm{D}\left(L_{h, \theta}^{*}\right)$. On the other hand, if $\psi \in \mathrm{D}\left(L_{h, \theta}^{*}\right)$ then the boundary term turns to be zero, that is

$$
[\phi(x) \overline{\psi(x)}]_{0}^{2 \pi}=0
$$

Equivalently,

$$
\left(\overline{\psi(0)}-h^{2 \pi} e^{2 \pi i \theta} \overline{\psi(2 \pi)}\right) \phi(0)=0, \quad \forall \phi \in \mathrm{D}\left(L_{h, \theta}\right),
$$

or $h^{-2 \pi} e^{2 \pi i \theta} \psi(0)=\psi(2 \pi)$. We conclude that the domain is given as in the statement.

To motivate the functional calculus of $L_{h, \theta}$, let us first briefly review the case of $L_{1,0}$. Notice that $L_{1,0}$ is self-adjoint and the boundary condition in the domain of definition becomes periodic, $\phi(0)=\phi(2 \pi)$.

If we denote by $C_{b}(\mathbb{R})$ the space of continuous and bounded complex valued functions then a functional calculus in $L^{2}(0,2 \pi)$ of $L_{1,0}$ can be defined from the eigenvalues $\lambda_{j}=j \in \mathbb{Z}$ and the normalised eigenfunctions $e_{j}(x)=e^{i j x} / \sqrt{2 \pi}$, which are the elements of the complex Fourier basis. In particular, for any function $\sigma \in$ $C_{b}(0,2 \pi), \sigma\left(L_{1,0}\right)$ is defined as the linear bounded operator in $L^{2}(0,2 \pi)$ by the formula

$$
\begin{equation*}
\sigma\left(L_{1,0}\right) f=\sum_{j \in \mathbb{Z}} \widehat{f}(j) \sigma(j) e_{j}, \quad f \in L^{2}(0,2 \pi) . \tag{7.6}
\end{equation*}
$$

Note that in the literature it is common to use the notation $\sigma(L)$ for the spectrum of a linear operator $L$. But here and everywhere below $\sigma$ will always be a function in $C_{b}(\mathbb{R})$.

Now, since $\{\sigma(j)\}_{j \in \mathbb{Z}}$ is a bounded complex valued sequence, then $\sigma\left(L_{1,0}\right)$ coincides with the Fourier multiplier associated to the sequence $\sigma=\sigma(j)$. For a concise
introduction to Fourier multipliers in the periodic setting see [54] and for extensions of the functional calculus (7.6) within the framework of periodic distributions see [55].

We now mention a couple of examples. Fix integer $r \geq 1$ and a real parameter $t \geq 0$, and consider the functions $\cos \left(x^{r} t\right)$ and $\sin \left(x^{r} t\right)$ for $x \in \mathbb{R}$. Then, from (7.6) we have the representations

$$
\cos \left(L_{1,0}^{r} t\right) f=\sum_{j \in \mathbb{Z}} \widehat{f}(j) \cos \left(j^{r} t\right) e_{j}, \quad \sin \left(L_{1,0}^{r} t\right) f=\sum_{j \in \mathbb{Z}} \widehat{f}(j) \sin \left(j^{r} t\right) e_{j},
$$

for all $f \in L^{2}(0,2 \pi)$. As we shall see in Section 7.3, linear combinations of these operators will be solution representations of the periodic poly-harmonic wave equation (7.3). The complex exponential function of $L_{1,0}$ provides another interesting example which will be considered in the next chapter.

Going back to the operator $L_{h, \theta}$, in order to define $\sigma\left(L_{h, \theta}\right)$ we follow the definition of Fourier multipliers (7.6). Thus, we determine the eigenvalues and the eigenfunctions of the operator $L_{h, \theta}$. For this, we solve the eigenvalue problem

$$
\begin{equation*}
-i \phi^{\prime}(x)+i \ln (h) \phi(x)=\lambda \phi(x), \quad\left(h e^{i \theta}\right)^{2 \pi} \phi(0)=\phi(2 \pi), \tag{7.7}
\end{equation*}
$$

on $[0,2 \pi]$. The solution to (7.7) is given by

$$
\begin{equation*}
\lambda_{j}=j+\theta, \quad \phi_{j}(x)=h^{x} e^{i \theta x} e_{j}(x), \quad j \in \mathbb{Z} \tag{7.8}
\end{equation*}
$$

From the expression of the eigenfunctions, we deduce that they form a Riesz basis of $L^{2}(0,2 \pi)$, since $h^{-x} e^{-i \theta x} \phi_{j}(x)=e_{j}(x)$. Therefore, any $f \in L^{2}(0,2 \pi)$ admits a generalised Fourier series in terms of the eigenfunctions

$$
f(x)=\sum_{j \in \mathbb{Z}}\left\langle f, \psi_{j}\right\rangle \phi_{j}(x),
$$

where the series converges in the mean and $\psi_{j}$ are the eigenfunctions of the adjoint operator $L_{h, \theta}^{*}$. These are given by

$$
\begin{equation*}
\psi_{j}(x)=h^{-x} e^{i \theta x} e_{j}(x), \quad j \in \mathbb{Z} \tag{7.9}
\end{equation*}
$$

and when paired with $\phi_{j}$, they form a biorthogonal system

$$
\left\langle\phi_{j}, \psi_{\ell}\right\rangle= \begin{cases}1, & j=\ell \\ 0, & j \neq \ell\end{cases}
$$

Now, if $\sigma \in C_{b}(\mathbb{R})$ and we take a function $f \in L^{2}(0,2 \pi)$, focusing on the generalised Fourier coefficients $\left\langle f, \psi_{j}\right\rangle \sigma\left(\lambda_{j}\right)$ and since $\sigma$ is bounded by a positive constant $C$, we have

$$
\begin{equation*}
\sum_{j \in \mathbb{Z}}\left|\left\langle f, \psi_{j}\right\rangle \sigma\left(\lambda_{j}\right)\right|^{2} \leq C^{2} \sum_{j \in \mathbb{Z}}\left|\left\langle f, \psi_{j}\right\rangle\right|^{2} \leq C^{2}\|f\|^{2}<\infty . \tag{7.10}
\end{equation*}
$$

Due to (7.10) and because $\left\{\phi_{j}\right\}_{j \in \mathbb{Z}}$ forms a Riesz basis in $L^{2}(0,2 \pi)$, from Lemma B.4(iii) follows that we can define a map denoted by $\sigma\left(L_{h, \theta}\right)$ and which takes any $f \in L^{2}(0,2 \pi)$ to

$$
\sigma\left(L_{h, \theta}\right) f=\sum_{j \in \mathbb{Z}}\left\langle f, \psi_{j}\right\rangle \sigma\left(\lambda_{j}\right) \phi_{j} \in L^{2}(0,2 \pi) .
$$

Proposition 7.2. Let $\sigma \in C_{b}(\mathbb{R})$. Consider the map $\sigma\left(L_{h, \theta}\right): L^{2}(0,2 \pi) \rightarrow L^{2}(0,2 \pi)$ defined by

$$
\begin{equation*}
\sigma\left(L_{h, \theta}\right) f=\sum_{j \in \mathbb{Z}}\left\langle f, \psi_{j}\right\rangle \sigma\left(\lambda_{j}\right) \phi_{j} . \tag{7.11}
\end{equation*}
$$

Then, we have the following.
(i) $\sigma\left(L_{h, \theta}\right)$ is a linear bounded operator in $L^{2}(0,2 \pi)$.
(ii) Each $\sigma\left(\lambda_{j}\right)$ is an eigenvalue of $\sigma\left(L_{h, \theta}\right)$ with associated eigenfunction $\phi_{j}$.

Proof. Linearity follows from the linearity of the inner product in its first argument. Now, let $f \in L^{2}(0,2 \pi)$. Then, from the hypothesis on $\sigma$ and Lemma B.4-(ii), there is a positive constant $C$ such that (7.10) holds. This shows $(i)$. For (ii), let $\ell$ be an arbitrary integer. Then, by biorthogonality, we have

$$
\sigma\left(L_{h, \theta}\right) \phi_{\ell}=\sum_{j \in \mathbb{Z}}\left\langle\phi_{\ell}, \psi_{j}\right\rangle \sigma\left(\lambda_{j}\right) \phi_{j}=\sigma\left(\lambda_{\ell}\right) \phi_{\ell} .
$$

Expression (7.11) describes the functional calculus generated by the linear op-
erator $L_{h, \theta}$ on the space of continuous and bounded functions and it will be used to formulate a revival functional calculus in Section 7.2. Before we proceed, we should mention that recently in [56], Ruzhanski and Tokmagambetov considered an arbitrary linear differential operator accompanied by fixed boundary conditions in a bounded domain $\Omega$ of $\mathbb{R}^{d}$. Under the assumption that the given operator has discrete spectrum and its eigenfunctions consist a Riesz basis of $L^{2}(\Omega)$ and do not have zeros in $\Omega$ (this was relaxed in [57] by the same authors), they developed a functional calculus of pseudo-differential operators generated by the given operator. In our case, the operator $L_{h, \theta}$ provides another example which falls into this framework and the functional calculus (7.11) is a special occurrence, which holds in $L^{2}(0,2 \pi)$, of their "L-quantization" representation [56, Theorem 9.2, Equation (9.2)].

We close this section with the following lemma which can be viewed as a transfer principle between the functional calculus of $L_{h, \theta}$ and the functional calculus of $L_{1,0}$. Meaning that, when proving a particular property for $L_{1,0}$, using the lemma below, we extend it to the case of $L_{h, \theta}$ for general $h$ and $\theta$. As an application of this procedure, in the next section we establish an abstract framework for the revival property.

Lemma 7.3. Fix $h>0$, with $h \neq 1$ and $\theta \in(0,1)$. Let $\sigma \in C_{b}(\mathbb{R})$ and denote by $\sigma_{\theta}$ the function $\sigma(x+\theta)$. Then, for any $f \in L^{2}(0,2 \pi)$

$$
\begin{equation*}
\sigma\left(L_{h, \theta}\right) f(x)=h^{x} e^{i \theta x} \sigma_{\theta}\left(L_{1,0}\right)\left(h^{-x} e^{-i \theta x} f(x)\right) \tag{7.12}
\end{equation*}
$$

Proof. Let $f \in L^{2}(0,2 \pi)$. Then, by (7.11)

$$
\sigma\left(L_{h, \theta}\right) f(x)=h^{x} e^{i \theta x} \sum_{j \in \mathbb{Z}} \int_{0}^{2 \pi} h^{-y} e^{-i \theta y} f(y) \overline{e_{j}(y)} d y \sigma(j+\theta) e_{j}(x) .
$$

Because $\sigma \in C_{b}(\mathbb{R})$, the function $\sigma_{\theta}=\sigma(x+\theta)$ also belongs in $C_{b}(\mathbb{R})$. Therefore, if we denote by $F(x)=h^{-x} e^{-i \theta x} f(x)$, then

$$
\begin{aligned}
\sigma\left(L_{h, \theta}\right) f(x) & =h^{x} e^{i \theta x} \sum_{j \in \mathbb{Z}}\left\langle F, e_{j}\right\rangle \sigma_{\theta}(j) e_{j}(x) \\
& =h^{x} e^{i \theta x} \sigma_{\theta}\left(L_{1,0}\right)\left(h^{-x} e^{-i \theta x} f(x)\right) .
\end{aligned}
$$

Remark 7.4. Lemma 7.3 implies the following. There exist a bounded linear operator $S: L^{2} \rightarrow L^{2}(0,2 \pi)$ given by $S f(x)=h^{x} e^{i \theta x} f(x)$, with bounded inverse $S^{-1} f(x)=h^{-x} e^{-i \theta x} f(x)$, such that the operator $\sigma\left(L_{h, \theta}\right)$ is given by

$$
\sigma\left(L_{h, \theta}\right)=S \sigma_{\theta}\left(L_{1,0}\right) S^{-1}
$$

This observation might allow further generalisations of the revival functional calculus presented next.

### 7.2 Functional Calculus for Revivals

In this section, we generalise Theorem 2.8 by showing that a special class of functions of the operator $L_{h, \theta}$ can be decomposed into finite linear copies of the operator

$$
\begin{equation*}
h^{-s} \exp \left(-i s L_{h, \theta}\right): L^{2}(0,2 \pi) \rightarrow L^{2}(0,2 \pi), \tag{7.13}
\end{equation*}
$$

where $s$ is a real number. The operator (7.13) corresponds to the quasi-periodic analogue of the periodic translation operator $\mathcal{T}_{s}$. More specifically, through the functional calculus of Proposition 7.2, we have that

$$
\exp \left(-i s L_{h, \theta}\right) f=\sum_{j \in \mathbb{Z}}\left\langle f, \psi_{j}\right\rangle e^{-i s(j+\theta)} \phi_{j}
$$

for any $f \in L^{2}(0,2 \pi)$. On the other hand, if we regard the quasi-periodic extension of an $L^{2}(0,2 \pi)$ function $f$ by

$$
\tilde{f}(x)=\left(h e^{i \theta}\right)^{m} f(x-2 \pi m), \quad 2 \pi m \leq x<2 \pi(m+1), \quad m \in \mathbb{Z},
$$

then the quasi-periodic translation of $f$ by a real number $s$, denoted by $f_{s}$, is

$$
f_{s}=\tilde{f}(x-s) .
$$

From the generalised Fourier coefficients of $f_{s}$ with respect to the adjoint eigenfunctions $\psi_{j}$, we find by a similar argument as in Lemma 2.7, that

$$
\left\langle f_{s}, \psi_{j}\right\rangle=h^{-s} e^{-i(j+\theta) s}\left\langle f, \psi_{j}\right\rangle
$$

which implies that

$$
f_{s}=h^{-s} \exp \left(-i s L_{h, \theta}\right) f
$$

To motivate the idea behind the main lemma below, we look at Theorem 2.8. There, we showed that the first-order in time problem (2.9) under periodic boundary conditions, exhibits pure revivals at rational times $t=2 \pi \frac{p}{q}$. In particular, translating the revival representation (2.14), in terms of complex exponential functions of the operator $L_{1,0}$ we have that

$$
\begin{equation*}
\sigma\left(L_{1,0}\right) f(x)=\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} \sigma(m) e_{m}\left(\frac{2 \pi k}{q}\right) \exp \left(-i \frac{2 \pi k}{q} L_{1,0}\right) f(x) \tag{7.14}
\end{equation*}
$$

with $\sigma(x)=e^{-i P(x) \frac{2 \pi p}{q}}$, where $P$ is a polynomial with integer coefficients and integer order $n \geq 2$ and $x \in \mathbb{R}$. One of the key features that lead to (7.14) was that for $m$ and $j$ integers such that $m \equiv j \bmod q$, then $\sigma(m)=\sigma(j)$. Based on this observation, we establish the following generalisation of Theorem 2.8. The proof consists of two steps. First, we extend the revival formula (7.14) in the context of the operator $L_{1,0}$ and through Lemma 7.3 we pass to the case of $L_{h, \theta}$. The following is one of the main results of this chapter.

Lemma 7.5. Fix $h>0, \theta \in[0,1)$ and consider the linear differential operator $L_{h, \theta}$ defined by (7.1) and (7.2). Let $s \in \mathbb{R}$ and $q$ be a positive integer. Consider a function $\sigma: \mathbb{R} \rightarrow \mathbb{C}$ in $C_{b}(\mathbb{R})$ satisfying the property

$$
\begin{equation*}
\sigma(m)=e^{i s(m-j)} \sigma(j), \quad \text { whenever } m \equiv j \bmod q, \quad m, j \in \mathbb{Z} \tag{7.15}
\end{equation*}
$$

Then, for any $f \in L^{2}(0,2 \pi)$, we have

$$
\begin{equation*}
\sigma\left(L_{h, \theta}\right) f(x)=\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \Gamma_{\theta}(k) h^{-\left(\frac{2 \pi k}{q}-s\right)} \exp \left(-i\left(\frac{2 \pi k}{q}-s\right) L_{h, \theta}\right) f(x), \tag{7.16}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma_{\theta}(k)=\sum_{m=0}^{q-1} \sigma(m+\theta) \phi_{m}\left(\frac{2 \pi k}{q}-s\right) . \tag{7.17}
\end{equation*}
$$

Proof. Step 1. Let $h=1$ and $\theta=0$. Recall that in this case the eigenpairs of the operator $L_{1,0}$ are $\left(j, e_{j}(x)\right)$ where $j$ runs over the integers and $e_{j}(x)=e^{i j x} / \sqrt{2 \pi}$ is the Fourier basis. By definition of the functional calculus for the operator $L_{1,0}$ we know that for any $f \in L^{2}(0,2 \pi)$

$$
\sigma\left(L_{1,0}\right) f(x)=\sum_{j \in \mathbb{Z}} \widehat{f}(j) \sigma(j) e_{j}(x),
$$

where equality is in $L^{2}(0,2 \pi)$.
Note that for any $s \in \mathbb{R}$,

$$
\exp \left(-i s L_{1,0}\right) f(x)=\sum_{j} \widehat{f}(j) e^{-i j s} e_{j}(x)=\mathcal{T}_{s} f(x)
$$

where $\mathcal{T}_{s}$ is the periodic translation operator. Therefore, the right hand side of (7.16) in this case corresponds to the representation

$$
\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \Gamma_{0}(k) \mathcal{T}_{\frac{2 \pi k}{q}-s} f(x), \quad \Gamma_{0}(k)=\sum_{m=0}^{q-1} \sigma(m) e_{m}\left(\frac{2 \pi k}{q}-s\right) .
$$

Calculating the Fourier coefficients of this representation, we have

$$
\begin{aligned}
\left\langle\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \Gamma_{0}(k) \mathcal{T}_{\frac{2 \pi k}{q}-s} f, e_{j}\right\rangle & =\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \Gamma_{0}(k) e^{-i j\left(\frac{2 \pi k}{q}-s\right)} \widehat{f}(j) \\
& =\frac{\widehat{f}(j)}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} e^{-i s(m-j)} \sigma(m) e^{i(m-j) \frac{2 \pi k}{q}} \\
& =\frac{\widehat{f}(j)}{q} \sum_{m=0}^{q-1} e^{-i s(m-j)} \sigma(m) \sum_{k=0}^{q-1} e^{i(m-j) \frac{2 \pi k}{q}} .
\end{aligned}
$$

By the hypothesis (7.15) on the function $\sigma$ and since

$$
\sum_{k=0}^{q-1} e^{i(m-j) \frac{2 \pi k}{q}}= \begin{cases}0, & m \not \equiv j \bmod q \\ q, & m \equiv j \bmod q\end{cases}
$$

we conclude that

$$
\left\langle\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \Gamma_{0}(k) \mathcal{T}_{\frac{2 \pi k}{q}-s} f, e_{j}\right\rangle=\widehat{f}(j) \sigma(j)=\left\langle\sigma\left(L_{1,0}\right) f, e_{j}\right\rangle .
$$

Hence,

$$
\sigma\left(L_{1,0}\right) f(x)=\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \Gamma_{0}(k) \mathcal{T}_{\frac{2 \pi k}{q}-s} f(x), \quad \Gamma_{0}(k)=\sum_{m=0}^{q-1} \sigma(m) e_{m}\left(\frac{2 \pi k}{q}-s\right),
$$

and so (7.16) holds whenever $h=1$.
Step 2. Let $h>0$, with $h \neq 1$, and $\theta \in(0,1)$. From Lemma 7.3 we have that

$$
\sigma\left(L_{h, \theta}\right) f(x)=h^{-x} e^{-i \theta x} \sigma_{\theta}\left(L_{1,0}\right) h^{x} e^{i \theta x} f(x),
$$

with $\sigma_{\theta}=\sigma(x+\theta)$. Moreover, from Step 1, we obtain that

$$
\begin{aligned}
\sigma\left(L_{h, \theta}\right) f(x) & =h^{-x} e^{-i \theta x} \frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} \sigma_{\theta}(m) e_{m}\left(\frac{2 \pi k}{q}-s\right) \mathcal{T}_{\frac{2 \pi k}{q}-s} h^{x} e^{i \theta x} f(x) \\
& =\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} \sigma(m+\theta) e_{m}\left(\frac{2 \pi k}{q}-s\right) h^{-x} e^{-i \theta x} \mathcal{T}_{\frac{2 \pi k}{q}-s} h^{x} e^{i \theta x} f(x) .
\end{aligned}
$$

However, we see that

$$
h^{-x} e^{-i \theta x} \mathcal{T}_{\frac{2 \pi k}{q}-s} h^{x} e^{i \theta x} f(x)=h^{-x} e^{-i \theta x} \exp \left(-i\left(\frac{2 \pi k}{q}-s\right) L_{1,0}\right) h^{x} e^{i \theta x} f(x)
$$

Now, consider the function defined on $\mathbb{R}$,

$$
g(x)=\exp \left(-i\left(\frac{2 \pi k}{q}-s\right) x+i\left(\frac{2 \pi k}{q}-s\right) \theta\right) .
$$

Then, if $g_{\theta}(x)=g(x+\theta)$, we have that

$$
g_{\theta}(x)=\exp \left(-i\left(\frac{2 \pi k}{q}-s\right) x\right) .
$$

Therefore, using Lemma 7.3 once more, we obtain

$$
h^{-x} e^{-i \theta x} \mathcal{T}_{\frac{2 \pi k}{q}-s} h^{x} e^{i \theta x} f(x)=h^{-x} e^{-i \theta x} g_{\theta}\left(L_{1,0}\right) h^{x} e^{i \theta x} f(x)=g\left(L_{h, \theta}\right) f(x)
$$

On the other hand, the functional calculus implies that

$$
g\left(L_{h, \theta}\right) f(x)=\sum_{j \in \mathbb{Z}}\left\langle f, \psi_{j}\right\rangle g(j+\theta) \phi_{j}(x)=e^{\left(\frac{2 \pi k}{q}-s\right) i \theta} \exp \left(-i\left(\frac{2 \pi k}{q}-s\right) L_{h, \theta}\right)
$$

Going back to $\sigma\left(L_{h, \theta}\right)$, we find that

$$
\sigma\left(L_{h, \theta}\right) f(x)=\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} \sigma(m+\theta) e_{m}\left(\frac{2 \pi k}{q}-s\right) e^{\left(\frac{2 \pi k}{q}-s\right) i \theta} \exp \left(-i\left(\frac{2 \pi k}{q}-s\right) L_{h, \theta}\right)
$$

Finally, due to (7.9), we know that $\phi_{m}(x)=h^{x} e^{i \theta x} e_{m}(x)$, which implies that

$$
\begin{aligned}
\sigma\left(L_{h, \theta}\right) f(x) & =\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} \sigma(m+\theta) \phi_{m}\left(\frac{2 \pi k}{q}-s\right) h^{-\left(\frac{2 \pi k}{q}-s\right)} \exp \left(-i\left(\frac{2 \pi k}{q}-s\right) L_{h, \theta}\right) f(x) \\
& =\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \Gamma_{\theta}(k) h^{-\left(\frac{2 \pi k}{q}-s\right)} \exp \left(-i\left(\frac{2 \pi k}{q}-s\right) L_{h, \theta}\right) f(x),
\end{aligned}
$$

where $\Gamma_{\theta}(k)$ is given as in the statement.
It follows from (7.16) that for any function $\sigma \in C_{b}(\mathbb{R})$ which satisfies (7.15), the operator $\sigma\left(L_{h, \theta}\right)$ can be expressed as a finite linear combination of the operators

$$
h^{-\left(\frac{2 \pi k}{q}-s\right)} \exp \left(-i\left(\frac{2 \pi k}{q}-s\right) L_{h, \theta}\right), \quad k=0,1, \ldots, q-1 .
$$

In turns this is a finite superposition of operators and an extension of the revival phenomenon in an theoretical operator setting.

