S-Dual of Maxwell–Chern-Simons Theory

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We discuss the dynamics of three-dimensional Maxwell theory coupled to a level-k Chern-Simons term. Motivated by S-duality in string theory, we argue that the theory admits an S-dual description. The S-dual theory contains a nongauge one-form field, previously proposed by Deser and Jackiw [Phys. Lett. **139B**, 371 (1984).] and a level-k $U(1)$ Chern-Simons term, $\mathcal{Z}_{\mathrm{MCS}} = \mathcal{Z}_{\mathrm{DJ}}\mathcal{Z}_{\mathrm{CS}}$. The couplings to external electric and magnetic currents and their string theory realizations are also discussed.

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Introduction.—Chern-Simons theory is vastly used in mathematical physics, in condensed-matter physics, and in string theory [1]. It was studied intensively in the past three decades, yet the dynamics of Yang-Mills–Chern-Simons theory is not fully understood in the strong coupling regime.

Four-dimensional S-duality is an exact duality between two $\mathcal{N} = 4$ super-Yang-Mills theories, enabling us to calculate quantities in the strong coupling regime using a dual weakly coupled theory. In the Abelian case, it reduces to the old electric-magnetic duality which swaps electric and magnetic fields:

$$F \leftrightarrow *F. \tag{1}$$

In 3D, Abelian S-duality relates the electric field to a dual scalar:

$$f \leftrightarrow *d\phi. \tag{2}$$

The purpose of this note is to extend S-duality to 3D Maxwell Chern-Simons (MCS) theory, with either a compact or noncompact $U(1)$ gauge group. It should hold on any spin manifold. The Lagrangian of the theory is given by

$$L = -\frac{1}{2g^2} da_e \wedge *da_e + \frac{k}{4\pi} a_e \wedge da_e. \tag{3}$$

MCS theory contains a vector boson of mass $M = (g^2k/2\pi)$. At low energies, the kinetic term is irrelevant, and the theory flows to a pure level-k Chern-Simons theory. As explained in the section on the derivation of the duality, the theory admits a global $\mathbb{Z}_k$ one-form symmetry generated by

$$G \equiv \exp\left(i \oint \left(a_e - \frac{1}{M} * da_e\right)\right). \tag{4}$$

When the theory is compactified on a torus, the global $\mathbb{Z}_k$ one-form symmetry is spontaneously broken, resulting in k degenerate vacua.

Several attempts were made to find the S-dual of Eq. (3). In Ref. [2], Deser and Jackiw proposed a "self-dual model" (SDM) which describes a massive vector. While SDM describes a massive vector, it does not admit a $\mathbb{Z}_k$ one-form symmetry, and neither does it flow to a pure Chern-Simons theory at low energies; hence, it cannot be an exact dual of MCS theory.

A closely related problem concerns the open string dynamics on a certain Hanany-Witten brane configuration. It is well known [3] that MCS theory lives on the left brane configuration shown in Fig. 1. Type-IIB S-duality maps the left configuration into the right configuration. Thus, knowing the field theory that lives on the right configuration will solve the problem of finding the S-dual. In early attempts [3,4], the authors found gauge theories with a fractional-level Chern-Simons term. While the theories

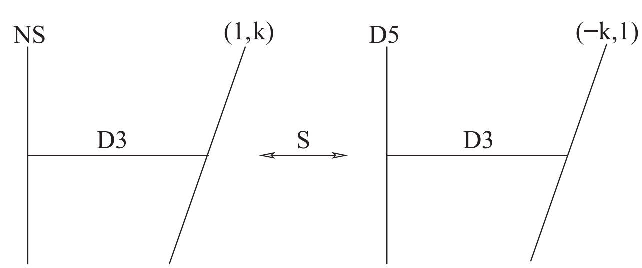

FIG. 1. The electric theory on the left brane configuration is Maxwell–Chern-Simons. The magnetic theory, obtained by type-IIB S-duality, lives on the right brane configuration.

they found are classically equivalent to MCS, it cannot be the full answer, as it does not admit the symmetries or the same dynamics as the electric theory.

Derivation of the duality.—We may use 4D S-duality between Maxwell theories to derive the 3D duality. In 4D, a pure Maxwell theory with a coupling g is dual to a pure Maxwell theory with a coupling $1/g$.

