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UNIQUENESS IN CAUCHY PROBLEMS FOR DIFFUSIVE REAL-VALUED STRICT LOCAL MARTINGALES

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ABSTRACT. For a real-valued one dimensional diffusive strict local martingale, we provide a set of smooth functions in which the Cauchy problem has a unique classical solution under a local $\frac{1}{2}$-Hölder condition. Under the weaker Engelbert-Schmidt conditions, we provide a set in which the Cauchy problem has a unique weak solution. We exemplify our results using quadratic normal volatility models and the two dimensional Bessel process.

1. INTRODUCTION

Consider a unique weak solution $\left(\mathbb{P}^x\right)_{x\in\mathbb{R}}$ of the time-homogenous diffusion

$$dX_t = \sigma(X_t)dB_t, \quad X_0 = x, \quad x \in \mathbb{R}, \tag{1.1}$$

where $B = (B_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion, $\sigma : \mathbb{R} \to \mathbb{R}$ is a Borel function, $\sigma^2(x) > 0$ for all $x \in \mathbb{R}$, and σ satisfies the necessary and sufficient *Engelbert-Schmidt conditions* for the existence and uniqueness of a weak solution of (1.1). For given continuous data $H : \mathbb{R} \to \mathbb{R}$, we consider the Cauchy problem

$$\begin{cases} h(0,x) = H(x), \quad x \in \mathbb{R}, \\ h_t(t,x) = \frac{1}{2}\sigma^2(x)h_{xx}(t,x), \quad t > 0, \quad x \in \mathbb{R}. \end{cases} \tag{1.2}$$

A function $h : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is said to be a *classical* solution of (1.2) if $h \in \mathcal{C}^{1,2}$ and satisfies (1.2). On the other hand, $h \in \mathcal{C}$ is said to be a *weak* solution of (1.2) if $h(0,x) = H(x)$, $x \in \mathbb{R}$, and

$$\int_0^\infty \int_{-\infty}^\infty \Big(\frac{2}{\sigma^2(x)} f_t(t,x) + f_{xx}(t,x)\Big) h(t,x)dxdt = 0, \tag{1.3}$$

for all $f : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ in $\mathcal{C}_c^{1,2}$. Appendix B shows this notion of weak solutions is equivalent to the notion of weak solutions used in Sawyer [42].

For σ in (1.1) with at most linear growth, the solution of (1.1) becomes a martingale. In that case, when the data H has at most polynomial growth, there exists at most one classical solution of (1.2) of polynomial growth (see Theorem 5.7.6 in

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Karatzas and Shreve [27] for when σ is additionally continuous). When σ in (1.1) satisfies the Engelbert-Schmidt conditions and ensures $X = (X_t)_{t\geq 0}$ is a positive martingale, Bayraktar and Xing [3] prove uniqueness in the set of at most linearly growing solutions of (1.2) for data H of at most linear growth. When σ is locally $\frac{1}{2}$-Hölder continuous and of at most linear growth and H is of at most polynomial growth, the function

$$h^*(t,x) := \mathbb{E}^x[H(X_t)], \quad t \geq 0, \quad x \in \mathbb{R}, \tag{1.4}$$

is the unique classical solution of (1.2) of at most polynomial growth (see Theorem 6.1 in Janson and Tysk [20]).

When the solution of (1.1) is a strict local martingale, the data $H(\xi) := \xi$ gives an example where uniqueness of (1.2) fails because both h^* defined in (1.4) and $h^\circ(t,x) := x$ are of at most linear growth and both h^* and h° solve (1.2).[1] Given a nonnegative strict local martingale with dynamics (1.1) and H of at most strict sublinear asymptotic growth as $\xi \uparrow \infty$, Theorem 4.3 in Ekström and Tysk [11] ensures that (1.4) is the unique classical solution of (1.2) in the class of strictly sublinearly growing functions under a local $\frac{1}{2}$-Hölder continuity hypothesis on σ. More recently, Theorem 6.2 in Çetin [6] allows H to be of at most linear growth and proves uniqueness in the class of strictly sublinearly growing classical solutions of (1.2) when the solution of (1.1) is nonnegative.[2]

Our contributions are: Under the standard Engelbert-Schmidt conditions on σ, we show that h^* in (1.4) is the unique weak solution of (1.2) in a suitable set $\mathcal{H}_\lambda$ of continuous functions satisfying nontrivial growth conditions as $\xi \to \pm\infty$ provided the data $H(\xi)$ satisfies certain growth conditions as $\xi \to \pm\infty$ (with $H(\xi) := \xi$ allowed). Under a stronger local $\frac{1}{2}$-Hölder-continuity condition on σ ensuring a unique strong solution to (1.1), we prove that h^* in (1.4) is the unique classical solution of (1.2) in $\mathcal{H}_\lambda$. Finally, in the last section, also under the local $\frac{1}{2}$-Hölder-continuity condition on σ, we provide a martingale regularization technique and use it to show uniqueness when $H(\xi)$ has polynomial growth as $\xi \to \pm\infty$ (provided an exogenous moment condition on the unique strong solution to (1.1)).