Indeed, first note that the revival functional calculus (7.16), extends the concept of the periodic revival operator $\mathcal{R}_{n}(p, q)$ of order $n \in \mathbb{N}$, recall Definition 3.4,

$$
\mathcal{R}_{n}(p, q)=\exp \left(-i \frac{2 \pi p}{q} L_{1,0}^{n}\right)
$$

where $p$ and $q$ are co-prime, positive integers. Furthermore, it includes Theorem 2.8
which holds for the operator

$$
\sigma\left(L_{1,0}\right)=\exp \left(-i \frac{2 \pi p}{q} P\left(L_{1,0}\right)\right) .
$$

Here $P$ is a polynomial of order $n \geq 2$ and integer coefficients. Hence, in some sense, any operator $\sigma\left(L_{h, \theta}\right)$ can be thought of as a revival operator whenever the function $\sigma$ satisfies the periodicity condition (7.15). In the following section more applications of Lemma 7.5 will be considered in the context of specific time evolution problems and the revival phenomenon.

In accordance with Remark 7.4, we could possibly take a step further and broaden the revival functional calculus (7.16) for any linear bounded operator $A$ in $L^{2}(0,2 \pi)$ given by a similarity transformation $S^{-1} \sigma\left(L_{1,0}\right) S$. Here, $S$ is some arbitrary bounded operator with bounded inverse and $\sigma$ in $C_{b}(\mathbb{R})$. If $\sigma$ satisfies the periodicity property (7.15), then we see that $A$ admits a linear decomposition in terms of the operators

$$
S \mathcal{T}_{\frac{2 \pi k}{q}-s} S^{-1}, \quad k=0,1, \ldots, q-1
$$

Finally, as an implication of Lemma 7.5 and for later convenience in applications, we extract the periodic case, corresponding to the operator $L_{1,0}$.

Lemma 7.6. Consider the linear differential operator $L_{1,0}$ defined by (7.1) and (7.2). Let $s \in \mathbb{R}$ and $q$ be a positive integer. Consider a function $\sigma: \mathbb{R} \rightarrow \mathbb{C}$ in $C_{b}(\mathbb{R})$ satisfying the property

$$
\sigma(m)=e^{i s(m-j)} \sigma(j), \quad \text { whenever } m \equiv j \bmod q, \quad m, j \in \mathbb{Z}
$$

Then, for any $f \in L^{2}(0,2 \pi)$, we have

$$
\sigma\left(L_{1,0}\right) f(x)=\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \Gamma_{0}(k) \mathcal{T}_{\frac{2 \pi k}{q}-s} f(x), \quad \Gamma_{0}(k)=\sum_{m=0}^{q-1} \sigma(m) e_{m}\left(\frac{2 \pi k}{q}-s\right) .
$$

### 7.3 The Even-Order Poly-harmonic Wave Equation

In the existing literature, the revival phenomenon seems to be restricted to firstorder in time evolution problems, consistent with the linear dispersive PDEs that we examined in the previous chapters. Here we consider second-order in time PDEs, and using Lemma 7.6, we will show that the weak revival effect is present in the poly-harmonic wave equation of even-order with periodic boundary conditions. This class of equations includes the wave equation and the bi-harmonic wave equations, both models of special interested in mathematics and physics.

Fix an integer $r \geq 1$ and consider the initial boundary value problem on $[0,2 \pi]$

$$
\begin{gather*}
\partial_{t}^{2} u(x, t)=-\left(-i \partial_{x}\right)^{2 r} u(x, t), \quad u(x, 0)=f(x), \quad \partial_{t} u(x, 0)=g(x),  \tag{7.18}\\
\partial_{x}^{m} u(0, t)=\partial_{x}^{m} u(2 \pi, t), \quad m=0,1,2, \ldots, 2 r-1 .
\end{gather*}
$$

In order to explore the phenomenon of revivals, we begin with the Fourier series representation of the solution as we did in the preceding cases. In particular, for any initial conditions $f$ and $g$ in $L^{2}(0,2 \pi)$, the Fourier method gives at any time $t \geq 0$ the $L^{2}(0,2 \pi)$ representation

$$
\begin{equation*}
u(x, t)=\sum_{j \in \mathbb{Z}} \widehat{f}(j) \cos \left(j^{r} t\right) e_{j}(x)+\langle g\rangle t+\sum_{0 \neq j \in \mathbb{Z}} \frac{\widehat{g}(j)}{j^{r}} \sin \left(j^{r} t\right) e_{j}(x), \tag{7.19}
\end{equation*}
$$

where $\langle g\rangle$ is the mean of $g$ over $[0,2 \pi]$,

$$
\langle g\rangle=\frac{1}{2 \pi} \int_{0}^{2 \pi} g(x) d x
$$

Note that (7.19) is a solution in a generalised sense. Indeed, $u(x, t)$ given by (7.19) defines a continuous function of the time parameter $t$ with respect to the norm of $L^{2}(0,2 \pi)$. Also, for every $t \in[0, \infty), u(x, t)$ is the limit in $L^{2}(0,2 \pi)$ as $n \rightarrow \infty$ of
the sequence of smooth solutions of (7.18) given by the partial sums

$$
u^{n}(x, t)=\langle f\rangle+\langle g\rangle t+\sum_{0 \neq j=-n}^{n}\left(\widehat{f}(j) \cos \left(j^{r} t\right)+\frac{\widehat{g}(j)}{j^{r}} \sin \left(j^{r} t\right)\right) e_{j}(x) .
$$

We can already observe a form of weak revival at a rational time $t=2 \pi \frac{p}{q}$ on each of the terms of (7.19). More specifically, using

$$
\cos \left(j^{r} t\right)=\frac{e^{i j^{r} t}+e^{-i j^{r} t}}{2}
$$

it follows that the first term represents a linear combination of the revival operators $\mathcal{R}_{r}(p, q)$ and $\mathcal{R}_{r}(-p, q)$ applied on $f$. However, as we see shortly, we can directly use Lemma 7.6 and deduce a pure revival effect. On the other hand, the second term, at each time, always gives a constant, so we can concentrate on the third term. For any $g \in L^{2}(0,2 \pi)$, note that

$$
\left|\frac{\widehat{g}(j)}{j^{r}} \sin \left(j^{r} t\right) e_{j}(x)\right| \leq\left|\frac{\widehat{g}(j)}{j^{r}}\right| .
$$

Moreover, by the Cauchy-Schwarz inequality, we have

$$
\sum_{0 \neq j \in \mathbb{Z}}\left|\frac{\widehat{g}(j)}{j^{r}}\right| \leq \sqrt{\sum_{0 \neq j \in \mathbb{Z}}|\widehat{g}(j)|^{2}} \sqrt{\sum_{0 \neq j \in \mathbb{Z}} \frac{1}{|j|^{2 r}}}<\infty
$$

Hence, it follows from the Weierstrass M-test (see for example [26, Theorem 3.27]), that the third term gives, at each time, a continuous $2 \pi$-periodic function in the space variable (otherwise we may invoke Lemma D.5).

The above indicate that at rational times we should expect the phenomenon of weak revivals, since the solution representation (7.19) will be given by the sum of a pure revival effect and a continuous function in space (recall Definition 2.16).

In the case of the periodic poly-harmonic wave equation, we can derive a weak revival representation utilizing Lemma 7.6, by explicitly characterising the continuous function given in terms of the Fourier series

$$
\sum_{0 \neq j \in \mathbb{Z}} \frac{\widehat{g}(j)}{j^{r}} \sin \left(j^{r} t\right) e_{j}(x)
$$

We do this by introducing a sequence of polynomials $\left\{h_{r}(x)\right\}_{r \in \mathbb{N}}$ defined on $[0,2 \pi)$ such that $h_{r}$ is of order $r$ and their Fourier coefficients are of the form $j^{-r}$, for $j \neq 0$.

Lemma 7.7. There exists a sequence of polynomials denoted by $h_{r}(x), r \in \mathbb{N}$, with $x \in[0,2 \pi)$, such that each $h_{r}(x)$ is of order $r$ and $\widehat{h}_{r}(j)=j^{-r}$ when $j \neq 0$. Concretely, for fixed integer $r \geq 1, h_{r}(x)$ is defined inductively by

$$
\begin{equation*}
h_{r}(x)=\frac{(-i)^{r}}{(-1)^{r-1}} \frac{\sqrt{2 \pi}}{2 \pi r!} x^{r}-\sum_{\ell=1}^{r-1} \frac{(-1)^{\ell-r}}{(-i)^{\ell-r}} \frac{(2 \pi)^{r-\ell}}{(r-\ell+1)!} h_{\ell}(x) . \tag{7.20}
\end{equation*}
$$

Proof. Fix integer $r \geq 1$ and consider the function $f_{r}(x)=x^{r}$ with $x \in[0,2 \pi)$. Then, for $j \neq 0$, the Fourier coefficients of $f_{r}(x)$ are given by

$$
\widehat{f}_{r}(j)=\left\langle f_{r}, e_{j}\right\rangle=\frac{r!}{\sqrt{2 \pi}} \sum_{k=0}^{r-1} \frac{(-1)^{k}}{(-i)^{k+1}} \frac{(2 \pi)^{r-k}}{(r-k)!} \frac{1}{j^{k+1}},
$$

Solving the above equation for $j^{-r}$ and setting $\ell=k+1$ in the sum over $k$, we find that

$$
\frac{1}{j^{r}}=\frac{(-i)^{r}}{(-1)^{r-1}} \frac{\sqrt{2 \pi}}{2 \pi r!} \widehat{f}_{r}(j)-\sum_{\ell=1}^{r-1} \frac{(-1)^{\ell-r}}{(-i)^{\ell-r}} \frac{(2 \pi)^{r-\ell}}{(r-\ell+1)!} \frac{1}{j^{\ell}}
$$

Let $h_{r}(x)$ be the function whose Fourier coefficients are given equal to $j^{-r}$ for $j \neq 0$. Then, the last equation above implies that

$$
\int_{0}^{2 \pi} h_{r}(x) \overline{e_{j}(x)} d x=\int_{0}^{2 \pi}\left[\frac{(-i)^{r}}{(-1)^{r-1}} \frac{\sqrt{2 \pi}}{2 \pi r!} x^{r}-\sum_{\ell=1}^{r-1} \frac{(-1)^{\ell-r}}{(-i)^{\ell-r}} \frac{(2 \pi)^{r-\ell}}{(r-\ell+1)!} h_{\ell}(x)\right] \overline{e_{j}(x)} d x .
$$

Therefore, $h_{r}(x)$ is the polynomial of order $r \geq 1$ given by (7.20).
Hence, we have another representation of the solution $u(x, t)$ to (7.18), in terms of convolutions with $h_{r}(x)$. Recall that the linear differential operator $L_{1,0}=-i \partial_{x}$ is accompanied by the periodic boundary condition on $[0,2 \pi]$.

Proposition 7.8. Fix integer $r \geq 1$ and let $f$ and $g$ be in $L^{2}(0,2 \pi)$. Then at every time $t \geq 0$ the solution to (7.18) admits the $L^{2}(0,2 \pi)$ representation

$$
\begin{equation*}
u(x, t)=\cos \left(L_{1,0}^{r} t\right) f(x)+\langle g\rangle t+\sin \left(L_{1,0}^{r} t\right)\left[\tilde{g} * h_{r}\right](x), \tag{7.21}
\end{equation*}
$$

where $\tilde{g}=g-\langle g\rangle$ and $h_{r}$ is given by (7.20).

Proof. From (7.19) and the definition of the functional calculus (7.11) for the operator $L_{1,0}$, we have that

$$
u(x, t)=\cos \left(L_{1,0}^{r} t\right) f(x)+\langle g\rangle t+\sum_{0 \neq j \in \mathbb{Z}} \frac{\widehat{g}(j)}{j^{r}} \sin \left(j^{r} t\right) e_{j}(x) .
$$

Let $\tilde{g}=g-\langle g\rangle$, then calculating its Fourier coefficients we found that

$$
\widehat{\tilde{g}}(j)=\widehat{g}(j), \quad \widehat{\tilde{g}}(0)=0 .
$$

Moreover, from Lemma 7.7 we know that for fixed integer $r \geq 1$ and when $j \neq 0$ the Fourier coefficients of $h_{r}(x)$ are equal to $j^{-r}$. Hence, by considering the $2 \pi$-periodic convolution

$$
\tilde{g} * h_{r}(x)=\frac{1}{\sqrt{2 \pi}} \int_{0}^{2 \pi} \tilde{g}^{*}(x-y) h^{*}(y) d y, \quad x \in[0,2 \pi],
$$

we obtain that

$$
\widehat{\tilde{g} * h_{r}}(j)= \begin{cases}0, & j=0, \\ \frac{\widehat{g}(j)}{j^{r}}, & j \neq 0 .\end{cases}
$$

Therefore, using the functional calculus again, the solution $u(x, t)$ admits the representation

$$
\begin{aligned}
u(x, t) & =\cos \left(L_{1,0}^{r} t\right) f(x)+\langle g\rangle t+\sum_{j \in \mathbb{Z}} \widehat{\tilde{g} * h_{r}}(j) \sin \left(j^{r} t\right) e_{j}(x) \\
& =\cos \left(L_{1,0}^{r} t\right) f(x)+\langle g\rangle t+\sin \left(L_{1,0}^{r} t\right)\left[\tilde{g} * h_{r}\right](x) .
\end{aligned}
$$

By combing Proposition 7.8 with Lemma 7.6, the next corollary gives the weak revival representation of the solution to the poly-harmonic wave equation at rational times.

Corollary 7.9. Fix integer $r \geq 1$ and let $f$ and $g$ be in $L^{2}(0,2 \pi)$. Then, at rational time $t=2 \pi \frac{p}{q}$, with $p$ and $q$ co-prime integers, the solution to (7.18) is given in

$$
\begin{align*}
u\left(x, 2 \pi \frac{p}{q}\right)=2 \pi\langle g\rangle \frac{p}{q} & +\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} \cos \left(2 \pi m^{r} \frac{p}{q}\right) e_{m}\left(\frac{2 \pi k}{q}\right) \mathcal{T}_{\frac{2 \pi k}{q}} f(x) \\
& +\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} \sin \left(2 \pi m^{r} \frac{p}{q}\right) e_{m}\left(\frac{2 \pi k}{q}\right) \mathcal{T}_{\frac{2 \pi k}{q}}\left[\tilde{g} * h_{r}\right](x) . \tag{7.22}
\end{align*}
$$

Proof. From Proposition 7.8, at rational time $t=2 \pi \frac{p}{q}$, the solution to (7.18) is given by

$$
u\left(x, 2 \pi \frac{p}{q}\right)=2 \pi \frac{p}{q}\langle g\rangle+\cos \left(2 \pi \frac{p}{q} L_{1,0}^{r}\right) f(x)+\sin \left(2 \pi \frac{p}{q} L_{1,0}^{r}\right)\left[\tilde{g} * h_{r}\right](x)
$$

However, we observe that both functions (of $m$ )

$$
\cos \left(2 \pi \frac{p}{q} m^{r}\right), \quad \sin \left(2 \pi \frac{p}{q} m^{r}\right)
$$

satisfy the hypothesis of Lemma (7.6) with $s=0$, which implies that representation (7.22) holds in $L^{2}(0,2 \pi)$.

As a consequence of (7.22), the even-order poly-harmonic wave equation under periodic boundary conditions exhibits the weak revival effect at rational times. More specifically, the second term in (7.22) corresponds to the pure revival of the initial function $f$, whereas the third term is the revival of the continuous on $[0,2 \pi]$ function $\tilde{g} * h_{r}$ and thus, together with the constant term, a continuous function on $[0,2 \pi]$. Hence, whenever the initial condition $f$ has a finite number of jump discontinuities, then the solution profile at rational time will exhibit finitely many jump discontinuities.

We now focus on the specific important models of the wave and bi-harmonic wave equation. In the case of the wave equation, we are going to compare the (weak) revival representation (7.22) with the classical D'Alembert's representation at rational times. For the bi-harmonic wave equation, we will establish a different approach to the revival effect. This approach stems from the special structure of the
bi-harmonic wave equation which can be factorised in terms of two linear Schrödinger equations.

### 7.3.1 The Wave Equation

When $r=1$, the corresponding PDE is the classical wave equation

$$
\begin{equation*}
\partial_{t}^{2} u(x, t)=\partial_{x}^{2} u(x, t) \tag{WA}
\end{equation*}
$$

which is posed under periodic boundary conditions on $[0,2 \pi]$ and with initial conditions $u(x, 0)=f(x)$ and $\partial_{t} u(x, 0)=g(x)$. For the one dimensional wave equation we classically know that the solution at any time $t \geq 0$ is also given by D'Alembert's formula

$$
\begin{equation*}
u(x, t)=\frac{1}{2}\left(\mathcal{T}_{t} f(x)+\mathcal{T}_{-t} f(x)\right)+\frac{1}{2} \int_{x-t}^{x+t} g^{*}(y) d y \tag{7.23}
\end{equation*}
$$

where recall that $g^{*}$ denotes the $2 \pi$-periodic extension of $g$.
Notice that D'Alembert's formula indicates that the weak revival effect is in fact present at all times for the wave equation. The pure revival of the initial condition comes from the finite linear combination of translations

$$
\frac{1}{2}\left(\mathcal{T}_{t} f(x)+\mathcal{T}_{-t} f(x)\right)
$$

and the continuous in space component is given by the other term

$$
\frac{1}{2} \int_{x-t}^{x+t} g^{*}(y) d y
$$

which is a continuous function of $x$. As a matter of fact, we can show that at rational times the weak revival representation (7.22), when $r=1$, reduces to D'Alembert's formula (7.23) evaluated at rational times.

Proposition 7.10. Let $r=1$. Then, the weak revival representation (7.22) of the solution to the wave equation reduces to D'Alembert's formula at rational time $t=2 \pi \frac{p}{q}$,

$$
u\left(x, 2 \pi \frac{p}{q}\right)=\frac{1}{2}\left(\mathcal{T}_{2 \pi \frac{p}{q}} f(x)+\mathcal{T}_{-2 \pi \frac{p}{q}} f(x)\right)+\frac{1}{2} \int_{x-2 \pi \frac{p}{q}}^{x+2 \pi \frac{p}{q}} g^{*}(y) d y
$$

Proof. For $r=1$, we look at the second term on the right hand-side of (7.22). Then, we have

$$
\begin{aligned}
& \frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} \cos \left(2 \pi m \frac{p}{q}\right) e_{m}\left(\frac{2 \pi k}{q}\right) \mathcal{T}_{\frac{2 \pi k}{q}} f(x) \\
& =\frac{1}{2 q} \sum_{k=0}^{q-1} f^{*}\left(x-2 \pi \frac{k}{q}\right)\left(\sum_{m=0}^{q-1}\left(e^{-i m 2 \pi \frac{p}{q}}+e^{i m 2 \pi \frac{p}{q}}\right) e^{i m 2 \pi \frac{k}{q}}\right) \\
& =\frac{1}{2 q}\left(\sum_{k=0}^{q-1} f^{*}\left(x-2 \pi \frac{k}{q}\right) \sum_{m=0}^{q-1} e^{i m \frac{2 \pi}{q}(k-p)}+\sum_{k=0}^{q-1} f^{*}\left(x-2 \pi \frac{k}{q}\right) \sum_{m=0}^{q-1} e^{i m \frac{2 \pi}{q}(k+p)}\right)
\end{aligned}
$$

which gives

$$
\begin{aligned}
& \frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} \cos \left(2 \pi m \frac{p}{q}\right) e_{m}\left(\frac{2 \pi k}{q}\right) \mathcal{T}_{\frac{2 \pi k}{q}} f(x) \\
& =\frac{1}{2}\left(f^{*}\left(x-2 \pi \frac{p}{q}\right)+f^{*}\left(x+2 \pi \frac{p}{q}\right)\right) \\
& =\frac{1}{2}\left(\mathcal{T}_{2 \pi \frac{p}{q}} f(x)+\mathcal{T}_{-2 \pi \frac{p}{q}} f(x)\right) .
\end{aligned}
$$

Therefore, when $r=1,(7.22)$ is

$$
\begin{aligned}
u\left(x, 2 \pi \frac{p}{q}\right)= & \frac{1}{2}\left(\mathcal{T}_{2 \pi \frac{p}{q}} f(x)+\mathcal{T}_{-2 \pi \frac{p}{q}} f(x)\right)+ \\
& \left(2 \pi\langle g\rangle \frac{p}{q}+\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} \sin \left(2 \pi m \frac{p}{q}\right) e_{m}\left(\frac{2 \pi k}{q}\right) \mathcal{T}_{\frac{2 \pi k}{q}}\left[(g-\langle g\rangle) * h_{1}\right](x)\right),
\end{aligned}
$$

where from (7.20), the polynomial $h_{1}$ is given by $h_{1}(x)=-i x / \sqrt{2 \pi}$, for $x \in[0,2 \pi)$.
Let

$$
G(x)=\left[(g-\langle g\rangle) * h_{1}\right](x), \quad x \in[0,2 \pi),
$$

and let

$$
\begin{aligned}
J\left(x, 2 \pi \frac{p}{q}\right) & =2 \pi\langle g\rangle \frac{p}{q}+\frac{\sqrt{2 \pi}}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} \sin \left(2 \pi m \frac{p}{q}\right) e_{m}\left(\frac{2 \pi k}{q}\right) \mathcal{T}_{\frac{2 \pi k}{q}} G(x) \\
& =2 \pi\langle g\rangle \frac{p}{q}+\frac{1}{q} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} \sin \left(2 \pi m \frac{p}{q}\right) e^{i m \frac{2 \pi k}{q}} G^{*}\left(x-\frac{2 \pi k}{q}\right) \\
& =2 \pi\langle g\rangle \frac{p}{q}+\frac{1}{2 i q} \sum_{k=0}^{q-1} G^{*}\left(x-\frac{2 \pi k}{q}\right) \sum_{m=0}^{q-1}\left(e^{i m \frac{2 \pi p}{q}}-e^{-i m \frac{2 \pi p}{q}}\right) e^{i m \frac{2 \pi k}{q}} .
\end{aligned}
$$

We have

$$
\begin{aligned}
& \sum_{k=0}^{q-1} G^{*}\left(x-\frac{2 \pi k}{q}\right) \sum_{m=0}^{q-1}\left(e^{i m \frac{2 \pi p}{q}}-e^{-i m \frac{2 \pi p}{q}}\right) e^{i m \frac{2 \pi k}{q}} \\
& =\left[\sum_{k=0}^{q-1} G^{*}\left(x-\frac{2 \pi k}{q}\right) \sum_{m=0}^{q-1} e^{i m \frac{2 \pi(k+p)}{q}}-\sum_{k=0}^{q-1} G^{*}\left(x-\frac{2 \pi k}{q}\right) \sum_{m=0}^{q-1} e^{i m \frac{2 \pi(k-p)}{q}}\right] \\
& =q\left[G^{*}\left(x+\frac{2 \pi p}{q}\right)-G^{*}\left(x-\frac{2 \pi p}{q}\right)\right]
\end{aligned}
$$

and thus, $J\left(x, 2 \pi \frac{p}{q}\right)$ becomes

$$
J\left(x, 2 \pi \frac{p}{q}\right)=2 \pi\langle g\rangle \frac{p}{q}+\frac{1}{2 i}\left[G^{*}\left(x+\frac{2 \pi p}{q}\right)-G^{*}\left(x-\frac{2 \pi p}{q}\right)\right] .
$$

By definition,

$$
\begin{aligned}
G(x)=(g-\langle g\rangle) * h(x) & =g * h(x)-\langle g\rangle * h(x) \\
& =\frac{1}{\sqrt{2 \pi}} \int_{0}^{2 \pi} h^{*}(x-y) g^{*}(y) d y-\frac{\langle g\rangle}{\sqrt{2 \pi}} \int_{0}^{2 \pi} h^{*}(x-y) d y .
\end{aligned}
$$

