E. Freitag, R. M. Hill *Res. Number Theory (2023)9:26*
https://doi.org/10.1007/s40993-022-00361-6

Research in Number Theory

RESEARCH

Modular forms on SU(2, 1) with weight $\frac{1}{3}$

Eberhard Freitag[1] and Richard M. Hill[2*]

*Correspondence:
r.m.hill@ucl.ac.uk
University College London, London, UK
Full list of author information is available at the end of the article

Abstract

In this note, we describe several new examples of holomorphic modular forms on the group SU(2, 1). These forms are distinguished by having weight $\frac{1}{3}$. We also describe a method for determining the levels at which one should expect to find such fractional weight forms.

Keywords: Fractional weight modular forms, Picard modular forms, Presentations of arithmetic groups, Borcherds products

Mathematics Subject Classification: 11F06, 11F55, 11F75

1 Introduction

In this note, we describe some new examples of holomorphic modular forms on the real Lie group $SU(2, 1)$. The forms have weight $\frac{1}{3}$, and their levels are certain congruence subgroups of $SU(2, 1)$. We recall that various half-integral weight forms on $SU(2, 1)$ have been studied in the past, but our examples seem to be the first whose weight is not half-integral. Indeed, they are not metaplectic forms, in the sense that they do not arise from automorphic forms on the metaplectic cover of an adèle group. This is because the adèlic metaplectic covers of forms of $SU(2, 1)$ are at most two-fold covers.

1.1 Notation

To state our results a little more precisely, we introduce some notation.

Consider the Hermitian form on $\mathbb{C}^3$ defined by $\langle v, w\rangle = \bar{v}^t J w$, where $J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$. Here and throughout the paper, we write $\bar{v}^t$ for the conjugate transpose of v. The group $SU(2, 1)$ consists of the matrices in $SL_3(\mathbb{C})$ which preserve this Hermitian form. Equivalently, the elements of $SU(2, 1)$ are the matrices g in $SL_3(\mathbb{C})$ which satisfy $\bar{g}^t J g = J$.[1]

The symmetric space associated to this Lie group may be identified with the following complex manifold:

[1]One could of course replace J by any Hermitian matrix of signature (2, 1). Our choice of J is convenient, because the upper triangular matrices of $SU(2, 1)$ form a Borel subgroup and the diagonal matrices form a maximal torus with this choice of J.

$$\mathcal{H} = \left\{ \underline{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \in \mathbb{C}^2 : \left\langle \begin{pmatrix} \tau_1 \\ \tau_2 \\ 1 \end{pmatrix}, \begin{pmatrix} \tau_1 \\ \tau_2 \\ 1 \end{pmatrix} \right\rangle < 0 \right\}.$$

Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an element of $\mathrm{SU}(2,1)$, where A is a 2×2 matrix, D is a complex number, etc. The action of g on $\mathcal{H}$ is given by

$$g * \underline{\tau} = \frac{1}{C\underline{\tau} + D} \cdot (A\underline{\tau} + B).$$

We also define

$$j(g, \underline{\tau}) = C\underline{\tau} + D. \tag{1}$$

The function $j(g, \underline{\tau})$ satisfies the usual condition of a multiplier system:

$$j(gh, \underline{\tau}) = j(g, h * \underline{\tau}) \cdot j(h, \underline{\tau}), \qquad g, h \in \mathrm{SU}(2,1), \quad \underline{\tau} \in \mathcal{H}. \tag{2}$$

Definition 1 Let Γ be an arithmetic subgroup of $\mathrm{SU}(2,1)$ and let $\frac{a}{b}$ be a rational number. A function $\ell : \Gamma \times \mathcal{H} \to \mathbb{C}^\times$ is called a *fractional weight multiplier system* of weight $\frac{a}{b}$ if it satisfies the following conditions:

1. ℓ is a multiplier system, i.e. $\ell(gh, \underline{\tau}) = \ell(g, h * \underline{\tau}) \cdot \ell(h, \underline{\tau})$ for all $g, h \in \Gamma$ and $\underline{\tau} \in \mathcal{H}$;
2. There exists a function $\chi : \Gamma \to \mathbb{C}^\times$ such that $\ell(g, \underline{\tau})^b = \chi(g) \cdot j(g, \underline{\tau})^a$ for all $g \in \Gamma$ and $\underline{\tau} \in \mathcal{H}$. Such a function χ must be a character of Γ by condition 1 and (2).
3. For each $g \in \Gamma$ the function $\underline{\tau} \mapsto \ell(g, \underline{\tau})$ is continuous (and hence holomorphic) on $\mathcal{H}$.

Half-integral weight multiplier systems have been known about for some time, and have been studied (for example) in [8,15]. The half-integral weight multiplier systems arise from the theory automorphic forms on the metaplectic double cover of a unitary adèle group. However, no other fractional weight multiplier systems on $\mathrm{SU}(2,1)$ have been shown to exist until recently.

By a *fractional weight modular form* of level Γ with weight $\frac{a}{b}$, we shall mean a holomorphic function $f : \mathcal{H} \to \mathbb{C}$, such that for all $\underline{\tau} \in \mathcal{H}$ and all $\gamma \in \Gamma$, we have

$$f(\gamma * \underline{\tau}) = \ell(\gamma, \underline{\tau}) \cdot f(\underline{\tau}),$$

where ℓ is a multiplier system with weight $\frac{a}{b}$. The usual growth condition in cusps is redundant for holomorphic modular forms on $\mathrm{SU}(2,1)$.

1.2 Statement of results

The purpose of this paper is to describe certain modular forms on $\mathrm{SU}(2,1)$ of weight $\frac{1}{3}$. The levels of these forms are congruence subgroups of the following group

$$\Gamma(1) = \mathrm{SU}(2,1) \cap \mathrm{SL}_3(\mathbb{Z}[\zeta]), \qquad \text{where } \zeta = e^{\frac{2\pi i}{3}}. \tag{3}$$

(The reason for focusing on subgroups of $\Gamma(1)$ is because we make use a presentation for $\Gamma(1)$ found in [3] and the description of integral weight forms in [5]). For a non-zero $\beta \in \mathbb{Z}[\zeta]$, we shall write $\Gamma(\beta)$ for the principal congruence subgroup of level β in $\Gamma(1)$. The levels of our forms are intermediate groups between $\Gamma(\sqrt{-3})$ and $\Gamma(3)$, all of which have index 3 in $\Gamma(\sqrt{-3})$.

One way in which one might expect to construct fractional weight forms is to use the theory of Borcherds products associated to the lattice $\mathbb{Z}[\zeta]^3$. Recall that such a Borcherds product is a meromorphic modular form $\Psi : \mathcal{H} \to \mathbb{C}\cup\{\infty\}$, whose divisor is a certain integer linear combination of Heegner divisors. Given a positive integer n and a congruency class in $\nu \in \frac{1}{\sqrt{-3}}\mathbb{Z}[\zeta]^3/\mathbb{Z}[\zeta]^3$, the Heegner divisor $D_{n,\nu}$ is defined by

$$D_{n,\nu} = \begin{cases} \displaystyle\sum_{w\in\mathbb{Z}[\zeta]^3+\nu\,:\,\langle w,w\rangle=n} w^{\perp} & \text{if } 2\nu \notin \mathbb{Z}[\zeta]^3 \\ \displaystyle\frac{1}{2}\sum_{w\in\mathbb{Z}[\zeta]^3+\nu\,:\,\langle w,w\rangle=n} w^{\perp} & \text{if } 2\nu \in \mathbb{Z}[\zeta]^3, \end{cases} \tag{4}$$

where $w^{\perp}$ is the set of $\underline{\tau} \in \mathcal{H}$ such that $\langle \begin{pmatrix} \tau \\ 1 \end{pmatrix}, w\rangle = 0$.

Notice that if w is a vector occurring in the sum (4), then ζw and $\zeta^2 w$ also arise in the sum, and we have $w^{\perp} = (\zeta w)^{\perp} = (\zeta^2 w)^{\perp}$. Hence each Heegner divisor $D_{n,\nu}$ is a multiple of 3. This implies that each Borcherds lift to SU(2, 1) must be the cube of a meromorphic function on $\mathcal{H}$. If a Borcherds lift has weight k, then its cube root will be a fractional weight form of weight $\frac{k}{3}$, implying that there is a multiplier system with weight $\frac{k}{3}$. The level of a Borcherds product associated with the lattice $\mathbb{Z}[\zeta]^3$ is the principal congruence subgroup $\Gamma(\sqrt{-3})$. In view of this, it makes sense to begin searching for third-integral weight forms at level $\Gamma(\sqrt{-3})$. In spite of this optimism, we obtain the following result:

Theorem 1 *Every multiplier system on the group $\Gamma(\sqrt{-3})$ has integral weight. Hence every modular form of level $\Gamma(\sqrt{-3})$ has integral weight.*

This result is proved in Theorem 5; it is a proof by contradiction, using a carefully chosen relation in the group $\Gamma(\sqrt{-3})$. Note that the theorem implies that the weight of each of the Borcherds product associated to the lattice $\mathbb{Z}[\zeta]^3$ must be a multiple of 3. Searching a little further, we do indeed find some fractional weight multiplier systems:

Theorem 2 *There are thirteen subgroups Γ of index 3 in $\Gamma(\sqrt{-3})$ and containing $\Gamma(3)$, such that Γ has a multiplier system of weight $\frac{1}{3}$.*

The result is proved computationally, by finding a presentation for each of the subgroups. The thirteen subgroups mentioned in the theorem are listed in Theorem 6.