Our uniqueness results complement [11] and [6] in the sense that we allow the solution X of (1.1) to take values on the entire real line whereas both [11] and [6] consider only nonnegative solutions. The uniqueness result in [6] is proved by a change-of-measure argument, in which the local martingale X acts as the Radon-Nikodym derivative process. However, because our solution X takes values in $\mathbb{R}$, it cannot be used as a Radon-Nikodym derivative process. Consequently, the change-of-measure technique of [6] is inapplicable. Nevertheless, our proof of uniqueness is similar in spirit and relies on a Radon-Nikodym derivative process based on a λ-harmonic function Φ_λ for a constant $\lambda > 0$. Thus, our proof can be seen as

[1] This lack of uniqueness produces several complications. For example, Monte-Carlo simulation cannot approximate h^* in (1.4) because all discretization schemes produce martingales.

[2] The existence and uniqueness results in [11] and [6] readily extend to the case where X_t takes values in $(-\infty, a)$ and (a, ∞) for $a \in \mathbb{R}$. For example, when $X_t \in (a,\infty)$, we can consider $\tilde{X}_t := X_t - a$ valued in $(0,\infty)$ with dynamics

$$d\tilde{X}_t = \sigma(\tilde{X}_t + a)dB_t.$$

Similarly, when $X_t \in (-\infty, a)$, we can consider $\tilde{X}_t := a - X_t$ valued in $(0,\infty)$ with dynamics

$$d\tilde{X}_t = \sigma(a - \tilde{X}_t)dB_t.$$

an extension of [6], where the change of measure in [6] uses the harmonic function $\Phi_0(\xi) := \xi$ for $\xi \geq 0$. Because there is no nonconstant harmonic function associated with (1.1), we use λ-harmonic functions with $\lambda > 0$ (λ-harmonic functions always exist). Theorem 2.2 in Urusov and Zervos [44] links the asymptotic behavior of minimal λ-harmonic functions ($\xi \to \pm\infty$) to whether the solution X of (1.1) is a true martingale or not. Identifying this asymptotic behavior when X is a strict local martingale was essential in constructing the set of functions $\mathcal{H}_\lambda$ in which we prove uniqueness.

While Feynman-Kac representations have a long history in stochastic analysis, our motivation comes from derivatives pricing in financial economics.[3] By allowing the solution X of (1.1) to be real valued, we can consider two widely used strict local martingale models: Quadratic normal volatility models (often used in financial economics to model stock bubbles, see, e.g., Zühlsdorff [45], Andersen [1], and Carr, Fisher, and Ruf [5]) and the logarithm of the two dimensional Bessel process. The class of data functions H covered by our result is given in terms of λ-harmonic functions and while our class always includes H of at most linear growth, our class typically also includes faster growing data H. For example, for the logarithm of the two dimensional Bessel example mentioned above, $H(\xi)$ can grow superlinearly as $\xi \uparrow \infty$.

The many applications of strict local martingales for modeling purposes in financial economics include: (i) Basak and Cuoco [2], Hugonnier [18], and Chabakauri [7] show that strict local martingales can appear endogenously in equilibrium theory. (ii) Cox and Hobson [8], Jarrow, Protter, and Shimbo [21], Heston, Loewenstein, and Willard [15], and Andersen [1] use strict local martingales for derivatives pricing. (iii) Stochastic portfolio theory as surveyed in Fernholz and Karatzas [14] uses strict local martingales to model relative arbitrage opportunities. (iv) Karatzas, Lehoczky, Shreve, and Xu [25], Kramkov and Schachermayer [33], and Lowenstein and Willard [36] exemplify that strict local martingales can appear as dual utility maximizers. More recent references based on nonnegative local martingales include Kardaras, Kreher, and Nikeghbali [28], Kramkov and Weston [34], and Kardaras and Ruf [29, 30]. Hulley and Platen [16] and Hulley and Ruf [17] are recent references based on real-valued local martingales.

