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Research Paper

Shapley allocation, diversification and services in operational risk

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ABSTRACT

In this paper, a method of allocating operational risk regulatory capital using a closed-form Shapley method, applicable to a large number of business units (BUs), is proposed. It is assumed that if BUs form coalitions, the value added to a coalition by a new entrant is a simple function of the value of that entrant. This function represents the diversification that can be achieved by combining operational risk losses. Two such functions are considered. The calculations account for a service that further reduces the capital payable by BUs. The results derived are applied to recent loss data.

Keywords: capital value; diversification; game theory; operational risk; service; Shapley.

1 INTRODUCTION AND MOTIVATION

In this paper, we expand on the ideas proposed in Mitic and Hassani (2016), in which we developed a Shapley allocation model with the following components: first, a set

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of business units (BUs), each having a measure of risk, which is used in the allocation process; second, two types of service. One type of service serves to mitigate risk, and thereby reduces the allocation to the BUs; the other increases risk, which serves to increase the capital allocation to the BUs. These types were termed a "receiver" and an "emitter" of capital, respectively. In order to simplify the service structure for this paper, an instance of a "receiver" service and an instance of an "emitter" service are treated as a net single service, which can be one type or the other, depending on the relative degrees of diversification that they provide. As risk decreases, diversification (the calculated measure of joint risk) increases. Effectively, risks from multiple BUs cancel rather than reinforce, and the degree to which they do so is the diversification. Diversification may be conveniently expressed as a percentage of a base risk measure. With the "net single service" simplification, we concentrate on formulations for the value that a new BU brings to an existing group of units. The principal idea behind the formulations presented here is that any value that can be added by a new BU diminishes with the size of the existing group. It is assumed that the new BU is not, in any sense, dominant. In business terms, a dominant BU is one able to manipulate policies of other BUs. In risk terms, dominance means that a BU is associated with much more risk than any others. Essentially, the existing group pays little attention to a new entrant if it is already large. Conversely, a new entrant is unable to significantly influence a large group. All of the natural language terms mentioned so far ("large", "diversification", "service", etc) will be defined precisely in the sections that follow.

The organization of this paper is as follows. In Section 2, we summarize the important elements in the analysis, and cast them in the language of game theory. The Shapley method is discussed briefly, as is its relevance to the scenario presented in this paper. In Section 3, the concept of a "service" model is introduced. This is followed by two theoretical models. The first ("constant diversification") is a simple linear model for the added value a new participant brings when it joins an existing group. In the second model ("diminishing diversification"), the added value depends nonlinearly on the size of the group. Expressions derived for the two models are then applied to operational risk loss data in Section 5, and the results are compared. Section 6 concludes the paper with a discussion of the results. Appendixes A–C are available online.

2 DIVERSIFICATION, ALLOCATION AND CAPITAL CHARGE

The term "business unit" can be used to refer to any meaningful subdivision of an organization, or, indeed, to one of a group of organizations. Simple examples are "retail sales Europe", "neurosurgery" or "Southern Banking Group". Each has an associated riskiness, which may be defined as an amount of money that could be lost in some given time period. In game theory terms, the BUs are known as players, and riskiness translates to value. When two or more players combine and cooperate, they form a coalition. Coalitions are an essential part of this analysis.

A general result from game theory, applicable in the context of risk management, is encapsulated in the following, which relates the separate values of two players to the value of the two players as a coalition. Call the players P_1 and P_2, and denote the coalition of these players by the "union" symbol: $P_1 \cup P_2$. In general, denote the value of a player by $v(\cdot)$, where the dot is a placeholder for any player or coalition. Then, provided that P_1 and P_2 cooperate in their coalition,

$$v(P_1 \cup P_2) \leqslant v(P_1) + v(P_2).$$

The direction of the inequality reflects "added value" through a reduction in risk for the coalition relative to the separate participants. In other contexts the inequality might be reversed if "added value" means a formal increase in $v(\cdot)$. See Owen (1995) for a discussion of this topic. The inequality as stated implies that the risk measure $v(\cdot)$ is subadditive. Value-at-risk (VaR) can be used, since it is generally subadditive, apart from the particular cases in which either there is a particular dependence structure, or the marginal value (ie, the difference between a coalition with and without a particular player) has infinite mean. Nešlehová *et al* (2006) make the point that VaR can grow at an exponential rate for infinite mean loss models. The difference between the left- and right-hand sides of the above equation represents the diversification introduced by forming the coalition. If, instead of measuring "value reduced" (ie, risk), we measure "value increased" (simply replace v by $-v$), the inequality is reversed. An example is where v measures sales or profit. Detailed discussions of this relationship in the context of portfolio management may be found in Wagner and Lau (1971) and Leippold and Vanin (2003). In the context of operational risk, Monti *et al* (2010) link this diversification to the dependency structure between the players concerned, estimated via correlations.

