OPERATOR-NORM RESOLVENT ASYMPTOTIC ANALYSIS OF CONTINUOUS MEDIA WITH HIGH-CONTRAST INCLUSIONS

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ABSTRACT. Using a generalisation of the classical notion of the Weyl m-function and the related formulae for the resolvents of boundary-value problems, we analyse the asymptotic behaviour of solutions to a "transmission problem" for a high-contrast inclusion in a continuous medium, for which we prove the operator-norm resolvent convergence to a limit problem of "electrostatic" type. In particular, our results imply the convergence of the spectra of high-contrast problems to the spectrum of the limit operator, with order-sharp convergence estimates. The approach developed in the paper is of a general nature and can thus be successfully applied in the study of other problems of the same type.

1. Introduction

Parameter-dependent problems for differential equations have traditionally attracted much interest within applied mathematics, by virtue of their potential for replacing complicated formulations with more straightforward, and often explicitly solvable, ones. This drive has led to a plethora of asymptotic techniques, from perturbation theory to multi-scale analysis, covering a variety of applications to physics, engineering, and materials science. It would be an insurmountable task to give a comprehensive review of the related literature. Notwithstanding the classical status of this subject area, problems that require new ideas continue emerging, often motivated by novel wave phenomena. One of the recent application areas of this kind is provided by composites and structures involving components with highly contrasting material properties (stiffness, density, refractive index). Mathematically, such problems lead to boundary-value formulations for classical operators (such as the Laplace operator), but with parameter-dependent coefficients. For example, problems of this kind have arisen in the study of periodic composite media with high contrast (or "large coupling") between the material properties of the components, see [15], [32], [8].

In the present work, we consider a prototype large-coupling transmission problem, posed on a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, see Fig. 1, containing a "low-index" (equivalently, "high propagation speed") inclusion Ω_-, located at a positive distance to the boundary $\partial\Omega$. Mathematically, this is modelled by a "weighted" Laplacian $-a_\pm\Delta$, where $a_+ = 1$ (the weight on the domain $\Omega_+ := \Omega \setminus \overline{\Omega}_-$), and $a_- \equiv a$ (the weight on the domain Ω_-) is assumed to be large, $a_- \gg 1$. This is supplemented by the Neumann boundary condition $\partial u/\partial n = 0$ on the outer boundary $\partial\Omega$, where n is the exterior normal to $\partial\Omega$, and "natural" continuity conditions on the "interface" $\Gamma := \partial\Omega_-$. For each a, we consider time-harmonic vibrations of the physical domain represented by Ω, described by the eigenvalue problem for an appropriate operator in $L^2(\Omega)$.

A formal asymptotic argument using expansions in powers of a^{-1} suggests that convergent eigenfunction sequences for the above eigenvalue problems should converge (as $a \to \infty$) to either a
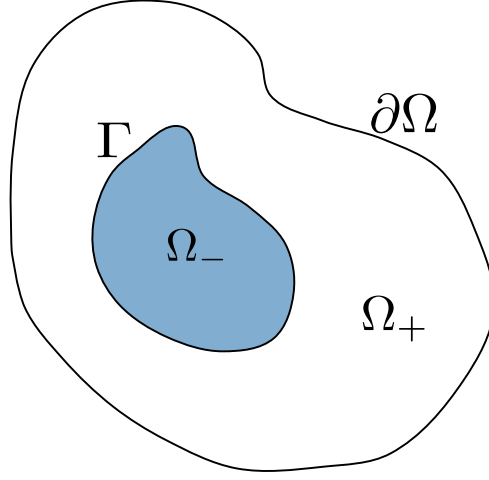
FIGURE 1. Domain with a "stiff" inclusion.

constant or a function of the form

$$v - \frac{1}{|\Omega|} \int_{\Omega_+} v,$$

where v satisfies the spectral boundary-value problem (BVP)

$$- \Delta v = z\left(v - \frac{1}{|\Omega|} \int_{\Omega_+} v\right) \quad \text{in } \Omega_+, \quad v|_\Gamma = 0, \quad \frac{\partial v}{\partial n}\bigg|_{\partial \Omega} = 0. \tag{1.1}$$

Here the spectral parameter z represents the ratio of the size of the original physical domain to the wavelength in its part represented by Ω_+.

The problem (1.1) is related to the so-called "electrostatic problem" discussed in [33, Lemma 3.4], see also [2] and references therein, namely the eigenvalue problem for the self-adjoint operator Q defined by the quadratic form

$$q(u, u) = \int_{\Omega_+} \nabla v \cdot \overline{\nabla v}, \quad u = v + c, \quad v \in H^1_{0,\Gamma} := \{v \in H^1(\Omega_+), \ v|_\Gamma = 0\}, \quad c \in \mathbb{C} \tag{1.2}$$

on the Hilbert space $L^2(\Omega_+) \dotplus \mathbb{C}$, treated as a subspace of $L^2(\Omega)$.

Indeed (see [33] for details), it is easily seen that the eigenvalue problem $Qu = zu$ is solvable either when $z = 0$, in which case $u = c$, or for $z > 0$ such that the problem (1.1) admits a non-trivial solution. Thus, the formal asymptotic argument suggests that the limiting spectrum is precisely that of the electrostatic problem.

Denote by A_0^+ the Laplacian $-\Delta$ on Ω_+, subject to the Dirichlet condition on Γ and the Neumann boundary condition on $\partial\Omega$ and write the function v in (1.1) in the form of an eigenfunction series

$$v = \sum_{j=1}^{\infty} d_j \phi_j^+,$$

where λ_j^+, ϕ_j^+, $j = 1, 2, \ldots$, are the eigenvalues and the corresponding orthonormal eigenfunctions, respectively, of A_0^+. Noticing that the function $\mathbb{1}_+(x) = 1$, $x \in \Omega_+$, can be written as

$$\mathbb{1}_+ = \sum_{j=1}^{\infty} \left(\int_{\Omega_+} \overline{\phi_j^+}\right) \phi_j^+,$$

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we obtain

$$\sum_{j=1}^{\infty} d_j \lambda_j^+ \phi_j^+ = \sum_{j=1}^{\infty} z \left(d_j - \frac{1}{|\Omega|} \int_{\Omega_+} v \int_{\Omega_+} \overline{\phi_j^+} \right) \phi_j^+,$$

and therefore

$$v = -\frac{z}{|\Omega|} \left(\int_{\Omega_+} v \right) \sum_{j=1}^{\infty} (\lambda_j^+ - z)^{-1} \left(\int_{\Omega_+} \overline{\phi_j^+} \right) \phi_j^+, \tag{1.3}$$

as long as $z \neq \lambda_j^+$, $j = 1, 2, \ldots$. Taking the integral over Ω_+ on both sides of (1.3) and assuming that the integral of v over Ω_+ does not vanish yields, by incorporation of $z = 0$ into the answer,

$$z \left[|\Omega| + z \sum_{j=1}^{\infty} (\lambda_j^+ - z)^{-1} \left| \int_{\Omega_+} \phi_j^+ \right|^2 \right] = 0. \tag{1.4}$$

Thus, the spectrum of the electrostatic problem is the union of two sets: a) the set of z solving the equation (1.4) and b) the set of those eigenvalues λ_j^+ for which the corresponding eigenfunction ϕ_j^+ has zero mean over Ω_+.