Finally, we mention that an alternative to explicitly pin down growth conditions as $\xi \to \pm\infty$ as we do in this paper, h in (1.4) can also be uniquely characterized via a smallness property (for details, see the references and results in Section 5.1 in Karatzas and Ruf [26]). From a numerical perspective, the smallness characterization can be difficult to implement (see [12]) whereas our growth conditions as $\xi \to \pm\infty$ are compatible with standard numerical procedures such as finite difference methods. For example, when the solution X of (1.1) is a strict local martingale, the solution of the Cauchy problem (1.2) can exhibit boundary layers. When X takes only nonnegative values, Corollary 6.1 in [6] produces a boundary layer as $\xi \to \infty$. However, in our case, there can be two boundary layers as $\xi \to \pm\infty$. For

[3]In financial economics, h^* in (1.4) serves as an arbitrage-free derivatives price. When the solution X of (1.1) is a strict local martingale, several arbitrage-free derivatives prices are available. For example, Cox and Hobson [8] call h^* in (1.4) the "fair" derivatives price whereas Andersen [1] calls h^* the "minimal" derivatives price.

$H \in \mathcal{C}^1(\mathbb{R})$, we illustrate that it is possible to have

$$\lim_{\xi \to \pm\infty} H'(\xi) \neq \lim_{t \downarrow 0} \lim_{\xi \to \pm\infty} h_x(t, \xi). \tag{1.5}$$

Unlike the smallness property characterization of h^*, such boundary layers are detected in our PDE characterization of h^* and produce explicit boundary values that can be used when truncating the state space $\mathbb{R}$ to a finite grid (needed for finite difference methods).

2. Assumptions and preliminaries

Let $\mathbb{R}_\Delta$ be the one-point compactification of $\mathbb{R}$ with Δ being the *point-at-infinity*. We consider the path space Ω of right continuous functions $\omega : [0, \infty) \to \mathbb{R}_\Delta$ which satisfy $\omega(t) = \Delta$ for all $t > s \geq 0$ whenever $\omega(s) = \Delta$ and we note that $\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \subset \mathcal{C}(\mathbb{R}_+, \mathbb{R}_\Delta) \subset \Omega$. We let X be the coordinate process on Ω and let $(\mathcal{F}_t^0)_{t \geq 0}$ be its natural filtration $\mathcal{F}_t^0 := \sigma(X_u)_{u \in [0,t]}$. As in, e.g., Section 4 in [43], we define the universal completion $\mathcal{F}_t^*$ of $\mathcal{F}_t^0$ by

$$\mathcal{F}_t^* := \cap_\mu \sigma(\mathcal{F}_t^0, \mu \text{ null sets in } \mathcal{F}_t^0), \tag{2.1}$$

where the intersection is taken over all finite measures μ on $\mathcal{F}_t^0$. We define the right-continuous extension of $\mathcal{F}_t^*$ as $\mathcal{F}_t := \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon}^*$. Finally, we define the σ-algebra $\mathcal{F} := \bigvee_{t \geq 0} \mathcal{F}_t$.

Assumption 2.1. The volatility function $\sigma : \mathbb{R} \to \mathbb{R}$ is Borel, $\sigma^2(x) > 0$ for all $x \in \mathbb{R}$, and for all $x \in \mathbb{R}$ there exists $\epsilon > 0$ such that $\int_{x-\epsilon}^{x+\epsilon} \frac{1}{\sigma^2(y)} dy < \infty$.

Assumption 2.1 is necessary and sufficient for the existence of a unique weak solution $(\mathbb{P}^x)_{x \in \mathbb{R}}$ of (1.1). This result is due to Engelbert and Schmidt [10] and can also be found in, e.g., Theorem 5.5.7 in [27]. Corollary 4.23 in [10] establishes that X is a strong Markov process under Assumption 2.1. Moreover, under Assumption 2.1, there are no absorbing states and therefore X in (1.1) is recurrent under $(\mathbb{P}^x)_{x \in \mathbb{R}}$ in the sense that $\mathbb{P}^x(T_y < \infty) = 1$ for all $(x, y) \in \mathbb{R}^2$ where

$$T_y := \inf\{t > 0 : X_t = y\}, \tag{2.2}$$

see, e.g., Proposition 5.5.22(a) in Karatzas and Shreve [27]. In particular, the lifetime of X defined by $\zeta := \inf\{t > 0 : X_t = \Delta\}$ satisfies $\mathbb{P}^x(\zeta < \infty) = 0$ for all $x \in \mathbb{R}$ (see, e.g., Problem 5.5.3 in Karatzas and Shreve [27]). Therefore, X is a regular diffusion on $\mathbb{R}$ with speed measure

$$m(d\xi) := \frac{2}{\sigma^2(\xi)} d\xi, \quad \xi \in \mathbb{R}. \tag{2.3}$$

2.1. Measure change on the path space.

Definitions 23.10 and 62.4 as well as the following paragraph in Sharpe [43] ensure that the path space Ω is projective. Thus, given a supermartingale multiplicative functional $Y = (Y_t)_{t \geq 0}$ as defined in Eqs. (54.1) and (54.7) in [43], Corollary 62.21 in [43] establishes the existence of Markov kernels $(\mathbb{P}^{Y,x})_{x \in \mathbb{R}_\Delta}$ on $(\Omega, \mathcal{F})$ such that for any stopping time T, we have

$$\mathbb{E}^{Y,x}[F 1_{T < \zeta}] = \mathbb{E}^x[F Y_T 1_{T < \zeta}], \quad F \in \mathbf{b}\mathcal{F}_T, \quad x \in \mathbb{R}.$$

