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Einstein's model of "the movement of small particles in a stationary liquid" revisited: finite propagation speed

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Abstract: The aforementioned celebrated model, though a breakthrough in Stochastic processes and a great step toward the construction of the Brownian motion, leads to a paradox: infinite propagation speed and violation of the 2nd law of thermodynamics. We adapt the model by assuming the diffusion matrix is dependent on the concentration of particles, rather than constant it was up to Einstein, and prove a finite propagation speed under the assumption of a qualified decrease of the diffusion for small concentrations. The method involves a nonlinear degenerated parabolic PDE in divergent form, a parabolic Sobolev-type inequality, and the Ladyzhenskaya-Ural'tseva iteration lemma.

Key words: Nonlinear partial differential equations, degenerate parabolic equations, Einstein paradigm, finite propagation speed

1. Introduction

In his celebrated paper [4] Einstein models the movement of a particle as a random walk of a step time τ and (random) displacement (free jump) Δ, of a symmetric distribution independent of a point and time of observation. The walk is restricted by the mass conservation law written for the concentration function. Using the argument eventually developed into the Ito formula, Einstein shows that the concentration function u, being a density of the distribution of particles, satisfies the classical heat equation (the forward one, since the random walk has a reversible law) (see [4] §4 or Section 2 below for details).

Though being a revolutionary paper in stochastic processes, and a decisive step toward the construction of Brownian motion, this model, however, leads to a physical paradox. Being a solution to the heat equation, the concentration u allows for a void volume instantly reach a positive concentration of particles. Moreover, as the free jumps process is reversible, the model allows for all particles, with a wonderful coherence, instantly concentrating in a small volume. This contradicts the second law of thermodynamics, as well as demonstrates an infinite propagation speed (see Remark 2.3).

The aim of this note is as follows: to adapt Einstein's model of free jumps, with a random walk replaced with a diffusion process, so as to get free of this paradox. We suppose that the villain of the piece is the assumption that the stationary liquid mentioned in the title, is passive in the subject, and hence the diffusion $a = \sigma^2/\tau$ is constant (σ^2 being the variance of Δ). Keeping with the Einstein assumption of an

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isotropic stationary media, we assume no drifts or sources, but the diffusion coefficient $a = a(u)$ depends on the concentration only, rather than the constant it was up to Einstein. Our aim is to choose $a(u)$ so that concentration function u demonstrates a finite propagation speed, that is, if a neighborhood of a point is void of the particles at time T, then a (smaller) neighborhood of the same point has been void of the particles for some time preceding T.

Finite propagation speed was demonstrated first by Barenblatt for a degenerate porous media equation (see [2, 6]). (The origin of the porous medium equation differs from the Einstein model.) The example of the porous media equation hints that a finite propagation speed appears if a small concentration implies small diffusion. Hence we assume diffusion coefficient a as a positive continuous function of concentration u, such that $a(0) = 0$ (see Hypothesis 3). This reflects the case of a higher medium resistance for small numbers of particles. One may think of hot particles heating the liquid when in numbers, and increasing its permeability. Alternatively, one can formulate it as the principle "the nature does not like crowds".

Note that concentration function u can be considered in two ways: first, as a function, and second, as a density of the distribution of the particles. The former approach leads to the backward Kolmogorov equation while the latter leads to the forward one. In this note we follow the first approach, considering a backward equation (2.8) for u. However, reversing the time brings (2.8) to a standard forward form:

$$u_t \le a(u)\Delta u \qquad \text{in } \Omega_T = \Omega \times (0, T), \tag{1.1}$$

with a spatial domain $\Omega \subset \mathbb{R}^N$ and a time horizon $T > 0$, and the inequality reflecting possible production of particles (see (3.1)-(3.3)).

Nevertheless, this is not the main trick we do with time. The main is a passage to an inner local time of the process, dependent on the concentration as well. In practice, it is the multiplication of (1.1) by some weight $h(u) > 0$, $h \in C(0, \infty)$, integrable at zero (meaning a local change of time $t \to \frac{t}{h(u)}$) so that $v \mapsto F(v) = h(v)a(v)$ is a monotone increasing locally Lipschitz continuous function on $(0, \infty)$, and $F(v) \to 0$ as $v \to +0$. Note that $h(u)u_t = \partial_t H(u)$ with $H(v) = \int_0^v h(r)dr$ (see Definition 3.2. This regularisation allows us to consider its divergent form and apply a rich technique of weak solutions. In particular, it allows us to replace (1.1) with its weak form

$$\iint\limits_{\Omega_T} \nabla u \nabla (F(u)\varphi)\, dxdt \le \iint\limits_{\Omega_T} H(u)\varphi_t\, dxdt, \qquad \varphi \in Lip_c(\Omega_T). \tag{1.2}$$

(Here the test functions space $Lip_c(\Omega_T)$ consists of Lipschitz continuous functions φ (of the pair of variables) such that $\operatorname{supp}\varphi \Subset \Omega_T$.)

The main result on finite speed of propagation is Theorem 5.1, considering an a priori finite propagation speed for a bounded positive weak solution to the concentration equation (1.2). It states that the concentration u demonstrates a finite propagation speed if the diffusion coefficient a defined in Hypothesis 3 satisfies the following.

$$\limsup_{u\to 0} a(u)I(u) < \infty, \tag{1.3}$$

$$\text{there exist } c, \mu > 0 \text{ such that } \; a(u)I^{\mu}(u) \ge c a(v)I^{\mu}(v),\; 0 < u < v < 1, \tag{1.4}$$

$$\text{with} \qquad I(u) = \int_u^{\infty} \frac{dv}{v\, a(v)}. \tag{1.5}$$

This assumptions hold for $a(u) = k\,u^\rho,\ k,\rho > 0$, or, more generally, for an O-regularly varying function with a strictly positive lower index (see Section 6 for more examples).