Theorem 2 gives us a list of levels where we might conceivably find a modular form of weight $\frac{1}{3}$. Using results of [5], we obtain a description of the ring of integral weight modular forms of level $\Gamma(3)$, as well as the action of $\Gamma(\sqrt{-3})/\Gamma(3)$ on this ring. We find a list of forms of level $\Gamma(3)$ and weight 1, all of whose divisors are multiples of 3. By considering the action of $\Gamma(\sqrt{-3})/\Gamma(3)$ on these forms, we discover that each of them is a modular form (with character) of some larger level. Each of these larger levels is one of the groups found in Theorem 2. The cube roots of these forms are forms of weight $\frac{1}{3}$. In this way, we prove the following:

Theorem 3 *For twelve of the thirteen groups Γ listed in Theorem 6, there exists a holomorphic modular form of weight $\frac{1}{3}$ and level Γ.*

1.3 Further comments and questions

Theorem 3 leaves an obvious open question: why is there no form of weight $\frac{1}{3}$ on the remaining group Γ in our list. We have no clear answer to this question.

We have investigated the multiplier systems on many other congruence subgroups of $\Gamma(1)$. For each congruence subgroup that we have investigated, the denominator of the weight of every multiplier system is a factor of 6. It is tempting to think that this might be true for all congruence subgroups of $\Gamma(1)$. One might even guess that if we replace the field $\mathbb{Q}(\zeta)$ involved in the construction of $\Gamma(1)$ by another CM field k, then a similar bound might be the number of roots of unity in k. However, we have not proved any such result; neither do we have any numerical evidence beyond the case $k = \mathbb{Q}(\zeta)$.

The situation for non-congruence subgroups is very different. For every positive integer n, there exists a (non-congruence) subgroup Γ_n of finite index in $\Gamma(1)$, such that Γ_n has a multiplier system of weight $\frac{1}{n}$. This fact follows from a more general result proved in [7], and independently in [14].

For $d > 2$, it is not known whether there exist any modular forms whose level is a congruence subgroup $SU(d, 1)$, and whose weight has denominator greater than 2. However, such forms certainly do exist at non-congruence levels by results in [7].

1.4 Organization of the paper

The paper is organized as follows. Suppose Γ is an arithmetic subgroup of $SU(2, 1)$. We prove (see Theorem 4) that there is an upper bound on the denominators of weights of multiplier systems on Γ. We define in Sect. 2 a natural number called the *weight denominator* of Γ. The weight denominator is the lowest common denominator of the weights of all multiplier systems on Γ.

Let $\tilde{\Gamma}$ be the pre-image of Γ in the universal cover of $SU(2, 1)$. One may easily calculate the weight denominator of Γ if one has a presentation of $\tilde{\Gamma}$. In Sect. 3, we recall some methods for obtaining such presentations. These methods give us a computational approach to determining the weight denominator of a given arithmetic group.

In Sect. 4 we describe in detail how the methods from Sect. 3 have been implemented for subgroups of finite index in the group $\Gamma(1)$. We calculate the weight denominators of various arithmetic groups, and we list in Sect. 4.4 thirteen congruence subgroups with weight denominator 3, all with index 3 in $\Gamma(\sqrt{-3})$.

In Sect. 5 we construct, for twelve of the thirteen groups found in Sect. 4.4, a modular form of weight $\frac{1}{3}$.

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

2 Background on SU(2,1)

In this section we shall define and investigate a positive integer, which we call the *weight denominator* of an arithmetic subgroup of $SU(2, 1)$.

Let $\widetilde{SU(2, 1)}$ be the universal cover of $SU(2, 1)$. For an arithmetic group Γ we write $\tilde{\Gamma}$ for the pre-image of Γ in $\widetilde{SU(2, 1)}$. The *weight denominator* of Γ is defined to be the order of the kernel of the projection map $\tilde{\Gamma}/[\tilde{\Gamma}, \tilde{\Gamma}] \to \Gamma/[\Gamma, \Gamma]$. We prove in Theorem 4 that the weight denominator is finite. We prove in Corollary 1 that Γ has a multiplier system of rational weight w if and only if the denominator of w is a factor of the weight denominator of Γ. Some other simple properties of the weight denominator are proved in Proposition 3. Methods for computing the weight denominator of an arithmetic group are discussed in Sects. 3 and 4.

2.1 The universal cover

The Lie group $\mathrm{SU}(2,1)$ has fundamental group $\mathbb{Z}$. We shall write $\widetilde{\mathrm{SU}(2,1)}$ for its universal cover. We therefore have a central extension of groups

$$1 \to \mathbb{Z} \to \widetilde{\mathrm{SU}(2,1)} \to \mathrm{SU}(2,1) \to 1.$$

Such extensions are classified by elements of the measurable cohomology group $H^2_{\mathrm{meas}}(\mathrm{SU}(2,1),\mathbb{Z})$ (see [11,12]). In order to perform calculations, it will be helpful to have a specific measurable 2-cocycle σ representing this group extension. We describe such a cocycle in Proposition 1 below. As a first step, we define a function $X : \mathrm{SU}(2,1) \to \mathbb{C}$ by

$$X\begin{pmatrix} * & * & * \\ * & * & * \\ a & b & c \end{pmatrix} = \begin{cases} -a & \text{if} a \neq 0, \\ c & \text{if} a = 0. \end{cases}$$

Lemma 1 *Let $g \in \mathrm{SU}(2,1)$ and $\underline{\tau} \in \mathcal{H}$. Then $X(g) \neq 0$ and the complex number $\frac{j(g,\underline{\tau})}{X(g)}$ has positive real part, where $j(g,\underline{\tau})$ is defined in (1).*

Proof Let $\begin{pmatrix} a & b & c \end{pmatrix}$ be the bottom row of the matrix g. Since $g \in \mathrm{SU}(2,1)$ we must have $\bar{a}c + \bar{b}b + \bar{c}a = 0$. We shall divide the proof into two cases depending on whether or not $a = 0$.

Consider first the case where $a = 0$. The equation above implies $b = 0$, and therefore $c \neq 0$. In such cases $X(g) = j(g,\underline{\tau}) = c$ for all $\underline{\tau} \in \mathcal{H}$ so the lemma is true in this case.

Suppose from now on that $a \neq 0$. In this case we have $X(g) = -a$, and in particular $X(g) \neq 0$. The group $\mathrm{SU}(2,1)$ contains the following torus:

$$T = \{t_z : z \in \mathbb{C}^\times\}, \quad \text{where } t_z = \begin{pmatrix} \frac{1}{\bar{z}} & & \\ & \frac{\bar{z}}{z} & \\ & & z \end{pmatrix}.$$

It is trivial to check that $X(t_z \cdot g) = z \cdot X(g)$ and $j(t_z \cdot g,\underline{\tau}) = z \cdot j(g,\underline{\tau})$. In view of this, it is sufficient to prove the lemma in the case $a = 1$, so that we have $2\Re(c) = -|b|^2$. We must check in this case that $j(g,\underline{\tau})$ has negative real part.

Let $\underline{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$. Since $\underline{\tau} \in \mathcal{H}$ we must have $2\Re(\tau_1) + |\tau_2|^2 < 0$. This implies:

$$\Re(j(g,\underline{\tau})) \;=\; \Re(\tau_1 + b\tau_2 + c) \;<\; -\frac{|\tau_2|^2}{2} + \Re(b\tau_2) - \frac{|b|^2}{2} \;=\; -\left|\tau_2 - \bar{b}\right|^2 \;\leq\; 0.$$

□

Lemma 1 allows us to define, for each $g \in \mathrm{SU}(2,1)$, a branch $\tilde{j}(g,-)$ of the logarithm of $j(g,-)$ as follows:

$$\tilde{j}(g,\underline{\tau}) = \log\left(\frac{j(g,\underline{\tau})}{X(g)}\right) + \log(X(g)), \tag{5}$$

where each logarithm in the right hand side of (5) is defined to be continuous away from the negative real axis and satisfy $-\pi < \Im(\log z) \leq \pi$. For each fixed $g \in \mathrm{SU}(2,1)$ the function $\tilde{j}(g,\underline{\tau})$ is continuous in $\underline{\tau}$ by Lemma 1.

The multiplier system condition (2) on $j(g, \underline{\tau})$ implies the following congruence for $\tilde{j}$:

$$\tilde{j}(gh, \underline{\tau}) \equiv \tilde{j}(g, h * \underline{\tau}) + \tilde{j}(h, \underline{\tau}) \quad \bmod 2\pi i \cdot \mathbb{Z}.$$

In particular, we may define for $g, h \in \mathrm{SU}(2,1)$ an integer $\sigma(g, h)$ by

$$\sigma(g,h) = \frac{1}{2\pi i}\Big(\tilde{j}(gh, \underline{\tau}) - \tilde{j}(g, h * \underline{\tau}) - \tilde{j}(h, \underline{\tau})\Big). \tag{6}$$

The right hand side of (6) is independent of $\underline{\tau}$, since it is a continuous $\mathbb{Z}$-valued function of $\underline{\tau}$.

Proposition 1 *The function σ defined in* (6) *is an inhomogeneous measurable 2-cocycle, whose cohomology class in $H^2_{\mathrm{meas}}(\mathrm{SU}(2,1), \mathbb{Z})$ corresponds to the universal cover of* $\mathrm{SU}(2,1)$.