The main result of the present paper, which is the norm-resolvent asymptotics for the operator of the BVP introduced above, yields in particular the description (1.4) for the limiting spectrum of the problem, together with an order-sharp estimate on the rate of the convergence, as $a \to +\infty$.

Relations similar to (1.4) appear in the analysis of periodic problems with micro-resonances ("metamaterials") [8], where they provide zeros of the functions describing the dispersion of waves propagating through media modelled by such problems.

The present paper is a development of the recent study [6, 7, 9, 8] aimed at implementing the ideas of the boundary triples theory as proposed by Ryzhov [24] (in its turn, this analysis heavily draws upon the celebrated Birman-Kreĭn-Višik theory [3, 16, 17, 31]) in the context of problems of materials science and wave propagation in inhomogeneous media. Our recent papers cited above have shown that the language of boundary triples is particularly fitting for the analysis of composite media, as one of the key difficulties in their analysis stems from the presence of interfaces (i.e., boundaries between individual material components) through which an exchange of energy between different components of the medium takes place. We point out that the papers [10, 11, 12] further demonstrate that an additional value of using the boundary triples approach is that the framework of functional models and the approach to scattering theory based thereupon ([22, 19]) can be formulated in the most natural terms of Dirichlet-to-Neumann maps pertaining to the interfaces.

An asymptotic analysis of the static (or "equilibrium") version of the above problem, where $z = 0$ and a forcing term is added to the right-hand side, has been carried out in [2], in the context of isotropic elasticity (which additionally implies that two material parameters are present, the so-called Lamé coefficients). The authors of [2], using the representation of solutions in terms of boundary layers, prove "strong" resolvent convergence of the original problem to the resolvent version of the "electrostatic" problem (1.1) (albeit framed in the context of linearised elasticity), still with $z = 0$. In the work [23] different methods were used to obtain the spectral convergence; however, neither the effective operator of the "limiting" medium nor the norm-resolvent convergence to it were discussed. We argue that the approach we present here allows one to improve such results in two respects: a) the new estimates are of the operator-norm resolvent type, implying, in particular, the control of the convergence of the associated spectra and the exponential groups; b) our estimates are uniform with respect to the "contrast" parameter a and are order-sharp, i.e. the rate of convergence in terms of $a \to 0$ cannot be improved further.

We briefly outline the contents of the paper. In Section 2, we recall the main points of the abstract construction of [24] and introduce the key tools for our analysis. These include a representation for the resolvents of a class of boundary-value problems in terms of the M-operator. Using these

general formulae, in Section 3 we study the asymptotic behaviour of the operators corresponding to transmission problems for two-component media with contrasting material properties, as described above. The asymptotic approximation of the spectra is discussed at the end of the paper, leading to the characterisation (1.4).

2. Ryzhov triples for boundary-value problems

In this section we follow [24] in outlining an operator framework suitable for dealing with boundary-value problems.

2.1. The boundary triple framework. The starting point of our construction is a self-adjoint operator A_0 in a separable Hilbert space $\mathcal{H}$ with $0 \in \rho(A_0)$, where $\rho(A_0)$, as usual, denotes the resolvent set of A_0. Alongside $\mathcal{H}$, we consider an auxiliary Hilbert space $\mathcal{E}$ and a bounded operator $\Pi : \mathcal{E} \to \mathcal{H}$ such that

$$\mathrm{dom}(A_0) \cap \mathrm{ran}(\Pi) = \{0\} \quad \text{and} \quad \ker(\Pi) = \{0\}.$$

Since Π has a trivial kernel, there is a left inverse Π^{-1}, so that $\Pi^{-1}\Pi = I_{\mathcal{E}}$. We define

$$\mathrm{dom}(A) := \mathrm{dom}(A_0) \dotplus \mathrm{ran}(\Pi),$$
$$A : A_0^{-1}f + \Pi\phi \mapsto f, \qquad f \in \mathcal{H}, \phi \in \mathcal{E}, \tag{2.1}$$

$$\mathrm{dom}(\Gamma_0) := \mathrm{dom}(A_0) \dotplus \mathrm{ran}(\Pi),$$
$$\Gamma_0 : A_0^{-1}f + \Pi\phi \mapsto \phi, \qquad f \in \mathcal{H}, \phi \in \mathcal{E}, \tag{2.2}$$

where neither A nor Γ_0 is assumed closed or indeed closable. The operator given in (2.1) is the null extension of A_0, while (2.2) is the null extension of Π^{-1}. Note also that

$$\ker(\Gamma_0) = \mathrm{dom}(A_0) \, .$$

For $z \in \rho(A_0)$, consider the *abstract* spectral BVP

$$\begin{cases} Au = zu, \\ \Gamma_0 u = \phi, \qquad \phi \in \mathcal{E}, \end{cases} \tag{2.3}$$

where the second equation is seen as a boundary condition. As it is asserted in [24, Thm. 3.1], there is a unique solution u of the BVP (2.3) for any $\phi \in \mathcal{E}$. Thus, there is an operator (clearly linear) which assigns to any $\phi \in \mathcal{E}$ the solution u of (2.3). This operator is called the solution operator for A and is denoted by[1] γ_z. An explicit expression for it in terms of A_0 and Π is obtained as

$$\gamma_z : \phi \mapsto (I + z(A_0 - zI)^{-1})\Pi\phi \tag{2.4}$$

for any $z \in \rho(A_0)$. Note that

$$I + z(A_0 - zI)^{-1} = (I - zA_0^{-1})^{-1}$$

and that (2.2) and (2.4) immediately imply

$$\Gamma_0\gamma_z = I_{\mathcal{E}} \, .$$

By (2.4) and a simple calculation, one has

$$\mathrm{ran}\,\gamma_z = \ker(A - zI) \, .$$

We remark that, since A is not required to be closed, $\mathrm{ran}\,\gamma_z$ is not necessarily a subspace. This is precisely the kind of situation that commonly occurs in the analysis of BVPs.

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[1]The function γ is sometimes referred to as the γ-field.

In what follows, we consider (abstract) BVPs of the form (2.3) associated with the operator A, with variable boundary conditions. To this end, for a self-adjoint operator Λ in $\mathcal{E}$, define

$$\mathrm{dom}(\Gamma_1) := \mathrm{dom}(A_0) \dotplus \Pi \, \mathrm{dom}(\Lambda),$$

$$\Gamma_1 : A_0^{-1} f + \Pi\phi \mapsto \Pi^* f + \Lambda\phi, \qquad f \in \mathcal{H}, \phi \in \mathrm{dom}(\Lambda). \tag{2.5}$$

The operator Λ can thus be seen as a parameter for the boundary operator Γ_1.