The article is organized as follows. In Section 2, we consider generalization of the classical Einstein model of an N-dim Brownian motion, with source term and general diffusion coefficients. We use the generic mass conservation law (2.7) to derive a partial differential inequality (3.3) for concentration u in Section 3. In the same section we pass to the inner local time in (3.6), and to weak solutions formulation in (3.7). In Section 4 we bring various auxiliary results necessary for the proof of Theorem 5.1. They include an ingenious nonlinear parabolic Sobolev-type inequality (4.7) and by the Ladyzhenskaya-Ural'tseva iteration lemma 4.6. In Section 5 we prove the finite propagation speed property. Section 6 focuses on various examples of generalized Einstein models satisfying conditions of the main theorem.

2. Generalized Einstein model

For a space-time point of observation $Z = (x,s) \in \mathbb{R}^N \times [0,\infty)$, consider an $\mathbb{R}^N$-valued random free jumps process $\vec{\Delta}^Z$,

$$\vec{\Delta}^Z \triangleq (\vec{\Delta}_t)^Z_{t\geq s},$$

describing a interaction-free displacement of a particle off Z (so $\vec{\Delta}^Z_s = 0$ a.s.). We assume the following extension of the axioms of classical Brownian motion.

Hypothesis 1 *1.* Free jumps process: $(Z, x+\vec{\Delta}^Z)_{Z\in\mathbb{R}^N\times\mathbb{R}_+}$ is a diffusion process with a state space $\mathbb{R}^N$ (see, e.g, [12] chapter 2 subsection 1.3). In particular it means that free jumps process is a Markov process with continuous trajectories and that the following assumptions hold:

(a) Trajectory continuity: for all $\epsilon > 0$, uniformly in $Z \in \mathbb{R}^N \times \mathbb{R}_+$,

$$\frac{1}{T}\sup_{0<t<T} \mathbb{P}\{\|\vec{\Delta}^Z_t\| > \epsilon\} \to 0 \quad \text{as} \quad T \to 0. \tag{2.1}$$

See [12] chapter 2 subsection 1.1.3 for the proof of continuity of almost all trajectories of the process, assumed (2.1).

(b) Diffusion coefficients: there exists a matrix-field $a(Z)$ and vector-field $b(z)$ such that for some (hence all) $\epsilon > 0$, uniformly in $Z \in \mathbb{R}^N \times \mathbb{R}_+$, for every $\xi, \eta \in \mathbb{R}^N$, one has the following.

$$\frac{1}{T}\sup_{0<t<T} \mathbb{E}\left[(\vec{\Delta}^Z_t\cdot\xi)(\vec{\Delta}^Z_t\cdot\eta)\mathbf{1}_{\{|\vec{\Delta}^Z_t|<\epsilon\}}\right] \to (a(Z)\xi)\cdot\eta, \tag{2.2}$$

$$\frac{1}{T}\sup_{0<t<T} \mathbb{E}\left[(\vec{\Delta}^Z_t\cdot\xi)\mathbf{1}_{\{|\vec{\Delta}^Z_t|<\epsilon\}}\right] \to b(Z)\cdot\xi, \tag{2.3}$$

as $T \to 0$. Matrix $a(Z) = \{a_{ij}(Z)\}^N_{ij=1}$ is referred as diffusion matrix, and vector $b(Z) = \{b_k(Z)\}^N_{k=1}$ is referred as drift coefficient (cf [12], part II, definition 1.3.1).

2. Whole universe axiom:

$$\mathbb{P}\{\vec{\Delta}^Z_t \in \mathbb{R}^N\} = 1, \quad t > 0. \tag{2.4}$$

3. The preceding axiom means no free jump sends a particle to infinity in finite time. However the free jump $\vec{\Delta}_t^Z$ is stopped at a (Markov) stopping time τ^Z, which is the last time before the particle gets into an action of a neighbour one.

   We assume τ^Z locally bounded from below: for every bounded domain $G \Subset \mathbb{R}^N \times \mathbb{R}_+$

$$\tau_G \triangleq \inf_{Z \in G} \tau^Z > 0. \tag{2.5}$$

   This property can be expressed in the terms of concentration function u which is assumed locally bounded.

An important property of a diffusion process is the existence of a correspondent diffusion differential operator, as it is shown in the following lemma (see [12], part II, lemma 1.3.1.)

Lemma 2.1 (diffusion operator) *Let φ be a bounded twice differentiable function on $\mathbb{R}^N$ (with a locally bounded second derivative). Then, locally uniformly in $Z \in \mathbb{R}^N \times \mathbb{R}_+$,*

$$\frac{1}{T} \sup_{0<t<T} \Big(\mathbb{E}\, \varphi(x + \vec{\Delta}_t^Z) - \varphi(x) \Big) \to \frac{1}{2} \sum_{i,j=1}^{N} a_{ij}(Z) \varphi_{x_i x_j}(x) - \sum_{k=1}^{N} b_k(Z) \varphi_{x_k}(x), \tag{2.6}$$
$$as \quad T \to 0.$$

In this lemma by a twice differentiable function one understands a function.

The second component of the Einstein paradigm is the mass (concentration) conservation law for a concentration function $u = u(x,s)$. The law reflects the idea of a spreading inkblot: the concentration $u(x, s-t)$ of particles at a space-time point $(x, s-t)$ equals to the total concentration of particles at points $(x+\vec{\Delta}_t^{(x,s-t)}, s)$ reached by particles by the time s plus/minus the number of particles consumed/produced in the way by the time period $[s-t, s]$, with a consumption-production reported by its flux M, $M(Z)$ being the quantity consumed/produced at a space-time point Z (a reader may think of a spreading and drying ink).

Hypothesis 2 (axiom of mass conservation)

$$u(x, s-t) = \mathbb{E}\, u(x + \vec{\Delta}_t^{(x,s-t)}, s) + \int_0^t \mathbb{E} M(x + \vec{\Delta}_r^{(x,s-t)}, s-t+r) dr \tag{2.7}$$

If we assume some regularity of u, we arrive at the following differential equation (see [12] chapter 2 theorem 1.3.1 for the idea of the proof).