In other words, the random variable Y_T acts as a Radon-Nikodym derivative for changing the measure from $\mathbb{P}^x$ to $\mathbb{P}^{Y,x}$ on $\mathcal{F}_T$ and $[T < \zeta]$. We emphasize that while $\mathbb{P}^x(\zeta < \infty) = 0$ for all $x \in \mathbb{R}$, the process Y determines whether or not $\mathbb{P}^{Y,x}(\zeta = \infty) < 1$ for all $x \in \mathbb{R}$.

A continuous function $u : \mathbb{R} \to (0, \infty)$ is called a *λ-harmonic function* if the process $e^{-\lambda t} u(X_t)$, $t \geq 0$, is a local martingale (see, e.g., Remark 2.3(iii) in Salminen and Ta [41]). For a λ-harmonic function u, the normalized process $Y_t := e^{-\lambda t} \frac{u(X_t)}{u(X_0)}$, $t \geq 0$, is a supermartingale multiplicative functional. Proposition A.1 in Appendix A ensures that u is both convex and λ-excessive.[4] Paragraph II.30 in [4] ensures that any λ-harmonic function can be written as a linear combination of the *minimal* λ-harmonic functions $\varphi_{\lambda\uparrow}$ and $\varphi_{\lambda\downarrow}$, where $\varphi_{\lambda\uparrow}$ is strictly increasing and $\varphi_{\lambda\downarrow}$ is strictly decreasing, strictly positive, and satisfy

$$\lambda \int_{[0,x)} u(\xi) m(d\xi) = u^-(x) - u^-(0), \quad x \in \mathbb{R}, \tag{2.4}$$

where u^- is the left derivative and m is from (2.3). Equivalently, $\varphi_{\lambda\uparrow}$ and $\varphi_{\lambda\downarrow}$ are uniquely given (up to multiplicative constants) by the Laplace transform properties:

$$\begin{aligned} \varphi_{\lambda\uparrow}(x) &= \varphi_{\lambda\uparrow}(y) \mathbb{E}^x[e^{-\lambda T_y}], \\ \varphi_{\lambda\downarrow}(y) &= \varphi_{\lambda\downarrow}(x) \mathbb{E}^y[e^{-\lambda T_x}], \end{aligned} \tag{2.5}$$

for $x, y \in \mathbb{R}$ with $x < y$, see, e.g., p.128 in Itô and McKean [19] and Paragraph II.10 in Borodin and Salminen [4].

The change of measure based on a λ-harmonic function and the corresponding characteristics of the resulting diffusion are given in Lemma 2.2. This lemma is used repeatedly in our proofs.

Lemma 2.2. *Suppose Assumption* 2.1 *holds and let $u : \mathbb{R} \to (0, \infty)$ be a λ-harmonic function for a constant $\lambda > 0$. Then, the following statements hold:*

(i) *There exist Markov kernels $(\mathbb{P}^{u,x})_{x \in \mathbb{R}_\Delta}$ such that for any stopping time T*

$$\mathbb{E}^{u,x}[F 1_{T<\zeta}] = \frac{1}{u(x)} \mathbb{E}^x\big[F e^{-\lambda T} u(X_T) 1_{T<\zeta}\big], \quad F \in \mathbf{b}\mathcal{F}_T, \quad x \in \mathbb{R}. \tag{2.6}$$

(ii) *u is strictly convex, λ-excessive, and satisfies* (2.4)*. If — additionally — σ is continuous, then $u \in \mathcal{C}^2$ and satisfies the* Sturm-Liouville ODE

$$\lambda u(x) = \frac{1}{2} \sigma^2(x) u''(x), \quad x \in \mathbb{R}. \tag{2.7}$$

(iii) *$(\mathbb{P}^{u,x})_{x \in \mathbb{R}_\Delta}$ is the law of a diffusion with values in $\mathbb{R}_\Delta$, $\mathbb{P}^{u,x}(T_y < \infty) > 0$ for all $x, y \in \mathbb{R}$, a null killing measure, and scale function and speed measure given by*

$$s_u(z) := \int_0^z \frac{1}{u^2(\xi)} d\xi, \quad z \in \mathbb{R}, \quad m_u(d\xi) := u^2(\xi) m(d\xi), \quad \xi \in \mathbb{R}. \tag{2.8}$$

(iv) *The mapping $(t, x) \mapsto \mathbb{P}^{u,x}(\zeta > t)$ is jointly continuous on $[0, \infty) \times \mathbb{R}$.*

Proof. (i) This property follows from [43, Corollary 62.21].