On the basis of (2.4), one obtains from (2.5) (see [24, Eq. 3.7]) that

$$\gamma_{\bar{z}}^* = \Gamma_1 (A_0 - zI)^{-1}.$$

Also, according to [24, Thm. 3.2], the following Green's type identity holds:

$$\langle Au, v \rangle_{\mathcal{H}} - \langle u, Av \rangle_{\mathcal{H}} = \langle \Gamma_1 u, \Gamma_0 v \rangle_{\mathcal{E}} - \langle \Gamma_0 u, \Gamma_1 v \rangle_{\mathcal{E}}, \qquad u, v \in \mathrm{dom}(\Gamma_1).$$

The above framework for triples (A_0, Λ, Π) stems from the Birman-Krein-Višik theory [3, 16, 17, 31], rather than the theory of boundary triples [14]. We employ it next to introduce the notion of an M-operator, generalising the well-known notion of a Dirichlet-to-Neumann map in the context of BVPs. This generalisation helps us achieve two goals: on the one hand, it allows us to treat a transmission (rather than a boundary-value) problem, and on the other hand, it enters the formulae for operators resolvents that we will use for obtaining operator-norm error estimates in the large-coupling asymptotic regime.

2.2. Definition and properties of the M-operator. Based on the notion of a triple, we now define the mentioned abstract version of the Dirichlet-to-Neumann map.

Definition 1. For a given triple (A_0, Λ, Π), define the operator-valued M-function associated with A_0 as follows. For any $z \in \rho(A_0)$, the operator $M(z)$ in $\mathcal{E}$ is defined by

$$M(z) : \phi \mapsto \Gamma_1 \gamma_z \phi, \qquad \phi \in \mathrm{dom}(M(z)) := \mathrm{dom}(\Lambda).$$

A detailed description of how one casts in the language of boundary triples classical boundary-value problems, such as the Dirichlet problem for the Laplace operator on a bounded domain with sufficiently regular boundary, can be found in [24, 12].

Taking into account (2.5), one concludes from Definition 1 that

$$M(z) = \Lambda + z\Pi^*(I - zA_0^{-1})^{-1}\Pi. \tag{2.6}$$

Also, due to the self-adjointness of Λ, one has

$$M^*(z) = M(\bar{z}).$$

Moreover, it is checked that M is an unbounded operator-valued Herglotz function, *i.e.*, $M(z) - M(0)$ is analytic and $\Im M(z) \geq 0$ whenever $z \in \mathbb{C}_+$. It is shown in [24, Thm. 3.3(4)] that

$$M(z)\Gamma_0 u_z = \Gamma_1 u_z \qquad \forall u_z \in \ker(A - zI) \cap \mathrm{dom}(\Gamma_1).$$

In this work we consider extensions (self-adjoint and non-selfadjoint) of the "minimal" operator

$$\widetilde{A} := A_0|_{\ker(\Gamma_1)} \tag{2.7}$$

that are restrictions of A. It is proven in [24, Sec. 5] that $\widetilde{A}$ is symmetric with equal deficiency indices. Moreover, [24, Prop. 5.1] asserts that $\widetilde{A}$ does not depend on the parameter operator Λ, contrary to what could be surmised from (2.7).

2.3. Resolvent formulae for general boundary-value problems. Still following [24], we let α and β be linear operators in the Hilbert space $\mathcal{E}$ such that $\operatorname{dom}(\alpha) \supset \operatorname{dom}(\Lambda)$ and β is bounded on $\mathcal{E}$. Additionally, assume that $\alpha + \beta\Lambda$ is closable and denote $\text{ß} := \overline{\alpha + \beta\Lambda}$. Consider the linear set

$$\mathcal{H}_\text{ß} := \left\{ A_0^{-1}f \dotplus \Pi\phi : f \in \mathcal{H}, \ \phi \in \operatorname{dom}(\text{ß}) \right\} \tag{2.8}$$

Following [24, Lem. 4.1], the identity

$$(\alpha\Gamma_0 + \beta\Gamma_1)(A_0^{-1}f + \Pi\phi) = \beta\Pi^*f + (\alpha + \beta\Lambda)\phi, \qquad f \in \mathcal{H}, \ \phi \in \operatorname{dom}(\Lambda),$$

implies that $\alpha\Gamma_0 + \beta\Gamma_1$ is correctly defined on $\operatorname{dom}(A_0) \dotplus \Pi\operatorname{dom}(\Lambda)$. The assumption that $\alpha + \beta\Lambda$ is closable is used to extend the domain of definition of $\alpha\Gamma_0 + \beta\Gamma_1$ to the set (2.8). Moreover, one shows that $\mathcal{H}_\text{ß}$ is a Hilbert space with respect to the norm

$$\|u\|_\text{ß}^2 := \|f\|_\mathcal{H}^2 + \|\phi\|_\mathcal{E}^2 + \|\text{ß}\phi\|_\mathcal{E}^2, \quad u = A_0^{-1}f + \Pi\phi.$$

It follows that the constructed extension $\alpha\Gamma_0 + \beta\Gamma_1$ is a bounded operator from $\mathcal{H}_\text{ß}$ to $\mathcal{E}$.

According to [24, Thm. 4.1], if the operator $\overline{\alpha + \beta M(z)}$ is boundedly invertible for $z \in \rho(A_0)$, then, on the one hand, the spectral BVP

$$\begin{cases} (A - zI)u = f, \\ (\alpha\Gamma_0 + \beta\Gamma_1)\,u = \phi, \qquad f \in \mathcal{H}, \ \phi \in \mathcal{E}, \end{cases}$$

has a unique solution $u \in \mathcal{H}_\text{ß}$, where, as above, $\alpha\Gamma_0 + \beta\Gamma_1$ is a bounded operator on $\mathcal{H}_\text{ß}$. On the other hand, it follows from [24, Thm. 5.1] that the function

$$(A_0 - zI)^{-1} - (I - zA_0^{-1})^{-1}\Pi[\overline{\alpha + \beta M(z)}]^{-1}\beta\Pi^*(I - zA_0^{-1})^{-1} \tag{2.9}$$

is the resolvent of a closed operator $A_{\alpha\beta}$ densely defined in $\mathcal{H}$. Moreover, $\widetilde{A} \subset A_{\alpha\beta} \subset A$ and $\operatorname{dom}(A_{\alpha\beta}) \subset \{u \in \mathcal{H}_\text{ß} : (\alpha\Gamma_0 + \beta\Gamma_1)u = 0\}$.