Theorem 2.2 *Assume that, for every $s > 0$, function $u(s, \cdot)$ is twice differentiable. Then u is absolutely continuous in time s and satisfies equation*

$$-u_s(Z) = \frac{1}{2} \sum_{i,j=1}^{N} a_{ij}(Z) u_{x_i x_j}(Z) + \sum_{k=1}^{N} b_k(Z) u_{x_k}(Z) + M(Z). \tag{2.8}$$

Remark 2.3 *Einstein in [4] made a strong assumption that the distribution of* free jump $\vec{\Delta}^Z$ *does not depend on* Z*. This yielded diffusion coefficients* a *and* b *being constant and allowed Einstein to write down* (2.7) *in the forward way (cf. [4], page 14). So by choice* $a = Id$*,* $b = 0$ *and* $M = 0$ *he reduced the problem to the standard heat equation. However, this leads to violation of the second law of thermodynamics. Indeed, with* $a = Id$*,* $b = 0$ *and* $M = 0$*, Equation* (2.8) *becomes* $-u_s = \frac{1}{2}\Delta u$*. Then a solution* u*,*

$$u(x,s) = (2\pi(t-s))^{-\frac{N}{2}} \exp\left\{-\frac{|x|^2}{2(t-s)}\right\}, \qquad x \in \mathbb{R}^N,\ 0 \le s < t$$

is positive on $\mathbb{R}^N \times [0,t)$ *while* $u(x,s) \to 0$ *as* $s \uparrow t$ *for all* $x \neq 0$*. Thus, all particles instantly concentrate at zero at time* t*, demonstrating not only an impossible coherence but also an infinite speed. This paradox is the main motivation of our study.*

3. Nonlinear degenerate inequality

In this section we make final assumptions on the structure of the generalised Einstein model and state the main problem of the note. Our aim is to establish some conditions on diffusion coefficients a and b and consumption-production flux M in (2.8) allowing us to escape the paradox described in Remark 2.3. First we bring (2.8) into a forward form, more used for specialists in parabolic PDEs. We will fix a domain Ω and a time-horizon $T > 0$, and consider (2.8) on $\Omega_T \triangleq \Omega \times (0,T)$. With a change of variables $t = T - s$, Equation (2.8) takes the following form:

$$u_t(Z) = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(Z)u_{x_i x_j}(Z) + \sum_{k=1}^{N} b_k(Z)u_{x_k}(Z) + M(Z), \qquad Z \in \Omega_T\ . \tag{3.1}$$

We seek to establish the following property of u.

Definition 3.1 (finite propagation speed) *A function* $u \ge 0$ *on* Ω_T *is said to enjoy a finite propagation speed if, for any open ball* $B \subset \Omega$ *and any* $\epsilon \in (0,1)$*, there exists* $T' \in (0,T]$ *(which might depend on* B*,* ϵ *and* u*), such that, given* $u(x,0) = 0$ *for all* $x \in B$*, one has* $u(x,t) = 0$ *for all* $(x,t) \in \epsilon B \times [0,T']$*.*

Obviously, if u enjoys a finite propagation speed, then the paradox of the Einstein model is resolved.

In this note we consider a simple model to study the very essence of the phenomenon. Namely we assume the following.

Hypothesis 3

No drift: $b = 0$*;*

No consumption: $M \le 0$ *(recall that* M *has come from* (2.7) *which is a backward equation);*

Basic diffusion matrix: $a_{ij} = 2a(u)\delta_{ij}$*,* $i,j = 1,2,\ldots,N$*, with some* $a \in C[0,\infty)$*,* $a(u) > 0$ *for* $u > 0$ *and* $a(0) = 0$*. Recall that Hypothesis 1(iv) suggests that* u *is locally bounded. Hence we are free in choosing a behaviour of* a *at infinity. Therefore, we assume* a *such that* $I(u)$ *defined as in* (1.5)*, is finite for all* $u > 0$*, and*

$$\limsup_{u\to\infty} a(u)I(u) < \infty. \tag{3.2}$$

Note that $I(u) \to \infty$ *as* $u \to 0$ *and that* (1.3),(3.2) *imply that* $u \mapsto a(u)I(u)$ *is a bounded continuous function.*

The concept of a lower diffusion speed for a lower concentration of particles (the only hope to obtain a finite propagation speed) reflects the case of a higher medium resistance for a smaller numbers of particles.

With Hypothesis 3 Equation (3.1) converts to the following inequality

$$u_t \leq a(u)\Delta u, \qquad \text{in} \quad \Omega_T. \tag{3.3}$$

We would like to transform (3.3) into a divergent form equation. Since $u \mapsto a(u)$ is not necessarily differentiable, we multiply the equation by a strictly positive function $h(u)$ of the following properties.

Definition 3.2

- *Let* $h \in C(0,\infty)$, $h > 0$, *integrable at 0, and such that* $u \mapsto h(u)a(u)$ *is a monotone increasing locally Lipschitz function,* $h(u)a(u) \to 0$ *as* $u \to 0$. *Let*

$$H(u) \triangleq \int_0^u h(s)\ ds. \tag{3.4}$$

- *Let*

$$F(u) \triangleq h(u)a(u)\ . \tag{3.5}$$

By the preceding, $F(0) = 0$, *and* F *is differentiable on* $(0,\infty)$ *with a locally bounded derivative* $F^{'} \geq 0$.

Mutplication of (3.3) by $h(u)$ corresponds to a local time change $t \to \frac{t}{h(u)}$. Hence we obtain an equivalent form of (3.3):

$$[H(u)]_t - F(u)\Delta u \leq 0 \quad \text{in } \Omega_T. \tag{3.6}$$

We do not require u neither differentiable in t not twice differentiable in x as it is not a classic solution. A weak solution u to (3.6) is a bounded nonnegative function such that $\nabla u \in L^2_{loc}(\Omega_T)$ and $u(t) \to u(t_0)$ locally in measure as $t \to t_0$.