Now, since $h(x)=-i x / \sqrt{2 \pi}$ in $[0,2 \pi)$, we find that

$$
\begin{aligned}
\int_{0}^{2 \pi} h^{*}(x-y) d y & =\int_{0}^{x} h^{*}(x-y) d y+\int_{x}^{2 \pi} h^{*}(x-y) d y \\
& =\int_{0}^{x} h(x-y) d y+\int_{x}^{2 \pi} h(x-y+2 \pi) d y \\
& =\int_{0}^{x}-i \frac{(x-y)}{\sqrt{2 \pi}} d y+\int_{x}^{2 \pi}-i \frac{(x-y+2 \pi)}{\sqrt{2 \pi}} d y=-i \pi \sqrt{2 \pi}
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\int_{0}^{2 \pi} h^{*}(x-y) g^{*}(y) d y & =\int_{0}^{x} h(x-y) g^{*}(y) d y+\int_{x}^{2 \pi} h(x-y+2 \pi) g(y) d y \\
& =\frac{-i}{\sqrt{2 \pi}}\left[2 \pi\langle g\rangle x+2 \pi \int_{x}^{2 \pi} g^{*}(y) d y-\int_{0}^{2 \pi} y g(y) d y\right]
\end{aligned}
$$

By substitution, we arrive at

$$
G(x)=-i\langle g\rangle f(x)-i \int_{f(x)}^{2 \pi} g^{*}(y) d y+\frac{i}{2 \pi} \int_{0}^{2 \pi} y g(y) d y+i \pi\langle g\rangle, \quad x \in[0,2 \pi),
$$

where $f(x)=x$ in $[0,2 \pi)$.
Going back to $J\left(x, 2 \pi \frac{p}{q}\right)$, the above calculations give that

$$
\begin{aligned}
J\left(x, 2 \pi \frac{p}{q}\right)= & 2 \pi\langle g\rangle \frac{a}{q}+\frac{1}{2 i}\left[G^{*}\left(x+\frac{2 \pi p}{q}\right)-G^{*}\left(x-\frac{2 \pi p}{q}\right)\right] \\
= & 2 \pi\langle g\rangle \frac{p}{q}+\frac{1}{2 i}\left[-i\langle g\rangle f^{*}\left(x+\frac{2 \pi p}{q}\right)-i \int_{f^{*}\left(x+\frac{2 \pi p}{q}\right)}^{2 \pi} g^{*}(y) d y\right. \\
& +\frac{i}{2 \pi} \int_{0}^{2 \pi} y g(y) d y+i \pi\langle g\rangle+i\langle g\rangle f^{*}\left(x-\frac{2 \pi p}{q}\right)+i \int_{f^{*}\left(x-\frac{2 \pi p}{q}\right)}^{2 \pi} g^{*}(y) d y \\
& \left.-\frac{i}{2 \pi} \int_{0}^{2 \pi} y g(y) d y-i \pi\langle g\rangle\right]
\end{aligned}
$$

or equivalently,

$$
J\left(x, 2 \pi \frac{p}{q}\right)=2 \pi\langle g\rangle \frac{p}{q}+\frac{\langle g\rangle}{2}\left[f^{*}\left(x-\frac{2 \pi p}{q}\right)-f^{*}\left(x+\frac{2 \pi p}{q}\right)\right]+\frac{1}{2} \int_{f^{*}\left(x-\frac{2 \pi p}{q}\right)}^{f^{*}\left(x+\frac{2 \pi p}{q}\right)} g^{*}(y) d y
$$

Recall that $f(x)=x$ in $[0,2 \pi)$ and since $2 \pi \frac{p}{q}$ is a positive real number we can write

$$
f^{*}\left(x-2 \pi \frac{p}{q}\right)=x-2 \pi \frac{p}{q}-2 \pi \ell, \quad f^{*}\left(x+2 \pi \frac{p}{q}\right)=x+2 \pi \frac{p}{q}-2 \pi n,
$$

for some $\ell$ and $n$ in $\mathbb{Z}$, with $n \geq 0$. Then, we have

$$
\begin{aligned}
J\left(x, 2 \pi \frac{p}{q}\right) & =2 \pi\langle g\rangle \frac{p}{q}+\frac{\langle g\rangle}{2}\left(x-\frac{2 \pi p}{q}-2 \pi \ell-x-\frac{2 \pi p}{q}+2 \pi n\right)+\frac{1}{2} \int_{x-\frac{2 \pi p}{q}-2 \pi \ell}^{x+\frac{2 \pi p}{q}-2 \pi n} g^{*}(y) d y \\
& =\pi\langle g\rangle(n-\ell)+\frac{1}{2}\left(\int_{x-\frac{2 \pi p}{q}-2 \pi \ell}^{x-2 \pi \frac{p}{q}} g^{*}(y) d y+\int_{x+2 \pi \frac{p}{q}}^{x+\frac{2 \pi p}{q}-2 \pi n} g^{*}(y) d y\right) \\
& +\frac{1}{2} \int_{x-\frac{2 \pi p}{q}}^{x+2 \pi \frac{p}{q}} g^{*}(y) d y .
\end{aligned}
$$

Because $n \geq 0$, using the periodicity of $g^{*}$, we see that

$$
\int_{x+2 \pi \frac{p}{q}}^{x+\frac{2 \pi p}{q}-2 \pi n} g^{*}(y) d y=-\int_{x+\frac{2 \pi p}{q}-2 \pi n}^{x+2 \pi \frac{p}{q}} g^{*}(y) d y=-n \int_{0}^{2 \pi} g^{*}(y) d y=-2 \pi\langle g\rangle n,
$$

which gives

$$
J\left(x, 2 \pi \frac{p}{q}\right)=\pi\langle g\rangle(n-\ell)+\frac{1}{2}\left(\int_{x-\frac{2 \pi p}{q}-2 \pi \ell}^{x-2 \pi \frac{p}{q}} g^{*}(y) d y-2 \pi\langle g\rangle n\right)+\frac{1}{2} \int_{x-\frac{2 \pi p}{q}}^{x+2 \pi \frac{p}{q}} g^{*}(y) d y
$$

To complete the proof, we distinguish the two cases $\ell \geq 0$ and $\ell<0$.
(i) If $\ell \geq 0$, then

$$
\int_{x-\frac{2 \pi p}{q}-2 \pi \ell}^{x-2 \pi \frac{p}{q}} g^{*}(y) d y=\ell \int_{0}^{2 \pi} g^{*}(y) d y=2 \pi\langle g\rangle \ell,
$$

giving

$$
J\left(x, 2 \pi \frac{p}{q}\right)=\pi\langle g\rangle(n-\ell)+\frac{1}{2}(2 \pi\langle g\rangle \ell-2 \pi\langle g\rangle n)+\frac{1}{2} \int_{x-\frac{2 \pi p}{q}}^{x+2 \pi \frac{p}{q}} g^{*}(y) d y=\frac{1}{2} \int_{x-\frac{2 \pi p}{q}}^{x+2 \pi \frac{p}{q}} g^{*}(y) d y .
$$

(ii) If $\ell<0$, then $\ell=-|\ell|$ and

$$
\int_{x-\frac{2 \pi p}{q}+2 \pi|\ell|}^{x-2 \pi \frac{p}{q}} g^{*}(y) d y=-|\ell| \int_{0}^{2 \pi} g^{*}(y) d y=-2 \pi\langle g\rangle|\ell|
$$

giving

$$
J\left(x, 2 \pi \frac{p}{q}\right)=\pi\langle g\rangle(n+|\ell|)+\frac{1}{2}(-2 \pi\langle g\rangle|\ell|-2 \pi\langle g\rangle n)+\frac{1}{2} \int_{x-\frac{2 \pi p}{q}}^{x+2 \pi \frac{p}{q}} g^{*}(y) d y=\frac{1}{2} \int_{x-\frac{2 \pi p}{q}}^{x+2 \pi \frac{p}{q}} g^{*}(y) d y .
$$

Therefore, in any case we conclude that

$$
J\left(x, 2 \pi \frac{p}{q}\right)=\frac{1}{2} \int_{x-\frac{2 \pi p}{q}}^{x+2 \pi \frac{p}{q}} g^{*}(y) d y .
$$

### 7.3.2 The Bi-harmonic Wave Equation

The existence of the weak revival in the poly-harmonic wave equation can be conjectured in the basis of its special structure. The bi-harmonic wave equation sets an interesting example for this analysis through a system of linear Schrödinger equations. When $r=2$, the initial boundary value problem (7.3) reduces to the periodic boundary value problem on $[0,2 \pi]$ for the bi-harmonic wave equation

$$
\begin{equation*}
\partial_{t}^{2} u(x, t)=-\partial_{x}^{4} u(x, t), \tag{BHW}
\end{equation*}
$$

with initial conditions $u(x, 0)=f(x)$ and $\partial_{t} u(x, 0)=g(x)$. This PDE is also known as the beam or Euler-Bernoulli equation and is of fundamental importance in engineering applications since it describes the free, absent to external loads, transverse vibrations of a homogeneous beam [58].

Notice that we can write (BHW) in the form

$$
\left(\partial_{t}+i \partial_{x}^{2}\right)\left(\partial_{t}-i \partial_{x}^{2}\right) u(x, t)=0
$$

Hence, setting

$$
w(x, t)=\left(\partial_{t}-i \partial_{x}^{2}\right) u(x, t),
$$

we see that the bi-harmonic wave equation can be equivalently written as the coupled system of linear Schrödinger equations

$$
\partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t)+w(x, t), \quad \partial_{t} w(x, t)=-i \partial_{x}^{2} w(x, t) .
$$

The solution $u(x, t)$ of the periodic by-harmonic wave equation can be obtained by solving the initial boundary value problems on $[0,2 \pi]$

$$
\begin{array}{lr}
\partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t)+w(x, t), & \partial_{t} w(x, t)=-i \partial_{x}^{2} w(x, t) \\
u(x, 0)=f(x), & w(x, 0)=g(x)-i \partial_{x}^{2} f(x),  \tag{7.24}\\
u(0, t)=u(2 \pi, t), & w(0, t)=w(2 \pi, t) \\
\partial_{x} u(0, t)=\partial_{x} u(2 \pi, t), & \partial_{x} w(0, t)=\partial_{x} w(2 \pi, t) .
\end{array}
$$

By looking at the inhomogeneous linear Schödinger equation

$$
\partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t)+w(x, t),
$$

we conjecture that, most likely, there will be a type of a revival phenomenon. Intuitively, this is a consequence of Duhamel's principle, which states that the solution to the inhomogeneous equation is obtained by the solution to the homogeneous equation with initial condition $f(x)$, denoted by $u_{f}(x, t)$, plus an integral, that is

$$
u(x, t)=u_{f}(x, t)+\int_{0}^{t} u_{w}(x, t-s) d s
$$

In our case the solution to the homogeneous equation is given by

$$
u_{f}(x, t)=\sum_{j \in \mathbb{Z}} \widehat{f}(j) e^{-i j^{2} t} e_{j}(x),
$$

and consequently whenever $f(x, t)$ is piecewise continuous, $u_{f}(x, t)$ would be piecewise continuous at rational times. From the Fourier method, we will show that the Duhamel integral corresponds to a continuous function in $x$, as expected from the
weak revival formula (7.22).
In the next statement, we will consider the domain of the linear operator $L_{1,0}^{2}=$ $-\partial_{x}^{2}$,

$$
\mathrm{D}\left(-\partial_{x}^{2}\right)=\left\{\phi \in H^{2}(0,2 \pi) ; \phi(0)=\phi(2 \pi), \phi^{\prime}(0)=\phi^{\prime}(2 \pi)\right\} .
$$

Theorem 7.11. Let $f \in D\left(-\partial_{x}^{2}\right)$ and $g \in L^{2}(0,2 \pi)$ and consider the system of periodic problems (7.24). Then, at any time $t \geq 0$, the solution $u(x, t)$ admits the following representations in $L^{2}(0,2 \pi)$
(i)

$$
\begin{equation*}
u(x, t)=\sum_{j \in \mathbb{Z}} \widehat{f}(j) \cos \left(j^{2} t\right) e_{j}(x)+\langle g\rangle t+\sum_{0 \neq j \in \mathbb{Z}} \frac{\widehat{g}(j)}{j^{2}} \sin \left(j^{2} t\right) e_{j}(x), \tag{7.25}
\end{equation*}
$$

(ii)

$$
\begin{align*}
u(x, t)= & \exp \left(-i L_{1,0}^{2} t\right) f(x)+\langle g\rangle t \\
& +\frac{1}{2 i}\left(\exp \left(i L_{1,0}^{2} t\right)-\exp \left(-i L_{1,0}^{2} t\right)\right)\left[(w(\cdot, 0)-\langle w(\cdot, 0)\rangle) * h_{2}(x)\right], \tag{7.26}
\end{align*}
$$

where $w(x, 0)=g(x)-i \partial_{x}^{2} f(x)$ and $h_{2}(x)$ is the polynomial defined in Lemma 7.7.
Proof. We begin by deriving a representation in $L^{2}(0,2 \pi)$ for $u(x, t)$ using the Fourier method. For this, we expand both $u(x, t)$ and $w(x, t)$ in Fourier series in $L^{2}(0,2 \pi)$,

$$
u(x, t)=\sum_{j \in \mathbb{Z}} \widehat{u}(j, t) e_{j}(x), \quad w(x, t)=\sum_{j \in \mathbb{Z}} \widehat{w}(j, t) e_{j}(x) .
$$

Since $w(x, t)$ is the solution to a free space linear Schrödinger equation, its Fourier coefficients are given by

$$
\widehat{w}(j, t)=\widehat{w}(j, 0) e^{i j^{2} t}
$$

and because $f \in \mathrm{D}\left(-\partial_{x}^{2}\right)$, we have that

$$
\widehat{w}(j, 0)=\widehat{g}(j)+i j^{2} \widehat{f}(j)
$$

Now, to compute the Fourier coefficients of $u(x, t)$, notice that if we multiply the
inhomogeneous equation

$$
\partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t)+w(x, t)
$$

by a function $\phi \in \mathrm{D}\left(-\partial_{x}^{2}\right)$, and integrade over $(0,2 \pi)$, then we obtain

$$
\left\langle\partial_{t} u(\cdot, t), \phi\right\rangle=i\left\langle\partial_{x}^{2} u(\cdot, t), \phi\right\rangle+\langle w(\cdot), \phi\rangle .
$$

After integration by parts twice in the first term of the right-hand side, and using the periodic boundary conditions satisfied by $u(x, t)$, we arrive at the weak form

$$
\frac{d}{d t}\langle u(\cdot, t), \phi\rangle=i\left\langle u(\cdot, t), \phi^{\prime \prime}\right\rangle+\langle w(\cdot, t), \phi\rangle .
$$

Choosing $\phi(x)=e_{j}(x)=e^{i j x} / \sqrt{2 \pi}$, the last equation above gives for each Fourier coefficient $\widehat{u}(j, t)$ the differential equation

$$
\frac{d}{d t} u(j, t)=-i j^{2} \widehat{u}(j, t)+\widehat{w}(j, t)
$$

Taking as initial conditions $\widehat{u}(j, 0)=\widehat{f}(j)$, the Fourier coefficients are obtained by

$$
\widehat{u}(j, t)=\widehat{f}(j) e^{-i j^{2} t}+\int_{0}^{t} \widehat{w}(j, s) e^{-i j^{2}(t-s)} d s
$$

However, we know the form of $\widehat{w}(j, t)$ from above, and thus

$$
\widehat{u}(j, t)=\widehat{f}(j) e^{-i j^{2} t}+\widehat{w}(j, 0) e^{-i j^{2} t} \int_{0}^{t} e^{2 i j^{2} s} d s
$$

So, $u(x, t)$ is given in $L^{2}(0,2 \pi)$ by

$$
\begin{aligned}
u(x, t) & =\sum_{j \in \mathbb{Z}} \widehat{f}(j) e^{-i j^{2} t} e_{j}(x)+\sum_{j \in \mathbb{Z}} \widehat{w}(j, 0) e^{-i j^{2} t} \int_{0}^{t} e^{2 i j^{2} s} d s e_{j}(x) \\
& =\sum_{j \in \mathbb{Z}} \widehat{f}(j) e^{-i j^{2} t} e_{j}(x)+\sum_{j \in \mathbb{Z}} \widehat{w}(j, 0) e^{-i j^{2} t} \int_{0}^{t} e^{2 i j^{2} s} d s e_{j}(x) \\
& =\sum_{j \in \mathbb{Z}} \widehat{f}(j) e^{-i j^{2} t} e_{j}(x)+\frac{\widehat{w}(0,0)}{\sqrt{2 \pi}} t+\frac{1}{2 i} \sum_{0 \neq j \in \mathbb{Z}} \frac{\widehat{w}(j, 0)}{j^{2}}\left(e^{-i j^{2} t}-e^{i j^{2} t}\right) e_{j}(x) .
\end{aligned}
$$

Observe that $\widehat{w}(0,0)=\widehat{g}(0)$, hence

$$
u(x, t)=\sum_{j \in \mathbb{Z}} \widehat{f}(j) e^{-i j^{2} t} e_{j}(x)+\langle g\rangle t+\frac{1}{2 i} \sum_{0 \neq j \in \mathbb{Z}} \frac{\widehat{w}(j, 0)}{j^{2}}\left(e^{-i j^{2} t}-e^{i j^{2} t}\right) e_{j}(x) .
$$

From the last representation we derive $(i)$ and (ii) as follows.
(i) Using $\widehat{w}(j, 0)=\widehat{g}(j)+i j^{2} \widehat{f}(j)$, the solution $u(x, t)$ becomes

$$
\begin{aligned}
u(x, t) & =\sum_{j \in \mathbb{Z}} \widehat{f}(j) \cos \left(j^{2} t\right) e_{j}(x)-i \sum_{j \in \mathbb{Z}} \widehat{f}(j) \sin \left(j^{2} t\right) e_{j}(x)+\langle g\rangle t \\
& +\sum_{0 \neq j \in \mathbb{Z}} \frac{\widehat{g}(j)}{j^{2}} \sin \left(j^{2} t\right) e_{j}(x)+i \sum_{0 \neq j \in \mathbb{Z}} \widehat{f}(j) \sin \left(j^{2} t\right) e_{j}(x)
\end{aligned}
$$

Hence,

$$
u(x, t)=\sum_{j \in \mathbb{Z}} \widehat{f}(j) \cos \left(j^{2} t\right) e_{j}(x)+\langle g\rangle t+\sum_{0 \neq j \in \mathbb{Z}} \frac{\widehat{g}(j)}{j^{2}} \sin \left(j^{2} t\right) e_{j}(x)
$$

(ii) From our previous calculation, using the functional calculus for the operator $L_{1,0}$, we have

$$
u(x, t)=\exp \left(-i L_{1,0}^{2} t\right) f(x)+\langle g\rangle t+\frac{1}{2 i} \sum_{0 \neq j \in \mathbb{Z}} \frac{\widehat{w}(j, 0)}{j^{2}}\left(e^{i j^{2} t}-e^{-i j^{2} t}\right) e_{j}(x)
$$

Moreover, recall from Lemma 7.7 that the polynomial

$$
h_{2}(x)=\frac{1}{2 \sqrt{2 \pi}}\left(x^{2}-2 \pi x\right), \quad x \in[0,2 \pi),
$$

has Fourier coefficients $j^{-2}$. Therefore, the Fourier coefficients of the $2 \pi$-periodic convolution

$$
(w(\cdot, 0)-\langle w(\cdot, 0)\rangle) * h_{2}(x)
$$

are given by

$$
\left\langle(w(\cdot, 0)-\langle w(\cdot, 0)\rangle) * h_{2}, e_{j}\right\rangle= \begin{cases}0, & j=0 \\ \frac{\widehat{w}(j, 0)}{j^{2}}, & j \neq 0\end{cases}
$$

Hence,

$$
\begin{aligned}
u(x, t)= & \exp \left(-i L_{1,0}^{2} t\right) f(x)+\langle g\rangle t \\
& +\frac{1}{2 i}\left(\exp \left(i L_{1,0}^{2} t\right)-\exp \left(-i L_{1,0}^{2} t\right)\right)\left[(w(\cdot, 0)-\langle w(\cdot, 0)\rangle) * h_{2}(x)\right]
\end{aligned}
$$

Part (i) from Theorem 7.11 shows that the solution to the bi-harmonic wave equation can be obtained through the system of linear Schrödinger equations (7.24). Indeed, as we noticed, solving (7.24) for $u(x, t)$ results into the same Fourier series representation in $L^{2}(0,2 \pi)$ as in (7.19) when $r=2$.

On the other hand, from the representation in part (ii) of Theorem 7.11, it follows that the solution to the periodic bi-harmonic wave equation exhibits weak revivals at rational times as anticipated earlier by the intuition given by the Duhamel principle. In particular, at rational times, we have

$$
\begin{align*}
u\left(x, 2 \pi \frac{p}{q}\right) & =\mathcal{R}_{2}(p, q)[f(x)] \\
& +2 \pi \frac{p}{q}\langle g\rangle+\frac{1}{2 i}\left(\mathcal{R}_{2}(-p, q)-\mathcal{R}_{2}(p, q)\right)\left[(w(\cdot, 0)-\langle w(\cdot, 0)\rangle) * h_{2}(x)\right] \tag{7.27}
\end{align*}
$$

which is exactly the definition of the weak revival, a pure revival effect perturbed by a continuous function.

The weak revival representations (7.27) and (7.22)-( $r=2$ ) is an attempt to a mathematical treatment of the revival effect in the context of the bi-harmonic wave equation. In accordance with this, only recently in [59] the revival phenomenon was experimentally observed in the context of the bi-harmonic wave equation in the two dimensional setting and under clamped boundary conditions.

Here, we note that the revival property can be extended to the case of simply supported boundary conditions on $[0, \pi]$

$$
u\left(x_{0}, t\right)=\partial_{x}^{2} u\left(x_{0}, t\right)=0, \quad x_{0}=0, \pi .
$$

This follows from the periodic problem by considering the initial conditions to be
odd functions with respect to $\pi$. Then, the solution representation (7.25) becomes
$u(x, t)=\frac{2}{\pi} \sum_{j=1}^{\infty}\left(\int_{0}^{\pi} f(y) \sin (j y) d y \cos \left(j^{2} t\right)+\frac{1}{j^{2}} \int_{0}^{\pi} g(y) \sin (j y) d y \sin \left(j^{2} t\right)\right) \sin (j x)$,
which when restricted to $L^{2}(0, \pi)$ corresponds to the solution of the bi-harmonic wave equation on $[0, \pi]$ subjected to simply supported boundary conditions.

Finally, to close this section, we remark that the poly-harmonic wave equation of order $2 r$, for any $r \geq 3$, can always be written in the form

$$
\left(\partial_{t}+i\left(-i \partial_{x}\right)^{r}\right)\left(\partial_{t}-i\left(-i \partial_{x}\right)^{r}\right) u(x, t)=0 .
$$

Consequently, a similar approach as in the bi-harmonic wave equation $(r=2)$ would be effective for the study of the weak revival. However, and more importantly, this approach indicates that properties of the first-order in time dispersive equations

$$
\partial_{t} u(x, t)=-i\left(-i \partial_{x}\right)^{r} u(x, t)
$$

are transferred to the poly-harmonic wave equation, with perhaps some perturbations. For example, the pure revival phenomenon in the first-order in time problems became weak revival in the second-order case. So, one can asks if the weak revival property will be present in the poly-harmonic wave equation under quasi-periodic boundary conditions. Also, recalling that the Airy PDE under quasi-periodic boundary conditions does not in general exhibit revivals, should we then expect a break of the weak revival in the poly-harmonic wave equation of order $2 \times 3$, under the same type of boundary conditions? In the next chapter, we will address these questions in detail.

## Chapter 8

## Revivals Under Self-adjoint Quasi-periodic Boundary Conditions

In this final chapter, we generalise various results obtained earlier. We consider both first-order and second-order in time evolution problems under, self-adjoint quasiperiodic boundary conditions. From Chapter 4, we know that, in contrast to the free space linear Schrödinger equation, the revival phenomenon in general breaks at rational times in the context of the Airy PDE. Here, we extend this result to higherorder linear differential operators using a Galilean transformation approach which allows to convert quasi-periodic boundary conditions to periodic. As an additional consequence of this approach, we extend the Talbot effect in the cubic non-linear Schrödinger equation with quasi-periodic boundary conditions. Moreover, a similar technique in second-order in time problems shows that in this case the weak revival phenomenon is present in the bi-harmonic wave equation, whereas for higher-order poly-harmonic wave equations this is no longer true.