3. Large-coupling asymptotics for a transmission problem

The aim of this section is to translate a problem familiar to the application-minded reader into the language of boundary triple theory presented above and obtain new results for this problem.

3.1. Problem formulation. Suppose that Ω is a bounded $C^{1,1}$ domain, and $\Gamma \subset \Omega$ is a closed $C^{1,1}$ curve, so that $\Gamma = \partial\Omega_-$ is the common boundary of domains Ω_+ and Ω_-, where Ω_- is strictly contained in Ω, such that $\overline{\Omega}_+ \cup \overline{\Omega}_- = \overline{\Omega}$, see Fig. 1.

For $a > 0$, $z \in \mathbb{C}$ we consider the "transmission" eigenvalue problem (*cf.* [25])

$$\begin{cases} -\Delta u_+ = z u_+ \ \text{ in } \Omega_+, \\ -a\Delta u_- = z u_- \ \text{ in } \Omega_-, \\ u_+ = u_-, \quad \dfrac{\partial u_+}{\partial n_+} + a\dfrac{\partial u_-}{\partial n_-} = 0 \ \text{ on } \Gamma, \\ \dfrac{\partial u_+}{\partial n_+} = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where $n_\pm$ denotes the exterior normal (defined a.e.) to the corresponding part of the boundary.[2] The above problem is understood in the strong sense, *i.e.* $u_\pm \in H^2(\Omega_\pm)$, the Laplacian differential expression Δ is the corresponding combination of second-order weak derivatives, and the boundary

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[2]The Neumann boundary condition on $\partial\Omega$ can be replaced by a Robin condition with an arbitrary coupling constant without affecting the analysis of this section.

values of $u_\pm$ and their normal derivatives are understood in the sense of traces according to the embeddings of $H^2(\Omega_\pm)$ into $H^s(\Gamma)$, $H^s(\partial\Omega)$, where $s = 3/2$ or $s = 1/2$.

3.2. A reformulation in the language of triples.

In order to make our framework applicable to (3.1), we first consider its weak formulation. We then apply the regularity theory for elliptic BVPs, see *e.g.* [25], to show that its solutions are in fact the solutions to (3.1). Indeed, the results of [24], see also (2.9), show that the problem (3.1) in the weak formulation is the eigenvalue problem for a self-adjoint operator of an appropriate class $A_{\alpha\beta}$ introduced in Section 2.3, and thus its solutions are in the domain of this operator. Then the result of [25] is used to show that these solutions have higher regularity, as required for the solvability of (3.1) in the strong sense.

We define the Dirichlet-to-Neumann map[3]

$$\Lambda : \phi \mapsto -\frac{\partial u_+}{\partial n_+} - a\frac{\partial u_-}{\partial n_-}, \qquad \phi \in H^1(\Gamma),$$

where $u_\pm$ are the harmonic functions in $\Omega_\pm$ subject to the above boundary condition on $\partial\Omega$ and the condition $u = \phi$ on Γ. Clearly $\Lambda = \Lambda^+ + a\Lambda^-$, where Λ^+, Λ^- are the Dirichlet-to-Neumann maps on each side of the interface Γ, which are self-adjoint operators in $L^2(\Gamma)$, with domain $H^1(\Gamma)$. For sufficiently large values of a, the operator Λ has the same properties as Λ^+ and Λ^-, see [8, Lemma 2.1].

Translating the spectral BVP (3.1) into the language of the abstract framework developed in Section 2.1, we define A_0 as the "Dirichet decoupling" $A_0^+ \oplus aA_0^-$, corresponding to the decomposition $L^2(\Omega_+) \oplus L^2(\Omega_-) = L^2(\Omega) =: \mathcal{H}$, where A_0^+ is the Laplace operator with Dirichlet condition on Γ and Neumann condition on $\partial\Omega$, and A_0^- is the Dirichlet Laplacian on Ω_-. Furthermore, in the context of the transmission problem (3.1), the boundary space is given by $L^2(\Gamma) =: \mathcal{E}$ and the abstract operator Π of Section 2.1 is simply the Poisson operator of harmonic lift from $\mathcal{E}$ to $\mathcal{H}$ (subject to the Neumann condition on $\partial\Omega$), while its left inverse is the operator of trace on Γ for functions that are harmonic on Ω_- and Ω_+, possessing square summable boundary values on Γ, and Γ_0 is the null extension of the latter to $(H^2(\Omega_-) \cap H_0^1(\Omega_-)) \dot{+} (H^2(\Omega_+) \cap H_{0,\Gamma}^1(\Omega_+)) \dot{+} \Pi L^2(\Gamma)$, where $H_{0,\Gamma}^1(\Omega_+)$ consists of functions in $H^1(\Omega_+)$ with zero trace on Γ, *cf.* (1.2).

The problem (3.1) is then written in the form $A_{\alpha\beta}u = zu$ with $\alpha = 0, \beta = I$, equivalently

$$Au = zu, \qquad \Gamma_1 u = 0,$$

where A is defined by (2.1) and Γ_1 is defined by (2.5).

Finally, the operator $M(z)$ of Definition 1 is the mapping

$$M(z) : \phi \mapsto -\frac{\partial u_+}{\partial n_+} - a\frac{\partial u_-}{\partial n_-}, \qquad \phi \in H^1(\Gamma),$$

where u_+, u_- solve

$$\begin{cases} -\Delta u_+ = zu_+ & \text{in } \Omega_+, \\ -a\Delta u_- = zu_- & \text{in } \Omega_-, \\ u_+ = u_- = \phi & \text{on } \Gamma, \\ \dfrac{\partial u_+}{\partial n_+} = 0 & \text{on } \partial\Omega, \end{cases}$$

and the formula (2.6) expresses $M(z)$ in terms of Λ and the decoupling $A_0^+ \oplus aA_0^-$. Recall (Section 2.2) that $M(z)$ is an (unbounded) operator-valued Herglotz function, analytic in $\mathbb{C} \setminus \mathbb{R}$ so that

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[3]For convenience, we define the Dirichlet-to-Neumann map via $-\partial u/\partial n|_{\partial\Omega}$ instead of the more common $\partial u/\partial n|_{\partial\Omega}$. As a side note, we mention that this is obviously not the only possible choice for the operator Λ. In particular, the trivial option $\Lambda = 0$ is always possible. Our choice of Λ is motivated by our interest in the analysis of classical boundary conditions.

$M(z) = M^*(\bar{z})$, $z \in \mathbb{C} \setminus \mathbb{R}$. In addition, for all values z outside a discrete set of points, $M(z)$ is invertible and its inverse is compact. Similarly, Definition 1 yields the M-operators $M^{\pm}$ corresponding to the components $\Omega_{\pm}$, so that

$$M^+(z) : \phi \mapsto -\frac{\partial u_+}{\partial n_+}, \qquad M^-(z) : \phi \mapsto -a\frac{\partial u_-}{\partial n_-}, \qquad \phi \in H^1(\Gamma),$$

where u_+ and u_- are as above.