The values for each player will be used in the calculation to allocate a fixed capital amount to the players. Dehez (2011) provides comprehensive accounts of fixed capital allocations, with useful theory for the Shapley method, and also gives a simple numerical example.

2.1 Shapley allocation

The methodology and problems associated with the Shapley allocation (Shapley 1952) in the general context of BUs supported by a service were discussed in Mitic and Hassani (2016). It is worth reiterating the need to formulate particular Shapley constructs in the current context. The Shapley allocation formula, giving the allocation ϕ_P for a player P, is

$$\phi_P = \frac{1}{n!} \sum_{C \in \mathrm{perm}(G)} [v(C) - v(C \setminus \{P\})],$$

where n is the total number of players in a set of all players G, C is a subset of G and $\text{perm}(G)$ is the set of permutations of members of G. The term $[v(C) - v(C \setminus \{P\})]$ is the difference in value between a coalition C with and without player P. The denominator, $n!$, is the number of permutations of n players. The formula then measures a mean marginal difference in value. The set of all players G is known as the "grand coalition". It is an axiom of the Shapley allocation that the value of the grand coalition is equal to the sum of Shapley allocation values, ie, $v(G) = \sum_P \phi_P$.

Barron (2008) gives some simple examples of three-player allocations using this formula directly, but it is easier to work with an algorithm implied by the formula. Using the algorithm means listing all possible coalitions and evaluating the marginal differences in value as each player enters the coalition. Step-by-step examples of the algorithm are rare in the literature, so in Section 3 we give some details of a three-player example. For n players, $n!$ separate evaluations of marginal differences are therefore necessary. It is not feasible to evaluate this number for large n. In the current context, "large" often means seven or more, and we would like to process hundreds of players.

In the context of operational risk, an even more significant problem is that of finding the actual values of coalitions. They are not readily available, so the strategy adopted in this paper will be to formulate implicit expressions for the added value that a new coalition member brings on joining, in terms of the stand-alone value of the joiner $v(i)$ and the current value of the coalition $v(s)$. In general terms, such an implicit formulation takes the form of a cost function:

$$v(C \cup P) = v(C) + f(v(P), v(C)),$$

where f is a function to be defined. A cost function in this context simply relates the value of a coalition to which a new player has been added to the value of the coalition before the new player joined.

2.2 Allocation applied to operational risk

The problems highlighted in the previous section may be addressed by modeling three assumptions.

(1) All players are equivalent with respect to value: none are dominant. The word "dominant" means the same when applied to "player" as it did when applied to "business unit" (see Section 1).

(2) When a new player joins a coalition, the new joiner introduces a diversification that is a function of the value of the new joiner and of the value of the coalition joined.

(3) The diversification introduced is less than the stand-alone value of the joiner.

The result is that when the coalition is large the joiner adds relatively little value to the coalition. It is assumed that the values of all players are of the same order. In contrast to the formulation in Mitic and Hassani (2016), we will cast these assumptions in terms of formal cost functions by defining more complex forms of the function f (introduced in Section 2.1).

Introducing an additional "service" player, with initial value zero, will model a net receiver of value. In real life the service might be a risk management department, whose task is to mitigate risk. The allocation process will measure its value to other, risky BUs by calculating the amount of allocation absorbed from those BUs.

The end result of this analysis is a closed-form expression for the Shapley allocation for each of the players, given the distinction between service and BU.

3 A SERVICE MODEL FOR BUSINESS UNITS

The intuition provided by the definitions in the previous section is that when a player joins a large coalition it adds some functions of its own value to the coalition, but not necessarily a large amount. In this section, we formalize that intuition.

3.1 Notation

Some basic notation has already been introduced, and we introduce specific notation to model the "serviced BUs" situation here.