### 8.1 Overview

Our purpose is to examine the revival property in the family of PDEs

$$
\partial_{t}^{m} u(x, t)=(-i)^{m}\left(-i \partial_{x}\right)^{m n} u(x, t),
$$

with initial conditions

$$
u(x, 0)=f(x), \quad \partial_{t}^{m-1} u(x, t)=g(x),
$$

and under the quasi-periodic boundary conditions on $[0,2 \pi]$

$$
e^{i 2 \pi \theta} \partial_{x}^{k} u(x, t)=\partial_{x}^{k} u(x, t), \quad k=0,1,2, \ldots, m n-1,
$$

for integers $n \geq 1$ and $m=1,2$ and $\theta \in(0,1)$.
This class of equations includes the main PDEs we have encountered in previous chapters. For example, when $m=1$, it includes the free space linear Schrödinger equation and the Airy PDE, whereas when $m=2$, it corresponds to the family of poly-harmonic wave equations of even order greater or equal than two.

In order to examine the revival property, we will develop a different approach in comparison with the analysis in the previous chapters and Remark 4.1. Instead of examining directly the quasi-periodic problems through their generalised Fourier series, we establish a correspondence between the solutions of the quasi-periodic problems and a class of periodic problems. Then, using the classical Fourier series representation of the solution to the periodic problems, will be able to deduce various conclusions for the quasi-periodic problems regarding the revival effect at rational times.

The main idea in the correspondence between quasi-periodic and periodic boundary conditions is encoded in the following transformation

$$
\begin{equation*}
f_{\theta}(x)=e^{-i \theta x} f(x) \tag{8.1}
\end{equation*}
$$

and a modification of this involving Galilean-type transformations in $x$ and $t$. Observe that if $f$ is quasi-periodic,

$$
e^{i 2 \pi \theta} f(0)=f(2 \pi),
$$

then $f_{\theta}$ is periodic.
In Chapter 4, a similar situation was encountered for the free linear Schrödinger equation with quasi-periodic boundary conditions, that is when $(m, n)=(1,2)$
above. Recall that, in this case, the solution $u(x, t)$ to the quasi-periodic problem is given by the representation (4.30) stating that

$$
u(x, t)=e^{-\theta^{2} t} e^{i \theta x} \mathcal{T}_{2 \theta t} v(x, t),
$$

where $v(x, t)$ denotes the solution to the free linear Schrödinger equation with periodic boundary conditions and initial condition $v(x, 0)=e^{-i \theta x} u(x, 0)$. The representation can be rewritten in terms of a Galilean transformation as

$$
\begin{equation*}
u(x, t)=e^{i\left(\theta x-\theta^{2} t\right)} v^{*}(x-2 \theta t, t) \tag{8.2}
\end{equation*}
$$

with $v^{*}$ denoting the $2 \pi$-periodic extension of $v$.
The expression (8.2) implies that, whenever $v(x, t)$ solves the linear Schrödinger equation with periodic boundary conditions and initial condition $e^{-i \theta x} u(x, 0)$, the transformed function $u(x, t)$ given by (8.2) solves the same equation with quasiperiodic boundary conditions and initial condition $u(x, 0)$. In particular, the free linear Schrödinger equation is invariant under the transformation (8.2), meaning that whenever $v(x, t)$ satisfies the equation, so does $u(x, t)$. This is known as the Galilean invariance of the linear Schrödinger equation, [60]. From (8.2) and as a direct consequence of the Galilean invariance, the revival effect in the quasi-periodic case follows immediately from the periodic case.

Following a generalised form of (8.2), we will show in Section 8.2 that firstorder in time evolution problems $(m=1)$ under quasi-periodic boundary conditions correspond to time evolution problems with periodic conditions. However, for $n \geq 3$, the PDE does not stay invariant under the transformation and the revival effect for the obtained periodic problems is present if and only if the quasi-periodic parameter $\theta$ is a rational number. This generalises the lack of revivals in the quasi-periodic Airy PDE. Another implication of this approach is for the cubic non-linear Schrödinger equation, as for the linear Schrödinger equation there is also a form of revival. Thus, in Section 8.3 we establish an extension of the (weak) revival and fractalisation effects from the periodic case by Erdoğan and Tzirakis [41] to the quasi-periodic case.

Finally, in Section 8.4, the quasi-periodic problem for the poly-harmonic wave equation ( $m=2$ ), will be transformed through (8.1) to a periodic problem. Again,
by considering the Fourier series representation of the periodic problem, we will deduce that the weak revival phenomenon holds in the quasi-periodic case for all $\theta \in(0,1)$ when $n=2$, that is for the bi-harmonic wave equation, whereas for $n \geq 3$ the revival persists only when $\theta$ is rational.

### 8.2 First-Order in Time Evolution Problems

In this section we consider the revival property in the initial boundary value problem on $[0,2 \pi]$

$$
\begin{align*}
& \partial_{t} u(x, t)=-i\left(-i \partial_{x}\right)^{n} u(x, t), \quad u(x, 0)=f(x),  \tag{8.3}\\
& e^{i 2 \pi \theta} \partial_{x}^{k} u(0, t)=\partial_{x}^{k} u(2 \pi, t), \quad k=0,1, \ldots, n-1, \quad \theta \in(0,1),
\end{align*}
$$

with $n \geq 2$ a fixed integer.
From Chapter 4, we know that the revival property at rational times $t=2 \pi \frac{p}{q}$, with $\frac{p}{q} \in \mathbb{Q}$, holds for any value of $\theta$ in $(0,1)$ when $n=2$, whereas when $n=3$ this is true only if $\theta \in \mathbb{Q}$. In both of these cases, the revival effect was implied by a representation at rational time of the solution of the quasi-periodic problem in terms of a solution of a periodic problem for the Schrödinger equation.

In contrast to this, and the arguments in Chapter 4 based on eigenfunction expansions through the Fourier method, here we provide an one-to-one correspondence between the quasi-periodic problem (8.3) and a periodic problem where the spatial differential operator is a polynomial of $-i \partial_{x}$ with real but, not necessarily rational coefficients. As a corollary, we find that the solution to (8.3) can be derived through another time evolution problem with periodic boundary conditions.

Lemma 8.1. Let $n \geq 2$ be an integer and $\theta \in(0,1)$. Consider the transformation

$$
\begin{equation*}
w(x, t)=e^{i\left(\theta^{n} t-\theta x\right)} u(x, t) \tag{8.4}
\end{equation*}
$$

Then, $w(x, t)$ satisfies the initial boundary value problem on $[0,2 \pi]$

$$
\begin{align*}
& \left(\partial_{t}+n \theta^{n-1} \partial_{x}\right) w(x, t)=-i\left(\left(-i \partial_{x}\right)^{n}+\sum_{k=2}^{n-1}\binom{n}{k} \theta^{n-k}\left(-i \partial_{x}\right)^{k}\right) w(x, t),  \tag{8.5}\\
& w(x, 0)=f_{\theta}(x)=e^{-i \theta x} f(x), \quad \partial_{x}^{k} w(0, t)=\partial_{x}^{k} w(a, t), \quad k=0,1, \ldots, n-1,
\end{align*}
$$

if and only if $u(x, t)$ satisfies the quasi-periodic problem (8.3).
Proof. We show that $w(x, t)$ satisfies the periodic problem when $u(x, t)$ satisfies the quasi-periodic problem. We note that the converse follows by similar calculations. Take the time derivative of $w(x, t)$. From the assumption on $u(x, t)$, we find that

$$
\begin{align*}
\partial_{t} w(x, t) & =i \theta^{n} w(x, t)+e^{i\left(\theta^{n} t-\theta x\right)} \partial_{t} u(x, t)  \tag{8.6}\\
& =i \theta^{n} w(x, t)+(-i)^{n+1} e^{i\left(\theta^{n} t-\theta x\right)} \partial_{x}^{n} u(x, t) .
\end{align*}
$$

We claim that

$$
\begin{equation*}
e^{i\left(\theta^{n} t-\theta x\right)} \partial_{x}^{m} u(x, t)=\left(\partial_{x}+i \theta\right)^{m} w(x, t), \quad \forall m \in \mathbb{N} . \tag{8.7}
\end{equation*}
$$

It is straightforward to see that the claim holds for $m=1$ and 2 , and we proceed by induction. Assuming that it is true for $m$ we examine the case $m+1$. We have

$$
\partial_{x}\left(e^{i\left(\theta^{n} t-\theta x\right)} \partial_{x}^{m} u(x, t)\right)=\partial_{x}\left(\left(\partial_{x}+i \theta\right)^{m} w(x, t)\right)
$$

which gives

$$
e^{i\left(\theta^{n} t-\theta x\right)} \partial_{x}^{m+1} u(x, t)=\left(\partial_{x}\left(\partial_{x}+i \theta\right)^{m}+i \theta\left(\partial_{x}+i \theta\right)^{m}\right) w(x, t) .
$$

Using the Binomial Theorem the last equation becomes

$$
\begin{aligned}
e^{i\left(\theta^{n} t-\theta x\right)} \partial_{x}^{m+1} u(x, t) & =\left(\partial_{x}\left(\sum_{k=0}^{m}\binom{m}{k}(i \theta)^{m-k} \partial_{x}^{k}\right)+i \theta\left(\sum_{k=0}^{m}\binom{m}{k}(i \theta)^{m-k} \partial_{x}^{k}\right)\right) w(x, t) \\
& =\left(\sum_{k=0}^{m}\binom{m}{k}(i \theta)^{m-k} \partial_{x}^{k+1}+\sum_{k=0}^{m}\binom{m}{k}(i \pi \theta)^{m-k+1} \partial_{x}^{k}\right) w(x, t) .
\end{aligned}
$$

We expand the sums over $k$ and find that

$$
\begin{aligned}
e^{i\left(\theta^{n} t-\theta x\right)} \partial_{x}^{m+1} u(x, t)= & \left(\partial_{x}^{m+1}+\binom{m}{m-1} i \theta \partial_{x}^{m}+\binom{m}{m-2}(i \theta)^{2} \partial_{x}^{m-1}+\ldots\right. \\
& +\binom{m}{0}(i \theta)^{m} \partial_{x}+\binom{m}{m} i \theta \partial_{x}^{m}+\binom{m}{m-1}(i \theta)^{2} \partial_{x}^{m-1}+\ldots \\
& \left.+\binom{m}{1}(i \theta)^{m} \partial_{x}+(i \theta)^{m+1}\right) w(x, t) \\
= & \left(\partial_{x}^{m+1}+\binom{m+1}{m} i \theta \partial_{x}^{m}+\binom{m+1}{m-1}(i \theta)^{2} \partial_{x}^{m-1}+\ldots\right. \\
& \left.+\binom{m+1}{1}(i \theta)^{m} \partial_{x}+(i \theta)^{m+1}\right) w(x, t) \\
= & \left(\sum_{k=0}^{m+1}\binom{m+1}{k}(i \theta)^{m+1-k} \partial_{x}^{k}\right) w(x, t) \\
= & \left(\partial_{x}+i \theta\right)^{m+1} w(x, t) .
\end{aligned}
$$

Hence, (8.7) holds true. Now substituting for $m=n$ in (8.6), we arrive at

$$
\begin{aligned}
\partial_{t} w(x, t) & =i \theta^{n} w(x, t)+(-i)^{n+1}\left(\partial_{x}+i \theta\right)^{n} w(x, t) \\
& =i \theta^{n} w(x, t)+(-i)^{n+1} \sum_{k=0}^{n}\binom{n}{k}(i \theta)^{n-k} \partial_{x}^{k} w(x, t) \\
& =\left(i+(-i)^{n+1} i^{n}\right) \theta^{n} w(x, t)+(-i)^{n+1} i^{n-1}\binom{n}{1} \theta^{n-1} \partial_{x} w(x, t) \\
& +(-i)^{n+1} \sum_{k=2}^{n-1}\binom{n}{k}(i \theta)^{n-k} \partial_{x}^{k} w(x, t)+(-i)^{n+1} \partial_{x}^{n} w(x, t)
\end{aligned}
$$

Now, for any $n$, we have that $i+(-i)^{n+1} i^{n}=0$. Also, since $(-i)^{n+1} i^{n-1}=-1$, the
above equation takes the form, after clearing,

$$
\left(\partial_{t}+n \theta^{n-1} \partial_{x}\right) w(x, t)=(-i)\left(\left(-i \partial_{x}\right)^{n}+\sum_{k=2}^{n-1}\binom{n}{k} \theta^{n-k}(-i)^{n} i^{n-k} \partial_{x}^{k}\right) w(x, t) .
$$

Observing that $(-i)^{n} i^{n-k} \partial_{x}^{k}=\left(-i \partial_{x}\right)^{k}$, we find that $w(x, t)$ satisfies the PDE

$$
\left(\partial_{t}+n \theta^{n-1} \partial_{x}\right) w(x, t)=(-i)\left(\left(-i \partial_{x}\right)^{n}+\sum_{k=2}^{n-1}\binom{n}{k} \theta^{n-k}\left(-i \partial_{x}\right)^{k}\right) w(x, t)
$$

which is exactly the one in the statement and with initial condition

$$
w(x, 0)=e^{-i \theta x} u(x, 0)=e^{-i \theta x} f(x)=f_{\theta}(x) .
$$

Finally, we need to show that $w(x, t)$ satisfies the periodic boundary conditions on the interval $[0,2 \pi]$. For any $m=0,1, \ldots, n-1$ we have from (8.7) at $x=2 \pi$

$$
\begin{aligned}
\partial_{x}^{m} w(2 \pi, t) & =e^{i \theta^{n} t} e^{-i 2 \pi \theta} \partial_{x}^{m} u(2 \pi, t)-\sum_{k=0}^{m-1}\binom{m}{k}(i \theta)^{m-k} \partial_{x}^{k} w(2 \pi, t) \\
& =e^{i \theta^{n} t} e^{-i 2 \pi \theta} \partial_{x}^{m} u(2 \pi, t)-\sum_{k=0}^{m-1}\binom{m}{k}(i \theta)^{m-k} \partial_{x}^{k} w(2 \pi, t) \\
& =e^{i \theta^{n} t} \partial_{x}^{m} u(0, t)-\sum_{k=0}^{m-1}\binom{m}{k}(i \theta)^{m-k} \partial_{x}^{k} w(2 \pi, t) .
\end{aligned}
$$

As $w(0, t)=w(2 \pi, t)$ and $\partial_{x} w(2 \pi, t)=\partial_{x} w(2 \pi, t)$, assuming that $\partial_{x}^{j} w(0, t)=$ $\partial_{x}^{j} w(2 \pi, t)$ holds for $j=0,1, \ldots, m-1$, the above equation gives

$$
\begin{equation*}
\partial_{x}^{m} w(2 \pi, t)=e^{i \theta^{n} t} \partial_{x}^{m} u(0, t)-\sum_{k=0}^{m-1}\binom{m}{k}(i \theta)^{m-k} \partial_{x}^{k} w(0, t)=\partial_{x}^{m} w(0, t) . \tag{8.8}
\end{equation*}
$$

We continue with a second transformation in the following lemma and recall that $\mathcal{T}_{s}$, for $s \in \mathbb{R}$, denotes the periodic translation operator (see Definition 3.2).

Lemma 8.2. Let integer $n \geq 2$ and $\theta \in(0,1)$. Consider the transformation

$$
\begin{equation*}
z(x, t)=e^{-i\left((n-1) \theta^{n} t+\theta x\right)} \mathcal{T}_{-n \theta^{n-1} t} u(x, t) \tag{8.9}
\end{equation*}
$$

Then, $z(x, t)$ satisfies the initial boundary value problem on $[0,2 \pi]$

$$
\begin{align*}
& \partial_{t} z(x, t)=-i\left(\left(-i \partial_{x}\right)^{n}+\sum_{k=2}^{n-1}\binom{n}{k} \theta^{n-k}\left(-i \partial_{x}\right)^{k}\right) z(x, t),  \tag{8.10}\\
& z(x, 0)=f_{\theta}(x)=e^{-i \theta x} f(x), \quad \partial_{x}^{k} z(0, t)=\partial_{x}^{k} z(2 \pi, t), \quad k=0,1, \ldots, n-1,
\end{align*}
$$

if and only if $u(x, t)$ satisfies the quasi-periodic problem (8.3).
Proof. From Lemma 8.1 we know that if $u(x, t)$ satisfies the quasi-periodic problem (8.3) then

$$
\begin{equation*}
w(x, t)=e^{i\left(\theta^{n} t-\theta x\right)} u(x, t) \tag{8.11}
\end{equation*}
$$

satisfies the periodic problem (8.5). We apply on $w(x, t)$ the Galilean transformation

$$
x=y+n \theta^{n-1} \tau, \quad t=\tau .
$$

Hence,

$$
\partial_{\tau} w=\left(\partial_{t}+\theta^{n-1} \partial_{x}\right) w, \quad \partial_{y}^{m} w=\partial_{x}^{m} w, \quad m=1, \ldots, n .
$$

Therefore the function

$$
z(y, \tau)=w\left(y+n \theta^{n-1} \tau, \tau\right)=e^{-i\left((n-1) \theta^{n} \tau+\theta y\right)} \mathcal{T}_{-n \theta^{n-1} \tau} u(y, \tau)
$$

satisfies the periodic problem (8.10).
For the converse, by applying on $z(x, t)$ the Galilean transformation

$$
x=y-n \theta^{n-1} \tau, \quad t=\tau,
$$

we find that the function $w(y, \tau)=\mathcal{T}_{n \theta^{n-1} \tau} z(y, \tau)$ satisfies the periodic problem (8.5). Therefore, by Lemma 8.1, the function

$$
u(y, \tau)=e^{-i\left(\theta^{n} \tau-\theta y\right)} w(y, \tau)=e^{-i\left(\theta^{n} \tau-\theta y\right)} \mathcal{T}_{n \theta^{n-1} \tau} z(y, \tau)
$$

satisfies the quasi-periodic problem (8.3).

The following corollary from Lemma 8.2 gives the solution to the quasi-periodic problem (8.3) through the solution to the periodic problem (8.10).

Corollary 8.3. Let integer $n \geq 2, \theta \in(0,1)$ and consider the quasi-periodic problem (8.3) for $u(x, t)$. Then, at a fixed time $t \geq 0$ we have

$$
\begin{equation*}
u(x, t)=e^{-i \theta^{n}} e^{i \theta x} \mathcal{T}_{n \theta^{n-1} t} z(x, t) \tag{8.12}
\end{equation*}
$$

where $z(x, t)$ satisfies the periodic problem (8.10).

Now that we have obtained the solution $u(x, t)$ of the quasi-periodic problem (8.3) in terms of the solution $z(x, t)$ of the periodic problem in Lemma 8.2, we can focus on $z(x, t)$ and study the revival property in the periodic setting. The behaviour of the quasi-periodic problem at rational (or irrational) times will be determined by the behaviour of the periodic problem. Whenever the periodic problem exhibits revivals at rational times, according to Corollary 8.3, the same will be true for the quasi-periodic one. In particular, we will show that for the periodic problem (8.10), when $n \geq 3$, the revival effect breaks if $\theta \notin \mathbb{Q}$, whereas as we already know it exists for all $\theta \in(0,1)$ when $n=2$. There are two main consequences of this statement which reflect one another.

First, as we said above, this will imply that all quasi-periodic problems of order $n \geq 3$ lack the revival property whenever $\theta \notin \mathbb{Q}$, generalising the result for the quasi-periodic Airy PDE from Chapter 4. Moreover, there exists a family of linear dispersive PDEs with spatial differential operator a polynomial of order $n \geq 2$ of $-i \partial_{x}$ and with real non-rational coefficients, for which under periodic boundary conditions the revival phenomenon breaks at rational times when the order of the operator is $n \geq 3$. In turn, this is in contrast to the classical revival result, Theorem 2.8, which shows that the revival property holds under periodic boundary conditions for any linear dispersive PDE with spatial differential operator a polynomial of order $n \geq 2$ and with integer coefficients.

We make all this precise and from now on we concentrate on the periodic problem (8.10). We replace $f_{\theta}$ by a general initial condition $z_{0}$, and so for a fixed $\theta \in(0,1)$
and integer $n \geq 2$, we consider the initial boundary value problem (8.10) on $[0,2 \pi]$, which is re-written in the following manner

$$
\begin{align*}
& \partial_{t} z(x, t)=-i P\left(-i \partial_{x}\right) z(x, t), \quad z(x, 0)=z_{0}(x),  \tag{8.13}\\
& \partial_{x}^{m} z(0, t)=\partial_{x}^{m} z(2 \pi, t), \quad m=0,1, \ldots, n-1 .
\end{align*}
$$

Here, in (8.13), $P$ is a polynomial of order $n \geq 2$ with real coefficients and it is given by

$$
\begin{equation*}
P(x)=\sum_{k=2}^{n}\binom{n}{k} \theta^{n-k} x^{k} . \tag{8.14}
\end{equation*}
$$

For the analysis of the revival phenomenon we rely on the Fourier series representation of the generalised solution to (8.13). Specifically, for any initial function $z_{0} \in L^{2}(0,2 \pi)$, the Fourier method yields at any fixed time $t \geq 0$ the representation

$$
\begin{equation*}
z(x, t)=\sum_{j \in \mathbb{Z}} \widehat{z_{0}}(j) e^{-i P(j) t} e_{j}(x), \tag{8.15}
\end{equation*}
$$

where $e_{j}(x)=e^{i j x} / \sqrt{2 \pi}$ denotes the elements of the Fourier basis, $\widehat{z_{0}}$ are the Fourier coefficients of $z_{0}$, and the convergence of the series holds in the norm of $L^{2}(0,2 \pi)$.

For any integer $n \geq 2$, the solution (8.15) can be expressed in terms of the generalised solutions of the simpler periodic problems on $[0,2 \pi]$

$$
\begin{align*}
& \partial_{t} v(x, t)=-i\left(-i \partial_{x}\right)^{k} v(x, t), \quad v(x, 0)=v_{0}(x),  \tag{8.16}\\
& \partial_{x}^{m} v(0, t)=\partial_{x}^{m} v(a, t), \quad m=0,1, \ldots, k-1, k=2, \ldots n .
\end{align*}
$$

For any fixed $t \geq 0$, the generalised solution of (8.16) in $L^{2}(0,2 \pi)$, which is given by

$$
v(x, t)=\sum_{j \in \mathbb{Z}} \widehat{v_{0}}(j) e^{-i j^{k} t} e_{j}(x),
$$

is isometric in $L^{2}(0,2 \pi)$ to $v_{0}$. We denote this isometry by $P_{k}(t)$. Concretely,

$$
\begin{equation*}
P_{k}(t): L^{2}(0,2 \pi) \rightarrow L^{2}(0,2 \pi), \quad P_{k}(t) v_{0}=\sum_{j \in \mathbb{Z}} \widehat{v}_{0}(j) e^{-i j^{k} t} e_{j} . \tag{8.17}
\end{equation*}
$$

Notice that the definition of $P_{k}(t)$ coincides with the definition of the operator
$\exp \left(-i L_{1,0}^{k} t\right)$ from the periodic functional calculus (7.6) in Section 7.1. Thus, it also defines a Fourier multiplier and sometimes it is referred as the solution operator of the initial boundary value problem (8.16).

Allowing negative values of $t$, then $P_{k}(t)$ can be also viewed as the family of operators parametrised by $t \in \mathbb{R}$. Using the definition (8.17) we see that the family $\left\{P_{k}(t)\right\}_{t \in \mathbb{R}}$ satisfies the following properties
(i) For any $t \in \mathbb{R}, P_{k}(t)$ is an isometry on $L^{2}(0,2 \pi)$. It is also surjective with the inverse operator given by $P_{k}(-t)$. Thus, $P_{k}(t)$ is a unitary operator for all $t \in \mathbb{R}$.
(ii) For any $t_{1}$ and $t_{2}$ real numbers we have the semi-group property

$$
P_{k}\left(t_{1}+t_{2}\right)=P_{k}\left(t_{1}\right) P_{k}\left(t_{2}\right) .
$$

(iii) For every $v_{0} \in L^{2}(0,2 \pi)$, the map $t \in \mathbb{R} \rightarrow P_{k}(t) v_{0}$ is continuous in $t$ with respect to the norm of $L^{2}(0,2 \pi)$.