By a similar argument to the one of [24, Theorem 3.3, Theorem 5.1], the following statement holds, *cf.* [5].

Proposition 3.1. *The spectrum of (3.1), i.e. the set of values z for which (3.1) has a nonzero solution in the strong sense as described above, coincides with the set of z to which the inverse of $\alpha + \beta M(z) \equiv M(z)$ does not admit an analytic continuation.*

The representation (2.6) applied to $M^-(z)$ implies that

$$M^-(z) = a\Lambda^- + z\Pi_-^* \left(I - a^{-1}z(A_0^-)^{-1} \right)^{-1} \Pi_- = a\Lambda^- + z\Pi_-^*\Pi_- + O(a^{-1}), \qquad (3.2)$$

where A_0^- is as above the Dirichlet Laplacian on Ω_-, and Π_- is the harmonic lift from Γ to Ω_-.

3.3. Asymptotic analysis and the main result. In what follows we analyse the resolvent of the operator A_a of the transmission problem (3.1), which coincides with the resolvent of the operator A_{0I}, in terms of the notation of Section 2. In particular, the spectrum of A_a coincides with the spectrum of (3.1). Our approach is based on the use of the Kreĭn formula (2.9) with $\alpha = 0$, $\beta = I$, where for the asymptotic analysis of $M(z)^{-1}$ we employ (3.2) and separate the singular and non-singular parts of Λ^-.

To this end, note first that the spectrum of Λ^- consists of the values μ ("Steklov eigenvalues") such that the problem

$$\begin{cases} \Delta u = 0, & u \in H^2(\Omega_-), \\ \dfrac{\partial u}{\partial n} = -\mu u & \text{on } \Gamma, \end{cases}$$

has a non-trivial solution. The least (by absolute value) Steklov eigenvalue is zero, and the associated normalised eigenfunction ("Steklov eigenvector") is $\psi_* := |\Gamma|^{-1/2}\mathbb{1}_\Gamma \in \mathcal{E} \equiv L^2(\Gamma)$. Introduce the corresponding orthogonal projection $P := \langle \cdot, \psi_* \rangle_{\mathcal{E}} \psi_*$, which is a spectral projection relative to Λ^-, and decompose the boundary space $\mathcal{E}$:

$$\mathcal{E} = P\mathcal{E} \oplus P^\perp \mathcal{E}, \qquad (3.3)$$

where $P^\perp := I - P$. This yields the following matrix representation for Λ^- :

$$\Lambda^- = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_\perp^- \end{pmatrix},$$

where $\Lambda_\perp^- := P^\perp \Lambda^- P^\perp$ is treated as a self-adjoint operator in $P^\perp \mathcal{E}$.

We write the operator $M(z)$ as a block-operator matrix relative to the decomposition (3.3), followed by an application of the Schur-Frobenius inversion formula, see [30, Theorem 2.3.3]. To this end, notice that for all $\psi \in \text{dom}\,\Lambda$ one has $P\psi \in \text{dom}\,\Lambda$, and therefore $P^\perp \Lambda P$ is well defined on $\text{dom}\,\Lambda$. Similarly, $P^\perp \psi = \psi - P\psi \in \text{dom}\,\Lambda$, and $P\Lambda P^\perp$ is also well defined. Furthermore, by the self-adjointness of Λ, one has $P\Lambda P^\perp \psi = \langle P^\perp \psi, \Lambda \psi_* \rangle \psi_*$, and therefore $\|P\Lambda P^\perp\|_{\mathcal{E} \to \mathcal{E}} \leq \|\Lambda \psi_*\|_{\mathcal{E}}$. It follows that $P\Lambda P^\perp$ is extendable to a bounded mapping on $P^\perp \mathcal{E}$. A similar calculation applied to $P^\perp \Lambda P$ and $P\Lambda P$ shows that these are extendable to bounded mappings on $P\mathcal{E}$. Therefore, for each $z \in \rho(A_0^+) \cap \rho(A_0^-)$ the operator $M(z)$ admits the representation

$$M(z) = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{E} & \mathbb{D} \end{pmatrix}, \qquad \mathbb{A}, \mathbb{B}, \mathbb{E} \text{ bounded.}$$

For evaluating $M(z)^{-1}$ we use the Schur-Frobenius inversion formula [26], [30, Theorem 2.3.3]

$$\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{E} & \mathbb{D} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbb{A}^{-1} + \mathbb{A}^{-1}\mathbb{B}\mathbb{S}^{-1}\mathbb{E}\mathbb{A}^{-1} & -\mathbb{A}^{-1}\mathbb{B}\mathbb{S}^{-1} \\ -\mathbb{S}^{-1}\mathbb{E}\mathbb{A}^{-1} & \mathbb{S}^{-1} \end{pmatrix}, \qquad \mathbb{S} := \mathbb{D} - \mathbb{E}\mathbb{A}^{-1}\mathbb{B}. \tag{3.4}$$

Using the fact that $\mathbb{S}^{-1} = (I - \mathbb{D}^{-1}\mathbb{E}\mathbb{A}^{-1}\mathbb{B})^{-1}\mathbb{D}^{-1}$, where $\|\mathbb{D}^{-1}\| \leq Ca^{-1}$, and therefore $\mathbb{S}$ is boundedly invertible with a uniformly small bound, we obtain (see [8] for details)

$$M(z)^{-1} = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{E} & \mathbb{D} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbb{A}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + O(a^{-1}). \tag{3.5}$$

Theorem 3.2. *Fix $\sigma > 0$ and a compact set $K \subset \mathbb{C}$, and denote $K_\sigma := \{z \in K : \text{dist}(z, \mathbb{R}) \geq \sigma\}$. There exist $C, a_0 > 0$ such that for all $z \in K_\sigma$, $a \geq a_0$ one has*

$$\left\| (A_a - z)^{-1} - (A_{P^\perp, P} - z)^{-1} \right\|_{L^2(\Omega) \to L^2(\Omega)} \leq Ca^{-1}.$$

Proof. We use (2.9) with $\alpha = 0$, $\beta = I$ for the resolvent $(A_a - z)^{-1}$ and with $\alpha = P^\perp$, $\beta = P$ for $(A_{P^\perp, P} - z)^{-1}$. In the former case we use (3.5) and in the latter case we write

$$\left(P^\perp + PM(z)\right)^{-1} P = P(PM(z)P)^{-1}P,$$

by the Schur-Frobenius inversion formula [30, Section 1.6], see (3.4).[4] The claim follows by comparing the obtained expressions for the two resolvents. □

We now rewrite the result of Theorem 3.2 in a block-matrix form relative to the decomposition $\mathcal{H} = P_-\mathcal{H} \oplus P_+\mathcal{H} = L^2(\Omega_-) \oplus L^2(\Omega_+)$, where P_-, P_+ are orthogonal projections from $L^2(\Omega)$ to $L^2(\Omega_-), L^2(\Omega_+)$, respectively. This allows us to express the asymptotics of $(A_a - z)^{-1}$ in terms of the generalised resolvent [20, 21] $R_a(z) := P_+(A_a - z)^{-1}P_+$, analysed next.