Consider the following partition function:

$$\mathcal{Z} = \int DF_m DA_e \exp i \int \left(-\frac{g^2}{2} F_m \wedge *F_m + F_m \wedge dA_e \right), \tag{5}$$

A_e is the "electric" gauge field, while F_m is a "magnetic" gauge-invariant two-form. g is the "electric" gauge coupling.

Upon integrating over F_m, we obtain the electric theory

$$\mathcal{Z} = \int DA_e \exp i \int \left(-\frac{1}{2g^2} dA_e \wedge *dA_e \right). \tag{6}$$

If instead we integrate over A_e, we obtain

$$\mathcal{Z} = \int D F_m \delta(dF_m) \exp i \int \left(-\frac{g^2}{2} F_m \wedge *F_m \right); \tag{7}$$

hence, it can be written in terms of A_m such that $F_m = dA_m$:

$$\mathcal{Z} = \int D A_m \exp i \left(-\frac{g^2}{2} dA_m \wedge *dA_m \right). \tag{8}$$

This is the magnetic theory dual to the electric theory.

Let us use the dimensional reduction of Eq. (5) in order to derive the 3D duality. Upon reducing to 3D, the 4D two-form F_m becomes a 3D two-form f_m and a one-form a_m. The 4D gauge field A_e becomes a 3D gauge field a_e and a scalar ϕ_e. The two-form f_m and the scalar ϕ_e decouple from the rest of the action and admit

$$\mathcal{Z} = \int D f_m D\phi_e \exp i \int \left(-\frac{g^2}{2} f_m \wedge *f_m + f_m \wedge d\phi_e \right), \tag{9}$$

which leads to the well-known S-duality

$$d\hat{a}_m \leftrightarrow *d\phi_e, \tag{10}$$

where $f_m = d\hat{a}_m$.

Let us focus on the duality between a_e and a_m, which is the prime purpose of this Letter. We add to the action a Chern-Simons term [5]. Our proposal is the following "master" partition function:

$$\mathcal{Z} = \int Da_m Da_e \exp i \int \left(-\frac{g^2}{2} a_m \wedge *a_m + a_m \wedge da_e + \frac{k}{4\pi} a_e \wedge da_e \right). \tag{11}$$

Note that a_m is a *gauge-invariant* one-form. Upon integration over a_m, we obtain the electric theory

$$\mathcal{Z} = \int Da_e \exp i \int \left(-\frac{1}{2g^2} da_e \wedge *da_e + \frac{k}{4\pi} a_e \wedge da_e \right), \tag{12}$$

namely Maxwell–Chern-Simons theory.

In order to derive the magnetic theory, we should use Eq. (11) and integrate over a_e. This is a subtle point. Instead, let us use a change of variables, $a_e = b - (2\pi/k) a_m$, to obtain the following partition function:

$$\mathcal{Z} = \int Da_m Db \exp i \int \left(-\frac{g^2}{2} a_m \wedge *a_m - \frac{\pi}{k} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db \right). \tag{13}$$

Equation (13) is our proposal for the S-dual of Maxwell–Chern-Simons theory. The partition function of the magnetic theory is a product of the Deser-Jackiw theory and a level-k Chern-Simons term,

$$\mathcal{Z}_{\mathrm{MCS}} = \mathcal{Z}_{\mathrm{DJ}} \mathcal{Z}_{\mathrm{CS}}. \tag{14}$$

Note that a_m is not a gauge field, and therefore the term $(\pi/k) a_m \wedge da_m$ is not ill-defined.

Both the electric and the magnetic theories describe a massive vector of mass $M = (g^2 k/2\pi)$ and a decoupled level-k Chern-Simons theory. Both theories exhibit a one-form $\mathbb{Z}_k$ symmetry.

Let us now provide another argument in favor of our proposal in Eq. (13). We begin with the magnetic brane configuration of Fig. 1. It was argued by Gaiotto and Witten [6] that the theory which lives on the intersection of the 3-brane and the tilted 5-brane (without a D5 brane) is

$$\mathcal{Z} = \int Da Dc \exp i \int \left(\frac{1}{2\pi} a \wedge dc + \frac{k}{4\pi} c \wedge dc \right). \tag{15}$$

In order to understand what happens when we add a D5 brane, let us assume that the terms that we need to add to the action are k-independent. Indeed, the information about k is encoded in the tilted 5-brane, not in the 3-brane. Let us use $k = 0$, since in this case the duality is well understood: the electric theory is pure Maxwell, and the magnetic (mirror) theory is a massless scalar. The brane realization of the duality was provided in the seminal work of Hanany and Witten [7].