Proof The function σ is evidently measurable, and it is a short exercise using (6) to verify the 2-cocycle relation:

$$\sigma(g_1, g_2) + \sigma(g_1g_2, g_3) = \sigma(g_1, g_2g_3) + \sigma(g_2, g_3).$$

By Calvin Moore's theory of measurable cohomology [11,12], there is a central extension of Lie groups corresponding to σ. It remains to show that this extension is the universal cover. For the moment, we'll write $\widetilde{\mathrm{SU}(2,1)}$ for the extension of $\mathrm{SU}(2,1)$ corresponding to σ. Explicitly, $\widetilde{\mathrm{SU}(2,1)}$ is the set $\mathrm{SU}(2,1) \times \mathbb{Z}$, with the group operation given by

$$(g, n) \cdot (g', n') = (gg', n + n' + \sigma(g, g')). \tag{7}$$

To prove that $\widetilde{\mathrm{SU}(2,1)}$ is the universal cover of $\mathrm{SU}(2,1)$, it's sufficient to prove that $\widetilde{\mathrm{SU}(2,1)}$ is connected. (Note that the topology on the Lie group $\widetilde{\mathrm{SU}(2,1)}$ is not the product topology on $\mathrm{SU}(2,1) \times \mathbb{Z}$; however the Borel measurable subsets of the Lie group $\widetilde{\mathrm{SU}(2,1)}$ coincide with the Borel measurable subsets of the product $\mathrm{SU}(2,1) \times \mathbb{Z}$).

Let $\tilde{T}$ be the pre-image in $\widetilde{\mathrm{SU}(2,1)}$ of the following torus in $\mathrm{SU}(2,1)$:

$$T = \left\{t_z : z \in \mathbb{C}^\times\right\}, \qquad \text{where } t_z = \begin{pmatrix} \frac{1}{\bar{z}} & & \\ & \frac{\bar{z}}{z} & \\ & & z \end{pmatrix}.$$

Note that on T, we have simply $\tilde{j}(t_z, \underline{\tau}) = \log(z)$, where as before we are taking $-\pi < \Im(\log z) \le \pi$. This easily implies that there is an isomorphism $\Psi : \tilde{T} \to \mathbb{C}$ given by

$$\Psi(t_z, n) = \log(z) - 2\pi i \cdot n. \tag{8}$$

The map Ψ and its inverse are measurable. Since measurable homomorphisms of Lie groups are continuous, it follows that Ψ is a homeomorphism. In particular $\tilde{T}$ is connected. Hence all elements of the subgroup $\mathbb{Z} \subset \widetilde{\mathrm{SU}(2,1)}$ are connected by paths in $\tilde{T}$ to the identity element. This implies that $\widetilde{\mathrm{SU}(2,1)}$ is connected. □

From now on, we shall refer to elements of $\widetilde{\mathrm{SU}(2,1)}$ as pairs $(g, n) \in \mathrm{SU}(2,1) \times \mathbb{Z}$, with the group operation given by (7). To find out whether a fractional weight multiplier system exists on a group Γ, we shall use the following result.

Proposition 2 *Let Γ be an arithmetic subgroup of* $\mathrm{SU}(2,1)$*; let $\tilde{\Gamma}$ be the pre-image of Γ in* $\widetilde{\mathrm{SU}(2,1)}$*, and let w be a rational number.*

For any multiplier system ℓ on Γ of weight w, there is a character $\Phi : \tilde{\Gamma} \to \mathbb{C}^\times$ defined by

$$\Phi(g, n) = \ell(g, \underline{\tau}) \cdot \exp\Big(w\left(2\pi i \cdot n - \tilde{j}(g, \underline{\tau})\right)\Big). \tag{9}$$

In particular, the right hand side of (9) *is independent of* $\underline{\tau} \in \mathcal{H}$.

Conversely, given any character $\Phi : \tilde{\Gamma} \to \mathbb{C}^\times$, *Eq.* (9) *defines a weight w multiplier system* $\ell(g, \underline{\tau})$ *for any rational number w satisfying* $\Phi(I_3, 1) = e^{2\pi i \cdot w}$.

Proof Let $\ell(g, \underline{\tau})$ be a multiplier system on Γ with weight $w = \frac{a}{b}$ and character χ. If we fix $(g, n) \in \tilde{\Gamma}$, then our formula (9) for $\Phi(g, n)$ is a continuous function of $\underline{\tau} \in \mathcal{H}$. Furthermore, $\Phi(g, n)^b = \chi(g)$, which does not depend on $\underline{\tau}$. Hence $\Phi(g, n)$ does not depend on $\underline{\tau}$. To show that Φ is a homomorphism, we calculate as follows:

$$\begin{aligned}\Phi\big((g, n) \cdot (h, m)\big) &= \Phi\big(gh, n + m + \sigma(g, h)\big)\\ &= \ell(gh, \underline{\tau}) \cdot \exp\Big(w\big(2\pi i(n + m + \sigma(g, h)) - \tilde{j}(gh, \underline{\tau})\big)\Big)\end{aligned}$$

Substituting the definition (6) of σ, we get

$$\Phi\big((g, n) \cdot (h, m)\big) = \ell(gh, \underline{\tau}) \cdot \exp\Big(w\big(2\pi i(n + m) - \tilde{j}(g, h * \underline{\tau}) - \tilde{j}(h, \underline{\tau})\big)\Big).$$

Using the multiplier system property of $\ell(g, \underline{\tau})$, we have

$$\begin{aligned}\Phi\big((g, n) \cdot (h, m)\big) &= \ell(g, h * \underline{\tau}) \cdot \ell(h, \underline{\tau}) \cdot \exp\Big(w\big(2\pi i(n + m) - \tilde{j}(g, h * \underline{\tau}) - \tilde{j}(h, \underline{\tau})\big)\Big)\\ &= \Phi(g, n) \cdot \Phi(h, m).\end{aligned}$$

Conversely, suppose $\Phi : \tilde{\Gamma} \to \mathbb{C}^\times$ is a character and $\Phi(I_3, 1) = e^{2\pi i \cdot w}$. The argument above may be reversed, to show that the function $\ell(g, \underline{\tau}) = \Phi(g, 0) \exp(w \cdot \tilde{j}(g, \underline{\tau}))$ satisfies the multiplier system relation. Assuming $w = \frac{a}{b}$, we have

$$\ell(g, \underline{\tau})^b = \Phi(g, 0)^b \cdot j(g, \underline{\tau})^a.$$

Therefore ℓ is a weight w multiplier system with character $\chi(g) = \Phi(g, 0)^b$. □

2.2 The weight denominator

Definition 2 Let Γ be an arithmetic subgroup of SU(2, 1) and let $\tilde{\Gamma}$ be the preimage of Γ in $\widetilde{\mathrm{SU}(2,1)}$. Furthermore, let $\tilde{\Gamma}^{\mathrm{ab}} = \tilde{\Gamma}/[\tilde{\Gamma}, \tilde{\Gamma}]$ be the abelianization of $\tilde{\Gamma}$. We define the *weight denominator* $\mathrm{Denom}(\Gamma)$ to be the order of the element $(I_3, 1)$ in $\tilde{\Gamma}^{\mathrm{ab}}$.

The following theorem is useful to know, although we do not use it in the rest of this paper.

Theorem 4 *For any arithmetic subgroup* Γ *of* SU(2, 1), *the weight denominator is finite.*

In the proof we'll use the following facts about the cohomology of a connected Lie group G:

- There is an isomorphism $H^2_{\mathrm{meas}}(G, \mathbb{Z}) \cong \mathrm{Hom}(\pi_1(G), \mathbb{Z})$ (see [11,12]). In particular $H^2_{\mathrm{meas}}(G, \mathbb{Z})$ is torsion-free, so in injects into $H^2_{\mathrm{meas}}(G, \mathbb{Z}) \otimes \mathbb{C}$.
- By [16], there is an isomorphism $H^\bullet_{\mathrm{meas}}(G, \mathbb{Z}) \otimes \mathbb{C} \cong H^\bullet_{\mathrm{cts}}(G, \mathbb{C})$, where $H^\bullet_{\mathrm{cts}}$ denotes continuous cohomology.
- If Γ is a cocompact arithmetic subgroup of G then by results in [2], the restriction maps $H^\bullet_{\mathrm{cts}}(G, \mathbb{C}) \to H^\bullet(\Gamma, \mathbb{C})$ are all in injective.

Combining these results, we see that if Γ is cocompact in G, then the map $H^2_{\mathrm{meas}}(G,\mathbb{Z}) \to H^2(\Gamma,\mathbb{C})$ is injective.

Proof Let us suppose, for the sake of argument, that $\mathrm{Denom}(\Gamma)$ is infinite. Hence the intersection of $\mathbb{Z}$ with $[\tilde\Gamma,\tilde\Gamma]$ is trivial, so $\mathbb{Z}$ injects into the finitely generated abelian group $\tilde\Gamma^{\mathrm{ab}}$. Choose a torsion-free subgroup L of finite index in $\tilde\Gamma^{\mathrm{ab}}$ containing $\mathbb{Z}$. The map $\mathbb{Z} \to L$ has a left inverse $\phi : L \to \mathbb{Z}$. We may inflate ϕ to a map $\tilde\Gamma' \to \mathbb{Z}$, where $\tilde\Gamma'$ is the pre-image of L in $\tilde\Gamma$. This implies $\tilde\Gamma' \cong \Gamma' \oplus \mathbb{Z}$, where Γ' is the image of $\tilde\Gamma'$ in Γ. Hence the restriction of σ to Γ' is a coboundary. We'll use this assertion to obtain a contradiction by proving that the image of σ in $H^2(\Gamma',\mathbb{C})$ is non-zero.