Proposition 3.3. *One has*

$$R_a(z) = (A_0^+ - z)^{-1} - \gamma_z^+\left(M^+(z) + M^-(z)\right)^{-1}(\gamma_{\bar{z}}^+)^*,$$

where $\gamma_z^+ : \phi \mapsto u$ is the solution operator of the BVP

$$\begin{cases} -\Delta u - zu = 0, & u \in \text{dom } A_0^+ \dotplus \text{ran } \Pi_+, \\ u|_\Gamma = \phi, \\ \dfrac{\partial u}{\partial n_+} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Proof. By the definition of M^+, M^- and a direct application of (2.9), as in [8, Lemma 3.2]. □

A comparison of the latter result with (2.9) taking into account (2.4) yields:

Corollary 3.4. *The generalised resolvent $R_a(z)$ is the solution operator for the BVP*

$$\begin{cases} -\Delta u - zu = f, & f \in L^2(\Omega_+), \\ \Gamma_1^+ u = -M^-(z)\Gamma_0^+ u, \\ \dfrac{\partial u}{\partial n_+} = 0 & \text{on } \partial\Omega. \end{cases}$$

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[4]We remark that $P^\perp + PM(z)$ is triangular ($\mathbb{A} = PM(z)P$, $\mathbb{B} = PM(z)P^\perp$, $\mathbb{E} = 0$, $\mathbb{D} = I$ in (3.4)) with respect to the decomposition $\mathcal{E} = P\mathcal{E} \oplus P^\perp\mathcal{E}$.

Theorem 3.2 now implies an operator-norm asymptotics for the generalised resolvents R_a as $a \to \infty$.

Theorem 3.5. *For all $z \in K_\sigma$ the operator $R_a(z)$ admits the asymptotics $R_a(z) = R_{\mathrm{eff}}(z) + O(a^{-1})$, as $a \to \infty$, in the operator-norm topology, where $R_{\mathrm{eff}}(z)$ is the solution operator for the BVP*

$$\begin{cases} -\Delta u - zu = f, & f \in L^2(\Omega_+), \\ \alpha(z)\Gamma_0^+ u + \beta\Gamma_1^+ u = 0, \\ \dfrac{\partial u}{\partial n_+} = 0 \ on \ \partial\Omega, \end{cases} \tag{3.7}$$

with $\alpha(z) = P^\perp + PM^-(z)P$ and $\beta = P$.

Proof. On the one hand, by Theorem 3.2, the resolvent $(A_a - z)^{-1}$ is $O(a^{-1})$-close to

$$\left(A_{P^\perp,P} - z\right)^{-1} = (A_0 - z)^{-1} - \gamma_z\left(\overline{P^\perp + PM(z)}\right)^{-1}P\gamma_{\bar z}^*,$$

and therefore

$$R_a(z) = P_+(A_0 - z)^{-1}P_+ - P_+\gamma_z\left(\overline{P^\perp + PM(z)}\right)^{-1}P\gamma_{\bar z}^*P_+ + O(a^{-1})$$

$$= (A_0^+ - z)^{-1} - \gamma_z^+\left(\overline{P^\perp + PM(z)}\right)^{-1}P(\gamma_{\bar z}^+)^* + O(a^{-1}) \tag{3.8}$$

$$= (A_0^+ - z)^{-1} - \gamma_z^+ P\left(PM(z)P\right)^{-1}P(\gamma_{\bar z}^+)^* + O(a^{-1}).$$

On the other hand, by (2.9) and applying the inversion formula (3.4), we obtain

$$R_{\mathrm{eff}}(z) = \left(A_0^+ - z\right)^{-1} - \gamma_z^+\left(\overline{P^\perp + PM^-(z)P + PM^+(z)}\right)^{-1}P(\gamma_{\bar z}^+)^*$$

$$= \left(A_0^+ - z\right)^{-1} - \gamma_z^+ P\left(PM^+(z)P + PM^-(z)P\right)^{-1}P(\gamma_{\bar z}^+)^*. \tag{3.9}$$

Comparing the right-hand sides of (3.8) and (3.9) completes the proof. □

Theorem 3.5 can be further clarified by considering the "truncated" boundary space[5] $\breve{\mathcal{E}} := P\mathcal{E}$. Introduce the truncated harmonic lift by $\breve\Pi_+ := \Pi_+|_{\breve{\mathcal{E}}}$ and Dirichlet-to-Neumann map $\breve\Lambda^+ := P\Lambda^+|_{\breve{\mathcal{E}}}$.

Theorem 3.6. *Denote $\breve{M}^+(z) := PM^+(z)|_{\breve{\mathcal{E}}} = \breve\Lambda^+ + z\breve\Pi_+^*\left(1 - z(A_0^+)^{-1}\right)^{-1}\breve\Pi_+$. The formula*

$$R_{\mathrm{eff}}(z) = (A_0^+ - z)^{-1} - \breve\gamma_z^+\left(\breve{M}^+(z) + PM^-(z)P\right)^{-1}(\breve\gamma_{\bar z}^+)^* \tag{3.10}$$

holds, where $\breve\gamma_z^+ : \phi \mapsto u_\phi$ is the solution operator of the problem

$$\begin{cases} -\Delta u_\phi - zu_\phi = 0, & u_\phi \in \mathrm{dom}\, A_0^+ \dotplus \mathrm{ran}\, \breve\Pi_+, \\ \Gamma_0^+ u_\phi = \phi, & \phi \in \breve{\mathcal{E}}. \end{cases}$$

Note that here $PM^-(z)P$, as opposed to $\breve{M}^+(z)$, is attributed the meaning of an operator defining nonlocal boundary conditions in the BVP.

Proof. By the definition of γ_z^+, one has

$$\gamma_z^+ = (I - z(A_0^+)^{-1})^{-1}\Pi_+, \tag{3.11}$$

and therefore $\breve\gamma_z^+ = \gamma_z^+|_{\breve{\mathcal{E}}}$. It follows that (3.9) is equivalent to (3.10), whence the assertion of Theorem follows. □

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[5]In what follows we consistently supply the (finite-dimensional) "truncated" spaces and operators pertaining to them by the breve overscript.