Properties $(i),(i i)$ and (iii) are what normally referred to as a strongly continuous one parameter unitary group of operators in $L^{2}(0,2 \pi)$, see [55]. The proof of the third property follows from the same argument as found in the proof of Theorem 2.4. Finally, notice that $P_{k}(0)=I$ either directly by the definition (8.17) or by taking $t_{1}=t_{2}=0$.

With the notation and definition above we will draw central conclusions regarding the revival property for the periodic problem (8.13) and subsequently for the quasi-periodic problem (8.3) through Corollary 8.3. The following proposition provides an alternative representation of the solution $z(x, t)$, in terms of a product (or composition) of the operators $P_{k}(t)$ as $k$ runs from 2 to $n \geq 2$.

Proposition 8.4. Fix integer $n \geq 2$ and $\theta \in(0,1)$. Then, for any $z_{0} \in L^{2}(0,2 \pi)$, the solution to (8.13) at each fixed time $t \geq 0$ admits the following representation

$$
\begin{equation*}
z(x, t)=\prod_{k=2}^{n} P_{k}\left(\binom{n}{k} \theta^{n-k} t\right) z_{0}(x) \tag{8.18}
\end{equation*}
$$

where $\prod$ denotes the product of operators.

Proof. Fix positive time $t$ and note that

$$
e^{-i P(j) t}=\prod_{k=2}^{n} \exp \left(-i j^{k}\binom{n}{k} \theta^{n-k} t\right) .
$$

Therefore, (8.15) becomes

$$
\begin{aligned}
z(x, t) & =\sum_{j \in \mathbb{Z}} \widehat{z}_{0}(j) \prod_{k=2}^{n} \exp \left(-i j^{k}\binom{n}{k} \theta^{n-k} t\right) e_{j}(x) \\
& =P_{2}\left(\binom{n}{2} \theta^{n-2} t\right)\left[\sum_{j \in \mathbb{Z}} \widehat{z}_{0}(j) \prod_{k=3}^{n} \exp \left(-i j^{k}\binom{n}{k} \theta^{n-k} t\right) e_{j}(x)\right] \\
& =\prod_{k=2}^{n-1} P_{k}\left(\binom{n}{k} \theta^{n-k} t\right)\left[\sum_{j \in \mathbb{Z}} \widehat{z}_{0}(j) e^{-i j^{n} t} e_{j}(x)\right] \\
& =\prod_{k=2}^{n} P_{k}\left(\binom{n}{k} \theta^{n-k} t\right) z_{0}(x) .
\end{aligned}
$$

Now, we combine Proposition 8.4 with Oskolkov's result from Theorem 2.12 or Theorem 2.14 and derive the following corollary for the revival property of the periodic problems (8.13). Recall that $\mathcal{R}_{n}(p, q)$ denotes the periodic revival operator of order $n \geq 2$, which is given by Definition 3.4.

Corollary 8.5. Let $n, p$ and $q$ be positive integers with $n \geq 2, p$ and $q$ co-prime and consider $\theta \in(0,1)$. Then, for any $z_{0} \in L^{2}(0,2 \pi)$, the solution $z(x, t)$ to (8.16) at a rational time $t=2 \pi \frac{p}{q}$ admits the representation

$$
\begin{equation*}
z\left(x, 2 \pi \frac{p}{q}\right)=\prod_{k=2}^{n-1} P_{k}\left(2 \pi\binom{n}{k} \frac{p}{q} \theta^{n-k}\right)\left[\mathcal{R}_{n}(p, q) z_{0}(x)\right] \tag{8.19}
\end{equation*}
$$

Moreover,
(i) if $\theta=d / m \in \mathbb{Q}$, then we have

$$
\begin{equation*}
z\left(x, 2 \pi \frac{p}{q}\right)=\prod_{k=2}^{n} \mathcal{R}_{k}\left(n!p d^{n-k}, k!(n-k)!q m^{n-k}\right) z_{0}(x) \tag{8.20}
\end{equation*}
$$

(ii) if $\theta \notin \mathbb{Q}, n \geq 3$ and $z_{0}$ is of bounded variation with finitely many jump discontinuities, then the solution $z\left(x, 2 \pi \frac{p}{q}\right)$ is a continuous function of $x$.

Proof. Representation (8.19) follows easily by substituting the rational times $t=$ $2 \pi \frac{p}{q}$ in (8.18), which gives

$$
z\left(x, 2 \pi \frac{p}{q}\right)=\prod_{k=2}^{n-1} P_{k}\left(2 \pi\binom{n}{k} \frac{p}{q} \theta^{n-k}\right)\left[P_{n}\left(2 \pi \frac{p}{q}\right) z_{0}(x)\right] .
$$

However, from the representation (8.17) of $P_{n}(t)$ and Lemma 3.6 we have

$$
P_{n}\left(2 \pi \frac{p}{q}\right) z_{0}(x)=\sum_{j \in \mathbb{Z}} \widehat{z_{o}} e^{-i j^{n} 2 \pi \frac{p}{q}} e_{j}(x)=\mathcal{R}_{n}(p, q) z_{0}(x) .
$$

Hence, we obtain (8.19). Now, we distinguish on rational or irrational values of $\theta \in(0,1)$.
(i) Let $\theta=\frac{d}{m}$ be a rational number in reduced form. Again from (8.17) and Lemma 3.6 on the revival operators we have for $k=2, \ldots, n-1$ that

$$
\begin{aligned}
P_{k}\left(2 \pi\binom{n}{k} \frac{p}{q} \frac{d^{n-k}}{m^{n-k}}\right) & {\left[\mathcal{R}_{n}(p, q) z_{0}(x)\right] } \\
& =\sum_{j \in \mathbb{Z}}\left\langle\mathcal{R}_{n}(p, q) z_{0}, e_{j}\right\rangle \exp \left(-i j^{k} 2 \pi\binom{n}{k} \frac{p}{q} \frac{d^{n-k}}{m^{n-k}}\right) e_{j}(x),
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
P_{k}\left(2 \pi\binom{n}{k} \frac{p}{q} \frac{d^{n-k}}{m^{n-k}}\right) & {\left[\mathcal{R}_{n}(p, q) z_{0}(x)\right] } \\
& =\sum_{j \in \mathbb{Z}}\left\langle\mathcal{R}_{n}(p, q) z_{0}, e_{j}\right\rangle \exp \left(-i j^{k} 2 \pi \frac{n!p d^{n-k}}{k!(n-k)!q m^{n-k}}\right) e_{j}(x) \\
& =\mathcal{R}_{k}\left(n!p d^{n-k}, k!(n-k)!q m^{n-k}\right)\left[\mathcal{R}_{n}(p, q) z_{0}(x)\right]
\end{aligned}
$$

Consequently, from the (8.19) it follows that

$$
z\left(x, 2 \pi \frac{p}{q}\right)=\prod_{k=2}^{n} \mathcal{R}_{k}\left(n!p d^{n-k}, k!(n-k)!q m^{n-k}\right) z_{0}(x) .
$$

(ii) On the other hand, if $\theta \notin \mathbb{Q}$, we re-write the solution as

$$
z\left(x, 2 \pi \frac{p}{q}\right)=P_{2}\left(2 \pi\binom{n}{2} \frac{p}{q} \theta^{n-2}\right)\left[\prod_{k=3}^{n-1} P_{k}\left(2 \pi\binom{n}{k} \frac{p}{q} \theta^{n-k}\right)\left[\mathcal{R}_{n}(p, q) z_{0}(x)\right]\right]
$$

Moreover, if $z_{0}$ is of bounded variation with finitely many jump discontinuities, then $\mathcal{R}_{n}(p, q) z_{0}$ is a function of the same class. Now, for any $k=3, \ldots, n-1$, the function

$$
P_{k}\left(2 \pi\binom{n}{k} \frac{p}{q} \theta^{n-k}\right)\left[\mathcal{R}_{n}(p, q) z_{0}(x)\right]
$$

will at least be of bounded variation with finitely many jump discontinuities. Finally, since the time

$$
t=2 \pi\binom{n}{2} \stackrel{p}{q} \theta^{n-2}
$$

corresponds to an irrational time for the free space linear Schrödinger equation, then due to Theorem 2.12, the solution $z\left(x, 2 \pi \frac{p}{q}\right)$ is a continuous function of $x$. Otherwise, we can directly invoke the first part of Theorem 2.14 and conclude that $z\left(x, 2 \pi \frac{p}{q}\right)$ is continuous in $x$.

From Corollary 8.5 we finally arrive at the following conclusions regarding the revival phenomenon for the periodic problem (8.13) and the quasi-periodic problem
(8.3). The following summarises our findings.

1. There exists a class of linear dispersive PDEs of order $n \geq 3$, given by

$$
\partial_{t} z(x, t)=-i \sum_{k=2}^{n}\binom{n}{k} \theta^{n-k}\left(-i \partial_{x}\right)^{k} z(x, t)
$$

for which the revival phenomenon under periodic boundary conditions breaks at rational times whenever $\theta$ is not rational in $(0,1)$. If $\theta$ is rational, then the revival phenomenon survives at rational times (as a pure revival effect). In general, subject to periodic boundary conditions the class of linear dispersive PDEs

$$
\partial_{t} z(x, t)=-i P\left(-i \partial_{x}\right) z(x, t)
$$

where $P$ is a polynomial of degree $n \geq 3$ and with real, but not rational coefficients, does not exhibit revivals at rational times.
2. Due to Corollary 8.3, for any integer $n \geq 3$, the quasi-periodic problem (8.3) exhibits revival if and only if the periodic problem (8.13) exhibits revival which holds if and only if $\theta$ is rational.

### 8.3 Cubic Non-linear Schrödinger Equation

As another application of the approach presented in the previous section we now consider the quasi-periodic problem for the cubic non-linear Schrödinger (NLS) equation over the interval $[0,2 \pi]$

$$
\begin{align*}
& \partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t)+i|u(x, t)|^{2} u(x, t), \quad u(x, 0)=f(x),  \tag{8.21}\\
& e^{i 2 \pi \theta} u(0, t)=u(2 \pi, t), \quad e^{i 2 \pi \theta} \partial_{x} u(0, t)=\partial_{x} u(2 \pi, t),
\end{align*}
$$

where the parameter $\theta$ lies in $(0,1)$.
Recall that in Chapter 2, Theorem 2.15, due to Erdoğan and Tzirakis [41], addresses that the revival and fractalisation effects occur accordingly at rational and irrational times in the cubic NLS equation under periodic boundary conditions. On the other hand, similar to the free space linear Schrödinger equation, the cubic NLS
equation is invariant under the transformation

$$
z(x, t)=e^{-i\left(\theta^{2} t+\theta x\right)} u(x, t+2 \theta t)
$$

with the property also being known as the Galilean invariance of the cubic NLS equation.

Therefore, following the same arguments as in the proof of Lemmas 8.1 and 8.2, it results that, if $u(x, t)$ solves the quasi-periodic problem (8.21), then it is given by

$$
u(x, t)=e^{-i \theta^{2} t} e^{i \theta x} \mathcal{T}_{2 \theta t} z(x, t)
$$

Here, $z(x, t)$ solves the cubic NLS with periodic boundary conditions on $[0,2 \pi]$ and with initial value $z(x, 0)=e^{-\theta x} f(x)=f_{\theta}(x)$.

Notice that if $f$ is of bounded variation, then $f_{\theta}$ is also of bounded variation as the product of the non-zero continuous function $e^{-i \theta x}$ and $f$. With this and the remarks above, we extend the revival and fractalisation phenomena in the case of the quasi-periodic problem (8.21) for the NLS equation.

Corollary 8.6. Fix $\theta \in(0,1)$ and consider the quasi-periodic problem (8.21) for the NLS equation. Assume that $f$ is of bounded variation on $[0,2 \pi]$. Then, we have the following
(i) If $t / 2 \pi$ is an irrational number, then the solution $u(x, t)$ is a continuous function of $x$.
(iii) For rational values of $t / 2 \pi$, the solution $u(x, t)$ is a bounded function with at most countably many discontinuities.
(iii) If $f$ is also continuous on $[0,2 \pi]$ and such that $e^{i 2 \pi \theta} f(0)=f(2 \pi)$, then $u(x, t)$ is continuous in $x$ and $t$.

### 8.4 Second-Order in Time Evolution Problems

In this last section, we turn our attention to second-order in time problems subject to quasi-periodic boundary conditions. More specifically for integer $n \geq 1$ and $\theta \in(0,1)$, we consider the initial boundary value problem for the poly-harmonic wave equation on $[0,2 \pi]$

$$
\begin{gather*}
\partial_{t}^{2} u(x, t)=-\left(-i \partial_{x}\right)^{2 n} u(x, t), \quad u(x, 0)=f(x), \quad \partial_{t} u(x, 0)=g(x),  \tag{8.22}\\
e^{2 \pi i \theta} \partial_{x}^{k} u(0, t)=\partial_{x}^{k} u(2 \pi, t), \quad k=0,1,2, \ldots, 2 n-1 .
\end{gather*}
$$

As we mentioned earlier, our approach here will be to transform the quasiperiodic boundary conditions into periodic boundary conditions. Then, the solution to (8.22) would be given through the solution of a periodic problem for which the revival property will be considered. The following lemma provides the desired correspondence.

Lemma 8.7. Let integer $n \geq 1, \theta \in(0,1)$. Consider the transformation

$$
\begin{equation*}
w(x, t)=e^{-i \theta x} u(x, t) . \tag{8.23}
\end{equation*}
$$

Then, $w(x, t)$ solves the initial boundary value problem on $[0,2 \pi]$

$$
\begin{align*}
& \partial_{t}^{2} w(x, t)=-\left(-i \partial_{x}+\theta\right)^{2 n} w(x, t), \\
& w(x, 0)=f_{\theta}(x)=e^{-i \theta x} f(x),  \tag{8.24}\\
& \partial_{t} w(x, 0)=g_{\theta}(x)=e^{-i \theta x} g(x), \\
& \partial_{x}^{k} w(0, t)=\partial_{x}^{k} w(2 \pi, t), \quad k=0,1,2, \ldots, 2 n-1 .
\end{align*}
$$

if and only if $u(x, t)$ solves the quasi-periodic problem (8.22).

Proof. Assume that $u(x, t)$ is the solution to the quasi-periodic problem (8.22) with initial conditions $f(x)$ and $g(x)$. Then, if $w(x, t)$ is given by (8.23), differentiating twice in time we obtain

$$
\partial_{t}^{2} w(x, t)=-e^{-i \theta x}\left(-i \partial_{x}\right)^{2 n} u(x, t) .
$$

Now, from the proof of Lemma 8.1, we have that

$$
e^{-i \theta x} \partial_{x}^{m} u(x, t)=\left(\partial_{x}+i \theta\right)^{m} w(x, t), \quad \forall m \in \mathbb{N} .
$$

Therefore, $w(x, t)$ satisfies the equation

$$
\partial_{t}^{2} w(x, t)=-\left(-i \partial_{x}+\theta\right)^{2 n} w(x, t) .
$$

Moreover, at time zero the initial values for $w$ and $\partial_{t} w$ are obtained through (8.23) and given as in (8.24). Again, as in Lemma 8.1, it follows by the transformation (8.23), that $w$ and $\partial_{x}^{k} w$ satisfy periodic boundary conditions on $[0,2 \pi]$. Finally, the complementary direction of the correspondence is obtained by a similar calculation.

According to Lemma 8.7, we can now concentrate on the periodic problem (8.24) and examine the revival property through its Fourier series representation. Then, via the transformation $u(x, t)=e^{i \theta x} w(x, t)$, we can extract any related information on the revival effect for the quasi-periodic problem (8.22).

Hence, for $\theta \in(0,1)$ and integer $n \geq 1$ fixed, we consider on $[0,2 \pi]$ the initial boundary value problem for $w(x, t)$,

$$
\begin{align*}
& \partial_{t}^{2} w(x, t)=-\left(-i \partial_{x}+\theta\right)^{2 n} w(x, t) \\
& w(x, 0)=w_{0}(x), \quad \partial_{t} w(x, 0)=w_{1}(x)  \tag{8.25}\\
& \partial_{x}^{k} w(0, t)=\partial_{x}^{k} w(2 \pi, t), \quad k=0,1,2, \ldots, 2 n-1 .
\end{align*}
$$

In order to study the revival phenomenon at rational times, we follow once again our general methodology outlined in Remark 4.1. We manipulate the Fourier series representation of the periodic problem (8.25) in order to decompose it in simpler components for which the classical periodic revival effect holds. As we shall see, when $n \geq 3$, the revival survives only under rational values of the parameter $\theta$, which is in alignment with the case of first-order in time dispersive PDEs from Section 8.2.

Suppose that the initial data $w_{0}$ and $w_{1}$ are in $L_{2}(0,2 \pi)$. Then, by the Fourier method, at a fixed positive time $t$, the (generalised) solution of the periodic problem
(8.25) is given by the $L^{2}(0,2 \pi)$ representation

$$
\begin{align*}
w(x, t)= & \sum_{j \in \mathbb{Z}}\left(\frac{\widehat{w_{0}}(j)}{2}+\frac{\widehat{w_{1}}(j)}{2 i(j+\theta)^{n}}\right) e^{i(j+\theta)^{n} t} e_{j}(x) \\
& +\sum_{j \in \mathbb{Z}}\left(\frac{\widehat{w_{0}}(j)}{2}-\frac{\widehat{w_{1}}(j)}{2 i(j+\theta)^{n}}\right) e^{-i(j+\theta)^{n} t} e_{j}(x), \tag{8.26}
\end{align*}
$$

where $\left\{e_{j}\right\}_{j \in \mathbb{Z}}$ is the classical orthonormal Fourier basis of $L^{2}(0,2 \pi)$ given by (2.6).
Recall that in Section 7.3, when the even-order poly-harmonic wave equation was considered under periodic boundary conditions, we identified in Lemma 7.7 the polynomials of order $n \geq 1$ with Fourier coefficients $j^{-n}$ for $j \neq 0$. This allowed us to derive an alternative solution representation from which the weak revival formula was obtained. Similarly, the next lemma identifies functions $h_{n}(x, \theta)$ whose Fourier coefficients are $(j+\theta)^{-n}$.

Lemma 8.8. Let $\theta \in(0,1)$. Then, there exists a sequence of smooth functions denoted by $h_{n}(x, \theta), n \in \mathbb{N}$, with $x \in[0,2 \pi)$ and such that $\widehat{h}_{n}(j, \theta)=(j+\theta)^{-n}$. For fixed integer $n \geq 1, h_{n}(x, \theta)$ is defined inductively by

$$
\begin{equation*}
h_{n}(x, \theta)=\frac{\sqrt{2 \pi} i^{n}}{1-e^{-i 2 \pi \theta}} \frac{x^{n-1} e^{-i \theta x}}{(n-1)!}-\frac{1}{1-e^{i 2 \pi \theta}} \sum_{\ell=1}^{n-1} \frac{(2 \pi i)^{n-\ell}}{(n-\ell)!} h_{\ell}(x, \theta) . \tag{8.27}
\end{equation*}
$$

Proof. Let $\theta \in(0,1), n \in \mathbb{N}$ and consider the function of $x$

$$
f_{n}(x, \theta)=x^{n-1} e^{-i \theta x},
$$

with $x \in[0,2 \pi)$. Then, for $j \in \mathbb{Z}$, the Fourier coefficients of $f_{n}(x, \theta)$ are given by

$$
\begin{aligned}
\widehat{f}_{n}(j, \theta)=\left\langle f_{n}(\cdot, \theta), e_{j}\right\rangle & =\sum_{\ell=1}^{n-1} \frac{(-1)^{\ell+1}}{(-i)^{\ell}} \frac{(2 \pi)^{n-\ell}}{\sqrt{2 \pi}} \frac{(n-1)!}{(n-\ell)!} \frac{e^{-i 2 \pi \theta}}{(j+\theta)^{\ell}} \\
& +\frac{(-1)^{n+1}}{\sqrt{2 \pi}} \frac{(n-1)!}{(-i)^{n}} \frac{\left(e^{-i 2 \pi \theta}-1\right)}{(j+\theta)^{n}} .
\end{aligned}
$$

Solving the above equation for $(j+\theta)^{-n}$ gives that

$$
\frac{1}{(j+\theta)^{n}}=\frac{\sqrt{2 \pi} i^{n}}{1-e^{-i 2 \pi \theta}} \frac{\widehat{f}_{n}(j, \theta)}{(n-1)!}-\frac{1}{1-e^{i 2 \pi \theta}} \sum_{\ell=1}^{n-1} \frac{(2 \pi i)^{n-\ell}}{(n-\ell)!} \frac{1}{(j+\theta)^{\ell}} .
$$

For $\ell \in\{1,2, \ldots, n\}$, let $h_{\ell}(x, \theta)$ be the function whose Fourier coefficients are equal to $(j+\theta)^{-\ell}$. Then, the last equation above implies that

$$
\begin{aligned}
& \int_{0}^{2 \pi} h_{n}(x, \theta) \overline{e_{j}(x)} d x \\
& \quad=\int_{0}^{2 \pi}\left[\frac{\sqrt{2 \pi} i^{n}}{1-e^{-i 2 \pi \theta}} \frac{x^{n-1} e^{-i \theta x}}{(n-1)!}-\frac{1}{1-e^{i 2 \pi \theta}} \sum_{\ell=1}^{n-1} \frac{(2 \pi i)^{n-\ell}}{(n-\ell)!} h_{\ell}(x, \theta)\right] \overline{e_{j}(x)} d x .
\end{aligned}
$$

Therefore, for every $n \in \mathbb{N}$, the function $h_{n}(x, \theta)$ is defined by (8.27).
We proceed by obtaining a representation of $w(x, t)$ in terms of the isometries $P_{k}(t)$ given by (8.17), with $k=2, \ldots, n$.

Proposition 8.9. Let integer $n \geq 1$ and $\theta \in(0,1)$. Then, for any initial conditions $w_{0}$ and $w_{1}$ in $L^{2}(0,2 \pi)$, the solution to (8.25) at fixed time $t \geq 0$ admits the following representation

$$
\begin{equation*}
w(x, t)=w^{+}(x, t)+w^{-}(x, t), \tag{8.28}
\end{equation*}
$$

where

$$
w^{ \pm}(x, t)=e^{ \pm i \theta^{n} t} \mathcal{T}_{\mp n \theta^{n-1} t} \prod_{k=2}^{n} P_{k}\left(\mp\binom{n}{k} \theta^{n-k} t\right)\left[w_{0}(x) \pm w_{1} * h_{n}(\cdot, \theta)(x)\right] .
$$

Proof. From (8.26), we write $w(x, t)$ as (8.28), where

$$
w^{ \pm}(x, t)=\sum_{j \in \mathbb{Z}}\left(\frac{\widehat{w_{0}}(j)}{2} \pm \frac{\widehat{w_{1}}(j)}{2 i(j+\theta)^{n}}\right) e^{ \pm i(j+\theta)^{n} t} e_{j}(x) .
$$

From Lemma 8.8 we have that

$$
\frac{\widehat{w_{1}}(j)}{2 i(j+\theta)^{n}}=\frac{1}{2 i}\left\langle w_{1} * h_{n}(\cdot, \theta), e_{j}\right\rangle, \quad \forall j \in \mathbb{Z} .
$$

Thus, if we set

$$
w_{0}^{ \pm}(x)=w_{0}(x) \pm w_{1} * h_{n}(\cdot, \theta)(x),
$$

then we can write $w^{ \pm}(x, t)$ as

$$
w^{ \pm}(x, t)=\sum_{j \in \mathbb{Z}} \widehat{w_{0}^{ \pm}}(j) e^{ \pm i(j+\theta)^{n} t} e_{j}(x) .
$$

Focusing on $w^{ \pm}(x, t)$, for any $t \geq 0$ we have

$$
e^{ \pm i(j+\theta)^{n} t}=\prod_{k=0}^{n} \exp \left( \pm i j^{k}\binom{n}{k} \theta^{n-k} t\right)
$$

which implies that

$$
w^{ \pm}(x, t)=\sum_{j \in \mathbb{Z}} \widehat{w_{0}^{ \pm}}(j) \prod_{k=0}^{n} \exp \left( \pm i j^{k}\binom{n}{k} \theta^{n-k} t\right) e_{j}(x) .
$$

Equivalently, the last representation is written as

$$
\begin{aligned}
w^{ \pm}(x, t) & =e^{ \pm i \theta^{n} t} \sum_{j \in \mathbb{Z}} \widehat{w_{0}^{ \pm}}(j) e^{ \pm i j n \theta^{n-1} t} \prod_{k=2}^{n} \exp \left( \pm i j^{k}\binom{n}{k} \theta^{n-k} t\right) e_{j}(x) \\
& =e^{ \pm i \theta^{n} t} \mathcal{T}_{\neq n \theta^{n-1} t}\left[\sum_{j \in \mathbb{Z}} \widehat{w_{0}^{ \pm}}(j) \prod_{k=2}^{n} \exp \left( \pm i j^{k}\binom{n}{k} \theta^{n-k} t\right) e_{j}(x)\right],
\end{aligned}
$$

where $\mathcal{T}_{\mp n \theta^{n-1} t}$ is the periodic translation operator.
Finally, for each $k=2, \ldots, n$ using the definition of the operators $P_{k}$ we find that

$$
w^{ \pm}(x, t)=e^{ \pm i \theta^{n} t} \mathcal{T}_{\mp n \theta^{n-1} t} \prod_{k=2}^{n} P_{k}\left(\mp\binom{n}{k} \theta^{n-k} t\right) w_{0}^{ \pm}(x) .
$$

From Proposition 8.9, we now address the behaviour of the solution at rational times and draw some conclusions on the revival property.