Note that $H(u), \nabla F(u) \in L^2_{loc}(\Omega_T)$ if u is as above. Hence we may understand $F(u)\Delta u$ and $[H(u)]_t$ in the weak sense: for every $\varphi \in Lip_c(\Omega_T)$,

$$-\int_{\Omega_T} \varphi F(u)\Delta u dx\, dt = \int_{\Omega_T} \nabla u \nabla\left(F(u)\varphi\right) dx\, dt$$

$$\int_{\Omega_T} \varphi [H(u)]_t dx\, dt = -\int_{\Omega_T} \varphi_t H(u) dx\, dt.$$

Thus, by a (positive bounded) solution u to (3.3) we mean u satisfying (3.6) in the following sense:

$$\int_{\Omega_T} \nabla u \nabla\left(F(u)\varphi\right) dx\, dt \leq \int_{\Omega_T} \varphi_t H(u) dx\, dt, \qquad \varphi \in Lip_c(\Omega_T),\ \varphi \geq 0. \tag{3.7}$$

Thus, we will look for solution in the following class.

Definition 3.3 *A weak positive bounded solution* u *to* (3.3) *is meant to be a positive* $u \in L^\infty(\Omega_T)$ *such that* $\nabla u \in L^2_{loc}(\Omega_T)$, *and* $u(t) \to u(t_0)$ *locally in measure whenever* $t \to t_0$, *satisfying* (3.7).

4. Auxiliary results

In this section we introduce some structure functions and bring auxiliary results necessary for proving a finite propagation speed for u satisfying (3.7).

In addition to structure functions F and H, let

$$G(s) \leq \int_0^s \sqrt{F'(\sigma)}\, d\sigma. \tag{4.1}$$

Therefore, $\sqrt{F'(u)} = G^{'}(u)$.

Remark 4.1 *By the Cauchy–Bunyakovsky–Schwarz inequality,*

$$0 \leq G(u) \leq \sqrt{u \int_0^u F'(s) ds} = \sqrt{uF(u)},$$

and all functions F, G, and H are increasing on closed interval.

Now we will make a set up of functions H and hence h, F and G.

Definition 4.2 *For some $\Lambda > 0$, and I as in (1.5), define*

$$H(s) \triangleq [\Lambda I(s)]^{-\frac{1}{\Lambda}} = \left(\Lambda \int_s^\infty \frac{1}{\tau a(\tau)} d\tau\right)^{-\frac{1}{\Lambda}}, \; s > 0. \tag{4.2}$$

Remark 4.3 *In (4.2), $H(s) \to 0$ as $s \to 0$. Moreover,*

$$h(s) = \frac{1}{sa(s)} [\Lambda I(s)]^{-\frac{1}{\Lambda}-1} = \frac{1}{sa(s)} H^{\Lambda+1}(s)\;,\; s > 0. \tag{4.3}$$

$$F(s) = h(s)a(s) = \frac{1}{s} H^{\Lambda+1}(s) = \left(\Lambda s^{\frac{\Lambda}{1+\Lambda}} I(s)\right)^{-\frac{1}{\Lambda}-1}. \tag{4.4}$$

Finally, with $\lambda \in (0, 2)$ such that $\Lambda + 1 = \dfrac{2}{\lambda}$ one has

$$(sF(s))^{\frac{\lambda}{2}} = H(s)\;. \tag{4.5}$$

Note that in general F defined by (4.4) is neither monotone increasing nor vanishes at zero. Thus, it does not automatically satisfy Definition 3.2. This property and parabolic Sobolev inequality is the subject of the following proposition.

Proposition 4.4 *Let (1.3) hold. Choose Λ,*

$$0 < \Lambda < \frac{1}{\sup\limits_u a(u) I(u)}, \tag{4.6}$$

and H, F, G as in (4.2), (4.4) and (4.1), respectively.

Then $F' > 0$ and $F(0) = 0$. Moreover, the following parabolic Sobolev-type inequality holds: for all

- *domain* $\Omega \subset \mathbb{R}^N$ *and* $T > 0$ *;*
- $\theta \in Lip_c(\Omega)$ *,* $0 \le \theta \le 1$ *and* $K \subset \{\theta = 1\}$ *;*
- $u \in L^\infty_{loc}(\Omega \times [0,T])$ *,* $\nabla u \in L^2_{loc}(\Omega \times (0,T))$ *and* $\nabla G(u) \in L^2_{loc}(\Omega \times [0,T])$ *;*

one has

$$\int_0^t \int_K G^2(u)dxdt \\ \le S^{k(1+j)} t^{1-(1+j)k} \left[\sup_{0\le\tau\le t} \int_\Omega \theta^2 H(u(x,\tau))dx + \int_0^t \int_\Omega |\nabla(\theta u)|^2 dxd\tau \right]^{1+jk}. \tag{4.7}$$

Here $j = \frac{2}{N-2}$ *,* $k = \frac{\Lambda}{\Lambda + j + j\Lambda}$ *, and* S *is a constant in the Sobolev inequality*

$$\|\psi\|^2_{L^{2+2j}} \le S \|\nabla\psi\|^2_{L^2}. \tag{4.8}$$

Proposition 4.4 is one of the three auxiliary results we need for demonstrating a finite propagation speed for solutions of (3.3). The second one deals with the left-hand side of (3.7).

Proposition 4.5 *Let* (1.3)*-*(1.4)*. Choose* Λ *as in* (4.6) *and* H, F, G *as in* (4.2)*,* (4.4) *and* (4.1)*, respectively. Then there exists* $C \ge 1$ *such that*

$$\nabla u \cdot \nabla\left(\theta^2 F(u)\right) \ge \frac{1}{2} |\nabla(\theta G(u))|^2 - C G^2(u) |\nabla\theta|^2. \tag{4.9}$$

for any measurable u *,* ∇u *and Lipschitz continuous* θ *.*

The last one is the celebrated Ladyzhenskaya-Ural'tseva iteration lemma (see [9] chapter 2 lemma 4.7).

