Bounds for Long Max-Plus Matrix Products

by

Arthur Kennedy-Cochran-Patrick

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Supervisor: Dr. Sergeĭ Sergeev

School of Mathematics

College of Engineering and Physical Sciences

University of Birmingham

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Abstract

We consider long matrix products over max-plus algebra and develop bounds on the transient of their length after which they admit a certain decomposition as the product length exceeds these bounds. First we build on the weak CSR approach for max-plus powers of a matrix by Merlet, Nowak, and Sergeev [68] and consider the case when the products are tropical matrix powers of just one matrix. For this case we obtain new bounds on the above mentioned transient that make use of the cyclicity of the associated digraph and the tropical factor rank. Next, we develop a CSR decomposition for tropical inhomogeneous matrix products and establish bounds in which certain matrix products become CSR. We also critically examine the limitations of the developed theory by presenting a number of counterexamples in the cases where no bound exists for a matrix product to be CSR.

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CHAPTER 1

INTRODUCTION AND PRELIMINARY INFORMATION

By max-plus algebra we mean the analogue of linear algebra based on the pair of operations $(\oplus, \otimes)$ where, for some a, b, $a \oplus b \coloneqq \max(a, b)$ and $a \otimes b \coloneqq a + b$. For both of these operations an identity needs to exist. For $\otimes$ the natural identity is 0 as it is the neutral element for addition. For $\oplus$ we set the identity to be $-\infty$ as all real numbers are strictly greater than $-\infty$. Therefore we need to include $-\infty$ on the set of real numbers as part of the semifield that these operations work on, which will be denoted $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$. These operations can be extended to matrices in the same manner as linear algebra and it can be shown with the following example:

$$\begin{pmatrix} -\infty & 2 \\ 1 & 4 \end{pmatrix} \otimes \begin{pmatrix} 2 & 4 \\ 3 & -\infty \end{pmatrix} = \begin{pmatrix} 5 & -\infty \\ 7 & 5 \end{pmatrix}$$

To elaborate, the first entry in the matrix on the RHS is calculated as $-\infty \otimes 2 \oplus 2 \otimes 3 = -\infty \oplus 5 = 5$. The other entries are calculated in the same manner. The second entry in the matrix on the RHS is calculated as $-\infty \otimes 4 \oplus 2 \otimes -\infty = -\infty \oplus -\infty = -\infty$.

1.1 Literature Review

1.1.1 Max-plus Algebra and Tropical Mathematics

Max-plus algebra is useful in many different areas of mathematics. Some notable examples include scheduling problems [58, 36, 51], cryptography [39], algebraic geometry [74], combinatorial optimisation [10] and mathematical physics [66, 55]. Note that we can replace the $\oplus$ operator to be $a \oplus b := \min(a, b)$ which is called min-plus algebra or tropical algebra [2, 57]. Alternatively we can replace the $\otimes$ operator to be $a \otimes b := a \times b$ and restrict ourselves to the set of nonnegative real numbers, then we will be working in max-times algebra [83, 28]. In this thesis we will focus entirely on max-plus algebra but many of the results presented here will have natural min-plus and max-times analogues as all these semirings are isomorphic to the max-plus semiring via the isomorphisms $f(x) = -x$ and $f(x) = \log(x)$ respectively.

One of the earliest examples of max-plus algebra being used was by Cuninghame-Green [19] in which it was shown that industrial processes could be modelled using matrix algebra by changing operations from linear to max-plus. In this paper he noted that they had been originally introduced in a previous work [20]. In those early works, Cunninghame-Green introduced the new arithmetic and showed that the most familiar laws of linear algebra still held under this new framework. It was then further popularised in a lecture series [21] which brought it to the attention to the mathematical community and also developed the connection between max-plus algebra and graph theory. This was then utilised further by U. Zimmermann [97] as well as Gondran and Minoux [35] by developing an approach to combinatorial optimisation that is based on idempotent semirings, with some of its most notable applications being methods to find optimal walks using matrices. The idea to express the Floyd-Warshall algorithm and some other shortest path algorithms as a kind of Gaussian elimination using max-plus semiring was considered by Carré [14]. For further developments of this idea, see the work of Litvinov, Rodionov, Sergeev and Sobolevskii [59]. Recently, progress was made in a preprint by Joswig and Schröter [47] where they develop parametrised versions of the Floyd-Warshall algorithm and Dijkstra's algorithm using tropical geometry, as well as using these to develop applications in real-world scenarios.

It should be noted that max-algebra is frequently called tropical mathematics, in honour of Simon [73], therefore we will use these terms interchangeably throughout the thesis. Since the beginning of 1990's several textbooks on tropical algebras and their relation to linear algebra have been written: by Baccelli, Cohen, Olsder and Quadrat [4], Heidergott, Olsder and van der Woude [43], Gondran and Minoux [35] and Butkovič [11]. A concise introduction to max-plus linear algebra can be found in any of these books. Notably, the book by Butkovič [11] is the basis of many of the key definitions and concepts used in this thesis. Many articles of linear algebra over the tropical semiring exist such as the survey by Akian, Bapat and Gaubert [1] and another which looks at its relation to control theory by Cohen, Gaubert and Quadrat [17]. One of the main defining features of max-plus mathematics, as well as tropical mathematics, is that for any $a \in \mathbb{R}_{\max}$, $a \oplus a = \max(a, a) = a$ thus it is part of idempotent mathematics.

This form of algebra also emerged independently on the other side of the iron curtain. Soviet works on extremal algebras and idempotent analysis (as it was called there) started with Vorobyov, a renowned game theorist. In the 1960's he published a number of works on max-plus algebra [90, 91, 92] in which he developed the theory of

$A \otimes x = b$ systems in max-plus algebra and derived the existence of tropical eigenvectors from Brouwer's fixed point theorem. Later, the academician Maslov started developing idempotent analysis, aiming to apply it to quasiclassical approximation in quantum physics [66], as well as some equations of mathematical economics and mathematical physics such as the Hamilton-Jacobi-Bellman and Burgers' equations [55]. In the late 1980's and early 1990's he led an informal group of mathematicians including Kolokoltsov, Yakovenko, Litvinov, Shpiz and Sobolevskiĭ working on idempotent mathematics. This group published many important and interesting works. One such example is the article by Litvinov and Maslov [60] where they stated that idempotent mathematics is the "classical shadow" of traditional mathematics over numerical fields, in the spirit of Bohr's correspondence principle between quantum and classical mechanics. In some other works, Litvinov, Maslov and Shpiz took an algebraic approach to idempotent functional analysis [62] and developed a link between idempotent mathematics and group representation theory [63]. Kolokoltsov and Maslov developed applications of idempotent mathematics to the Hamilton-Jacobi-Bellman equations as well as Burgers' equation [55], see also a work by Khanin and Sobolevskiĭ [52]. Yet another area, which was studied by this group and is closer to the main topic of this thesis, was the applications of idempotent mathematics in turnpike theory developed by Kontorer and Yakovenko [56], which was also further explored by Kolokoltsov and Maslov [55]. Litvinov also wrote many surveys to popularise max-plus algebra and tropical mathematics, with a notable example being the survey [61] which serves as a concise introduction to idempotent and tropical mathematics.

Tropical and idempotent mathematics has played a large part in developing semiring theory in works such as the book by Golan [34]. Similar semirings have been developed, both independently and in relation to the max-plus semiring. An example is the max-min semiring, also known as the bottleneck semiring, in which the real numbers are adjoined with $\pm\infty$ to accompany the addition operation max and the multiplication operation min. The powers of max-min matrices were thoroughly studied, in particular, the monograph by Gavalec [32], the paper by Gavalec and Plávka [33] and Semančíková [78, 79], who used cyclic classes in the study of max-min matrix powers and their periodicity transients. The same approach was later taken by Butkovič [11], Sergeev and H.Schneider [81, 83] in their study of max-plus matrix powers and their ultimate periodicity (with an immediate connection to this thesis).

The Boolean semiring holds close similarities to work in this thesis and the research area in general, particularly, powers of a single Boolean matrix. A key concept, the CSR decomposition, contains a Boolean matrix representing the critical subgraph and the development of the powers of this matrix are crucial to the decomposition. Many key results from works in Boolean algebra can be extrapolated to this setting such as the ideas from the book by Brualdi and Ryser [9, Sections 3.4 and 3.5], which was partly motivated by its use in nonnegative matrix theory and Perron-Frobenius theory as referenced in the book by Berman and Plemmons [8]. Other works of note come from Zhang [95], Gregory, Kirkland and Pullman [38], as well as de Moor and de Schutter [23], who studied periodicity transients of Boolean matrix powers and showed how their results apply to periodicity transients of tropical matrix powers and discrete event systems in max-plus algebra.

Naturally one can evolve the Boolean semiring to hold values between 1 and 0 inclusive. These are known as fuzzy sets and have been the study of many mathematicians in this field such as Kim and Roush [54] or their extension with Cao [13] to fuzzy sets by adjoining the incline property, $x \oplus xy = x$ for any $x, y \in \{0, 1\}$ to the usual idempotent semiring axioms.

A notable area of tropical mathematics is tropical convexity, which is a geometric counterpart of tropical linear algebra pioneered by K. Zimmermann [96] and later developed and popularised by Develin and Sturmfels [26], Joswig [46], Litvinov [61] and by Cohen, Gaubert, Quadrat and Singer [18]. Let us also mention the successes of tropical algebraic geometry: in particular, the works of Mikhalkin [70] on counting algebraic curves and Viro [88] who developed the patchworking method. A famous use of the patchworking method was the application of it to the sixteenth Hilbert problem by Viro [89]. Although tropical algebraic geometry and tropical convexity do not have any direct connection to this thesis, some of the most powerful techniques and fascinating results were developed and obtained in those areas of tropical mathematics.

For powers of matrices, the seminal monograph by Brualdi and Ryser [9] gives two bounds on the exponent of a primitive (cyclicity is equal to 1) nonnegative matrix. This is the smallest natural number k such that A^t is positive for all $t \geq k$. While this is defined for linear algebra the crucial link here is that it is shown in the monograph [9] that the exponent of a primitive matrix A depends on the digraph associated with such matrix $\mathcal{D}(A)$. In other words it is equal to the smallest natural number k such that for any ordered pair of nodes from $\mathcal{D}(A)$ there exists a walk connecting those two nodes and thus a walk of any length greater than said k. These bounds will play a key role in both Chapter 2 and Chapter 3 in developing bounds on walks on digraphs.

The first bound is the Wielandt bound [93] defined as,

$$\mathrm{Wi}(d) = \begin{cases} 0 & \text{if } d = 1 \\ (d-1)^2 + 1 & \text{for } d > 1, \end{cases}$$

where d is the number of nodes in the given digraph. The second bound is the Dulmage-Mendelsohn bound [27] defined as,

$$\mathrm{DM}(d, g) = d + g(d - 2),$$

where d is the number of nodes and g is the minimal length of a cycle in $\mathcal{D}(A)$.

Naturally as these bounds were for primitive matrices there was a desire to generalise them by introducing a non-trivial cyclicity γ which led to two new bounds, the Schwarz and Kim bounds. The Schwarz bound [77] is an improvement of the Wielandt bound and is defined as,

$$\mathrm{Sch}(\gamma, d) = \gamma \,\mathrm{Wi}\left(\left\lfloor \frac{d}{\gamma} \right\rfloor\right) + d(\operatorname{mod} \gamma),$$

where $d(\operatorname{mod} \gamma)$ is the lowest positive value p such that $p = d + l\gamma$. The Kim bound [53] is an improvement of the Dulmage-Mendelsohn bound and is defined as,

$$\mathrm{Kim}(\gamma, g, d) = g\left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 2\right) + d,$$

where g is the length of the shortest cycle of $\mathcal{D}(A)$. These four bounds are crucial for the following work in this thesis. The proof of Wielandt's bound on exponents was given in full by H.Schneider [75], where he transcribed and translated Wielandt's personal diaries.

1.1.2 Thesis Outline and Closely Related Publications

This thesis will be focusing on transients on max-plus matrix products in two areas. The first area will be powers of a single matrix. Inspired by the earlier works of Nachtigall [72] and Molnárová [71], Sergeev and H.Schneider [83] developed the CSR decomposition, where a tropical matrix power can be decomposed into $A^t = \lambda^t \otimes C \otimes S^t \otimes R$, where C, S and R are matrices developed from A and λ is the maximum max-plus eigenvalue of A. Without loss of generality we will assume $\lambda = 0$ as this can be made possible by scaling A appropriately which is an old idea used by Cunninghame-Green and other pioneers. In the same paper [83] some bounds on T were given after which this property appears for all $t \geq T$. Based on this idea and the previous research on tropical matrix powers periodicity transients, Merlet, Nowak and Sergeev developed more accurate bounds for this property and on the periodicity transient of tropical matrix powers [68]. One method they employed in achieving this was by introducing a graph theoretical term known as the cycle removal threshold, which is a bound T in which certain walks with length greater than T can have cycles added or removed to develop an associated walk with length less than T. While these bounds are useful we found them to be lacking, especially in the use of cyclicity and the potential of introducing the tropical factor rank into the bounds. Therefore in Chapter 2 we will take the bounds proposed by Merlet et al. [68] and refine them further using the cyclicity of A as well as introducing new bounds that use the factor rank of A where applicable. Some of these bounds will be proved directly and some of them will be proved using new bounds on the cycle removal threshold using cyclicity and factor rank. Theorems 2.3.3, 2.5.4, 2.5.7 and 2.6.3 in Chapter 2 were published in the joint paper with Merlet, Nowak and Sergeev [48]. Theorems 2.5.5 and 2.6.5 are results that have not been previously published and are the bounds that involve the tropical factor rank.

The second area will be working with inhomogeneous matrix products from semigroups. We will be looking for bounds on the transients, in which the matrix products exhibit a factor rank property. To achieve this we create a product analogue of the CSR decomposition and develop bounds in which matrix products become CSR as well as show some cases where this does not happen. This work has been submitted as a preprint co-authored with Sergeev [49] and there is a section devoted to the special case, which was explored originally in collaboration with Berežný and Sergeev [50].

A lot of exploration has been done on periodicity transients of tropical matrix powers, such as the work by Nachtigall [72] who developed expansions based on the periodicity of a single matrix power. Hartmann and Arguelles [42] also wrote a key work in this area by looking at transients for long walks over digraphs and making use of the max-balancing scaling introduced by H.Schneider and M.Schneider [76]. Their ideas and results were crucial to Soto y Koelemeijer [87] who improved their bounds on the transient, as well as to Merlet et al. [68], who formalised the notion of Hartmann-Arguelles expansion and improved their bounds further. Akian, Gaubert and Walsh developed local bounds for the transient on individual entries of tropical matrix powers with infinite dimensions [3], an idea that was also used by Merlet et al. [68]. Using the work by Nachtigall [72], H.Schneider and Sergeev [83] developed the CSR decomposition as described earlier. As this decomposition plays a key role throughout the entire thesis we will take some time to look at two papers which develop the decomposition: the seminal work by H.Schneider and Sergeev [83], as well as the paper by Merlet et al. [68] which serves as a basis for Chapter 2.

The first paper [83] introduced the CSR decomposition as a method to show a matrix power is periodic after $O(d^4 \log d)$ operations. Here the authors gave the initial bound of $3d^2$ in which a matrix power A^t where $t \geq 3d^2$ can be expressed in CSR terms. It is worth noting that this bound does not depend on the entries of A but on the size of the matrix A itself. They also proved some elementary results for CSR as well as develop a link between the CSR decomposition and certain walks on digraphs which will be stated in Chapter 2. Merlet, Nowak, Sergeev and H.Schneider [67] explored bounds on weighted digraphs further, looking at the Wielandt, Dulmage-Mendelsohn, Schwarz and Kim bounds as well as bounds in Boolean algebra by Gregory, Kirkland and Pullman [38] and generalising them into max-plus algebra.

In parallel, Merlet, Nowak and Sergeev [68] took a particular version of the CSR decomposition, which they called the weak CSR decomposition. In that weak CSR decomposition, one introduces a subordinate matrix B such that $A^t = CS^tR \oplus B^t$. This subordinate matrix can be constructed using three different decomposition schemes, where the first scheme stemmed from the decomposition developed by Nachtigall [72], the second stemmed from the transient work by Hartmann and Arguelles [42] and the third was a completely new concept. They also defined two transients T_1 and T_2 where for all $t \geq T_1(A, B)$, $A^t = CS^tR \oplus B^t$ and for all $t \geq T_2(A, B)$, $CS^tR \geq B^t$. Therefore for any $t \geq (T_1(A, B), T_2(A, B))$, A^t has a CSR decomposition as explored in the paper by H.Schneider and Sergeev [83]. By essentially splitting the transient into two steps using the weak CSR decomposition they developed bounds separately which allowed them to greatly refine the original bound presented in the article [83] as well as the bounds of Hartmann and Arguelles [42], Soto y Koelemeijer [87] and some other works. They also introduced the term cycle removal threshold and developed bounds relating to both T_1 and T_2 using this term, as well as bounds for the cycle removal threshold itself. These bounds, along with optimal walk representation results will be given in Chapter 2. These papers provide the groundwork on which Chapter 2 and the joint paper [48] (with Merlet, Nowak and Sergeev) are based. By introducing the cyclicity of the associated digraph we improved the bounds in many cases, particularly by introducing the Schwarz bound and the Kim bound. Using the factor rank of A we also develop bounds that have the potential to be much smaller than their non factor rank counterparts. For some of the decomposition schemes we will use the cycle removal threshold to develop these new bounds using cyclicity and factor rank. Note that in a further paper, Merlet, Nowak and Sergeev [69] characterised the matrices which attain the T_1 generalisations of the Wielandt and Dulmage-Mendelsohn bounds.

The other area of focus will be inhomogenous matrix products and much work has been done in this field both inside and outside of max-plus mathematics. The two books by Hartfiel [41] and Seneta [80] offer a comprehensive look at inhomogenous matrix products in a linear algebra setting. In both books the authors focus on the idea of matrix products converging to various states as the product length tends to infinity. Weighted automata are closely related to inhomogenous products as discussed by Daviaud, Guillon and Merlet [22] as well as Zhang [95] who worked in the Boolean semiring. Some results on the Lyapunov exponents of max-plus inhomogeneous stochastic products were obtained by Goverde, Heidergott and Merlet [37]. A number of theoretical results have been also proved concerning the tropical matrix semigroups and behaviour of inhomogeneous tropical matrix products. In particular, Gaubert [30] proved that the tropical matrix semigroups have the Burnside property and, with Katz [31], he studied the decidability of the following reachability problem: given a tropical matrix semigroup with r generators and entries from a semiring, is there a product of these matrices which attains a prescribed matrix. As an application, they show that this problem is undecidable for the max-plus semiring when $r = 2$. Johnson and Kambites [45] provided a systematic study of the algebraic structure of 2×2 tropical matrices under multiplication. In relation to the CSR decomposition, Izhakian, Johnson and Kambites [44] described general groups that could be found within a tropical matrix semigroup, in which CSR's can form one of these groups.

A common concept throughout this thesis is the notion of factor rank. By factor rank we mean the value which is the smallest natural number r such that a matrix can be expressed as the max-plus product of a matrix with r columns and a matrix with r rows. In max-plus algebra is is also known as the Barvinok rank as it first appeared in the paper by Barvinok, Johnson, Woeginger and Woodroofe [5]. However there exist other forms of rank in max-plus algebra as discussed by Akian, Gaubert and Guterman [2] as well as by Develin, Santos and Sturmfels [25]. A closely related notion of tropical ultimate rank was introduced by Guillon, Izhakian, Mairesse and Merlet [40] in which this notion exists for powers of tropical matrices. A key theorem from this paper is that the ultimate rank of a matrix A, which is the common value when several non-coinciding notions of matrix rank are equal to each other, is equal to the sum of the cyclicities of all strongly connected components (s.c.c.s) of the critical subgraph of $\mathcal{D}(A)$. What is interesting here is that a result from this thesis, Theorem 3.3.12, which is proved for inhomogenous products of matrices, is similar to the result of [40, Theorem 5.2].

Before we delve into the core background of Chapter 3 we need to take a look at a paper by Butkovič, H.Schneider and Sergeev [84] which introduces the concept of visualisation scaling. This is a method in which one can, by a change of base, scale a max-plus matrix A in such a way so that the entries associated with the edges on the critical graph of $\mathcal{D}(A)$ are equal to the maximal eigenvalue. This also ensures that any other edge that is not critical has a value that is less than or equal to said eigenvalue. This concept is very useful in Chapter 3 as it allows us to scale the generators in the semigroup to allow them to work with the main results given within the chapter. It can also be noted that Litvinov, Sergeev and Shpiz [85] study common eigenvectors of particular semigroups of matrices in tropical algebra, which opens up to the idea of a common visualisation, a concept which is important for Chapter 3.

Let us take some time to explore the background for Chapter 3 the paper by Shue, Anderson and Dey [86], which inspired its development, as well as the two papers on which this chapter is based [50, 49]. The paper by Shue, Anderson and Dey [86] investigated the asymptotic properties of certain inhomogeneous max-plus matrix products. By restricting the matrices to having just one critical loop they showed that as the product becomes long enough it exhibits a factor rank one property. Upon reading the paper we discovered a gap in one of the proofs where cycles were removed from walks associated with the product. Another main idea that came to us was to look for bounds in which these factor rank properties appear. This led to the development of the first publication in collaboration with Sergeev and Barežný [50], where both an implicit and a weaker explicit bound on the length of inhomogeneous product were developed, after which a very particular type of matrix product becomes rank one. Naturally, as this was a very particular case, we looked to develop it into a more general setting as well as build on the theory around it. Chapter 3 presents the results of developing the initial case into a more general setting by introducing an inhomogenous product analogue of CSR with the two definitions 3.3.1 and 3.3.2, in which the latter definition incorporates disjoint components of the critical subgraph. Some preliminary results are given with this analogue, such as: showing both definitions are equivalent in Proposition 3.3.5; giving optimal walk interpretations of both definitions in Lemma 3.3.7; and showing that the factor rank of a matrix that is CSR is bounded above by the sum of cyclicites over all s.c.c.s in the critical graph (see Theorem 3.3.12).Notably we show the relation to the CSR decomposition from H.Schneider and Sergeev [83] (see Proposition 3.3.8) which opens it up to the possibility of relating results from various papers [83, 68, 69] to this new concept. The majority of these results are written in the preprint [49] but there exist some novel results exclusively in this thesis.

There are many applications of max-plus mathematics to real world processes as noted earlier but we will explore two areas briefly. The first application is turnpike theory which is studied as an infinite horizon optimal control problem [94]. Informally this is the concept that an optimal walk will traverse a certain area or certain points as its length grows. This has many applications in economics such as capital growth model as noted by Yakovenko and Kontorer [56] or stochastic processes as explored by Marimon [65]. It was also explored by Kolokoltsov and Maslov in their book on idempotent analysis [55], in relation to ideas put forward by Yakovenko.

Another widely known application of max-plus algebra is in railway scheduling as the close association to weighted digraphs allows for easy modelling of some railway scheduling problems. Interestingly, many of the key works on this application were written by academics at the Technical University of Delft, such as the PhD Theses of Goverde [36], Soto y Koelemeijer [87], Kersbergen [51] and the textbook of Heidergott, Olsder and van der Woude [43] which is an excellent introduction to this kind of application of max-plus mathematics. We can relate inhomogeneous matrix products to max-plus switching systems which have been studied in depth at the same university by Kerbergen [51] and De Schutter alongside van den Boom in [24]. For further exploration, it would be interesting to connect the work done by these experts to the results of the present thesis.

The principal plan of the thesis is as follows. For the rest of Chapter 1 we will look at preliminary definitions in max-plus algebra, weighted digraphs and inhomogeneous matrix products. In Chapter 2 we will study powers of a single matrix and develop new bounds on the T_1 and T_2 parts of the periodicity transient, involving cyclicity and factor rank. For T_1 we will, using cyclicity and the cycle removal threshold, develop bounds that employ the Kim and Schwarz bounds, for T_2 we will improve on the bounds from the paper [68] by using the cycle removal threshold and for both T_1 and T_2 we will also introduce bounds that feature the factor rank of the matrix. Finally in Chapter 3 we will look at inhomogenous matrix products and develop bounds on the length of inhomogeneous matrix products, for which the product assumes a CSR form and its factor rank becomes bounded by the cyclicity of the critical graph. This will be done in a special case, for which the product ultimately assumes a CSR form. Then we will provide a number of counterexamples showing that the conditions of that special case are necessary to impose and that in a more general case a different approach has to be taken.

1.2 Max-Plus Algebra

1.2.1 Basic Definitions

We begin the preliminaries with a formal definition of the max-plus semiring.

Definition 1.2.1 (Max-plus semiring)**.** *Let* $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$*. The* max-plus semiring $(\mathbb{R}_{\max}, \oplus, \otimes, -\infty, 0)$ *is the semiring with the operations* $\oplus$ *and* $\otimes$ *defined for any two* $a, b \in \mathbb{R}_{\max}$*:*

- $\oplus$ *: The additive operator* $\oplus$ *where* $a \oplus b := \max(a, b)$*;*

- $\otimes$ *: The multiplicative operator* $\otimes$ *where* $a \otimes b := a + b$ *where* $+$ *is the usual addition defined for the real numbers. We also assume that* $-\infty \otimes a = a \otimes -\infty = -\infty$ *for any* $a \in \mathbb{R}_{\max}$*.*

With these operations, the additive identity is $-\infty$*, which we denote* ε *and the multiplicative identity is* 0*.*

The operations of this semiring are naturally extended to matrices and vectors and this is known as max-plus algebra. To define max-plus multiplication for $A = (a_{i,j})$ and $B = (b_{i,j})$ with entries from $\mathbb{R}_{\max}$ of appropriate sizes, we will use the following rule,

$$(A \otimes B)_{i,j} = \bigoplus_{1 \leq k \leq n} a_{i,k} \otimes b_{k,j} \ = \max_{1 \leq k \leq d} a_{i,k} + b_{k,j}.$$

In particular, the tth max-plus power of a square matrix A is defined as

$$A^{\otimes t} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{(t \text{ times})}.$$

For completeness we must also state that for any matrix $A \in \mathbb{R}_{\max}^{d\times d}$, $A^{\otimes 0} = I$, where I is the tropical identity matrix, i.e. $I = \operatorname{diag}(0)$.

A common concept that will be used throughout the thesis are block matrices as defined below.

Definition 1.2.2 (Block Matrices)**.** *A* block matrix *is an $d_1 \times d_2$ matrix F that can be represented as several submatrices or* blocks*, i.e.*

$$F = \begin{pmatrix} F_{1,1} & F_{1,2} & \ldots & F_{1,J} \\ F_{2,1} & F_{2,2} & \ldots & F_{2,J} \\ \vdots & \vdots & \vdots & \vdots \\ F_{I,1} & F_{I,2} & \ldots & F_{I,J} \end{pmatrix}$$

Here $F_{i,j}$, for $i \in \{1, 2, \ldots, I\}$ and $j \in \{1, 2, \ldots, J\}$, are matrices such that the dimensions for two distinct blocks are not necessarily the same. However the sum of row dimensions of the submatrices over the same j is equal to d_1 and the sum of the column dimension of the submatrices over the same i is equal to d_2. Finally we write $F_{i,j} = -\infty$ if every entry in that block is equal to ε.

The following definitions will play crucial roles in both Chapters 2 and 3. In Chapter 2 we will use this definition to derive more refined bounds on the periodicity transient. In Chapter 3 the bounds derived in the chapter are the transients in which a matrix product will exhibit this property for some given value.

Definition 1.2.3 (Factor rank)**.** *Let* $A \in \mathbb{R}^{d_1 \times d_2}$*. The* Factor Rank r *of* A *is the smallest* $r \in \mathbb{N}$ *such that* $A = UL$ *where* $U \in \mathbb{R}^{d_1 \times r}$ *and* $L \in \mathbb{R}^{r \times d_2}$*.*

Naturally there is an analogous definition in the max-plus semiring.

Definition 1.2.4 (Tropical Factor rank)**.** *Let* $A \in \mathbb{R}_{\max}^{d_1 \times d_2}$*. The* Tropical Factor Rank r *of* A *is the smallest* $r \in \mathbb{N}$ *such that* $A = U \otimes L$ *where* $U \in \mathbb{R}_{\max}^{d_1 \times r}$ *and* $L \in \mathbb{R}_{\max}^{r \times d_2}$

As most of this work is in the max-plus semiring we will use the tropical factor rank always omitting the adjective "tropical".

Using the matrices U and L from the above definition we also define the matrix,

$$F = \begin{pmatrix} -\infty & U \\ L & -\infty \end{pmatrix}. \tag{1.1}$$

Upon squaring this block matrix we have $F^2 = \begin{pmatrix} A & -\infty \\ -\infty & \check{A} \end{pmatrix}$, where $\check{A} = L \otimes U$ is an r by r max-plus matrix. Hence there exist similarities between A, $\check{A}$ and F which will be explored in Chapter 2.

Definition 1.2.5 (Subordinate matrix)**.** *Let* $A \in \mathbb{R}_{\max}^{d \times d}$*. We say* B *is* subordinate *if* B *can be constructed from* A *by setting some entries in* A *to* $\varepsilon = -\infty$*.*

1.2.2 Weighted Digraphs and Max-Plus Matrices

This subsection presents some concepts and notation expressing the connection between tropical matrices and weighted digraphs. Monographs [11, 43] are our basic references for such definitions.

Definition 1.2.6 (Weighted digraphs)**.** *A* directed graph (digraph) *is a pair* (N, E) *where* N *is a finite set of nodes and* $E \subseteq N \times N = \{(i, j) \colon i, j \in N\}$ *is the set of edges, where* (i, j) *is a directed edge from node* i *to node* j*.*

A weighted digraph *is a digraph with associated weights* $w_{i,j} \in \mathbb{R}_{\max}$ *for each edge* (i, j) *in the digraph.*

A digraph associated with a square matrix A *is a weighted digraph* $\mathcal{D}(A) = (N_A, E_A)$ *where the set* N_A *has the same number of elements as the number of rows or columns in the matrix* A*. The set* $E_A \subseteq N_A \times N_A$ *is the set of directed edges in* $\mathcal{D}(A)$*, where* (i, j) *is an edge if and only if* $a_{i,j} \neq \varepsilon$*, and in this case the weight of* (i, j) *equals the corresponding entry in the matrix* A*, i. e.,* $w_{i,j} = a_{i,j} \in \mathbb{R}$.

Definition 1.2.7 (Walks, paths and weights)**.** *A sequence of nodes* $W = (i_0, \dots, i_l)$ *is called a* walk on a weighted digraph $\mathcal{D} = (N, E)$ *if* $(i_{s-1}, i_s) \in E$ *for each* $s \colon 1 \leq s \leq l$*. This walk is a* cycle *if the start node* i_0 *and the end node* i_l *are the same and the cycle is* elementary *if the start and end nodes are the only time in the walk that a repeated node appears and a cycle of length* 1 *is called a* loop*. It is a* path *if no two nodes in* $i_0, \dots, i_l$ *are the same. The* length *of* W *is* $l(W) = l$*.*
The weight *of* W *is defined as the max-plus product (i. e., the usual arithmetic sum) of the weights of each edge* (i_{s-1}, i_s) *traversed throughout the walk, and it is denoted by* $p_{\mathcal{D}}(W)$*. Note that a sequence* $W = (i_0)$ *is also a walk (without edges), and we assume that it has weight and length* 0*.*
The mean weight *of* W *is defined as the ratio* $p_{\mathcal{D}}(W)/l(W)$*.*

For a digraph, being strongly connected is a particularly useful property.

Definition 1.2.8 (Strongly connected)**.** *A digraph is* strongly connected*, if for any two nodes* i *and* j *there exists a walk connecting* i *to* j*.*

Definition 1.2.9 (Reducible, irreducible)**.** *A square matrix A is* reducible *if there exists some permutation matrix P such that the matrix $P^T AP$ is block upper-triangular. If no such P exists then the matrix is* irreducible*.*

Note that, any strongly connected digraph is irreducible as shown with this following lemma.

Lemma 1.2.10. *Let A be a matrix in $\mathbb{R}_{\max}^{d\times d}$ with an associated digraph $\mathcal{D}_A$. Then A is irreducible if and only if $\mathcal{D}_A$ is strongly connected.*

A proof of this lemma is contained in [9, Theorem 3.2.1]. Finally a digraph is called *completely reducible*, if it consists of a number of s.c.c.s, such that no two nodes of any two different components can be connected to each other by a walk.

The following more refined notions are crucial in the study of ultimate periodicity of tropical matrix powers.

Definition 1.2.11 (Cyclicity and cyclic classes)**.** *The* cyclicity *of a strongly connected digraph is the highest common factor of the lengths of cycles within the graph. The cyclicity of a completely reducible digraph is the lowest common multiple of the cyclicities from each s.c.c. making up the digraph. Both will be denoted by γ.*

For two nodes $i, j \in N$ we say that i and j are in the same cyclic class *if there exists a walk whose length is a multiple of γ, connecting i to j or j to i. This splits the set of nodes into γ cyclic classes: $\mathcal{C}_0, \ldots, \mathcal{C}_{\gamma-1}$. The notation $\mathcal{C}_l \to_k \mathcal{C}_m$ means that some (and hence all) walks connecting nodes of $\mathcal{C}_l$ to nodes of $\mathcal{C}_m$ have lengths congruent to k modulo γ. The cyclic class containing i will be also denoted by $[i]$ and for any i, j we will use $[i] \to_t [j]$ to say that there is a walk of length t connecting a node in $[i]$ to an node in $[j]$.*

The correctness of the above definition of cyclic classes follows, for example, from [9, Lemma 3.4.1]: in fact, every walk from i to j on $\mathcal{D}$ has the same length modulo γ.

In tropical algebra, we often have to deal with two digraphs: 1) the digraph associated with A and 2) the critical digraph of A. The latter digraph (being a subdigraph of the first) is defined below.

Definition 1.2.12 (Maximum cycle mean and critical digraph)**.** *For a square matrix A, the* maximum cycle mean *of $\mathcal{D}(A)$ denoted as $\lambda(A)$ (equivalently, the maximum cycle mean of A) is the biggest mean weight of all cycles of $\mathcal{D}(A)$.*

A cycle in $\mathcal{D}(A)$ is called critical *if its mean weight is equal to the maximum cycle mean (i.e., if its mean weight is maximal).*

The critical digraph *of $\mathcal{D}(A)$, denoted by $\mathcal{G}^c(A)$, is the subdigraph of $\mathcal{D}(A)$ whose node set $\mathcal{N}_c$ and edge set $\mathcal{E}_c$ consist of nodes and edges that belong to all critical cycles (i.e., that are critical).*

Note that any critical digraph is completely reducible. The seminal Cyclicity Theorem proved by Cohen, Dubois, Quadrat and Viot [16, 15] states that the cyclicity of critical digraph of A is the ultimate period (see definition below) of the tropical matrix powers sequence $\{A^{\otimes t}\}_{t\geq 1}$, provided that A is irreducible and $\lambda(A) = 0$. See also Butkovič [11] and Sergeev [81] for more detailed analysis of the ultimate periodicity of this sequence.

Definition 1.2.13 (Threshold of Ultimate Periodicity)**.** *Let σ be the cyclicity of the critical subgraph of $\mathcal{D}(A)$. The* threshold of ultimate periodicity *of powers of A, is threshold of ultimate periodicity, denoted by $T(A)$, is the smallest T with the property that $\forall k \geq T(A)$, $\lambda^{\otimes\sigma} \otimes A^{\otimes k} = A^{\otimes(k+\sigma)}$. We refer to σ as the* ultimate period *of A.*

The following graph theoretical notion will be useful in Chapter 2.

Definition 1.2.14 (Girth and Max-Girth)**.** *The* girth*, denoted g, of a strongly connected digraph is the length of the shortest elementary cycle. The* max-girth*, denoted* $\hat{\mathrm{g}}$*, is the maximum over the girths of all the s.c.c.s of a digraph.*

Below we will use notation for walk sets and their maximal weights that is similar to that of Merlet et al. [68].

Definition 1.2.15 (Sets of walks)**.** *Let $\mathcal{D} = (N, E)$ be a weighted digraph and let $i, j \in N$. The three sets $\mathcal{W}_{\mathcal{D}}(i \to j)$, $\mathcal{W}^k_{\mathcal{D}}(i \to j)$ and $\mathcal{W}_{\mathcal{D}}(i \xrightarrow{\mathcal{N}} j)$, where $\mathcal{N} \subseteq N$ is a subset of nodes, are defined as follows:*

$\mathcal{W}_{\mathcal{D}}(i \to j)$ is the set of walks over $\mathcal{D}$ connecting i to j;

$\mathcal{W}^k_{\mathcal{D}}(i \to j)$ is the set of walks over $\mathcal{D}$ of length k connecting i to j;

$\mathcal{W}_{\mathcal{D}}(i \xrightarrow{\mathcal{N}} j)$ is the set of walks over $\mathcal{D}$ connecting i to j that traverse at least one node of $\mathcal{N}$.

The supremum of the weights in any set of walks $\mathcal{W}$ will be denoted by $p(\mathcal{W})$.

Definition 1.2.16 (Geometrical equivalence)**.** *Let the matrices A and B have their respective digraphs $\mathcal{D}(A) = (N_A, E_A)$ and $\mathcal{D}(B) = (N_B, E_B)$. We say that A and B are* weakly geometrically equivalent *if $N_A = N_B$ and $E_A = E_B$, and they are* strongly geometrically equivalent *if they are weakly geometrically equivalent and $\mathcal{G}^c(A) = \mathcal{G}^c(B)$.*

Before we move onto visualisation scaling we need to briefly discuss the matrix inverse. The existence of an identity matrix implies the existence of multiplicative inverse matrices, which were originally studied by Cunninghame-Green [21]. He showed that, generally speaking, for a matrix A, there exists a matrix B such that $A \otimes B = I = B \otimes A$ if and only if A is a generalised permutation matrix. This type of matrix can be formed by permuting a diagonal matrix, which is a matrix $X = \text{diag}(x)$ with entries $X_{ii} = x_i \in \mathbb{R}$ on the diagonal and $X_{ij} = \varepsilon$ off the diagonal (i.e., if $i \neq j$). For a diagonal matrix X, its tropical inverse X^- can be found by changing the signs on all the diagonal entires of X. The concept of visualisation scaling was studied by Butkovič, H.Schneider and Sergeev [84], see also a paper by Sergeev [82] and references therein for more background on this scaling.

Definition 1.2.17 (Visualisation)**.** *Matrix B is called a* visualisation *of $A \in \mathbb{R}^{d\times d}_{\max}$ if there exists a diagonal matrix $X = \text{diag}(x)$, such that $B = X^- \otimes A \otimes X$, where X^- is the tropical inverse of X, and B satisfies the following conditions: $B_{ij} = \lambda(B)$ for $(i, j) \in \mathcal{E}_c(B)$ and $B_{ij} \leq \lambda(B)$ for $(i, j) \notin \mathcal{E}_c(B)$.*

Once $\lambda(A) \neq \varepsilon$, a visualisation of A always exists and, moreover, vectors x providing a visualisation by means of diagonal matrix scaling $A \mapsto X^- \otimes A \otimes X$ are precisely the tropical subeigenvectors of A, i.e., vectors satisfying $A \otimes x \leq \lambda(A) \otimes x$.