If we let $t=2 \pi \frac{p}{q}$ be a rational time, then each component $w^{ \pm}(x, t)$ in (8.28) becomes

$$
\begin{aligned}
& w^{ \pm}\left(x, 2 \pi \frac{p}{q}\right) \\
& =e^{ \pm 2 \pi i \theta^{n} \frac{p}{q}} \mathcal{T}_{\mp 2 \pi n \theta^{n-1} \frac{p}{q}} \prod_{k=2}^{n-1} P_{k}\left(\mp 2 \pi\binom{n}{k} \frac{p}{q} \theta^{n-k}\right)\left[\mathcal{R}_{n}(\mp p, q)\left(w_{0}(x) \pm\left(w_{1} * h_{n}(\cdot, \theta)\right)(x)\right],\right.
\end{aligned}
$$

or due to linearity

$$
\begin{align*}
& w^{ \pm}\left(x, 2 \pi \frac{p}{q}\right)=e^{ \pm 2 \pi i \theta^{n} \frac{p}{q}} \mathcal{T}_{\mp 2 \pi n \theta^{n-1} \frac{p}{q}}^{n} \prod_{k=2}^{n-1} P_{k}\left(\mp 2 \pi\binom{n}{k} \frac{p}{q} \theta^{n-k}\right)\left[\mathcal{R}_{n}(\mp p, q)\left(w_{0}(x)\right]\right. \\
& \quad \pm e^{ \pm 2 \pi i \theta^{n} \frac{p}{q}} \mathcal{T}_{\mp 2 \pi n \theta^{n-1} \frac{p}{q}} \prod_{k=2}^{n-1} P_{k}\left(\mp 2 \pi\binom{n}{k} \frac{p}{q} \theta^{n-k}\right)\left[\mathcal{R}_{n}(\mp p, q)\left(\left(w_{1} * h_{n}(\cdot, \theta)\right)(x)\right] .\right. \tag{8.29}
\end{align*}
$$

Now, we first notice that for any $\theta \in(0,1)$ the second component corresponds to a continuous function on $[0,2 \pi]$. Indeed, $w_{1} * h_{n}(\cdot, \theta)$ is a $2 \pi$-periodic smooth function in $\mathbb{R}$, so when we apply the revival operator $\mathcal{R}_{n}(\mp p, q)$, we obtain a continuous function on $[0,2 \pi]$ and the same is true after the action of $P_{k}(\cdot)$ for all $k=2, \cdots n-1$.

On the other hand, the behaviour of the first component in (8.29) depends on the value of $n$ and the parameter $\theta$. In particular, observe that the expression

$$
\begin{equation*}
\prod_{k=2}^{n-1} P_{k}\left(\mp 2 \pi\binom{n}{k} \frac{p}{q} \theta^{n-k}\right)\left[\mathcal{R}_{n}(\mp p, q)\left(w_{0}(x)\right]\right. \tag{8.30}
\end{equation*}
$$

is the same, for the plus sign, as the representation (8.19) of Corollary 8.5. If $n=2$, it reduces to $\mathcal{R}_{2}( \pm p, q) w_{0}(x)$ and if $n=1$ we just obtain $w_{0}$. So, when $n \leq 2$, it provides a pure revival representation for any $\theta \in(0,1)$. However, if $n \geq 3$, the expression (8.30) reduces to a revival formula only when $\theta$ is rational. If otherwise, it becomes a continuous function on $[0,2 \pi]$ even if $w_{0}$ exhibits jump discontinuities. Similar considerations apply to the minus sign in (8.30).

Summarising the above, we find that representation (8.29) implies that, at rational times, the two components $w^{ \pm}(x, t)$ are given by a pure revival effect perturbed by a continuous function for all $\theta \in(0,1)$ when $n=1$ or $n=2$. For values $n \geq 3$, this weak revival property holds if and only if $\theta$ is a rational number in $(0,1)$. Moreover, because the solution to the periodic problem (8.25) is given by

$$
w(x, t)=w^{+}(x, t)+w^{-}(x, t),
$$

it follows that, when $n \leq 2$, the periodic problem (8.25) exhibits weak revival at rational times, whereas if $n \geq 3$ the weak revival phenomenon breaks whenever $\theta$ is
not rational. Finally, from Lemma 8.7, the same conclusion applies to the solution $u(x, t)$ at rational times of the even-order poly-harmonic wave equation under quasiperiodic boundary conditions on $[0,2 \pi]$.

## Chapter 9

## Conclusion and Further Directions

### 9.1 Conclusion

The main goal of this work was to examine the revival phenomenon in a variety of time evolution problems for linear dispersive PDEs subject to boundary conditions posed on a finite interval. We extended the classical revival Theorem 2.8 in several time evolution problems. Theorem 2.8 describes analytically the revival phenomenon, as a pure revival (see Definition 2.10), in the family of first-order in time, linear dispersive PDEs with integer coefficients when subject to periodic boundary conditions on $[0,2 \pi]$. In this thesis, we considered two main directions. The influence of the boundary conditions on the revival phenomenon and the persistence of the revivals in time evolution problems with second-order derivatives in time.

In Chapters 4 and 6, we showed that the free space linear Schrödinger equation exhibits revivals under two different types of boundary conditions. Pseudo-periodic conditions which couple the two endpoints of $[0,2 \pi]$ in Chapter 4, and Robin-type boundary conditions posed separately at the two ends of $[0, \pi]$ in Chapter 6. From Theorem 4.8 and Corollary 4.9 on the pseudo-periodic problem, we concluded that the non-self-adjointness of the boundary conditions does not affect the revival effect in this case. On the other hand, by the examination of the Robin problem we established Theorem 6.7 and Corollary 6.9, which showed that the revival phenomenon appears in this case due to the weak revival effect defined as a perturbation of a pure revival effect by a continuous function in space (see Definition 2.16).

In comparison to the FSLS equation, we saw in Chapter 5 that there is a strong
influence of the boundary conditions on the revival property for the Airy PDE. In particular, Theorem 5.2 implied that the Airy PDE does not in general exhibit revivals under quasi-periodic boundary conditions on $[0,2 \pi]$, even though they are self-adjoint. This indicated that outside the classical periodic setting of Theorem 2.8, the revival phenomenon could break in PDEs with polynomial dispersion relation and higher than two order derivatives in space, when we consider quasi-periodic boundary conditions. We confirmed this assertion in Section 8.2 where we generalised the lack of revivals in Airy's quasi-periodic problem to quasi-periodic problems with monomial dispersion relations of order higher than two. Furthermore, we were able to conclude that periodic problems for first-order in time PDEs with dispersion relation a polynomial with real, non-rational coefficients, do not, in general, exhibit revivals.

With regards to second-order in time evolution problems, our main model was the even-order poly-harmonic wave equation. For this family of equations, we analysed the revival effect under periodic and quasi-periodic boundary conditions on $[0,2 \pi]$ in Chapters 7 and 8 , respectively. For periodic boundary conditions, we found that the poly-harmonic wave equation exhibit revivals at rational times and thus we extended Theorem 2.8 to second-order in time periodic problems. The revival phenomenon in this case resulted due to the weak revival effect as established by Corollary 7.9 and formulated using the revival functional calculus given in Lemma 7.6. Lemma 7.6 is a special case of the more general revival functional calculus given in Lemma 7.5. Both lemmas imply the validity of Theorem 2.8 and provide an abstract operatorbased framework for the revival phenomenon. The quasi-periodic problem for the even-order poly-harmonic wave equation was consider in Section 8.4. Similar to the first-order in time problems, we were able to deduce that the (weak) revival phenomenon survives in the case of the bi-harmonic wave equation (fourth-order in space), whereas for higher-order space derivatives, in general, breaks.

For any revival phenomenon to manifest, the periodicity and number-theoretic properties of the Fourier series representation of periodic solutions seem to be essential. Our analysis strongly supports this conjecture. Indeed, for a given time evolution problem, in the spirit of Remark 4.1 and the overview given in Section 8.1, our approach on the examination of the revival phenomenon beyond the classical
theory of Theorem 2.8 was to identify the individual periodic components that ensure the existence of pure or weak revivals. We were able to follow this idea either by decomposing the solutions representations to simpler periodic components and utilise the special transformations from Chapter 3, or to perform a suitable transformation that converts non-periodic boundary conditions to periodic, such as a Galilean-type transformation. In the future, we aim to confirm this claim by considering more general time evolution problems which could require a generalisation of the methods here or a completely different approach. Possible directions on the revival phenomenon are included in the next section.

### 9.2 Open Problems

Below, we list a few open problems for future consideration and indicate further directions for the study of the revival phenomenon.

### 9.2.1 Linear Perturbations and More General Boundary Conditions

Recall from Section 2.5, that due to the results of Cho, Kim, Kim, Kwon and Seo in [44] and Rodnianski in [45], the linear Schrödinger equation with a potential $V(x)$,

$$
\partial_{t} u(x, t)=-i\left(-\partial_{x}^{2} u(x, t)+V(x) u(x, t)\right),
$$

exhibits weak revivals (see Definition 2.16) at rational times $t=2 \pi \frac{p}{q}$, under periodic boundary conditions on $[0,2 \pi]$. We believe that the weak revival phenomenon is also present in the linear Schrödinger equation with separated boundary conditions such that the underlying eigenvalue problem is of the regular Sturm-Liouville type.

Conjecture 9.1. Let $h$ and $H$ be real numbers. The IBVP for the linear Schrödinger equation,

$$
\begin{align*}
& \partial_{t} u(x, t)=-i\left(-\partial_{x}^{2} u(x, t)+V(x) u(x, t)\right), \quad u(x, 0)=u_{0}(x)  \tag{9.1}\\
& \partial_{x} u(0, t)-h u(0, t)=0, \quad \partial_{x} u(\pi, t)+H u(\pi, t)=0,
\end{align*}
$$

on $[0, \pi]$, exhibits the phenomenon of weak revivals at rational times $t=2 \pi \frac{p}{q}$, where $p$ and $q$ are co-prime positive integers.

As a first step, we may consider the potential $V(x)$ to be a twice, continuously differentiable function on $[0, \pi]$. Observe that Conjecture 9.1 holds true for the time evolution problem (6.12) from Chapter 6, which is a special case of the time evolution problem (9.1) when $V(x)=0$ for all $x \in[0, \pi]$. However, the eigenvalues of the more general problem (9.1) can not be computed explicitly, with either $V(x)=0$ or $V(x) \neq 0$. Thus, a completely similar approach as in Chapter 6 is not possible.

A possible direction could be to utilise the precise asymptotic behaviour of the eigenvalues $\lambda_{j}$ and then decompose the solution into a continuous and a discontinuous part at rational times. It is known that the eigenvalues behave like $j^{2}$ for large $j$ when $V \in C^{2}[0, \pi]$, see [61] by Levitan and Sargsian. For more singular potentials, the work of Fulton and Pruess, [62], on the asymptotic behaviour of the eigenpairs to regular Sturm-Liouville problems should be valuable.

We should further mention that a rigorous proof of Conjecture 9.1 will strongly support the general conjecture of Chen and Olver in [38]. Indeed, in [38, page 12], Chen and Olver suggested that dispersion relations with polynomial growth should lead the solution of a given time evolution problem to exhibit revivals. Additional numerical examples by Olver, Sheils and Smith in [15], on the free linear Schrödinger equation, are in favour of an affirmative answer.

In [15], the authors derived the generalised Fourier series representation of the solution to the FSLS equation

$$
\partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t),
$$

with an initial condition $u(x, 0)=u_{0}(x)$ and subject to the general linear, homogeneous boundary conditions

$$
\begin{array}{r}
\beta_{11} \partial_{x} u(2 \pi, t)+\beta_{12} u(2 \pi, t)+\beta_{13} \partial_{x} u(0, t)+\beta_{14} u(0, t)=0, \\
\beta_{22} u(2 \pi, t)+\beta_{23} \partial_{x} u(0, t)+\beta_{24} u(0, t)=0,
\end{array}
$$

on $[0,2 \pi]$, where the coefficients are, in general, complex numbers. They performed
numerical experiments for a variety of boundary conditions and observe that there are cases for which the revival and fractalisation effects occur. However, complete analytical results have not been established yet.

Another model of notable interest is the bi-harmonic wave equation with clamped boundary conditions on $[0, \pi]$,

$$
\begin{align*}
& \partial_{t}^{2} u(x, t)=-\partial_{x}^{4} u(x, t), \quad u(x, 0)=u_{0}(x),  \tag{9.2}\\
& u(0, t)=u(\pi, t)=0, \quad \partial_{x} u(x, t)=\partial_{x}(\pi, t)=0 .
\end{align*}
$$

Contrast to the periodic or simply supported boundary conditions considered in Section 7.3 and Subsection 7.3.2, in the case of (9.2), the eigenvalues satisfy a transcendental equation for which a closed form solution is not possible. Thus, a direct derivation of the revival effect can not follow based on the methods of Chapter 7 or observed by simply plotting the truncated eigenfuction representation.

Accurate numerical approximation of the solution will first allow the observation of the revivals at rational times. For example, by employing the shooting method for the numerical solution of the underlying eigenvalue problem, see [63] by Guenther and Lee. Furthermore, an analytical approach could follow by deriving the precise asymptotic behaviour of the eigenpairs in order to extract the periodic components of the solution. The approach here will be in the same lines as for the study of Conjecture 9.1. We speculate that if any revival phenomenon is present in the time evolution problem (9.2), this is because of the weak revival effect, resembling the case of periodic and simply supported boundary conditions.

Conjecture 9.2. The IBVP (9.2) for the bi-harmonic wave equation on $[0, \pi]$ exhibits weak revivals at rational times $t=2 \pi \frac{p}{q}$, where $p$ and $q$ are co-prime positive integers.

As mentioned in Subsection 7.3.2, recently, in [59], Dubois, Lefebvre and Sebbah provided experimental data for the observation of the revival effect in the context of the two-dimensional bi-harmonic wave equation with clamped boundaries. Therefore, a rigorous analysis of Conjecture 9.2, will provide, in the one-dimensional regime, strong theoretical support to [59].

Finally, in a different direction, as we have seen throughout the thesis, the anal-
ysis of the revival phenomenon has been on the base of eigenfunction expansions, see also Remark 4.1-1. Still, for the Airy equation

$$
\partial_{t} u(x, t)=\partial_{x}^{3} u(x, t)
$$

there exist boundary conditions for which the associate spatial differential operator does not admit a basis of eigenfunctions. For example, the pseudo-Dirichlet boundary conditions

$$
u(0, t)=u(2 \pi, t)=\partial_{x} u(2 \pi, t)=0
$$

on $[0,2 \pi]$, do not yield a generalised Fourier series representation. Nonetheless, an integral representation of the solution is available via the Unified Transform Method. For a detailed analysis of the Airy PDE with pseudo-Dirichlet conditions we refer to [53], [64] by Fokas and Pelloni. Therefore, for these types of boundary conditions the investigation of the revival phenomenon requires new ideas. Moreover, it would be interesting to identify other boundary conditions for which the revival effect breaks at rational times in the Airy PDE, similar to the quasi-periodic conditions in Chapter 5.

### 9.2.2 Applications of the Revival Functional Calculus

In Chapter 7, based on the non-self-adjoint differential operator $L_{h, \theta}$, defined by (7.1) and (7.2), we developed a functional calculus for the revival phenomenon. This was the context of Lemma 7.5. However, the main application of the revival functional calculus was in the framework of Lemma 7.6, which corresponds to the periodic self-adjoint case of the operator $L_{1,0}$. This allowed us to derive the weak revival representation (7.22) of the solution to the periodic problem for the evenorder poly-harmonic wave equation.

Ideally, we would like to identify situations where the general form of the revival functional calculus, that is Lemma 7.5, can be utilised. As a starting point in this
direction, the following initial boundary value problem on $[0,2 \pi]$,

$$
\begin{array}{r}
\partial_{t} u(x, t)=\left(i \partial_{x}^{2}-2 i \ln (h) \partial_{x}+i \ln ^{2}(h)\right) u(x, t), \quad u(x, 0)=u_{0}(x),  \tag{9.3}\\
\left(h e^{i \theta}\right)^{2 \pi} u(0, t)=u(2 \pi, t), \quad\left(h e^{i \theta}\right)^{2 \pi} \partial_{x} u(0, t)=\partial_{x} u(2 \pi, t),
\end{array}
$$

seems to be a suitable example. In (9.3), $h$ and $\theta$ are fixed real numbers such that $h>0$ and $\theta \in[0,1)$.

### 9.2.3 Non-linear Equations

In this thesis, we mainly considered the revival effect in linear dispersive PDEs. In terms of non-linear equations, we were able to extend the Talbot effect from the periodic problem for the cubic NLS equation, Theorem 2.15 due to Erdoğan and Tzirakis [41], to the quasi-periodic problem (8.21), Theorem 8.6. We expect that it is likely to generalise the Talbot effect to the more general non-linear, pseudoperiodic problem

$$
\begin{aligned}
& \partial_{t} u(x, t)=i \partial_{x}^{2} u(x, t)+i|u(x, t)|^{2} u(x, t), \quad u(x, 0)=f(x), \\
& \beta_{0} u(0, t)=u(2 \pi, t), \quad \beta_{1} \partial_{x} u(0, t)=\partial_{x} u(2 \pi, t),
\end{aligned}
$$

given on $[0,2 \pi]$ and where the complex numbers $\beta_{0}, \beta_{1}$ satisfy Assumption 4.3. The main idea here is to combine the decomposition derived in Theorem 4.8 for the linear problem in terms of periodic problems together with the analysis in [41] for the periodic non-linear equation.

In the non-linear regime, a further direction is to consider the revival and fractalisation effects in the Korteweg-de Vries (KdV) equation with quasi-periodic boundary conditions,

$$
\begin{align*}
& \partial_{t} u(x, t)=-\partial_{x}^{3} u(x, t)-2 u(x, t) \partial_{x} u(x, t), \quad u(x, 0)=u_{0}(x),  \tag{9.4}\\
& e^{i 2 \pi \theta} \partial_{x}^{m} u(0, t)=\partial_{x}^{m} u(2 \pi, t), \quad m=0,1,2, \quad \theta \in[0,1),
\end{align*}
$$

on $[0,2 \pi]$. For $\theta=0$, the boundary conditions become periodic, and we know that in this case the Talbot effect persists due to the results in [40] by Erdoğan and

Tzirakis.
For $\theta \neq 0$, compared to the quasi-periodic problem for the cubic NLS equation, the KdV equation is not invariant under the transformation (8.9), for $n=3$, which transforms the quasi-periodic boundary conditions to periodic. This suggests that new ideas needed to tackle this problem.

Further, we remark that, to our knowledge, there is no result that indicates the well-posedness of the IBVP (9.4) when $\theta \neq 0$. Therefore, a first direction would be to analyse this aspect, starting with the case when $\theta$ is a rational number. Then, it might be feasible to adopt the arguments in [27], [40] for the periodic case into the case of the quasi-periodic problem (9.4). We suspect that the framework provided by Ruzhanski and Tokmagambeto in [56] should be valuable for the analysis of (9.4), either for rational or generic $\theta$ in $(0,1)$.

Finally, if we assume that there exists a well-posed setting for the quasi-periodic problem (9.4), then a natural question that arises is if the revival phenomenon breaks whenever $\theta$ is irrational, similar to the quasi-periodic problem for the Airy PDE (Theorem 5.2). In other words, if we fix $\theta$ to be an irrational number in $(0,1)$, then is there a smoothing effect on the solution of (9.4) at rational times $t=2 \pi \frac{p}{q}$ ?

### 9.2.4 Revivals in Higher Dimensions

In the existing literature, the revival phenomenon in higher dimensions seems to be restricted to the free space linear Schrödinger equation

$$
\begin{equation*}
\partial_{t} u(x, t)=i \Delta u(x, t) . \tag{9.5}
\end{equation*}
$$

In (9.5), $\Delta$ denotes the Laplace operator and $x$ is regarded as the space co-ordinate of dimension $d$. Extensions of the revival property on higher dimensional spheres and tori were obtained by Taylor in [12]. Moreover, in [65], Bělín, Horsley and Tyc recently derived a surprising result. They showed that the revival effect persists in the Schrödinger equation (9.5) when posed on the surface of a regular tetrahedron.

It should be possible to extend the revival phenomenon in the case of zero Dirichlet (and/or Neumann) boundary conditions on specific two-dimensional, triangular domains for the linear Schrödinger equation (9.5). Similar to the one-dimensional
case, in order to examine the revival phenomenon an investigation of the spectral problem of the Laplace operator is required. In the case of a half-square (right triangle with two equal sides), the eigenpairs can be obtained by the spectral problem on the square under rigid transformation methods. From our end, preliminary results show that at rational times the revivals persist in this case. On the other hand, the eigenvalues of the Dirichlet-Laplacian on an equilateral triangle are known since the work of Lamé in the 19th Century and since then many researchers have re-derived them together with the completeness of the associated eigenfunctions, see for example [66] by McCartin. In the future, based on these results we plan to examine the revival phenomenon. Further directions might include the case of an isosceles or an arbitrary triangle and regular polygonal domains.

Another, perhaps more ambitious, direction in higher dimensions is the examination of the revival phenomenon, with either numerical or analytical methods, in the two-dimensional, bi-harmonic wave equation

$$
\begin{equation*}
\partial_{t}^{2} u(x, t)=-\Delta^{2} u(x, t), \tag{9.6}
\end{equation*}
$$

on the square $[0, \pi] \times[0, \pi]$. In (9.6), the symbol $\Delta^{2}$ denotes the bi-harmonic operator. We pose the equation under either simply supported or clamped boundary conditions. Both cases set physical problems of special interest for the theoretical treatment of the revival effect, since they may lead to full justification of the experimental revival observations in [59]. Simply supported boundary conditions could be considered as a first step. In this case, we may approach the problem either by utilizing the one-dimensional results or the equivalence of the bi-harmonic wave equation with a system of linear Schrödinger equations. For the case of clamped boundaries, which corresponds to the experimental study [59], a numerical examination of the revival could be based on spectral methods for the underlying eigenvalue problem of the bi-harmonic operator, see [67] by Trefethen.