Let there be n players, denoted by $P_1, P_2, \dots, P_n$. One of them will be the service, and without loss of generality we will label the service P_1. When expedient, the service will also be referred to as S. The symbol C will denote a coalition comprising a generic subset of $\{P_1, P_2, \dots, P_n\}$, and the number of players in C is denoted by $|C|$. The value of any single player or a coalition given in an argument is denoted by $v(\cdot)$. A player that is not a service will, when convenient, be referred to as a "nonservice" or "nonservice player".

The marginal allocation to a player P will be denoted by $M(P)$, sometimes with a subscript when appropriate. This is the difference in the value of an existing coalition before and after P joins.

The Shapley value of player P_r is denoted by $\mathrm{SH}(n, r)$. The pro rata allocation for P_r, denoted by $\mathrm{PR}(n, r)$, is a share of some fixed amount A:

$$\mathrm{PR}(n, r) = \frac{Av(P_r)}{\sum v(P_i)}.$$

The amount A could be the sum of the values of all players, $\sum v(P_i)$ (in which case it can be compared directly with the Shapley allocation), or any other agreed amount.

3.2 Constant diversification

The essence of a service, as distinct from a business function, is that it does not generate income or activity-based risk in its own right, and hence has zero “riskiness”. Setting its value to zero would be to treat the service as a dummy player (one that adds zero value to a coalition). As such it would receive zero allocation in the Shapley process. This is not what is intended. An easy way to avoid treating a service as a dummy player in our analyses is to make an initial minimal transfer of value to the service from all business functions that use the service. In practice such a transfer makes a negligible difference to the value of the final allocation result, since the amount of the transfer can be as little as one penny or cent. To do the calculations it is not even necessary to formally transfer anything, provided that it is clear that a service is not to be treated as a dummy player. After the transfer, the allocation process can proceed such that the service receives a positive capital allocation. The business functions then receive reduced allocations to compensate.

We now state a definition of “service” and propose rules that apply when a single service interacts with a coalition.

DEFINITION 3.1 A service (or service player) S is a player that satisfies the following.

(1) $0 < v(S) \ll v(P)$ for all other players P (ie, the value of the service player is much smaller than that of any other player).

(2) Whenever S enters a coalition C, the value of the resulting coalition is given by $v(C \cup S) = v(C) + v(S) + g(d, C)$, where $g(\cdot) > 0$ is some function of a constant diversification factor d and the values of the members of C (the important points being that g is positive and is added, not subtracted).

Condition (1) of Definition 3.1 requires that a service is assigned a minimal value in advance of any allocation. This is the technicality of the Shapley analysis already referred to, and it ensures that the service is not treated as a dummy player. The diversification term $g(d, C)$ in condition (2) should not be set so large that the resulting diversification is negative. The intuition behind condition (2) is that whenever S enters a coalition there is an effective transfer of value to S from nonservice players in C. Value added to S comes from value subtracted from the nonservice players. The organizational strategy is that if you add a service department, the overall exposure (or potential loss) should decrease. However, this service department has a cost that must be supported by the other departments, as it does not generate external profit. The only customers for a service department are internal. However, that cost must be at least equal to the exposure reduction obtained using the service department. In many cases the cost exceeds the exposure reduction, and that is the situation modeled here.

Precise forms for the function g will be given in the examples that follow. In order to satisfy condition (1), we first populate P_1 with a nominal value by transferring a small amount ε from each of the other players. Picking a very small ε (compared with the v_r), let the values of the players be

$$\left.\begin{aligned} v(P_r) &= v_r - \varepsilon, \quad 2 \leqslant r \leqslant n, \\ v(P_1) &= (n-1)\varepsilon. \end{aligned}\right\} \tag{3.1}$$

The intention is that S receives a small value ε from each of the other players, such that the total transfer to S, $(n-1)\varepsilon$, should be much smaller than the values of all the other players. With this initialization, we obtain the following easy result for the marginal allocation to S when S enters a coalition. From Definition 3.1,

$$M(S) = v(CUS) - v(C) = v(S) + g(d, C) = (n-1)\varepsilon + g(d, C). \tag{3.2}$$

The diversification in Definition 3.1 will be made explicit. The following is a simple example for which calculations can easily be made explicit. For n players, let

$$g(d, C) = (n-1)dm, \tag{3.3}$$

where m is the median of the values of the members of C, and d $(0 < d < 1)$ is the constant diversification factor. Although the values of most members of C are generally of the same order, the existence of outliers can have an adverse effect if m is taken as the mean instead of the median. If the mean is used, the term $(n-1)dm$ can become excessively large, leading to an unreasonable situation where the service would appear to have transferred far too much value to itself from the other players. We choose the median rather than the mean because it is less sensitive to outliers. (In that case g depends on C only implicitly via the median.)