In the case that Γ' is cocompact in $\mathrm{SU}(2,1)$, the discussion above shows that the image of σ in $H^2(\Gamma',\mathbb{C})$ is non-zero, and we are done. In the case that Γ' has cusps we need to be a little more careful. In this case, we may choose a Lie subgroup $G \subset \mathrm{SU}(2,1)$ isomorphic to $\mathrm{SU}(1,1)$, such that $\Gamma' \cap G$ is cocompact in G (see page 590-591 of [13]). The subgroup G contains a conjugate of the matrix

$$t_{-1} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

Although t_{-1} has order 2, every pre-image of t_{-1} has infinite order in $\widetilde{\mathrm{SU}(2,1)}$ by the isomorphism $\tilde T \cong \mathbb{C}$ in (8). This shows that σ does not split on G. Again by the discussion above, the image of σ in $H^2(\Gamma'\cap G,\mathbb{C})$ is non-zero. The map $H^2_{\mathrm{meas}}(\mathrm{SU}(2,1),\mathbb{Z}) \to H^2(G\cap\Gamma',\mathbb{C})$ factors as follows:

$$H^2_{\mathrm{meas}}(\mathrm{SU}(2,1),\mathbb{Z}) \to H^2(\Gamma',\mathbb{C}) \to H^2(G\cap\Gamma',\mathbb{C}).$$

Hence the image of σ in $H^2(\Gamma',\mathbb{C})$ is non-zero. □

Corollary 1 *Let Γ be an arithmetic subgroup of* $\mathrm{SU}(2,1)$ *and let w be a rational number. There exists a weight w multiplier system on Γ if and only if the denominator of w is a factor of* $\mathrm{Denom}(\Gamma)$.

Proof By Proposition 2, the existence of a weight w multiplier system is equivalent to the existence of a character $\Phi : \tilde\Gamma \to \mathbb{C}^\times$ satisfying $\Phi(I_3,1) = e^{2\pi i\cdot w}$. Any such character Φ would factor as a map $\Phi : \tilde\Gamma^{\mathrm{ab}} \to \mathbb{C}^\times$. By Pontryagin duality, there exists a homomorphism $\Phi : \tilde\Gamma^{\mathrm{ab}} \to \mathbb{C}^\times$ satisfying $\Phi(I_3,1) = e^{2\pi i\cdot w}$ if and only if the denominator of w is a factor of the order of $(I_3,1)$ in $\tilde\Gamma^{\mathrm{ab}}$. □

Proposition 3 *Let Γ and Γ' be arithmetic subgroups of* $\mathrm{SU}(2,1)$ *with $\Gamma' \subset \Gamma$, and let Z be the centre of* $\mathrm{SU}(2,1)$.

1. $\mathrm{Denom}(\Gamma)$ *is a factor of* $\mathrm{Denom}(\Gamma')$.
2. $\mathrm{Denom}(\Gamma')$ *is a factor of* $n \cdot \mathrm{Denom}(\Gamma)$, *where n is the index of Γ' in Γ.*
3. $\mathrm{Denom}(\Gamma \cdot Z) = \mathrm{Denom}(\Gamma)$.

Proof
1. By definition, $\mathrm{Denom}(\Gamma)$ is the smallest positive integer n, such that $(I_3,n) \in [\tilde\Gamma,\tilde\Gamma]$. The first statement follows because $[\tilde\Gamma',\tilde\Gamma']$ is a subgroup of $[\tilde\Gamma,\tilde\Gamma]$.
2. Recall that if H is a subgroup of finite index in a group G, then there is a transfer (or Verlagerung) homomorphism $\mathrm{Verl} : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$. For elements z in the centre

of G, the transfer map is given by $\mathrm{Verl}(z) = z^n$, where n is the index of H in G. We therefore have a homomorphism $\mathrm{Verl} : \tilde{\Gamma}^{\mathrm{ab}} \to \tilde{\Gamma}'^{\,\mathrm{ab}}$, which takes $(I_3, 1)$ to (I_3, n). Since $(I_3, \mathrm{Denom}(\Gamma))$ is the identity element in $\tilde{\Gamma}^{\mathrm{ab}}$, it follows that $(I_3, n \cdot \mathrm{Denom}(\Gamma))$ is the identity element in $\tilde{\Gamma}'^{\,\mathrm{ab}}$.

3. Let $\tilde{Z}$ be the pre-image of Z in $\widetilde{\mathrm{SU}(2,1)}$. By a general result on covering groups of Lie groups, $\tilde{Z}$ is the centre of $\widetilde{\mathrm{SU}(2,1)}$. Hence the commutator of an element of $\tilde{Z}$ with any other group element is trivial. This implies $[\widetilde{\Gamma \cdot Z}, \widetilde{\Gamma \cdot Z}] = [\tilde{\Gamma}, \tilde{\Gamma}]$, from which our result immediately follows. □

3 Calculus of presentations

To calculate the weight denominators of arithmetic groups Γ, we need some methods for producing presentations of the groups $\tilde{\Gamma}$. In this section, we shall describe some methods for doing this. We have used [9] as our main source for this section. The methods described here apply to arbitrary finitely presented groups.

3.1 Covering groups

Suppose $\tilde{G}$ is a central extension of a group G by a cyclic group $\langle z \rangle$ of the form

$$1 \to \langle z \rangle \to \tilde{G} \to G \to 1.$$

Assume that we have a presentation of G:

$$G = \langle g_1, \ldots, g_n | r_1, \ldots, r_m \rangle,$$

where each relation r_i is a word in the generators g_i. We shall describe a presentation of $\tilde{G}$.

For each generator g_i in G, we choose a pre-image $\hat{g}_i$ of g_i in $\tilde{G}$. The group $\tilde{G}$ is generated by the elements

$$\hat{g}_1, \ldots, \hat{g}_n, z.$$

There are three obvious kinds of relation in $\tilde{G}$:

1. For each relation r_i in our presentation of G, we let $\hat{r}_i$ be the word in the generators of $\tilde{G}$, obtained by replacing each g_i with the corresponding element $\hat{g}_i$ in $\tilde{G}$. Since $r_i = 1$ in G, it follows that $\hat{r}_i \in \langle z \rangle$, so we have a relation $\hat{r}_i = z^{c_i}$ in $\tilde{G}$.
2. Since the extension is assumed to be central, the element z is in the centre of $\tilde{G}$, so we have a relation $[z, \hat{g}_i]$ for each generator g_i in G.
3. If z has finite order n, then we have the relation $z^n = 1$.

The generators and relations listed above give a presentation of $\tilde{G}$.

3.2 Subgroups

Let H be a subgroup of finite index in G. Assume that we have a finite presentation of G. The method for constructing a finite presentation of H is called the Reidemeister – Schreier algorithm (see for example Chapter 9 of [9]).

Typically, the Reidemeister–Schreier algorithm produces a presentation with many generators, when far fewer generators are actually needed. The *rewriting* process allows us to reduce the number of generators.

Suppose that we have a group presentation:

$$H = \langle g_1, \ldots, g_r | r_1, \ldots, r_t \rangle,$$

and we also have a second set of generators $\{h_1, \ldots, h_s\}$ for H. We'll explain now how to find a presentation using $h_1, \ldots, h_s$ as generators. This is achieved through a series of Tietze transformations (see [9, Sect. 4.4]). The steps are as follows:

1. Find an expression for each element h_i as a word in the generators g_j:

$$h_i = w_i(g_1, \ldots, g_r).$$

   Then we can the add the generators h_i to the presentation of G, together with the relations $h_i = w_i(g_1, \ldots, g_r)$.
2. Find an expression for each g_i as a word in the generators h_j:

$$g_i = x_i(h_1, \ldots, h_s).$$

   We can add the relations $g_i = x_i(h_1, \ldots, h_s)$ to the presentation of G.
3. In each relation apart from the relation $g_i = x_i(h_1, \ldots, h_s)$, we replace each occurrence of g_i by the word $x_i(h_1, \ldots, h_s)$.
4. Finally, we remove the generators g_i and the relations $g_i = x_i(h_1, \ldots, h_s)$.

4 Weight denominators of certain arithmetic groups

This section concerns computer calculations. The code for these calculations runs on sage version 9.0, and is available at [6].

As before [see (3)], we shall write $\Gamma(1)$ for the subgroup of $\mathrm{SU}(2,1)$ consisting of matrices whose entries are in the ring $\mathbb{Z}[\zeta]$ of Eisenstein integers, where $\zeta = e^{2\pi i/3}$. Let $\Gamma(\sqrt{-3})$ denote the principal congruence subgroup in $\Gamma(1)$ of level $\sqrt{-3}$. As mentioned in the introduction, the group $\Gamma(\sqrt{-3})$ is a level where one might expect to find forms of third-integral weight. In spite of this, we show in Theorem 5 that $\Gamma(\sqrt{-3})$ has weight denominator 1.

Looking just a little further up the tower of congruence subgroups, we do indeed find groups with weight denominator 3. The group $\Gamma(3)$ has weight denominator 3. Furthermore, there are precisely 13 subgroups of index 3 in $\Gamma(\sqrt{-3})$ containing $\Gamma(3)$, which have weight denominator 3 (see Sect. 4.4).

4.1 The subgroup Υ

Rather than dealing with the group $\Gamma(\sqrt{-3})$ directly, it will be slightly more convenient to consider a certain subgroup Υ of index 3 in $\Gamma(\sqrt{-3})$. This is because Υ has a slightly smaller presentation (see Proposition 4 below).