Definition 1.2.18 (Max-balancing)**.** *Let $A \in \mathbb{R}^{d\times d}_{\max}$ and partition the set of nodes into two disjoint sets $N = N_1 \cup N_2$. The matrix A is* max-balanced *if*

$$\max_{i\in N_1, j\in N_2} a_{i,j} = \max_{i\in N_2, j\in N_1} a_{i,j}$$

for any disjoint N_1 and N_2.

As in the case of visualisation, by [76, Corollary 9], for any $A \in \mathbb{R}^{d\times d}_{\max}$ there exists a vector x with real entries that provides a max-balancing by means of diagonal matrix scaling $A \mapsto X^{-1}AX$. Thus the definitions can be combined to give.

Definition 1.2.19 (Max-balancing scaling)**.** *Let* $A \in \mathbb{R}_{\max}^{d\times d}$. *The* max-balancing scaling *of* A *is a vector with real entries* x *such that the matrix* $X^{-1}AX$ *(where* $X = \operatorname{diag}(x)$*) is max-balanced.*

Note that, by definition, a max-balanced matrix is always visualised.

The following definition comes from the book by Butkovič [11] and is a staple in max-plus algebra.

Definition 1.2.20 (Kleene Star and Metric Matrix)**.** *Let* $A \in \mathbb{R}_{\max}^{d\times d}$. *The* Kleene star A^* *is the matrix defined by the following infinite sum,*

$$A^* = I \oplus A \oplus A^{\otimes 2} \oplus A^{\otimes 3} \oplus \dots .$$

Here I *is the tropical identity matrix. The* metric matrix A^+ *is a similar infinite sum as the Kleene star with the exclusion of* I*, thus*

$$A^+ = A \oplus A^{\otimes 2} \oplus A^{\otimes 3} \oplus \dots .$$

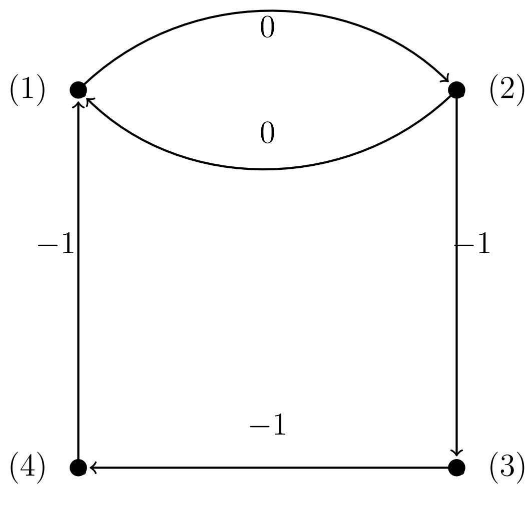
Note that in order for these infinite sums to be truncated we require $\lambda(A) \leq 0$ otherwise we end up with matrix powers that grow with their power therefore the summation would not converge. In relation to weighted walks on $\mathcal{D}(A)$ the Kleene star and metric matrix represent the optimal weighted walks of any length over $\mathcal{D}(A)$, i.e.,

$$((A)^*)_{ij} = \bigoplus_{l \in \mathbb{N} \cup \{0\}} \left(p\left(\mathcal{W}^l_{\mathcal{D}(A)}(i \to j)\right)\right)$$

$$((A)^+)_{ij} = \bigoplus_{l \in \mathbb{N}} \left(p\left(\mathcal{W}^l_{\mathcal{D}(A)}(i \to j)\right)\right).$$

For the rest of the thesis we refer to the representation of optimal weighted walks in this manner as the *optimal walk interpretation*. To help understand some of the definitions presented in this section we conclude with an example. Let $A \in \mathbb{R}^{4\times 4}_{\max}$ with entries and associated weighted digraph:

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon \\ 0 & \varepsilon & -1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & -1 \\ -1 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$



By definition for any entry $a_{i,j} = \varepsilon$ the associated digraph does not have a weighted edge connecting $i \to j$ and vice versa. For example, the entry $A_{3,4} = -1$ has the associated edge $3 \to 4$ with weight -1. Now we can calculate the cyclicity of the digraph. Note that there are only two elementary cycles in this digraph, $1 \to 2 \to 1$ of length 2 and $1 \to 2 \to 3 \to 4 \to 1$ of length 4. Hence $\gamma = \gcd(4, 2) = 2$ thus the cyclicity is 2 where $[1] = [3] = \{1, 3\}$ and $[2] = [4] = \{2, 4\}$ are the distinct cyclic classes. Also note that the shortest cycle is of length 2 therefore the girth of $\mathcal{D}(A)$ is equal to 2. Since the cycle $1 \to 2 \to 1$ has mean weight of 0 then it is the critical cycle and as every other edge in the digraph is either equal to or less than 0 then the associated matrix must be visualised. Finally we need to check that the matrix is max-balanced and to do that we can check the associated digraph. See that there exist 7 partitions of interest to check, four partitions of the form $\{a\}\{b, c, d\}$ and three partitions in the form $\{a, b\}, \{c, d\}$. We exclude the partition of $\{1, 2, 3, 4\}$, as it is trivial. For the partition of the first form the max weight of arcs leaving and entering the partition is either 0 if $a = 1$ or $a = 2$ or -1 if $a = 3$ or $a = 4$. For the partitions of the second form if $a = 1$ and $b = 2$ or vise versa, the max weight of arcs leaving and entering the partition is -1, if $a = 1$ and $b = 4$ or vice versa the max weight of arcs leaving and entering the partition is 0 and if $a = 1$ and $b = 3$ or vise versa the max weight of arcs leaving and entering the partition is 0. Therefore as all partitions have the same weight entering and leaving the partition the matrix A is max-balanced.

1.2.3 Inhomogeneous Products

This section will introduce some definitions required for Chapter 3. When referring to an *inhomogeneous product* we are talking about the matrix $\Gamma(k) = A_1 \otimes A_2 \otimes \ldots \otimes A_k$ where $A_i \in \mathbb{R}_{\max}^{d \times d}$ are generators from the, in general, infinite matrix set $\mathcal{X}$. We also say that this product has *length* k and this leads us to our first definition.

Definition 1.2.21 (Words). *The* word *associated with the matrix product* $\Gamma(k)$ *is the string of characters* i *from* $A_i \in \mathcal{X}$ *that make up said* $\Gamma(k)$*.*

The following is designed for the use of representing an inhomogeneous product of geometrically equivalent matrices in digraph form.

Definition 1.2.22 ([50],Definition 2.5). *[Trellis Digraph]* The *trellis digraph* $\mathcal{T}_{\Gamma(k)} = (\mathcal{N}, \mathcal{E})$ associated with the product $\Gamma(k) = A_1 \otimes A_2 \otimes \ldots \otimes A_k$ is the digraph with the set of nodes $\mathcal{N}$ and the set of edges $\mathcal{E}$, where:

(1) $\mathcal{N}$ consists of $N+1$ copies of the set of nodes making up any A_l denoted $N_0, \ldots, N_k$, and the nodes in N_l for each $0 \le l \le k$ are denoted by $1 : l, \ldots, d : l$ where $d = |N_l|$;

(2) $\mathcal{E}$ is defined by the following rules:

a) there are edges only between N_l and N_{l+1} for each l,

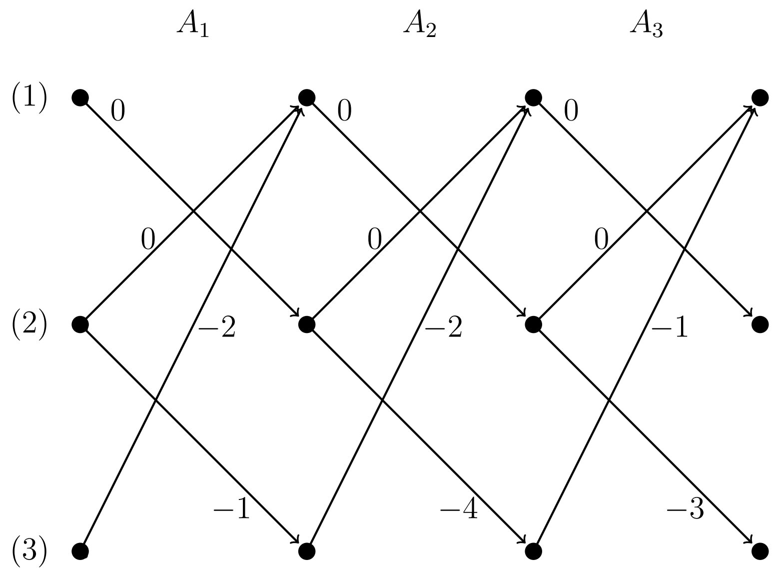
b) we have $(i : (l-1), j : l) \in \mathcal{E}$ if and only if (i, j) is an edge of $\mathcal{D}_{A_l}$, and the weight of that edge is $(A_l)_{i,j}$.

The weight of a walk W on $\mathcal{T}_{\Gamma(k)}$ is denoted by $p_{\mathcal{T}}(W)$.

We now give an example to illustrate the notion of the trellis digraph. Let $\Gamma(k) = A_1 \otimes A_2 \otimes A_3$, where

$$A_1 = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ 0 & \varepsilon & -1 \\ -2 & \varepsilon & \varepsilon \end{pmatrix}, A_2 = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ 0 & \varepsilon & -4 \\ -2 & \varepsilon & \varepsilon \end{pmatrix}, A_3 = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ 0 & \varepsilon & -3 \\ -1 & \varepsilon & \varepsilon \end{pmatrix}. \tag{1.2}$$

Therefore we can draw the trellis digraph as follows:



Here we can see that there are 4 copies of the set of nodes $\{1, 2, 3, 4\}$ and between each set of nodes there exists the edges given in the respective matrix A_i with their associated weights. Note that here the matrices A_1,A_2 and A_3 are strongly geometrically equivalent to each other as the edges connecting the copies of nodes are the same.

The final definition is designed to describe walks on the trellis digraph with weights associated to the inhomogeneous matrix product $\Gamma(k)$.

Definition 1.2.23 ([50],Definition 2.6)**.** *[Walks on $\mathcal{T}_{\Gamma(k)}$]* Consider a trellis digraph $\mathcal{T}_{\Gamma(k)}$.

By an *initial walk* connecting i to j on $\mathcal{T}_{\Gamma(k)}$ we mean a walk on $\mathcal{T}_{\Gamma(k)}$ connecting node $i : 0$ to $j : m$ where m is such that $0 \leq m \leq k$.

By a *final walk* connecting i to j on $\mathcal{T}_{\Gamma(k)}$ we mean a walk on $\mathcal{T}_{\Gamma(k)}$ connecting node $i : l$ to $j : k$, where l is such that $0 \leq l \leq k$.

By a *strict initial walk to the critical nodes* connecting i to j on $\mathcal{T}_{\Gamma(k)}$ we mean a walk on $\mathcal{T}_{\Gamma(k)}$ connecting node $i : 0$ to $j : m$ where m is the first and the last time the walk arrives at node $j \in \mathcal{N}_c$ and is such that $0 \leq m \leq k$.

By a *strict final walk from the critical nodes* connecting i to j on $\mathcal{T}_{\Gamma(k)}$ we mean a walk on $\mathcal{T}_{\Gamma(k)}$ connecting node $i : l$ to $j : k$, where l is the first and the last time the walk leaves node $i \in \mathcal{N}_c$ and is such that $0 \leq l \leq k$. We denote this with $i \rightarrow_{\mathcal{T}} j$.

A *full walk* connecting i to j on $\mathcal{T}_{\Gamma(k)}$ is a walk on $\mathcal{T}_{\Gamma(k)}$ connecting node $i : 0$ to $j : k$.

CHAPTER 2

BOUNDS OF KIM AND SCHWARZ FOR THE PERIODICITY THRESHOLD OF THE TROPICAL MATRIX POWERS

2.1 Introduction

This chapter is based on the results of the joint paper with Merlet, Nowak and Sergeev [48] which are bounds on the ultimate periodicity of a sequence of max-plus matrix powers $\{A^t\}_{t\geq 1}$, developing and improving the bounds obtained earlier by Merlet, Nowak and Sergeev [68]. This work will make use of the CSR decomposition, defined in 2.2.1. The idea of this decomposition is to approximate tropical matrix powers with large exponents by products of the form CS^tR, where C and R are extracted from the Kleene star of the matrix $(A \otimes -\lambda)$ raised to the power equal to the cyclicity of its critical graph, and S is diagonally similar to the associated max-plus Boolean matrix of the critical subgraph of $\mathcal{D}(A)$. This decomposition was formally introduced by H.Schneider and Sergeev [83] and two partial periodicity transients were introduced: $T_1(A, B)$ which we call the weak CSR threshold, and $T_2(A, B)$ which we call the Strong CSR threshold, where A is the given matrix and B is a subordinate matrix developed following the Nachtigall, Hartman-Arguelles or Cycle Threshold schemes, all of these will be formally defined later in this section. As shown by Merlet et al. [68], the periodicity transient $T(A)$ is then bounded by $\max(T_1(A, B), T_2(A, B))$.

The results in this chapter are published in the paper [48] and, for most of them, the formulations were obtained as a result of the joint work with the co-authors and the proofs are the results of independent work. There are also results that were obtained exclusively by the co-authors and these will be explicitly stated.

The chapter will proceed as follows. For the rest of the introduction, the definition of the CSR decomposition will be given with some key properties from the original work by H.Schneider and Sergeev [83]. The results from Merlet et al. [68] will also be presented as they are the initial bounds which the results of this chapter stem from. In Section 2.3 bounds using the Nachtigall decomposition will be presented. Then in Section 2.4 the Cycle Removal Threshold will be introduced and some bounds on it will be developed. The Cycle Removal Threshold bounds and other results are then utilised in Section 2.5 to develop bounds on $T_1(A, B)$. Then the Cycle Removal Threshold results are used to develop new bounds for $T_2(A, B)$ in Section 2.6. Finally we will present an example which shows a practical application of the bounds developed in this chapter. In this final example, the bounds developed in this Chapter help to show that the tropical matrix powers (in the case considered) are periodic from the very beginning, while the previous bounds of [48, 68] fail to imply this property.

2.2 Preliminary results

To begin, a formal introduction of the CSR decomposition is required.

Definition 2.2.1. *Let $A \in \mathbb{R}_{\max}^{n\times n}$ with $\lambda(A) = \lambda$. Consider the critical graph $\mathcal{C} = (\mathcal{N}_c, \mathcal{E}_c)$ of the digraph of A. Let $G = (g_{i,j})$ be the Kleene star $((A \otimes \lambda^-)^\sigma)^*$ (by Definition 1.2.20) where σ is the cyclicity of the critical graph. Define the matrices $C = (c_{i,j}), S = (s_{i,j}), R = (r_{i,j}) \in \mathbb{R}_{\max}^{n\times n}$ as:*

$$c_{i,j} = \begin{cases} g_{i,j}, & \text{if } j \in \mathcal{N}_c \\ \varepsilon & \text{otherwise;} \end{cases}$$

$$s_{i,j} = \begin{cases} a_{i,j}, & \text{if } (i,j) \in \mathcal{E}_c \\ \varepsilon & \text{otherwise;} \end{cases}$$

$$r_{i,j} = \begin{cases} g_{i,j}, & \text{if } i \in \mathcal{N}_c \\ \varepsilon & \text{otherwise.} \end{cases}$$

Then, for any subordinate matrix B we say that A^t admits a weak CSR expansion with this B if for some integer T we have

$$A^t := C \otimes S^t \otimes R \oplus B^t, \quad \forall t \geq T.$$

We denote the product $C \otimes S^t \otimes R$ derived from A as $CS^tR[A]$.

This definition can be extended to the s.c.c.s of the critical graph of A.

Definition 2.2.2. *[68] Let $\mathcal{G}_1, \ldots, \mathcal{G}m$ be the distinct s.c.c.s of $\mathcal{G}^c(A)$ with node sets $N_1, \ldots, N_m$ such that $N_1 \cup \ldots \cup N_m = \mathcal{N}_c$. We define the* CSR decomposition with respect to $\mathcal{G}_\nu$ *as the CSR decomposition following Definition 2.2.1 using the subordinate matrix* $A^{(\nu)}$*, which sets all entries in the columns and rows associated with the nodes* $N_1, \ldots, N_{\nu-1}$ *to* ε*. We denote the three matrices made using this decomposition as* C_ν*,* S_ν *and* R_ν*.*

Naturally this has the following result in association with Definition 2.2.1.

Lemma 2.2.3. *[68, Corollary 6.3] If* $\mathcal{G}_1, \ldots, \mathcal{G}_m$ *are the s.c.c.s of* $\mathcal{G}^c(A)$*, then we have:*

$$CS^tR = \bigoplus_{\nu=1}^{m} C_\nu S_\nu^t R_\nu$$

It should be noted that many of the following properties, while developed for Definition 2.2.1, have their analogues for Definition 2.2.2. When A is irreducible and using [68, Theorem 2.2] we have the following statement about CSR:

Lemma 2.2.4 ([48, Lemma 2.2])**.** *Let* $A \in \mathbb{R}_{\max}^{d\times d}$ *such that* $\lambda(A) = 0$*. Then for any natural* t *we have*

$$\lim_{k\to\infty} A^{t+\gamma k} = CS^tR[A] \tag{2.1}$$

where σ *is the cyclicity of* $\mathcal{G}(A)$*.*

We now present some key CSR properties from H.Schneider and Sergeev [83] and we denote $\mathcal{P}^{(t)} := CS^tR$ for CSR as defined in Definition 2.2.1. The proofs will be omitted here and can be found in the article [83].

Theorem 2.2.5 ([83, Theorem 3.3])**.** *Let* $A \in \mathbb{R}_{\max}^{n\times n}$ *have* $\lambda(A) = 0$ *and let* $T \geq 0$ *be such that the sequence* $\{S^{\otimes t}\}_{t\geq T}$*, where* S *is defined from from 2.2.1, is periodic with period* γ*.*

1. For $t \geq 0$,

$$p\left(\mathcal{W}^t(i \xrightarrow{\mathcal{N}_c} j)\right) \leq \mathcal{P}_{i,j}^{(t)} \quad \forall i, j. \tag{2.2}$$

2. For $t \geq T + 2\tau(n-1)$,

$$p\left(\mathcal{W}^t(i \xrightarrow{\mathcal{N}_c} j)\right) \geq \mathcal{P}_{i,j}^{(t)} \quad \forall i, j, \tag{2.3}$$

where τ is the maximal cyclicity of the strongly connected components of $\mathcal{C}(A)$.

The previous theorem provides an optimal walk interpretation of CSR. However, it starts to work only after a certain bound. A different approach is taken by Merlet et al. [68], where a different optimal walk interpretation was established not involving any bounds. Note that we are giving here a slightly simplified version of the original result of in the article [68].

Theorem 2.2.6 ([68, Theorem 6.1]). *Let $A \in \mathbb{R}_{\max}^{n\times n}$ be a matrix with $\lambda(A) = 0$ and C, S and R be defined as in Definition 2.2.1.*

Let γ be an integer multiple of $\gamma(\mathcal{G}^c)$ and $\mathcal{N}$ be a set of critical nodes that contains at least one node from each s.c.c of $\mathcal{G}^c$.

Then we have, for any i, j and $t \in \mathbb{N}$:

$$\mathcal{P}_{i,j}^{(t)} = p\left(\mathcal{W}^{t,\gamma}(i \xrightarrow{\mathcal{N}} j)\right) \tag{2.4}$$

where $\mathcal{W}^{t,\gamma}(i \xrightarrow{\mathcal{N}} j) := \{W \in \mathcal{W}(i \xrightarrow{\mathcal{N}} j) | l(W) \equiv t (\operatorname{mod} \gamma)\}$.

The following properties show that CSR products form a cyclic group. Here γ is the same as in the previous theorem.

Proposition 2.2.7 ([83, Proposition 3.2])**.** *$\mathcal{P}^{(t+\gamma)} = \mathcal{P}^{(t)}$ for all $t \geq 0$.*

Theorem 2.2.8 ([83, Theorem 3.4])**.** *$\mathcal{P}^{(t_1+t_2)} = \mathcal{P}^{(t_1)}\mathcal{P}^{(t_2)}$ for all t_1,$t_2 \geq 0$.*

Next, we also have the following properties, which describe the role of cyclic classes and state that the non-critical rows and columns can be obtained as tropical linear combinations of rows and columns with indices in $\mathcal{N}_c$.

Theorem 2.2.9 ([83, Theorem 3.6])**.** *Let $A \in \mathbb{R}_{\max}^{n\times n}$ have $\lambda(A) = 0$. If $[i] \to_t [j]$, then*

$$\mathcal{P}_{i\cdot}^{(t+s)} = \mathcal{P}_{j\cdot}^{(s)} \ , \ \mathcal{P}_{\cdot i}^{(s)} = \mathcal{P}_{\cdot j}^{(t+s)}, \tag{2.5}$$

where $\mathcal{P}_{i\cdot}^{(t)}$ is the i^{th} row of $\mathcal{P}^{(t)}$ and $\mathcal{P}_{\cdot j}^{(t)}$ is the j^{th} column of $\mathcal{P}^{(t)}$.

Corollary 2.2.10 ([83, Corollary 3.7])**.** *Let $A \in \mathbb{R}_{\max}^{n\times n}$ have $\lambda(A) = 0$. Then $\mathcal{P}_{i\cdot}^{(t)} = (S^tR)_{i\cdot}$ and $\mathcal{P}_{\cdot i}^{(t)} = (CS^t)_{\cdot i}$ for all $i \in \mathcal{N}_c$.*

Corollary 2.2.11 ([83, Corollary 3.8])**.** *Let $A \in \mathbb{R}_{\max}^{n\times n}$ have $\lambda(A) = 0$. For each $k = 1, \ldots, n$ there exist α_{ik} and β_{ki} where $i \in \mathcal{N}_c$, such that*

$$\mathcal{P}_{\cdot k}^{(t)} = \bigoplus_{i\in\mathcal{N}_c} \alpha_{ik}\mathcal{P}_{\cdot i}^{(t)} \ , \ \mathcal{P}_{k\cdot}^{(t)} = \bigoplus_{i\in\mathcal{N}_c} \beta_{ki}\mathcal{P}_{i\cdot}^{(t)} \tag{2.6}$$

We now define the two threshold types which will be explored throughout the chapter.

Definition 2.2.12. *Let $A \in \mathbb{R}_{\max}^{d\times d}$ and let B be any subordinate matrix to A. The* Weak CSR Threshold *denoted by $T_1(A, B)$ least number T satisfying,*

$$\forall\, t \geq T, \quad A^t = CS^tR[A] \oplus B^t. \tag{2.7}$$

Definition 2.2.13. *Let* $A \in \mathbb{R}_{\max}^{d\times d}$ *be irreducible and let* B *be any subordinate matrix to* A. *The* Strong CSR Threshold *denoted by* $T_2(A,B)$ *is the smallest number* T *satsfying,*

$$\forall\, t \geq T, \quad CS^tR[A] \geq B^t. \tag{2.8}$$

The subordinate matrix B can be defined in a different ways which we refer to as schemes. For all of the schemes the subordinate matrix is created from a well-defined subgraph $\mathcal{G}$ of $\mathcal{D}(A)$ under the following rule,

$$b_{i,j} = \begin{cases} \varepsilon & \text{if } i \text{ or } j \text{ is a node of } \mathcal{G} \\ a_{i,j} & \text{otherwise.} \end{cases} \tag{2.9}$$

The following three schemes come from Merlet et al. [68] and will be used throughout the chapter.

(B_N) *Nachtigall Scheme:* For this scheme we set $\mathcal{G} = \mathcal{G}^c(A)$ (The critical subgraph of $\mathcal{D}(A)$). Then B_N is defined using (2.9).

(B_{HA}) *Hartman-Arguelles Scheme:*Let V be the max-balanced form of A (see Definition 1.2.18). For some $\mu \in \mathbb{R}_{\max}$ the Hartman-Arguelles threshold graph $\mathcal{D}_{HA}(\mu)$ is the digraph induced from the edges in $\mathcal{D}(V)$ where $v_{i,j} \geq \mu$. In the case when $\mu = \lambda(A)$ then $\mathcal{D}_{HA}(\mu) = \mathcal{G}^c(A)$. If $\mu = \varepsilon$ then $\mathcal{D}_{HA}(\mu) = \mathcal{D}(A)$. Let μ^{ha} be the maximal $\mu \leq \lambda(A)$ such that $\mathcal{D}_{HA}(\mu^{ha})$ has a s.c.c. that does not contain any s.c.c.s of $\mathcal{G}^c(A)$. Then the digraph $\mathcal{G}$ is the union of the s.c.c.s of $\mathcal{D}_{HA}(\mu^{ha})$ that contain the components of $\mathcal{G}^c(A)$ and B_{HA} is defined by 2.9.

(B_{CT}) *Cycle Threshold Scheme:* Let $\mu \in \mathbb{R}_{\max}$, define the Cycle Threshold graph

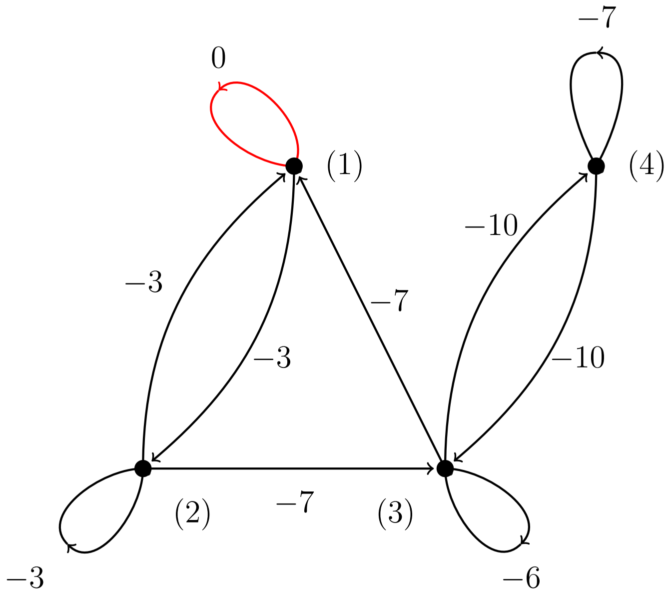
$\mathcal{D}_{CT}(\mu)$ induced by the edges belonging to the cycles of $\mathcal{D}(A)$ with mean weight greater than or equal to μ. As with the Harman-Arguelles scheme if $\mu = \lambda(A)$ then $\mathcal{D}_{CT}(\mu) = \mathcal{G}^c(A)$ and if $\mu = \varepsilon$ then $\mathcal{D}_{CT}(\mu) = \mathcal{D}(A)$. Let μ^{ct} be the maximal $\mu \leq \lambda(A)$ such that $\mathcal{D}_{CT}(\mu^{ct})$ has a s.c.c. that does not contain any s.c.c.s of $\mathcal{G}^c(A)$. Then the digraph $\mathcal{G}$ is the union of s.c.c.s of $\mathcal{D}_{CT}(\mu^{ct})$ that contain the components of $\mathcal{G}^c(A)$ and B_{CT} is defined by 2.9.

The subdigraphs defined by the Harmann-Arguelles scheme and the Cycle Threshold scheme will be denoted by $\mathcal{G}^{ha}$ and $\mathcal{G}^{ct}$.

We now present a brief example of these decomposition schemes at work. Let $A \in \mathbb{R}_{\max}^{4\times 4}$ be a max-balanced matrix with entries

$$A = \begin{pmatrix} 0 & -3 & \varepsilon & \varepsilon \\ -3 & -3 & -7 & \varepsilon \\ -7 & \varepsilon & -6 & -10 \\ \varepsilon & \varepsilon & -10 & -7 \end{pmatrix}$$

and its associated weighted digraph.



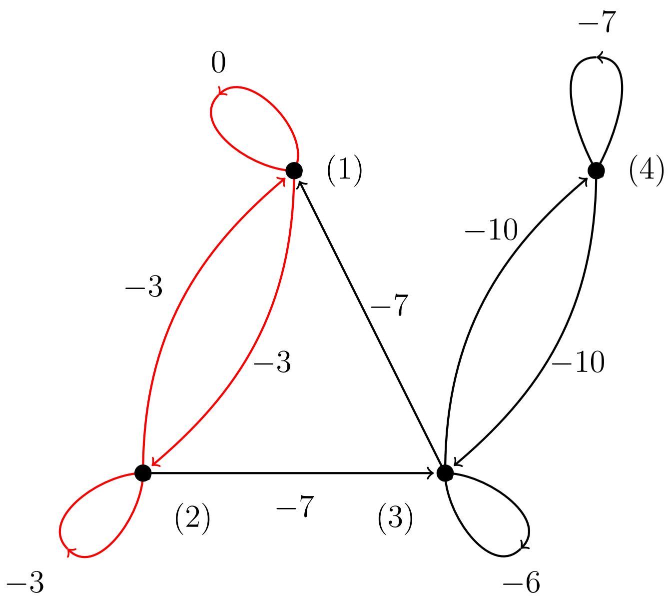
We can see that $\mathcal{G}^c(A)$ is the loop (in red) $1 \to 1$ (as that cycle has the largest mean weight which is 0) and that defines the Nachtigall decomposition for A as

$$B_N = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & -3 & -7 & \varepsilon \\ \varepsilon & \varepsilon & -6 & -10 \\ \varepsilon & \varepsilon & -10 & -7 \end{pmatrix}.$$

Now looking at the Hartman-Arguelles scheme we set $\mu^{ha} = -6$ which gives a subgraph of the loop $3 \to 3$ which is a s.c.c that does not contain $\mathcal{G}^c(A)$. Following the definition, we have the decomposition as

$$B_{HA} = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & -6 & -10 \\ \varepsilon & \varepsilon & -10 & -7 \end{pmatrix}$$

with $\mathcal{G}^{ha}$ (in red) shown in the following digraph:

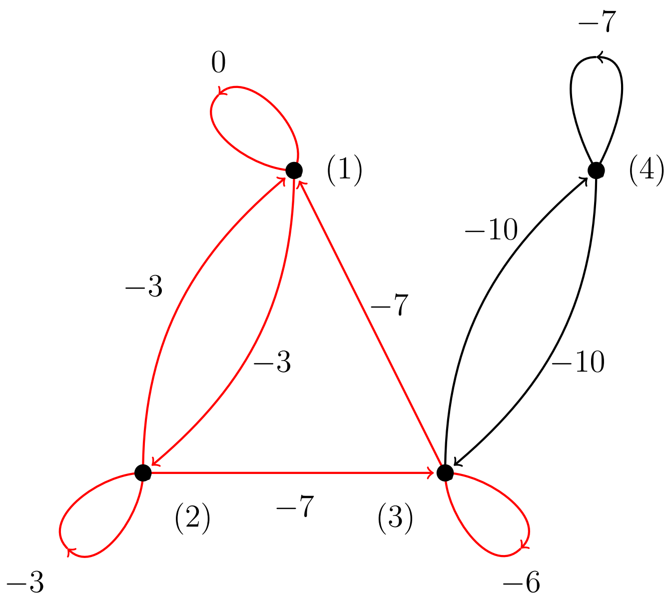


This happens as node 3 is connected to nodes 1 and 2 by two edges with weights equal to -7. Therefore the loop $3 \to 3$ is disconnected when $\mu^{ha} = -6$. Finally for the Cycle Threshold scheme we can set $\mu^{ct} = -7$ and that gives us a subgraph that includes the cycle $4 \to 4$ which is a s.c.c that does not contain $\mathcal{G}^c(A)$. Hence the Cycle

Threshold decomposition of A is

$$B_{CT} = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & -7 \end{pmatrix}.$$

with $\mathcal{G}^{ct}$ (in red) shown in the following digraph:



For this decomposition if we had $\mu^{ct} = -6$ the cycle $1 \to 2 \to 3 \to 1$ has mean weight $-\frac{17}{3}$ which is just larger than -6 so the subgraph $\mathcal{G}^{ct}$ would contain $\mathcal{G}^{c}(A)$. Therefore we move $\mu^{ct} = -7$ which now contains the cycle $4 \to 4$. Since the edges connecting nodes 3 to 4 both have weight 10 then this s.c.c does not contain $\mathcal{G}^{c}(A)$ and we get the distinct decomposition.

The following statement holds in the particular case of the Nachtigall expansion and was proved by my co-authors in the paper [48].

Lemma 2.2.14 ([48, Lemma 2.3]). *Let A be irreducible. Then $A^t \geq CS^tR[A]$ if and only if $t \geq T_{1,N}(A)$.*

We also present some previous results for the bounds of $T_1(A, B)$ from the paper by Merlet, Nowak and Sergeev [68]. The results for $T_2(A, B)$ from the same paper will be presented later in this chapter.

Theorem 2.2.15 ([68, Theorem 4.1]). *For any matrix $A \in \mathbb{R}_{\max}^{d\times d}$ and for $B = B_N$, $B = B_{HA}$ or $B = B_{CT}$ we have the following bound*

$$T_1(A, B) \leq \mathrm{Wi}(d) = (d-1)^2 + 1 \qquad \text{for } d > 1 \tag{2.10}$$

If $B = B_N$ or $B = B_{HA}$, we have the following bounds

$$T_1(A, B) \leq \mathrm{DM}(d, \hat{\mathrm{g}}) = \hat{\mathrm{g}}(d-2) + d \tag{2.11}$$

$$T_1(A, B) \leq (\hat{\mathrm{g}} - 1)(\mathrm{cr} - 1) + (\hat{\mathrm{g}} + 1)\mathrm{cd} \tag{2.12}$$

where $\hat{\mathrm{g}} = \hat{\mathrm{g}}(\mathcal{G}^c(A))$ is the max-girth of $\mathcal{G}^c(A)$, $\mathrm{cr} = \mathrm{cr}(\mathcal{D}(A))$ is the length of the longest elementary cycle in the associated digraph of A, $\mathrm{cd} = \mathrm{cd}(\mathcal{D}(A))$ is the length of the longest path in the associated digraph of A.

Theorem 2.2.16 ([68, Theorem 4.4]). *For any matrix $A \in \mathbb{R}_{\max}^{d\times d}$, we have the following bounds*

$$T_1(A, B_{CT}) \leq \mathrm{Wi}(d) \tag{2.13}$$

$$T_1(A, B_{CT}) \leq (d-1)\mathrm{cr} + \min(d, \mathrm{cd} + \mathrm{cr} + 1) \tag{2.14}$$

$$T_1(A, B_{CT}) \leq (\mathrm{cd} + \mathrm{cr} - 1)\mathrm{cr} + \mathrm{cd} + 1 \tag{2.15}$$

where $\mathrm{cr} = \mathrm{cr}(\mathcal{D}(A))$ *and* $\mathrm{cd} = \mathrm{cd}(\mathcal{D}(A))$.

When speaking about $T_1(A, B)$ we will also use the following simplified notation:

$$T_{1,HA}(A) = T_1(A, B_{HA}(A)),\ T_{1,CT}(A) = T_1(A, B_{CT}(A)).$$

Let $A_1, A_2, \cdots A_\gamma$ be matrices with entries in $\mathbb{R}_{\max}$, such that the product $A_i A_{i+1}$ is well defined for each i (with the indices considered modulo γ) and define

$$A := \begin{pmatrix} -\infty & A_1 & -\infty & \cdots & -\infty \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & A_{\gamma-2} & -\infty \\ -\infty & \cdots & \cdots & -\infty & A_{\gamma-1} \\ A_\gamma & -\infty & \cdots & \cdots & -\infty \end{pmatrix}. \tag{2.16}$$

This form can be written for any tropical matrix, with γ as the cyclicity of the associated graph.This can be done by grouping the nodes into distinct cyclic classes. As the edges moving from $[i]$ to $[i+1]$ represent the edges making up a single block then by reordering the columns/rows one can write the matrix in the form (2.16).

2.3 The case of Nachtigall expansion

Throughout the chapter we will be interested in the tropical powers of A and their limits. These matrices always have a block decomposition compatible with (2.16) and at most one non-zero block on each row. We denote by A_i the only possibly non-zero block of A on row i, and all indices in what follows are always considered modulo γ. This is consistent with (2.16) and for instance, A^γ is block diagonal with $A_i^\gamma = A_i \cdots A_{i+\gamma-1}$. This is a version of the more general form of identity $A_i^v = A_i \ldots A_{i+v-1}$ for any $v \geq 0$ which is the same as saying that the i^{th} block in A^v is equal to the product of matrix blocks $A_i \otimes A_{i+1} \otimes \ldots \otimes A_{i+v-1}$.

By construction, the nodes of $\mathcal{D}(A)$, where A is of the form (2.16), can be split into γ sets N_i that are cyclically ordered such that an arc always goes from one set to the next one and an arc from N_i to N_{i+1} is labelled by an entry of A_i. Therefore, the cycles on $\mathcal{D}(A)$ contain nodes from each N_i and the length is an integer multiple of γ. Moreover, a cycle with length l with maximal average weight on $\mathcal{D}(A)$ gives a cycle with length l/γ with maximal average weight on each $\mathcal{D}(A_i^\gamma)$. This is the idea behind the proof of the following proposition. Note that its proof was omitted in the joint paper [48], but we will give it here in full.

Proposition 2.3.1 ([48],Proposition 2.1)**.** *Let γ be such that A admits a block decomposition as in (2.16):*

(1) $\lambda(A_i^\gamma) = \gamma \cdot \lambda(A)$*;*

(2) $\hat{\mathrm{g}}(\mathcal{G}^c(A_i^\gamma)) = \frac{\hat{\mathrm{g}}(\mathcal{G}^c(A))}{\gamma}$*;*

(3) $c(\mathcal{G}(A_i^\gamma)) = \frac{c(\mathcal{G}(A))}{\gamma}$.

Here $\lambda(B)$ is the maximal average weight of cycles (or circuits) on $\mathcal{D}(B)$ for any $B \in \mathbb{R}_{\max}^{d\times d}$, which is the largest tropical eigenvalue of B, $\hat{\mathrm{g}}(\mathcal{D})$ the maximum of the girths (length of shortest cycle) of its s.c.c.s, and $c(\mathcal{D})$ is the cyclicity for arbitrary digraph $\mathcal{D}$.

Proof. The proof of (1) is given in [11, Theorem 4.5.10] and will be omitted. For the proof of (2) we show the following two inequalities:

$$\hat{\mathrm{g}}(\mathcal{G}^c(A_i^\gamma)) \geq \frac{\hat{\mathrm{g}}(\mathcal{G}^c(A))}{\gamma} \tag{2.17}$$

$$\hat{\mathrm{g}}(\mathcal{G}^c(A_i^\gamma)) \leq \frac{\hat{\mathrm{g}}(\mathcal{G}^c(A))}{\gamma} \tag{2.18}$$

To prove (2.17) we choose the elementary cycle V with length l from $\mathcal{G}^c(A_i^\gamma)$ that attains the maximal girth over all elementary cycles in $\mathcal{G}^c(A_i^\gamma)$. By [7, Lemma 2.8], the critical graph of a power of a matrix is the same as the power of the critical graph of the matrix. Therefore there exists a cycle on $\mathcal{G}^c(A)$ that has the same weight, traverses all nodes of V (albeit also with nodes from other A_i in between), and has length γl. This cycle can be split into a number of elementary cycles and the length of each of them does not exceed $\gamma\hat{\mathrm{g}}(\mathcal{G}^c(A_i^\gamma))$. Therefore we have (2.17).