3.2.1 Three-player example

In this section, we give an example of a Shapley allocation for three players: A, B and a service S. Using (3.1), their stand-alone values are

$$v(A) = v_a - \varepsilon, \qquad v(B) = v_b - \varepsilon, \qquad v(S) = 2\varepsilon.$$

The factor 2 in the equation for $v(S)$ comes from the number of nonservice players. Effectively, an amount dm is transferred to the service from each nonservice. When a nonservice player P enters a coalition C, the cost function is set to

$$v(C \cup P) = v(C) + v(P) - dv(P) - dm. \tag{3.4}$$

This cost function says that when P enters C it provides a value from two components: one proportional to its stand-alone value, and the other proportional to the median

TABLE 1 Shapley analysis with one service and constant diversification factor.

Permutation	Allocation to S	Allocation to A	Allocation to B
$S\ A\ B$	2ε	$v_a - \varepsilon - dv_a - dm$	$v_b - \varepsilon - dv_b - dm$
$S\ B\ A$	2ε	$v_a - \varepsilon - dv_a - dm$	$v_b - \varepsilon - dv_b - dm$
$A\ B\ S$	$2\varepsilon + 2dm$	$v_a - \varepsilon$	$v_b - \varepsilon - dv_b - dm$
$A\ S\ B$	$2\varepsilon + 2dm$	$v_a - \varepsilon$	$v_b - \varepsilon - dv_b - dm$
$B\ A\ S$	$2\varepsilon + 2dm$	$v_a - \varepsilon - dv_a - dm$	$v_b - \varepsilon$
$B\ S\ A$	$2\varepsilon + 2dm$	$v_a - \varepsilon - dv_a - dm$	$v_b - \varepsilon$
Sum	$12\varepsilon + 8dm$	$6v_a - 6\varepsilon - 4dv_a - 4dm$	$6v_b - 6\varepsilon - 4dv_b - 4dm$
Shapley value	$2\varepsilon + \frac{4}{3}dm$	$v_a - \varepsilon - \frac{2}{3}dv_a - \frac{2}{3}dm$	$v_b - \varepsilon - \frac{2}{3}dv_b - \frac{2}{3}dm$

coalition value. In this case the median is simply given by $m = \frac{1}{2}(v_a + v_b)$. Table 1 shows the Shapley analysis.

Table 1 makes it clear that the sum of the allocations to the three players is different for each permutation. It should be stressed that none of them is the value of the grand coalition, which is the sum of the Shapley values of all players. The value of the grand coalition is not shown explicitly in Table 1, but it can be deduced as follows.

There are six permutations of the players $\{S, A, B\}$, and in four of these a value $2dm$ is transferred to the service. So the mean transfer to the service is a measure of the total diversification in the system, and its value is $(\frac{4}{6})2dm = \frac{2}{3}d(v_a + v_b)$. Before diversification, the total value in the system was $(v_a + v_b)$. Therefore, after diversification the total value in the system is the prediversification value minus the diversification value: $(1 - \frac{2}{3}d)(v_a + v_b)$. This (from Table 1) is the sum of Shapley values.

The results in Table 1 give solid clues as to how to extend the process to more players, which is the subject of the next section.

3.3 n-player closed-form solution with constant diversification

The example of three players (one of which is a service) indicates how to analyze the case where there are many more (nonservice) players. The cases where a particular player is the first to enter a coalition should be treated separately from other cases because the first player initializes the coalition with its own value. The details are given in the enumerated cases in Appendix B online.

The stand-alone values of $P_1, P_2, \dots, P_n$ (where P_1 is the service) are defined by (3.1). The overall cost function is given by Definition 3.1, with the additional definition for the diversification in (3.4). To compensate for adding value to the service, players have their marginal allocations reduced by an amount dm. In this case m can be expressed directly in terms of the values v_r, but it is convenient to continue to use the symbol m. The closed-form result for the Shapley allocation for each player is given in the following proposition.