Lemma 2 *The group $\Gamma(\sqrt{-3})$ decomposes as a direct sum: $\Gamma(\sqrt{-3}) = \Upsilon \oplus Z$, where Z is the centre of $\mathrm{SU}(2,1)$ generated by $\zeta \cdot I_3$, and Υ is a subgroup of index 3 defined as follows:*

$$\Upsilon = \{(g_{i,j}) \in \Gamma(\sqrt{-3}) : g_{1,1} \equiv 1 \bmod 3\}.$$

If $(g_{i,j})$ is any element of Υ then we have $g_{1,1} \equiv g_{2,2} \equiv g_{3,3} \equiv 1 \bmod 3$.

Note that by Proposition 3 the groups Υ and $\Gamma(\sqrt{-3})$ have the same weight denominator, since they are the same modulo the centre.

Proof Let $g = (g_{i,j})$ be an element of $\Gamma(\sqrt{-3})$. If v and w are the first and third columns of g, then we must have $\langle v, w\rangle = 1$. Reducing this identity modulo 3, and using the fact that the off-diagonal entries of g are multiples of $\sqrt{-3}$, we find that $\bar{g}_{1,1}g_{3,3} \equiv 1 \bmod 3$.

If g is in Υ, then this implies $g_{3,3} \equiv 1 \bmod 3$. Using the fact that $\det g = 1$, we obtain the other congruence $g_{2,2} \equiv 1 \bmod 3$.

We'll next show that the subset Υ defined in the lemma is a subgroup. Suppose g and h are elements of Υ then

$$(gh)_{1,1} = g_{1,1}h_{1,1} + g_{1,2}h_{2,1} + g_{1,3}h_{3,1}.$$

We have congruences $g_{1,1} \equiv h_{1,1} \equiv 1 \bmod 3$. Furthermore $g_{1,2}$, $g_{1,3}$, $h_{2,1}$ and $h_{3,1}$ are all multiples of $\sqrt{-3}$. Therefore $(gh)_{1,1} \equiv 1 \bmod 3$, and so $gh \in \Upsilon$. Since $g^{-1} = J\bar{g}^t J$, we have $(g^{-1})_{1,1} = \bar{g}_{3,3}$. The congruence $g_{3,3} \equiv 1 \bmod 3$ implies $g^{-1} \in \Upsilon$.

It is clear that Υ and Z are subgroups of $\Gamma(\sqrt{-3})$ which commute with each other and have trivial intersection, so it only remains to show that $\Gamma(\sqrt{-3}) = \Upsilon \cdot Z$. To see why this is the case, suppose $g \in \Gamma(\sqrt{-3})$. We have $g_{1,1} \equiv 1 \bmod \sqrt{-3}$. Hence there is a cube root of unity ζ^r, such that $g_{1,1} \equiv \zeta^r \bmod 3$. It follows that $\zeta^{-r} g \in \Upsilon$, so $g \in \Upsilon \cdot Z$. □

We shall find a presentation for Υ. We start by finding a small generating set. It will be useful to have the following notation for upper-triangular elements of Υ:

$$n(z,x) = \begin{pmatrix} 1 & \sqrt{-3}\cdot z & \frac{-3\,\mathrm{N}(z) + x\sqrt{-3}}{2} \\ 0 & 1 & \sqrt{-3}\cdot \bar{z} \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \mathbb{Z}[\zeta], \quad x \in \mathbb{Z}, \quad x \equiv \mathrm{N}(z) \bmod 2.$$

Here and later we write N and Tr for the norm and trace maps from $\mathbb{Q}(\zeta)$ to $\mathbb{Q}$.

Lemma 3 *The group Υ is generated by the elements $n(z,x)$ and their transposes $n(z,x)^t$, where $z \in \mathbb{Z}[\zeta]$ and $x \in \mathbb{Z}$ satisfy $x \equiv \mathrm{N}(z) \bmod 2$.*

Proof Let g be an element of Υ, and let

$$g = \begin{pmatrix} a & * & * \\ b & * & * \\ c & * & * \end{pmatrix}, \qquad a \equiv 1 \bmod 3, \quad b \equiv c \equiv 0 \bmod \sqrt{-3}.$$

Define $H(g) = \mathrm{N}(a) + \mathrm{N}(c)$. Since $g \in \mathrm{SU}(2,1)$ we have $\mathrm{Tr}(a\bar{c}) + \mathrm{N}(b) = 0$. In particular, a and c cannot both be zero, so $H(g)$ is a positive integer. We shall prove by induction on $H(g)$, that g may be expressed as a product of elements of the form $n(z,x)$ and $n(z,x)^t$.

Assume first that $H(g) = 1$. Since $\mathrm{N}(c)$ is a multiple of 3, we must have $\mathrm{N}(a) = 1$ and $c = 0$. This implies $b = 0$. The congruence $a \equiv 1 \bmod 3$ implies $a = 1$. Using the fact that $g \in \mathrm{SU}(2,1)$ we deduce that $g = n(z,x)$ for suitable z and x.

Assume now that $H(g) > 1$. Our congruence conditions on a and c imply that $\mathrm{N}(a) \neq \mathrm{N}(c)$. For the inductive step, we must prove the following:

1. If $\mathrm{N}(a) > \mathrm{N}(c)$ then there exists an element $n(z,x)$ such that $H(n(z,x)g) < H(g)$.
2. If $\mathrm{N}(a) < \mathrm{N}(c)$ then there exists an element $n(z,x)^t$ such that $H(n(z,x)^t g) < H(g)$.

We shall prove statement 1 in detail; the proof of statement 2 is similar.

Assume that $\mathrm{N}(a) > \mathrm{N}(c)$. One may check that the closed hexagon in $\mathbb{C}$ with vertices at the roots of unity $\pm 1, \pm\zeta, \pm\zeta^2$ is a fundamental domain for the lattice $\sqrt{-3}\cdot\mathbb{Z}[\zeta]$. If we choose $z \in \mathbb{Z}[\zeta]$ so that $\sqrt{-3}\bar{z}$ is as near as possible to $-\frac{b}{c}$, then $\frac{b}{c} + \sqrt{-3}\bar{z}$ is in this closed hexagon. In particular $\mathrm{N}(\frac{b}{c} + \sqrt{-3}\bar{z}) \leq 1$. We have

$$n(z,x_0)\cdot g = \begin{pmatrix} a' & * & * \\ b' & * & * \\ c & * & * \end{pmatrix}, \qquad \text{where } b' = b + \sqrt{-3}\bar{z}c.$$

Here we have chosen x_0 to be any integer congruent to $\mathrm{N}(z)$ modulo 2; this choice does not change b'. Our choice of z implies

$$\mathrm{N}\left(\frac{b'}{c}\right) \leq 1. \tag{10}$$

Next we choose $x \in \mathbb{Z}$ so that $\sqrt{3}x$ is as near as possible to $\Im(\frac{a'}{c})$. With this choice we have $|\Im(\frac{a'}{c} - \sqrt{-3}x)| \leq \frac{\sqrt{3}}{2}$. Consider the matrix

$$n(z, x_0 - 2x) \cdot g = \begin{pmatrix} a'' & * * \\ b' & * * \\ c & * \ * \end{pmatrix}, \qquad \text{where } a'' = a' - \sqrt{-3}xc.$$

Since this matrix is in $\mathrm{SU}(2,1)$ we must have $a''\bar{c} + b'\bar{b}' + c\bar{a}'' = 0$. This implies $\Re(\frac{a''}{c}) = -\frac{1}{2}\mathrm{N}(\frac{b'}{c})$, so by (10) we have $\left|\Re(\frac{a''}{c})\right| \leq \frac{1}{2}$. Our choice of x implies $|\Im(\frac{a''}{c})| \leq \frac{\sqrt{3}}{2}$. Combining these bounds, we obtain

$$\mathrm{N}\left(\frac{a''}{c}\right) = \Re\left(\frac{a''}{c}\right)^2 + \Im\left(\frac{a''}{c}\right)^2 \leq 1.$$

In particular, $\mathrm{N}(a'') \leq \mathrm{N}(c) < \mathrm{N}(a)$ and therefore $H(n(z, x_0 - 2x) \cdot g) < H(g)$. □

4.2 The weight denominators of Υ and $\Gamma(\sqrt{-3})$

To calculate the weight denominators of Υ and $\Gamma(\sqrt{-3})$ we shall use a presentation of Υ.

Proposition 4 *The group Υ has a presentation with the following five generators:*

$$n_1 = n(1,1), \quad n_2 = n(\zeta, 1), \quad n_3 = n(0,2), \quad n_4 = n_1^t, \quad n_5 = n_3^t,$$

and the following thirteen relations:

$$\begin{gathered}
[n_1, n_3], \qquad [n_2, n_3], \qquad [n_4, n_5], \qquad (n_3 n_5)^3, \\
n_3 n_2 n_1 n_2^{-1} n_3 n_1^{-1} n_3, \qquad (n_1^{-1} n_3 n_4^{-1})^3, \\
n_5^{-1} n_2 n_5 n_4^{-1} n_1^{-1} n_2^{-1} n_3 n_4 n_3^{-1} n_1, \\
n_4^{-1} n_1^{-1} n_3 n_5 n_2 n_1 n_5^{-1} n_4 n_2^{-1} n_3^{-1}, \\
n_5^{-1} n_4 n_1 n_5 n_3^{-1} n_1^{-1} n_2^{-1} n_4^{-1} n_3^{-2} n_2, \\
n_5^{-1} n_2 n_1 n_5^{-1} (n_4 n_1)^2 n_5^{-1} n_4 n_2^{-1}, \\
n_3 n_5 n_1 n_4 n_5^{-1} n_2^{-1} n_4 n_3^{-1} n_1 n_4 n_2 n_1, \\
n_3^{-1} n_1 n_4 n_2 n_3 n_1 n_5^{-1} n_1^{-1} n_4^{-1} n_5 n_1^{-1} n_2^{-1}, \\
n_4^{-1} n_3^{-1} n_5 n_3 n_1^{-1} n_4^{-1} n_2 n_1 n_3^{-1} n_4 n_1 n_5^{-1} n_4 n_2^{-1} n_1^{-1} n_3.
\end{gathered}$$

Proof We begin by showing that the matrices $n_1, \ldots, n_5$ generate Υ. It is easy to see that any matrix $n(z, x)$ in Υ may be expressed as a product of the elements n_1, n_2, n_3 raised to appropriate powers. Similarly, any matrix $n(z, x)^t$ may be expressed in terms of n_4, n_5, n_2^t. In view of Lemma 3, this shows that Υ is generated by $n_1, n_2, n_3, n_4, n_5, n_2^t$. The generator n_2^t may be eliminated using the relation $n_2^t = n_3^{-1} n_1 n_4 n_1 n_3^{-2} n_2$.