We may write the theory of a free massless scalar as follows:

$$\mathcal{Z} = \int Da\,Dc\,\exp i\int\left(a\wedge *a + \frac{1}{2\pi}a\wedge dc\right), \quad (16)$$

with a being a gauge-invariant one-form. The equation of motion for c is $*da = 0$—namely, that $a = d\chi$. Thus, for $k = 0$ we obtain a theory of a free scalar $(d\chi)^2$, as expected.

We find that adding a term $a\wedge *a$ to the action yields a theory that describes the correct dual of Maxwell theory. We propose that $a\wedge *a$ is the missing term in Eq. (15)—namely, that by adding it to Eq. (15) we obtain the dual of MCS for any k. Note that

$$\mathcal{Z} = \int Da\,Dc\,\exp i\int\left(a\wedge *a + \frac{1}{2\pi}a\wedge dc + \frac{k}{4\pi}c\wedge dc\right) \quad (17)$$

is almost identical to Eq. (11). An important difference is that Gaitto and Witten introduced a *gauge field* a, whereas in Eq. (17) we added a term that breaks gauge invariance. We may reintroduce gauge invariance in Eq. (17) by transforming the fixed gauge vector a into a gauge-invariant term by adding a scalar η as follows:

$$\mathcal{Z} = \int DaDcD\eta\,\exp i\int\Big((a-d\eta)\wedge *(a-d\eta) + \frac{1}{2\pi}a\wedge dc + \frac{k}{4\pi}c\wedge dc\Big), \quad (18)$$

such that under a gauge transformation $a\to a + d\lambda$, $\eta\to\eta+\lambda$, with a being a $U(1)$ gauge field. Equation (17) may be viewed as the fixed-gauge version of Eq. (18) with $d\eta = 0$.

Our proposal in Eq. (13) passes all the requirements from a dual theory: it admits a $\mathbb{Z}_k$ global symmetry, it flows to pure Chern-Simons theory in the IR, it contains a massive vector of mass M, and finally, when $k = 0$, it agrees with the results of Hanany and Witten [7]. As we shall see, the brane realizations of both electric and magnetic theories predict the existence of k-degenerate vacua.

We summarize this section by writing the precise map between the electric and magnetic variables using Eq. (11):

$$-g^2 a_m = *da_e, \quad (19)$$

$$b = a_e - \frac{1}{M} * da_e, \quad (20)$$

or

$$a_e = b - \frac{2\pi}{k}a_m. \quad (21)$$

Comments on $\mathbb{Z}_k$.—Let us introduce a Wilson loop in MCS theory. We wish to measure the $\mathbb{Z}_k$ charge of the loop—namely, the number of fundamental strings, n, that pass through a certain contour C. We will define an operator G such that

$$GW_n = \exp\left(i\frac{2\pi n}{k}\right)W_n, \quad (22)$$

with W_n being a Wilson loop of charge n, $W_n = \exp(in\oint a_e)$.

In order to define G, let us consider the equation of motion in MCS:

$$d\left(\frac{1}{g^2} * da_e - \frac{k}{2\pi}a_e\right) = j_e \equiv dJ_e, \quad (23)$$

where J_e is the integral of the electric current j_e over a disk D such that $C = \partial D$. The setup is depicted in Fig. 2. By integrating Eq. (23), we learn that

$$\frac{1}{g^2} * da_e - \frac{k}{2\pi}a_e = J_e. \quad (24)$$

We can therefore define a generator of a $\mathbb{Z}_k$ symmetry as follows:

$$G = \exp\left(i\frac{2\pi}{k}\oint_C\left(\frac{k}{2\pi}a_e - \frac{1}{g^2} * da_e\right)\right) = \exp\left(i\oint_C b\right). \quad (25)$$

Note that the implication of $\mathbb{Z}_k$ symmetry is a symmetry $n\to n+k$: namely, that a collection of k strings is topologically isomorphic to a singlet—namely, to no strings at all. This is supported by string theory: suppose that we attempt to place the end points of k coincident strings on the worldvolume of the D3 brane. The collection of k fundamental strings can transform itself into an anti-D-string and a $(k,1)$ string. Instead of ending on the