Looking to (2.18) we choose the cycle of length l that attains the maximal girth over $\mathcal{G}^c(A)$. Upon powering up the matrix A γ times, the cycle will split into γ disjoint cycles, each of length $\frac{l}{\gamma}$. These cycles will be critical in A^γ as the power of a critical graph of a matrix is the critical graph of a power of the same matrix. If all of these cycles are elementary, then clearly $\hat{\mathrm{g}}(A_i^\gamma) \leq \frac{\hat{\mathrm{g}}(\mathcal{G}^c(A))}{\gamma}$. If some of these cycles are not elementary then we can take one and break it down into elementary cycles of either smaller or equal length and this also gives $\hat{\mathrm{g}}(A_i^\gamma) \leq \frac{\hat{\mathrm{g}}(\mathcal{G}^c(A))}{\gamma}$. Hence for either situation we have (2.18).

Therefore as both (2.17) and (2.18) hold then equality (2) holds.

For the proof of (3) we use [12, Theorem 2.1 (ii)]. This states that the node set of each component of A^γ consists of $\frac{c(\mathcal{D}(A))}{\gcd(\gamma,c(\mathcal{D}(A))}$ cyclic classes of A. It can be seen that, if γ is such that A admits block decomposition (2.16), then the length of all cycles of $\mathcal{D}(A)$ is a multiple of γ, therefore $\gcd(\gamma, c(\mathcal{D}(A))) = \gamma$ and the result follows. $\square$

We have the following straightforward relations where A_i and A_j are arbitrary blocks from (2.16).

$$A_i^{\gamma k+s+t} = A_i^s (A_{i+s}^\gamma)^k A_{i+s}^t \tag{2.19}$$

$$(A_i^\gamma)^{k+1} = A_i^{j-i} (A_j^\gamma)^k A_j^{i-j} \tag{2.20}$$

Using Lemma 2.2.4 we obtain the following identities as limits of (2.19) and (2.20):

$$CS^{\gamma k+s+t} R[A]_i = A_i^s CS^k R[A_{i+s}^\gamma] A_{i+s}^t \tag{2.21}$$

$$CS^{k+1} R[A_i^\gamma] = A_i^{j-i} CS^k R[A_j^\gamma] A_j^{i-j} \tag{2.22}$$

The following Lemma, Theorem 2.3.3 and Theorem 2.3.4 were hypothesised and proved by the coauthors [48] therefore we will state the results and omit the proofs where necessary. We direct the reader to the paper [48] for the complete proofs. Without loss of generality we can assume that $\lambda(A) = 0$ and by Proposition 2.3.1 we also have $\lambda(A_i^\gamma) = 0$ for every i.

Lemma 2.2.14 is used to deduce the following.

Lemma 2.3.2 ([48],Lemma 3.1)**.** *The following two relations hold for all* $i, j \in \{1, \ldots, \gamma\}$ *where* γ *is of* (2.16)*:*

(i) $T_{1,N}(A) \leq \gamma \max_i T_{1,N}(A_i^\gamma)$,

(ii) $|T_{1,N}(A_i^\gamma) - T_{1,N}(A_j^\gamma)| \leq 1$ *for arbitrary* i, j.

It should be reiterated that the proof given below was written by the co-authors [48], but it will be included to highlight the use of Proposition 2.3.1.

Theorem 2.3.3 ([48],Theorem 3.2)**.** *Let $A \in \mathbb{R}_{\max}^{d\times d}$ be irreducible. Denote by γ the cyclicity of $\mathcal{D}(A)$ and by $\hat{\mathrm{g}}$ the maximal girth of s.c.c.s of $\mathcal{G}^c(A)$. We denote by $d \operatorname{rem} \gamma$ the remainder of the Euclidean division of d by γ. The following upper bounds on $T_{1,N}(A)$ hold:*

(i) $\gamma \cdot \mathrm{Wi}\left(\left\lfloor \frac{d}{\gamma} \right\rfloor\right) + (d \operatorname{rem} \gamma)$*;*

(ii) $\hat{\mathrm{g}} \cdot \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 2\right) + d$*.*

Proof. The bounds follow from the application of bounds of Theorem 2.2.15 to the A_i^γ with minimal size. This size m is at most $\left\lfloor \frac{d}{\gamma} \right\rfloor$. When it is at most $\left\lfloor \frac{d}{\gamma} \right\rfloor - 1$, the bounds follow from the inequalities of Lemma 2.3.2. When $m = \left\lfloor \frac{d}{\gamma} \right\rfloor$, we use the fact that at most $d \operatorname{rem} \gamma$ blocks have a strictly larger size (otherwise the total size would be larger than d). In this case, we set

$$k = \max_{A_i^\gamma \text{ has size } m} T_{1,N}(A_i^\gamma).$$

Using (2.19) and (2.22) with k as above and $s+t = d \operatorname{rem} \gamma$ and applying Lemma 2.2.14 we obtain

$$A_i^{\gamma k+s+t} = A_i^s (A_{i+s}^\gamma)^k A_{i+s}^t \geq A_i^s C S^k R[A_{i+s}^\gamma] A_{i+s}^t = C S^{\gamma k+s+t} R[A]_i.$$

In the above, we can select s in such a way that A_{i+s}^γ has size $m = \left\lfloor \frac{d}{\gamma} \right\rfloor$. Applying

Lemma 2.2.14 again, we obtain that

$$T_{1,N}(A) \leq \gamma \max_{A_i^\gamma \text{ has size } m} T_{1,N}(A_i^\gamma) + d \operatorname{rem} \gamma.$$

Using Wielandt and Dulmage-Mendelsohn bounds for such blocks together with Proposition 2.3.1 we obtain that

$$\begin{aligned} T_{1,N}(A) &\leq \gamma \operatorname{Wi}\left(\left\lfloor \frac{d}{\gamma} \right\rfloor\right) + d \operatorname{rem} \gamma. \\ T_{1,N}(A) &\leq \gamma \left(\frac{\hat{\mathrm{g}}}{\gamma}\left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 2\right) + \left\lfloor \frac{d}{\gamma} \right\rfloor\right) + d \operatorname{rem} \gamma \\ &= \hat{\mathrm{g}} \cdot \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 2\right) + d, \end{aligned}$$

which concludes the proof. □

The following theorem is the immediate result of applying Lemma 2.3.2[(ii)] to the matrix F defined in (1.1) using the bounds in Theorem 2.2.15 and Theorem 2.3.3. Recall the notion of factor rank introduced in Definition 1.2.4.

Theorem 2.3.4 ([48],Theorem 3.4). *Let $A \in \mathbb{R}_+^{d \times d}$ be irreducible. Let r be the factor rank of A, γ the cyclicity of $\mathcal{D}(A)$, and $\hat{\mathrm{g}}$ the max-girth of $\mathcal{G}^c(A)$.*

The following upper bounds on $T_{1,N}$ hold:

(i) $\operatorname{Wi}(r) + 1$;

(ii) $\hat{\mathrm{g}} \cdot (r-2) + r + 1$.

(iii) $\gamma . \operatorname{Wi}\left(\left\lfloor \frac{r}{\gamma} \right\rfloor\right) + (r \operatorname{rem} \gamma) + 1$;

(iv) $\hat{\mathrm{g}} \cdot \left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 2\right) + r + 1$.

The first two bounds apply to reducible matrices as well.

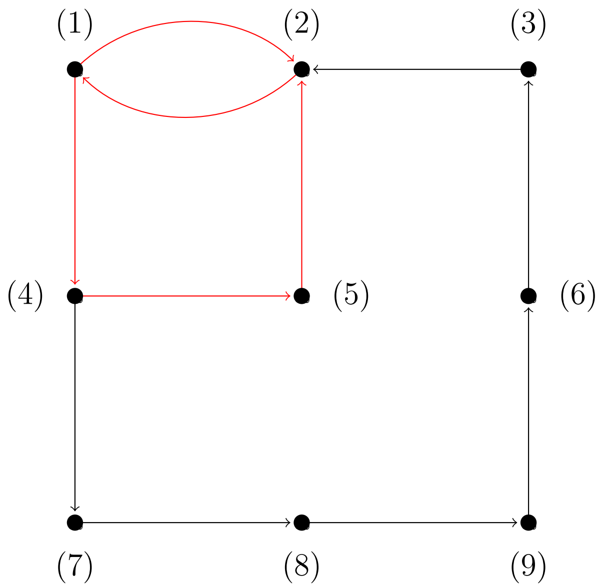
2.4 Bounds for the Cycle Removal Threshold

In this section we introduce the concept of the cycle removal threshold and prove some new bounds for it to be used throughout the chapter.

Definition 2.4.1 ([48],Definition 4.1)**.** *Let $\mathcal{G}$ be a subgraph of $\mathcal{D}(A)$ and $\gamma \in \mathbb{N}$.*

The cycle removal threshold $T_{cr}^{\gamma}(A, \mathcal{G})$, *of $\mathcal{G}$ is the smallest nonnegative integer T for which the following holds: for all walks $W \in \mathcal{W}(i \xrightarrow{\mathcal{G}} j)$ with length $\geq T$, there is a walk $V \in \mathcal{W}(i \xrightarrow{\mathcal{G}} j)$ obtained from W by possibly removing cycles of W and possibly inserting cycles of $\mathcal{G}$ such that $l(V) \equiv l(W) \pmod{\gamma}$ and $l(V) \leq T$.*

We can give an example to explain this definition. Let $\mathcal{D}(A)$ be the digraph shown below.



If we define the subgraph of critical nodes $\mathcal{G}$ to be the subgraph in red then we start by calculating $T^2_{cr}(A, \mathcal{G})$. As we can separate the nodes into critical and non-critical sets then we can look at certain types of walks, namely critical to critical walks, critical to non-critical walks (and vice versa) and non-critical to non-critical walks. We can group the first three types of walks into walks with at least one critical node at the ends and we can rename the final group into walks with no critical nodes at either end. We will start with a walk with at least one end that is critical. If this walk contains any cycle then we can delete it as all cycles have length modulo 2 and the walk will still traverse $\mathcal{G}$. Therefore the longest possible walk must be the longest possible path where one end is non-critical and the other is critical. This will have length 8 ($7 \to 5$) and acts as a lower bound for $T^2_{cr}(A, \mathcal{G})$. Turning our attention to walks with no critical nodes at either end we suggest the walk consisting of the cycle of length 8 from node 7 to itself (denoted by $7 \to 7$ for brevity) and the path $7 \to 8 \to 9 \to 6 \to 3$ (further denoted by $7 \to 3$). This walk does contain a cycle of length 8 which we cannot delete as if it was removed, then the path $7 \to 3$ would not traverse $\mathcal{G}$. This is the longest possible walk of this nature as $7 \to 3$ is the longest path not containing any critical nodes. This walk has length 12 and also acts as a lower bound for $T^2_{cr}(A, \mathcal{G})$. Any walk of length greater than 12 starting and ending on a non-critical node must contain at least two cycles of length 8 or multiple cycles of length 2 or 4 and a cycle of length 8. Therefore, one of those cycles can be removed to give a walk of length less than or equal to 12. As 12 is the larger value over the two groups then $T^2_{cr}(A, \mathcal{G}) = 12$.

Looking at $T^4_{cr}(A, \mathcal{G})$ we can take the same look at the groups of walks. Starting with walks with at least one critical node at an end we have to now account for a potential odd number of cycles of length 2 as only two cycles of length 2 can be removed for $T_{cr}^4(A, \mathcal{G})$. We can now consider the walk 7 to 5 which comprises of a path from 7 to 5 and a cycle of length 2 from 1 to 1 inserted in the path. This walk has length 10 and as there is only one cycle of length 2 it cannot be removed. This is the largest possible walk of this nature and any walk of length greater than 10 with at least one end at a critical node will either have to contain a cycle of length 4, more than two cycles of length 2 or a cycle of length 8. All of these cycles can be removed to give a walk of length less than or equal to 10. Looking at walks with no critical nodes at either end we can take the same walk $7 \to 3$ considered above and add in a single cycle of length 2 when the walk traverses those nodes. This gives a walk of length 14 and no number of cycles of total length being a multiple of 4 can be removed to give a walk that traverses $\mathcal{G}$. On the other hand, if we have any walk of length greater than 14 connecting two non critical nodes then we have at least two cycles of length 8 or multiple cycles of length 2 (in pairs)/4 and a cycle of length 8. Therefore these cycles can be removed to give a walk that is less than or equal to 14 which implies that T_{cr}^4 does not exceed $\max(10, 14) = 14$. Hence $T_{cr}^4(A, \mathcal{G}) = 14$.

We bound $T_{cr}^{\gamma_l}(\mathcal{G}_l)$ thanks to the following proposition.

Proposition 2.4.2. *[68, Proposition 9.5] Let $A \in \mathbb{R}_{\max}^{d \times d}$ and $\mathcal{G}$ be a subgraph of $\mathcal{D}(A)$ with d_1 nodes. Then*

$$\forall \gamma \in \mathbb{N}, \quad T_{cr}^{\gamma}(A, \mathcal{G}) \leq \gamma d + d - d_1 - 1.$$

In the paper [48] this bound was developed further. The formulation of the theorem below was worked out in collaboration with the co-authors [48], and the proof was written independently.

Theorem 2.4.3 ([48],Proposition 4.5)**.** *Let $A \in \mathbb{R}_{\max}^{d\times d}$ be irreducible and let $\mathcal{G}$ be a strongly connected subgraph of $\mathcal{D}(A)$. Then*

$$T_{cr}^{\sigma}(A,\mathcal{G}) \le \sigma \left\lfloor \frac{d}{\gamma} \right\rfloor + d - \sigma - 1, \tag{2.23}$$

where γ is the cyclicity of $\mathcal{D}(A)$ and σ is the cyclicity of $\mathcal{G}$.

Proof. Let m be the size of the smallest cyclic class of $\mathcal{D}(A)$.

Let us consider a walk $W \in \mathcal{W}^{t,\sigma}(i \xrightarrow{\mathcal{G}} j)$. If W does not go through all nodes of $\mathcal{G}$, then we can insert cycles from $\mathcal{G}$ in it so that the new walk contains all nodes of $\mathcal{G}$ and still belongs to $\mathcal{W}^{t,\sigma}(i \xrightarrow{\mathcal{G}} j)$.

Let C_k be the first cyclic class of size m encountered by W. The digraph $\mathcal{D}(A^{\gamma})$ consists of γ isolated s.c.c.s, whose node sets are the cyclic classes of $\mathcal{D}(A)$. Denote by A_k^{γ} the submatrix of A^{γ} whose node set is C_k Let us call $\mathcal{G}_k^{\gamma}$ the digraph which consists of all nodes and edges of $\mathcal{G}^{\gamma}$ that belong to $\mathcal{D}(A_k^{\gamma})$.

Then, W can be decomposed into $W = W_1 V W_2$ where W_1 only has its last node in C_k and W_2 only has its first node in C_k. By construction, there is a walk $\tilde{V}$ on $\mathcal{D}(A_k^{(\gamma)})$ with same weight, start and end node as V and $l(V) = \gamma l(\tilde{V})$. As W goes through all nodes of $\mathcal{G}$, $\tilde{V}$ goes through all (and hence some) nodes of $\mathcal{G}_k^{\gamma}$.

Applying Proposition 2.4.2 to $\tilde{V}$ and the subgraph $\mathcal{G}_k^{\gamma}$ of $\mathcal{D}(A_k^{\gamma})$, we build a walk $\tilde{V}_1$ with length at most $\frac{\sigma}{\gamma}m + m - d_1 - 1$, where d_1 is the number of nodes in $\mathcal{G}_k^{\gamma}$ and $l(\tilde{V}_1) \equiv l(\tilde{V}) \pmod{\frac{\sigma}{\gamma}}$. As $d_1 \ge l(Z)/\gamma \ge \sigma/\gamma$ where Z is any cycle of $\mathcal{G}$, we also have $l(\tilde{V}_1) \le \frac{\sigma}{\gamma}m + m - \frac{\sigma}{\gamma} - 1$. This walk can be developed into a walk V_2 on $\mathcal{D}(A)$ with length at most $\sigma m + \gamma m - \sigma - \gamma$ and such that $l(V_2) \equiv l(V) \pmod{\sigma}$. To bound

$l(W_1V_2W_2)$, we consider two cases.

If $m < \left\lfloor \frac{d}{\gamma} \right\rfloor$, we just use that $l(W_1) \leq \gamma - 1$ and $l(W_2) \leq \gamma - 1$ to get

$$l(W_1V_2W_2) \leq 2\,(\gamma - 1) + (\gamma + \sigma)\left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 1\right) - \sigma - \gamma < \sigma \left\lfloor \frac{d}{\gamma} \right\rfloor - \sigma + d - 1.$$

If $m = \left\lfloor \frac{d}{\gamma} \right\rfloor$, we use that $l(W_2) \leq \gamma - 1$ and $l(W_1) \leq d \operatorname{rem} \gamma$ to get

$$l(W_1V_2W_2) \leq (\gamma - 1) + d \operatorname{rem} \gamma + (\gamma + \sigma) \left\lfloor \frac{d}{\gamma} \right\rfloor - \sigma - \gamma = \sigma \left\lfloor \frac{d}{\gamma} \right\rfloor - \sigma + d - 1.$$

Thus, we proved (2.23).

□

When the subgraph $\mathcal{G}$ is a cycle we obtain the following result:

Corollary 2.4.4 ([48],Corollary 4.6)**.** *For $A \in \mathbb{R}_{\max}^{d \times d}$ and Z a cycle of $\mathcal{D}(A)$, we have:*

$$T_{cr}^{l(Z)}(A, Z) \leq l(Z) \left\lfloor \frac{d}{\gamma} \right\rfloor + d - l(Z) - 1 \tag{2.24}$$

where γ is the cyclicity of $\mathcal{D}(A)$.

When the cycle of $\mathcal{D}(A)$ has the maximal possible length, which is $\gamma \left\lfloor \frac{d}{\gamma} \right\rfloor$, we also have

Proposition 2.4.5 ([68, Proposition 9.4])**.** *For $A \in \mathbb{R}_{\max}^{d \times d}$ and Z a cycle with length d of $\mathcal{D}(A)$, we have $T_{cr}^{d}(A, Z) \leq d^2 - d + 1$.*

In the paper [48] this bound was developed further in the case where $\mathcal{D}(A)$ has cyclicity γ. The formulation of the theorem below was worked out in collaboration with the co-authors [48] and the proof was developed independently.

Theorem 2.4.6 ([48],Proposition 4.8)**.** *For $A \in \mathbb{R}_{\max}^{d\times d}$ with γ being the cyclicity of $\mathcal{D}(A)$ and Z an elementary cycle with length $\gamma \left\lfloor \frac{d}{\gamma} \right\rfloor$ of $\mathcal{D}(A)$, we have $T_{cr}^{\gamma\lfloor d/\gamma \rfloor}(A,Z) \leq \gamma\left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 1\right)^2 + \gamma + d - 1$.*

Proof. We first observe that the number of nodes in the smallest cyclic class is $m = \left\lfloor \frac{d}{\gamma} \right\rfloor$, for otherwise we have $m < \left\lfloor \frac{d}{\gamma} \right\rfloor$ and this case there is no elementary cycle Z with the length $\gamma \left\lfloor \frac{d}{\gamma} \right\rfloor$. Indeed, such cycle would have to contain exactly $\left\lfloor \frac{d}{\gamma} \right\rfloor$ nodes in each cyclic class, and all these nodes would have to be different since the cycle is elementary, in contradiction with $m < \left\lfloor \frac{d}{\gamma} \right\rfloor$.

So let $m = \left\lfloor \frac{d}{\gamma} \right\rfloor$ be the size of the smallest cyclic class of $\mathcal{D}(A)$.

Consider a walk $W \in \mathcal{W}^{t,l(Z)}(i \xrightarrow{Z} j)$. If W does not go through all nodes of Z, then we insert a copy of Z in it.

Let C_k be the first cyclic class of size m encountered by W. The nodes of C_k are the nodes of $\mathcal{D}(A_k^\gamma)$. Let us call Z_k the cycle on $\mathcal{D}(A_k^\gamma)$ corresponding to Z and containing nodes from C_k. This cycle is elementary with length $\left\lfloor \frac{d}{\gamma} \right\rfloor$.

We decompose W into $W = W_1 V W_2$ where W_1 has only an end node in C_k and W_2 only has a start node in C_k. This can be done since W contains all nodes of Z. Walk V can be contracted to a walk $\tilde{V}$ on $\mathcal{D}(A^\gamma)$ with the same weight, start and end node as V. Since W contains all nodes of Z, walk $\tilde{V}$ contains all nodes of Z_k. We also have $l(V) = \gamma l(\tilde{V})$. Applying Proposition 2.4.5 to $\tilde{V}$ and Z_k on $\mathcal{D}(A_k^\gamma)$, builds a walk $\tilde{V_1}$ with length $l(\tilde{V_1}) \leq m^2 - m + 1$ and $l(\tilde{V_1}) \equiv l(\tilde{V}) \pmod{m}$, which can be developed into a walk V_2 on $\mathcal{D}(A)$ with length at most $\gamma m^2 - \gamma m + \gamma$ and $l(V_2) \equiv l(V)\gamma m$. To bound $l(W_1 V_2 W_2)$, we can use $l(W_1) \leq d \operatorname{rem} \gamma$, $l(W_2) \leq \gamma - 1$,

$m = \left\lfloor \frac{d}{\gamma} \right\rfloor$, and $d = \gamma \left\lfloor \frac{d}{\gamma} \right\rfloor + (d \operatorname{rem} \gamma)$ to obtain

$$\begin{aligned} l(W_1V_2W_2) &\le (\gamma - 1) + d \operatorname{rem} \gamma + \gamma \left\lfloor \frac{d}{\gamma} \right\rfloor^2 - \gamma \left\lfloor \frac{d}{\gamma} \right\rfloor + \gamma \\ &= \gamma - 1 + d + \gamma \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 1 \right)^2. \end{aligned}$$

Thus, we proved the claim. □

The following two results are not included in the paper [48], as we chose to pursue a different approach to including factor rank in the bounds.

Now we will obtain some bounds on the cycle removal threshold that involve the factor rank r. Being the results of independent work, these bounds are new with respect to the paper [48] and they have not been published previously. We will use the notation $\check{A}$ and F introduced in (1.1).

From [9][Theorem 3.4.5], the square of any strongly connected subgraph $\mathcal{G}$ of $\mathcal{D}(F)$ with cyclicity σ consists of two s.c.c.s with the same cyclicity $\frac{\sigma}{2}$, which are called the *children* of their *parent* $\mathcal{G}$ and are said to be *related* to one another. One of them is a subgraph of $\mathcal{D}(A)$ and the other is a subgraph of $\mathcal{D}(\check{A})$. In particular, if $\mathcal{G}$ is a s.c.c. of $\mathcal{G}^c(F)$ then one of its children is a s.c.c. of $\mathcal{G}^c(A)$ and the other an s.c.c. of $\mathcal{G}^c(\check{A})$. This principle can be also applied to any elementary cycle on $\mathcal{D}(F)$ of length l, whose children are two elementary cycles of length $\frac{l}{2}$, where one of them is in $\mathcal{D}(A)$ and the other is in $\mathcal{D}(\check{A})$.

Theorem 2.4.7. *Let an irreducible $A \in \mathbb{R}_{\max}^{d\times d}$ have factor rank r and cyclicity γ and* *let $\mathcal{G}$ be a strongly connected subgraph of $\mathcal{D}(F)$ with cyclicity σ. Then*

$$T_{cr}^{\sigma}(F,\mathcal{G}) \leq \sigma\left(\left\lfloor\frac{r}{\gamma}\right\rfloor - 1\right) + 2r. \tag{2.25}$$

Proof. Consider a walk $W \in \mathcal{W}^{t,\sigma}(i \xrightarrow{\mathcal{G}} j)$ on $\mathcal{D}(F)$. By adding cycles of $\mathcal{G}$ if necessary we can assume that W contains at least some nodes of $\check{\mathcal{G}}$, which is the child of $\mathcal{G}$ in $\mathcal{D}(\check{A})$. Let k and l be the first and the last node in W which belong to $\{n+1,\ldots,n+r\}$, which are nodes of $\mathcal{D}(\check{\mathcal{G}})$, and let $\tilde{W}$ be the walk between them. Clearly, k is the second node of W and l is the penultimate node of W. Contract $\tilde{W}$ to $\check{V}$, a walk on $\mathcal{D}(\check{A})$. As $\tilde{W}$ contains some nodes of $\mathcal{D}(\check{A})$, so does $\check{V}$.

By Theorem 2.4.3, there exists $\check{V}_1$ with same start and end node as $\check{V}$, going through a node of $\check{\mathcal{G}}$, with length satisfying $l(\check{V}_1) \leq \frac{\sigma}{2}\left\lfloor\frac{r}{\gamma}\right\rfloor + r - \frac{\sigma}{2} - 1$ and $l(\check{V}_1) \equiv l(\check{V}) \operatorname{rem} \frac{\sigma}{2}$. Walk $\check{V}_1$ is obtained from walk $\check{V}$ by removing cycles and possibly inserting cycles from $\check{\mathcal{G}}$.

The walk $\check{V}_1$ is then developed to walk $\tilde{W}_1$ on $\mathcal{D}(F)$ connecting k to l, also resulting in a walk W_1 connecting i to j on $\mathcal{D}(F)$ with length at most $\sigma\left(\left\lfloor\frac{r}{\gamma}\right\rfloor - 1\right) + 2r$. As this walk contains a node of $\mathcal{G}$, and is obtained from W by removing cycles and possibly inserting cycles from $\mathcal{G}$, the claim follows. $\square$

We can also give the following results in relation to what we have before. Note that the proof for the second statement is analogous to the proof of Theorem 2.4.6.

Proposition 2.4.8. *Let irreducible $A \in \mathbb{R}_{\max}^{d\times d}$ have factor rank r and cyclicity γ and let $\hat{Z}$ be a cycle of $\mathcal{D}(\check{A})$ with length l (see* (1.1)*). Then*

(i) $T_{cr}^{l}(F,Z) \leq l\left(\left\lfloor\frac{r}{\gamma}\right\rfloor - 1\right) + 2r$,

(ii) $T^l_{cr}(F,Z) \leq 2\gamma\left(\left\lfloor \frac{r}{\gamma}\right\rfloor - 1\right)^2 + 2\gamma + 2r$ *if* Z *is elementary with length* $l = 2\gamma\left\lfloor \frac{r}{\gamma}\right\rfloor$.

Proof. Regarding (i) the proof follows immediately from substituting l into Theorem 2.4.7 as the cyclicity of a cycle is the length of said cycle.

Regarding (ii) assume $m = \left\lfloor \frac{r}{\gamma}\right\rfloor$ be the size of the smallest cyclic class of $\mathcal{D}(\check{A})$. Otherwise we have $m < \left\lfloor \frac{r}{\gamma}\right\rfloor$ but, in F^2, Z splits into two elementary cycles Z_1 and $\check{Z}_1$, both of length $\gamma\lfloor\frac{r}{\gamma}\rfloor$, which means that $\check{Z}_1$ contains $\lfloor\frac{r}{\gamma}\rfloor$ nodes in each cyclic class of $\mathcal{D}(\check{A})$, which is a contradiction. Therefore let m be the size of the smallest cyclic class of $\mathcal{D}(F)$.

Consider a walk $W \in \mathcal{W}^{t,l(Z)}(i \xrightarrow{Z} j)$. If this walk does not traverse every node of Z then we insert a copy of Z into it.

Let C_k be the first cyclic class of $\mathcal{D}(\check{A})$ size m encountered by W. The nodes of C_k are the nodes of $\mathcal{D}(\check{A}^\gamma_k)$ Let Z_k be the cycle on $\mathcal{D}(\check{A}^\gamma_k)$ corresponding to Z, containing the nodes from C_k. It is elementary with length $m = \lfloor\frac{r}{\gamma}\rfloor$.

Now decompose W into $W = W_1VW_2$ where W_1 only has an end node in C_k, W_2 only has its start node in C_k, and V is the walk connecting W_1 to W_2. We can do it since W contains all nodes of Z. Walk V can be contracted to a walk $\tilde{V}$ on $\mathcal{D}(\check{A}^\gamma)$ with the same weight and the same start and end nodes. As V contains all the nodes of Z then $\tilde{V}$ contains all the nodes of Z_k. We also have $l(V) = 2\gamma l(\tilde{V})$. By applying Proposition 2.4.5 to $\tilde{V}$ and Z_k, we build a walk $\tilde{V}_1$ with length $l(\tilde{V}_1) \leq m^2 - m + 1$ and $l(\tilde{V}_1) \equiv l(\tilde{V}) \pmod{m}$. This can be further developed into a walk V_2 on $\mathcal{D}(F)$ with length at most $2\gamma m^2 - 2\gamma m + 2\gamma$ such that $l(V_2) \equiv l(V) \pmod{2\gamma m}$. Finally to bound $l(W_1V_2W_2)$, we use $l(W_1) \leq 2r \operatorname{rem} \gamma + 1$, $l(W_2) \leq 2\gamma - 1$, $m = \left\lfloor \frac{r}{\gamma}\right\rfloor$ and

$r = \gamma \left\lfloor \frac{r}{\gamma} \right\rfloor + (r \operatorname{rem} \gamma)$ to give

$$\begin{aligned} l(W_1 V_2 W_2) \leq & 2r \operatorname{rem} \gamma + 1 + 2\gamma - 1 + 2\gamma \left\lfloor \frac{r}{\gamma} \right\rfloor^2 - 2\gamma \left\lfloor \frac{r}{\gamma} \right\rfloor + 2\gamma \\ \leq & 2\gamma \left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 1 \right)^2 + 2\gamma + 2r. \end{aligned}$$

□

2.5 Bounds for $T_1(A, B)$ using the Cycle Removal Threshold

2.5.1 Bounds of Schwarz and Kim

In this section, we deduce the bounds of Schwarz and Kim and their factor rank versions using the bounds for Cycle Removal Threshold which we established in the previous section.

For this we use the following link between the cycle removal threshold and $T_{1,B}$. The statement will require the following notion, introduced Merlet et al. [68].

Definition 2.5.1. *Let $\mathcal{D}$ be a subgraph of $\mathcal{D}(A)$ and $\gamma \in \mathbb{N}$.*

The exploration penalty $\mathrm{ep}^\gamma(i)$ *of a node $i \in \mathcal{D}$ is the least $T \in \mathbb{N}$ such that for any multiple t of γ greater than or equal to T, there is a cycle on $\mathcal{D}$ with length t starting at i.*

The exploration penalty $\mathrm{ep}^\gamma(\mathcal{D})$ *of $\mathcal{D}$ is the maximum of the* $\mathrm{ep}^\gamma(i)$ *for $i \in \mathcal{D}$.*

We will use the following bound for $T_1(A,B)$:

$$T_1(A,B) \leq \max_l \left(T_{cr}^{\gamma_l}(A,\mathcal{G}_l) - \gamma_l + 1 + \mathrm{ep}^{\gamma_l}(\mathcal{G}_l)\right). \tag{2.26}$$

Here $\mathcal{G}_1, \cdots, \mathcal{G}_m$ are the s.c.c.s of a representing subgraph $\mathcal{G}$ of $\mathcal{G}^c(A)$. A representing subgraph $\mathcal{G}$ of $\mathcal{G}^c(A)$ is a completely reducible subgraph of $\mathcal{G}^c(A)$ such that every s.c.c. of $\mathcal{G}^c(A)$ contains exactly one s.c.c. of $\mathcal{G}$. This concept was originally defined by Merlet et al. [68]. We also denote γ_l as the cyclicities of $\mathcal{G}_l$ for $l \in [m]$.

Proposition 2.5.2. *[68, Proposition 6.5] Bound* (2.26) *holds when $B = B_N(A)$ or $B = B_{HA}(A)$.*

This proposition asserts that (2.26) holds not only for the Nachtigall but also for the Hartmann-Arguelles version of the weak CSR expansion. We will show that (2.26) will suffice for obtaining the bounds of Schwarz and Kim by means of the results of Section 2.4. Note also that Lemma 2.2.14 does not hold in the case of the Hartmann-Arguelles expansion. The formulation of the following proposition and theorem were suggested by the co-authors [48]. The proofs were written independently, based on the ideas given by the co-authors.

Bound (2.26) will be used only with $\mathcal{G}_l$ being cycles, and in this case $\gamma_l = l(\mathcal{G}_l)$ and $\mathrm{ep}^{\gamma_l}(\mathcal{G}_l) = 0$ for $l = 1, \ldots, m$.

Let us first pay attention to the case $\left\lfloor \frac{d}{\gamma} \right\rfloor = 1$, for which we will not use Proposition 2.5.2.

Proposition 2.5.3 ([48],Proposition 5.2)**.** *If $d < 2\gamma$, where γ is the cyclicity of $\mathcal{G}^c(A)$, then for any $A \in \mathbb{R}_+^{d\times d}$ such that $\lambda(A) \neq -\infty$, and any $t \geq d \operatorname{rem} \gamma$, we have $A^t = CS^tR$.*

Proof. Without loss of generality, we assume that $\lambda(A) = 0$.

Let us first notice that all cycles of $\mathcal{D}(A)$ have length γ, since their length is less than 2γ and divisible by γ. In particular, $\mathcal{G}^c(A)$ has cyclicity γ and all cycles of $\mathcal{G}^c(A)$ have length γ. Moreover, at most $d \operatorname{rem} \gamma$ cyclic classes of $\mathcal{D}(A)$ have more than one node, so that there is a class with only one node, $\mathcal{G}^c(A)$ is strongly connected, and the nodes in those classes are critical.

Proof of $(CS^tR)_{ij} \leq A^t_{ij}$**.**
Let us take an optimal walk $W \in \mathcal{W}^{t,\gamma}(i \xrightarrow{\mathcal{G}^c(A)} j)$, i.e., such that $p(W) = (CS^tR)_{ij}$.

First assume that $t \geq l(W)$. Then, since W traverses a critical node and $t \equiv l(W) \pmod{\gamma}$, we can form a walk of length t by possibly inserting a number of critical cycles into W (recall that all of them have length γ). By doing so we obtain $p(W) \leq (A^t)_{ij}$.

Now let $t < l(W)$. We have $t \geq d \operatorname{rem} \gamma$ and $l(W) \geq t + \gamma$. Since $l(W) > d \operatorname{rem} \gamma$, there is a path P_1 which is a prefix of W and which connects i to the first occurrence of the only node k of a cyclic class with 1 element. Next we find the last occurrence of k in W and take the path P_2, the suffix of W which connects k to j. Thus we obtain a decomposition

$$W = P_1P_2 + \sum_{l \in I} C_l,$$

where C_l for $l \in I$ are cycles going through k. Note that all these cycles have length γ by above arguments, and that all of them are critical, or this contradicts with the optimality of W.

We have $l(P_1) \leq d \operatorname{rem} \gamma$ and $l(P_2) \leq \gamma - 1$, hence $l(P_1P_2) < \gamma + d \operatorname{rem} \gamma$. This also implies $I \neq \emptyset$ and we have $p(W) = p(P_1P_2)$.

Furthermore, as $l(P_1P_2) \equiv t \operatorname{rem} \gamma$, $t \geq d \operatorname{rem} \gamma$ and $l(P_1P_2) < \gamma + d \operatorname{rem} \gamma$, we conclude that $l(P_1P_2) > t$ is impossible, so $l(P_1P_2) \leq t$ with $t - l(P_1P_2)$ being a multiple of γ. Applying the result for the first case to P_1P_2, we get $p(W) = p(P_1P_2) \leq (A^t)_{ij}$.

Proof of $A^t_{ij} \leq (CS^tR)_{ij}$.

Since at most $d \operatorname{rem} \gamma$ cyclic classes of $\mathcal{D}(A)$ have more than one node, all walks on $\mathcal{D}(A)$ with length $t \geq d \operatorname{rem} \gamma$ meet a cyclic class with only one node, which is critical. Hence $A^t_{ij} \leq (CS^tR)_{ij}$.

$\square$

We now prove that the bounds of Theorem 2.3.3 apply to $T_{1,B}$ whenever we have (2.26), and in particular for $B = B_{HA}$.

Theorem 2.5.4 ([48],Theorem 5.3)**.** *Suppose that B is defined in such a way that* (2.26) *is satisfied. Then the following bounds on $T_1(A.B)$ hold:*

(i) $\gamma \cdot \mathrm{Wi}\left(\left\lfloor \frac{d}{\gamma} \right\rfloor\right) + (d \operatorname{rem} \gamma)$*;*

(ii) $\hat{\mathrm{g}}.\left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 2\right) + d$

where γ is the cyclicity of $\mathcal{D}(A)$. In particular these bounds apply when $B = B_{HA}$.

Proof. Let $\mathcal{G}_1, \ldots, \mathcal{G}_m$ be the s.c.c. of $\mathcal{G}^c(A)$ and let $Z_1, \ldots, Z_m$ be the cycles of minimal length in those components. Using Corollary 2.4.4 with $l = l(Z_k)$ for any $k \in \{1, \ldots, m\}$, we have

$$\begin{aligned} T^{l(Z_k)}_{cr}(A, Z_k) &\leq l(Z_k) \left\lfloor \frac{d}{\gamma} \right\rfloor + d - l(Z_k) - 1 \\ T^{l(Z_k)}_{cr}(A, Z_k) - l(Z_k) + 1 &\leq l(Z_k) \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 2\right) + d \end{aligned}$$

for all $k = 1, \ldots, m$. Combining with Proposition 2.5.2 it becomes,

$$\begin{aligned} T_1(A, B) \leq \max_{k=1}^{m}(T_{cr}^{l(Z_k)} - l(Z_k) + 1) \leq \max_{k=1}^{m} l(Z_k) \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 2 \right) + d \\ = \hat{g} \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 2 \right) + d. \end{aligned}$$

Therefore $T_1(A, B)$ satisfies the second bound of Theorem 2.3.3.

Taking this further, when $\frac{l(Z_k)}{\gamma} \leq \left\lfloor \frac{d}{\gamma} \right\rfloor - 1$, we obtain

$$\begin{aligned} l(Z_k) \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 2 \right) + d &= \gamma \left(\frac{l(Z_k)}{\gamma} \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 2 \right) + \frac{n}{\gamma} \right) \\ &\leq \gamma \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 1 \right) \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 2 \right) + (d \operatorname{rem} \gamma) + \gamma \left\lfloor \frac{d}{\gamma} \right\rfloor \\ &= \gamma \left(\left\lfloor \frac{d}{\gamma} \right\rfloor^2 - 3 \left\lfloor \frac{d}{\gamma} \right\rfloor + 2 \right) + \gamma \left\lfloor \frac{d}{\gamma} \right\rfloor + (d \operatorname{rem} \gamma) \\ &= \gamma \operatorname{Wi} \left(\left\lfloor \frac{d}{\gamma} \right\rfloor \right) + (d \operatorname{rem} \gamma), \end{aligned}$$

which gives the first bound in Theorem 2.3.3 in this case. Otherwise, in the case when $\frac{l(Z_k)}{\gamma} = \left\lfloor \frac{d}{\gamma} \right\rfloor$ for some k we use Theorem 2.4.6 to obtain

$$\begin{aligned} T_{cr}^{\gamma \left\lfloor \frac{d}{\gamma} \right\rfloor}(A, Z_k) - l(Z_k) + 1 &\leq \gamma \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 1 \right)^2 + \gamma + d - 1 - \gamma \left\lfloor \frac{d}{\gamma} \right\rfloor + 1 \\ &= \gamma \operatorname{Wi} \left(\left\lfloor \frac{d}{\gamma} \right\rfloor \right) + d \operatorname{rem} \gamma. \end{aligned}$$

Thus treating these two cases yields the first bound in Theorem 2.3.3 in the case $\left\lfloor \frac{d}{\gamma} \right\rfloor > 1$. The remaining case $\left\lfloor \frac{d}{\gamma} \right\rfloor = 1$ was considered in Proposition 2.5.3. $\square$

2.5.2 Bounds using factor rank

The results of this subsection are new and were not published previously. The statement of Theorem 2.5.5 was worked out in collaboration with Sergeev and the proof was written independently. Recall that if A has factor rank r, then we have the matrices F and $\breve{A}$ introduced in (1.1). It is easy to see that $T(A) \leq \left\lceil \frac{T(F)}{2} \right\rceil$ for the periodicity thresholds, and therefore we have $T(A) \leq \max\left(\left\lceil \frac{T_1(F,B)}{2} \right\rceil, \left\lceil \frac{T_2(F,B)}{2} \right\rceil \right)$ for arbitrary B. This motivates us to seek bounds for $\left(\left\lceil \frac{T_1(F,B)}{2} \right\rceil \right)$ (in this section) and for $T_2(F,B)$ (later).