(ii) The first two statements follow from Proposition A.1, Paragraph II.30 in [4], and that $\varphi_{\lambda\downarrow}$ and $\varphi_{\lambda\uparrow}$ satisfy (2.4). The remaining assertion follows from the continuity of σ and (2.4).

(iii) The formulas in (2.8) can be found in Paragraph II.31 in [4], Theorem 8.3 in Langer and Schenk [35], and Theorem 6.2 in Evans and Hening [13]. The killing measure under $\mathbb{P}^{u,x}$ is null because u is a λ-harmonic function.

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[4]The local martingale property only implies that u is superaveraging in the sense that $u(x) \geq \mathbb{E}^x[u(X_t)]$ for $t \geq 0$. However, as we show in Proposition A.1(ii) in Appendix A, $\lim_{t \downarrow 0} \mathbb{E}^x[u(X_t)] = u(x)$ holds automatically in our setting.

(iv) We define the process $\tilde{X}_t := s_u(X_t)$ for $t \geq 0$. Then, $\tilde{X}$ is a local martingale under $\mathbb{P}^{u,x}$ with volatility coefficient $s'_u\big(s_u^{-1}(\tilde{X}_t)\big)\sigma\big(s_u^{-1}(\tilde{X}_t)\big)$. Proposition 4.3 in Karatzas and Ruf [26] gives continuity of $(t,x) \mapsto \mathbb{P}^{u,x}(\tilde{\zeta} > t)$ where $\tilde{\zeta} := \inf\big\{t > 0 : \tilde{X}_t \in \{s_u(-\infty), s_u(\infty)\}\big\}$. Then, (iii) follows because $\tilde{\zeta} = \inf\{t > 0 : X_t = \Delta\} = \zeta$. ☐

2.2. **Asymptotics of $\varphi_{\lambda\downarrow}$ and $\varphi_{\lambda\uparrow}$.** The next result describes the asymptotic behavior of the minimal solutions $\varphi_{\lambda\downarrow}$ and $\varphi_{\lambda\uparrow}$ and is a consequence of Theorem 2.2 in [44].

Theorem 2.3 (Urusov and Zervos [44])**.** *Under Assumption* 2.1*, the following statements are equivalent:*

(i) $\int_0^\infty \xi m(d\xi) < \infty$ *(resp.* $\int_{-\infty}^0 |\xi| m(d\xi) < \infty$*).*
(ii) $\varphi_{\lambda\uparrow}(\xi)$ *(resp.* $\varphi_{\lambda\downarrow}(\xi)$*) has linear growth at* $\xi = \infty$ *(resp.* $\xi = -\infty$*).*
(iii) *The process* $\big(e^{-\lambda t}\varphi_{\lambda\uparrow}(X_t)\big)_{t\geq 0}$ *(resp.* $\big(e^{-\lambda t}\varphi_{\lambda\downarrow}(X_t)\big)_{t\geq 0}$*) is a strict local martingale under* $\mathbb{P}^x$ *for all* $x \in \mathbb{R}$*.*

Proof. First, the boundary point ∞ is inaccessible because

$$\int_0^\infty \int_0^y m(d\xi)dy = \infty. \tag{2.9}$$

Furthermore, the boundary point ∞ is a natural or entrance boundary point depending on whether

$$\begin{aligned}\int_0^\infty m((y,\infty))dy &= \int_0^\infty \int_0^\xi dy m(d\xi)\\ &= \int_0^\infty \xi m(d\xi)\end{aligned} \tag{2.10}$$

is infinite or not. Thus, the claimed equivalences follow from [44, Theorem 2.2]. ☐

In the setting of the logarithm of the two dimensional Bessel process, Example 2.4 gives $\varphi_{\lambda\uparrow}$ and $\varphi_{\lambda\downarrow}$ explicitly.