Lemma 4.6 *Let sequence* y_n *for* $n = 0, 1, 2, \ldots,$ *be nonnegative sequence satisfying the recursion inequality,* $y_{n+1} \le c\ b^n\ y_n^{1+\delta}$ *with some constants* $c, \delta > 0$ *and* $b \ge 1$ *. Then*

$$y_n \le c^{\frac{(1+\delta)^n - 1}{\delta}}\ b^{\frac{(1+\delta)^n - 1}{\delta^2} - \frac{n}{\delta}}\ y_0^{(1+\delta)^n}.$$

In particular if $y_0 \le \theta_L = c^{\frac{-1}{\delta}}\ b^{\frac{-1}{\delta^2}}$ *and* $b > 1$ *, then* $y_n \le \theta\ b^{\frac{-n}{\delta}}$ *and consequently,*

$$y_n \to 0 \text{ when } n \to \infty.$$

By the end of the section we will be proving and commenting Propositions 4.4 and 4.5. The reader not interested in the technique can pass to the next section.

The proofs will be split into several lemmas, to facilitate the reading.

Lemma 4.7 *Let* (1.3) *hold. Choose* Λ *as in* (4.6) *and* F *as in* (4.4)*. Then* $F' > 0$ *,* $F(0) = 0$ *.*

Proof Note that (4.6) implies

$$\frac{\Lambda+1}{\Lambda} > \sup_s a(s)I(s).$$

Let F be as in (4.4). We show that F is increasing. It is equivalent to demonstrating that function

$$s \mapsto s^{\frac{\Lambda}{1+\Lambda}} I(s),$$

decreases. And so it is since

$$\frac{d}{ds}\left(s^{\frac{\Lambda}{1+\Lambda}} I(s)\right) = \left(\frac{\Lambda}{\Lambda+1}\frac{1}{a(s)}\right)\left(a(s)I(s) - \frac{\Lambda+1}{\Lambda}\right) s^{-\frac{1}{\Lambda+1}} < 0.$$

Next, we show that $s^{\frac{\Lambda}{1+\Lambda}} I(s) \to \infty$ as $s \to 0$, which implies $\lim_{s\to 0} F(s) = 0$. By (1.3), $\limsup_{s\to 0} a(s)I(s) = \alpha < \infty$. Hence, for every $\epsilon > 0$ there exists $s_\epsilon > 0$ such that $a(s)I(s) < \alpha + \epsilon$ for $s \in (0, s_\epsilon)$. This yields the following inequalities.

$$I(s) < (\alpha+\epsilon)\frac{1}{a(s)} = -(\alpha+\epsilon)sI'(s),$$

$$\frac{d}{ds}\ln I(s) < -\frac{1}{\alpha+\epsilon}\cdot\frac{1}{s},$$

$$I(s) > I(s_\epsilon)\left(\frac{s_\epsilon}{s}\right)^{\frac{1}{\alpha+\epsilon}} \text{ for } 0 < s < s_\epsilon,$$

$$s^{\frac{\Lambda}{\Lambda+1}} I(s) \geq I(s_\epsilon) s_\epsilon^{\frac{1}{\alpha+\epsilon}} \times s^{\frac{\Lambda}{\Lambda+1} - \frac{1}{\alpha+\epsilon}}.$$

Since $\frac{\Lambda+1}{\Lambda} > \sup a(s)I(s) \geq \alpha$, one has $\frac{\Lambda}{\Lambda+1} - \frac{1}{\alpha+\epsilon} < 0$ for small enough ϵ. Hence $s^{\frac{\Lambda}{\Lambda+1}} I(s) \to \infty$ as $s \to 0$. □

Lemma 4.8 *Inequality* $(sF(s))^{\frac{\lambda}{2}} \leq H(s)$ *implies* (4.7).

Proof Let $\lambda = \frac{2}{\Lambda+1}$. By Remark 4.1 and (4.5), $G(s)^\lambda \leq H(s)$. Note that

$$\lambda = \frac{2 - 2(1+j)k}{1-k}.$$

Therefore,

$$G^2(u) \leq G^{2(1+j)k}(u) H^{1-k}(u).$$

Integrate both side of the latter over $K \times (0,t)$ to obtain the following.

$$\begin{aligned}
\int_0^t \int_K G^2(u)dxdt &\leq \int_0^t \int_K G^{2(1+j)k}(u)H^{1-k}(u)dxd\tau \\
&\leq \int_0^t \int_\Omega \left(|\theta G(u)|^{2(1+j)}\right)^k \left(\theta^2 H(u)\right)^{1-k} dxd\tau \\
&\leq \int_0^t \left[\int_\Omega |\theta G(u)|^{2(1+j)}dx\right]^k \left[\int_\Omega \theta^2 H(u)dx\right]^{1-k} d\tau \\
&\leq S^{k(1+j)} \int_0^t \left[\int_\Omega |\nabla(\theta G(u))|^2 dx\right]^{(1+j)k} d\tau \left[\sup_{0\leq\tau\leq t}\int_\Omega \theta^2 H(u)dx\right]^{1-k},
\end{aligned}$$

by (4.8). Note the inequality $x^v y^w \leq (x+y)^{v+w}$; $x, y, v, w > 0$. Indeed, by the Young inequality,

$$x^{\frac{v}{v+w}} y^{\frac{w}{v+w}} \leq \frac{v}{v+w} x + \frac{w}{v+w} y < x + y$$

. We apply these together with the Holder inequality for time integral, to get the following

$$\int_0^t \int_K G^2(u) dx dt \leq S^{k(1+j)} \left[\sup_{0 \leq \tau \leq t} \int_\Omega \theta^2 H(u) dx \right]^{1-k} t^{1-k(1+j)} \left[\int_0^t \int_\Omega |\nabla(\theta G(u))|^2 dx d\tau \right]^{(1+j)k}$$

$$\leq S^{k(1+j)} t^{1-k(1+j)} \left[\sup_{0 \leq \tau \leq t} \int_\Omega \theta^2 H(u) dx + \int_0^t \int_\Omega |\nabla(\theta G(u))|^2 dx d\tau \right]^{1+jk}.$$

□

The core of the proof of Proposition 4.5 is the following lemma.