PROPOSITION 3.2 *The Shapley values for the nonservice players P_r $(2 \leqslant r \leqslant n)$ and the service P_1 are given by*

$$\left.\begin{aligned} \mathrm{SH}(n,r) &= v_r - \varepsilon - dv_r\left(1-\frac{1}{n}\right) - dm\left(1-\frac{1}{n}\right), \quad 2 \leqslant r \leqslant n, \\ \mathrm{SH}(n,1) &= (n-1)\varepsilon + (n-1)dm\left(1-\frac{1}{n}\right). \end{aligned}\right\} \tag{3.5}$$

The proof is given in Appendix A online. This proposition follows from the proof of Proposition 4.1 in Appendix B online. Technically, the proof of Proposition 3.2 is much simpler, but it is a useful check to be able to derive the result from a more general case. There is a swap of allocation between the nonservices and the service, which gains additional allocation. An example of the use of (3.5) with operational risk loss data will be given in Section 5.

4 *n*-PLAYER CLOSED-FORM SOLUTION WITH DIMINISHING DIVERSITY

In this section, we extend the idea of the previous closed-form solution to model a diminishing diversification effect. The intuition behind this is that, for a large number of players, adding an extra one makes increasingly less difference to the value of the augmented coalition. It is assumed, as in Section 2.2, that the values of all players are of the same order. In this revised model, the added value when a new member joins decreases with increasing coalition size. We have selected a geometrically decreasing function of the coalition size, but, in principle, any appropriate means of reducing diversification should be satisfactory. The geometric function has the property that, if its parameters are chosen suitably, decay with increasing coalition size is rapid. A geometrically decreasing function is also a simple case that allows us to obtain explicit allocation results. This is an appropriate model for the case when a new player entering a coalition only has an impact on the coalition if the coalition size is small. For example, a player can have a significant impact when it forms a coalition with one other player. As more players join, each has an increasingly smaller impact on the coalition's value. After the coalition has reached a certain size, which may be quite small, new players add almost nothing to the coalition.

Using the same notation as for Proposition 3.2, we amend the cost function (3.4) to account for the number of players already in the coalition when a new member joins. The new cost function for nonservice players (where m is the median of the values $v(P_2), \dots, v(P_n)$ and $0 < d < 1$) is given by

$$v(C \cup P) = v(C) + v(P) - d^{|C|}v(P) - d^{|C|}m. \tag{4.1 a}$$

The corresponding cost function for a service player P_1 with value $(n-1)\varepsilon$ is

$$v(C \cup P_1) = v(C) + (n-1)\varepsilon + (n-1)d^{|C|}m. \tag{4.1b}$$

Using (4.1 a) and (4.1 b), we can state the following.

PROPOSITION 4.1 *The Shapley values for the nonservice players P_r $(2 \leqslant r \leqslant n)$ and the service P_1 under conditions of diminishing diversification are given by*

$$\left.\begin{aligned} \mathrm{SH}(n,r) &= v_r - \varepsilon - \frac{(m+v_r)D_n}{n}, \quad 2 \leqslant r \leqslant n, \\ \mathrm{SH}(n,1) &= (n-1)\varepsilon + \frac{(n-1)mD_n}{n}, \end{aligned}\right\} \tag{4.2}$$

where $D_n = d + d^2 + \cdots + d^{n-1}$.

The proof is given in Appendix B online. An example of the use of (4.2) with operational risk loss data will be given in Section 5. For small d, $D_n \sim d$. The interpretation of this approximation when applied to loss data is that diversification effects are effectively zero for coalitions of size $n \geqslant 3$. In practice the terms $d^{|C|}$ in (4.1 a) have exactly the decay properties referred to in the first paragraph of this section. When $|C| > 4$, the values of $d^{|C|}$ for the values of d that we have observed are close to zero. For example, for typical values of d varying between 0.04 and 0.15, $d^{|C|}$ varies between 0.00000256 and 0.00050625. The terms $d^{|C|}v(P)$ and $d^{|C|}m$ in (4.1 a) are then very small compared with $v(C) + v(P)$.