## Appendix A

## Linear Dispersive Partial

## Differential Equations

One possible classification of a partial differential equation in one space dimension is based on the attached dispersion relation. Following [25], consider a linear PDE

$$
\begin{equation*}
P\left(\partial_{t}, \partial_{x}\right) u(x, t)=0, \tag{A.1}
\end{equation*}
$$

where $P$ is a polynomial in the partial derivatives and with complex constant coefficients. The variable $t$ denotes time and $x$ serves as the space co-ordinate of dimension one. By fixing plane wave solutions of the form

$$
\begin{equation*}
u(x, t)=e^{i(k x-\omega t)} \tag{A.2}
\end{equation*}
$$

we obtain by substitution the dispersion relation

$$
\begin{equation*}
P(-i \omega, i k)=0, \tag{A.3}
\end{equation*}
$$

which gives the time frequency $\omega(k)$ as a function of the spatial frequency or the wave number $k$ (assuming that we can solve (A.3) for $\omega$ ).

In physical terms, see [25] or [26], a linear dispersive equation describes the fact that plane wave solutions of different wave numbers $k$ propagate with a different
phase velocity $w(k) / k$. Thus, for example, the transport equation

$$
\partial_{t} u(x, t)=-\partial_{x} u(x, t),
$$

with dispersion relation $\omega=k$, is not dispersive. Similarly, the classical wave equation

$$
\partial_{t}^{2} u(x, t)=\partial_{x}^{2} u(x, t),
$$

is not dispersive. On the other hand, a typical example of a linear dispersive PDE is the linear Schrödinger equation (FSLS)

$$
\begin{equation*}
\partial_{t}(x, t)=i \partial_{x}^{2} u(x, t), \tag{A.4}
\end{equation*}
$$

which has dispersion relation $\omega=k^{2}$. Another case is the Airy PDE (AI)

$$
\begin{equation*}
\partial_{t} u(x, t)=\partial_{x}^{3} u(x, t), \tag{A.5}
\end{equation*}
$$

with the dispersion relation given by $\omega=k^{3}$.
In rigorous terms, for a given linear PDE of the form (A.1), the term dispersive amounts for the following definition, [25].

Definition A.1. The PDE (A.1) is called dispersive if for each wave number $k$, the frequency $\omega(k)$ is real and the second derivative of $\omega$ with respect to $k$ does not vanish identically ( $\omega^{\prime \prime} \not \equiv 0$ ).

The condition $\omega^{\prime \prime} \not \equiv 0$ eliminates some controversial cases, such as the class of equations with dispersion relation $\omega(k)=a k+b$, where $a$ and $b$ are some fixed real numbers. For example, the equation

$$
\partial_{t} u(x, t)=-\partial_{x} u(x, t)-i u(x, t)
$$

has dispersion relation $\omega(k)=k+1$, which implies that the phase velocity is not constant. Thus, in a sense it is a dispersive equation. However, by a standard Fourier-transform argument, it follows that an initial waveform $f(x)$ evolves, at time $t>0$, to

$$
u(x, t)=e^{i t} f(x-t)
$$

Therefore, although the initial function $f(x)$ is modified by the propagation, there is no dispersion in the sense that the shape of $f(x)$ stays essentially unaltered and does not spread out.

In particular, $\omega^{\prime \prime} \not \equiv 0$ implies that the group velocity $\omega^{\prime}(k)$, which in the onedimensional setting has a more significant role than the phase velocity, is not a constant. We refer to [25] for further details on the group velocity in linear dispersive equations.

Regarding our investigation of the revival phenomenon, equations (A.4) and (A.5) are the main points of reference for first-order in time evolution problems. Moreover, the bi-harmonic wave equation

$$
\partial_{t}^{2} u(x, t)=-\partial_{x}^{4} u(x, t),
$$

gives a model of a second-order in time linear dispersive equation, with dispersion relation $\omega^{2}=k^{2}$.

## Appendix B

## Bases in Hilbert Space

In many instances, the solution to an initial boundary value problem on a finite interval can be expressed as an infinite series in terms of a basis of square integrable functions. For example, when applicable, this is the result of the classical Fourier method. Hence, to set some agreed-upon terminology, in this appendix we briefly revise the notion of a basis in a Hilbert space and provide also relevant properties useful for the solution to initial boundary value problems.

Following [68], see also [69], a countable family $\left\{\phi_{j}\right\}_{j=1}^{\infty}$ of elements of a complex, separable, infinite-dimensional Hilbert space $\mathcal{H}$ is called a Schauder basis or simply a basis if any $f \in \mathcal{H}$ has a unique expansion of the form

$$
\begin{equation*}
f=\sum_{j=1}^{\infty} f_{j} \phi_{j}:=\lim _{n \rightarrow \infty} \sum_{j=1}^{n} f_{j} \phi_{j} \tag{B.1}
\end{equation*}
$$

for some (unique) $f_{j} \in \mathbb{C}$.
The convergence in (B.1) is with respect to the norm of $\mathcal{H}$ defined as usual by $\|f\|=\sqrt{\langle f, f\rangle}$, with $\langle f, g\rangle$ the inner product on $\mathcal{H}$. The numbers $f_{j}$ in (B.1) are called generalised Fourier coefficients or simply Fourier coefficients of $f$ with respect to $\left\{\phi_{j}\right\}_{j=1}^{\infty}$ and the expansion is called the generalised Fourier series or Fourier series of $f$ with respect to $\left\{\phi_{j}\right\}_{j=1}^{\infty}$.

We say that $\left\{\phi_{j}\right\}_{j=1}^{\infty}$ forms an orthonormal set in $\mathcal{H}$ if it satisfies the condition

$$
\left\langle\phi_{j}, \phi_{k}\right\rangle= \begin{cases}1, & j=k  \tag{B.2}\\ 0, & j \neq k\end{cases}
$$

If it is a basis, then $\left\{\phi_{j}\right\}_{j=1}^{\infty}$ is said to be an orthonormal basis in $\mathcal{H}$. The next, elementary lemma gives equivalent conditions for an orthonormal set to be a basis, see [32, Theorem 5.27].

Lemma B.1. Let $\left\{\phi_{j}\right\}_{j=1}^{\infty}$ be an orthonormal set in $\mathcal{H}$. Then, the following are equivalent.
(i) $\left\{\phi_{j}\right\}_{j=1}^{\infty}$ is an orthonormal basis.
(ii) For every $f \in \mathcal{H}$ the expansion

$$
f=\sum_{j=1}^{\infty}\left\langle f, \phi_{j}\right\rangle \phi_{j},
$$

holds with the respect to the norm of $\mathcal{H}$.
(iii) For every $f \in \mathcal{H}$, we have Parseval's identity

$$
\|f\|^{2}=\sum_{j=1}^{\infty}\left|\left\langle f, \phi_{j}\right\rangle\right|^{2} .
$$

(iv) If $\left\langle f, \phi_{j}\right\rangle=0$ for all $j$, then $f=0$.

Apart form orthonormal bases, in the analysis of time evolution problems we may encountered other types of bases. For particular interest to us, are the Riesz bases defined as follows.

Definition B. 2 ([68],[69]). A sequence $\left\{\phi_{j}\right\}_{j=1}^{\infty}$ is called a Riesz basis in a Hilbert space $\mathcal{H}$ if there exists a bounded linear operator $B: \mathcal{H} \rightarrow \mathcal{H}$, with bounded inverse, and such that the sequence $\left\{B \phi_{j}\right\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$.

Notice that if $\left\{\phi_{j}\right\}_{j=1}^{\infty}$ is a Riesz basis then it is a basis. Indeed, for every $f \in \mathcal{H}$ we have

$$
B f=\sum_{j=1}^{\infty}\left\langle B f, B \phi_{j}\right\rangle B \phi_{j} .
$$

Because $B$ is a bounded linear operator we can exchange its action and take the limit. Hence,

$$
B f=B\left(\sum_{j=1}^{\infty}\left\langle B f, B \phi_{j}\right\rangle \phi_{j}\right)
$$

Then, we take the inverse on both sides to obtain the expansion

$$
f=\sum_{j=1}^{\infty}\left\langle B f, B \phi_{j}\right\rangle \phi_{j}=\sum_{j=1}^{\infty} f_{j} \phi_{j} .
$$

Note that the generalised Fourier coefficients are given by

$$
f_{j}=\left\langle B f, B f_{j}\right\rangle \in \mathbb{C},
$$

or in terms of $B$ and its adjoint $B^{*}$ by

$$
f_{j}=\left\langle f, B^{*} B f_{j}\right\rangle .
$$

In particular, for a given Riesz basis we obtain a sequence $\left\{\psi_{j}\right\}_{j=1}^{\infty}$ defined by

$$
\psi_{j}=B^{*} B \phi_{j}
$$

for which the biorthogonality condition

$$
\left\langle\phi_{j}, \psi_{k}\right\rangle=\left\langle B \phi_{j}, B \phi_{k}\right\rangle= \begin{cases}1, & j=k \\ 0, & j \neq k\end{cases}
$$

holds. This motivates the following classical definition.

Definition B.3. Two countable sets $\left\{\phi_{j}\right\}_{j=1}^{\infty},\left\{\psi_{j}\right\}_{j=1}^{\infty}$ in a separable, infinite dimensional Hilbert space $\mathcal{H}$ are called biorthogonal if the following condition holds

$$
\left\langle\phi_{j}, \psi_{k}\right\rangle= \begin{cases}1, & j=k  \tag{B.3}\\ 0, & j \neq k\end{cases}
$$

We note that $\left\{\psi_{j}\right\}_{j=1}^{\infty}$ is also a Riesz basis when $\left\{\phi_{j}\right\}_{j=1}^{\infty}$ is. The next lemma is the analogue to Lemma B. 1 for Riesz bases, see [68, Theorem 3.4.5]

Lemma B.4. Given a countable set $\left\{\phi_{j}\right\}_{j=1}^{\infty}$ in $\mathcal{H}$ the following conditions are equivalent.
(i) $\left\{\phi_{j}\right\}_{j=1}^{\infty}$ is a Riesz basis.
(ii) There is a positive constant $c$ such that

$$
c^{-1}\|f\|^{2} \leq \sum_{j=1}^{\infty}\left|\left\langle f, \phi_{j}\right\rangle\right|^{2} \leq c\|f\|^{2},
$$

for all $f$ in $\mathcal{H}$.
(iii) Given complex numbers $\left\{f_{j}\right\}_{j=1}^{\infty}$, the series

$$
\sum_{j=1}^{\infty} f_{j} \phi_{j}
$$

is norm convergent and defines an element in $\mathcal{H}$ if and only if

$$
\sum_{j=1}^{\infty}\left|f_{j}\right|^{2}<\infty
$$

We close with a specific criterion for a sequence to form a Riesz basis in a Hilbert space. It allows us to derive a generalised Fourier series representation to the solution of the free linear Schrödinger equation under the class of pseudo-periodic boundary conditions in Chapter 4. We were not able to find a specific reference for this result, so we include a self-contained proof.

Lemma B.5. Let $\left\{m_{j}\right\}_{j=1}^{\infty}$ and $\left\{\ell_{j}\right\}_{j=1}^{\infty}$ be orthonormal bases in a Hilbert space $\mathcal{H}$ and let $a \in \mathbb{C}$ such that $|a| \neq 1$. Then, the sequence

$$
\phi_{j}=m_{j}+a \ell_{j}, \quad j \in \mathbb{N}
$$

forms a Riesz basis in $\mathcal{H}$.
Proof. Consider the linear map $S: \mathcal{H} \rightarrow \mathcal{H}$ given by

$$
S f=\sum_{j=1}^{\infty}\left\langle f, m_{j}\right\rangle \ell_{j} .
$$

Then, since $\left\{m_{j}\right\}_{j=1}^{\infty}$ and $\left\{\ell_{j}\right\}_{j=1}^{\infty}$ are both orthonormal bases, by Parseval's identiy, we have $\|S f\|^{2}=\|f\|^{2}$, which implies that $S$ is an isometry of $\mathcal{H}$. Moreover, $S$ is
also one-to-one and onto with inverse given by

$$
S^{-1} f=\sum_{j=1}^{\infty}\left\langle f, \ell_{j}\right\rangle m_{j}, \quad \forall f \in \mathcal{H} .
$$

Clearly, $S^{-1}$ is also bounded as it is also an isometry. Moreover, for each $j \in \mathbb{N}$,

$$
S m_{j}=\sum_{n=1}^{\infty}\left\langle m_{j}, m_{n}\right\rangle \ell_{n}=\ell_{j} .
$$

Now assume that $|a|<1$. Then, $\|a S f\|<|a|\|f\|$ for any $f \in \mathcal{H}$. Thus, if $I$ denotes the identity operator, then the operator operator $T=I+a S$ has a bounded inverse $T^{-1}$ (see [70, Theorem 4.40]). Because $T m_{j}=\phi_{j}$, we see that $T^{-1} \phi_{j}=m_{j}$, which implies that $\phi_{j}$ is a Riesz basis.

If $|a|>1$, then we can write $\phi_{j}=a h_{j}$, with $h_{j}=\ell_{j}+a^{-1} m_{j}$. However, $\left|a^{-1}\right|<1$, and so $\left\{h_{j}\right\}_{j=1}^{\infty}$ is a Riesz basis and the same is true for $\phi_{j}$.

## Appendix C

## Linear Opeators and <br> Eigenfunction Expansions

In time evolution partial differential equations the spatial part of the equation is usually viewed as a linear operator, in general unbounded, defined on a suitable subset of a Hilbert space. For example, the linear Schrödinger equation (FSLS) or the Airy PDE (AI) can be rewritten as

$$
\partial_{t} u(x, t)=-i L u(x, t)
$$

where $L$ is either $-\partial_{x}^{2}$ or $i \partial_{x}^{3}$. When we require $u(x, t)$ to satisfy a number of specific boundary conditions on a finite interval, then $L$ can be defined on a space of sufficiently continuously differentiable functions satisfying the boundary conditions. Therefore, in this appendix we introduce the standard concepts around the theory of (unbounded) linear operators. A detailed exposition of the material here can be found in [71, sections 1.1 and 1.2].

Definition C.1. A linear operator on the Hilbert space $\mathcal{H}$ is a linear map $L$ : $\mathrm{D}(L) \rightarrow \mathcal{H}$, where $\mathrm{D}(L)$ is a linear dense subspace of $\mathcal{H}$. We call $\mathrm{D}(L)$ the domain of the operator $L$.

The problem of funding all complex numbers $\lambda$ and non-zero $f \in \mathrm{D}(L)$ such that

$$
L f=\lambda f,
$$

is called the eigenvalue problem for the linear operator $L$. If such a pair $(\lambda, f)$ exists then we call $\lambda$ an eigenvalue of $L$ and $f$ an eigenfunction (associated with the eigenvalue $\lambda$ ). Sometimes we refer to the pair $(\lambda, f)$ as an eigenpair of the operator $L$.

In the analysis of initial boundary value problems, the underlying spatial linear differential operator often has a family of eigenfuctions which forms a Riesz basis of the underlying Hilbert space $\mathcal{H}$. In these cases, for an arbitrary element $f \in \mathcal{H}$ its unique expansion with respect to the basis of eigenfunctions is known as an eigenfunction expansion.

The set of eigenvalues is a subset of the spectrum of an operator $L$, which is denoted by $\operatorname{Spec}(\mathrm{L})$ and defined below, where $I: \mathcal{H} \rightarrow \mathcal{H}$ is the identity operator.

Definition C.2. The spectrum of a linear operator $L: D(L) \rightarrow \mathcal{H}$ is the set Spec $(L)$ containing all complex numbers $z$ such that the operator $(z I-L): D(L) \rightarrow \mathcal{H}$ does not have a bounded inverse.

Eigenvalue problems for linear differential operators correspond to particular types of boundary value problems for linear differential equations. In such problems, we often encounter the special classes of self-adjoint operators.

Definition C.3. Let $L: D(L) \rightarrow \mathcal{H}$ be a linear operator. Define $D\left(L^{*}\right) \subset \mathcal{H}$ to be the set of all $g \in \mathcal{H}$ such that there exist $h \in \mathcal{H}$ so that

$$
\langle L f, g\rangle=\langle f, h\rangle, \quad \forall f \in D(L)
$$

The adjoint operator $L^{*}: D\left(L^{*}\right) \rightarrow \mathcal{H}$ is defined by $L^{*} g=h$. If $L=L^{*}$, meaning that $D(L)=D\left(L^{*}\right)$ and $L f=L^{*} f$ for all $f \in D(L)$, then $L$ is called self-adjoint.

Due to the density of $\mathrm{D}(L)$ in $\mathcal{H}$ the adjoint $L^{*}$ is a well-defined linear operator. Indeed, let $g \in \mathrm{D}\left(L^{*}\right)$ and $h_{1}, h_{2}$ be two elements of $\mathcal{H}$ such that

$$
\langle L f, g\rangle=\left\langle f, h_{1}\right\rangle=\left\langle f, h_{1}\right\rangle
$$

for all $f \in \mathrm{D}(L)$. Then, $\left\langle f, h_{1}-h_{2}\right\rangle=0$, for all $f \in \mathrm{D}(L)$, which implies that $h_{1}=h_{2}$. Moreover, because Hilbert spaces are reflexive Banach spaces, it follows that the adjoint is a linear map with dense domain and hence a linear operator.

A related class of operators is that of symmetric operators.
Definition C.4. A linear operator $L: D(L) \rightarrow \mathcal{H}$ is called symmetric if for every $f$ and $g$ in $D(L)$ we have

$$
\langle L f, g\rangle=\langle f, L g\rangle .
$$

Notice, that self-adjoint operators are particular cases of symmetric operators. It is well known that the eigenvalues of a symmetric operator are always real. This can be seen by considering an eigenpair $(\lambda, f)$ with $\|f\|=1$. Then, we have

$$
\lambda=\lambda\|f\|^{2}=\lambda\langle f, f\rangle=\langle\lambda f, f\rangle=\langle L f, f\rangle=\langle f, L f\rangle=\langle f, \lambda f\rangle=\bar{\lambda}\langle f, f\rangle=\bar{\lambda}
$$

where $\overline{(\cdot)}$ denotes complex conjugation.
A symmetric operator $L$ is always closable, meaning that there exists an extension $\bar{L}$ of $L$ (that is $\bar{L} f=L f$, for all $f$ in $\mathrm{D}(L)$ and $\mathrm{D}(L) \subset \mathrm{D}(\bar{L})$ ) which is closed. In turn, $\bar{L}$ is said to be closed, if for a sequence $f_{n} \in \mathrm{D}(\bar{L})$ with limit $f \in \mathcal{H}$ and for $g \in \mathcal{H}$ such that $\bar{L} f_{n} \rightarrow g$ as $n \rightarrow \infty$, it follows that $f \in \mathrm{D}(\bar{L})$ and $\bar{L} f=g$. For a given closable operator there always exists a closed extension with minimal domain among all closed extensions, which is called the closure. Self-adjoint operators are always closed.

Regarding the applications of interest in this thesis, for initial boundary value problems sometimes it is more convenient to work with domains for which the differential operators are not necessarily closed, but nonetheless they are symmetric. An intermediate class between symmetry and self-adjointness is known as essential self-adjointness.

Definition C.5. A linear operator $L: D(L) \rightarrow \mathcal{H}$ is called essentially self-adjoint if it is symmetric and its closure $\bar{L}$ is self-adjoint.

In practise, working with an essentially self-adjoint operator is almost the same as working with a self-adjoint operator, since we can use the density of the domain to argue. The next lemma provides a useful criterion, suited for the framework of the thesis, which establishes when a symmetric operator is essentially self-adjoint, see [71, Lemma 1.2.2].

Lemma C.6. Let $L: D(L) \rightarrow \mathcal{H}$ be a linear symmetric operator with a family of eigenfuctions $\left\{f_{n}\right\}_{n=1}^{\infty} \subset D(L)$ forming an orthonormal basis of $\mathcal{H}$. Then $L$ is essentially self-adjoint and the spectrum of its closure $\bar{L}$ is the closure of the set of eigenvalues $\left\{\lambda_{n}\right\}_{n=1}^{\infty}$ of $L$ in $\mathbb{R}$.

Observe that, if a linear operator is symmetric and has an orthonormal basis of eigenfunctions, then Lemma C. 6 implies that its closure is self-adjoint.

## Appendix D

## Sobolev Spaces

In this appendix we recall the standard $L^{2}(0,2 \pi)$ based Sobolev spaces. First, we give the definition of the Sobolev space of positive integer order over the open interval $(0,2 \pi)$ and outline some of their properties. For this part we follow Grubb [72, Chapter 4]. Then, we consider the case of $2 \pi$-periodic Sobolev spaces of real non-negative order. For this part see either Kress [73, Chapter 8] or Iorio and Iorio [55, Chapter 3]. In the former, the approach is directly in the setting of Fourier series in $L^{2}(0,2 \pi)$, whereas in the latter, the approach is in the more general framework of periodic distributions.

Below, we use the notation $\phi^{\prime}, \phi^{\prime \prime}, \phi^{(m)}$ for the derivatives of a function and we denote by $C_{c}^{\infty}(0,2 \pi)$ the space of smooth functions with compact support in $(0,2 \pi)$. We begin with the notion of the weak derivative in $L^{2}(0,2 \pi)$.

Definition D.1. Let $m \in \mathbb{N}$. We say that a function $f \in L^{2}(0,2 \pi)$ is $m$ times weakly differentiable in $L^{2}(0,2 \pi)$ if there exist functions $g_{n} \in L^{2}(0,2 \pi), n=1, \ldots, m$, such that

$$
\int_{0}^{2 \pi} f(x) \overline{\phi^{(n)}(x)} d x=(-1)^{n} \int_{0}^{2 \pi} g_{n}(x) \overline{\phi(x)} d x
$$

for all for all $\phi \in C_{c}^{\infty}(0,2 \pi)$ and $j=1, \ldots, m$. We call $g_{n}$ the $n$-th weak derivative of $f$ and write $g_{n}=f^{(n)}$.

The $L^{2}(0,2 \pi)$ based Sobolev space $H^{m}(0,2 \pi)$ can now be defined as follows.

Definition D. 2 ([72]). The first-order Sobolev space $H^{1}(0,2 \pi)$ consists of functions $f$ in $L^{2}(0,2 \pi)$ which have a weak derivative $f^{\prime} \in L^{2}(0,2 \pi)$. For integer $m \geq 2$, the

Sobolev space $H^{m}(0,2 \pi)$ is defined inductively by

$$
H^{m}(0,2 \pi)=\left\{f \in H^{m-1}(0,2 \pi) ; f^{\prime} \in H^{m-1}(0,2 \pi)\right\} .
$$

For any $m \in \mathbb{N}$, it is known that $H^{m}(0,2 \pi)$ is a complex Hilbert space under the inner product

$$
\langle f, g\rangle_{m}=\sum_{n=0}^{m} \int_{0}^{2 \pi} f^{(n)}(x) \overline{g^{(n)}(x)} d x, \quad \forall f, g \in H^{m}(0,2 \pi) .
$$

Also, it is readily seen that $H^{m}(0,2 \pi)$ contains $C^{m}[0,2 \pi]$, i.e. the space of all $m$ times continuously differentiable functions on $[0,2 \pi]$. In particular, as we see in the next proposition, one-dimensional Sobolev spaces can be characterised in terms of continuously differentiable functions. Moreover, the formula of integration by parts extends in the Sobolev space setting. For the proof of each part we refer to [72, Theorem 4.17 and 4.14].

Proposition D.3. Let $m \in \mathbb{N}$. Then $H^{m}(0,2 \pi)$ admits the following properties.
(i) $H^{1}(0,2 \pi)$ consists of continuous functions on $[0,2 \pi]$ which are weakly differentiable. For any integer $m \geq 2, H^{m}(0,2 \pi)$ can be characterised as follows

$$
H^{m}(0,2 \pi)=\left\{f \in C^{m-1}[0,2 \pi] ; f^{(m-1)} \in H^{1}\right\} .
$$

(ii) If $f, g \in H^{1}(0,2 \pi)$, then the product $f g$ belongs in $H^{1}(0,2 \pi)$ with $(f g)^{\prime}=$ $f^{\prime} g+f g^{\prime}$ and for any $a, b \in[0,2 \pi]$ the integration by parts formula

$$
\int_{a}^{b} f^{\prime}(x) g(x) d x=[f(x) g(x)]_{a}^{b}-\int_{a}^{b} f(x) g^{\prime}(x) d x
$$

holds.