Finding the relations in Υ is much harder, and we have used a computer for this (see [6] for the code). We'll explain briefly how the calculation was done. Consider the following groups:

$$\mathrm{U}(2,1)(\mathbb{Z}[\zeta]) = \{g \in \mathrm{GL}_3(\mathbb{Z}[\zeta]) : \bar{g}^t J g = J\}, \quad \mathrm{PU}(2,1)(\mathbb{Z}[\zeta]) = \mathrm{U}(2,1)(\mathbb{Z}[\zeta])/Z_6,$$

where Z_6 is the centre of $\mathrm{U}(2,1)(\mathbb{Z}[\zeta])$, which is a cyclic group of order 6. In [3], Falbel and Parker have obtained a presentation for the group $\mathrm{PU}(2,1)(\mathbb{Z}[\zeta])$ with three generators and five relations. By the method described in Sect. 3.1 we may use their result to obtain a presentation of $\mathrm{U}(2,1)(\mathbb{Z}[\zeta])$. The group Υ is a subgroup of $\mathrm{U}(2,1)(\mathbb{Z}[\zeta])$ of finite index, so we may use the Reidemeister–Schreier algorithm to obtain a presentation of Υ. The resulting presentation of Υ is rather large (52 generators and 223 relations). The next step is to use the method described in Sect. 3.2 to rewrite the presentation in terms of the five generators n_1, n_2, n_3, n_4, n_5 found above. Finally, we simplify the presentation using GAP; this reduces the number of relations to the thirteen given above. □

Theorem 5 *The groups Υ and $\Gamma(\sqrt{-3})$ have weight denominator* 1*.*

Proof These two groups are the same modulo their centre, so by Proposition 3 it is sufficient to prove the result for Υ.

We must show that $(I_3, 1)$ is in the commutator subgroup $[\tilde{\Upsilon}, \tilde{\Upsilon}]$. We shall write $r_1, \dots, r_{13}$ for the thirteen relations in Υ found in the previous result. We shall regard relations as elements of the free group on the generators $n_1, \dots, n_5$.

The relation $R = r_4^{-1} r_9^{-1} r_{10}^{-1} r_{11}$ is given by:

$$\begin{aligned} R = {} & (n_5^{-1} n_3^{-1})^3 n_2^{-1} n_3^2 n_4 n_2 n_1 n_3 n_5^{-1} n_1^{-1} n_4^{-1} n_5 n_2 n_4^{-1} n_5 (n_1^{-1} n_4^{-1})^2 \\ & n_5 n_1^{-1} n_2^{-1} n_5 n_3 n_5 n_1 n_4 n_5^{-1} n_2^{-1} n_4 n_3^{-1} n_1 n_4 n_2 n_1. \end{aligned}$$

Notice that the relation R has an interesting property: the number of times that each generator n_i occurs in R is equal to the number of times that n_i^{-1} occurs. Equivalently, R is in the commutator subgroup of the free group on $n_1, \dots, n_5$.

For each generator n_i, we may choose a lift $\hat{n}_i$ to $\tilde{\Upsilon}$. Let $\hat{R}$ be the element of $\tilde{\Upsilon}$ which we obtain by replacing each generator n_i in the word R by the lift $\hat{n}_i$. In fact $\hat{R}$ does not depend on the choices of lift $\hat{n}_i$ since the total degree of n_i in R is zero. A short calculation using a computer shows that $\hat{R} = (I_3, -1)$. Furthermore, $\hat{R}$ is in the commutator subgroup $[\tilde{\Upsilon}, \tilde{\Upsilon}]$, since every occurrence of a generator $\hat{n}_i$ is balanced out be an occurrence of $\hat{n}_i^{-1}$. □

Although Theorem 5 was found using a computer, it would be possible to verify the result by hand (given a few days). One needs only to calculate the element $\hat{R}$ in $\tilde{\Upsilon}$, and note that this element is in the commutator subgroup. It would however be difficult to find the relation R without the presentation of Proposition 4.

4.3 Subgroups of Υ

There is code (see [6]) which calculates the number $\mathrm{Denom}(\Gamma)$ for certain arithmetic subgroups Γ of Υ. The method of calculation is as follows.

1. We already have a presentation of Υ (Proposition 4). Using the Reidemeister–Schreier algorithm, we find a presentation of Γ:

$$\Gamma = \langle \gamma_1, \dots, \gamma_r | \delta_1 = \cdots = \delta_s = 1 \rangle.$$

2. For each generator γ_i we let $\hat{\gamma}_i = (\gamma_i, 0) \in \tilde{\Gamma}$. For each relation δ_j, we let $\hat{\delta}_j$ be the element of $\tilde{\Gamma}$ obtained by replacing each generator γ_i in δ_j by its lift $\hat{\gamma}_i$. In $\tilde{\Gamma}$ we have relations

$$\hat{\delta}_j \cdot (I_3, n_j) = 1, \qquad n_j \in \mathbb{Z}. \tag{11}$$

The integers n_j are calculated by multiplying in $\tilde{\Gamma}$ using (7).

3. To each of the relations (11) we form a row vector in $\mathbb{Z}^{r+1}$, where the first r entries are the multiplicities of the generators $\hat{\gamma}_i$ and the last entry is n_j. If we let M be the matrix formed of these rows, then we have a presentation

$$\tilde{\Gamma}^{\mathrm{ab}} \cong \mathbb{Z}^{r+1}/(\mathbb{Z}^s \cdot M).$$

4. The subgroup $\mathbb{Z}^s \cdot M$ is unchanged by integral row operations on M. We may transform M by a sequence of such operations to a matrix M' in Hermite normal form. The last non-zero row of M' must have the form

$$\begin{pmatrix} 0 & \cdots & 0 & n \end{pmatrix}, \quad n > 0.$$

Indeed if this were not the case, then the generator $(I_3, 1)$ would have infinite order in $\tilde{\Gamma}^{\mathrm{ab}}$, contradicting Theorem 4.

5. The positive integer n is the order of $(I_3, 1)$ in $\tilde{\Gamma}^{\mathrm{ab}}$, which is by definition the weight denominator of Γ.

The speed of the computation depends on the index of the subgroup Γ in Υ. The current version of the code is able to handle subgroups of index up to around 500 on a home computer. The most time consuming step is currently the row reduction in step 4. There is an opportunity (using Proposition 3) to speed up this step by row-reducing modulo the prime powers which divide the index of Γ in Υ, rather than row-reducing over $\mathbb{Z}$. There is also scope for simplifying the presentation of Γ at the end of step 1, using a smaller set of generators (as we did for Υ in Proposition 4). At the moment, this is not implemented.

4.4 Congruence subgroups with weight denominator 3

The principal congruence subgroup $\Gamma(3)$ is a normal subgroup of Υ with index 81. One easily checks using the code at [6] that $\Gamma(3)$ has weight denominator 3. This means that $\Gamma(3)$ has a multiplier system with weight $\frac{1}{3}$. If any reader would like to run this code and experiment further, then there are some instructions online. The sage code for this particular calculation is

```
sage: load(``SU21.sage'')
sage: G = Gamma(3)
sage: print(G.weight_denominator())
```

However, one does not need to go as far as $\Gamma(3)$ to find such a multiplier system. We shall discuss the intermediate groups between Υ and $\Gamma(3)$

Lemma 4 *There is an isomorphism $F : \Upsilon/\Gamma(3) \to \mathbb{F}_3^4$. given by*

$$F\begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,1} & g_{2,2} & g_{2,3} \\ g_{3,1} & g_{3,2} & g_{3,3} \end{pmatrix} = \begin{pmatrix} \frac{g_{1,2}}{\sqrt{-3}} & \frac{g_{1,3}}{\sqrt{-3}} & \frac{g_{2,1}}{\sqrt{-3}} & \frac{g_{3,1}}{\sqrt{-3}} \end{pmatrix}.$$

(Here we are regarding the numbers $\frac{g_{i,j}}{\sqrt{-3}} \in \mathbb{Z}[\zeta]$ as elements of $\mathbb{F}_3$ by reducing modulo $\sqrt{-3}$.)

Proof We'll first show that the formula for F gives a homomorphism from Υ to $\mathbb{F}_3^4$. Suppose $g, h \in \Upsilon$. We therefore have $g = I + \sqrt{-3}X$ and $h = I + \sqrt{-3}Y$ for suitable matrices $X, Y \in M_3(\mathbb{Z}[\zeta])$. This implies $gh \equiv I + \sqrt{-3}(X + Y) \mod 3$. Since $F(gh)$ depends only on gh modulo 3, it follows easily that $F(gh) = F(g) + F(h)$.