Lemma 4.9 *Assume that there exists $c > 0$ such that*

$$F(s) \leq c G'(s) G(s) \ . \tag{4.10}$$

Then the assertion of Proposition 4.5 holds.

Proof By a direct computation,

$$\begin{aligned} \nabla u \cdot \nabla \left(\theta^2 F(u) \right) &= \theta^2 F'(u) |\nabla u|^2 + 2F(u) \nabla u \cdot \theta \nabla \theta \\ &\geq \theta^2 |\nabla G(u)|^2 - 2c\theta |\nabla G(u)| \, G(u) |\nabla \theta|. \end{aligned} \tag{4.11}$$

By the Cauchy-Bunyakovsky-Schwarz inequality,

$$2C\theta |\nabla G(u)| \, G(u) |\nabla \theta| \leq \frac{1}{4} \theta^2 |\nabla G(u)|^2 + 4c^2 G(u)^2 |\nabla \theta|^2$$

and

$$\theta^2 |\nabla G(u)|^2 = |\nabla(\theta G(u)) - G(u)\theta|^2 \geq \frac{2}{3} |\nabla(\theta G(u))|^2 - 4G(u)^2 |\nabla \theta|^2.$$

Thus,

$$\nabla u \cdot \nabla \left(\theta^2 F(u) \right) \geq \frac{1}{2} |\nabla(\theta G(u))|^2 - 4(1+c^2) G(u)^2 |\nabla \theta|^2.$$

□

To show that (1.4) implies (4.10), we will recall the definition of equivalent functions.

Definition 4.10 *Functions f and g on a set E are equivalent ($f \asymp g$) on E, if there exists a constant $k \geq 1$ such that $k^{-1} g(x) \leq f(x) \leq k\, g(x)$ for all $x \in E$.*

Proposition 4.11 *Assumption (1.4) implies (4.10).*

Proof By direct computation, it follows from (4.4) that

$$F(s) \asymp s^{-1} I^{-\frac{1}{\Lambda}-1}(s) \tag{4.12}$$

$$F^{'}(s) \asymp \frac{F(s)}{s}[a(s)I(s)]^{-1}. \tag{4.13}$$

Using (4.12), (4.13) and Remark 4.1

$$\frac{F(s)}{[G(s)G'(s)]} = \frac{F(s)}{\left(\int_0^s \sqrt{F'(t)}\, dt\right)\left(\sqrt{F'(s)}\right)} \tag{4.14}$$

$$\asymp \frac{F(s)}{\left(\int_0^s \sqrt{\frac{F(t)}{t}[a(t)I(t)]^{-1}}\, dt\right)\left(\sqrt{\frac{F(s)}{s}[a(s)I(s)]^{-1}}\right)} \tag{4.15}$$

$$= \frac{[I(s)]^{-\frac{1}{2\Lambda}-\frac{1}{2}}[a(s)I(s)]^{\frac{1}{2}}}{\int_0^s t^{-1}[I(t)]^{-\frac{1}{2\Lambda}-\frac{1}{2}}[a(t)I(t)]^{-\frac{1}{2}}\, dt} \tag{4.16}$$

$$= \frac{[I(s)]^{-\frac{1}{2\Lambda}-\frac{\mu}{2}}[a(s)I^{\mu}(s)]^{\frac{1}{2}}}{\int_0^s t^{-1}[I(t)]^{-\frac{1}{2\Lambda}-1}[a(t)]^{-\frac{1}{2}}\, dt}\ . \tag{4.17}$$

By the Cauchy's mean value theorem, there exists $t \in (0, s)$ such that

$$\frac{[I(s)]^{-\frac{1}{2\Lambda}-\frac{\mu}{2}}}{\int_0^s t^{-1}[I(t)]^{-\frac{1}{2\Lambda}-1}[a(t)]^{-\frac{1}{2}}\, dt} = \left(\frac{1}{2\Lambda}+\frac{\mu}{2}\right)\frac{[I(t)]^{-\frac{1}{2\Lambda}-1-\frac{\mu}{2}}[ta(t)]^{-1}}{t^{-1}[I(t)]^{-\frac{1}{2\Lambda}-1}[a(t)]^{-\frac{1}{2}}} \asymp [a(t)I^{\mu}(t)]^{-\frac{1}{2}}\ .$$

Since (1.4) holds, one has

$$\frac{F(s)}{G(s)G'(s)} \asymp \left[\frac{a(s)I^{\mu}(s)}{a(t)I^{\mu}(t)}\right]^{\frac{1}{2}} \leq \frac{1}{\sqrt{c}}\ ;\ 0 < t < s. \tag{4.18}$$

□

A sufficient condition for (1.4) is given in the next remark.