Example 2.4. For $\sigma(\xi) := e^{-\xi}$, $\xi \in \mathbb{R}$, the dynamics (1.1) become those of the logarithm of the two dimensional Bessel process that solves the SDE

$$dX_t = e^{-X_t}dB_t, \quad t \geq 0, \quad X_0 = x \in \mathbb{R}. \tag{2.11}$$

Eq. (2.11) has a unique strong solution for a given Brownian motion $B = (B_t)_{t\geq 0}$. For a constant $\lambda > 0$, the corresponding minimal solutions of (2.4) are

$$\begin{aligned}\varphi_{\lambda\uparrow}(\xi) &:= I_0(e^\xi\sqrt{2\lambda}), \quad \xi \in \mathbb{R}, \quad \lim_{\xi\uparrow\infty}\frac{\varphi_{\lambda\uparrow}(\xi)}{\xi} = \infty,\\ \varphi_{\lambda\downarrow}(\xi) &:= K_0(e^\xi\sqrt{2\lambda}), \quad \xi \in \mathbb{R}, \quad \lim_{\xi\downarrow-\infty}\frac{\varphi_{\lambda\downarrow}(\xi)}{\xi} = -1,\end{aligned} \tag{2.12}$$

see, e.g., Jeanblanc, Yor, and Chesney [22, p.279]. In (2.12), the functions I_0 and K_0 are modified Bessel functions. The superlinear growth limit in (2.12) (as $\xi \uparrow \infty$) and Theorem 2.3 ensure that the local martingale $\big(e^{-\lambda t}\varphi_{\lambda\uparrow}(X_t)\big)_{t\geq 0}$ is a martingale. The linear growth limit in (2.12) (as $\xi \downarrow -\infty$) and Theorem 2.3 ensure that the local martingale $\big(e^{-\lambda t}\varphi_{\lambda\downarrow}(X_t)\big)_{t\geq 0}$ is a strict local martingale.

2.3. **Asymptotics of the expectation of $\varphi_{\lambda\downarrow}$ and $\varphi_{\lambda\uparrow}$.** In the next section, we need the following result.

Lemma 2.5. *Suppose Assumption* 2.1 *holds.*

(i) *If $\int_0^\infty \xi m(d\xi) < \infty$, the strict local martingale*

$$Y_t := e^{-\lambda t}\varphi_{\lambda\uparrow}(X_t), \quad t \geq 0, \quad \lambda > 0, \tag{2.13}$$

has the asymptotic expectation

$$\lim_{n\uparrow\infty} \frac{\mathbb{E}^n[Y_{s_n}]}{\varphi_{\lambda\uparrow}(n)} = \lim_{n\uparrow\infty} \frac{\mathbb{E}^n[Y_{s_n}]}{n} = 0, \quad \infty > s_n \geq s_{n+1}, \quad s_\infty := \lim_{n\uparrow\infty} s_n > 0. \tag{2.14}$$

(ii) *If $\int_{-\infty}^0 |\xi| m(d\xi) < \infty$, the strict local martingale*

$$Y_t := e^{-\lambda t}\varphi_{\lambda\downarrow}(X_t), \quad t \geq 0, \quad \lambda > 0, \tag{2.15}$$

has the asymptotic expectation

$$\lim_{n\uparrow\infty} \frac{\mathbb{E}^n[Y_{s_n}]}{\varphi_{\lambda\downarrow}(-n)} = \lim_{n\uparrow\infty} \frac{\mathbb{E}^n[Y_{s_n}]}{n} = 0, \quad \infty > s_n \geq s_{n+1}, \quad s_\infty := \lim_{n\uparrow\infty} s_n > 0. \tag{2.16}$$

Proof. (i) **Step 1/3:** From Theorem 2.3, we know that (2.13) is a strict local martingale and $\varphi_{\lambda\uparrow}(\xi)$ is of linear growth as $\xi \uparrow \infty$. It follows from Lemma 2.2 that there exist Markov kernels $(\mathbb{P}^{\varphi,x})_{x\in\mathbb{R}_\Delta}$ such that (2.6) holds with $u := \varphi_{\lambda\uparrow}$. Moreover,

$$s_{\lambda\uparrow}(z) := \int_0^z \frac{1}{\varphi^2_{\lambda\uparrow}(\xi)} d\xi, \quad z \in \mathbb{R}, \quad m_{\lambda\uparrow}(d\xi) := 2\frac{\varphi^2_{\lambda\uparrow}(\xi)}{\sigma^2(\xi)} d\xi, \quad \xi \in \mathbb{R}, \tag{2.17}$$

are a scale function and a speed measure for X under $(\mathbb{P}^{\varphi,x})_{x\in\mathbb{R}_\Delta}$. Since $s_{\lambda\uparrow}(-\infty) = -\infty$ and $s_{\lambda\uparrow}(\infty) < \infty$, the limit X_∞ exists and $\mathbb{P}^{\varphi,x}(X_\infty = \infty) = 1$ for all $x \in \mathbb{R}$ (see, e.g., Proposition 5.5.22(c) in [27]).