Note that we have

$$F(n(z,x)) = \begin{pmatrix} z & \frac{x}{2} & 0 & 0 \end{pmatrix}, \qquad F(n(z,x)^t) = \begin{pmatrix} 0 & 0 & z & \frac{x}{2} \end{pmatrix}.$$

This shows that F is surjective.

If $g \in \Gamma(3)$ then clearly $g \in \ker(F)$. Conversely, suppose $g \in \Upsilon$ satifies $F(g) = 0$. Since $g \in \Upsilon$, the diagonal entries of g must be congruent to 1 modulo 3 (by Lemma 2). Since $F(g) = 0$, we have the congruence

$$g \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \sqrt{-3}x \\ 0 & \sqrt{-3}y & 1 \end{pmatrix} \mod 3.$$

Substituting this congruence into the equation $\bar{g}^t J g = J$, we obtain $\sqrt{-3}x \equiv \sqrt{-3}y \equiv 0 \mod 3$, which implies $g \in \Gamma(3)$. □

In view of the lemma, the intermediate group between Υ and $\Gamma(3)$ correspond to subgroups of $\mathbb{F}_3^4$. In particular, for $a, b, c, d \in \mathbb{F}_3$ we define a subgroup of Υ by

$$\Upsilon_{\text{index } 3}(a, b, c, d) = \{g \in \Upsilon : a \cdot g_{1,2} + b \cdot g_{1,3} + c \cdot g_{2,1} + d \cdot g_{3,1} \equiv 0 \mod 3\}.$$

If v is a non-zero vector in $\mathbb{F}_3^4$ then $\Upsilon_{\text{index } 3}(v)$ has index 3 in Υ. Every subgroup of index 3 in Υ containing $\Gamma(3)$ has this form. Two of these subgroups $\Upsilon_{\text{index } 3}(v)$ and $\Upsilon_{\text{index } 3}(v')$ are equal if and only if $v \equiv \pm v' \mod 3$. We therefore have 40 subgroups of index 3 containing $\Gamma(3)$. By Proposition 3, each of these subgroups has weight denominator either 1 or 3. We have calculated all of these weight denominators. For example, to calculate the weight denominator of $\Upsilon_{\text{index } 3}(0, 0, 1, 0)$, the command is

```
sage: load(``SU21.sage'')
sage: G = Index3congruence(0,0,1,0)
sage: print(G.weight_denominator())
```

The results of our calculations are as follows.

Theorem 6 *The following thirteen groups have weight denominator* 3*:*

$$\begin{gathered}\Upsilon_{\text{index } 3}(0,0,1,0), \quad \Upsilon_{\text{index } 3}(0,0,1,1), \quad \Upsilon_{\text{index } 3}(0,0,1,2), \quad \Upsilon_{\text{index } 3}(0,1,1,0), \\ \Upsilon_{\text{index } 3}(0,1,2,0), \quad \Upsilon_{\text{index } 3}(1,0,0,0), \quad \Upsilon_{\text{index } 3}(1,0,0,1), \\ \Upsilon_{\text{index } 3}(1,0,0,2), \quad \Upsilon_{\text{index } 3}(1,0,2,0), \quad \Upsilon_{\text{index } 3}(1,1,0,0), \\ \Upsilon_{\text{index } 3}(1,1,2,2), \quad \Upsilon_{\text{index } 3}(1,2,0,0), \quad \Upsilon_{\text{index } 3}(1,2,2,1).\end{gathered}$$

The other 27 groups of the form $\Upsilon_{\text{index } 3}(v)$ *all have weight denominator* 1.

Note that for each of the groups Γ listed above, the group $\Gamma \cdot Z$ is a subgroup of index 3 in $\Gamma(\sqrt{-3})$, and also has weight denominator 3 by Proposition 3.

5 Modular forms

Multiplier systems arise when one wants to consider modular forms of non integral weight. It is a natural problem to realize a given multiplier system by a modular form. In some cases the whole rings of modular forms of integral weight have been determined and we can use these results to construct forms of non integral weight. Recall that a modular form on an arithmetic subgroup $\Gamma \subset \mathrm{U}(n,1)$ and with respect to a multiplier system $\ell(g,\tau)$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ with the property

$$f(g\tau) = \ell(g,\tau)f(\tau), \quad g \in \Gamma,$$

where in the case $n = 1$ the usual regularity condition at the cusps has to be added.

Notice that we work in this section with the group $\mathrm{U}(n,1)$ instead of $\mathrm{SU}(n,1)$. This is due to the fact that in the theory of modular forms reflections play an important role. Of course Definition 1 works literally in the case $\mathrm{U}(n,1)$. A general result using Poincaré series or compactification theory states that for every multiplier system of weight $\frac{a}{b}$ there exists a natural number $k \gg 0$ such that the multiplier system $\ell(g,\tau)j(g,\tau)^k$ of weight $\frac{a}{b} + k$ admits a non zero modular form. We will see that 12 of the above 13 multiplier systems admit a modular form of weight $\frac{1}{3}$.

5.1 Rings of modular forms

We consider the $(n+1) \times (n+1)$ matrix

$$S = \begin{pmatrix} 0 & -1 & & & \\ -1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

It defines a Hermitian form of signature $(n,1)$,

$$\langle z, w\rangle = \bar{z}^t S w, \quad z, w \in \mathbb{C}^{n+1} \text{ (columns)}.$$

We denote by G_n the group of all $(n+1) \times (n+1)$-matrices g with coefficients in the ring of Eisenstein integers with the property $\bar{g}^t S g = S$. Notice that we do not assume that $\det(g) = 1$.

In the case $n = 2$ it is related to the form J from Sect. 1 by

$$J = \bar{V}^t S V, \quad V = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

So the group $\Gamma(1)$ of Sect. 4 is embedded into G_2,

$$\Gamma(1) \longrightarrow G_2, \quad g \longmapsto h = VgV^{-1}.$$

For each non zero Eisenstein integer β we can consider the principal congruence subgroups $\Gamma(\beta)$ and $G_n(\beta)$. The group $\Gamma(\beta)$ can be identified with the subgroup of $G_2(\beta)$ of matrices with determinant one. Rings of modular forms of integral weight have been determined in the literature for $G_4(\sqrt{-3})$, [1,4], for $G_3(\sqrt{-3})$, [5,10] and for the group $G_3(3)$ [5].

The ring of modular form for $G_3(3)$ is rather complicated. We can use its structure to derive the rings for $G_2(2)$ (and also for $G_1(3)$). We formulate now these results and give some hints how to get them.

First we recall the definition of the ring of modular forms. Here we use the unitary group for the Hermitian form S and denote it from now on by $\mathrm{U}(1,n)$. So $G_n(\ell)$ is a subgroup of $\mathrm{U}(1,n)$. We have to replace the homogeneous space $\mathcal{H}$ by the space $\mathcal{H}_n$ of all columns

$z \in \mathbb{C}^n$ with the property

$$\begin{pmatrix} 1 \\ \bar{z} \end{pmatrix}^t S \begin{pmatrix} 1 \\ z \end{pmatrix} < 0.$$

This means

$$\mathcal{H}_n = \{z \in \mathbb{C}^n; \quad \Re z_1 > |z_2|^2 + \cdots + |z_n|^2\}.$$

The action of $\mathrm{U}(n, 1)$ is given by

$$g * z = \frac{1}{A + Bz} \cdot (C + Dz).$$

Here g is a block matrix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A is a complex number, D a 2×2-matrix etc. We also define

$$j(g, z) = A + Bz. \tag{12}$$

The definition of a multiplier system is the same as in Definition 1 and the notion of a modular form has to be defined as in the beginning of Sect. 5. The ring of modular forms $A(G_n(l))$ is the direct sum of the spaces of all modular forms with respect to integral powers of $j(g, z)$. From compactification theory one knows that this is a finitely generated algebra whose associated projective variety is a compactification (by finitely many points) of $\mathcal{H}_n/G_n(\beta)$. The embedding

$$\mathcal{H}_n \longrightarrow \mathcal{H}_{n+1}, \quad z \longmapsto (z, 0),$$

induces a homomorphism

$$A(G_{n+1}(\beta)) \longrightarrow A(G_n(\beta)).$$

From compactification theory follows that $A(G_n(\beta))$ is the normalization of the image.

5.2 Examples of rings of modular forms

We apply the discussion above to determine the ring of modular forms for $G_2(3)$. We want to use the known ring for $G_3(3)$. This ring is rather complicated. There are 15 basic modular forms $B_1, \dots, B_{15}$ of weight 1. They have been constructed in [5] as Borcherds products. The algebra $\mathbb{C}[B_1, \dots, B_{15}]$ agrees with $A(G_3(3))$ in weight ≥ 7. The zeros of the B_i are located at Heegner divisors which can be considered as $G_3(\sqrt{-3})$-translates of $\mathcal{H}_2$ which can be identified with the subspace of $\mathcal{H}_3$ defined by $z_3 = 0$. A finite system of generators of the ideal of relations has been determined [5]. Using these results it is easy to prove that the restrictions of

$$B_1, B_2, B_3, B_4 + B_5, B_6 + B_7, B_8 - B_9, B_{10} - B_{11}, B_{12} + B_{13}, B_{14} + B_{15}$$

to $\mathcal{H}_2$ are zero. The quotient of $\mathbb{C}[B_1, \dots, B_{15}]$ by the ideal that is generated by these relations turns out to be normal and of Krull dimension 3. Hence this ideal is the kernel of the homomorphism $\mathbb{C}[B_1, \dots, B_{15}] \to A(G_2(3))$ and this homomorphism must be surjective. In this way we can determine $A(G_2(3))$. We denote the restrictions of

$$B_5,\ B_{11},\ B_{15},\ B_7,\ B_{13},\ B_9$$

to $\mathcal{H}_2$ in the same order by

$$X_1,\ X_2,\ X_3,\ X_4,\ X_5,\ X_6.$$

Proposition 5 *The algebra $A(G_2(3))$ is generated by $X_1, \dots, X_6$. Defining relations are*

$$X_1^3 + X_2^3 - X_3^3 = 0,\ X_4^3 - X_2^3 - X_5^3 = 0,\ X_6^3 - X_4^3 + X_3^3 = 0.$$

The group G_2 acts on the forms X_i by permutations combined with multiplication with 6th roots of unity.