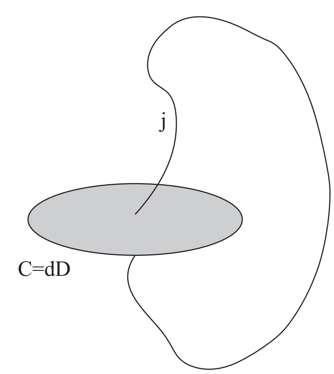

FIG. 2. A Wilson loop passing through a domain D (shaded region) whose boundary is $C = \partial D$.

worldvolume of the D3 brane, the D-string can end on an NS5 brane, and a $(k, 1)$ string can end on a $(1, k)$ 5-brane. Thus, string theory predicts that a collection of k strings can be removed from the worldvolume of the 3D gauge theory. A similar phenomenon happens in the magnetic dual if we attempt to introduce k coincident D-strings in the worldvolume of the magnetic theory.

When the theory is defined on the torus, the $\mathbb{Z}_k$ symmetry is broken, resulting in k vacua [8]. An intuitive explanation is as follows: The level-k $U(1)$ Chern-Simons theory is equivalent (using level-rank duality) to a level-$1SU(k)$ theory that admits a $\mathbb{Z}_k$ center symmetry. When it is defined on the torus, the $SU(k)$ theory deconfines, resulting in k degenerate vacua, parametrized by the eigenvalues of the 't Hooft loop.

The k vacua manifest themselves in both the electric and magnetic brane configurations as follows: the D3 brane may end on any of the k constituents of the 5-branes. Each one of the k choices corresponds to a vacuum.

Coupling to external sources, Wilson and magnetic loops.—Consider the coupling of the electric gauge field to a source j_e, namely $a_e j_e$. It translates to the coupling $(b - (\pi/k)a_m)j_e$ in the magnetic side. We therefore suggest that the Wilson loop

$$W_e = \exp i \oint a_e \tag{26}$$

in the electric side is mapped into a magnetic loop of the form

$$M_m = \exp i \oint \left(b - \frac{2\pi}{k} a_m\right) \tag{27}$$

in the magnetic side.

We may use the above map between the Wilson loop in the electric side and its magnetic counterpart to study the dynamics of 3D QED-CS. Using the worldline formalism [9], we can write the partition function of MCS theory coupled to N_f massless fields as follows:

$$\mathcal{Z}_{\text{QED-CS}} = \int Da_e \exp(iS_{\text{MCS}}) \sum_n \frac{(N_f \Gamma_e)^n}{n!}, \tag{28}$$

with

$$\Gamma_e = \int \frac{dt}{t^{\frac{5}{2}}} \int Dx \exp\left(-\int_0^t d\tau (\dot{x})^2\right) \exp i \oint a_e. \tag{29}$$

The duality yields the following partition function:

$$\mathcal{Z}_{\text{magnetic}} = \int Da_m Db \exp(iS_{\text{DJ-CS}}) \sum_n \frac{(N_f \Gamma_m)^n}{n!}, \tag{30}$$

with

$$\Gamma_m = \int \frac{dt}{t^{\frac{5}{2}}} \int Dx \exp\left(-\int d\tau (\dot{x})^2\right) \exp i \oint \left(b - \frac{2\pi}{k} a_m\right). \tag{31}$$

This suggests that the dynamics of QED with N_f massless flavors is captured by a dual DJ-CS theory coupled to N_f massless "monopoles." The precise coupling of a_m and b to the monopoles is given by Eq. (31). We may write the dual magnetic theory in a more "standard" form:

$$\mathcal{Z} = \int Da_m Db\, D\bar{\psi}_m D\psi_m \exp i\, S_{\text{magnetic}}, \tag{32}$$

with S_{magnetic} given by

$$S_{\text{magnetic}} = \int \left(-\frac{g^2}{2} a_m \wedge * a_m - \frac{\pi}{k} a_m \wedge da_m + \frac{k}{4\pi} b \wedge db + \bar{\psi}_m \gamma \wedge \star \left(i\partial + b - \frac{2\pi}{k} a_m\right) \psi_m\right). \tag{33}$$

It is interesting to note that the QED-CS theory is mapped to a theory of interacting massless magnetic "monopoles," with a coupling $2\pi/gk$. Thus, when the electrons couple strongly to a_e, the "monopoles" couple weakly to a_m, and we may use perturbation theory in the magnetic side to study the strongly coupled electric theory.