Step 2/3: Define the stopping times

$$T_n := \inf\{t > 0 : X_t \geq n\}, \quad n \in \mathbb{N}, \tag{2.18}$$

and observe that T_n increases to ζ under $\mathbb{P}^{\varphi,x}$ as $n \uparrow \infty$. Moreover, $\mathbb{P}^x(T_n < \infty) = 1$ for all $n \in \mathbb{N}$ due to recurrence of X. For $t \in [0, \infty)$, Lemma 2.2 yields

$$\begin{aligned} \mathbb{P}^{\varphi,x}(t < T_n) &= \tfrac{1}{Y_0}\mathbb{E}^x[Y_{T_n} 1_{t<T_n}] \\ &= \tfrac{1}{\varphi_{\lambda\uparrow}(x)}\mathbb{E}^x[Y_t 1_{t<T_n}], \end{aligned} \tag{2.19}$$

where the last equality is due to the fact that $(Y_{t\wedge T_n})_{t\geq 0}$ is a uniformly integrable martingale for $n \in \mathbb{N}$. Because $\mathbb{P}^x(\lim_{n\uparrow\infty} T_n = \infty) = 1$, the dominated convergence theorem applied to (2.19) gives us

$$w(t, x) = \tfrac{1}{\varphi_{\lambda\uparrow}(x)}\mathbb{E}^x[Y_t], \tag{2.20}$$

where we have defined the function

$$w(t, x) := \mathbb{P}^{\varphi,x}(t < \zeta), \quad t \geq 0, \quad x \in \mathbb{R}. \tag{2.21}$$

Lemma 2.2(iv) ensures that the function w in (2.21) is jointly continuous.

Step 3/3: For $n \in \mathbb{N}$, let T_n be defined in (2.18) and let $t \in [0,\infty)$. We then have

$$\begin{aligned}\mathbb{P}^{\varphi,x}(\zeta > t) &= \mathbb{P}^{\varphi,x}(\zeta > t, T_n \le t) + \mathbb{P}^{\varphi,x}(\zeta > t, T_n > t)\\ &= \mathbb{E}^{\varphi,x}\Big[1_{T_n\le t}\mathbb{P}^{\varphi,X_{T_n}}(\zeta > t-u)|_{u=T_n}\Big] + \mathbb{P}^{\varphi,x}(T_n > t)\\ &= \mathbb{E}^{\varphi,x}\Big[1_{T_n\le t}w\big(t-T_n,n\big)\Big] + \mathbb{P}^{\varphi,x}(T_n > t),\end{aligned} \tag{2.22}$$

where the second line follows from the strong Markov property of X under $\mathbb{P}^{\varphi,x}$ in view of Lemma 2.2 (see, e.g., Exercise 6.12 in [43]).
Let $(s_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be a nonincreasing sequence with a positive limit $s_\infty \in (0,\infty)$. By replacing t with s_n in (2.22), we can use the dominated convergence theorem when passing $n \uparrow \infty$ in (2.22) to see

$$\begin{aligned}0 &= \lim_{n\uparrow\infty}\mathbb{E}^{\varphi,x}\Big[1_{T_n\le s_n}w\big(s_n-T_n,n\big)\Big]\\ &= \mathbb{E}^{\varphi,x}\Big[1_{\zeta\le s_\infty}\lim_{n\uparrow\infty}w\big(s_n-T_n,n\big)\Big].\end{aligned} \tag{2.23}$$

Because $w \ge 0$, we have

$$1_{\zeta\le s_\infty}\lim_{n\uparrow\infty}w\big(s_n-T_n,n\big) = 0, \quad \mathbb{P}^{\varphi,x}\text{-a.s.} \tag{2.24}$$

By using (2.20) and the strict local martingale property of (2.13), we see that $w(t,x) < 1$ for $t \in (0,\infty)$ and $x \in \mathbb{R}$. Because $s_\infty \in (0,\infty)$, we then have

$$\mathbb{P}^{\varphi,x}(\zeta \le s_\infty) = 1 - w(s_\infty, x) > 0. \tag{2.25}$$

Consequently, the set $(\zeta \le s_\infty)$ in (2.24) is not $\mathbb{P}^{\varphi,x}$-null. Because s_n is nonincreasing with limit $s_\infty \in (0,\infty)$ and $T_n(\omega)$ is nondecreasing with limit $\zeta(\omega)$ for $\omega \in (\zeta \le s_\infty)$, we have

$$T_n(\omega) \le \zeta(\omega) \le s_\infty \le s_n, \quad \omega \in (\zeta \le s_\infty).$$