Remark 4.12 *Assume that there exists* $\tilde{a} \in C^1(0,\infty)$ *such that* $a \asymp \tilde{a}$ *on* $(0,\infty)$, *and*

$$\limsup_{s\to 0} s\tilde{I}(s)\tilde{a}'(s) < \infty, \tag{4.19}$$

where $\tilde{I}(s) \triangleq \int_s^{\infty} \frac{dt}{t\tilde{a}(t)}$. *Since we a free in choosing a behaviour of* a *and* $\tilde{a}$ *at infinity, we may assume* $\limsup_{s\to\infty} s\tilde{I}(s)\tilde{a}'(s) < \infty$ *as well, so function* $s \mapsto s\tilde{I}(s)\tilde{a}^{'}(s)$ *is bounded above,*

$$B = \sup_{s>0} s\tilde{I}(s)\tilde{a}'(s).$$

Fix $\mu \geq B$. *Then the function* $Q(s) = \tilde{a}(s)\tilde{I}^{\mu}(s)$ *is nonincreasing since*

$$Q'(s) = \tilde{a}^{'}(s)\tilde{I}^{\mu}(s) - \mu s^{-1}\tilde{I}^{\mu-1}(s) = s^{-1}\tilde{I}^{\mu-1}(s)\left[s\tilde{I}(s)\tilde{a}^{'}(s) - \mu\right] \leq 0.$$

Finally, $aI^{\mu} \asymp \tilde{a}\tilde{I}^{\mu}$.

5. A proof of finite propagation speed

Theorem 5.1 *Let* a *be as in Hypothesis 3, and let* (1.3) *and* (1.4) *hold. Choose* Λ *as in* (4.6) *and* H, F, G *as in* (4.2), (4.4), *and* (4.1), *respectively.*

For a domain $\Omega \in \mathbb{R}^N$ *and* $T > 0$, *let* u *be a positive bounded weak solution to* (3.3) *in* Ω_T, *i.e.* u *be as in Definition 3.3 satisfying* (3.7).

Then u *enjoys a finite propagation speed property in the sense of* (3.1).

Proof Let $B \Subset \Omega$ be an open ball and let $\epsilon \in (0,1)$. Assume that $u(x,0) = 0$ for all $x \in B$. We are to construct $T' \in (0,T)$ such that $u = 0$ on $\epsilon B \times [0, T']$.

We start from proving the following estimate. With C as is (4.9), for every $\theta \in Lip_c(B)$, one has

$$\int_B \theta^2 H\left(u(x,t)\right)\, dx + \frac{1}{2}\int_0^t \int_B |\nabla(\theta G(u))|^2 \, dxd\tau \leq C \int_0^t \int_B G^2(u)|\nabla\theta|^2 \, dxd\tau. \tag{5.1}$$

Indeed, by Proposition 4.5, due to assumption (1.4), we can apply (4.9) to the left hand side of (3.7) with $\varphi(x,\tau) = \theta^2(x)\zeta(\tau)$, where $\zeta \in Lip_c(0,t)$, $0 \leq \zeta \leq 1$, approximating $\mathbf{1}_{(0,t)}$ in $BV[0,T]$. Then

$$\frac{1}{2}\int_0^t \int_B |\nabla(\theta G(u))|^2\zeta \, dxd\tau \leq C\int_0^t \int_B G^2(u)|\nabla\theta|^2\zeta \, dxd\tau + \int_0^t \int_B \theta^2 H(u)\, dx\, \zeta' d\tau.$$

Note that, by Definition 3.3, map $\tau \mapsto \int_B \theta^2 H\big(u(\tau)\big)\, dx$ is continuous, and that $\zeta' d\tau \to \delta_0 - \delta_t$. Hence

$$\int_0^t \int_B \theta^2 H(u)\, dx\, \zeta' d\tau \to \int_B \theta^2 H\big(u(x,0)\big)\, dx - \int_B \theta^2 H\big(u(x,t)\big)\, dx = -\int_B \theta^2 H\big(u(x,\tau)\big)\, dx.$$

(The last equality is yielded by $u(x,0) = 0$ for $x \in B$.) Hence (5.1).

In particular, (5.1) implies that $\nabla G(u) \in L^2_{loc}(\Omega \times [0,T])$. Therefore, by Proposition 4.4, due to assumption (1.3), we can apply (4.7) to the right hand side of (5.1) and get the following. For $\tilde{\theta}, \hat{\theta} \in Lip_c(B)$, $\hat{\theta} = 1$ on $\operatorname{supp}\tilde{\theta}$, one has

$$\begin{aligned}
&\int_B \tilde{\theta}^2 H\big(u(x,t)\big)\, dx + \frac{1}{2}\int_0^t \int_\Omega |\nabla(\tilde{\theta}G(u))|^2 \, dxd\tau \\
&\qquad \leq D\|\nabla\tilde{\theta}\|_\infty^2 t^{1-k(1+j)} \left[\sup_{0\leq\tau\leq t}\int_B \hat{\theta}^2 H\big(u(x,\tau)\big)dx + \int_0^t \int_B |\nabla(\hat{\theta}G(u))|^2 dxd\tau\right]^{1+jk},
\end{aligned}$$

with $D = CS^{k(1+j)}$. Take supremum in $t \in (0,s)$ for $0 < s \leq T$, to obtain

$$\begin{aligned}
&\sup_{0\leq\tau\leq s}\int_B \tilde{\theta}^2 H\big(u(x,\tau)\big)dx + \int_0^s \int_B |\nabla(\tilde{\theta}G(u))|^2 dxd\tau \\
&\qquad \leq D\|\nabla\tilde{\theta}\|_\infty^2 s^{1-k(1+j)} \left[\sup_{0\leq\tau\leq s}\int_B \hat{\theta}^2 H\big(u(x,\tau)\big)dx + \int_0^s \int_B |\nabla(\hat{\theta}G(u))|^2 dxd\tau\right]^{1+jk}.
\end{aligned}$$

Multiply the latter with s^β with $\beta > 0$ such that

$$\beta + 1 - (1+j)k = \beta(1+kj) \quad \left(\beta \triangleq \frac{1-(1+j)k}{kj}\right).$$

Then we get

$$\begin{aligned} &s^\beta \sup_{0\le\tau\le s} \int_B \tilde{\theta}^2 H\big(u(x,\tau)\big)dx + s^\beta \int_0^s \int_B |\nabla(\tilde{\theta}G(u))|^2 dxd\tau \\ &\le D\|\nabla\tilde{\theta}\|_\infty^2 \left[s^\beta \sup_{0\le\tau\le s} \int_B \hat{\theta}^2 H\big(u(x,\tau)\big)dx + s^\beta \int_0^s \int_B |\nabla(\hat{\theta}G(u))|^2 dxd\tau\right]^{1+jk}. \end{aligned} \tag{5.2}$$

Estimate (5.2) is vehicle of the following iteration procedure.