Corollary 3.7. *The operator $R_{\text{eff}}(z)$ is the solution operator of the problem*

$$\begin{cases} -\Delta u - zu = f, \quad f \in L^2(\Omega_+), \quad u \in \operatorname{dom} A_0^+ \dotplus \operatorname{ran} \breve{\Pi}_+, \\ P\Gamma_1^+ u = -PM^-(z)P\Gamma_0^+ u. \end{cases}$$

Equipped with Theorems 3.5 and 3.6, we provide a more convenient representation for the asymptotics of $(A_a - z)^{-1}$ of Theorem 3.2.

Theorem 3.8. *Denote $\mathfrak{K}_z := \Gamma_0^+|_{\operatorname{ran}(\gamma_z^+ P)}$, $z \in \mathbb{C}_\pm$. For the resolvent $(A_a - z)^{-1}$ one has*

$$(A_a - z)^{-1} = \begin{pmatrix} R_{\text{eff}}(z) & \big(\mathfrak{K}_{\bar{z}}\big[R_{\text{eff}}(\bar{z}) - (A_0^+ - \bar{z})^{-1}\big]\big)^* \Pi_-^* \\ \Pi_- \mathfrak{K}_z\big[R_{\text{eff}}(z) - (A_0^+ - z)^{-1}\big] & \Pi_- \mathfrak{K}_z \big(\mathfrak{K}_{\bar{z}}\big[R_{\text{eff}}(\bar{z}) - (A_0^+ - \bar{z})^{-1}\big]\big)^* \Pi_-^* \end{pmatrix} + O(a^{-1}),$$

where the operator matrix is written with respect to the decomposition $L^2(\Omega) = L^2(\Omega_+) \oplus L^2(\Omega_-)$.

Proof. First, we note that since $\operatorname{ran}(\gamma_z^+ P)$ is one-dimensional, the operator $\mathfrak{K}_z$ is well defined as a bounded linear operator from $\operatorname{ran}(\gamma_z^+ P)$ to $\mathcal{E}$, where the former is equipped with the standard norm of $L^2(\Omega_+)$. We proceed by representing the operator $(A_{P^\perp,P} - z)^{-1}$, see Theorem 3.2, in a block-operator matrix form relative to the orthogonal decomposition $\mathcal{H} = L^2(\Omega_+) \oplus L^2(\Omega_-)$. The upper left matrix entry $P_+(A_{P^\perp,P} - z)^{-1}P_+$ due to Theorem 3.5 is $O(a^{-1})$-close to $R_{\text{eff}}(z)$. Next we consider the lower left matrix entry:

$$P_-\big(A_{P^\perp,P} - z\big)^{-1}P_+ = -\gamma_z^- P\big(PM^+(z)P + PM^-(z)P\big)^{-1}P(\gamma_{\bar{z}}^+)^*$$

$$= -\gamma_z^- \Gamma_0^+ \gamma_z^+ P\big(PM^+(z)P + PM^-(z)P\big)^{-1}P(\gamma_{\bar{z}}^+)^*$$

$$= \gamma_z^- \Gamma_0^+ \big[R_{\text{eff}}(z) - (A_0^+ - z)^{-1}\big] = \gamma_z^- \mathfrak{K}_z\big[R_{\text{eff}}(z) - (A_0^+ - z)^{-1}\big],$$

where $\gamma_z^- : \phi \mapsto u_\phi$ is the solution operator of the BVP (*cf.* (3.6))

$$\begin{cases} -a\Delta u_\phi - zu_\phi = 0, \quad u_\phi \in \operatorname{dom} A_0^- \dotplus \operatorname{ran} \Pi_-, \\ \Gamma_0^- u_\phi = \phi. \end{cases}$$

Here in the second equality we use the fact that $\Gamma_0^+ \gamma_z^+ = I$, and in the third equality we use (2.9), see also (3.9). Passing over to the top-right entry, we write

$$P_+\big(A_{P^\perp,P} - z\big)^{-1}P_- = -\gamma_z^+ P\big(PM^+(z)P + PM^-(z)P\big)^{-1}P(\gamma_{\bar{z}}^-)^*$$

$$= \big(\mathfrak{K}_{\bar{z}}\big[R_{\text{eff}}(\bar{z}) - (A_0^+ - \bar{z})^{-1}\big]\big)^*(\gamma_{\bar{z}}^-)^*$$

$$= \big(\mathfrak{K}_{\bar{z}}\big[R_{\text{eff}}(\bar{z}) - (A_0^+ - \bar{z})^{-1}\big]\big)^* \Pi_-^* \big(1 - a^{-1}z(A_0^-)^{-1}\big)^{-1},$$

and the claim pertaining to the named entry follows by a virtually unchanged argument. Finally, for the bottom-right entry we have

$$P_-\big(A_{P^\perp,P} - z\big)^{-1}P_- = (aA_0^- - z)^{-1} + \gamma_z^- \mathfrak{K}_z \big(\mathfrak{K}_{\bar{z}}\big[R_{\text{eff}}(\bar{z}) - (A_0^+ - \bar{z})^{-1}\big]\big)^*(\gamma_{\bar{z}}^-)^*,$$

which completes the proof taking into account the equality (cf. (3.11))

$$\gamma_z^- = (I - a^{-1}z(A_0^-)^{-1})^{-1}\Pi_- = \Pi_- + O(a^{-1}).$$

$\square$

The representation for $R_{\text{eff}}(z)$ given by Theorem 3.6 allows us to further simplify the asymptotics of $(A_a - z)^{-1}$, using the fact that

$$PM^-(z)P = P\Lambda^- P + zP\Pi_-^* \Pi_- P + O(a^{-1}) = z\breve{\Pi}_-^* \breve{\Pi}_- + O(a^{-1}), \quad \breve{\Pi}_- := \Pi_-|_{\breve{\mathcal{E}}}.$$

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As a result, one has[6]

$$R_{\text{eff}}(z) = \widetilde{R}_{\text{eff}}(z) + O(a^{-1}), \qquad \widetilde{R}_{\text{eff}}(z) := (A_0^+ - z)^{-1} - \breve{\gamma}_z^+ \big(\breve{M}^+(z) + z \breve{\Pi}_-^* \breve{\Pi}_- \big)^{-1} (\breve{\gamma}_{\bar{z}}^+)^*,$$

and hence the following result holds.

Theorem 3.9. *The resolvent $(A_a - z)^{-1}$ has the following asymptotics in the operator-norm topology:*

$$(A_a - z)^{-1} = \begin{pmatrix} \widetilde{R}_{\text{eff}}(z) & \big(\mathfrak{K}_{\bar{z}} [\widetilde{R}_{\text{eff}}(\bar{z}) - (A_0^+ - \bar{z})^{-1}] \big)^* \breve{\Pi}_-^* \\ \breve{\Pi}_- \mathfrak{K}_z [\widetilde{R}_{\text{eff}}(z) - (A_0^+ - z)^{-1}] & \breve{\Pi}_- \mathfrak{K}_z \big(\mathfrak{K}_{\bar{z}} [\widetilde{R}_{\text{eff}}(\bar{z}) - (A_0^+ - \bar{z})^{-1}] \big)^* \breve{\Pi}_-^* \end{pmatrix} + O(a^{-1}),$$

$$(3.12)$$

where the operator matrix is written with respect to the decomposition $L^2(\Omega) = L^2(\Omega_+) \oplus L^2(\Omega_-)$.