We now turn our attention to the periodic setting and define the $2 \pi$-periodic Sobolev Spaces of real non-negative order.

Definition D. 4 ([73], [55]). Let $s \geq 0$. The $2 \pi$-periodic Sobolev space $H_{\text {per }}^{s}(0,2 \pi)$
of order $s$ is defined as the following subspace of $L^{2}(0,2 \pi)$

$$
H_{\text {per }}^{s}(0,2 \pi)=\left\{f \in L^{2}(0,2 \pi) ; \sum_{j \in \mathbb{Z}}\left(1+|j|^{2}\right)^{s}|\widehat{f}(j)|^{2}<\infty\right\}
$$

where $\widehat{f}(j)$ are the Fourier coefficients of $f$.
To give an intuition behind Definition D. 4 and establish a connection with Definition D.2, let $f \in H_{\text {per }}^{1}(0,2 \pi)$. We show that $f$ belongs in $H^{1}(0,2 \pi)$ and satisfies the periodic boundary condition $f(0)=f(2 \pi)$. First, note that since $f$ belongs in $L^{2}(0,2 \pi)$, it admits a Fourier series. Thus, we have

$$
f=\sum_{j \in \mathbb{Z}} \widehat{f}(j) e_{j},
$$

where $e_{j}(x)=e^{i j x} / \sqrt{2 \pi}$ and the hats denote the Fourier coefficients. Also, note that $\left|\widehat{f}(j) e_{j}(x)\right|=|\widehat{f}(j)|$. Moreover, by the Cauchy-Schwarz inequality in $\ell^{2}(\mathbb{Z})$ (the Hilbert space of square-summable sequences) and the hypothesis on $f$, we have that

$$
\begin{aligned}
\sum_{j \in \mathbb{Z}}|\widehat{f}(j)| & =\sum_{j} \frac{\left(1+|j|^{2}\right)^{1 / 2}|\widehat{f}(j)|}{\left(1+|j|^{2}\right)^{1 / 2}} \\
& \leq \sqrt{\sum_{j \in \mathbb{Z}}\left(1+|j|^{2}\right)|\widehat{f}(j)|^{2}} \sqrt{\sum_{j \in \mathbb{Z}} \frac{1}{\left(1+\left|j^{2}\right|\right)}}<\infty .
\end{aligned}
$$

Now, by Weierstrass M-test, the Fourier series of $f$ converges absolutely and uniformly to a continuous $2 \pi$-periodic function and coincides with $f$ (since $f \in L^{2}(0,2 \pi)$ ). Thus, $f \in C[0,2 \pi]$ and $f(0)=f(2 \pi)$.

To show that $f$ has a first weak derivative in $L^{2}(0,2 \pi)$, let $\phi \in C_{c}^{\infty}(0,2 \pi)$. By the uniform converge of the Fourier series of $f$ we can exchange integration with summation below and have the following

$$
\begin{aligned}
\left\langle f, \phi^{\prime}\right\rangle=\int_{0}^{2 \pi} \sum_{j \in \mathbb{Z}} \widehat{f}(j) e_{j}(x) \overline{\phi^{\prime}(x)} d x & =\sum_{j \in \mathbb{Z}} \widehat{f}(j) \int_{0}^{2 \pi} e_{j}(x) \overline{\phi^{\prime}(x)} d x \\
& =-\sum_{j \in \mathbb{Z}} j \widehat{f}(j) \overline{\widehat{\phi}(j)} .
\end{aligned}
$$

The function

$$
g=\sum j \widehat{f}(j) e_{j}
$$

belongs in $L^{2}(0,2 \pi)$ due to the hypothesis

$$
\sum_{j \in \mathbb{Z}}|j \widehat{f}(j)|^{2}<\infty .
$$

Therefore, there is $g \in L^{2}(0,2 \pi)$ such that

$$
\left\langle f, \phi^{\prime}\right\rangle=-\sum_{j \in \mathbb{Z}} \widehat{g}(j) \overline{\widehat{\phi}(j)}=-\langle g, \phi\rangle, \quad \forall \phi \in C_{c}^{\infty}(0,2 \pi),
$$

where the second equality above follows from the isometry $f \rightarrow \widehat{f}(n)$ between $L^{2}(0,2 \pi)$ and $\ell^{2}(\mathbb{Z})$, see [50]. It follows that $f$ has a weak derivative in $L^{2}(0,2 \pi)$.

Now, the converse is also true. Indeed, let $f \in H^{1}(0,2 \pi)$ such that $f(0)=f(2 \pi)$. Since, both $f$ and its weak derivative $f^{\prime}$ are in $L^{2}(0,2 \pi)$, they admit a Fourier series representation

$$
f=\sum_{j \in \mathbb{Z}} \widehat{f}(j) e_{j} \quad \text { and } \quad f^{\prime}=\sum_{j \in \mathbb{Z}} \widehat{f}^{\prime}(j) e_{j} .
$$

The boundary condition implies that for any $j \in \mathbb{Z}$,

$$
\widehat{f}^{\prime}(j)=\int_{0}^{2 \pi} f^{\prime}(x) \overline{e_{j}(x)} d x=-i j \int_{0}^{2 \pi} f(x) \overline{e_{j}(x)} d x=-i j \widehat{f}(j) .
$$

Moreover, by Parseval's identity we have that

$$
\langle f, f\rangle_{1}=\|f\|^{2}+\left\|f^{\prime}\right\|^{2}=\sum_{j \in \mathbb{Z}}\left(1+|j|^{2}\right)|\widehat{f}(j)|^{2} .
$$

In particular, we notice that for any $f \in H^{1}(0,2 \pi)$ such that $f(0)=f(2 \pi)$, the series on the right-hand side above converges.

Consequently, we see that the periodic boundary conditions are encoded in the definition of $H_{\mathrm{per}}^{1}(0,2 \pi)$. Similar considerations yield, for any positive integer $m$, the following identification

$$
H_{\mathrm{per}}^{m}(0,2 \pi)=\left\{f \in H^{m}(0,2 \pi) ; f^{(n)}(0)=f^{(n)}(2 \pi), \quad n=0,1, \cdots, m-1\right\} .
$$

For $s \geq 0$, the Space $H_{\text {per }}^{s}(0,2 \pi)$ is also a Hilbert space with the inner-product given by

$$
\langle f, g\rangle_{s}=\sum_{j \in \mathbb{Z}}\left(1+|j|^{2}\right)^{s} \widehat{f}(j) \overline{\widehat{g}(j)}, \quad \forall f, g \in H_{\mathrm{per}}^{s}(0,2 \pi) .
$$

For $s=0, H_{\text {per }}^{s}(0,2 \pi)$ is identified with $L^{2}(0,2 \pi)$. Finally, the next lemma, known also as Sobolev embedding, holds true (see [73, Theorem 8.4] or [55, Theorem 3.195]).

Lemma D.5. Let $s>1 / 2$ and $f \in H_{p e r}^{s}(0,2 \pi)$. Then, the Fourier series of $f$ converges absolutely and uniformly. Its limit is a continuous $2 \pi$-periodic function which coincides with $f$ almost everywhere.

## Appendix E

## Two Proofs

In this appendix we prove two statements from Chapter 2.
Proof of Theorem 2.4. We know that a fixed initial function $u_{0}$ in $L^{2}(0,2 \pi)$ admits the $L^{2}(0,2 \pi)$ Fourier series representation

$$
u_{0}(x)=\sum_{j \in \mathbb{Z}} \widehat{u_{0}}(j) e_{j}(x)
$$

To prove the theorem, we first construct a sequence of smooth solutions. In particular, for $n \in \mathbb{N}$, the bounded, smooth functions

$$
\begin{equation*}
u^{n}(x, t)=\sum_{j=-n}^{n} \widehat{u_{0}}(j) e^{-i j^{2} t} e_{j}(x), \tag{E.1}
\end{equation*}
$$

form a sequence of smooth solutions to the periodic problem (2.3) with the initial conditions given by the partial Fourier sums

$$
u_{0}^{n}(x)=\sum_{j=-n}^{n} \widehat{u_{0}}(j) e_{j}(x) .
$$

The construction (E.1) of these smooth solutions follows from the Fourier method illustrated along the subsequent lines.

Assume that $u^{n}(x, t)$ is a smooth solution to (2.3) with initial condition $u_{0}^{n}(x)$. Then, for a fixed time $t \geq 0$, from Fourier series theory, the solution can be written as an absolutely and uniformly convergent Fourier series of the space variable, see
[20],

$$
u^{n}(x, t)=\sum_{j \in \mathbb{Z}} \widehat{u^{n}}(j, t) e_{j}(x) .
$$

Since $u^{n}(x, t)$ satisfies the IBVP (2.3) point-wise, we can uniquely determine the Fourier coefficients $\widehat{u^{n}}(j, t)$. Indeed, for each fixed $j \in \mathbb{Z}$, taking the inner-product with $e_{j}(x)$ we have that

$$
\int_{0}^{2 \pi} \partial_{t} u^{n}(x, t) \overline{e_{j}(x)} d x=i \int_{0}^{2 \pi} \partial_{x}^{2} u^{n}(x, t) \overline{e_{j}(x)} d x
$$

On the left-hand side, we can exchange differentiation with integration by the Dominated Convergence Theorem since $u^{n}(x, t)$ and $\partial_{t} u^{n}(x, t)$ are bounded and continuous functions of $t$ (see Theorem 2.27 in [32]). On the right-hand side we perform integration by parts twice using the periodic boundary conditions. Thus, for each $j \in \mathbb{Z}, \widehat{u^{n}}(j, t)$ satisfies the differential equation with respect to the time variable

$$
\frac{d}{d t} \widehat{u^{n}}(j, t)=-i j^{2} \widehat{u^{n}}(j, t),
$$

with initial condition $\widehat{u^{n}}(j, 0)=\widehat{u_{0}^{n}}(j)$. Hence, for each $j \in \mathbb{Z}$,

$$
\widehat{u^{n}}(j, t)=\widehat{u_{0}^{n}}(j) e^{-i j^{2} t},
$$

and the solution $u^{n}(x, t)$ takes the form

$$
u^{n}(x, t)=\sum_{j \in \mathbb{Z}} \widehat{u_{0}^{n}}(j) e^{-i j^{2} t} e_{j}(x)=\sum_{j=-n}^{n} \widehat{u}_{0}(j) e^{-i j^{2} t} e_{j}(x) .
$$

This finishes the construction of the smooth solutions and we continue by showing that (2.7) satisfies Definition 2.3 of a generalised solution.

Observe that for any $t \geq 0,\left|\widehat{u_{0}}(j) e^{-i j^{2}} t\right|^{2}=\left|\widehat{u_{0}}(j)\right|^{2}$ and so

$$
\sum_{j=-n}^{n}\left|\widehat{u_{0}}(j) e^{-i j^{2} t}\right|^{2}=\sum_{j=-n}^{n}\left|\widehat{u_{0}}(j)\right|^{2} .
$$

Since $u_{0} \in L^{2}(0,2 \pi)$, then as $n \rightarrow \infty$ we see that the series in the right-hand side of the expression above converges to $\left\|u_{0}\right\|^{2}$ (Parseval's identity). Hence, the sequence of the partial sums $u^{n}(x, t)$ converges uniformly in $t$ with respect to the $L^{2}(0,2 \pi)$
norm, and so the map

$$
u(\cdot, t)=\sum_{j \in \mathbb{Z}} \widehat{u_{0}}(j) e^{-i j^{2} t} e_{j}
$$

takes every $t \in[0, \infty)$ into $L^{2}(0,2 \pi)$.
To prove continuity in $t$ with respect to the norm of $L^{2}(0,2 \pi)$, we fix $t \geq 0$ and let $h>0$. Then, using Parseval's identity we have that

$$
\|u(\cdot, t+h)-u(\cdot, t)\|^{2}=\sum_{j \in \mathbb{Z}}\left|e^{-i j^{2} h}-1\right|^{2}\left|\widehat{u_{0}}(j)\right|^{2}
$$

However, because $\left|e^{-i j^{2} h}-1\right|^{2}\left|\widehat{u_{0}}(j)\right|^{2} \leq 4\left|u_{0}\right|^{2}$ and also $u_{0} \in L^{2}(0,2 \pi)$, the series in the right-hand side above converges absolutely and uniformly with respect to $h$ by the Weierstrass M-test. Moreover, since

$$
\left|e^{-i j^{2} h}-1\right|^{2}=2\left(1-\cos \left(j^{2} h\right)\right)
$$

is continuous as a function of $h$ and goes to zero as $h$ goes to zero we see that

$$
\lim _{h \rightarrow 0}\|u(\cdot, t+h)-u(\cdot, t)\|^{2}=0
$$

proving continuity from the right. To show continuity from the left, we fix $t>0$ and pick $h \in(0, t)$. Then, we have

$$
\|u(\cdot, t)-u(\cdot, t-h)\|^{2}=\sum_{j \in \mathbb{Z}}\left|1-e^{-i j^{2} h}\right|^{2}\left|\widehat{u_{0}}(j)\right|^{2},
$$

and a similar argument as before shows that

$$
\lim _{h \rightarrow 0}\|u(\cdot, t)-u(\cdot, t-h)\|^{2}=0
$$

Finally, note that the uniqueness of the generalised solution is implied from the uniqueness of the Fourier coefficients or equivalently from the basis property of the periodic eigenfunctions $e_{j}(x)$.

Proof of Proposition 2.6. For an initial function $u_{0}$ in $L^{2}(0,2 \pi)$, we consider the generalised solution $u(x, t)$ of (2.3) given for each fixed $t \geq 0$ by the Fourier series

Let $t, t_{1} \geq 0$. Then by the Cauchy-Schwarz inequality in $L^{2}(0,2 \pi)$ we find that for any $\phi \in \mathrm{D}(L)$

$$
\left|\langle u(\cdot, t), \phi\rangle-\left\langle u\left(\cdot, t_{1}\right), \phi\right\rangle\right|=\left|\left\langle u(\cdot, t)-u\left(\cdot, t_{1}\right), \phi\right\rangle\right| \leq\left\|u(\cdot, t)-u\left(\cdot, t_{1}\right)\right\|\|\phi\| .
$$

Hence, the continuity of the function $\langle u(\cdot, t), \phi\rangle$ for any $t \geq 0$ follows from the continuity of the map $u(\cdot, t):[0, \infty) \rightarrow L^{2}(0,2 \pi)$.

To show (2.8), first notice that since every partial sum

$$
u^{n}(x, t)=\sum_{j=-n}^{n} \widehat{u_{0}}(j) e^{-i j^{2} t} e_{j}(x), \quad n \in \mathbb{N}
$$

is a smooth solution with initial condition the $n$-th partial sum of the Fourier series of $u_{0}$, it satisfies

$$
\frac{d}{d s}\left\langle u^{n}(\cdot, s), \phi\right\rangle=i\left\langle u^{n}(\cdot, s), \phi^{\prime \prime}\right\rangle
$$

for any $\phi \in \mathrm{D}(L)$. This implies that

$$
\left\langle u^{n}(\cdot, t), \phi\right\rangle-\left\langle u^{n}(\cdot, \delta), \phi\right\rangle=i \int_{\delta}^{t}\left\langle u^{n}(\cdot, s), \phi^{\prime \prime}\right\rangle d s
$$

where $0<\delta<t$. We want to take the limit as $n \rightarrow \infty$.
Observe that by the Cauchy-Schwartz inequality once more, we have

$$
\left|\langle u(\cdot, t), \phi\rangle-\left\langle u^{n}(\cdot, t), \phi\right\rangle\right| \leq\left\|u(\cdot, t)-u^{n}(\cdot, t)\right\|\|\phi\|,
$$

and

$$
\left|\int_{\delta}^{t}\left\langle u(\cdot, s), \phi^{\prime \prime}\right\rangle d s-\int_{\delta}^{t}\left\langle u^{n}(\cdot, s), \phi^{\prime \prime}\right\rangle d s\right| \leq(t-\delta)\left\|u(\cdot, t)-u^{n}(\cdot, t)\right\|\left\|\phi^{\prime \prime}\right\| .
$$

Because for any $t>0, u^{n}(\cdot, t)$ converges in $L^{2}(0,2 \pi)$ to $u(\cdot, t)$ as $n \rightarrow \infty$, letting $n \rightarrow \infty$ we obtain that

$$
\langle u(\cdot, t), \phi\rangle-\langle u(\cdot, \delta), \phi\rangle=i \int_{\delta}^{t}\left\langle u(\cdot, s), \phi^{\prime \prime}\right\rangle d s
$$

which gives

$$
\frac{d}{d t}\langle u(\cdot, t), \phi\rangle=i\left\langle u(\cdot, t), \phi^{\prime \prime}\right\rangle .
$$

## Appendix F

## Numerical Examples of Revivals and Fractalisation

This appendix contains numerical examples which illustrate the revival and fractalisation phenomena in three time evolution problems considered in the thesis. In all the examples, we start with a step initial condition with the jump discontinuity placed in the middle of the interval of definition. In Section F.1, we display the numerical solutions to the pseudo-periodic problem for the free linear Schrödinger equation analysed in Chapter 4. Both non-self-adjoint and self-adjoint boundary conditions are treated. Numerical examples for the Airy PDE with quasi-periodic boundary conditions are given in Section F.2. This was the problem examined in Chapter 5. Finally, in Section F.3, we display numerical examples which correspond to the free linear Schrödinger equation under the Robin boundary conditions considered in Chapter 6. All graphs were plotted in Octave by summing over 4000 terms of the generalised Fourier series representations (4.14), (5.8), (6.17).

## F. 1 Free Linear Schrödinger Equation with PseudoPeriodic Boundary Conditions

Here, we illustrate the phenomenon of revivals and fractalisation in the pseudoperiodic problem (4.1) for the free linear Schrödinger equation with initial condition

$$
u_{0}(x)= \begin{cases}0, & 0 \leq x \leq \pi \\ 1, & \pi<x \leq 2 \pi\end{cases}
$$

In Figures F. 1 and F. 2 we plot the graph of the solution (4.14) at rational and irrational times for non-self-adjoint boundary conditions specified by the choice of parameters $\beta_{0}=1 / 5$ and $\beta_{1}=2$. In the first figure, the solution is evaluated at rational times. We clearly notice that the real and imaginary parts are piecewise constant functions, confirming the revival effect as obtained by Corollary 4.9. On the other hand, in Figure F.2, the solution profiles at generic times appear to be continuous, nowhere differentiable functions. Thus in both figures, the behaviour of the solution is in accordance with the consequences of Theorem 4.8, which gives at any time, the solution to the pseudo-periodic problem in terms of the solutions to specific periodic problems.


Figure F.1: Real (blue) and imaginary (red) parts of the solution of the pseudo-periodic (4.1) for the FSLS equation with $\beta_{0}=1 / 5, \beta_{1}=2$ at rational times $t=2 \pi p / q$.


Figure F.2: Real (blue) and imaginary (red) parts of the solution of the pseudo-periodic (4.1) for the FSLS equation with $\beta_{0}=1 / 5, \beta_{1}=2$ at generic times.


Figure F.3: Real (blue) and imaginary (red) parts of the solution of the pseudo-periodic (4.1) for the FSLS equation with $\beta_{0}=\beta_{1}=e^{i 2 \pi \theta}, \theta=\sqrt{2} / 3$, at rational times $t=2 \pi p / q$.


Figure F.4: Real (blue) and imaginary (red) parts of the solution of the pseudo-periodic (4.1) for the FSLS equation with $\beta_{0}=\beta_{1}=e^{i 2 \pi \theta}, \theta=\sqrt{2} / 3$, at generic times.

Similarly, in Figures F. 3 and F.4, the revival and fractalisation phenomena manifest in the simpler case of self-adjoint boundary conditions. In these two figures, the boundary parameters are fixed by choosing $\beta_{0}=\beta_{1}=e^{i 2 \pi \theta}$, with $\theta=\sqrt{2} / 3$. In the next section, the same choice of quasi-periodic boundary conditions shows the lack of revivals in Airy's PDE at rational times.

## F. 2 Airy's Partial Differential Equation with SelfAdjoint Quasi-Periodic Boundary Conditions

In this section, we display the numerical solutions of the quasi-periodic problem (5.4) for Airy's PDE, where the boundary conditions determined by $\beta=e^{2 \pi i \theta}$ with $\theta \in(0,1)$. We use the generalised Fourier series (5.8) with the step initial condition

$$
u_{0}(x)= \begin{cases}0, & 0 \leq x \leq \pi \\ 1, & \pi<x \leq 2 \pi\end{cases}
$$

and illustrate the phenomenon of revivals and fractalisation for two choices of the boundary parameter $\theta$, one rational and one irrational. The figures clearly demonstrate the extra dichotomy posed on the revival effect by rational and irrational values of $\theta$.

In Figures F. 5 and F.6, we plot the solution profile in space when $\theta$ is rational. In the first figure, the time is set to be rational and we clearly see the reappearance of the initial jump discontinuity. In the second figure, the time is irrational and the real and imaginary parts of the solution are both continuous. As expected, the discontinuity has been smoothed out.

Now, in Figures F. 7 and F. 8 we choose $\theta$ irrational. At any time, either rational or irrational, no discontinuities appear in the solution. In particular, in Figure F.7, there is no revival at rational times. This is in agreement with Theorem 5.2, which gives the solution at rational times in terms of the solution of a periodic problem for the free linear Schrödinger equation at an irrational time. Moreover, note that the lack of revivals in Airy's PDE is in contrast to the case of the FSLS equation for which the revival effect persists under these quasi-periodic boundary conditions, as this was illustrated in Figures F. 3 and F.4.


Figure F.5: Real (blue) and imaginary (red) parts of the solution of Airy's problem (5.4) with $\theta=1 / 4$ at rational times $t=2 \pi p / q$.


Figure F.6: Real (blue) and imaginary (red) parts of the solution of Airy's problem (5.4) with $\theta=1 / 4$ at generic times.





Figure F.7: Real (blue) and imaginary (red) parts of the solution of Airy's problem (5.4) with $\theta=\sqrt{2} / 3$ at rational times $t=2 \pi p / q$.


Figure F.8: Real (blue) and imaginary (red) parts of the solution of Airy's problem (5.4) with $\theta=\sqrt{2} / 3$ at generic times.

## F. 3 Free Linear Schrödinger Equation with Robintype Boundary Conditions

In this section, we display numerical experiments which correspond to the initial boundary value problem (6.12). As an initial condition we consider the piecewise constant function

$$
u_{0}(x)= \begin{cases}0, & 0 \leq x \leq \frac{\pi}{2} \\ 1, & \frac{\pi}{2}<x \leq \pi\end{cases}
$$

Picking different values of the parameter $b \in[0,1]$, we plot at generic (irrational) and rational times the real and imaginary part of the solution $u(x, t)$ as given by (6.17). In figures (F.9) and (F.11), we observe that at rational times the solution evolves to, not exactly, only translations (and/or reflections) of the initial profile. However, the revival of the discontinuities is preserved as predicted by Corollary 6.9. This is the weak revival effect (Definition 2.16). On the other hand, see figures F. 10 and F.12, at generic times the solution profile is clear of discontinuities.


Figure F.9: Real (blue) and imaginary (red) part of the solution of the IBVP (6.12) with $b=0.35$ at rational times $t=2 \pi p / q$.


Figure F.10: Real (blue) and imaginary (red) parts of the solution of the IBVP (6.12) with $b=0.35$ at generic times.






Figure F.11: Real (blue) and imaginary (red) parts of the solution of the IBVP (6.12) with $b=0.6$ at rational times $t=2 \pi p / q$.


Figure F.12: Real (blue) and imaginary (red) parts of the solution of the IBVP (6.12) with $b=0.6$ at generic times.

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