Rewriting (2.26) for F, we obtain

$$T_1(F,B) \leq \max_l \left(T_{cr}^{\gamma_l}(F,\mathcal{G}_l) - \gamma_l + 1 + \mathrm{ep}^{\gamma_l}(\mathcal{G}_l) \right). \tag{2.27}$$

Here $\mathcal{G}_1, \cdots, \mathcal{G}_m$ are the s.c.c.s of a representing subgraph $\mathcal{G}$ of $\mathcal{G}^c(F)$ and γ_l are the cyclicities of $\mathcal{G}_l$ for $l \in [m]$. By Proposition 2.5.2, bound (2.27) works for $B = B_N(F)$ or $B = B_{HA}(F)$.

Theorem 2.5.5. *Suppose that $T_1(F,B)$ satisfies* (2.27)*. Then the following bounds on $T_{1,N}(A)$ also apply to $\left\lceil \frac{T_1(F,B)}{2} \right\rceil$ under the same assumptions.*

(i) $\mathrm{Wi}(r) + 1$;

(ii) $\hat{\mathrm{g}}.\,(r-2) + r + 1$.

(iii) $\gamma.\,\mathrm{Wi}\left(\left\lfloor \frac{r}{\gamma} \right\rfloor \right) + (r \,\mathrm{rem}\, \gamma) + 1$;

(iv) $\hat{\mathrm{g}}.\left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 2 \right) + r + 1$

where γ is the cyclicity of $\mathcal{D}(A)$.

Proof. It suffices to prove (iii) and (iv), since (i) and (ii) follow from (iii) and (iv) respectively for purely arithmetic reasons (recalling that $\gamma \le r$)

(iv): Let Z_k, for $k = 1, \ldots, m$, be elementary cycles with minimal lengths in s.c.c. $\mathcal{G}_1, \ldots, \mathcal{G}_m$ of the critical graph $\mathcal{G}^c(F)$.

By Proposition 2.4.8 part (i) we have

$$T_{cr}^{l(Z_k)}(F, Z_k) \le l(Z_k)\left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 1\right) + 2r \tag{2.28}$$

We now combine this inequality with the bound (2.27):

$$\begin{aligned} T_1(F,B) &\le \max_k \left(T_{cr}^{F,l(Z_k)}(Z_k) - l(Z_k) + 1\right) \\ &\le \max_k l(Z_k)\left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 2\right) + 2r + 1 = 2\hat{\mathrm{g}}\left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 2\right) + 2r + 1, \end{aligned}$$

where we used that the maximal girth of $\mathcal{G}^c(F)$ is twice the maximal girth of $\mathcal{G}^c(A)$. Using the above inequality we obtain the desired bound for $\left\lceil \frac{T_{1,B}(F)}{2} \right\rceil$

(iii): We split the proof of the Schwarz bound into three cases: (a): When $\frac{\hat{\mathrm{g}}}{\gamma} \le \lfloor \frac{r}{\gamma} \rfloor - 1$; (b): When $\frac{\hat{\mathrm{g}}}{\gamma} = \left\lfloor \frac{r}{\gamma} \right\rfloor > 1$; (c) When $\frac{\hat{\mathrm{g}}}{\gamma} = \left\lfloor \frac{r}{\gamma} \right\rfloor = 1$.

For case (a): we have that

$$\hat{\mathrm{g}}\left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 2\right) + r + 1 \le \gamma\left(\left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 1\right)^2 + 1\right) + r \operatorname{rem} \gamma + 1 \tag{2.29}$$

when $\frac{\hat{\mathrm{g}}}{\gamma} \le \lfloor \frac{r}{\gamma} \rfloor - 1$.

For case (b): Let Z_k, for $k = 1, \ldots, m$, be elementary cycles with minimal lengths in s.c.c. $\mathcal{G}_1, \ldots, \mathcal{G}_m$ of the critical graph $\mathcal{G}^c(F)$. Let M_1 be the set of indices p such that $\frac{l(Z_p)}{2\gamma} = \left\lfloor \frac{r}{\gamma} \right\rfloor$. Then for all $p \in M_1$ by Proposition 2.4.8 part (ii) we have

$$T_{cr}^{2\gamma\left\lfloor \frac{r}{\gamma} \right\rfloor}(F, Z_p) \leq 2\gamma\left(\left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 1\right)^2 + 1\right) + 2r. \tag{2.30}$$

Note that for all $p \notin M_1$ we have $\frac{l(Z_p)}{2\gamma} \leq \left\lfloor \frac{r}{\gamma} \right\rfloor - 1$.

Combining (2.28) and (2.30) with bound (2.27) we obtain

$$\begin{aligned}
T_1(F,B) &\leq \max_k \left(T_{cr}^{l(Z_k)}(F, Z_k) - l(Z_k) + 1\right) \\
&= \max\left(\max_{k \notin M_1}\left(T_{cr}^{l(Z_k)}(F, Z_k) - l(Z_k) + 1\right), \max_{k \in M_1}\left(T_{cr}^{l(Z_k)}(F, Z_k) - l(Z_k) + 1\right)\right) \\
&\leq \max\left(2\gamma\left(\left\lfloor \frac{r}{\gamma} \right\rfloor^2 - 3\left\lfloor \frac{r}{\gamma} \right\rfloor + 2\right) + 2r + 1, 2\gamma\left(\left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 1\right)^2 + 1\right) + 2r \operatorname{rem} \gamma + 1\right) \\
&= 2\gamma\left(\left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 1\right)^2 + 1\right) + 2r \operatorname{rem} \gamma + 1.
\end{aligned}$$

where we also used inequality (2.29). From this we obtain the desired bound for $\left\lceil \frac{T_1(F,B)}{2} \right\rceil$.

The final case to check is when $\frac{\hat{\mathrm{g}}}{\gamma} = \left\lfloor \frac{r}{\gamma} \right\rfloor = 1$. To begin we use Proposition 2.5.3 with $d = r$ on $\breve{A}$ that says that for $r < 2\gamma$ we have

$$\breve{A}^t = CS^t R[\breve{A}] \quad \text{for } t \geq r \operatorname{rem} \gamma.$$

Recalling that $A = UL$ and $\breve{A} = LU$, we have

$$A^{t+1} = U\breve{A}^t L$$

Using (2.22), we obtain

$$A^{t+1} = UCS^t R[\check{A}]L = CS^{t+1}R[A] \quad \text{for } t \geq r \operatorname{rem} \gamma.$$

Therefore, for $t \geq r \operatorname{rem} \gamma + 1$ we have $A^t = CS^t R[A]$, and the Schwarz bound (iii) also holds in this case.

□

2.5.3 The case of cycle threshold expansion

The formulations and the proofs in this subsection are results of independent work, based on the ideas given by the co-authors [48]. In this section we obtain a new bound for the Cycle Threshold scheme using the bounds for the cycle removal threshold obtained previously. It will use the following bound on $T_1(A, B)$:

$$T_1(A, B) \leq \max \left\{ T_{cr}^{l(Z)}(Z) + 1 \mid Z \text{ cycle in } \mathcal{G} \right\} \tag{2.31}$$

Here $\mathcal{G}$ is a subgraph of $\mathcal{D}(A)$.

Proposition 2.5.6. *[68, Proposition 6.5] When $B = B_{CT}$, bound* (2.31) *holds with $\mathcal{G} = \mathcal{G}^{ct}$.*

For the definition of $\mathcal{G}^{ct}$ see the description of the Cycle Threshold scheme in Subsection 2.2.

Theorem 2.5.7 ([48],Theorem 5.5)**.** *If bound* (2.31) *holds, then we also have the following bound:*

$$T_1(A, B) \leq \gamma \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 1 \right)^2 + d + \gamma$$

where γ is the cyclicity of $\mathcal{D}(A)$.

Before we prove this theorem, we first introduce the following lemma

Lemma 2.5.8. *Let $A \in \mathbb{R}_{\max}^{d\times d}$ and Z be an elementary cycle of $\mathcal{D}(A)$ of length $l(Z)$ which is not maximal, and let $\mathcal{D}(A)$ have cyclicity γ. Then*

$$l(Z)\left(\left\lfloor\frac{d}{\gamma}\right\rfloor-1\right)+d \leq \gamma\left(\left\lfloor\frac{d}{\gamma}\right\rfloor-1\right)^2+d+\gamma. \tag{2.32}$$

Proof. As we bound the length of non maximal cycles by $l(Z) \leq \gamma\left\lfloor\frac{d}{\gamma}\right\rfloor-\gamma$, we substitute it into the LHS of the inequality (2.32) to give

$$\begin{aligned} l(Z)\left(\left\lfloor\frac{d}{\gamma}\right\rfloor-1\right)+d &\leq \left(\gamma\left\lfloor\frac{d}{\gamma}\right\rfloor-\gamma\right)\left(\left\lfloor\frac{d}{\gamma}\right\rfloor-1\right)+d \\ &\leq \gamma\left(\left\lfloor\frac{d}{\gamma}\right\rfloor-1\right)^2+d+\gamma, \end{aligned}$$

as required. □

Proof of Theorem 2.5.7. We can split this proof into two distinct cases, the first is when there is a cycle of maximal length, which is $l(Z)=\gamma\left\lfloor\frac{d}{\gamma}\right\rfloor$, and the second is when every cycle has length that is smaller than the maximal possible length, i.e., $l(Z)<\gamma\left\lfloor\frac{d}{\gamma}\right\rfloor$.

For the first case we can use Theorem 2.4.6 for maximal cycle length to give

$$T_{cr}^{\gamma\left\lfloor\frac{d}{\gamma}\right\rfloor}(A,Z) \leq \gamma\left(\left\lfloor\frac{d}{\gamma}\right\rfloor-1\right)^2+d+\gamma-1 \tag{2.33}$$

Turning to the second case, we can use Corollary 2.4.4, which means that

$$T_{cr}^{l(Z)}(A,Z) \leq l(Z)\left\lfloor \frac{d}{\gamma} \right\rfloor + d - l(Z) - 1$$
$$T_{cr}^{l(Z)}(A,Z) + 1 \leq l(Z)\left\lfloor \frac{d}{\gamma} \right\rfloor + d - l(Z) = l(Z)\left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 1\right) + d.$$

We can use Lemma 2.5.8 to bound this from above to get,

$$T_{cr}^{l(Z)}(A,Z) + 1 \leq l(Z)\left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 1\right) + d \leq \gamma\left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 1\right)^2 + d + \gamma. \tag{2.34}$$

Using (2.33) and (2.34) we obtain

$$T_1(A,B) \leq \max_Z \left\{ T_{cr}^{\gamma\lfloor \frac{d}{\gamma} \rfloor}(A,Z) + 1 \right\} \leq \gamma\left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 1\right)^2 + d + \gamma.$$

□

2.6 Bounds for $T_2(A,B)$

In this section we develop new bounds for $T_2(A,B)$, where B is a subordinate to A, i.e., a matrix obtained from A by setting some entries of A to ε and keeping all other entries the same as in A. In particular, $B_N(A)$, $B_{HA}(A)$ and $B_{CT}(A)$ are subordinate matrices.

In the paper [68], multiple bounds were developed for T_2 using bounds for the cycle removal threshold (from the same paper). We are going to improve the following bounds:

Proposition 2.6.1. *[68, Theorem 4.5] Let $A \in \mathbb{R}_{\max}^{d\times d}$ be irreducible and let B be* *subordinate to A. Denote by $\mathrm{cd}_B = \mathrm{cd}(\mathcal{D}(B))$ the biggest length of a path in the associated digraph of B and by $\hat{\sigma}$ the maximal cyclicity of the components of $\mathcal{G}^c(A)$. If $\lambda(B) = \varepsilon$, then $T_2(A, B) \leq cd_B + 1 \leq n_B$. Otherwise we have the following bounds*

$$T_2(A, B) \leq \frac{(d^2 - d + 1)(\lambda(A) - \min_{ij} a_{ij}) + \mathrm{cd}_B(\max_{ij} b_{ij} - \lambda(B))}{\lambda(A) - \lambda(B)} \tag{2.35}$$

$$T_2(A, B) \leq \frac{(\hat{\sigma}(d - 1) + d)(\lambda(A) - \min_{ij} a_{ij}) + \mathrm{cd}_B(\max_{ij} b_{ij} - \lambda(B))}{\lambda(A) - \lambda(B)} \tag{2.36}$$

The formulation of Proposition 2.6.2 and Theorem 2.6.3 below were developed in collaboration with the co-authors [48] and the proofs were written independently.

Proposition 2.6.2. *Let A be an irreducible matrix, $\mathcal{G}$ be a representing subgraph of $\mathcal{G}^c(A)$ with s.c.c.s $\mathcal{G}_1, \ldots, \mathcal{G}_m$ and let γ_l be the cyclicity of $\mathcal{G}_l$. Let B be subordinate to A such that $\lambda(B) \neq \varepsilon$. Then*

$$T_2(A, B) \leq \frac{\max_i(T_{cr}^{\gamma_i}(\mathcal{G}_i))(\lambda(A) - \min_{ij} a_{ij}) + \mathrm{cd}_B(\max_{ij} b_{ij} - \lambda(B)).}{\lambda(A) - \lambda(B)}. \tag{2.37}$$

This proposition is inspired by [68, Theorem 10.1] but it is different since we need to have the maximum of T_{cr} over subgraphs $\mathcal{G}_l$ in the bound. Therefore it will require a proof.

Proof. Assume that t is greater than the RHS of (2.37). We need to prove that

$$t\lambda(A) \otimes (CS^t R[\tilde{A}])_{ij} \geq t\lambda(B) \otimes \tilde{b}_{ij}^{(t)} \tag{2.38}$$

holds for all i,j, where $\tilde{A} = A - \lambda(A)$ and $\tilde{B} = B - \lambda(B)$. Before we begin this, we need to bound $(CS^tR[\tilde{A}])_{ij}$. Using [68, Theorem 6.1] and [68, Corollary 6.2] we have that

$$(CS^tR[\tilde{A}])_{ij} = \max_{\nu=1,\ldots,l} \left(p \left(\mathcal{W}^{t,\gamma_\nu} \left(i \xrightarrow{\mathcal{N}_\nu} j \right) \right) \right).$$

where $\mathcal{N}_\nu$ are the node sets of the s.c.c. of $\mathcal{G}^c(A)$, γ_ν are the cyclicities of these components, and the weights are computed in $\mathcal{D}(A)$. If $(CS^tR[\tilde{A}])_{ij}$ is finite then one of the sets $\mathcal{W}^{t,\gamma_\nu}\left(i \xrightarrow{\mathcal{N}_\nu} j\right)$ is non-empty. Let it be non-empty for $\nu = \mu$ for some μ, then we have:

$$(CS^tR[\tilde{A}])_{ij} \geq p\left(\mathcal{W}^{t,\gamma_\mu}\left(i \xrightarrow{\mathcal{N}_\mu} j\right)\right) \geq T_{cr}^{\gamma_\mu}(\mathcal{G}_\mu) \min_{k,l} \tilde{a}_{kl},$$

as $\mathcal{W}^{t,\gamma_\mu}\left(i \xrightarrow{\mathcal{N}_\mu} j\right)$ contains a walk whose length does not exceed $T_{cr}^{\gamma_\mu}(\mathcal{G}_\mu)$ and as $\min_{k,l} \tilde{a}_{kl}$ is non-positive. We further obtain that

$$(CS^tR[\tilde{A}])_{ij} \geq \min_\nu \left(T_{cr}^{\gamma_\nu}(\mathcal{G}_\nu) \min_{kl} \tilde{a}_{kl} \right). \tag{2.39}$$

By [68, Lemma 10.2] if the entry $(CS^tR[A])_{ij}$ is not finite then neither is $\tilde{b}_{ij}^t$ and there is nothing to prove so we assume that $(CS^tR[A])_{ij}$ (and, equivalently, $(CS^tR[\tilde{A}])_{ij}$) is finite. Passing to $A = \lambda(A) \otimes \tilde{A}$, we then use (2.39) to argue that the inequality

$$t\lambda(A) + \min_\nu \left(T_{cr}^{\gamma_\nu}(\mathcal{G}_\nu) \left(\min_{kl} a_{kl} - \lambda(A) \right) \right) \geq t\lambda(B) + \mathrm{cd}\left(\mathcal{D}(B)\right) \left(\max_{kl} b_{kl} - \lambda(B) \right) \tag{2.40}$$

guarantees (2.38). Rearranging the last inequality we obtain

$$t(\lambda(A)-\lambda(B)) \geq \max_{\nu} T_{cr}^{\gamma_\nu}(\mathcal{G}_\nu)\left(\lambda(A)-\min_{kl} a_{kl}\right)+\operatorname{cd}(\mathcal{D}(B))\left(\max_{kl} b_{kl}-\lambda(B)\right), \tag{2.41}$$

Since $(\lambda(A)-\min_{kl} a_{kl})$ does not depend on ν and $\lambda(A) \geq \lambda(B)$, dividing this inequality by $\lambda(A)-\lambda(B)$ we end up with (2.37). Therefore any t greater than (2.37) will satisfy (2.38) as well, thus completing the proof. □

Using this proposition along with Corollary 2.4.4 and Theorem 2.4.6 we obtain a theorem for the bounds for T_2.

Theorem 2.6.3. *Let $A \in \mathbb{R}_{\max}^{d\times d}$ be irreducible with cyclicity γ and let B be subordinate to A such that $\lambda(B) \neq \varepsilon$. Then the following bounds on $T_2(A,B)$ hold.*

$$T_2(A,B) \leq \frac{\left(\gamma\left(\left\lfloor\frac{d}{\gamma}\right\rfloor-1\right)^2+\gamma+d-1\right)(\lambda(A)-\min_{ij} a_{ij})+\operatorname{cd}_B(\max_{ij} b_{ij}-\lambda(B))}{\lambda(A)-\lambda(B)} \tag{2.42}$$

$$T_2(A,B) \leq \frac{\left(\hat{\sigma}\left(\left\lfloor\frac{d}{\gamma}\right\rfloor-1\right)+d-1\right)(\lambda(A)-\min_{ij} a_{ij})+\operatorname{cd}_B(\max_{ij} b_{ij}-\lambda(B))}{\lambda(A)-\lambda(B)}, \tag{2.43}$$

where $\hat{\sigma}$ is the greatest cyclicity of the s.c.c.s of $\mathcal{G}^c(A)$.

Proof. For the first bound we recall that the length of each cycle does not exceed $\gamma\left\lfloor\frac{d}{\gamma}\right\rfloor$, and the second largest length does not exceed $\gamma\left(\left\lfloor\frac{d}{\gamma}\right\rfloor-1\right)$. If a component $\mathcal{G}_\nu$ has a cycle of the maximal length $\gamma\left\lfloor\frac{d}{\gamma}\right\rfloor$ then denoting it by Z_ν and using Theorem 2.4.6

we have

$$T_{cr}^{l(Z_\nu)}(A, Z_\nu) \leq \gamma \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 1 \right)^2 + d + \gamma - 1.$$

If it does not have such cycle, then using Corollary 2.4.4 we obtain

$$T_{cr}^{l(Z_\nu)}(A, Z_\nu) \leq l(Z_\nu) \left\lfloor \frac{d}{\gamma} \right\rfloor - l(Z_\nu) + d - 1.$$

We can bound this above using Lemma 2.5.8 to get again that

$$T_{cr}^{l(Z_\nu)}(A, Z_\nu) \leq \gamma \left(\left\lfloor \frac{d}{\gamma} \right\rfloor - 1 \right)^2 + d + \gamma - 1.$$

Substituting this into Proposition 2.6.2 we get the first bound.

For the second bound, using Theorem 2.4.3 we obtain

$$T_{cr}^{\gamma_\nu}(A, \mathcal{G}_\nu) \leq \left(\sigma_\nu \left\lfloor \frac{d}{\gamma} \right\rfloor - \sigma_\nu + d - 1 \right)$$

where $\mathcal{G}_\nu$ is a component of $\mathcal{G}^c(A)$ and σ_ν is the cyclicity of this component.

Substituting this into Proposition 2.6.2 we get the second bound. □

With these bounds it remains to check that they are better bounds than the previous ones. Obviously, (2.43) is better than (2.36), and it remains to compare (2.42) with (2.35). This is achieved in the following

Remark 2.6.4. *For any irreducible matrix $A \in \mathbb{R}_{\max}^{d\times d}$ with subordinate matrix B, the bound* (2.42) *is smaller than the bound* (2.35).

Proof. Upon comparing the two bounds the inequality simplifies down to trying to prove that

$$\gamma\left(\left\lfloor\frac{d}{\gamma}\right\rfloor - 1\right)^2 + \gamma + d - 1 \leq d^2 - d + 1.$$

This simplifies down to

$$\left\lfloor\frac{d}{\gamma}\right\rfloor^2 - 2\left\lfloor\frac{d}{\gamma}\right\rfloor + 2 \leq \frac{d^2}{\gamma} - 2\frac{d}{\gamma} + \frac{2}{\gamma} \tag{2.44}$$

We can prove this by using induction on d. We begin by simplifying the LHS of the statement to something easier to prove.

$$\left\lfloor\frac{d}{\gamma}\right\rfloor^2 - 2\left\lfloor\frac{d}{\gamma}\right\rfloor + 2 \leq \left(\frac{d}{\gamma}\right)^2 - 2\frac{(d-1)}{\gamma} + 2$$

Comparing this with the RHS of (2.44) it becomes,

$$\left(\frac{d}{\gamma}\right)^2 - 2\frac{(d-1)}{\gamma} + 2 \leq \frac{d^2}{\gamma} - 2\frac{d}{\gamma} + \frac{2}{\gamma}$$
$$\frac{d^2}{\gamma} + 2\gamma \leq d^2.$$

We begin the induction by taking the base case of $d = \gamma \in \mathbb{N}$. This gives,

$$\frac{d^2}{d} + 2d \leq d^2$$
$$\Rightarrow 3 \leq d.$$

This means that for $d = \gamma \geq 3$ the base case works. We will need to check for the cases $(d, \gamma) = (d, 1)$ and $(d, \gamma) = (d, 2)$ so they will be addressed after the induction.

We now assume that if $d = \gamma + k$ for $k \in \mathbb{N} \cup \{0\}$ then,

$$\frac{(\gamma + k)^2}{\gamma} + 2\gamma \leq (\gamma + k)^2. \tag{2.45}$$

Looking at the inductive step we set $d = \gamma + k + 1$. Then

$$\begin{aligned}
\frac{(\gamma + k + 1)^2}{\gamma} + 2\gamma &= \frac{\gamma^2 + 2k\gamma + 2\gamma + 2k + 1 + k^2}{\gamma} + 2\gamma \\
&= \frac{(\gamma + k)^2}{\gamma} + 2\gamma + \frac{2\gamma + 2k + 1}{\gamma} \\
&\leq (\gamma + k)^2 + \frac{2\gamma + 2k + 1}{\gamma} \\
&\leq \gamma^2 + 2k\gamma + k^2 + 2\gamma + 2k + 1 \\
&= (\gamma + k + 1)^2.
\end{aligned}$$

Therefore, by induction, the equation (2.44) holds for $d, \gamma \geq 3$. To finalise this proof we need to check the outlying cases, when $(d, \gamma) = (d, 1)$, $(d, \gamma) = (d, 2)$ and $(d, \gamma) = (2, 2)$.

Case 1: $(d, \gamma) = (d, 1)$ Setting $\gamma = 1$ in (2.44) gives the inequality,

$$d^2 - 2d + 2 \leq d^2 - 2d + 2.$$

which holds with equality for all values of d.

Case 2: $(d, \gamma) = (d, 2)$ Setting $\gamma = 2$ in (2.44) gives the inequality,

$$\left\lfloor \frac{d}{2} \right\rfloor^2 - 2 \left\lfloor \frac{d}{2} \right\rfloor + 2 \leq \frac{d^2}{2} - d + 1$$

We can bound the LHS to give an easier statement to prove, which is,

$$\left(\frac{d}{2}\right)^2 - 2\frac{(d-1)}{2} + 2 \le \frac{d^2}{2} - d + 1$$
$$\left(\frac{d}{2}\right)^2 + 2 \le \frac{d^2}{2}$$
$$d \ge 2\sqrt{2}$$

Which means that the statement is true for $d \ge 3$. It remains to show it is true for the final case;

Case 3: $(d, \gamma) = (2, 2)$ We can plug in the values $d = 2$ and $\gamma = 2$ into (2.44) and it gives,

$$\left\lfloor \frac{2}{2} \right\rfloor^2 - 2\left\lfloor \frac{2}{2} \right\rfloor + 2 \le \frac{2^2}{2} - 2\frac{2}{2} + \frac{2}{2}$$
$$1 \le 1$$

as required. Therefore (2.44) holds for all $d, \gamma \in \mathbb{N}$. □

We note that in the collaborative paper [48], a simplified version of the proof is written that was developed by the coauthors.

Now we also present a development of this theorem in the case when A has a nontrivial factor rank r. The proof is similar to the proof of Theorem 2.6.3, the only difference is that we formulate the bound for F and use Theorem 2.4.7 and Proposition 2.4.8 instead of Theorem 2.4.6 and related results.

Theorem 2.6.5. *Let $A \in \mathbb{R}_{\max}^{d\times d}$ be irreducible with cyclicity γ and factor rank r. Let B be a subordinate to F. If $\lambda(B) = \varepsilon$, then $T_2(F, B) \le cd_B + 1$. If $\lambda(B) > \varepsilon$, then the* *following bounds on $T_2(F,B)$ hold.*

$$T_2(F,B) \leq \frac{2\left(\gamma\left(\left\lfloor\frac{r}{\gamma}\right\rfloor - 1\right)^2 + \gamma + r\right)(\lambda(F) - \min_{ij} f_{ij}) + \mathrm{cd}_B(\max_{ij} b_{ij} - \lambda(B))}{\lambda(F) - \lambda(B)} \tag{2.46}$$

$$T_2(F,B) \leq \frac{2\left(\hat{\sigma}\left(\left\lfloor\frac{r}{\gamma}\right\rfloor - 1\right) + r\right)(\lambda(F) - \min_{ij} f_{ij}) + \mathrm{cd}_B(\max_{ij} b_{ij} - \lambda(B))}{\lambda(F) - \lambda(B)}, \tag{2.47}$$

where $\hat{\sigma}$ is the greatest cyclicity of the s.c.c.s of $\mathcal{G}^c(A)$.

Proof. If $\lambda(B) = \varepsilon$ then powers of B will become $-\infty$ starting from $\mathrm{cd}_B + 1$ at most. Thus the bound follows. So we now assume that $\lambda(B) > \varepsilon$.

For the first bound we know that the length of each elementary cycle on $\mathcal{D}(F)$ cannot exceed $2\gamma\lfloor\frac{r}{\gamma}\rfloor$, and the length of the second largest cycle does not exceed $2\gamma\left(\lfloor\frac{r}{\gamma}\rfloor - 1\right)$. If the s.c.c.s $\mathcal{G}^c_\nu$ of $\mathcal{G}^c(F)$ contains an elementary cycle of maximal length $2\gamma\lfloor\frac{r}{\gamma}\rfloor$ then denoting it by Z_ν and using Proposition 2.4.8 we have

$$T_{cr}^{l(Z_\nu)}(F, Z_\nu) \leq 2\gamma\left(\left\lfloor\frac{r}{\gamma}\right\rfloor - 1\right)^2 + 2\gamma + 2r.$$

If no such cycle exists, then with Proposition 2.4.8

$$T_{cr}^{l(Z_\nu)}(F, Z_\nu) \leq l(Z_\nu)\left(\left\lfloor\frac{r}{\gamma}\right\rfloor - 1\right) + 2r.$$

If we bound $l(Z_\nu) \leq 2\gamma\left(\lfloor \frac{r}{\gamma} \rfloor - 1\right)$ and substitute that in we again have

$$T_{cr}^{l(Z_\nu)}(F, Z_\nu) \leq 2\gamma\left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 1\right)^2 + 2\gamma + 2r.$$

Finally using Proposition 2.6.2 we have the first bound.

For the second bound we use Theorem 2.4.7 to get

$$T_{cr}^{\sigma_\nu}(F, \mathcal{G}_\nu^c) \leq \sigma_\nu\left(\left\lfloor \frac{r}{\gamma} \right\rfloor - 1\right) + 2r,$$

where σ_ν is the cyclicity of the s.c.c. $\mathcal{G}_\nu^c$ of $\mathcal{G}^c(F)$. As the maximal cyclicity of $\mathcal{G}^c(A)$ is $\hat{\sigma}$, the maximal cyclicity of a component of $\mathcal{G}^c(F)$ is $2\hat{\sigma}$. Substituting this into Proposition 2.6.2 gives the second bound. □

2.7 Conclusion

The two main results of this Chapter that are published in [48] showed the validity of the Schwarz and Kim bounds on T_1, which originate from the works by Schwarz [77] and Kim [53] respectively, for the Nachtigall scheme in Theorem 2.3.3, and for the Hartman-Arguelles scheme in Theorem 2.5.4. Making use of the cyclicity of the digraph, these results can yield better bounds than the previously published bounds of [68]. Another result, Theorem 2.3.4 showed that by introducing a factor rank r then the Weilandt number [93], Dulmage-Mendelsohn number [27], Schwarz bound, and Kim bound apply to T_1 using the tropical factor rank of A, r, in place of the number of nodes d with a negligible penalty of adding 1. As with the other results, if the associated matrix A has some factor rank $r < d$, then there will be sharper bounds developed in that case. For T_2, using the cycle removal threshold introduced in Definition 2.4.1, new bounds are developed in Theorem 2.6.3. These bounds, especially the ones involving factor rank, can yield much better results. It should be noted that the factor rank bounds on T_2 depend on the entries of the subordinate matrix of F, B, which in most tested cases resulted in $B = -\infty$. This is an ideal outcome to yield minimal bounds. However it raises the question of the existence of an initial matrix A that produces a subordinate matrix of F, following the three schemes, that does not equal $-\infty$. This presents itself as a potential area of study.

There is some difference between the paper [48] and this chapter. In the paper [48], no bound was obtained on $T_1(A, B)$ that involved the factor rank, in the case of the Hartmann-Arguelles expansion. As a consequence of this the results of the paper [48] do not yield a bound on $T(A)$ that would involve the factor rank, in the case of that scheme. However here we took a different approach by bounding $T(A) \leq \max\left(\left\lceil \frac{T_1(F,B)}{2} \right\rceil, \left\lceil \frac{T_2(F,B)}{2} \right\rceil\right)$ recall that F is the matrix defined in (1.1) such that its square consist of two diagonal blocks: A, and an r by r matrix $\check{A}$, where r is the tropical factor rank of A as stated previously. Then, we obtained the bounds involving factor rank for $\left\lceil \frac{T_1(F,B)}{2} \right\rceil$ in Theorem 2.5.5 and $\left\lceil \frac{T_2(F,B)}{2} \right\rceil$ in Theorem 2.6.5.

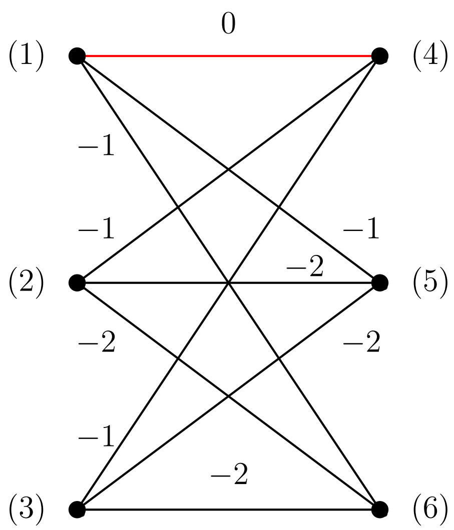
2.8 An Example

To end this chapter we will go through an example to show the effectiveness of the new bounds presented in the chapter.

Let A be the matrix in $\mathbb{R}^{6\times 6}_{\max}$, in the form (2.16), with entries

$$A = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & 0 & -1 & -1 \\ \varepsilon & \varepsilon & \varepsilon & -1 & -2 & -2 \\ \varepsilon & \varepsilon & \varepsilon & -1 & -2 & -2 \\ 0 & -1 & -1 & \varepsilon & \varepsilon & \varepsilon \\ -1 & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \\ -1 & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

The associated weighted graph of A is:



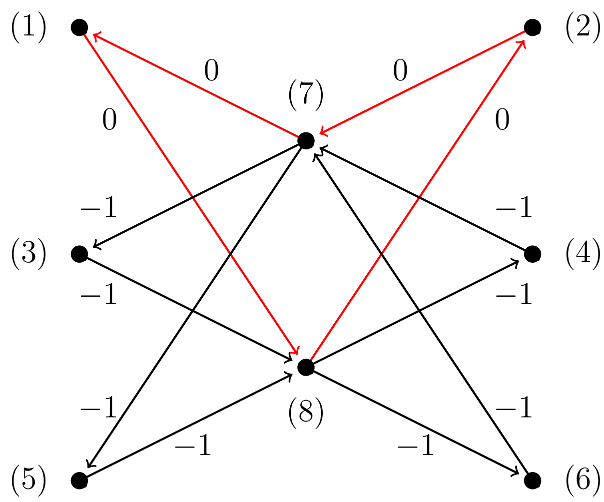
We do not show the directions of edges here as A is symmetrical therefore each edge from i to j has a reverse edge from j to i. It can be easily that the $\gamma = 2$ as the digraph is made up of cycles of length 2. We can also see that as there are no smaller cycles then $\hat{\mathrm{g}} = 2$ and obviously $d = 6$. Note that A has factor rank 2 and can be made up of the product

$$\begin{pmatrix} \varepsilon & 0 \\ \varepsilon & -1 \\ \varepsilon & -1 \\ 0 & \varepsilon \\ -1 & \varepsilon \\ -1 & \varepsilon \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 & -1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & -1 & -1 \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & 0 & -1 & -1 \\ \varepsilon & \varepsilon & \varepsilon & -1 & -1 & -2 \\ \varepsilon & \varepsilon & \varepsilon & -1 & -2 & -2 \\ 0 & -1 & -1 & \varepsilon & \varepsilon & \varepsilon \\ -1 & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \\ -1 & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

Therefore the matrix F from (1.1) is

$$F = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -1 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -1 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -1 & \varepsilon \\ 0 & \varepsilon & -1 & \varepsilon & -1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & -1 & \varepsilon & -1 & \varepsilon & \varepsilon \end{pmatrix},$$

with the associated digraph:



Now we will calculate the subordinate matrix B_N of A.

$$B_N = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -1 & -2 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -2 & -2 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

The subordinate matrices B_{HA} and B_{CT} of A only have entries equal to ε, and it follows from the associated digraph of F that all three subordinate matrices of F are equal to $-\infty$. Now we will present tables for T_1 and T_2 separately with the calculated bounds.

T_1	N, HA	Source	CT	Source	$N, HA(r)$	Source
Wi	26	(2.10)	26	(2.10)	3	Thm 2.5.5(i)
DM	14	(2.11)	N/A	N/A	3	Thm 2.5.5(ii)
Sch	8	Thm 2.5.4(i)	16	Thm 2.5.7	1	Thm 2.5.5(iii)
Kim	8	Thm 2.5.4(ii)	N/A	N/A	1	Thm 2.5.5(iv)

T_2	[68]	Source	[48]	Source	$T_2(F, B)$	Source
1	31	(2.35)	15	(2.42)	1	(2.46)
2	16	(2.36)	9	(2.43)	1	(2.47)

It can be seen from the table, for T_1, that not only do the Kim and Schwarz bounds prove better than the Weilandt and Dulmage-Mendelsohn bounds but the factor rank versions are vastly better in this example. For T_2 the same is also true. This means that by choosing the factor rank bounds for the Nachtigall or Hartman-Arguelles decomposition schemes we have $T(A) = \max(1, 1) = 1$ hence A^t is CSR for $t \geq 1$. More importantly it shows that the sequence of matrix powers of A is periodic with period 2 as seen below, which would not have been shown using the previous bounds.

$$A^2 = \begin{pmatrix} 0 & -1 & -1 & \varepsilon & \varepsilon & \varepsilon \\ -1 & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \\ -1 & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & -1 & -1 \\ \varepsilon & \varepsilon & \varepsilon & -1 & -2 & -2 \\ \varepsilon & \varepsilon & \varepsilon & -1 & -2 & -2 \end{pmatrix} \quad A^3 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & 0 & -1 & -1 \\ \varepsilon & \varepsilon & \varepsilon & -1 & -2 & -2 \\ \varepsilon & \varepsilon & \varepsilon & -1 & -2 & -2 \\ 0 & -1 & -1 & \varepsilon & \varepsilon & \varepsilon \\ -1 & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \\ -1 & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & -1 & -1 & \varepsilon & \varepsilon & \varepsilon \\ -1 & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \\ -1 & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & -1 & -1 \\ \varepsilon & \varepsilon & \varepsilon & -1 & -2 & -2 \\ \varepsilon & \varepsilon & \varepsilon & -1 & -2 & -2 \end{pmatrix} \quad A^5 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & 0 & -1 & -1 \\ \varepsilon & \varepsilon & \varepsilon & -1 & -2 & -2 \\ \varepsilon & \varepsilon & \varepsilon & -1 & -2 & -2 \\ 0 & -1 & -1 & \varepsilon & \varepsilon & \varepsilon \\ -1 & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \\ -1 & -2 & -2 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

CHAPTER 3

EXTENDING CSR DECOMPOSITION TO TROPICAL INHOMOGENEOUS MATRIX PRODUCTS

3.1 Introduction

In this chapter, instead of studying max-plus powers of a single matrix, we will consider a max-plus inhomogeneous matrix product of the form $A_1 \otimes A_2 \otimes \ldots \otimes A_k$ where matrices $A_1, \ldots, A_k$ are taken from an infinite matrix set $\mathcal{X}$, where each generating matrix in this set shares the same associated critical digraph. Denoting γ_ν as the cyclicity of a s.c.c. of the critical digraph for $\nu = \{1, \ldots, m\}$ and making use of some core assumptions that are to be outlined, we will derive some bounds for the *rank-*$\sum_{\nu=1}^{m} \gamma_\nu$ *transient* of inhomogeneous products of matrices from $\mathcal{X}$, which is the minimal K such that $A_1 \otimes A_2 \otimes \ldots \otimes A_k$ for any $k \geq K$ can be represented as a max-plus outer product of a column matrix of size $d \times \sum_{\nu=1}^{m} \gamma_\nu$ and a row matrix of size $\sum_{\nu=1}^{m} \gamma_\nu \times d$, where γ_ν is the cyclicity of the s.c.c. of the critical subgraph $\mathcal{G}^c_\nu$ for $\nu = \{1, \ldots, m\}$, which will depend on the matrix product.