As we mentioned these results are consequences of the paper [5] where the case G_3 has been treated. For the action of G_3 on the B_i we refer to Lemma 8.3 in this paper. The action of G_2 on the X_i is a consequence.

5.3 Modular forms of weight $\frac{1}{3}$

Let $f \in A(G_2(3))$ be a modular form of weight r. Since the structure of $A(G_2(3))$ is known and simple enough, one can determine the primary decomposition of the principal ideal (f),

$$(f) = fA(G_2(3)) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m.$$

If one has luck, the multiplicities of the $\mathfrak{q}_i$ are multiples of 3. Then the multiplicities of the zero components of f in $\mathcal{H}_2$ are also multiples of 3. So wa can take a holomorphic cube root of f to produce a form of weight $\frac{r}{3}$. Hence our construction of forms of weight $1/3$ rests on the knowledge of the structure of the ring of modular forms of *integral* weight. We give an example.

We consider the form $X_1 + X_2$ of weight one. It has been found by trial and error. Its primary decomposition can be computed by means of MAGMA. It turns out that (X_1+X_2) is the intersection of three primary ideals

$$\mathfrak{q}_1 = (X_1 + X_2, X_4 + (\zeta + 1)X_6, X_3^3, X_2^3 + X_5^3 - X_6^3), \tag{13}$$

$$\mathfrak{q}_2 = (X_1 + X_2, X_4 - \zeta X_6, X_3^3, X_2^3 + X_5^3 - X_6^3), \tag{14}$$

$$\mathfrak{q}_3 = (X_1 + X_2, X_4 - X_6, X_3^3, X_2^3 + X_5^3 - X_6^3). \tag{15}$$

The associated prime ideals (which are the radicals) $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are obtained if one replaces in each case X_3^3 by X_3. For example

$$\mathfrak{p}_3 = (X_1 + X_2, X_4 - X_6, X_3, X_2^3 + X_5^3 - X_6^3).$$

This defines the elliptic curve

$$X_2^3 + X_5^3 - X_6^3 = 0$$

in $P^2\mathbb{C}$. Its j-invariant is 0. All three curves are isomorphic.

Lemma 5 *The form $X_1 + X_2$ on $\mathcal{H}_2$ with respect to the group $G_2(3)$ vanishes along three elliptic curves with multiplicity three and has no other zero. Hence it is the third power of a modular form of weight $\frac{1}{3}$.*

In [5] the action of G_3 on $A_3(G_3)$ has been determined. Using this we can determine the invariance group of $X_1 + X_2$ to verify the following result.

Lemma 6 *The invariance group of the form $X_1 + X_2$ is the subgroup of $G_2(\sqrt{-3})$ that is defined through the congruence $h_{13} + h_{21} \equiv 0 \mod 3$. This group is an extension of index 3 of its intersection with* $\mathrm{SU}(2,1)$. *This intersection corresponds to the group $\Upsilon_{\text{index }3}(0,1,1,0)$ (subsection 4.4).*

The form $X_1 + X_2$ is a special example of a form of weight 1 which admits a holomorphic third root. To get more such forms, we tried also the forms $X_i \pm X_j$. It turned out that for each pair (i, j), $i \neq j$, there is one of the two signs such that the form has third root of unity.

The following table contains forms $X_i \pm X_j$ which have the same property. They belong to a subgroup defined by a congruence $L(h) \equiv 0 \mod 3$. The linear form L is in the first column. The second column contains the form and the third column contains the vector ν such that intersection with SU(2, 1) corresponds to $\Upsilon_{\text{index } 3}(\nu)$.

12 Congruence groups that admit a modular form of weight $\frac{1}{3}$

Congruence L	third power of a form of weight 1/3	$\Upsilon_{\text{index } 3}(\nu)$ $\nu :=$
h_{13}	$X_2 - X_3$	(0, 0, 1, 0)
$h_{12} + h_{13}$	$X_3 - X_4$	(0, 0, 1, 1)
$h_{12} - h_{13}$	$X_2 - X_4$	(0, 0, 1, 2)
$h_{13} + h_{21}$	$X_1 + X_2$	(0, 1, 1, 0)
$h_{13} - h_{21}$	$X_1 - X_3$	(0, 1, 2, 0)
h_{31}	$X_5 - X_6$	(1, 0, 0, 0)
$h_{12} + h_{31}$	$X_1 - X_5$	(1, 0, 0, 1)
$h_{12} - h_{31}$	$X_1 + X_6$	(1, 0, 0, 2)
$h_{21} + h_{31}$	$X_4 - X_5$	(1, 1, 0, 0)
$h_{12} + h_{13} - h_{21} - h_{31}$	$X_4 - X_6$	(1, 1, 2, 2)
$h_{21} - h_{31}$	$X_3 + X_6$	(1, 2, 0, 0)
$-h_{12} + h_{13} + h_{21} - h_{31}$	$X_2 + X_5$	(1, 2, 2, 1)

These forms are not uniquely determined. In fact, one can replace each form $X_i \pm X_j$ in the table by $X_i \pm \zeta^\nu X_j$, $0 \leq \nu \leq 2$ and in this way we get forms with respect to the same group that admit also third roots of unity.

This means that we constructed 36 forms of weight $\frac{1}{3}$. So we have proved that 12 of the above 13 groups in Sect. 4.4 admit three modular forms of weight $\frac{1}{3}$.

One can show that this is false for the remaining group which is given through $h_{13} \equiv h_{31} \mod 3$. Its intersection with SU(2, 1) corresponds to $\nu = (1, 0, 2, 0)$. In this case one can prove that no form of weight one which belongs to this congruence group and which admits a holomorphic third root can exist. By general arguments there must exist in $A(G_2(3))$ a modular of weight $3r + 1$, $r \in \mathbb{Z}$ suitable, which admits a holomorphic third root of unity.

So far we do not know an example.

Author details
Heidelberg University, Heidelberg, Germany, University College London, London, UK.

Received: 2 March 2022 Accepted: 24 July 2022
Published online: 28 March 2023

References

1. Allcock, D., Freitag, E.: Cubic surfaces and Borcherds products. Commentarii Math. Helv. **77**(2), 270–296 (2002). https://doi.org/10.1007/s00014-002-8340-4
2. Borel, A., Wallach, N.: *Continuous cohomology, discrete subgroups, and representations of reductive groups.* Second edition. Mathematical Surveys and Monographs, 67. American Mathematical Society, Providence, RI, (2000)
3. Falbel, E., Parker, J.R.: The geometry of the Eisenstein–Picard modular group. Duke Math. J. **131**(2), 249–289 (2006). https://doi.org/10.1215/S0012-7094-06-13123-X
4. Freitag, E.: A graded algebra related to cubic surfaces. Kyushu J. Math. **56**(2), 299–312 (2002). https://doi.org/10.2206/kyushujm.56.299

5. Freitag, E., Salvati Manni, R.: A three dimensional ball quotient. Math. Zeitschrift **276**, 345–370 (2014). https://doi.org/10.1007/s00209-013-1203-4
6. Hill, R. M.: sage code. https://github.com/rmhi/SU21Eisenstein
7. Hill, R. M.: Fractional weight multiplier systems on SU(d,1). arxiv:2108.04538
8. Jalal, L.: Calculation of a half-integral weight multiplier system on SU(2,1). PhD thesis, University College London (2011). arxiv:1107.1939
9. Johnson, D.L.: Presentations of Groups. Cambridge University Press, Cambridge (1990)
10. Kondo, S.: K3 and enriques surfaces. Arithmetic and geometry of K3-surfaces and Calabi-Yau threefolds. Fields Inst. Commun. **67**, 3–28 (2013). https://doi.org/10.1007/978-1-4614-6403-7
11. Moore, C.C.: Extensions and low dimensional cohomology theory of locally compact groups. I. Trans. Am. Math. Soc. **113**, 40–63 (1964). https://doi.org/10.2307/1994090
12. Moore, C.C.: Extensions and low dimensional cohomology theory of locally compact groups II. Trans. Am. Math. Soc. **113**, 64–86 (1964). https://doi.org/10.2307/1994091
13. Shimura, G.: Automorphic forms and the periods of abelian varieties. J. Math. Soc. Jpn. **31**(3), 561–592 (1979). https://doi.org/10.2969/jmsj/03130561
14. Stover, M., Toledo, D.: Residually finite lattices in $\widetilde{\mathrm{PU}(2,1)}$ and fundamental groups of smooth projective surfaces. (2021) arxiv:2105.12772
15. Wang, H., Williams, B.: Borcherds products of half-integral weight. J. Number Theory (2021). https://doi.org/10.1016/j.jnt.2021.10.008
16. Wigner, D.: Algebraic cohomology of topological groups. Trans. AMS **178**, 83–93 (1973). https://doi.org/10.1090/S0002-9947-1973-0338132-7

Publisher's Note