5 APPLICATION TO LOSS DATA: CAPITAL VALUE ALLOCATION

In this section, we apply the closed-form Shapley formulas to sets of operational risk losses for which it would be impossible to calculate exact Shapley values. The data sets comprise losses for eleven players, each with losses ranging from very small to millions of euro. Business units do not cooperate in practice, so any "cooperation" has to be measured by considering loss data alone. In this context, cooperation takes the form of calculating diversification, which should reduce regulatory capital.

The data sets are labeled $P_2, P_3, \dots, P_{12}$ for convenience. To these we add a service P_1 that has no measured losses. The service is nominally labeled "risk department". It develops and implements risk-mitigation practice. With this notation, the "players" in previous sections are synonymous with the data sets that represent them.

The stand-alone capital values $\{v_r\}$ are calculated in a standard way, using the loss distribution approach (LDA) described by Frachot *et al* (2001). A summary of the LDA algorithm is given in Appendix C online. Using this method, the capital values listed in Table 2 were obtained. These values are the v_r of Propositions 3.2 and 4.1. In principle, any reasonable assessment of value (eg, the median loss) could be used, and such values are effectively weights that reflect the relative importance of the players. The value for v_1 arises from transfers of €1000 from each of the eleven nonservice players to P_1. This is done to prevent the service from being treated as a dummy player (see Section 3.2). This amount is small enough to have a minimal effect on the v_r values while allowing the Shapley process to proceed as required.

5.1 Application to loss data: calculation of the diversification factor

In order to assess the diversification factor, we evaluate the effect of any one player on the others. A useful way to do this is to first aggregate the losses from all players to give a capital value for the grand coalition G. Then, each player is successively removed, and the capital value, C_r', for the remaining players (ie, the grand coalition without the player that was removed) is calculated. The method is described in detail in Milliman (2009). The diversification factor is then calculated by finding the median value of the percentage deviation of the value of each (nonservice) player from that of the grand coalition. Specifically, the percentage deviations, $v_r - v(G)$, were

$$\{-8.33\%, -26.70\%, 1.91\%, -81.20\%, 0.74\%, -3.28\%, -2.55\%, \\ 44.93\%, -13.12\%, -8.10\%, -4.17\%\}.$$

The value of the grand coalition was 36.50. The use of the median ensures that any extreme values do not influence the result unduly. The result for the data in Table 2 was $d = 4.17\%$, used in the form 0.0417. We have found that assessing diversification in this way is easy to explain to business users, as it uses concepts with which they are familiar: median, deviation and the effect of extreme values.

The process of successively removing players and recalculating capital value is not feasible if the total number of players is huge. We have found that it is possible to apply the method to several thousand players. Beyond that, we suggest assessing the value of each player by simpler means, such as total, mean or median loss.

Using 10 000 trials in the LDA algorithm (see Appendix C online) took about two seconds per run with a computer with a 2.6 GHz i5 processor and 4 GB RAM. The actual time depends on the number of elements in each data set. The figures quoted apply for typical sizes up to $n = 300$. Processing data for 1000 players therefore takes in the region of 30–60 minutes, which is not too onerous. More reliable results can be obtained by increasing the number of Monte Carlo trials to about 50 000.

5.2 Calculation of the Shapley allocations

Having determined the constant factor diversification factor, d, calculating the Shapley allocations is an easy matter of applying (3.5) for the constant diversification case and (4.2) for the diminishing diversification case. The results are shown in Table 2.

TABLE 2 Shapley allocations: SH(12, r) (all figures in millions of €).

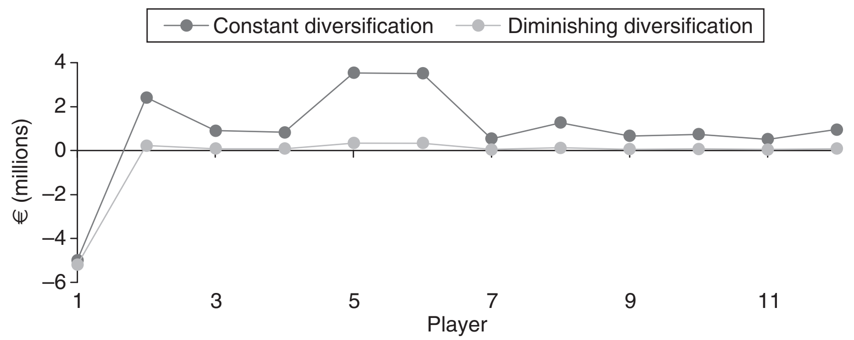
Player r	Constant diversification (3.5)	Diminishing diversification (4.2)	Capital v_r
1	4.98	5.20	0.011
2	49.34	51.52	51.75
3	11.02	11.83	11.92
4	9.06	9.81	9.89
5	77.67	80.86	81.20
6	77.07	80.25	80.58
7	1.83	2.32	2.37
8	20.29	21.44	21.56
9	4.84	5.44	5.50
10	6.77	7.44	7.51
11	1.09	1.55	1.60
12	12.21	13.07	13.16