The main results of the chapter are in some relation to the bounds on the ultimate periodicity of the sequence of max-plus matrix powers $\{A^t\}_{t\geq 1}$, similar to those established by Merlet et al. [68, 67], and in Chapter 2. However the ideas of Shue, Anderson and Dey [86], where the steady state properties of certain max-plus inhomogeneous matrix products were considered, are used as a base to develop the theory of the chapter. Their aim was to prove that, under certain assumptions, a sufficiently long max-plus matrix product is rank-one meaning that it can be written as the outer max-plus product of two vectors [86]. Components of these vectors are optimal weights of walks going to and from node 1 respectively. However, there is an oversight in [86, Corollary 3.1] with the removal of cycles from walks associated with the product. The results by Shue et al. [86] are also proved for a sufficient k that is large enough but no concrete bounds are established. This invited us to look for a bound on the length of a max-plus inhomogeneous matrix product, after which, it becomes an outer product of two vectors and the matrix product is rank-one. This is what inspired the development of joint paper with Sergeev and Berežný [50] and will be presented in Section 3.6, albeit with some improvement using newer developed theory. The rank-one case requires the associated critical graph to be a single loop which is restrictive, therefore generalising the critical digraph was the next aim and became the driving force behind the development of the main part of this chapter, namely Sections 3.4 and 3.5. Rather than directly proving the factor rank property from an inhomogeneous product, a CSR analogue is used, as explained earlier, which changes the aim to develop bounds on CSR transients rather than factor rank transients. Upon showing that the new definition of CSR exhibits similar properties to the original CSR (see Definition 2.2.1) then we can use similar proof methods and results from Merlet, Nowak, Schneider and Sergeev [67] as well as Brualdi and Ryser [9] to develop the key result, which is Theorem 3.5.10, together with Corollary 3.5.12, which gives an explicit bound on the length of the product after which it becomes CSR. However there are limitations to this approach, namely, where it can be shown for other cases that no bound exists for the CSR transient, and then we cannot guarantee a factor rank property. The special case of a single loop critical subgraph is then revisited in Section 3.6, which gives some improvemnt of the results obtained in collaboration with Sergeev and Berežný in the paper [50]. In Section 3.7 three cases where CSR does not work are given along with the counterexamples that demonstrate this. In all these counterexamples we present families of words of infinite length, in which the product made using such a word is not CSR.

Most of the main results presented in this Chapter are enhanced versions of those obtained in collaboration with Sergeev [49] and with Sergeev and Berežný [50](Section 3.6). While the formulations of these results were worked out together with Sergeev (and Berežný, for Section 3.6), the proofs of all of them are the results of independent work.

3.2 Assumptions and Notation

3.2.1 Main assumptions

In this subsection, we set out the main assumptions about $\mathcal{X}$ and the matrices A_α that are drawn from this set. Firstly, recall that $\mathcal{D}(A_\alpha)$ is the same for all α. Secondly, it is not realistic to assume that the maximum cycle mean of each $A_\alpha \in \mathcal{X}$ is zero therefore we normalise each matrix to give the new set of matrices $\mathcal{Y}$, where

$$\mathcal{Y} = \{A'_\alpha : A'_\alpha = A_\alpha \otimes \lambda^{-}(A_\alpha),\ A_\alpha \in \mathcal{X}\}.$$

For simplicity we will assume that the set $\mathcal{X}$ has already been normalised and will take this going forwards. With this normalised set we introduce the supremum and infimum matrices of $\mathcal{X}$.

Notation 3.2.1 ($A^{\sup}$ and $A^{\inf}$)**.**

> $A^{\sup}$*: entrywise supremum of all matrices in* $\mathcal{X}$*. In formula,* $A^{\sup} = \bigoplus_{\alpha:\, A_\alpha \in \mathcal{X}} A_\alpha$.
>
> $A^{\inf}$*: entrywise infimum of all matrices in* $\mathcal{X}$*.*

If the generating set $\mathcal{X}$ is finite then the non-ε entries of $A^{\sup}$ are also finite. However if $\mathcal{X}$ is infinite then an upper bound must be placed in order for the non-ε entries to also be finite. Therefore we will assume that either the generating set is finite or, in the case it is infinite, there exists a finite upper bound on the entries in the generators of $\mathcal{X}$. Note that the latter requirement is implicit in Assumption $\mathcal{B}$ written below. The concept of $A^{\sup}$ has been used before for various purposes. Gursoy, Mason and Sergeev [7] use $A^{\sup}$ to find a common subeigenvector for a semigroup of matrices (from which $A^{\sup}$ is defined), which is a technique we will use later on. Gursoy and Mason [6] also use $A^{\sup}$ and $\lambda(A^{\sup})$ to develop bounds for the max-eigenvalues over a set of matrices.

Assumption $\mathcal{A}$. *Any matrix* $A_\alpha \in \mathcal{X}$ *is irreducible.*

Assumption $\mathcal{B}$. *Any two matrices* $A_\alpha, A_\beta \in \mathcal{X}$ *are strongly geometrically equivalent, to each other and to* $A^{\sup}$ *(Definition 1.2.16), which has all entries in* $\mathbb{R}_{\max}$.

Notation 3.2.2. *The common associated digraph of the matrices from $\mathcal{X}$ will be denoted by $\mathcal{D}(\mathcal{X}) = (N, E)$, and the common critical digraph by $\mathcal{G}^c(\mathcal{X}) = (\mathcal{N}_c, \mathcal{E}_c)$. In general, this critical digraph has $m \geq 1$ s.c.c.s, denoted by $\mathcal{G}^c_\nu$, for $\nu = 1, \ldots, m$.*

Assumption $\mathcal{C}$. *Any matrix $A_\alpha \in \mathcal{X}$ is weakly geometrically equivalent to $A^{\inf}$. In other words, for each $(i, j) \in E$, we have $(A^{\inf})_{i,j} \neq -\infty$.*

Assumption $\mathcal{D}$1. *For the matrix A^{sup}, we have $\lambda(A^{sup}) = 0$.*

The first three assumptions come from the previous works by Shue et al. [86] and Kennedy-Cochran-Patrick et al. [50]. However, unlike in those works, we will no longer assume that the critical graph consists just of one loop.

The final Assumption $\mathcal{D}$2 will be inspired by the visualisation scaling, Definition 1.2.17 and its connection to tropical subeigenvectors. Before stating Assumption $\mathcal{D}$2 we first prove the following simple lemma. Note that as $\lambda(A^{\sup}) = 0$ we can find a subeigenvector of $A^{\sup}$ by taking any column from the Kleene star $(A^{\sup})^*$. The claim below also follows from [7, Theorem 3.1].

Lemma 3.2.3 ([49],Lemma 2.11)**.** *Suppose that the vector x satisfies $A^{\sup}x \leq x$. Then x provides a simultaneous visualisation for all matrices of $\mathcal{X}$.*

Proof. Let x be the vector that satisfies $A^{\sup}x \leq x$. By construction, $A^{\sup}$ is the supremum matrix of all the normalised generators in $\mathcal{X}$. Therefore $A_\alpha \leq A^{\sup}$ for all these normalised generators A_α. Hence the vector x also satisfies $A_\alpha x \leq x$ and it can be used to visualise A_α. As this applies for all α then all A_α can be simultaneously visualised. □

This is referred to as the set of matrices having a *common visualisation*, therefore, without loss of generality we assume that we have performed this common visualisation on all of the matrices in $\mathcal{X}$ to give the final core assumption.

Assumption $\mathcal{D}$2. *For all $A_\alpha \in \mathcal{X}$, we have $(A_\alpha)_{i,j} = 0$ and $(A^{\text{sup}})_{i,j} = 0$ for $(i,j) \in \mathcal{E}_c$, and $(A_\alpha)_{i,j} \leq 0$ and $(A^{\text{sup}})_{i,j} \leq 0$ for $(i,j) \notin \mathcal{E}_c$.*

From now on we will use Assumption $\mathcal{D}$2 instead of Assumption $\mathcal{D}$1. By Lemma 3.2.3 this can be done without loss of generality.

3.2.2 Extension to inhomogeneous products

Recall now that we have a set of matrices $\mathcal{X}$, from which we can select matrices to make arbitrary products.

Below we will need to use initial walks, final walks, strict initial walks to the critical nodes, strict final walks from the critical nodes, and full walks as defined in Definition 1.2.7.

This leads to the following notation which we will mostly work with the following sets of walks on $\mathcal{T}$.

Notation 3.2.4 (Walk sets on $\mathcal{T}(\Gamma(k))$)**.**

> $\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to j)$, $\mathcal{W}^l_{\mathcal{T},\text{init}}(i \to j)$ *and* $\mathcal{W}^l_{\mathcal{T},\text{final}}(i \to j)$: *set of full walks (of length k), and sets of initial and final walks of length l on $\mathcal{T}$ connecting i to j.*
>
> $\mathcal{W}^k_{\mathcal{T},\text{full}}(i \xrightarrow{\mathcal{N}_c} j)$, $\mathcal{W}^l_{\mathcal{T},\text{init}}(i \xrightarrow{\mathcal{N}_c} j)$ *and* $\mathcal{W}^l_{\mathcal{T},\text{final}}(i \xrightarrow{\mathcal{N}_c} j)$: *set of full walks (of length k), and sets of initial and final walks of length l on $\mathcal{T}$ traversing a critical node and connecting i to j;*

$\mathcal{W}_{\mathcal{T},\mathrm{init}}(i \to \mathcal{N}_c\|)$: *set of strict initial walk to the critical nodes connecting i to a node in $\mathcal{N}_c$ so that this node of $\mathcal{N}_c$ is the only node of $\mathcal{N}_c$ that is visited by the walk and it is visited only once;*

$\mathcal{W}_{\mathcal{T},\mathrm{final}}(\|\mathcal{N}_c \to j)$: *set of strict final walks from the critical nodes connecting a node in $\mathcal{N}_c$ to j so that this node of $\mathcal{N}_c$ is the only node of $\mathcal{N}_c$ that is visited by the walk and it is visited only once.*

$i \to_{\mathcal{T}} j$: *this denotes the situation where $i:0$ can be connected to $j:k$ on $\mathcal{T}$ by a full walk.*

Recall that $p(\mathcal{W})$ denotes the optimal weight of a walk in a set of walks $\mathcal{W}$. The *optimal walk interpretation* of entries of $\Gamma(k)$ in terms of walks on $\mathcal{T} = \mathcal{T}(\Gamma(k))$ is now apparent:

$$\Gamma(k)_{i,j} = p\left(\mathcal{W}^k_{\mathcal{T},\mathrm{full}}(i \to j)\right). \tag{3.1}$$

We will also need special notation for the optimal weights of walks in the sets $\mathcal{W}_{\mathcal{T},\mathrm{init}}(i \to \mathcal{N}_c\|)$ and $\mathcal{W}_{\mathcal{T},\mathrm{final}}(\|\mathcal{N}_c \to j)$ introduced above.

Notation 3.2.5 (Optimal weights of walks on $\mathcal{T}(\Gamma(k))$)**.**

$w^*_{i,\mathcal{N}_c} = p(\mathcal{W}_{\mathcal{T},\mathrm{init}}(i \to \mathcal{N}_c\|))$: *the maximal weight of walks in $\mathcal{W}_{\mathcal{T},\mathrm{init}}(i \to \mathcal{N}_c\|)$,*

$v^*_{\mathcal{N}_c,j} = p(\mathcal{W}_{\mathcal{T},\mathrm{final}}(\|\mathcal{N}_c \to j))$: *the maximal weight of walks in $\mathcal{W}_{\mathcal{T},\mathrm{final}}(\|\mathcal{N}_c \to j)$.*

The following notation is for optimal values of various optimisation problems involving paths and walks on $\mathcal{D}(A^{\sup})$, $\mathcal{D}(A^{\inf})$, which will be used in our factor rank bounds.

Notation 3.2.6 (Optimal weights of walks on $\mathcal{D}(A^{\sup})$ and $\mathcal{D}(A^{\inf})$)**.**

$\alpha_{i,\mathcal{N}_c}$ *: the weight of the optimal walk on* $\mathcal{D}(A^{\sup})$ *connecting node* i *to a node in* $\mathcal{N}_c$*;*

$\beta_{\mathcal{N}_c,j}$ *: the weight of the optimal walk on* $\mathcal{D}(A^{\sup})$ *connecting a node in* $\mathcal{N}_c$ *to node* j*;*

$\gamma_{i,j}$ *: the weight of the optimal path on* $\mathcal{D}(A^{\sup})$ *connecting node* i *to node* j *without traversing any node in* $\mathcal{N}_c$*.*

$w_{i,\mathcal{N}_c}$ *: the weight of the optimal walk on* $\mathcal{D}(A^{\inf})$ *connecting node* i *to a node in* $\mathcal{N}_c$*;*

$v_{\mathcal{N}_c,j}$ *: the weight of the optimal walk on* $\mathcal{D}(A^{\inf})$ *connecting a node in* $\mathcal{N}_c$ *to node* j*;*

$u_{i,j}^k$ *: the weight of the optimal walk on* $\mathcal{D}(A^{\inf})$ *of length* k *connecting node* i *to node* j*.*

Note that $\gamma_{i,j}$ is the only notation here that strictly represents a path. This is the case because the walk W such that $\gamma_{i,j} = p(W)$ must not contain any non-critical cycles otherwise it would not be optimal.

Remark 3.2.7. *The Kleene star* $(A^{\sup})^*$ *can be used to express* $\alpha_{i,\mathcal{N}_c}$ *and* $\beta_{\mathcal{N}_c,j}$*. Similarly the Kleene star* $(A^{\inf})^*$ *can be used to express* $w_{i,\mathcal{N}_c}$ *and* $v_{\mathcal{N}_c,j}$*.*

Before we proceed to the next remark, let us introduce the following piece of notation, inspired by the weak CSR expansion of Merlet et al. [68]:

Notation 3.2.8 (B^{sup} and λ_*)**.** *Denote*

$$(B^{\text{sup}})_{i,j} = \begin{cases} \varepsilon, & \textit{if } i \in \mathcal{N}_c \textit{ or } j \in \mathcal{N}_c, \\ (A^{\text{sup}})_{i,j}, & \textit{otherwise} \end{cases}$$

and by λ_* the maximum cycle mean of B^{sup}.

Using B^{sup} we have the following remark.

Remark 3.2.9. *Metric matrices and Kleene stars of B (see Definition 1.2.20) can be used to express $\gamma_{i,j}$ and all other parameters defined in Notation 3.2.6. To calculate them one needs shortest path algorithms such as the Floyd-Warshall [29] algorithm which can also be used to compute whole metric matrices and Kleene stars.*

Let us end this section with the following observation, which follows from the geometric equivalence (Assumptions $\mathcal{B}$ and $\mathcal{C}$)

Lemma 3.2.10. *The following are equivalent:* (i) $i \to_{\mathcal{T}} j$*;* (ii) $(\Gamma(k))_{i,j} > \varepsilon$*;* (iii) $u_{i,j}^k > \varepsilon$*.*

Proof. We begin by assuming (i) to be true. This is the same as saying there exists a walk on $\mathcal{T}(\Gamma(k))$ for any $\Gamma(k) = A_1 \otimes \ldots A_k$, $A_i \in \mathcal{X}$, connecting i to j with length k and weight $p > \varepsilon$. Since this is true for any word making up $\Gamma(k)$ then it is true for the word that gives the minimal weight for p which we denote $p' > \varepsilon$. This is the same weight as that of the walk on $\mathcal{D}(A^{\text{inf}})$ connecting i to j of length k. Hence $u_{i,j}^k > \varepsilon$ and (iii) holds.

Since p' is minimal then for any $\Gamma(k)$, $\Gamma(k)_{i,j} \geq u_{i,j}^k > \varepsilon$ so (ii) holds.

Finally as $\Gamma(k) \geq \varepsilon$ then on the associated trellis digraph $\mathcal{T}(\Gamma(k))$, by definition, a walk must exist connecting i to j of length k hence we are back to (i) and the statements are equivalent. $\square$

3.3 CSR products

In this section we introduce CSR decomposition of inhomogeneous products and study the properties of this decomposition. We will give two definitions of the CSR decomposition of $\Gamma(k)$ and prove their equivalence.

The threshold of ultimate periodicity (Definition 1.2.13) is required to develop the CSR decomposition for $\Gamma(k)$ as seen in the following definitions.

Definition 3.3.1 ([49],Definition 3.2)**.** *Let $\Gamma(k) = A_1 \otimes \ldots \otimes A_k$ be a matrix product of length k. Define C, S and R as follows:*

$S = (s_{i,j})$ is the matrix associated with the critical graph, i.e.

$$s_{i,j} = \begin{cases} a_{i,j} & \text{if } (i,j) \in \mathcal{E}_c \\ \varepsilon & \text{otherwise.} \end{cases} \tag{3.2}$$

Let γ be the cyclicity of critical graph, and t be a big enough integer, such that $t\gamma \geq T(S)$, where $T(S)$ is the threshold of ultimate periodicity of (the powers of) S.

C and R are defined by the following formulae:

$$C = \Gamma(k) \otimes S^{(t+1)\gamma - k(\bmod \gamma)}, \quad R = S^{(t+1)\gamma - k(\bmod \gamma)} \otimes \Gamma(k).$$

The product of C, $S^{k(\bmod \gamma)}$ and R will be denoted by $CS^{k(\bmod \gamma)}R[\Gamma(k)]$. We say that $\Gamma(k)$ is CSR *if $CS^{k(\bmod \gamma)}R[\Gamma(k)]$ is equal to $\Gamma(k)$.*

In the next definition, we prefer to define CSR terms corresponding to the components of the critical graph.

Definition 3.3.2 ([49],Definition 3.3)**.** *Let $\Gamma(k) = A_1 \otimes \ldots \otimes A_k$ be a matrix product of length k, and let $\mathcal{G}^c_\nu$, for $\nu = 1, \ldots, m$ (for some $m \geq 1$) be the components of $\mathcal{G}^c(\mathcal{X})$. For each $\nu = 1, \ldots, m$ define C_ν, S_ν and R_ν as follows:*

$S_\nu = (s^\nu_{i,j}) \in \mathbb{R}^{d\times d}_{\max}$ is the matrix associated with the s.c.c. $\mathcal{G}^c_\nu$ of the critical graph, i.e.,

$$s^\nu_{i,j} = \begin{cases} a_{i,j} & \text{if } (i,j) \in \mathcal{G}^c_\nu, \\ \varepsilon & \text{otherwise.} \end{cases} \tag{3.3}$$

Let γ_ν be the cyclicity of the critical component $\mathcal{G}^c_\nu$, and t_ν be a big enough integer, such that $t_\nu\gamma_\nu \geq T(S_\nu)$, where $T(S_\nu)$ is the threshold of ultimate periodicity of (the powers of) S_ν.

C_ν and R_ν are defined by the following formulae:

$$C_\nu = \Gamma(k) \otimes S_\nu^{(t_\nu+1)\gamma_\nu - k(\bmod \gamma_\nu)}, \quad R_\nu = S_\nu^{(t_\nu+1)\gamma_\nu - k(\bmod \gamma_\nu)} \otimes \Gamma(k).$$

The product of C_ν*,* $S_\nu^{k(\mathrm{mod}\,\gamma_\nu)}$ *and* R_ν *will be denoted by* $C_\nu S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} R_\nu[\Gamma(k)]$*. We say that* $\Gamma(k)$ *is* CSR *if*

$$\Gamma(k) = \bigoplus_{\nu=1}^{m} C_\nu S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} R_\nu[\Gamma(k)].$$

Using the definitions given above, we can write out the CSR terms more explicitly:

$$\begin{aligned} CS^{k(\mathrm{mod}\,\gamma)}R[\Gamma(k)] &= \Gamma(k) \otimes S^{(t+1)\gamma - k(\mathrm{mod}\,\gamma)} \otimes S^{k(\mathrm{mod}\,\gamma)} \otimes S^{(t+1)\gamma - k(\mathrm{mod}\,\gamma)} \otimes \Gamma(k) \\ &= \Gamma(k) \otimes S^{2(t+1)\gamma - k(\mathrm{mod}\,\gamma)} \otimes \Gamma(k), \\ C_\nu S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} R_\nu[\Gamma(k)] &= \Gamma(k) \otimes S_\nu^{2(t_\nu+1)\gamma_\nu - k(\mathrm{mod}\,\gamma_\nu)} \otimes \Gamma(k), \end{aligned}$$

Since the powers of S are ultimately periodic with period γ and the powers of S_ν are ultimately periodic with period γ_ν, and since we also have $t\gamma \geq T(S)$ and $t_\nu\gamma_\nu \geq T(S_\nu)$, we can reduce the exponents of S and S_ν to $(t+1)\gamma - k(\mathrm{mod}\,\gamma)$ and $(t_\nu+1)\gamma_\nu - k(\mathrm{mod}\,\gamma_\nu)$, respectively, and thus

$$CS^{k(\mathrm{mod}\,\gamma)}R[\Gamma(k)] = \Gamma(k) \otimes S^{v} \otimes \Gamma(k), \quad C_\nu S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} R_\nu[\Gamma(k)] = \Gamma(k) \otimes S_\nu^{v_\nu} \otimes \Gamma(k),$$

$$\text{for } v = (t+1)\gamma - k(\mathrm{mod}\,\gamma),\ v_\nu = (t_\nu+1)\gamma_\nu - k(\mathrm{mod}\,\gamma_\nu),\ t\gamma \geq T(S),\ t_\nu\gamma_\nu \geq T(S_\nu). \tag{3.4}$$

Below we will also need the following elementary observation.

Lemma 3.3.3 ([49],Lemma 3.4). *Let* $v = (t+1)\gamma - k(\mathrm{mod}\,\gamma)$*, where* $t\gamma \geq T(S)$*. Then, for any* ν*, we can find* t_ν *such that* $v = (t_\nu+1)\gamma_\nu - k(\mathrm{mod}\,\gamma_\nu)$ *and* $t_\nu\gamma_\nu \geq T(S_\nu)$*.*

Proof. The existence of t_ν such that $v = (t_\nu+1)\gamma_\nu - k(\mathrm{mod}\,\gamma_\nu)$ follows since γ is a multiple of γ_ν, and then we also have $t_\nu\gamma_\nu \geq t\gamma \geq T(S) \geq T(S_\nu)$. □

This lemma allows us to also write

$$C_\nu S_\nu^{k(\operatorname{mod}\gamma_\nu)} R_\nu[\Gamma(k)] = \Gamma(k) \otimes S_\nu^v \otimes \Gamma(k), \tag{3.5}$$

with v as in (3.4).

We also have some direct identities between Definition 3.3.1 and Definition 3.3.2.

Lemma 3.3.4. *We have the following identities:*

$$\begin{gathered} C = \bigoplus_{\nu=1}^m C_\nu, \qquad S = \bigoplus_{\nu=1}^m S_\nu, \qquad R = \bigoplus_{\nu=1}^m R_\nu, \\ C \otimes S^{k(\operatorname{mod}\gamma)} = \bigoplus_{\nu=1}^m C_\nu \otimes S_\nu^{k(\operatorname{mod}\gamma_\nu)}, \quad S^{k(\operatorname{mod}\gamma)} \otimes R = \bigoplus_{\nu=1}^m S_\nu^{k(\operatorname{mod}\gamma_\nu)} \otimes R_\nu. \end{gathered} \tag{3.6}$$

Proof. Observe that, as each component $\mathcal{G}_\nu^c$ is distinct, then the node sets of components $\mathcal{G}_{\nu_1}^c$ and $\mathcal{G}_{\nu_2}^c$ are disjoint for any distinct $\nu_1, \nu_2 \in \{1, \dots, m\}$. For S this means that $S_{\nu_1} \otimes S_{\nu_2} = -\infty$ where $-\infty$ is an abuse of notation representing an $d \times d$ matrix with entries all equal to ε. Hence we can stack each block together to give $S = \bigoplus_{\nu=1}^m S_\nu$.

By Definition 3.3.1 we have $C = \Gamma(k) \otimes S^{(t+1)\gamma - k(\operatorname{mod}\gamma)}$. We can raise $S = \bigoplus_{\nu=1}^m S_\nu$ to any power so raise it to $(t+1)\gamma - k(\operatorname{mod}\gamma)$. By Lemma 3.3.3 there exists a sequence of t_ν for $\nu = 1 \dots m$ such that $(t+1)\gamma - k(\operatorname{mod}\gamma) = (t_\nu + 1)\gamma_\nu - k(\operatorname{mod}\gamma_\nu)$ for every

ν. Therefore

$$\begin{aligned}
C &= \Gamma(k) \otimes S^{(t+1)\gamma - k(\operatorname{mod}\gamma)} \\
&= \Gamma(k) \otimes \bigoplus_{\nu=1}^{m} S_{\nu}^{(t_\nu+1)\gamma_\nu - k(\operatorname{mod}\gamma_\nu)} \\
&= \bigoplus_{\nu=1}^{m} \Gamma(k) \otimes S_{\nu}^{(t_\nu+1)\gamma_\nu - k(\operatorname{mod}\gamma_\nu)} = \bigoplus_{\nu=1}^{m} C_\nu.
\end{aligned}$$

Note that the penultimate step can happen as k is independent of ν and the final step comes from Definition 3.3.2.

For $C \otimes S^{k(\operatorname{mod}\gamma)}$ we can use Definition 3.3.1 and we have $C \otimes S^{k(\operatorname{mod}\gamma)} = \Gamma(k) \otimes S^{(t+1)\gamma}$. As before we can use the identity for S and raise both sides to the power $(t+1)\gamma$. As $\gamma = p_\nu\gamma_\nu$ then we can substitute $t_\nu = tp_\nu + p_\nu - 1$ for every ν and this gives

$$\begin{aligned}
C \otimes S^{k(\operatorname{mod}\gamma)} &= \Gamma(k) \otimes S^{(t+1)\gamma} \\
&= \Gamma(k) \otimes \bigoplus_{\nu=1}^{m} S_{\nu}^{(t+1)\gamma} \\
&= \bigoplus_{\nu=1}^{m} \Gamma(k) \otimes S_{\nu}^{(t_\nu+1)\gamma_\nu} = \bigoplus_{\nu=1}^{m} C_\nu \otimes S_{\nu}^{k(\operatorname{mod}\gamma_\nu)}.
\end{aligned}$$

Again the penultimate step happens as k is independent of ν and the final step comes from using Definition 3.3.2.

By symmetry we also have the identities for R and $S^{k(\operatorname{mod}\gamma)} \otimes R$. □

Some of these identities can be used for the following proposition.

Proposition 3.3.5 ([49],Proposition 3.5). $\Gamma(k)$ *is CSR by Definition 3.3.1 if and only if it is CSR by Definition 3.3.2.*

Proof. We need to show that

$$CS^{k(\operatorname{mod}\gamma)}R[\Gamma(k)] = \bigoplus_{\nu=1}^{m} C_\nu S_\nu^{k(\operatorname{mod}\gamma_\nu)} R_\nu[\Gamma(k)] \tag{3.7}$$

for arbitrary k. Using (3.4) and (3.5) we can rewrite (3.7) equivalently as

$$\Gamma(k) \otimes S^{(t+1)\gamma-k(\operatorname{mod}\gamma)} \otimes \Gamma(k) = \Gamma(k) \otimes \left(\bigoplus_{\nu=1}^{m} S_\nu^{(t+1)\gamma-k(\operatorname{mod}\gamma)}\right) \otimes \Gamma(k) \tag{3.8}$$

with $t\gamma \geq T(S)$. Using $S = \bigoplus_{\nu=1}^{m} S_\nu$ from Lemma 3.3.4, we can raise both sides to the same power to give us $S^t = \bigoplus_{\nu=1}^{m} S_\nu^t$ for any t. This shows (3.8), and the claim follows. □

This version of CSR is designed for products of matrices rather than powers of a single matrix as given Definition 2.2.1 by Schneider and Sergeev [83].

To give an optimal walk interpretation of CSR for inhomogeneous products, we will need to define the trellis graph corresponding to these terms, by modifying Definition 1.2.22.

Definition 3.3.6 ([49],Definition 3.6)**.** *Let $v = (t+1)\gamma - k(\operatorname{mod}\gamma)$, where t is a large enough number such that $t\gamma \geq T(S)$.*
Define $\mathcal{T}'(\Gamma(k))$ as the digraph $\mathcal{T}' = (\mathcal{N}', \mathcal{E}')$ with the set of nodes $\mathcal{N}'$ and edges $\mathcal{E}'$, such that:

(1) $\mathcal{N}'$ consists of $2k+v+1$ copies of N which are denoted $N_0, \ldots, N_{2k+v}$ and the nodes for N_l for each $0 \leq l \leq 2k+v$ are denoted by $1:l, \ldots, d:l$;

(2) $\mathcal{E}'$ is defined by the following rules:

a) there are edges only between N_l and N_{l+1},

b) for $1 \leq l \leq k$ we have $(i : l-1, j : l) \in \mathcal{E}'$ if and only if $(i,j) \in E(\mathcal{X})$ and the weight of the edge is $(A_l)_{i,j}$,

c) for $k+v+1 \leq l \leq 2k+v$ we have $(i : l-1, j : l) \in \mathcal{E}'$ if and only if $(i,j) \in E(\mathcal{X})$ and the weight of the edge is $(A_{l-k-v})_{i,j}$,

d) for $k < l < k+v+1$ we have $(i : l-1, j : l) \in \mathcal{E}'$ if and only if $(i,j) \in \mathcal{G}^c(\mathcal{X})$ and the weight of the edge is 0.

The weight of a walk on $\mathcal{T}'(\Gamma(k))$ is denoted by $p_{\mathcal{T}'}(W)$.

We will refer to this as symmetric extension of the trellis graph associated with CSR from now on. The following optimal walk interpretation of CSR terms on $\mathcal{T}'$ is now obvious.

Lemma 3.3.7 ([49],Lemma 3.7)**.** *The following identities hold for all i, j*

$$\begin{aligned}(CS^{k(\operatorname{mod}\gamma)}R[\Gamma(k)])_{i,j} &= p\left(\mathcal{W}^{2k+v}_{\mathcal{T}',\text{full}}(i \to j)\right), \\ (C_\nu S_\nu^{k(\operatorname{mod}\gamma_\nu)}R_\nu[\Gamma(k)])_{i,j} &= p\left(\mathcal{W}^{2k+v}_{\mathcal{T}',\text{full}}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right),\end{aligned} \tag{3.9}$$

where $v = (t+1)\gamma - k(\operatorname{mod}\gamma)$, with $t\gamma \geq T(S)$.

Proof. With (3.4) in mind, the first identity follows from the optimal walk interpretation of $\Gamma(k) \otimes S^v \otimes \Gamma(k)$, and the second identity follows from (3.5) and the optimal walk interpretation of $\Gamma(k) \otimes S_\nu^v \otimes \Gamma(k)$. □

We can show that given a matrix product consisting of the same matrix, i.e. $\Gamma(k) = A^k$, then the original CSR definition is equivalent to the new one. Note that this result is not contained in the preprint [49].

Proposition 3.3.8. *Let* $\Gamma(k) = A^k$ *for some* $A \in \mathcal{X}$ *and* $k \geq \max_\nu (T_{cr}^{\gamma_\nu}(A, \mathcal{G}_\nu^c))$ *where* γ_ν *is the cyclicity of the strongly connected critical component* $\mathcal{G}_\nu^c$ *of* $\mathcal{D}(A)$*. If* $\Gamma(k)$ *is CSR by Definition 3.3.1 then it is equal to* CS^kR *from Definition 2.2.1.*

Proof. Assume that $\Gamma(k) = A^k = CS^{k(\operatorname{mod}\gamma)}R[\Gamma(k)]$. To show that $CS^kR = CS^{k(\operatorname{mod}\gamma)}R[\Gamma(k)]$, by Corollary 2.2.3 and Proposition 3.3.5, we will show that $C_\nu S_\nu^k R_\nu = C_\nu S_\nu^{k(\operatorname{mod}\gamma_\nu)} R_\nu[\Gamma(k)]$ for all $\nu = 1, \ldots, m$. To achieve this we need to use the optimal walk representations for both products. By the analogue of (2.4) for $C_\nu S_\nu R_\nu$, $(C_\nu S_\nu^k R_\nu)_{i,j} = p\left(\mathcal{W}^{k,\gamma_\nu}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right)$ and by Lemma 3.3.7 we have $(C_\nu S_\nu^{k(\operatorname{mod}\gamma_\nu)} R_\nu[\Gamma(k)])_{i,j} = p\left(\mathcal{W}_{\mathcal{T}',\mathrm{full}}^{2k+v}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right)$ where $v = (t+1)\gamma_\nu - k(\operatorname{mod}\gamma_\nu)$ for t described in Definition 3.3.2. Now all that is required is to show that these two representations are equal. We will achieve this by proving the following two inequalities,

$$p\left(\mathcal{W}^{k,\gamma_\nu}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right) \geq p\left(\mathcal{W}_{\mathcal{T}',\mathrm{full}}^{2k+v}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right) \tag{3.10}$$

and

$$p\left(\mathcal{W}^{k,\gamma_\nu}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right) \leq p\left(\mathcal{W}_{\mathcal{T}',\mathrm{full}}^{2k+v}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right). \tag{3.11}$$

We will tackle inequality (3.10) first. Let W be the optimal walk over $\mathcal{T}'$ of length $2k+v$ connecting i to j traversing $\mathcal{N}_c^\nu$ such that $W \in \mathcal{W}_{\mathcal{T}',\mathrm{full}}^{2k+v}(i \xrightarrow{\mathcal{N}_c^\nu} j)$ and, more importantly, $p(W) = p\left(\mathcal{W}_{\mathcal{T}',\mathrm{full}}^{2k+v}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right)$. Expanding v we have $l(W) \equiv 2k + (t+1)\gamma_\nu - k(\operatorname{mod}\gamma_\nu) \equiv k(\operatorname{mod}\gamma_\nu)$. For this reason as all edges of W belong to $\mathcal{D}(A)$ and W traverses a critical node, we can associate with W a walk of the same weight in $\mathcal{W}^{k,\gamma_\nu}(i \xrightarrow{\mathcal{N}_c^\nu} j)$, and then inequality (3.10) follows.

Now we give a proof for inequality (3.11). Let W' be the optimal walk of length congruent to $k(\operatorname{mod}\gamma_\nu)$ connecting i to j and traversing $\mathcal{N}_c^\nu$. We have $W' \in \mathcal{W}^{k,\gamma_\nu}(i \xrightarrow{\mathcal{N}_c^\nu} j)$

and $p(W') = p\left(\mathcal{W}^{k,\gamma_\nu}(i \to j)\right)$. Now we need to develop this walk into another walk W'' of length $2k+v$ with the same weight.

Assume that k is greater than or equal to $T_{cr}^{\gamma_\nu}(A, \mathcal{G}_\nu)$. Then a walk V can be constructed such that $p(W') = p(V)$ and $l(V) \leq k$ so we choose a node $l \in \mathcal{N}_c^\nu$ that the walk V traverses. As non-elementary, critical cycles exist for all t such that $t\gamma_\nu \geq T(S_\nu)$ then we add a critical cycle of length $(t+1)\gamma_\nu - k(\bmod\,\gamma_\nu) + k \equiv 0(\bmod\,\gamma_\nu)$ to V at node l. As such a cycle exists then we obtain the walk V', such that $p(V') = p(V)$, and $l(V') = 2k+v$, and as the walk has a fixed length we can put it on the trellis digraph associated with the matrix A^{2k+v}. Split this new walk into three $V' = W_1W_2W_3$ where W_1 is the first k steps of the walk, W_2 is the subsequent v steps of the walk and W_3 are the final k steps of the walk. As $l(V) \leq k$, and $l(W_1) = l(W_3) = k$, then walk W_2 has critical edges only. Hence we can associate with V' a walk with the same weight on $\mathcal{T}'$ and we have $p(V') \leq p\left(\mathcal{W}_{\mathcal{T}',\text{full}}^{2k+v}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right)$ hence the inequality follows.

As both inequalities are true then for $k \geq \max_\nu \left(T_{cr}^{\gamma_\nu}(A, \mathcal{G}_\nu^c)\right)$, $p\left(\mathcal{W}^{k,\gamma_\nu}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right) = p\left(\mathcal{W}_{\mathcal{T}',\text{full}}^{2k+v}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right)$ for all ν and $CS^kR = CS^{k(\bmod\,\gamma_\nu)}R[\Gamma(k)]$. Therefore as $\Gamma(k) = CS^{k(\bmod\,\gamma_\nu)}R[\Gamma(k)]$, then $\Gamma(k) = CS^kR$.

□

In what follows, we mostly work with Definition 3.3.2, but we can switch between the equivalent definitions if we find it convenient. Our next aims will be 1) to obtain a bound on the factor rank of CSR decomposition and 2) to obtain inhomogenous analogues of some properties of the ”one-matrix” CSR products, listed above in Theorem 2.2.9, Corollaries 2.2.10 and 2.2.11.

We now present a useful lemma that shows equality for columns of C_ν and rows of R_ν with indices in the same cyclic class.

Lemma 3.3.9 ([49],Lemma 3.8)**.** *For any i and for any two nodes x and y in the same cyclic class of the critical component $\mathcal{G}_\nu^c$ we have*

$$(C_\nu)_{i,x} = (C_\nu)_{i,y} \quad \textit{and} \quad (R_\nu)_{x,i} = (R_\nu)_{y,i} \tag{3.12}$$

Proof. We prove the lemma for columns, as the case of the rows is similar.

For any i, j, denote $(C_\nu)_{i,j}$ by $c_{i,j}$. From the definition of C_ν, it follows that $c_{i,x}$ is the weight of an optimal walk in $\mathcal{W}_{\mathcal{T}',\mathrm{init}}^{k+(t_\nu+1)\gamma_\nu-k(\mathrm{mod}\,\gamma_\nu)}(i \xrightarrow{\mathcal{N}_c^\nu} j)$ where $t_\nu\gamma_\nu \geq T(S_\nu)$, and such walk consists of two parts. The first part is a full walk on $\mathcal{T}$ connecting i to the critical subgraph at some node s. The second part is a walk over the critical subgraph of length $(t_\nu + 1)\gamma_\nu - k(\mathrm{mod}\,\gamma_\nu)$ connecting s to x with weight zero. As the length of the second walk is greater than $T(S_\nu)$, a walk connecting s to x exists if and only if $[s] \to_{-k(\mathrm{mod}\,\gamma_\nu)} [x]$. If a full walk connecting i to $[s]$ on $\mathcal{T}$ exists then, for arbitrary x, y in the same cyclic class, $c_{i,x}$ and $c_{i,y}$ are both equal to the optimal weight of all walks connecting i to $[s]$ on $\mathcal{T}$, where $[s] \to_{-k(\mathrm{mod}\,\gamma_\nu)} [x]$, otherwise both $c_{i,x}$ and $c_{i,y}$ are equal to ε. This shows that $c_{i,x} = c_{i,y}$.

The case of rows of R_ν is considered similarly, but instead of initial walks one has to use final walks on $\mathcal{T}'$. □

We can use this to prove the same property for C and R of Definition 3.3.1.