3.4. An out-of-space extension and the "electrostatic" problem. Consider the space $H_{\text{eff}} = L^2(\Omega_+) \oplus \mathbb{C}$ and the following linear subset of $L^2(\Omega)$:

$$\text{dom}\, \mathcal{A}_{\text{eff}} = \left\{ \begin{pmatrix} u_+ \\ \eta \end{pmatrix} \in H_{\text{eff}} : \ u_+ \in H^2(\Omega_+), \ u_+|_\Gamma = \frac{\eta}{\sqrt{|\Omega_-|}} \mathbb{1}_\Gamma, \ \frac{\partial u_+}{\partial n_+}\Big|_{\partial \Omega} = 0 \right\}, \qquad (3.13)$$

where $u|_\Gamma$ is the trace of the function u and $\mathbb{1}_\Gamma$ is the unity function on Γ. On $\text{dom}\, \mathcal{A}_{\text{eff}}$ we set the action of the operator $\mathcal{A}_{\text{eff}}$ by the formula

$$\mathcal{A}_{\text{eff}} \begin{pmatrix} u_+ \\ \eta \end{pmatrix} = \begin{pmatrix} -\Delta u_+ \\ \dfrac{1}{\sqrt{|\Omega_-|}} \displaystyle\int_\Gamma \frac{\partial u_+}{\partial n_+} \end{pmatrix}.$$

Remark 1. Note that the resolvent of $\mathcal{A}_{\text{eff}}$ is the so-called Strauss dilation [28, 29] of the generalised resolvent $\widetilde{R}_{\text{eff}}(z)$.

Next we use [8, Theorem 4.4], which in view of Theorem 4.6 implies the following theorem.

Theorem 3.10. *The resolvent $(\mathcal{A}_{\text{eff}} - z)^{-1}$ is unitary equivalent to the block operator matrix on the right hand side of (3.12), , treated as an operator in $L^2(\Omega_+) \oplus \Pi_- \breve{\mathcal{E}}$.*

We remark that as it is easily seen the block operator matrix on the right hand side of (3.12) is equal to zero in the orthogonal complement to the subspace $L^2(\Omega_+) \oplus \Pi_- \breve{\mathcal{E}}$.

Together with the spectral theorem for self-adjoint operators, the result obtained in Theorem 3.10 immediately yields (see, e.g., [4]) the following corollary.

Corollary 3.11. *The spectra of the operators A_a converge in the sense of Hausdorff, as $a \to \infty$, with an order $O(a^{-1})$ error estimate, uniformly on compact subsets of $\mathbb{C}$, to the spectrum of the operator $\mathcal{A}_{\text{eff}}$.*

An explicit representation for the spectrum of $\mathcal{A}_{\text{eff}}$, *i.e.* the set of $z \in \mathbb{C}$ for which the problem

$$\mathcal{A}_{\text{eff}} \begin{pmatrix} u_+ \\ \eta \end{pmatrix} = z \begin{pmatrix} u_+ \\ \eta \end{pmatrix}, \qquad (3.14)$$

has a nontrivial solution $(u_+, \eta)^\top$, can be obtained as follows. We represent $u_+ \in H^2(\Omega_+)$ in the form $u_+ = v + c$, where $c \in \mathbb{C}$ is related to η by the formula $c\sqrt{|\Omega_-|} = \eta$, cf. (3.13), and v solves the problem $-\Delta v = z(v + c)$ subject to the Neumann boundary condition $(\partial v / \partial n_+)|_{\partial \Omega} = 0$ and the Dirichlet boundary condition $v|_\Gamma = 0$, or equivalently $v = zc(A_0^+ - z)^{-1} \mathbb{1}_{\Omega_+}$, where $\mathbb{1}_{\Omega_+}$ is the unity

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[6]We remark that the generalised resolvent $\widetilde{R}_{\text{eff}}(z)$ is thus the solution operator of a spectral BVP of a special class. Namely, the dependence of its boundary condition on the spectral parameter is linear with respect to the latter. Such spectral BVPs were considered e.g. in [34].

function on Ω_+. Therefore, in terms of the pair $(v, c)^\top$ the eigenvalue problem (3.14) admits the form:

$$\begin{pmatrix} -\Delta v \\ \dfrac{1}{\sqrt{|\Omega_-|}} \displaystyle\int_\Gamma \dfrac{\partial v}{\partial n_+} \end{pmatrix} = z \begin{pmatrix} v + c \\ c\sqrt{|\Omega_-|} \end{pmatrix}, \tag{3.15}$$

and so its solvability is easily seen to be equivalent to the boundary equation appearing in the second component of (3.15):

$$\frac{1}{\sqrt{|\Omega_-|}} \int_\Gamma \frac{\partial v}{\partial n_+} = zc\sqrt{|\Omega_-|}. \tag{3.16}$$

Suppose first that $c \neq 0$.

Denoting as in Introduction above by λ_j^+, $j = 1, 2, \ldots$, and ϕ_j^+, $j = 1, 2, \ldots$ the eigenvalues and the corresponding normalised eigenfunctions of the operator A_0^+, the relation (3.16) is reduced to $c = -|\Omega|^{-1} \int_{\Omega_+} v_+$ using the Green's formula and the equality $-\Delta v = z(v + c)$. Therefore, based on the calculation contained in Introduction, the solvability of (3.14) turns out to be equivalent to (1.4). Alternatively, if $c = 0$, which corresponds to the case when $\eta = 0$, the function u_+ is clearly an eigenfunction of A_0^+, and it can easily be shown to have zero mean[7] over Ω_+.

Remark 2. It has been conjectured [27] (and established rigorously in the case of the Dirichlet Laplace operator in two dimensions [13]) that the eigenfunctions of a BVP on a sufficiently regular bounded domain Ω can be "nearly extended" (see [13] for rigorous details) to generalised eigenfunctions of the whole-space problem, corresponding to the same eigenvalue. This can be interpreted as an effect of "transparency" of the domain Ω to the waves of certain wavelengths. The set of values described by (1.4) can therefore be interpreted as the set of wavelengths at which a similar effect of transparency occurs for problems (3.1), as $a \to \infty$, in other words for problems with low-index dielectric inclusions. A rigorous proof of this observation requires the development of scattering theory for high-contrast transparent obstacles, and it will be addressed in a separate publication.

Remark 3. The spectrum of A_0^+, contrary to what would seem from Theorem 3.9, in the generic case does not enter the spectrum of (3.14). The mechanism for this is described in [18].

Acknowledgements

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