Following Itzhaki [10], let us define a magnetic ("disorder") loop in the electric theory

$$M_e = \exp\left(i \oint_C \left(k a_e - \frac{2\pi}{g^2} * da_e\right)\right), \tag{34}$$

which is mapped into the electric loop in the magnetic side,

$$W_m = \exp\left(ik \oint_C b\right). \tag{35}$$

The magnetic loop in the electric side and the electric (Wilson) loop in the magnetic side are trivial [10].

We suggest that a rectangular Wilson loop (or magnetic loop) should be identified with the end points of an F-string and an anti-F-string (or a D-string and an anti-D-string) that end on the 3-branes of Fig. 3.

A D-string can end on an NS5 brane instead of a 3-brane; hence, the magnetic loop in the electric theory should be trivial. Similarly, a F-string can end on a D5 brane instead of a 3-brane; hence, a Wilson loop in the magnetic theory should be trivial. This is consistent with our definitions of the magnetic loop [Eq. (34)] and the Wilson loop [Eq. (35)].

Summary.—The purpose of this Letter is to find the S-dual of MCS theory. We found that the dual theory [Eq. (13)] contains a nongauge vector of mass M and a decoupled pure TQFT. The magnetic theory nicely captures

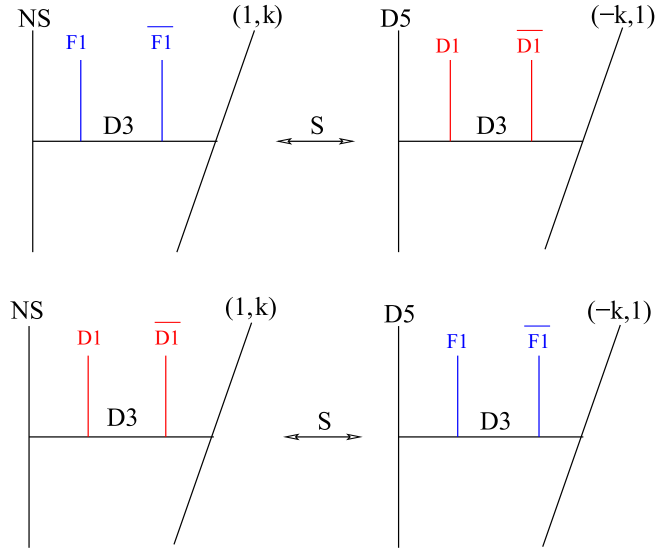

FIG. 3. Rectangular Wilson loops can be realized in string theory by ending a pair of an F-string and an anti-F-string on the 3-brane. Similarly, rectangular 't Hooft loops can be realized by ending a pair of a D-string and an anti-D-string on the 3-brane. The end of the F-string represents a heavy quark, whereas the end of the D-string represents a heavy monopole.

the dynamics of the electric theory: a theory with a mass gap that flows in the IR to a TQFT. The duality we uncovered in this Letter is a precise manifestation of the duality between a topological insulator and a topological superconductor outlined in Ref. [11].

It will be interesting to find the S-dual of the non-Abelian $U(N)$ theory that lives on a collection of N coincident D3 branes, suspended between tilted 5-branes. The master field of that theory may be obtained by replacing the Abelian one-forms of Eq. (11) with non-Abelian one-forms as follows [12]:

$$\mathcal{Z} = \int Da_m Da_e \exp i \,\mathrm{tr} \int \Big(-\frac{g^2}{2} a_m \wedge *a_m + a_m \wedge (da_e + a_e \wedge a_e) + \frac{k}{4\pi}\Big(a_e \wedge da_e + \frac{2}{3} a_e \wedge a_e \wedge a_e \Big)\Big), \tag{36}$$

together with $a_e = b - (2\pi/k)a_m$. Other dualities that involve SO/Sp (and an orientifold in string theory) could also be derived. The generalization to supersymmetric QED or QCD theories with a CS term [4] is also interesting and can be written down using the worldline formalism, as in the previous section on coupling. The duality found in this Letter is useful for studying the strong coupling regime of those theories.

Finally, it is well known that MCS theory admits Seiberg duality. The manifestation of the duality using an exchange of 5-branes in the magnetic theory might teach us about 5-branes' dynamics.

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