Corollary 3.3.10 ([49],Corollary 3.9)**.** *For any i and for any two nodes x and y in the same critical component and the same cyclic class of said critical component, we have*

$$C_{i,x} = C_{i,y} \quad \textit{and} \quad R_{x,i} = R_{y,i} \tag{3.13}$$

Proof. We will prove only the first identity, as the proof of the second identity is similar. Let x, y belong to the same component $\mathcal{G}^c_\mu$ of $\mathcal{G}^c(\mathcal{X})$, and let them belong to the same cyclic class of that component. By Lemma 3.3.9 we have $(C_\mu)_{i,x} = (C_\mu)_{i,y}$, and we also have $(C_\nu)_{i,x} = (C_\nu)_{i,y} = \varepsilon$ for any $\nu \neq \mu$. Using these identities and (3.6), we have

$$C_{i,x} = \left(\bigoplus_{\nu=1}^{m} C_\nu\right)_{i,x} = (C_\mu)_{i,x} = (C_\mu)_{i,y} = \left(\bigoplus_{\nu=1}^{m} C_\nu\right)_{i,y} = C_{i,y}.$$

$\square$

The next theorem explains why CSR is useful for inhomogeneous products. Note that in the proof of it we use the CSR structure rather than the $\Gamma(k) \otimes S^v \otimes \Gamma(k)$ representation that was used above.

Theorem 3.3.11 ([49],Theorem 3.10)**.** *The factor rank of each $C_\nu S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} R_\nu[\Gamma(k)]$ is no more than γ_ν, for $\nu = 1, \ldots, m$, and the factor rank of $CS^{k(\mathrm{mod}\,\gamma)} R[\Gamma(k)]$ is no more than $\sum_{\nu=1}^{m} \gamma_\nu$.*

Proof. For each $\nu = 1, \ldots, m$, take all the nodes from $\mathcal{G}_\nu$ and order them into cyclic classes $\mathcal{C}_0^\nu, \ldots, \mathcal{C}_{\gamma_\nu - 1}^\nu$. Take two columns with indices $x, y \in \mathcal{C}_i^\nu$ from the matrix C_ν. As they are in the same cyclic class, by Lemma 3.3.9 the columns are equal to each other. This means that we can take a column representing a single node from each cyclic class and since there are γ_ν distinct classes then there will be γ_ν distinct columns of C_ν. The same also holds for any two rows of R_ν: if the row indices are in the same cyclic class, then the rows are equal, so that we have γ_ν distinct rows.

Let us now check that the same holds for $S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} \otimes R_\nu$. By the construction of

$S_\nu^{k(\mathrm{mod}\,\gamma_\nu)}$ we know that if $(S_\nu^{k(\mathrm{mod}\,\gamma_\nu)})_{i,j} \neq 0$ then $[i] \to_{k(\mathrm{mod}\,\gamma_\nu)} [j]$. Therefore

$$(S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} \otimes R_\nu)_{i,\cdot} = \bigoplus_{j \in N_c} (S_\nu^{k(\mathrm{mod}\,\gamma_\nu)})_{i,j} \otimes (R_\nu)_{j,\cdot} = \bigoplus_{j\colon [i] \to_{k(\mathrm{mod}\,\gamma_\nu)} [j]} (S_\nu^{k(\mathrm{mod}\,\gamma_\nu)})_{i,j} \otimes (R_\nu)_{j,\cdot} = (R_\nu)_{j,\cdot}.$$

This means that for a row i such that $[i] \to_{k(\mathrm{mod}\,\gamma_\nu)} [j]$ we have $(S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} \otimes R_\nu)_{i,\cdot} = (R_\nu)_{j,\cdot}$ and all such rows of $S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} \otimes R_\nu$ are equal to each other.

Our next aim is to define, for each ν, matrices C'_ν and R'_ν with γ_ν rows and γ_ν columns, such that $C_\nu S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} R_\nu[\Gamma(k)] = C'_\nu \otimes R'_\nu$. To form matrix C'_ν, we select a node of $\mathcal{G}_\nu^c$ from each cyclic class $\mathcal{C}_0^\nu, \ldots, \mathcal{C}_{\gamma_\nu - 1}^\nu$ and define the column of C'_ν whose index is the number of this node to be the column of C_ν with the same index. The rest of the columns of C'_ν are set to $-\infty$. To form matrix R'_ν, we use the same selected nodes, but this time (instead of taking columns of C_ν and making them columns of C'_ν) we take the rows from $S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} \otimes R_\nu$ whose indices are the numbers of selected nodes and make them rows of R'_ν. The rest of the rows of R'_ν are set to $-\infty$. Since the columns of C_ν with indices in the same cyclic class are equal to each other and the same is true about the rows of $S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} \otimes R_\nu$, we have $C_\nu S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} R_\nu[\Gamma(k)] = C'_\nu \otimes R'_\nu$, thus the factor rank of any of these terms is no more than γ_ν.

We next form the matrices $C' = \bigoplus_{\nu=1}^m C'_\nu$ and $R' = \bigoplus_{\nu=1}^m R'_\nu$. Obviously, $C'_{\nu_1} \otimes R'_{\nu_2} = -\infty$ for $\nu_1 \neq \nu_2$ and therefore

$$C' \otimes R' = \bigoplus_{\nu=1}^m C'_\nu \otimes R'_\nu = \bigoplus_{\nu=1}^m C_\nu S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} R_\nu[\Gamma(k)] = C S^{k(\mathrm{mod}\,\gamma)} R[\Gamma(k)].$$

Finally, as C' and, respectively, R' have $\sum_{\nu=1}^m \gamma_\nu$ columns with finite entries and, respectively, rows with finite entries with the same indices, $CS^{k(\mathrm{mod}\,\gamma)} R[\Gamma(k)] = C' \otimes R'$

has factor rank at most $\sum_{\nu=1}^{m} \gamma_\nu$. $\square$

The next result follows immediately from Theorem 3.3.11 and underpins the key factor rank aspect of the CSR decomposition.

Theorem 3.3.12 ([49],Corollary 3.11)**.** *If* $\Gamma(k)$ *is CSR, then its rank is no more than* $\sum_{\nu=1}^{m} \gamma_\nu$.

Let us also prove the following results that are similar to [83, Corollary 3.7].

Proposition 3.3.13 ([49],Proposition 3.12)**.** *For each* $\nu = 1, \ldots, m$

$$(C_\nu \otimes S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} \otimes R_\nu)_{\cdot,j} = (C_\nu \otimes S_\nu^{k(\mathrm{mod}\,\gamma_\nu)})_{\cdot,j} \quad \textit{for} \quad j \in \mathcal{N}_c^\nu$$
$$(C_\nu \otimes S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} \otimes R_\nu)_{i,\cdot} = (S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} \otimes R_\nu)_{i,\cdot} \quad \textit{for} \quad i \in \mathcal{N}_c^\nu.$$

Proof. As the proofs are very similar for both statements we will only prove the first and omit the proof for the second statement. We begin by observing that

$$(C_\nu \otimes S_\nu^{k(\mathrm{mod}\,\gamma_\nu)})_{i,j} = p\left(\mathcal{W}_{\mathcal{T}',\mathrm{init}}^{k+t_\nu\gamma_\nu}(i \to j)\right),$$

where we used the definitions of C_ν and S_ν and the identity $S_\nu^{(t_\nu+1)\gamma_\nu} = S_\nu^{t_\nu\gamma_\nu}$ (since $t_\nu\gamma_\nu \geq T(S_\nu)$). Here it is convenient to choose t_ν that satisfies $(t_\nu+1)\gamma_\nu - k(\mathrm{mod}\,\gamma_\nu) = (t+1)\gamma - k(\mathrm{mod}\,\gamma) = v$, with t used in the definition of $\mathcal{T}'$. With this choice $(t_\nu+1)\gamma_\nu \leq (t+1)\gamma$ but $t_\nu\gamma_\nu \geq t\gamma$ and this can be shown with Figure 3.1.

In Figure 3.1 we can see that, as $k(\mathrm{mod}\,\gamma) < \gamma$, v must sit somewhere on this line segment in between $t\gamma$ and $(t+1)\gamma$. The same also holds true for $t_\nu\gamma_\nu$ and $(t_\nu+1)\gamma_\nu$. Since $\gamma_\nu \leq \gamma$ then the segment $(t_\nu\gamma_\nu, (t_\nu+1)\gamma_\nu)$ must be smaller than $(t\gamma, (t+1)\gamma)$ and both inequalities follow.

Using (3.9), all we need to show is that $p\left(\mathcal{W}^{2k+v}_{\mathcal{T}',\text{full}}(i \xrightarrow{\mathcal{N}^\nu_c} j)\right) = p\left(\mathcal{W}^{k+t_\nu\gamma_\nu}_{\mathcal{T}',\text{init}}(i \to j)\right)$, where $v = (t+1)\gamma - k(\operatorname{mod}\gamma)$. We will achieve this by proving these two inequalities:

$$p\left(\mathcal{W}^{2k+v}_{\mathcal{T}',\text{full}}(i \xrightarrow{\mathcal{N}^\nu_c} j)\right) \geq p\left(\mathcal{W}^{k+t_\nu\gamma_\nu}_{\mathcal{T}',\text{init}}(i \to j)\right), \quad p\left(\mathcal{W}^{2k+v}_{\mathcal{T}',\text{full}}(i \xrightarrow{\mathcal{N}^\nu_c} j)\right) \leq p\left(\mathcal{W}^{k+t_\nu\gamma_\nu}_{\mathcal{T}',\text{init}}(i \to j)\right) \tag{3.14}$$

To prove the first inequality of (3.14) we first consider $\mathcal{W}^{k+t_\nu\gamma_\nu}_{\mathcal{T}',\text{init}}(i \to j')$, where j' is an arbitrary node of $[j]$. Optimal walk in any of these sets can be decomposed into 1) an optimal full walk on $\mathcal{T}$ connecting i to a node of $[j]$, and 2) a walk of weight 0 and length $t_\nu\gamma_\nu$ on $\mathcal{G}^c_\nu$ connecting that node of $[j]$ to j', whose existence follows since $t_\nu\gamma_\nu \geq T(S_\nu)$. This decomposition implies that the weights of all these optimal walks are equal. One of these walks denoted by W_1 can be concatenated with a walk W_2 on $\mathcal{G}^c_\nu$ of length $k - k(\operatorname{mod}\gamma_\nu) + \gamma$ and ending in j. We see that $p(W_1W_2) = p(W_1)$ and $W_1W_2 \in \mathcal{W}^{2k+v}_{\mathcal{T}',\text{full}}(i \xrightarrow{\mathcal{N}^\nu_c} j)$.

To prove the second inequality of (3.14) we take a walk in $\mathcal{W}^{2k+v}_{\mathcal{T}',\text{full}}(i \xrightarrow{\mathcal{N}^\nu_c} j)$ and decompose it into 1) a walk in $\mathcal{W}^{k+t_\nu\gamma_\nu}_{\mathcal{T}',\text{init}}(i \to j')$, where $j' \in [j]$, 2) a walk in $\mathcal{W}^{k-k(\operatorname{mod}\gamma_\nu)+\gamma_\nu}_{\mathcal{T}',\text{final}}(j' \to j)$. The weight of the first walk is bounded by $p\left(\mathcal{W}^{k+t_\nu\gamma_\nu}_{\mathcal{T}',\text{init}}(i \to j)\right)$, and the weight of the second walk is bounded by 0, thus the second inequality also holds. $\square$

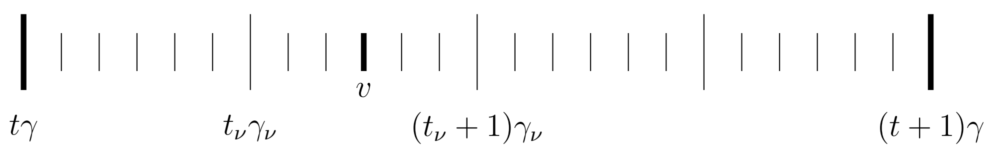


Figure 3.1: Visual proof of $(t_\nu + 1)\gamma_\nu \leq (t+1)\gamma$ and $t_\nu\gamma_\nu \geq t\gamma$

Corollary 3.3.14 ([49],Corollary 3.13)**.** *For CSR as defined in Definition 3.3.1 we* *have,*

$$(C \otimes S^{k(\operatorname{mod}\gamma)} \otimes R)_{\cdot,j} = (C \otimes S^{k(\operatorname{mod}\gamma)})_{\cdot,j} \quad \textit{for} \quad j \in \mathcal{N}_c$$
$$(C \otimes S^{k(\operatorname{mod}\gamma)} \otimes R)_{i,\cdot} = (S^{k(\operatorname{mod}\gamma)} \otimes R)_{i,\cdot} \quad \textit{for} \quad i \in \mathcal{N}_c.$$

Proof. The proofs for both statements are similar so we will only prove the first one.

Let $j \in \mathcal{N}_c$. As all nodes from $\mathcal{N}_c$ can be sorted into $\mathcal{N}_c^\nu$ for some $\nu = 1, \ldots, m$, assume without loss of generality that $j \in \mathcal{N}_c^\mu$.

Taking the RHS of the first statement and using (3.6), we have

$$(C \otimes S^{k(\operatorname{mod}\gamma)})_{\cdot,j} = \left(\bigoplus_{\nu=1}^{m} C_\nu \otimes S_\nu^{k(\operatorname{mod}\gamma_\nu)} \right)_{\cdot,j}.$$

By Definition 3.3.2, if $j \in \mathcal{N}_c^\mu$ then for all $\nu \neq \mu$, $(C_\nu \otimes S_\nu^{k(\operatorname{mod}\gamma_\nu)})_{\cdot,j} = -\infty$. Therefore, for every ν, $(C_\nu \otimes S_\nu^{k(\operatorname{mod}\gamma_\nu)})_{\cdot,j}$ will be dominated by $(C_\mu \otimes S_\mu^{k(\operatorname{mod}\gamma_\mu)})_{\cdot,j}$. Hence,

$$\left(\bigoplus_{\nu=1}^{m} C_\nu \otimes S_\nu^{k(\operatorname{mod}\gamma_\nu)} \right)_{\cdot,j} = (C_\mu \otimes S_\mu^{k(\operatorname{mod}\gamma_\mu)})_{\cdot,j}. \tag{3.15}$$

Turning our attention to the LHS of the first statement, by (3.6) we get

$$(C \otimes S^{k(\operatorname{mod}\gamma)} \otimes R)_{\cdot,j} = \left(\bigoplus_{\nu=1}^{m} C_\nu \otimes S_\nu^{k(\operatorname{mod}\gamma_\nu)} \otimes R_\nu \right)_{\cdot,j}.$$

Now we must show that, for $j \in \mathcal{N}_c^\mu$ and for all ν, $(C_\nu \otimes S_\nu^{k(\operatorname{mod}\gamma_\nu)} \otimes R_\nu)_{\cdot,j} \leq$

$(C_\mu \otimes S_\mu^{k(\mathrm{mod}\,\gamma_\mu)} \otimes R_\mu)_{\cdot,j}$. By (3.9) this is the same as saying

$$p\left(\mathcal{W}^{2k+v}_{\mathcal{T}',\mathrm{full}}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right) \leq p\left(\mathcal{W}^{2k+v}_{\mathcal{T}',\mathrm{full}}(i \xrightarrow{\mathcal{N}_c^\mu} j)\right)$$

for arbitrary node i. Let W be the walk of length $2k+v$ connecting i to j that traverses $\mathcal{N}_c^\nu$, such that $p(W) = p\left(\mathcal{W}^{2k+v}_{\mathcal{T}',\mathrm{full}}(i \xrightarrow{\mathcal{N}_c^\nu} j)\right)$. As $j \in \mathcal{N}_c^\mu$ then W is also a walk of length $2k+v$ connecting i to j that traverses $\mathcal{N}_c^\mu$, hence $W \in \mathcal{W}^{2k+v}_{\mathcal{T}',\mathrm{full}}(i \xrightarrow{\mathcal{N}_c^\mu} j)$ and the inequality holds.

Therefore, as with the RHS, we have

$$\left(\bigoplus_{\nu=1}^{m} C_\nu \otimes S_\nu^{k(\mathrm{mod}\,\gamma_\nu)} \otimes R_\nu\right)_{\cdot,j} = (C_\mu \otimes S_\mu^{k(\mathrm{mod}\,\gamma_\mu)} \otimes R_\mu)_{\cdot,j}. \tag{3.16}$$

Finally the first statement of Proposition 3.3.13 gives us equality between (3.15) and (3.16). As j was chosen arbitrarily, this holds for any $j \in \mathcal{N}_c$ and the result follows. □

3.4 General results

This section presents some results that hold for general inhomogeneous products satisfying the assumptions set out in Section 3.2.1.

Notation 3.4.1 (q)**.** *We will denote by q the number of critical nodes, i.e., $q = |\mathcal{N}_c|$.*

The following results develop bounds for strict initial walks to the critical nodes and strict final walks from the critical nodes for any given critical subgraph. Observe that, under Assumptions $\mathcal{B}$ and $\mathcal{D}2$, we have $\lambda_* < 0$, so that the bounds in the following lemmas make sense. Recall the sets of walks $\mathcal{W}_{\mathcal{T},\mathrm{init}}(i \to \mathcal{N}_c\|)$ and $\mathcal{W}_{\mathcal{T},\mathrm{final}}(\|\mathcal{N}_c \to j)$ introduced in Notation 3.2.4.

Lemma 3.4.2 ([49],Lemma 4.3)**.** *Let $W_{i,\mathcal{N}_c}$ be an optimal walk in $\mathcal{W}_{\mathcal{T},\mathrm{init}}(i \to \mathcal{N}_c\|)$, so that $p(W_{i,\mathcal{N}_c}) = w^*_{i,\mathcal{N}_c}$. Then we have the following bound on the length of $W_{i,\mathcal{N}_c}$:*

$$l(W_{i,\mathcal{N}_c}) \leq \begin{cases} d-q, & \text{if } \lambda_* = \varepsilon, \\ \frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (d-q), & \text{if } \lambda_* > \varepsilon \end{cases} \tag{3.17}$$

Proof. If $\lambda_* = \varepsilon$, then any walk in $\mathcal{W}_{\mathcal{T},\mathrm{init}}(i \to \mathcal{N}_c\|)$ has to be a path, and its length is bounded by $d-q$. Now let $\lambda_* > \varepsilon$. As $\lambda_* < 0$, the weight of the walk $W_{i,\mathcal{N}_c}$ connecting i to a node in $\mathcal{N}_c$ is less than or equal to that of a path $P_{i,\mathcal{N}_c}$ on $\mathcal{D}(A^{\sup})$ connecting i to a node in $\mathcal{N}_c$ plus the remaining length multiplied by λ_*. The remaining length is bounded from above by $d-q$, since all intermediate nodes in $W_{i,\mathcal{N}_c}$ are non-critical. Hence

$$p_{\mathcal{T}}(W_{i,\mathcal{N}_c}) \leq p_{\sup}(P_{i,\mathcal{N}_c}) + (l(W_{i,\mathcal{N}_c}) - (d-q))\lambda_*.$$

We can bound $p_{\sup}(P_{i,\mathcal{N}_c}) \leq \alpha_{i,\mathcal{N}_c}$, so

$$p_{\mathcal{T}}(W_{i,\mathcal{N}_c}) \leq \alpha_{i,\mathcal{N}_c} + (l(W_{i,\mathcal{N}_c}) - (d-q))\lambda_*. \tag{3.18}$$

Now assuming for contradiction that $l(W_{i,\mathcal{N}_c}) > \frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (d-q)$. This is equivalent to

$$\alpha_{i,\mathcal{N}_c} + (l(W_{i,\mathcal{N}_c}) - (d-q))\lambda_* < w^*_{i,\mathcal{N}_c}. \tag{3.19}$$

In combining (3.18) and (3.19) we get $p_{\mathcal{T}}(W_{i,\mathcal{N}_c}) < w^*_{i,\mathcal{N}_c}$ meaning that $W_{i,\mathcal{N}_c}$ is not optimal, a contradiction.Therefore, for any $l \in \mathcal{N}_c$

$$l(W_{i,\mathcal{N}_c}) \leq \frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (d-q).$$

The proof is complete. $\square$

Lemma 3.4.3. *Let $W_{\mathcal{N}_c,j}$ be an optimal walk in $\mathcal{W}_{\mathcal{T},\text{final}}(\|\mathcal{N}_c \to j)$, so that $p(W_{\mathcal{N}_c,j}) = v^*_{\mathcal{N}_c,j}$. Then we have the following bound on the length of $W_{\mathcal{N}_c,j}$:*

$$l(W_{\mathcal{N}_c,j}) \leq \begin{cases} d-q, & \text{if } \lambda_* = \varepsilon, \\ \frac{v^*_{\mathcal{N}_c,j} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + (d-q), & \text{if } \lambda^* > \varepsilon. \end{cases} \tag{3.20}$$

As the proof of this lemma is analogous to the proof of Lemma 3.4.2 it is omitted.

Remark 3.4.4. *Observe that $d-q$ is the limit of the expressions on the RHS of* (3.17) *and* (3.20) *as $\lambda_* \to \varepsilon$, hence we will not consider this case separately in the rest of the chapter. If $i \in \mathcal{N}_c$ or $j \in \mathcal{N}_c$ then the length of the walk is, by definition, zero. Therefore we shall use the adjusted bounds*

$$l(W_{i,\mathcal{N}_c}) \leq \overline{\delta}(i, \mathcal{N}_c) \cdot \left(\frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (d-q) \right) \tag{3.21}$$

$$l(W_{\mathcal{N}_c,j}) \leq \overline{\delta}(\mathcal{N}_c, j) \cdot \left(\frac{v^*_{\mathcal{N}_c,j} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + (d-q) \right) \tag{3.22}$$

where

$$\bar{\delta}(i, \mathcal{N}_c) = \begin{cases} 0 & \text{if } i \in \mathcal{N}_c \\ 1 & \text{otherwise.} \end{cases} \tag{3.23}$$

$$\bar{\delta}(\mathcal{N}_c, j) = \begin{cases} 0 & \text{if } j \in \mathcal{N}_c \\ 1 & \text{otherwise.} \end{cases} \tag{3.24}$$

The following result is a bound designed for walks avoiding a subset of nodes, which uses a nominal weight ω.

Lemma 3.4.5. *If $\gamma_{i,j} = \varepsilon$, then any full walk connecting i to j on $\mathcal{T}(\Gamma(k))$ traverses a node in $\mathcal{N}_c$.*
If $\gamma_{i,j} > \varepsilon$, let

$$k > \frac{\omega - \gamma_{i,j}}{\lambda_*} + (d - q) \tag{3.25}$$

for some $\omega \in \mathbb{R}$. Then any full walk W connecting i to j on $\mathcal{T}(\Gamma(k))$ that does not go through any node $l \in \mathcal{N}_c$ has weight smaller than ω.

Proof. In the case when $\gamma_{i,j} = \varepsilon$, the claim follows by the definition of $\gamma_{i,j}$ and by the geometric equivalence between A^{sup} and the matrices from $\mathcal{X}$. So we assume that $\gamma_{i,j} > \varepsilon$. Any walk W that does not traverse any node in $\mathcal{N}_c$ can be decomposed into a path P connecting i to j avoiding $\mathcal{N}_c$ and a number of cycles. Hence we have the following bound:

$$p_{\mathcal{T}}(W) \leq p_{\text{sup}}(P) + (k - (d - q))\lambda_*.$$

We can further bound $p_{\sup}(P) \leq \gamma_{i,j}$ so

$$p_{\mathcal{T}}(W) \leq \gamma_{i,j} + (k - (d - q))\lambda_*. \tag{3.26}$$

Now (3.25) can be rewritten as

$$\gamma_{i,j} + (k - (d - q))\lambda_* < \omega. \tag{3.27}$$

By combining (3.26) with (3.27) we have $p_{\mathcal{T}}(W) < \omega$, which completes the proof. $\square$

Using this bound we can obtain a bound after which the CSR term becomes a valid upper bound for $\Gamma(k)$.

Theorem 3.4.6. *If $\gamma_{i,j} = \varepsilon$ then $\Gamma(k)_{i,j} \leq \left(CS^{k(\mathrm{mod}\,\gamma)}R[\Gamma(k)]\right)_{i,j}$.*
If $\gamma_{i,j} > \varepsilon$, let

$$k > \max_{i,j\colon i \to_{\mathcal{T}} j, \gamma_{i,j} > \varepsilon} \left(\frac{\Gamma(k)_{i,j} - \gamma_{i,j}}{\lambda_*} + (d - q) \right). \tag{3.28}$$

Then $\Gamma(k)_{i,j} \leq (CS^{k(\mathrm{mod}\,\gamma)}R[\Gamma(k)])_{i,j}$ for all $i, j \in N$.

Proof. If $i \not\to_{\mathcal{T}} j$, then $(\Gamma(k))_{i,j} = \varepsilon$. In this case, obviously, $\Gamma(k)_{i,j} \leq (CS^{k(\mathrm{mod}\,\gamma)}R[\Gamma(k)])_{i,j}$.

If $i \to_{\mathcal{T}} j$, then $(\Gamma(k))_{i,j} \neq \varepsilon$. Let W^* be an optimal walk of length k on $\mathcal{T}(\Gamma(k))$ connecting i to j with weight $\Gamma(k)_{i,j}$. If k is greater than the bound (3.28) then, by Lemma 3.4.5, for the walk to have weight equal to $\Gamma(k)_{i,j}$, it must traverse at least one node in $\mathcal{N}_c$ The same is true when $\gamma_{i,j} = \varepsilon$ and in this case, the expression (3.28) is equal to $d - q$. Hence this walk belongs to the set $\mathcal{W}^k_{\mathcal{T}}(i \xrightarrow{\mathcal{N}_c} j)$ and further $\Gamma(k)_{i,j} = p(W^*) \leq p\left(\mathcal{W}^k_{\mathcal{T}}(i \xrightarrow{\mathcal{N}_c} j)\right)$.

Let $f \in \mathcal{N}_c$ be the first critical node in the first critical s.c.c $\mathcal{G}^c_\nu$, with cyclicity γ_ν, that W^* traverses. We can split the walk into $W^* = W_1 W_3$ where W_1 is a walk connecting i to f of length r and W_3 is a walk connecting f to j of length $k-r$. We have $p(W^*) = p(W_1) + p(W_3)$.

Let $\mathcal{T}'$ be the trellis extension for the matrix product $CS^{k(\bmod \gamma)}R[\Gamma(k)]$ with length $2k+v$ where $v = (t+1)\gamma - k(\bmod \gamma)$ as described in Definition 3.3.6.

We now introduce the new walk $W' = W_1W_2W_3$ on $\mathcal{T}'$. Here W_1 and W_3 are the subwalks from W^* introduced before, where W_1 is viewed as an initial walk on $\mathcal{T}'$ and W_3 as a final walk on $\mathcal{T}'$, and W_2 is a cycle of length $k+v$ that starts and ends at f. Since $k+v \equiv 0(\bmod \gamma_\nu)$ and $k+v \geq T(S) \geq T(S_\nu)$, this cycle exists and can be entirely made up of edges from $\mathcal{G}^c_\nu$. This means the walk W' is of length $2k+v$ and it traverses the set of nodes $\mathcal{N}^\nu_c$ therefore $W' \in \mathcal{W}^{2k+v}_{\mathcal{T}'}(i \xrightarrow{\mathcal{N}^\nu_c} j)$.

As W_2 is made entirely from critical edges, we have $p(W_2) = 0$ and $p(W^*) = p(W') \leq p\left(\mathcal{W}^{2k+v}_{\mathcal{T}'}(i \xrightarrow{\mathcal{N}^\nu_c} j)\right)$, and using (3.37) gives us

$$\Gamma(k)_{i,j} = p(W^*) \leq (C_\nu S_\nu^{k(\bmod \gamma_\nu)} R_\nu[\Gamma(k)])_{i,j} \leq (CS^{k(\bmod \gamma)}R[\Gamma(k)])_{i,j},$$

where the last inequality is due to Proposition 3.3.5. The claim follows.

$\square$

This bound is implicit, as it requires $\Gamma(k)$ to be calculated in order to generate the transient. However, we can use $A^{\inf}$ and $u_{i,j}$ to develop an explicit bound.

Corollary 3.4.7. *Let*

$$k > \max_{i,j\,:\, i \to_{\mathcal{T}} j, \gamma_{i,j} > \varepsilon} \left(\frac{u^k_{i,j} - \gamma_{i,j}}{\lambda_*} + (d-q) \right). \tag{3.29}$$

Then $\Gamma(k) \leq CS^{k(\bmod \gamma)}R[\Gamma(k)]$.

Proof. By Lemma 3.2.10, $i \to_{\mathcal{T}} j$ is equivalent to $u^k_{i,j} > \varepsilon$, so maximum in (3.29) is taken over i, j for which $u^k_{i,j}$ and $\gamma_{i,j}$ are finite. We also have $u^k_{i,j} \leq (\Gamma(k))_{i,j}$ by the definition of A^{inf}.

Further, as $\lambda_* < 0$, then any k that satisfies (3.29) will also satisfy (3.28). The claim now follows from Theorem 3.4.6. $\square$

Remark 3.4.8. *All the results in this section do not require common visualisation scaling on the matrices from $\mathcal{X}$, but we need $\lambda_* < 0$ and we require all critical edges to have weight zero in all matrices of $\mathcal{X}$.*

3.5 The case where CSR works

In this section we present the results to the case when $\mathcal{D}(\mathcal{X})$ and $\mathcal{G}^c(\mathcal{X})$ satisfy the following assumption, in addition to the assumptions that were set out in Section 3.2.1.

Assumption 3.5.1. *$\mathcal{G}^c(\mathcal{X})$ is strongly connected and its cyclicity γ is equal to the cyclicity of $\mathcal{D}(\mathcal{X})$.*

The equality between cyclicities means that the associated digraph $\mathcal{D}(\mathcal{X})$ has the same number of cyclic classes γ as $\mathcal{G}^c(\mathcal{X})$.

Notation 3.5.2. *The cyclic classes of $\mathcal{D}(\mathcal{X})$ are denoted by $\mathcal{C}'_0, \ldots, \mathcal{C}'_{\gamma-1}$.*
For a node $i \in N$, the cyclic class of this node with respect to $\mathcal{D}(\mathcal{X})$ will be denoted by $[i]'$.

For a node $i \in \mathcal{N}_c$, we will use both $[i]$ (the cyclic class with respect to $\mathcal{G}^c(\mathcal{X})$) and $[i]'$ (the cyclic class with respect to $\mathcal{D}(\mathcal{X})$), and an obvious inclusion relation between them: $[i] \subseteq [i]'$.

One of the ideas is to combine Lemmas 3.4.2 and 3.4.3 together with Schwarz's bound. To define this bound, following the work by Merlet et al. [68], we require the Schwarz number (1.1.1)

$$\mathrm{Sch}(\gamma, d) = \gamma \, \mathrm{Wi}\left(\left\lfloor \frac{d}{\gamma} \right\rfloor\right) + d (\mathrm{mod}\, \gamma).$$

Here $\mathrm{Wi}(d)$ is the Wielandt's number (1.1.1)

Let us now prove the following lemma.

Lemma 3.5.3 ([49],Lemma 5.2). *Let*

$$k \geq \overline{\delta}(i, \mathcal{N}_c) \cdot \left(\frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (d-q) \right) + \mathrm{Sch}(\gamma, q) + \overline{\delta}(\mathcal{N}_c, j) \cdot \left(\frac{v^*_{\mathcal{N}_c,j} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + (d-q) \right). \tag{3.30}$$

Then

(i) If $[i]' \not\to_k [j]'$ *then there are no full walks connecting* i *to* j *on* $\mathcal{T}(\Gamma(k))$ *(i.e.,* $i \not\to_{\mathcal{T}} j$*).*

(ii) If $[i]' \to_k [j]'$*, then there is a full walk* W *connecting* i *to* j *on* $\mathcal{T}(\Gamma(k))$ *and going through a critical node, and we have* $p_{\mathcal{T}}(W) = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$ *if* W *is optimal.*

Proof. The property $[i]' \not\to_k [j]'$ implies that there is no full walk W connecting i to j on $\mathcal{T}(\Gamma(k))$.

In the case $[i]' \to_k [j]'$, we construct a walk $W' = W_{i,\mathcal{N}_c} W_c W_{\mathcal{N}_c,j}$ of length k, where $W_{i,\mathcal{N}_c}$ be an optimal walk in $\mathcal{W}_{\mathcal{T},\mathrm{init}}(i \to \mathcal{N}_c \|)$ (see Lemma 3.4.2) , $W_{\mathcal{N}_c,j}$ be an optimal walk in $\mathcal{W}_{\mathcal{T},\mathrm{final}}(\|\mathcal{N}_c \to j)$ (see Lemma 3.4.3), and W_c is a walk that connects the end of $W_{i,\mathcal{N}_c}$ to the beginning of $W_{\mathcal{N}_c,j}$ and such that all edges of W_c are critical (the existence of such W_c is yet to be proved). Without loss of generality set $[i]' = \mathcal{C}'_0$ and $[j]' = \mathcal{C}'_{p_3}$: the cyclic classes of $\mathcal{D}(\mathcal{X})$ to which i and j belong. Let x be the final node of $W_{i,\mathcal{N}_c}$ and let y be the first node of $W_{\mathcal{N}_c,j}$. Set $[x]' = \mathcal{C}'_{p_1}$ and $[y]' = \mathcal{C}'_{p_2}$.

By [9, Lemma 3.4.1.iv] $l(W_{i,\mathcal{N}_c}) \equiv p_1 (\operatorname{mod} \gamma)$, $l(W_{\mathcal{N}_c,j}) \equiv (p_3 - p_2)(\operatorname{mod} \gamma)$. Hence the congruence of the walk W_c to be inserted is $(p_3 - p_1 - (p_3 - p_2))(\operatorname{mod} \gamma) \equiv (p_2 - p_1)(\operatorname{mod} \gamma)$. As the cyclicity of the critical subgraph is the same as that of the digraph, the cyclic classes of the critical subgraph are $\mathcal{C}_0, \dots, \mathcal{C}_{\gamma-1}$ and we can assume that the numbering is such that $\mathcal{C}_0 \subseteq \mathcal{C}'_0, \dots,$ $\mathcal{C}_{\gamma-1} \subseteq \mathcal{C}'_{\gamma-1}$. Then $x \in \mathcal{C}_{p_1}$ and $y \in \mathcal{C}_{p_2}$ and by [9, Lemma 3.4.1.iv] there exists a walk on the critical subgraph of length congruent to $(p_2 - p_1)(\operatorname{mod} \gamma)$. Moreover, all walks connecting x to y have such length and by Schwarz's bound if $k - l(W_{i,\mathcal{N}_c}) - l(W_{\mathcal{N}_c,j}) \geq \operatorname{Sch}(\gamma, q)$ then there is a walk of length equal to $l(W') - l(W_{i,\mathcal{N}_c}) - l(W_{\mathcal{N}_c,j})$. According to Lemmas 3.4.2, 3.4.3, and Remark 3.4.4; $l(W_{i,\mathcal{N}_c}) \leq \overline{\delta}(i, \mathcal{N}_c) \cdot \left(\frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (d-q)\right)$, $l(W_{\mathcal{N}_c,j}) \leq \overline{\delta}(\mathcal{N}_c, j) \cdot \left(\frac{v^*_{\mathcal{N}_c,j} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + (d-q)\right)$, therefore k is a sufficient length for $k - l(W_{i,\mathcal{N}_c}) - l(W_{\mathcal{N}_c,j})$ to satisfy Schwarz's bound, so a walk of the form $W' = W_{i,\mathcal{N}_c} W_c W_{\mathcal{N}_c,j}$ exists and $p(W') = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$.

Let now W be an optimal full walk connecting i to j on $\mathcal{T}$ that passes through $\mathcal{N}_c$ at least once. As it passes through the critical nodes then the walk can be decomposed into $W = \tilde{W}_{i,\mathcal{N}_c} \tilde{W}_c \tilde{W}_{\mathcal{N}_c,j}$ where $\tilde{W}_{i,\mathcal{N}_c}$ is a walk in $\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c \|)$, and $\tilde{W}_{\mathcal{N}_c,j}$ is a walk in $\mathcal{W}_{\mathcal{T},\text{final}}(\| \mathcal{N}_c \to j)$, and $\tilde{W}_c$ connects the end of $\tilde{W}_{i,\mathcal{N}_c}$ to the beginning of $\tilde{W}_{\mathcal{N}_c,j}$ on $\mathcal{T}(\Gamma(k))$. We then have $p_{\mathcal{T}}(\tilde{W}_{i,\mathcal{N}_c}) \leq p_{\mathcal{T}}(W_{i,\mathcal{N}_c})$ and $p_{\mathcal{T}}(\tilde{W}_{\mathcal{N}_c,j}) \leq p_{\mathcal{T}}(W_{\mathcal{N}_c,j})$ and also $p_{\mathcal{T}}(\tilde{W}_c) \leq p(W_c) = 0$. Since W is optimal then all of these inequalities hold with equality, and $p_{\mathcal{T}}(W) = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$, as claimed. □

Remark 3.5.4. *It follows from the proof that, under the conditions of this lemma* *and in the case $[i]' \to_k [j]'$, there is an optimal full walk connecting i to j on $\mathcal{T}_{\Gamma(k)}$ and traversing a critical node that can be decomposed as $W = W_{i,\mathcal{N}_c} W_c W_{\mathcal{N}_c,j}$, where $W_{i,\mathcal{N}_c}$ is an optimal walk in $\mathcal{W}_{\mathcal{T},\mathrm{init}}(i \to \mathcal{N}_c\|)$ and $W_{\mathcal{N}_c,j}$ is an optimal walk in $\mathcal{W}_{\mathcal{T},\mathrm{final}}(\|\mathcal{N}_c \to j)$, and W_c consists of edges solely in the critical subgraph. If semigroup's generators are also strictly visualised in the sense of the article by Butkovič, Schneider and Sergeev [84], then any such optimal full walk has to be of this form.*

Lemma 3.5.3 gives us the first part of the final bound for the case. In order to be able to use this lemma we must ensure that the walk must traverse $\mathcal{N}_c$ hence we can use Lemma 3.4.5 in conjunction with Lemma 3.5.3 to give us the following theorem. For compactness we define the following notation.