The limiting value case $\varepsilon \to 0$ makes no difference in practice since ε is within the limits of stochastic variation of the LDA. The values, v_r, are before deduction if the nominal is €1000. All calculations use $d = 0.0417$, as above.

Comparing the results in Table 2 (before and after allocation, respectively), it is clear that each (nonservice) BU has had its capital value reduced with respect to the corresponding stand-alone values, which are also the pro rata allocations. The service (P_1) has gained considerably in relative terms, but this gain is small compared with the gains of the other players. The diminishing diversification capital is only marginally less than the stand-alone capital. The constant factor diversification capital is substantially smaller than the stand-alone capital. This reflects the rapid decay of the factor $d^{|C|}$ when $|C| > 3$. In some cases, the diminishing diversification allocation is very near to the pro rata allocation. In practice, the allocation to the service would not be formally allocated and would be made available for investment elsewhere in the business. Figure 1 shows a direct comparison (the difference between the pro rata and Shapley allocations) of the two calculations. The solid line shows the difference (pro rata minus Shapley) for constant diversification and the dashed line shows the difference for diminishing diversification.

It is possible that some of the Shapley allocations calculated by either of the methods in Figure 1 can be negative. Table 2 shows that the allocations to players 7 and 11 are nearer zero than the other players listed. Further calculations show that if the diversification factor, d, is increased to about 9%, negative allocations do result. This is undesirable, from both a theoretical and a "political" point of view. A BU would

FIGURE 1 Comparison of constant and diminishing diversification factor



not like to see that another BU was receiving allocation funding rather than paying it, because the accepted practice is for BUs to pay allocation capital. In practice the allocation would be set at an appropriate level, perhaps €1.0 million or €0.5 million (tentative exemplar amounts, small percentages of calculated capital) for the values in Table 2.

6 CONCLUSION AND FURTHER WORK

We have proposed an allocation methodology that is applicable to a large number of players, including a service in the form of a support function. The closed-form allocation formulas developed solve the combinatorial problems always associated with Shapley methods, and also define the interaction when a new player joins a coalition. Two such interactions are proposed. The first uses a constant diversification factor, so that each new player adds a value proportional to its stand-alone value when it enters a coalition. Refining the diversification assumption, in the second interaction the added value on joining a coalition decreases as the size of the coalition increases. This models the case where any new entrant to a large coalition adds increasingly minimal value, and is more applicable for very large coalitions. The second interaction produces allocations that are very similar to pro rata allocations because of the short relaxation time for the size of diversification factor used. In this case, pro rata allocations would be more convenient, since they are simpler and more intuitive. However, the small differences in allocations produced by the pro rata and Shapley methods might still be significant for an organization that wishes to optimize its allocation as much as possible. For example, for player 12 in Table 2, the difference between €13.16 million (pro rata) and €13.07 million (Shapley) is €90 000. This is not an amount to be thrown away lightly.

In principle, it appears possible to generalize the methodology used to derive Proposition 4.1 by replacing the terms $d^{|C|}$ in the cost function by the generic function $g(d, C)$ used in Definition 3.1. This would provide a more generic form for the allocations, assuming that terms in the necessary enumerations are separable in the way they are in the proof of Proposition 4.1 in Appendix B online (see cases 2 and 4 in that proof). An interesting result might be obtained if, instead of using a decreasing sequence such as $\{d, d^2, \dots, d^{n-1}\}$ associated with joining a coalition of size $1, 2, \dots, n-1$, an increasing sequence is used instead. This might model a peculiar situation in which new joiners to the coalition strengthen the coalition significantly rather than minimally.

DECLARATION OF INTEREST

The opinions, ideas and approaches expressed or presented are those of the authors and do not necessarily reflect Santander's position. The values presented are just illustrations and do not represent Santander data. The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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