Choose $b > 2$ such that

$$\frac{b-2}{b-1} = \epsilon \quad \left(b = 1 + \frac{1}{1-\epsilon}\right)$$

. Define

$$\epsilon_n = \frac{b-2+b^{-n}}{b-1}, \qquad n = 0,1,2,\ldots$$

Obviously,

$$\epsilon_0 = 1, \quad \epsilon_\infty = \epsilon, \quad \epsilon_n - \epsilon_{n+1} = b^{-(n+1)}.$$

Therefore, for $n = 0,1,2,\ldots,$ we can choose $\theta_n \in Lip_c(\epsilon_n B)$ such that $\theta_n = 1$ on $\epsilon_{n+1}B$ and

$$\|\nabla\theta_n\|_\infty^2 \le Kb^{2(n+1)}$$

with the same constant K for all $n = 0,1,2,\ldots$

Define

$$Y_n[s] \triangleq s^\beta \sup_{0\le\tau\le s} \int_B \theta_n^2 H(u(x,\tau))dx + s^\beta \int_0^s \int_B |\nabla(\theta_n G(u))|^2 dxd\tau. \tag{5.3}$$

Then (5.2) yields the iterative inequality

$$Y_{n+1}[s] \le DKb^4 \cdot (b^2)^n \, Y_n^{1+kj}[s]. \tag{5.4}$$

Finally, choose $s > 0$ such that

$$Y_0[s] \le (DKb^4)^{-\frac{1}{kj}} b^{-\frac{2}{k^2j^2}}. \tag{5.5}$$

Then by Lemma 4.6,

$$s^\beta \sup_{0\le\tau\le s} \int_{\epsilon B} H(u(x,\tau))dx \le \lim_{n\to\infty} Y_n[s] = 0.$$

□

Remark 5.2 *The use of the Ladyzhenskaya-Ural'tseva iterative lemma for the proof of finite speed of propagation for degenerate parabolic equation was first used in [13].*

6. Models for degeneracy

Without loss of generality, in this section, we will assume that $u \in (0,1]$, and illustrate some generic examples of the function a, for which hold all constrains on the functions F, G and H, with the following summarized remark.

Remark 6.1 *Let $a \in C^1(0,\infty)$ be as in Hypothesis 3. Assume that*

$$\limsup_{u\to 0} a(u)I(u) < \infty, \tag{6.1}$$

$$\limsup_{u\to 0} sa'(u)I(u) < \infty. \tag{6.2}$$

Then Remark 4.12 yields Theorem 5.1 on finite speed of propagation.

Proposition 6.2 *Let a is given as follows:*

$$a(u) = \exp\left\{-\int_u^1 \frac{dv}{v\,\xi(v)}\right\}, \quad u \in (0,1), \tag{6.3}$$

where $\xi \in C(0,1)$, $0 < \xi < c$ and

$$\sup_{u<v<2u} \xi(v) \le c\,\xi(u), \quad u \in (0,1),$$

with some $c > 0$.

Then there exists $M > 0$ such that

$$a(u)I(u) \le M\xi(u), \quad u \in (0,1). \tag{6.4}$$

Thus, (6.1)-(6.2) hold, together with Theorem 5.1 on the finite speed of propagation.

Proof Note that, for $u < \frac{1}{2}$,

$$\begin{aligned} I(u) =& I(1) + \int\limits_{2u}^{1} \xi(w) \exp\left\{\int_w^1 \frac{dv}{v\,\xi(v)}\right\} \frac{dw}{w\,\xi(w)} \\ &+ \int\limits_{u}^{2u} \xi(w) \exp\left\{\int_w^1 \frac{dv}{v\,\xi(v)}\right\} \frac{dw}{w\,\xi(w)}. \end{aligned}$$

We will use $\xi(w) \le c$ for the first integral, and $\xi(w) \le c\,\xi(u)$, for the second one. Hence,

$$I(u) \le I(1) + c\exp\left\{\int_{2u}^1 \frac{dv}{v\,\xi(v)}\right\} + c\,\xi(u)\exp\left\{\int_u^1 \frac{dv}{v\,\xi(v)}\right\}.$$

Now observe that inequality $e^x \ge x$ implies

$$\frac{\xi(u)\exp\left\{\int_u^1 \frac{dv}{v\,\xi(v)}\right\}}{\exp\left\{\int_{2u}^1 \frac{dv}{v\,\xi(v)}\right\}} = \xi(u)\exp\left\{\int_u^{2u} \frac{dv}{v\,\xi(v)}\right\} \ge \int_u^{2u} \frac{\xi(u)}{\xi(v)}\frac{dv}{v} \ge \frac{1}{c}\ln 2.$$

Therefore, (6.4), and hence (6.1) hold.

Finally, to show (6.1), observe that

$$u\,a'(u)I(u) = u\frac{1}{u\,\xi(u)}a(u)I(u) \leq M, \quad u \in (0,1).$$

□

Example 6.3

- $\xi(u) = \beta > 0\,,\ a(u) = u^{\frac{1}{\beta}}\,.$
- $\xi(u) = \frac{\beta}{\alpha+1}|\ln u|^{-\alpha}\,,\ a(u) = u^{\frac{1}{\beta}|\ln u|^{\alpha}}\,,\ \alpha, \beta > 0\,.$
- $\xi(u) = \frac{1}{\alpha}u^{\alpha}\,,\ a(u) = \exp\{1 - u^{-\alpha}\}\,,\ \alpha > 0\,.$

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