Notation 3.5.5. *Given the set $\mathcal{X}$, inhomogeneous matrix product $\Gamma(k)$ and indices i and j, we define*

$$\begin{aligned} T_{\alpha\beta}(\mathcal{X}, \Gamma(k), i, j) &:= \overline{\delta}(i, \mathcal{N}_c) \cdot \left(\frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (d-q) \right) + \mathrm{Sch}(\gamma, q) \\ &\quad + \overline{\delta}(\mathcal{N}_c, j) \cdot \left(\frac{v^*_{\mathcal{N}_c,j} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + (d-q) \right) \\ T_{\gamma}(\mathcal{X}, \Gamma(k), i, j) &:= \frac{w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j} - \gamma_{i,j}}{\lambda_*} + (d-q+1) \end{aligned}$$

Theorem 3.5.6 ([49],Theorem 5.4)**.** *Let*

$$k \geq \max\left(T_{\alpha\beta}(\mathcal{X},\ \Gamma(k), i, j), T_{\gamma}(\mathcal{X}, \Gamma(k), i, j)\right) \tag{3.31}$$

if $\gamma_{i,j} > \varepsilon$ or just

$$k \geq T_{\alpha\beta}(\mathcal{X}, \Gamma(k), i, j), \tag{3.32}$$

if $\gamma_{i,j} = \varepsilon$, for some $i, j \in N$. Then

(i) *If $[i]' \not\to_k [j]'$ then $\Gamma(k)_{i,j} = \varepsilon$,*

(ii) *If $[i]' \to_k [j]'$ then $\Gamma(k)_{i,j} = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$.*

Proof. We only need to prove the second part. By Lemma 3.4.5 and taking $\omega = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$, if

$$k > \frac{w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j} - \gamma_{i,j}}{\lambda_*} + (d-q)$$

then any walk on $\mathcal{T}(\Gamma(k))$ that does not traverse the nodes in $\mathcal{N}_c$ will have weight smaller than $w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$, or such walk will not exist if $\gamma_{i,j} = \varepsilon$. Using Lemma 3.5.3, if

$$k \geq \overline{\delta}(i, \mathcal{N}_c) \cdot \left(\frac{w^*_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (d-q) \right) + \mathrm{Sch}(\gamma, q) + \overline{\delta}(\mathcal{N}_c, j) \cdot \left(\frac{v^*_{\mathcal{N}_c,j} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + (d-q) \right)$$

and $[i]' \to_k [j]'$ then the weight of any optimal full walk on $\mathcal{T}(\Gamma(k))$ connecting i to j and traversing a critical node will be equal to $w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$. If $\gamma_{i,j} = \varepsilon$, $[i]' \to_k [j]'$ and the above inequality holds, or if $\gamma_{i,j} > \varepsilon$, k satisfies both inequalities and $[i]' \to_k [j]'$, then any optimal full walk traverses nodes in $\mathcal{N}_c$ and has weight

$$\Gamma(k)_{i,j} = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}.$$

□

Our next aim is to rewrite Theorem 3.5.6 in a CSR form, and we first want to look at the optimal walk representations of $w^*_{i,\mathcal{N}_c}$ and $v^*_{\mathcal{N}_c,j}$. This leads us to the following lemma.

Lemma 3.5.7 ([49],Lemma 5.5)**.** *We have*

$$w^*_{i,\mathcal{N}_c} = p(\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c)), \quad v^*_{\mathcal{N}_c,j} = p(\mathcal{W}^k_{\mathcal{T},\text{full}}(\mathcal{N}_c \to j)). \tag{3.33}$$

Proof. We will prove only the first of these two equalities, as the second one can be proved in a similar way.

Let $W_{i,\mathcal{N}_c}$ be an optimal walk in $\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)$, with weight $w^*_{i,\mathcal{N}_c}$. We are required to prove that

$$p\left(\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)\right) = p\left(\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c)\right), \tag{3.34}$$

where on the right we have the set of full walks connecting i to a critical node on $\mathcal{T}(\Gamma(k))$. We split (3.34) into two inequalities,

$$p\left(\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)\right) \leq p\left(\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c)\right), \quad p\left(\mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)\right) \geq p\left(\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c)\right) \tag{3.35}$$

For the first inequality in (3.35), observe that we can concatenate $W_{i,\mathcal{N}_c}$ with a walk V on the critical graph which has length $l(V) = k - l(W_{i,\mathcal{N}_c})$. The resulting walk $W_{i,\mathcal{N}_c}V$ belongs to $\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c)$ and has weight $w^*_{i,\mathcal{N}_c}$, which proves the first inequality. For the second inequality, take an optimal walk $W^* \in \mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c)$, whose weight is $p(\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c))$. By observing the first occurrence of a critical node in this walk, we represent $W^* = WV$, where $W \in \mathcal{W}_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c\|)$. We then have $p(W^*) = p(W) + p(V) \leq p(W) \leq w^*_{i,\mathcal{N}_c}$ proving the second inequality. Combining both inequalities gives the equality (3.34) and finishes the proof of $w^*_{i,\mathcal{N}_c} = p(\mathcal{W}^k_{\mathcal{T},\text{full}}(i \to \mathcal{N}_c))$. The second part of the claim is proved similarly. □

Remark 3.5.8. *In the previous lemma, the length of the walks on the RHS does not have to be restricted to k. We can obtain the following results:*

$$w^*_{i,\mathcal{N}_c} = p(\mathcal{W}^l_{\mathcal{T},\text{init}}(i \to \mathcal{N}_c)) \quad \text{for any } l \geq \min\left(\overline{\delta}(i,\mathcal{N}_c)\cdot\left(\frac{w^*_{i,\mathcal{N}_c}-\alpha_{i,\mathcal{N}_c}}{\lambda_*}+(d-q)\right),k\right)$$
$$v^*_{\mathcal{N}_c,j} = p(\mathcal{W}^m_{\mathcal{T},\text{final}}(\mathcal{N}_c \to j)) \quad \text{for any } m \geq \min\left(\overline{\delta}(\mathcal{N}_c,j)\cdot\left(\frac{v^*_{\mathcal{N}_c,j}-\beta_{\mathcal{N}_c,j}}{\lambda_*}+(d-q)\right),k\right). \tag{3.36}$$

We now establish the connection between the previous Lemma and CSR.

Lemma 3.5.9 ([49],Lemma 5.7)**.** *We have one of the following cases:*

(i) $(CS^{k(\bmod\gamma)}R[\Gamma(k)])_{i,j}=\varepsilon$ *if* $[i]'\not\to_k [j]'$,

(ii) $(CS^{k(\bmod\gamma)}R[\Gamma(k)])_{i,j}=w^*_{i,\mathcal{N}_c}+v^*_{\mathcal{N}_c,j}$ *if* $[i]'\to_k [j]'$.

Proof. By Lemma 3.3.7 we have $p\left(\mathcal{W}^{2k+v}_{\mathcal{T}',\text{full}}(i\to j)\right) = (CS^{k(\bmod\gamma)}R[\Gamma(k)])_{i,j}$, where $v=(t+1)\gamma-k(\bmod\gamma)$ and $t\gamma\geq T(S)$, and let $W\in\mathcal{W}^{2k+v}_{\mathcal{T}',\text{full}}(i\to j)$ be optimal. W can be decomposed as $W_1W_2W_3$ where W_1 is a full walk (of length k) connecting i to some $l\in\mathcal{N}_c$ on $\mathcal{T}$, W_3 is a (full) walk of length k connecting some $m\in\mathcal{N}_c$ to j and W_2 is a walk on the critical graph of length v connecting the end of W_1 to the beginning of W_3. In formula,

$$\begin{aligned}(CS^{k(\bmod\gamma)}R[\Gamma(k)])_{i,j} = \max\{&p(W_1)+p(W_2)+p(W_3)\colon\\ &W_1\in\mathcal{W}^k_{\mathcal{T},\text{full}}(i\to l),\ W_2\in\mathcal{W}^v_{\mathcal{G}^c}(l\to m),\ W_3\in\mathcal{W}^k_{\mathcal{T},\text{full}}(m\to j),\ l,m\in\mathcal{N}_c\}\end{aligned} \tag{3.37}$$

If the weights of W_1, W_2 and W_3 in (3.37) are finite then $[i]'\to_k [l]'$, $[l]'\to_v [m]'$ and $[m]'\to_k [j]'$, hence $[i]'\to_k [j]'$. Thus $(CS^tR[\Gamma(k)]_{i,j})>\varepsilon$ implies $[i]'\to_k [j]'$ proving (i).

As the cyclicity of the associated graph is the same as the cyclicity of the critical graph, Lemma 3.5.7 implies that

$$w^*_{i,\mathcal{N}_c} = p(\mathcal{W}^k_{\mathcal{T}}(i \to \mathcal{C}_{i,k})), \quad v^*_{\mathcal{N}_c,j} = p(\mathcal{W}^k_{\mathcal{T}}(\mathcal{C}_{k,j} \to j)), \tag{3.38}$$

where $\mathcal{C}_{i,k} = \mathcal{C}'_{i,k} \cap \mathcal{N}_c$ is the cyclic class of $\mathcal{G}^c(\mathcal{X})$ that can be found by intersecting with critical nodes $\mathcal{N}_c$ the cyclic class $\mathcal{C}'_{i,k}$ of $\mathcal{D}$ defined by $[i]' \to_k \mathcal{C}'_{i,k}$. Similarly, $\mathcal{C}_{k,j} = \mathcal{C}'_{k,j} \cap \mathcal{N}_c$ is the cyclic class of $\mathcal{G}^c(\mathcal{X})$ that can be found by intersecting with critical nodes $\mathcal{N}_c$ the cyclic class $\mathcal{C}'_{k,j}$ of $\mathcal{D}$ defined by $\mathcal{C}'_{k,j} \to_k [j]'$.

Now note that in (3.37) we can similarly restrict l to $\mathcal{C}_{i,k}$ and m to $\mathcal{C}_{k,j}$, which transforms it to

$$\begin{aligned}(CS^{k(\mathrm{mod}\,\gamma)}R[\Gamma(k)])_{i,j} = \max\{&p(W_1)+p(W_2)+p(W_3)\colon \\ &W_1 \in \mathcal{W}^k_{\mathcal{T}}(i \to l),\ W_2 \in \mathcal{W}^v_{\mathcal{G}^c}(l \to m),\ W_3 \in \mathcal{W}^k_{\mathcal{T}}(m \to j),\ l \in \mathcal{C}_{i,k},\ m \in \mathcal{C}_{k,j}\}\end{aligned} \tag{3.39}$$

Note that if a walk W_2 exists between any $l \in \mathcal{C}_{i,k}$ and $m \in \mathcal{C}_{k,j}$ then using (3.38) we immediately obtain $(CS^{k(\mathrm{mod}\,\gamma)}R[\Gamma(k)])_{i,j} = w^*_{i,\mathcal{N}_c} + v^*_{\mathcal{N}_c,j}$. Thus it remains to show existence of $W_2 \in \mathcal{W}^v_{\mathcal{G}^c}(l \to m)$ between any $l \in \mathcal{C}_{i,k}$ and $m \in \mathcal{C}_{k,j}$. For this note that since $v = (t+1)\gamma - k(\mathrm{mod}\,\gamma) \geq T(S)$, either $\mathcal{C}_{i,k} \to_{(\gamma-k(\mathrm{mod}\,\gamma))} \mathcal{C}_{k,j}$ and a walk on $\mathcal{G}^c(\mathcal{X})$ of length v exists between each pair of nodes in $\mathcal{C}_{i,k}$ and $\mathcal{C}_{k,j}$, or $\mathcal{C}_{i,k} \not\to_{(\gamma-k(\mathrm{mod}\,\gamma))} \mathcal{C}_{k,j}$ and then no such walk exists. We thus have to check that $\mathcal{C}_{i,k} \to_{(\gamma-k(\mathrm{mod}\,\gamma))} \mathcal{C}_{k,j}$ on $\mathcal{D}$. But this follows since we have $[i]' \to_k [j]'$, and since in the sequence $[i]' \to_k \mathcal{C}'_{i,k} \to_l \mathcal{C}'_{k,j} \to_k [j]'$ we then must have $l \equiv (\gamma - k(\mathrm{mod}\,\gamma))(\mathrm{mod}\,\gamma)$.

$\square$

Combining Theorem 3.5.6 and Lemma 3.5.9 we obtain the following result.

Theorem 3.5.10 ([49],Theorem 5.8)**.** *Let k be greater than or equal to*

$$k \geq \max\left(\max_{i,j} T_{\alpha\beta}(\mathcal{X}, \Gamma(k), i, j),\ \max_{i,j:\gamma_{i,j}>\varepsilon} T_{\gamma}(\mathcal{X}, \Gamma(k), i, j)\right)$$

Then $\Gamma(k) = CS^{k(\mathrm{mod}\,\gamma)}R[\Gamma(k)]$.

As with Theorem 3.4.6 this bound requires $\Gamma(k)$ in order to calculate the bound, which makes it implicit, but as with Corollary 3.4.7 we can use $w_{i,\mathcal{N}_c} \leq w^*_{i,\mathcal{N}_c}$ and $v_{\mathcal{N}_c,j} \leq v^*_{\mathcal{N}_c,j}$ to give us an explicit bound. This also leads to some analogous notation in relation to Notation 3.5.5.

Notation 3.5.11. *Given the set $\mathcal{X}$ and indices i and j, we define:*

$$\begin{aligned} T'_{\alpha\beta}(\mathcal{X}, i, j) &:= \overline{\delta}(i, \mathcal{N}_c) \cdot \left(\frac{w_{i,\mathcal{N}_c} - \alpha_{i,\mathcal{N}_c}}{\lambda_*} + (d-q)\right) + \mathrm{Sch}(\gamma, q) \\ &\quad + \overline{\delta}(\mathcal{N}_c, j) \cdot \left(\frac{v_{\mathcal{N}_c,j} - \beta_{\mathcal{N}_c,j}}{\lambda_*} + (d-q)\right) \\ T'_{\gamma}(\mathcal{X}, i, j) &:= \frac{w_{i,\mathcal{N}_c} + v_{\mathcal{N}_c,j} - \gamma_{i,j}}{\lambda_*} + (d-q+1) \end{aligned}$$

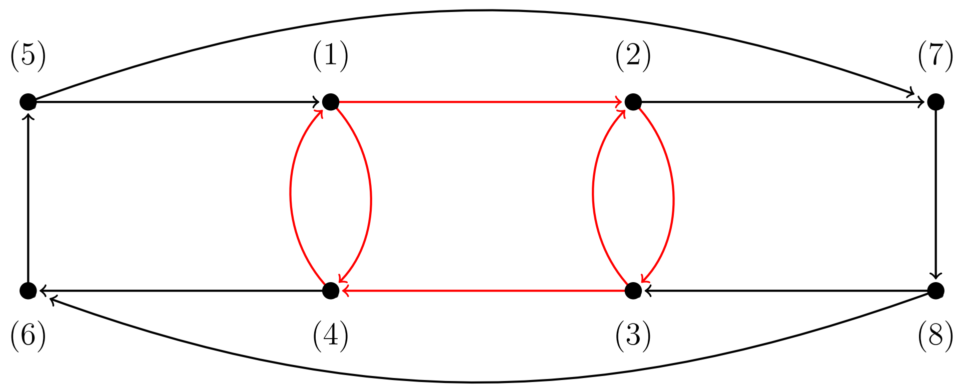
Corollary 3.5.12 ([49],Corollary 5.9)**.** *Let k be greater than or equal to*

$$k \geq \max\left(\max_{i,j} T'_{\alpha\beta}(\mathcal{X}, i, j),\ \max_{i,j:\gamma_{i,j}>\varepsilon} T'_{\gamma}(\mathcal{X}, i, j)\right)$$

Then $\Gamma(k) = CS^{k(\mathrm{mod}\,\gamma)}R[\Gamma(k)]$.

We will now present an example of this bound in action.

Let $\mathcal{D}(G)$ be the eight node digraph with the following structure:



along with the associated weight matrix.

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & a_{2,7} & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & a_{4,6} & \varepsilon & \varepsilon \\ a_{5,1} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & a_{5,7} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & a_{6,5} & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & a_{7,8} \\ \varepsilon & \varepsilon & a_{8,3} & \varepsilon & \varepsilon & a_{8,6} & \varepsilon & \varepsilon \end{pmatrix}$$

There are three critical cycles in this digraph; one cycle $1 \to 2 \to 3 \to 4$ of length 4, and two cycles $1 \to 4 \to 1$ and $2 \to 3 \to 2$ of length 2. There are also cycles of length 4, 6 and 8 which means that the cyclicity of the whole digraph is 2, which is the same cyclicity of the critical subgraph. Therefore Assumption 3.5.1 is satisfied and we can continue. The semigroup of matrices $\mathcal{X}$ used by this example will be generated by these five matrices:

$$A_1 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -16 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -6 & \varepsilon & \varepsilon \\ -11 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -14 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -18 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -20 \\ \varepsilon & \varepsilon & -11 & \varepsilon & \varepsilon & -3 & \varepsilon & \varepsilon \end{pmatrix}, A_2 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -3 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -6 & \varepsilon & \varepsilon \\ -17 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -6 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -17 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -5 \\ \varepsilon & \varepsilon & -19 & \varepsilon & \varepsilon & -7 & \varepsilon & \varepsilon \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -4 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -6 & \varepsilon & \varepsilon \\ -13 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -10 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -8 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -17 \\ \varepsilon & \varepsilon & -12 & \varepsilon & \varepsilon & -11 & \varepsilon & \varepsilon \end{pmatrix}, A_4 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -19 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -6 & \varepsilon & \varepsilon \\ -16 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -16 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -8 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -12 \\ \varepsilon & \varepsilon & -2 & \varepsilon & \varepsilon & -2 & \varepsilon & \varepsilon \end{pmatrix},$$

$$A_5 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -16 & \varepsilon & \varepsilon \\ -19 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -3 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -12 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -10 \\ \varepsilon & \varepsilon & -1 & \varepsilon & \varepsilon & -7 & \varepsilon & \varepsilon \end{pmatrix}.$$

Using these matrices we can calculate A^{sup} and A^{inf},

$$A^{\text{sup}} = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -3 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -6 & \varepsilon & \varepsilon \\ -11 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -3 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -8 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -5 \\ \varepsilon & \varepsilon & -1 & \varepsilon & \varepsilon & -2 & \varepsilon & \varepsilon \end{pmatrix}, A^{\text{inf}} = \begin{pmatrix} \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & -19 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -16 & \varepsilon & \varepsilon \\ -19 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -16 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -18 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -20 \\ \varepsilon & \varepsilon & -19 & \varepsilon & \varepsilon & -11 & \varepsilon & \varepsilon \end{pmatrix}$$

as well as $\alpha_{i,\mathcal{N}_c}$, $\beta_{\mathcal{N}_c,j}$, $\gamma_{i,j}$, $w_{i,\mathcal{N}_c}$ and $v_{\mathcal{N}_c,j}$:

$$\alpha_{i,\mathcal{N}_c} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -9 \\ -17 \\ -6 \\ -1 \end{pmatrix}, \qquad \beta^T_{\mathcal{N}_c,j} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -14 \\ -6 \\ -3 \\ -8 \end{pmatrix}, \qquad \gamma_{i,j} = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -18 & -10 & -3 & -8 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -18 & -10 & -3 & -8 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -15 & -7 & -18 & -5 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & -10 & -2 & -13 & -18 \end{pmatrix}$$

$$w^T_{i,\mathcal{N}_c} = \begin{pmatrix} 0 & 0 & 0 & 0 & -19 & -37 & -39 & -19 \end{pmatrix}, v_{\mathcal{N}_c,j} = \begin{pmatrix} 0 & 0 & 0 & 0 & -34 & -16 & -19 & -39 \end{pmatrix}.$$

Note that by definition $\lambda^* = -\frac{18}{4}$. With all the pieces ready we can now form the bound of Corollary 3.5.12,

$$k \geq \max\left(\begin{pmatrix} 4 & 4 & 4 & 4 & 12.4 & 10.2 & 11.6 & 14.9 \\ 4 & 4 & 4 & 4 & 12.4 & 10.2 & 11.6 & 14.9 \\ 4 & 4 & 4 & 4 & 12.4 & 10.2 & 11.6 & 14.9 \\ 4 & 4 & 4 & 4 & 12.4 & 10.2 & 11.6 & 14.9 \\ 10.2 & 10.2 & 10.2 & 10.2 & 18.7 & 16.4 & 17.8 & 21.1 \\ 12.4 & 12.4 & 12.4 & 12.4 & 20.9 & 18.7 & 20 & 23.3 \\ 15.3 & 15.3 & 15.3 & 15.3 & 23.8 & 21.6 & 22.9 & 26.2 \\ 12 & 12 & 12 & 12 & 20.4 & 18.22 & 19.6 & 22.9 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12.8 & 10.6 & 12.8 & 16.1 \\ 0 & 0 & 0 & 0 & 19 & 12.8 & 15 & 18.3 \\ 0 & 0 & 0 & 0 & 17.9 & 15.7 & 13.9 & 21.2 \\ 0 & 0 & 0 & 0 & 14.6 & 12.3 & 10.6 & 13.9 \end{pmatrix}\right)$$

$\Rightarrow k \geq 23.8.$

Therefore by Corollary 3.5.12 if the length of a product using the matrices from $\mathcal{X}$ is greater than or equal to 24 then the resulting product will be CSR. We will show such a product. Let $\Gamma(24)$ be the inhomogeneous matrix product made using the word $P = 551541235515535135454155$ which gives us:

$$\Gamma(24) = \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \\ 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \\ \varepsilon & -19 & \varepsilon & -19 & -47 & \varepsilon & \varepsilon & -40 \\ -31 & \varepsilon & -31 & \varepsilon & \varepsilon & -47 & -42 & \varepsilon \\ -11 & \varepsilon & -11 & \varepsilon & \varepsilon & -27 & -22 & \varepsilon \\ \varepsilon & -1 & \varepsilon & -1 & -29 & \varepsilon & \varepsilon & -22 \end{pmatrix}.$$

This matrix product is indeed CSR and by Definition 3.3.1 we have,

$$\Gamma(24) = \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ 0 & \varepsilon & 0 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & -19 & \varepsilon & -19 \\ -31 & \varepsilon & -31 & \varepsilon \\ -11 & \varepsilon & -11 & \varepsilon \\ \varepsilon & -1 & \varepsilon & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \\ 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \end{pmatrix}$$

$$\Gamma(24) = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \\ 0 & \varepsilon \\ \varepsilon & 0 \\ \varepsilon & -19 \\ -31 & \varepsilon \\ -11 & \varepsilon \\ \varepsilon & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon & \varepsilon & -16 & -11 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 & -28 & \varepsilon & \varepsilon & -21 \end{pmatrix}.$$

We can see that, for the C matrix, columns 3 and 4 are copies of columns 1 and 2 respectively. The same is also true for the rows of the R matrix so they can be deleted. As $24 (\bmod 2) = 0$ we replace the S matrix with the tropical identity matrix which shows us that the matrix product $\Gamma(24)$ using the word P is indeed CSR and it has factor rank equal to 2.

3.6 The one loop special case

This case was initially explored by Kennedy-Cochran-Patrick et al. [50] as a precursor to the results in the previous section. However we can use the general results from this chapter to refine the bounds found in the paper [50]. We will use the following assumption on the critical digraph of matrices from $\mathcal{X}$

Assumption 3.6.1. $\mathcal{G}^c(\mathcal{X})$ *is a single loop situated at node* 1 *of length* 1*.*

The following three corollaries are the one loop case version of the initial bound, final bound and the bound avoiding $\mathcal{N}_c$. The claims follow immediately from Lemmas 3.4.2, 3.4.3, Remark 3.4.4, and Lemma 3.4.5. These statements are improved versions of [50, Lemma 3.1, Lemma 3.2, and Lemma 3.4]. In what follows, we will use the following simplified version of (3.23) and (3.24):

$$\overline{\delta}(i,j) = \begin{cases} 0 & \text{if } i,j \in \mathcal{N}_c \\ 1 & \text{otherwise.} \end{cases}$$

From Remark 3.4.4.

Corollary 3.6.2. *Let* W_1 *be an optimal strict initial walk to the critical nodes on trellis digraph* $\mathcal{T}_{\Gamma(k)}$ *connecting* i *to* 1*. Then we have the following upper bound on its length:*

$$l(W_1) \leq \overline{\delta}(i,1)\left(\frac{w^*_{i,1} - \alpha_{i,1}}{\lambda^*} + (d-1)\right). \tag{3.40}$$

Corollary 3.6.3. *Let* W_2 *be an optimal strict final walk from the critical nodes on trellis digraph* $\mathcal{T}_{\Gamma(k)}$ *connecting* 1 *to* j*. Then we have the following upper bound on its* *length:*

$$l(W_2) \leq \overline{\delta}(1,j)\left(\frac{v^*_{1,j} - \beta_{1,j}}{\lambda^*} + (d-1)\right). \tag{3.41}$$

Corollary 3.6.4. *If $\gamma_{i,j} = \varepsilon$ then any full walk connecting i to j on $\mathcal{T}(\Gamma(k))$ traverses node 1. If $\gamma_{i,j} > \varepsilon$, let*

$$k \geq \frac{w^*_{i,1} + v^*_{1,j} - \gamma_{i,j}}{\lambda^*} + d. \tag{3.42}$$

Then any full walk W connecting i to j on $\mathcal{T}_{\Gamma(k)}$ that does not go through node 1 has weight smaller than $w^*_{i,1} + v^*_{1,j}$.

The claim below is an improved version of [50, Lemma 3.3]. Note that it follows from Lemma 3.5.3 using that $\mathrm{Sch}(1,1) = 0$. However, we also give a complete proof based on Corollaries 3.6.2 and 3.6.3 written above.

Lemma 3.6.5. *Let*

$$k \geq \overline{\delta}(i,1)\left(\frac{w^*_{i,1} - \alpha_{i,1}}{\lambda^*} + (d-1)\right) + \overline{\delta}(1,j)\left(\frac{v^*_{1,j} - \beta_{1,j}}{\lambda^*} + (d-1)\right). \tag{3.43}$$

Then any optimal full walk W connecting i to j on $\mathcal{T}_{\Gamma(k)}$ and going through node 1 is decomposed as, $W = W_1 C W_2$ where W_1 is an optimal strict initial walk from i to 1 and W_2 is an optimal strict final walk from 1 to j which satisfy

$$l(W_1) \leq \overline{\delta}(i,1)\left(\frac{w^*_{i,1} - \alpha_{i,1}}{\lambda^*} + (d-1)\right),$$
$$l(W_2) \leq \overline{\delta}(1,j)\left(\frac{v^*_{1,j} - \beta_{1,j}}{\lambda^*} + (d-1)\right),$$

C is empty or consists of a number of loops $1 \to 1$ and

$$p_{\mathcal{T}}(W) = w^*_{i,1} + v^*_{1,j}.$$

Proof. Let W be an optimal full walk connecting i to j that traverses node 1 at least once. Note first that all edges between the first and the last occurrence of 1 in W can be replaced with the copies of $(1,1)$, since these edges are present in every matrix X_α from $\mathcal{X}$. Assumption $\mathcal{D}1$ implies that this leads to a strict increase of the weight, therefore we must have $W = \tilde{W}_1\tilde{C}\tilde{W}_2$, where $\tilde{C}$ consists of $l(C) \geq 0$ edges $(1,1)$, $\tilde{W}_1$ is a strict initial walk from i to 1 and $\tilde{W}_2$ is a strict final walk from 1 to j. We have $p_{\mathcal{T}}(\tilde{C}) = 0$, so $p_{\mathcal{T}}(W) = p_{\mathcal{T}}(\tilde{W}_1) + p_{\mathcal{T}}(\tilde{W}_2)$.

Now we note that by Corollaries 3.6.2 and 3.6.3 the length k is sufficient for constructing a walk $W' = V_1C'V_2$ where V_1 is an optimal strict initial walk from i to 1, C' consists of $l(C') \geq 0$ edges of $(1,1)$ and V_2 is an optimal strict final walk from 1 to j. The weight of this walk is $w^*_{i,1} + v^*_{1,j}$.

By the optimality of V_1 and V_2 we have $p_{\mathcal{T}}(\tilde{W}_1) \leq p_{\mathcal{T}}(V_1)$ and $p_{\mathcal{T}}(\tilde{W}_2) \leq p_{\mathcal{T}}(V_2)$. Since W is optimal, both inequalities should hold with equality.

That is, $\tilde{W}_1$ is an optimal strict initial walk connecting i to 1 and $\tilde{W}_2$ is an optimal strict final walk connecting 1 to j, so that $\tilde{W}_1$, $\tilde{W}_2$ and $\tilde{C}$ can be taken for W_1, W_2 and C respectively.

The final step is to check the outlying case when $i = j = 1$. This means that, by the definition of the strict initial walks to critical nodes and strict final walks from critical nodes, they will consist a single node at 1. For the strict initial walk this is because it has reached 1 and thus it will stop. For the strict final walk it is because it cannot go back to 1 if it leaves therefore it must stay at 1. The lengths of these walks will be 0 therefore their weights will, by definition, be zero. Since there can also $l(C)$ loops over C then any $k \geq 0$ will suffice in length to construct an optimal walk from $1 \to 1$. The proof is complete. $\square$

We can now combine the previous lemma and corollaries into a theorem that does not require the CSR expansion to show a rank property. Theorem 3.6.6 and Corollary 3.6.7 are improved versions of [50, Theorem 4.1, Corollary 4.2, and Corollary 4.4], and they can also be obtained as corollaries of Theorem 3.5.6.

Theorem 3.6.6. *Let $\Gamma(k)$ be an inhomogeneous max-plus matrix product $\Gamma(k) = A_1 \otimes A_2 \otimes \ldots \otimes A_k$ with k satisfying*

$$k \geq \max\left(\frac{w^*_{i,1} + v^*_{1,j} - \gamma_{i,j}}{\lambda^*} + d, \overline{\delta}(i,1)\left(\frac{w^*_{i,1} - \alpha_{i,1}}{\lambda^*} + (d-1)\right) + \overline{\delta}(1,j)\left(\frac{v^*_{1,j} - \beta_{1,j}}{\lambda^*} + (d-1)\right)\right) \tag{3.44}$$

for all $i, j \in N$, then $\Gamma(k)$ is rank one, more precisely we have $\Gamma(k)_{i,j} = w^*_{i,1} + v^*_{1,j}$ for all i and j, and

$$\Gamma(k) = \begin{bmatrix} \Gamma(k)_{1,1} \\ \Gamma(k)_{2,1} \\ \vdots \\ \Gamma(k)_{d,1} \end{bmatrix} \otimes \begin{bmatrix} \Gamma(k)_{1,1} & \Gamma(k)_{1,2} & \ldots & \Gamma(k)_{1,d} \end{bmatrix}.$$

Proof. As seen by Lemma 3.6.4, if $\gamma_{i,j} = \varepsilon$ then any full walk connecting i to j on

$\mathcal{T}(\Gamma(k))$ traverses node 1. If $\gamma_{i,j} > \varepsilon$ and if

$$k \geq \frac{w^*_{i,1} + v^*_{1,j} - \gamma_{i,j}}{\lambda^*} + d$$

then any walk on $\mathcal{T}_{\Gamma(k)}$ not going through node 1 will have weight smaller than $w^*_{i,1}+v^*_{1,j}$. By Lemma 3.6.5, if

$$k \geq \overline{\delta}(i,1)\left(\frac{w^*_{i,1} - \alpha_{i,1}}{\lambda^*} + (d-1)\right) + \overline{\delta}(1,j)\left(\frac{v^*_{1,j} - \beta_{1,j}}{\lambda^*} + (d-1)\right)$$

then any optimal full walk going through node 1 will consist of the three parts W_1, W_2 and C as defined in the Lemma and its weight will be $w^*_{i,1} + v^*_{1,j}$. Hence if k satisfies both inequalities then any optimal full walk goes through node 1 and has weight

$$\Gamma(k)_{i,j} = w^*_{i,1} + v^*_{1,j}$$

Observe that by definition the strict initial walk connecting $1 \to 1$ and the strict final walk connecting $1 \to 1$ will have lengths equal to zero as shown in Lemma 3.6.5. Therefore $w^*_{1,1}$ and $v^*_{1,1}$ are equal to 0 hence,

$$\Gamma(k)_{i,1} = w^*_{i,1} + v^*_{1,1} = w^*_{i,1},$$
$$\Gamma(k)_{1,j} = w^*_{1,1} + v^*_{1,j} = v^*_{1,j}.$$

Then for all $i, j \in N$,

$$\begin{aligned}\Gamma(k)_{i,j} &= w^*_{i,1} + v^*_{1,j} \\ &= w^*_{i,1} + v^*_{1,1} + w^*_{1,1} + v^*_{1,j} \\ &= \Gamma(k)_{i,1} + \Gamma(k)_{1,j}.\end{aligned}$$

If k satisfies the condition (3.44) for all $i, j \in N$ then

$$\Gamma(k)_{i,j} = \Gamma(k)_{i,1} + \Gamma(k)_{1,j}.$$

Since this applies for all $i, j \in N$, $\Gamma(k)_{i,1}$ and $\Gamma(k)_{1,j}$ can be written as vectors in $\mathbb{R}^d$. Using the max-plus outer product of these two vectors it becomes

$$\Gamma(k) = \begin{bmatrix}\Gamma(k)_{1,1} \\ \Gamma(k)_{2,1} \\ \vdots \\ \Gamma(k)_{d,1}\end{bmatrix} \otimes \begin{bmatrix}\Gamma(k)_{1,1} \\ \Gamma(k)_{1,2} \\ \vdots \\ \Gamma(k)_{1,d}\end{bmatrix}^\top$$

thus the proof is complete. □

The bound of Theorem 3.6.6 is implicit, and as we did for Corollary 3.5.12, we can use $w_{i,\mathcal{N}_c} \leq w^*_{i,\mathcal{N}_c}$ and $v_{\mathcal{N}_c,j} \leq v^*_{\mathcal{N}_c,j}$ to give us an explicit bound.

Corollary 3.6.7. *Let $\Gamma(k)$ be an inhomogenous max-plus matrix product $\Gamma(k) = A_1 \otimes A_2 \otimes \ldots \otimes A_k$ with k satisfying*

$$k \geq \max_{i,j \in N}\left(\frac{w_{i,1} + v_{1,j} - \gamma_{i,j}}{\lambda^*} + d, \overline{\delta}(i,1)\left(\frac{w_{i,1} - \alpha_{i,1}}{\lambda^*} + (d-1)\right) + \overline{\delta}(1,j)\left(\frac{v_{1,j} - \beta_{1,j}}{\lambda^*} + (d-1)\right)\right)$$

then $\Gamma(k)$ *is rank one and*

$$\Gamma(k) = \begin{bmatrix} \Gamma(k)_{1,1} \\ \Gamma(k)_{2,1} \\ \vdots \\ \Gamma(k)_{d,1} \end{bmatrix} \otimes \begin{bmatrix} \Gamma(k)_{1,1} & \Gamma(k)_{1,2} & \dots & \Gamma(k)_{1,d} \end{bmatrix}.$$

To illustrate this bound in action, let us consider an example. Let D_A be a digraph consisting of five nodes with the generalised associated weight matrix,

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \varepsilon & \varepsilon \\ a_{2,1} & \varepsilon & \varepsilon & \varepsilon & a_{2,5} \\ \varepsilon & \varepsilon & \varepsilon & a_{3,4} & \varepsilon \\ \varepsilon & a_{4,2} & \varepsilon & \varepsilon & \varepsilon \\ a_{5,1} & \varepsilon & \varepsilon & a_{5,4} & \varepsilon \end{bmatrix},$$

where $a_{i,j} \in \mathbb{R}_{\max}$. Consider the set $\mathcal{X} = \{A_1, A_2, A_3\}$ where

$$A_1 = \begin{bmatrix} 0 & -1 & -2 & \varepsilon & \varepsilon \\ -3 & \varepsilon & \varepsilon & \varepsilon & -3 \\ \varepsilon & \varepsilon & \varepsilon & -4 & \varepsilon \\ \varepsilon & -5 & \varepsilon & \varepsilon & \varepsilon \\ -6 & \varepsilon & \varepsilon & -5 & \varepsilon \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & -4 & -3 & \varepsilon & \varepsilon \\ -4 & \varepsilon & \varepsilon & \varepsilon & -3 \\ \varepsilon & \varepsilon & \varepsilon & -2 & \varepsilon \\ \varepsilon & -1 & \varepsilon & \varepsilon & \varepsilon \\ -1 & \varepsilon & \varepsilon & 1 & \varepsilon \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 2 & -4 & \varepsilon & \varepsilon \\ -5 & \varepsilon & \varepsilon & \varepsilon & -6 \\ \varepsilon & \varepsilon & \varepsilon & -4 & \varepsilon \\ \varepsilon & -3 & \varepsilon & \varepsilon & \varepsilon \\ -2 & \varepsilon & \varepsilon & 2 & \varepsilon \end{bmatrix}.$$

It can be seen that these satisfy the assumptions with the top left entry of each matrix being zero. Using these we can calculate the coarser bound of Corollary 3.6.7. In order to do that we need A^{sup} and A^{inf}, which are

$$A^{\text{sup}} = \begin{bmatrix} 0 & 2 & -2 & \varepsilon & \varepsilon \\ -3 & \varepsilon & \varepsilon & \varepsilon & -3 \\ \varepsilon & \varepsilon & \varepsilon & -2 & \varepsilon \\ \varepsilon & -1 & \varepsilon & \varepsilon & \varepsilon \\ -1 & \varepsilon & \varepsilon & 2 & \varepsilon \end{bmatrix} \quad \text{and} \quad A^{\text{inf}} = \begin{bmatrix} 0 & -4 & -4 & \varepsilon & \varepsilon \\ -5 & \varepsilon & \varepsilon & \varepsilon & -6 \\ \varepsilon & \varepsilon & \varepsilon & -4 & \varepsilon \\ \varepsilon & -5 & \varepsilon & \varepsilon & \varepsilon \\ -6 & \varepsilon & \varepsilon & -5 & \varepsilon \end{bmatrix}.$$

We now begin to calculate the bounds of Corollary 3.6.7. The only cycle that does not go through node 1 is $(2 \to 5 \to 4 \to 2)$ which has average weight $\lambda^* = -\frac{2}{3}$. Using

A^{sup} we get $\alpha_{i,1}$, $\beta_{1,j}$ and $\gamma_{i,j}$ as the entries of

$$\alpha = \begin{bmatrix} 0 \\ -3 \\ -6 \\ -4 \\ -1 \end{bmatrix}, \; \beta = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \\ -1 \end{bmatrix}, \; \gamma = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & -2 & \varepsilon & -1 & -3 \\ \varepsilon & -3 & \varepsilon & -2 & -6 \\ \varepsilon & -1 & \varepsilon & -2 & -4 \\ \varepsilon & 1 & \varepsilon & 2 & -2 \end{bmatrix}.$$

Using A^{inf} we can also calculate $w_{i,1}$ and $v_{1,j}$ as the entries of

$$w = \begin{bmatrix} 0 \\ -5 \\ -14 \\ -10 \\ -6 \end{bmatrix}, \; v = \begin{bmatrix} 0 \\ -4 \\ -4 \\ -8 \\ -10 \end{bmatrix}.$$

With these pieces we can construct the bounds for k for each combination of i and j:

$$k \geq \max_{i,j \in N} \left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 15.5 & 0 & 23 & 23 \\ 0 & 27.5 & 0 & 35 & 32 \\ 0 & 24.5 & 0 & 29 & 29 \\ 0 & 21.5 & 0 & 29 & 26 \end{bmatrix}, \begin{bmatrix} 0 & 13 & 7 & 17.5 & 17.5 \\ 7 & 20 & 14 & 24.5 & 24.5 \\ 16 & 29 & 23 & 33.5 & 33.5 \\ 13 & 26 & 20 & 30.5 & 30.5 \\ 11.5 & 24.5 & 18.5 & 29 & 29 \end{bmatrix} \right) \Leftrightarrow k \geq 35.$$

This means that if a matrix product $\Gamma(k)$ has length greater then 35 then it will be rank-one. Let us now take a random product of length 44:

$$\begin{aligned}\Gamma(k) = A_1 &\otimes A_3 \otimes A_1 \otimes A_2 \otimes A_3 \otimes A_1 \otimes A_2 \otimes A_2 \otimes A_1 \otimes A_3 \otimes A_1 \\ &\otimes A_2 \otimes A_1 \otimes A_2 \otimes A_3 \otimes A_3 \otimes A_1 \otimes A_2 \otimes A_1 \otimes A_1 \otimes A_3 \otimes A_2 \\ &\otimes A_3 \otimes A_2 \otimes A_2 \otimes A_3 \otimes A_1 \otimes A_1 \otimes A_2 \otimes A_3 \otimes A_2 \otimes A_1 \otimes A_3 \\ &\otimes A_1 \otimes A_2 \otimes A_3 \otimes A_1 \otimes A_3 \otimes A_3 \otimes A_1 \otimes A_2 \otimes A_2 \otimes A_1 \otimes A_1.\end{aligned}$$

We obtain that

$$\Gamma(k) = \begin{bmatrix} 0 & -1 & -2 & -6 & -4 \\ -3 & -4 & -5 & -9 & -7 \\ -10 & -11 & -12 & -16 & -14 \\ -10 & -11 & -12 & -16 & -14 \\ -6 & -7 & -8 & -12 & -10 \end{bmatrix}.$$

We see that $\Gamma(k) = w^*_{i,1} \otimes (v^*_{1,j})^\top = \Gamma(k)_{i,1} \otimes (\Gamma(k)_{1,j})^\top$ where

$$w^* = \begin{bmatrix} 0 \\ -3 \\ -10 \\ -10 \\ -6 \end{bmatrix}, \; v^* = \begin{bmatrix} 0 \\ -1 \\ -2 \\ -6 \\ -4 \end{bmatrix}.$$

Note that the bound appearing in Corollary 3.6.6 is equal to

$$\max_{i,j\in N}\left(\begin{bmatrix}0&0&0&0&0\\0&8&0&17&11\\0&17&0&26&17\\0&20&0&26&20\\0&17&0&26&17\end{bmatrix},\begin{bmatrix}0&8.5&4&14.5&8.5\\4&12.5&8&18.5&12.5\\10&18.5&14&24.5&18.5\\13&21.5&17&27.5&21.5\\11.5&20&15.5&26&20\end{bmatrix}\right)=27.5,$$

which is indeed smaller than the coarser bound 35.

3.7 Counterexamples

Here we present a number of counterexamples for the different cases of digraph structure. These counterexamples present families of products which are not CSR, and we construct them in such a way that they have no upper bound on their length.

3.7.1 The ambient graph is primitive but the critical graph is not

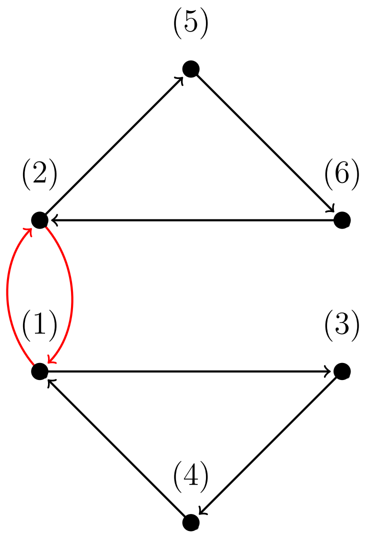
First we will look at two cases where we are unable to create a bound for matrix products to become CSR. For the first case we will be looking at digraphs that are primitive but have a critical subgraph with a non-trivial cylicity. Therefore we have the following assumption:

Assumption $\mathcal{P}$1. *$\mathcal{D}(\mathcal{X})$ is primitive (i.e., $\gamma(\mathcal{D}(\mathcal{X}))=1$) and the critical subgraph $\mathcal{G}^c(\mathcal{X})$, which is a single s.c.c., has cyclicity $\gamma(\mathcal{G}^c(\mathcal{X}))=\gamma>1$.*

Using this assumption we can now present the a counterexample which shows that no bound for k in terms of A^{sup} and A^{inf} can exist that ensures that $\Gamma(k)$ is CSR.

First Counterexample

Let $\mathcal{D}(G)$ be the five node digraph with the following structure:



This digraph will have the following associated weight matrix.

$$A = \begin{pmatrix} \varepsilon & 0 & a_{1,3} & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & a_{2,5} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & a_{3,4} & \varepsilon & a_{3,6} \\ a_{4,1} & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & a_{5,6} \\ \varepsilon & a_{6,2} & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

There is a critical subgraph consisting of the cycle between nodes 1 and 2. There also exist two cycles, $1 \to 3 \to 4 \to 1$ and $2 \to 5 \to 6 \to 2$, both of length 3 which makes $\mathcal{D}(A)$ primitive. We aim to present a family of words with infinite length such that the products made up using these words are not CSR. Since the cyclicity of the critical subgraph is 2 then we will have to create two classes of words, one of even length and one of odd length to define the family.

The semigroup of matrices we will use is generated by the two matrices:

$$A_1 = \begin{pmatrix} \varepsilon & 0 & -100 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & -100 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & -100 & \varepsilon & \varepsilon \\ -100 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -100 \\ \varepsilon & -100 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

$$A_2 = \begin{pmatrix} \varepsilon & 0 & -100 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & -1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & -100 & \varepsilon & \varepsilon \\ -1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -100 \\ \varepsilon & -100 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

Let us first consider the class of words $(1)^{2t}2$ where $t \geq 2$, and let $U = (A_1)^{2t}A_2$ for arbitrary such t. We will first examine entries $u_{6,1}$, $u_{2,5}$, $u_{6,2}$ and $u_{1,5}$.

The entry $u_{6,1}$ can be obtained as the weight of the walk $6\underbrace{(21)(21)\ldots(21)}_{t-1}341$, which is -301. This is the sum of the edge $6 \to 2$ on A_1 of weight -100, $t-1$ cycles of $2 \to 1 \to 2$ of weight zero minus the final edge to end at node 1, the edges $1 \to 3$

and $3 \to 4$ on A_1 with weights equal to -100, and finally the edge $4 \to 1$ in A_2 with weight -1. For this observe that the walk 621 has an even length and therefore we need to use one of the three-cycles to make it odd, and using the southern three-cycle in the end of the walk is the most profitable way to do so. The entry u_{25} is equal to -1, as there is a walk that mostly rests on the critical cycle and only in the end jumps to node 5. We also have $u_{6,2} = -100$ (go to node 2 and remain on the critical cycle) and $u_{1,5} = -301$ (use the southern triangle once, then dwell on the critical cycle and in the end jump to node 5). Note that in the case of $u_{1,5}$ we again need to use one of the triangles to create a walk of an odd length.

We then compute

$$(CSR)[U]_{6,5} = (US^3U)_{6,5} = \max(u_{6,1} + u_{2,5},\ u_{6,2} + u_{1,5}) = -301 - 1 = -302.$$

However, $u_{6,5}$ results from the walk $6\underbrace{(21)(21)\ldots(21)}_{t-1}2562$, with weight -401, needing to use the northern triangle to make a walk of odd length.

The following an example of U and $CS^{2t+1}R[U]$ for $t = 10$:

$$U = \begin{pmatrix} -201 & 0 & -100 & -500 & -301 & -200 \\ 0 & -300 & -400 & -200 & -1 & -500 \\ -401 & -200 & -300 & -700 & -501 & -400 \\ -100 & -400 & -500 & -300 & -101 & -600 \\ -200 & -500 & -600 & -400 & -201 & -700 \\ -301 & -100 & -200 & -600 & -401 & -300 \end{pmatrix}$$

$$CS^{21(\mathrm{mod}\,2)}R[U] = \begin{pmatrix} -201 & 0 & -100 & -401 & -202 & -200 \\ 0 & -300 & -400 & -200 & -1 & -500 \\ -401 & -200 & -300 & -601 & -402 & -400 \\ -100 & -400 & -500 & -300 & -101 & -600 \\ -200 & -500 & -600 & -400 & -201 & -700 \\ -301 & -100 & -200 & -501 & -302 & -300 \end{pmatrix}$$

We now consider the class of words $(1)^{2t+1}2$ where $t \geq 1$, and let $V = (A_1)^{2t+1}A_2$ for arbitrary such t. We will first examine entries $v_{2,1}$, $v_{1,5}$, $v_{2,2}$ and $v_{2,5}$.

The entry $v_{2,1} = -201$ is obtained as the weight of the walk $2\underbrace{(12)(12)\ldots(12)}_{t-1}341$: it is necessary to use one of the triangles to create a walk of even length, and using the southern triangle once in the end of the walk is the most profitable way to do so. The walk 125 already has an even length, and we only have to augment it with enough copies of the critical cycle and use the arc $2 \to 5$ in the end of the walk, thus getting $v_{1,5} = -1$. Obviously, $v_{2,2} = 0$: we just stay on the critical cycle. The entry $v_{2,5} = -301$ is obtained as the weight of the walk $\underbrace{(21)(21)\ldots(21)}_{t-1}5625$, where we have to use the northern triangle in the end of the walk to create a walk of even walk and minimise the loss.

We then find

$$(CS^2R[V])_{2,5} = (VS^2V)_{2,5} = \max(v_{2,1} + v_{1,5}, v_{2,2} + v_{2,5}) = v_{2,1} + v_{1,5} = -202,$$

which is bigger than $v_{2,5} = -301$.

The case for $v_{2,5}$ is one for connecting a critical node to a non critical node. For completeness we should also look at a walk connecting two non critical nodes, namely the walk representing $v_{4,5}$. To do this we will need to also look at the entries $v_{4,1}$ and $v_{4,2}$. For $v_{4,1} = -301$ the entry is obtained as the weight of the walk $4\underbrace{(12)(12)\ldots(12)}_{t-1}341$. As the walk 41 has odd length, one of the triangles is required to make the walk even so choosing the southern triangle is the most profitable way to achieve an even length walk. The walk 412 already has an even length so we can augment it with enough copies of the critical cycle to give us the desired length for the walk representing the entry $v_{4,2} = -100$. Using $v_{1,5}$ and $v_{2,5}$ discussed earlier we calculate

$$(CS^2R[V])_{4,5} = (VS^2V)_{4,5} = \max(v_{4,1} + v_{1,5}, v_{4,2} + v_{2,5}) = v_{4,1} + v_{1,5} = -302,$$

which is bigger than $v_{4,5} = -401$.

We now show an example of V for $t = 10$:

$$V = \begin{pmatrix} 0 & -300 & -400 & -200 & -1 & -500 \\ -201 & 0 & -100 & -500 & -301 & -200 \\ -200 & -500 & -600 & -400 & -201 & -700 \\ -301 & -100 & -200 & -600 & -401 & -300 \\ -401 & -200 & -300 & -700 & -501 & -400 \\ -100 & -400 & -500 & -300 & -101 & -600 \end{pmatrix}$$

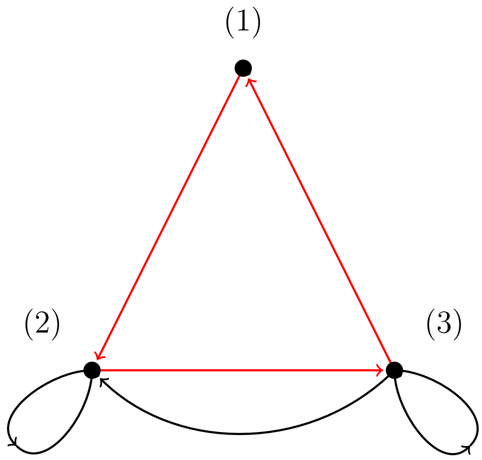
$$CS^{22 (\mathrm{mod}\, 2)}R[V] = \begin{pmatrix} 0 & -300 & -400 & -200 & -1 & -500 \\ -201 & 0 & -100 & -401 & -202 & -200 \\ -200 & -500 & -600 & -400 & -201 & -700 \\ -301 & -100 & -200 & -501 & -302 & -300 \\ -401 & -200 & -300 & -601 & -402 & -400 \\ -100 & -400 & -500 & -300 & -101 & -600 \end{pmatrix}$$

Combining both classes we have a family of words covering all lengths greater than 29 such that any product made using these words will not be CSR. Therefore there cannot be a transient for this case as there is no upper limit to the lengths of these words.

Second Counterexample

There also exists another counterexample in the primitive case which shows that even walks connecting two nodes from the same critical subgraph can not be CSR.

Let $\mathcal{D}(G)$ be the three node digraph with the following structure:



The digraph has the following associated weight matrix.

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & a_{2,2} & 0 \\ 0 & a_{3,2} & a_{3,3} \end{pmatrix}.$$

For this example there is a single critical cycle of length 3 traversing all of the nodes. There also exists two loops $2 \to 2$ and $3 \to 3$ and a cycle $2 \to 3 \to 2$ of length 2. Like the previous example this digraph is primitive but the critical subgraph has cyclicity 3. As the cyclicity is greater than one we need to present three different classes of words making up a family of words such that any product $\Gamma(k)$ made using these words will not be CSR.

The semigroup of matrices that we will use is again generated only by two matrices:

$$A_1 = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & -100 & 0 \\ 0 & -100 & -100 \end{pmatrix} \quad A_2 = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & -1 & 0 \\ 0 & -100 & -1 \end{pmatrix}$$

Let the first class of words be $(1)^{3t+2}2$ for $t \geq 0$, and let $M = (A_1)^{3t+2}A_2$ for any arbitrary t. We will now examine the entries $m_{1,1}$, $m_{1,2}$, $m_{2,2}$ $m_{1,3}$ and $m_{3,2}$.

Since all the walks are of length 0 modulo 3 then any walk connecting i to i will have weight zero as we can simply use the critical cycle. This gives $m_{1,1} = m_{2,2} = 0$. The entry $m_{1,2}$ can be obtained as the weight of the walk $(123)^{t+1}2$ which is -100. In this entry observe that the walk 12 is of length 1 modulo 3 therefore we need to use the two cycle $2 \to 3 \to 2$ to give us a walk of the desired length. The entry $m_{1,3}$ is equal to the weight of the walk $(123)^{t+1}3$ and the entry $m_{3,2}$ is equal to the weight of the walk $(312)^{t+1}2$. For these entries observe that the walks 123 and 312 are both of length 2 modulo 3 therefore we require a loop for both walks to give us the required length. The most profitable time to use these loops are right at the end of the walk.

We then compute

$$(CSR)[M]_{1,2} = (MS^3M)_{1,2} = \max(m_{1,1}+m_{1,2}, m_{1,2}+m_{2,2}, m_{1,3}+m_{3,2}) = -1-1 = -2.$$

However, as seen earlier the entry $m_{1,2}$ has weight -100 which is less than the CSR suggestion.

The following is an example of M and $CS^{3t+3}R[M]$ for $t = 10$:

$$M = \begin{pmatrix} 0 & -100 & -1 \\ -100 & 0 & -100 \\ -100 & -1 & 0 \end{pmatrix} \quad CS^{33 \pmod 3}R[M] = \begin{pmatrix} 0 & -2 & -1 \\ -100 & 0 & -100 \\ -100 & -1 & 0 \end{pmatrix}$$

For efficiency we will simply present the final two classes and omit the in-depth analysis of them:

For walks of length 1 modulo 3 we have the class of words $(1)^{3t+3}2$ for $t \geq 0$.

For walks of length 2 modulo 3 we have the class of words $(1)^{3t+4}2$ for $t \geq 0$.

We will also present examples of products and their CSR counterparts made using these words for $t = 10$ where $N = (A_1)^{3t+3}A_2$ and $P = (A_1)^{3t+4}A_2$.

$$N = \begin{pmatrix} -100 & 0 & -100 \\ -100 & -1 & 0 \\ 0 & -100 & -1 \end{pmatrix} \quad CS^{34(\bmod 3)}R[N] = \begin{pmatrix} -100 & 0 & -100 \\ -100 & -1 & 0 \\ 0 & -2 & -1 \end{pmatrix}$$
$$P = \begin{pmatrix} -100 & -1 & 0 \\ 0 & -100 & -1 \\ -100 & 0 & -100 \end{pmatrix} \quad CS^{35(\bmod 3)}R[P] = \begin{pmatrix} -100 & -1 & 0 \\ 0 & -2 & -1 \\ -100 & 0 & -100 \end{pmatrix}.$$

The combination of these three classes create a family of words such that any product $\Gamma(k)$ made using these words is not CSR and as all the nodes are critical then there exist walks connecting them that are not CSR.

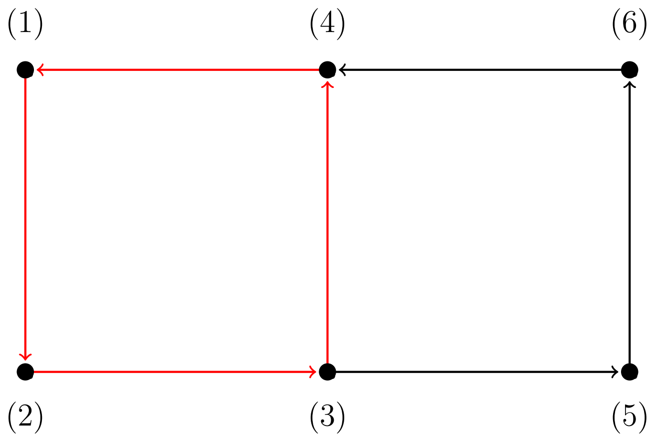
We now extend these counterexamples to a more general form where we consider digraphs with non-trivial cyclicity r along with critical subgraphs with cyclicity γ which is greater than r. This leads to the following assumptions.

3.7.2 More general case

Assumption $\mathcal{P}$2. *$\mathcal{D}(\mathcal{X})$ has cyclicity r and the critical subgraph $\mathcal{G}^c(\mathcal{X})$, which is strongly connected, has cyclicity $\gamma > r$.*

In a similar method to the primitive example above, using the new assumptions, we can now describe a counterexample that shows that no bound for k in terms of $A^{\sup}$ and $A^{\inf}$ can exist that ensures $\Gamma(k)$ is CSR.

Let $\mathcal{D}(\mathcal{X})$ be a six node digraph with the following structure:



along with the following associated weight matrix,

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & a_{3,5} & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & a_{5,6} \\ \varepsilon & \varepsilon & \varepsilon & a_{6,4} & \varepsilon & \varepsilon \end{pmatrix}$$

Here the critical cycle traverses nodes $1 \to 2 \to 3 \to 4 \to 1$ however there also exists another non-critical cycle of length six traversing $1 \to 2 \to 3 \to 5 \to 6 \to 4 \to 1$. This means that while the cyclicity of the critical subgraph is 4 the cyclicity of $\mathcal{D}(G)$ is 2. Therefore the digraph structure satisfies the assumptions and we can develop a family of words with infinite length such that any $\Gamma(k)$ made using these words will not be

CSR. As the cyclicity of the critical subgraph is 4 then we will require four classes of words to fully define the family.

The semigroup of matrices that will be used is generated by two matrices:

$$A_1 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & -100 & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -100 \\ \varepsilon & \varepsilon & \varepsilon & -100 & \varepsilon & \varepsilon \end{pmatrix} \quad A_2 = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -100 \\ \varepsilon & \varepsilon & \varepsilon & -1 & \varepsilon & \varepsilon \end{pmatrix}$$

Let us begin with the first class of words $(1)^{4t}2$ where $t \geq 2$, and let $L = (A_1)^{4t}A_2$ for arbitrary such t. We will begin by examining the entries $l_{1,2}$, $l_{1,5}$, $l_{1,4}$ and $l_{3,5}$.

The entry $l_{1,2}$ can be obtained as the weight of the walk $\underbrace{(1234)}_{t}12$, which is 0. As the walk 12 has length congruent to $1(\bmod 4)$ then a walk exists on the critical cycle connecting these nodes. The entry $l_{1,5}$ is obtained from the weight of the walk $\underbrace{(1234)}_{t-2}1235641235$, which is -301. As the walk 1235 has length congruent to $3(\bmod 4)$ then we need to add on the six cycle with weight -300 to give us a walk of length congruent to $1(\bmod 4)$ and finally the last step of the walk is to go from 3 to 5 with weight -1. For the entry $l_{1,4} = -201$ which is the weight of the walk $\underbrace{(1234)}_{t-1}123564$ and the entry $l_{3,5} = -1$ comes from the weight of the walk $\underbrace{(3412)}_{t}35$. Note that in the case of $l_{1,4}$ we used the six cycle to give us the desired length of walk.

We then compute

$$(CSR)[L]_{1,5} = (L \otimes S^3 \otimes L)_{1,5} = \max(l_{1,2} + l_{1,5},\ l_{1,4} + l_{3,5}) = -201 - 1 = -202.$$

However $l_{1,5}$, as explained earlier, results from a walk with weight -301.

The following is an example of L and $CS^{4t+1}R[L]$ for $t = 10$

$$L = \begin{pmatrix} \varepsilon & 0 & \varepsilon & -201 & -301 & \varepsilon \\ -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ -500 & \varepsilon & -200 & \varepsilon & \varepsilon & -601 \\ \varepsilon & -400 & \varepsilon & -100 & -101 & \varepsilon \end{pmatrix}$$

$$CS^{41 \pmod 4}R[L] = \begin{pmatrix} \varepsilon & 0 & \varepsilon & -201 & -202 & \varepsilon \\ -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ -500 & \varepsilon & -200 & \varepsilon & \varepsilon & -601 \\ \varepsilon & -400 & \varepsilon & -100 & -101 & \varepsilon \end{pmatrix}$$

We now consider the second class of words $(1)^{4t+1}2$ where $t \geq 2$, and let $M = (A_1)^{4t+1}A_2$ for arbitrary such t. We will examine the entries $m_{4,2}$, $m_{4,5}$, $m_{4,4}$ and $m_{2,5}$:

$m_{4,2}$ is the weight of the walk $\underbrace{(4123)}_{t}412$, which is 0;

$m_{4,5}$ is the weight of the walk $\underbrace{(4123)}_{t-2}41235641235$, which is -301;

$m_{4,4}$ is the weight of the walk $\underbrace{(4123)}_{t-1}4123564$, which is -201;

$m_{2,5}$ is the weight of the walk $\underbrace{(2341)}_{t}235$, which is -1.

Looking at the entries $m_{4,5}$ and $m_{4,4}$, the walks 41235 and 41234 both have length congruent to $0(\bmod 4)$ then we must include the six cycle to give us length congruent to $2(\bmod 4)$ as required. Using these entries we can calculate,

$$(CSR[M])_{4,5} = (MS^2M)_{4,5} = \max(m_{4,2} + m_{4,5}, m_{4,4} + m_{2,5}) = -201 - 1 = -202.$$

However, as explained above, $m_{45} = -301$ which is less than $(CSR[M])_{4,5}$. We finish this class with an example of M and $CS^{4t+2}R[M]$ for $t = 10$.

$$M = \begin{pmatrix} -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ \varepsilon & 0 & \varepsilon & -201 & -301 & \varepsilon \\ \varepsilon & -500 & \varepsilon & -200 & -201 & \varepsilon \\ -100 & \varepsilon & -400 & \varepsilon & \varepsilon & -201 \end{pmatrix}$$

$$CS^{42(\bmod 4)}R[M] = \begin{pmatrix} -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ \varepsilon & 0 & \varepsilon & -201 & -202 & \varepsilon \\ \varepsilon & -500 & \varepsilon & -200 & -201 & \varepsilon \\ -100 & \varepsilon & -400 & \varepsilon & \varepsilon & -201 \end{pmatrix}$$

Moving on to the third class of words $(1)^{4t+2}2$ where $t \geq 2$, and let $N = (A_1)^{4t+2}A_2$ for arbitrary such t. We now examine the entries $n_{3,2}$, $n_{6,2}$, $n_{3,5}$, $n_{3,4}$, $n_{6,4}$ and $n_{1,5}$:

$n_{3,2}$ is the weight of the walk $\underbrace{(3412)}_{t}3412$, which is 0;

$n_{6,2}$ is the weight of the walk $6\underbrace{(4123)}_{t-2}412$, which is -100;

$n_{3,5}$ is the weight of the walk $\underbrace{(3421)}_{t-1}35641235$, which is -301;

$n_{3,4}$ is the weight of the walk $\underbrace{(3412)}_{t}3564$, which is -201;

$n_{6,4}$ is the weight of the walk $6\underbrace{(4123)}_{t-1}4123564$, which is -301;

$n_{1,5}$ is the weight of the walk $\underbrace{(1234)}_{t}1235$, which is -1.

As before with the entries $n_{3,5}$, $n_{3,4}$ and $n_{6,4}$ the walks 35, 34 and 64 all have length congruent to $1 \pmod 4$ therefore we must include the six cycle to give us a length congruent to $3 \pmod 4$ as desired. As we have six entries we can look at two separate calculations, starting with

$$(CSR[N])_{3,5} = (NSN)_{3,5} = \max(n_{3,2} + n_{3,5}, n_{3,4} + n_{1,5}) = -201 - 1 = -202.$$

As we can see from above the entry $n_{3,5}$ results from a walk of weight -301 which is smaller than $(CSR[N])_{3,5}$. For the second calculation

$$(CSR[N])_{6,5} = (NSN)_{6,5} = \max(n_{6,2} + n_{3,5}, n_{6,4} + n_{1,5}) = -301 - 1 = -302,$$

which is bigger than the walk that results from $n_{6,5}$ which is $6\underbrace{(4123)}_{t-1}5641235$, which has weight -401.

We now give an example of N and $CS^{4t+3}R[N]$ for $t = 10$,

$$N = \begin{pmatrix} \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ \varepsilon & 0 & \varepsilon & -201 & -301 & \varepsilon \\ -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ -200 & \varepsilon & -500 & \varepsilon & \varepsilon & -301 \\ \varepsilon & -100 & \varepsilon & -301 & -401 & \varepsilon \end{pmatrix}$$

$$CS^{43 \pmod 4}R[N] = \begin{pmatrix} \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ \varepsilon & 0 & \varepsilon & -201 & -202 & \varepsilon \\ -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ -200 & \varepsilon & -500 & \varepsilon & \varepsilon & -301 \\ \varepsilon & -100 & \varepsilon & -301 & -302 & \varepsilon \end{pmatrix}$$

We end by considering the final class of words $(1)^{4t+3}2$ where $t \geq 2$, and let $R = (A_1)^{4t+3}A_2$ for arbitrary such t. As with the third class we consider the six entries $r_{2,2}$, $r_{5,2}$, $r_{2,5}$, $r_{2,4}$, $r_{5,4}$ and $r_{4,5}$:

$r_{2,2}$ is the weight of the walk $\underbrace{(2341)}_{t}23412$, which is 0;

$r_{5,2}$ is the weight of the walk $56\underbrace{(4123)}_{t-2}412$, which is -200;

$r_{2,5}$ is the weight of the walk $\underbrace{(2341)}_{t-1}235641235$, which is -301;

$r_{2,4}$ is the weight of the walk $\underbrace{(2341)}_{t}23564$, which is -201;

$r_{5,4}$ is the weight of the walk $56\underbrace{(4123)}_{t-1}4123564$, which is -401;

$r_{4,5}$ is the weight of the walk $\underbrace{(4123)}_{t}41235$, which is -1.

For the entries $r_{2,5}$,$r_{2,4}$ and $r_{5,4}$ the walks 235, 234 and 564 all have length congruent to $2 \pmod 4$ hence in order to get a length congruent to $0 \pmod 4$ we must include the six cycle for those walks.

With these six entries we can calculate two entries from $CSR[R]$,

$$(CSR[N])_{2,5} = (R^2)_{2,5} = \max(r_{2,2} + n_{25}, r_{2,4} + r_{4,5}) = -201 - 1 = -202,$$

$$(CSR[N])_{5,5} = (R^2)_{5,5} = \max(r_{5,2} + r_{2,5}, r_{5,4} + r_{4,5}) = -401 - 1 = -402.$$

We can see that both calculations are larger than $r_{2,5}$ and $r_{5,5}$ respectively. We know the walk that results from $r_{2,5}$ and the walk that results from $r_{5,5}$ is $56\underbrace{(4123)}_{t-1}5641235$, which has weight -501.

We end the final class with an example of R and $CS^{4t+4}R[R]$ for $t = 10$,

$$R = \begin{pmatrix} 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ \varepsilon & 0 & \varepsilon & -201 & -301 & \varepsilon \\ -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ \varepsilon & -200 & \varepsilon & -401 & -501 & \varepsilon \\ -400 & \varepsilon & -100 & \varepsilon & \varepsilon & -501 \end{pmatrix}$$

$$CS^{44(\mathrm{mod}\,4)}R[R] = \begin{pmatrix} 0 & \varepsilon & -300 & \varepsilon & \varepsilon & -101 \\ \varepsilon & 0 & \varepsilon & -201 & -202 & \varepsilon \\ -300 & \varepsilon & 0 & \varepsilon & \varepsilon & -401 \\ \varepsilon & -300 & \varepsilon & 0 & -1 & \varepsilon \\ \varepsilon & -200 & \varepsilon & -401 & -402 & \varepsilon \\ -400 & \varepsilon & -100 & \varepsilon & \varepsilon & -501 \end{pmatrix}$$

Combining all these classes gives us a family of words covering all lengths greater than 9 such that any product made using these words will not be CSR. Therefore no transient can exist as there is no upper limit to the lengths of these words.

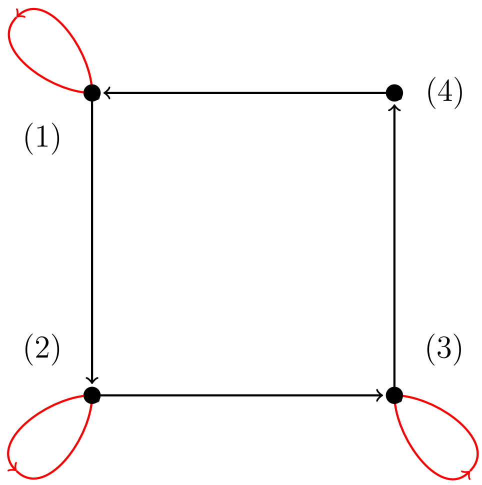
3.7.3 Critical graph is not connected

For this counterexample we now consider a digraph with multiple critical components $\mathcal{G}_1^c, \ldots, \mathcal{G}_m^c$ which are each s.c.c.s with respective cyclicities $\gamma_1, \ldots, \gamma_m$.

Assumption $\mathcal{P}3$. *$\mathcal{G}^c(\mathcal{X})$ is composed of multiple s.c.c.s $\mathcal{G}_1^c, \ldots, \mathcal{G}_m^c$ where the component $\mathcal{G}_i^c$ has cyclicity γ_i. The cyclicity of $\mathcal{D}(\mathcal{X})$ is $lcm_i(\gamma_i)$, which is the same as the* *cyclicity of* $\mathcal{G}^c(\mathcal{X})$.

Let us now show a counterexample, which demonstrates that, for the case of several critical components, we cannot have any bounds after which the product becomes CSR in terms of A^{sup} and A^{inf}. The reason is that the non-critical parts of optimal walks whose weights are the entries of C and R cannot be separated in time: in general, they will use the same letters, and such walks on the symmetric extension of $\mathcal{T}(\Gamma(k))$ cannot be transformed back to the walks on $\mathcal{T}(\Gamma(k))$.

Let $\mathcal{D}(\mathcal{X})$ be the four node digraph with the following structure:



along with the following associated weight matrix

$$A = \begin{pmatrix} 0 & a_{1,2} & \varepsilon & \varepsilon \\ \varepsilon & 0 & a_{2,3} & \varepsilon \\ \varepsilon & \varepsilon & 0 & a_{3,4} \\ a_{4,1} & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

For this digraph we have a critical subgraph comprised of three separate loops at nodes 1,2 and 3. There is also a cycle of length 4 which means the cyclicity of the digraph is 1. We are going to present a class of words of infinite length such that the matrix generated by this class of words is not CSR.

We introduce a semigroup of tropical matrices with two generators $\mathcal{X} = \{A_1, A_2\}$ where A_1 to A_2 are

$$A_1 = \begin{pmatrix} 0 & -100 & \varepsilon & \varepsilon \\ \varepsilon & 0 & -100 & \varepsilon \\ \varepsilon & \varepsilon & 0 & -100 \\ -100 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 & \varepsilon & \varepsilon \\ \varepsilon & 0 & -1 & \varepsilon \\ \varepsilon & \varepsilon & 0 & -100 \\ -100 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

and the class of the words that we will consider is $(1)^t 2$, where $t \geq 2$. In other words we will consider a set of matrices of the form $U = (A_1)^t A_2$ (the actual value of $t \geq 2$ will not matter to us).

We have: $u_{1,2} = -1$ (as the weight of the walk $\underbrace{11\ldots1}_{t+1}2$), $u_{2,3} = -1$ (as the weight of the walk $\underbrace{22\ldots2}_{t+1}3$),and therefore $(CS^{t+1}R[U])_{1,3} = (U^2)_{1,3} = u_{1,2} \otimes u_{2,3} = -2$, but $u_{1,3} = -101$ (as the weight of the walk $1\underbrace{22\ldots2}_{t}3$).

Similarly, we can also look at the entry $u_{4,3}$. Then we have $u_{4,2} = -101$ (as the weight of the walk $4\underbrace{11\ldots1}_{t}2$), $u_{2,3} = -1$ and hence $(CS^{t+1}R)_{4,3} = (USU)_{4,3} = u_{4,2} \otimes u_{2,3} = -102$, but $u_{4,3} = -201$ (as the weight of the walk $41\underbrace{22\ldots2}_{t-1}3$).

Here is an example of the word from the class for $t = 10$ and the corresponding

CSR

$$W = \begin{pmatrix} 0 & -1 & -101 & -300 \\ -300 & 0 & -1 & -200 \\ -200 & -201 & 0 & -100 \\ -100 & -101 & -201 & -400 \end{pmatrix}, \qquad CS^{11(\bmod 1)}R[W] = \begin{pmatrix} 0 & -1 & -2 & -201 \\ -201 & 0 & -1 & -101 \\ -200 & -201 & 0 & -100 \\ -100 & -101 & -102 & -301 \end{pmatrix}.$$

Therefore any matrix product of length greater than 3 which has been made following this word will not be CSR. Hence there can be no upper bound to guarantee the CSR decomposition in this case.

3.8 Conclusion

In order to achieve the key results an inhomogeneous product analogue of CSR (Definition 2.2.1) was introduced in Definition 3.3.1, and in Theorem 3.3.12 it was shown that any matrix product that is CSR has rank at most $\sum_{\nu=1}^{m} \gamma_\nu$. By creating this new definition we have developed a product analogue for the CSR decomposition and, due to Proposition 3.3.8, there could exist some scope in bringing previous results on the CSR decomposition of matrix powers to inhomogenous matrix products. In Theorem 3.4.6 a condition on the length of the product was established in which a product with length satisfying the condition is bounded above by its CSR decomposition. If we assume that for every A_i making up the product, the associated digraph is strongly connected and the cyclicity of its critical digraph is equal to the cyclicity of the associated digraph of A_i, then in Theorem 3.5.10 we established another condition in which a product of length satisfying the condition, and the assumptions, becomes

CSR. In Theorem 3.6.6 a condition for the case when the critical subgraph is a single loop is developed, in which the CSR definition is forgone and any product with length satisfying this transient exhibits a rank-one property. In Corollaries 3.4.7, 3.5.12 and 3.6.7 we deduced explicit bounds on the length of an inhomogeneous product, after which the product is bounded by its CSR decomposition, is equal to its CSR decomposition, and exhibits the rank-one property, respectively. These corollaries are deduced from Theorems 3.4.6, 3.5.10, and 3.6.6 respectively, and they make use of A^{inf}, the infimum matrix. For these cases we now have both implicit and explicit bounds on not only a CSR property but a factor rank property as well due to Theorem 3.3.12. However there are also more general cases in which a CSR property cannot be found and the set of counterexamples showing this were presented in Section 3.7.

Finally in Section 3.7 the three cases where CSR does not work were as counterexamples. The first case was when the digraph is primitive ($\gamma = 1$) but the critical subgraph is not, the second case was a more general version of the first case, where the critical subgraph is strongly connected, but the cyclicity of the ambient digraph is strictly less than the cyclicity of the critical subgraph, and the final case was when the critical subgraph is made up of more than one distinct s.c.c. (each of them being a loop). In all the counterexamples we presented families of words of infinite length, in which the product made using such a word was not CSR. These counterexamples give insight into where this new definiton of CSR does not work and could give rise to potential restrictions require in order to produce working examples in these cases.

CHAPTER 4

CONCLUSION AND DISCUSSION

In this chapter we will briefly summarise the results presented in this thesis as well as outline some directions for further research.

In Chapter 2 we not only showed the validity of the Schwarz and Kim bounds on $T_1(A, B)$ for the Nachtigall and Hartman-Arguelles decomposition schemes (see Theorems 2.3.3 and 2.5.4 respectively), but we also refined the bounds on $T_1(A, B)$ for the Cycle Threshold decomposition scheme in Theorem 2.5.7 by introducing the cyclicity into the bounds. For $T_2(A, B)$ we improved the bounds of Proposition 2.6.3 by introducing cyclicity to give the new bounds in Theorem 2.6.3. All these results have been published in the paper [48]. In Theorems 2.3.4 and 2.5.5 we also developed new bounds on $T_1(A, B)$ for both the Nachtigall and Hartman-Arguelles schemes using a factor rank property, as well as developing bounds on $T_2(F, B)$ involving the factor rank in Theorem 2.6.5. These results appear in this thesis for the first time and, using the example at the end of Chapter 2, we show how effective these new bounds can be.

These results lie atop a long history of periodicity transients for matrix powers but that is not to say that it will be the end for them. There will always be scope to improve the bounds by introducing more graph theoretical terms into the bounds or by refining them further. It should be noted that the paper by Merlet et al. [69] characterises the matrices that attain the Wielandt and Dulmage-Mendelsohn bounds which, for a natural extension, means that similar theory could be developed for both the Kim and Schwarz bounds as well as the factor rank bounds.

In Chapter 3 we develop the inhomogeneous product analogue of CSR, for which we proved a factor rank property of matrix products that are CSR (Theorem 3.3.12). We also show the link between this new product analogue of CSR and the original definition of CSR that is used for Chapter 2 which is an original result for this thesis. Using the new CSR definition we then develop a bound, in Theorem 3.5.10, in which for a certain case, matrix products become CSR and naturally exhibit a factor rank property which is currently in the preprint [49]. We also outline a more strict, rank-1, case in Theorem 3.6.6 where we forgo the use of the CSR definition to show the property directly. This case was published in the paper [50]. For the cases which did not work we presented counterexamples in which families of words are given where no bound exists on products using these words becoming CSR.

Naturally there are many directions in which to expand on this research. The first, and maybe most important, direction is to find a way to develop bounds on the cases where counterexamples exist. This could be achieved in many ways with some simple examples being altering the CSR definition in some way, or to exert some control over the word making up the product. It should also be noticed that in every counterexample only a select few entries were not CSR which introduces the potential of an approximation to CSR in a similar vein to the infinite horizon case explored by Akian, Gaubert, and Walsh [3]. Another direction for further exploration would be to develop applications of these results. For instance these results tie in with discrete turnpike theory. This can be seen from [3, Theorem 7.5] in which the authors show that turnpikes are equivalent to walks of maximal weight traversing the critical nodes. There are also applications in optimal control optimisation with distinct applications in railway networks such as the work done by Soto y Koelemeijer [87] or the textbook by Heidergott, Olsder and van der Woude [43]. By using the discrete system $x(k) = A_k \otimes x(k-1)$, where A_k is some prescribed matrix and $x(k)$ is the result vector, one can recursively substitute the previous state to give $x(k) = A_k \otimes A_{k-1} \otimes \ldots \otimes A_1 \otimes x(0) = \Gamma(k) \otimes x(0)$. Naturally, our results on tropical inhomogenous products can be utilised to develop applications on optimal control systems where such products are involved. Examples of such systems include the process industry using model-predictive control as explored by De Schutter and van den Boom [24] and legged locomotion of robots, which was explored by Lopes, Kersbergen, De Schutter, van den Boom, and Babuška [64].

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