This property gives us

$$1_{\zeta\le s_\infty}\lim_{n\uparrow\infty}w\big(s_n-T_n,n\big) \ge 1_{\zeta\le s_\infty}\lim_{n\uparrow\infty}w\big(s_n,n\big), \tag{2.26}$$

where the inequality uses that $t \to w(t,x)$ is nonincreasing for all $x \in \mathbb{R}$. Combining (2.24) and (2.26) yields $\lim_{n\uparrow\infty} w\big(s_n,n\big) = 0$ because the set $(\zeta \le s_\infty)$ is not $\mathbb{P}^{\varphi,x}$-null. Finally, by using (2.20), we get

$$0 = \lim_{n\uparrow\infty} w(s_n,n) = \lim_{n\uparrow\infty}\tfrac{1}{\varphi_{\lambda\uparrow}(n)}\mathbb{E}^n[Y_{s_n}].$$

(ii) This proof is similar to the proof of (i) and is omitted.

□

In the setting of Section 6 of Çetin [6], the function $\varphi_0(x) := x$, $x \ge 0$, is a minimal harmonic function (unique up to a multiplicative constant) because X is a nonnegative local martingale. A special case of Corollary 6.1 in [6] shows that $\lim_{n\to\infty}\frac{\mathbb{E}^n[\varphi_0(X_t)]}{n} = 0$, $t > 0$, which is the analogue of the limits (2.14) and (2.16) associated with the minimal λ-harmonic functions $\varphi_{\lambda\uparrow}$ and $\varphi_{\lambda\downarrow}$ for $\lambda > 0$ and $s_n := t$ for $n \in \mathbb{N}$.

Example 2.6 shows that the limits (2.14) and (2.16) do not hold for arbitrary positive strict local martingales.

Example 2.6. Let $Y^y = (Y_t^y)_{t\geq 0}$ be the inverse three dimensional Bessel process with dynamics

$$dY_t = -Y_t^2 dB_t, \quad t \in (0,\infty), \quad Y_0 = y > 0. \tag{2.27}$$

Eq. (2.27) has a unique strong solution for any initial value $y \in (0,\infty)$ and for a given Brownian motion $B = (B_t)_{t\geq 0}$. This is the classical example due to Johnson and Helms [23] of a strict local martingale. For $t \geq 0$ and $\tilde{y} > 0$, the strict local martingale defined by $\tilde{Y}_t := \tilde{y} Y_t^1$ satisfies

$$\lim_{\tilde{y}\uparrow\infty} \frac{\mathbb{E}^{\mathbb{P}}[\tilde{Y}_t]}{\tilde{y}} = \mathbb{E}^{\mathbb{P}}[Y_t^1] \in (0,1), \quad t \in (0,\infty). \tag{2.28}$$

The dependence on the initial value $\tilde{y} > 0$ in the dynamics $d\tilde{Y}_t = -\frac{1}{\tilde{y}}\tilde{Y}_t^2 dB_t$ implies that Corollary 6.1 in Çetin [6] cannot be applied.

3. Main results and examples

In the first subsection, our first result relies only on a unique weak solution of the SDE (1.1). The second result uses a stronger local $\frac{1}{2}$-Hölder continuity and the resulting unique strong solution of (1.1). The second subsection gives two examples.

3.1. Main results.

Our first main result gives uniqueness of weak and classical solutions and existence of a weak solution of (1.2). The class of functions in which we establish uniqueness is constructed based on Theorem 1 in Kotani [31] and Lemma 2.5. The former result ensures that when the solution of (1.1) is a strict local martingale, at least one of (i) $\int_0^\infty \xi m(d\xi) < \infty$ and (ii) $\int_{-\infty}^0 |\xi| m(d\xi) < \infty$ holds (see also Theorem 1.6 in Delbaen and Shirakawa [9] for the case when the solution of (1.1) is nonnegative). In terms of the functions $\varphi_{\lambda\uparrow}$ and $\varphi_{\lambda\downarrow}$ from (2.5), we can define the nonnegative convex function

$$\Phi_\lambda := \varphi_{\lambda\downarrow} + \varphi_{\lambda\uparrow}, \tag{3.1}$$

which is uniformly bounded away from zero. Because both $\varphi_{\lambda\downarrow}$ and $\varphi_{\lambda\uparrow}$ are λ-harmonic functions, so is the function Φ_λ.

We write $f \in \mathcal{H}_{\lambda\uparrow}$ if $f : [0,\infty)\times\mathbb{R} \to \mathbb{R}$ is continuous and satisfies $f(0,x) = H(x)$, and $|f(t,x)| \leq K e^{\lambda t}\Phi_\lambda(x)$ for some constant $K \geq 0$ (K can vary with f), as well as the boundary condition

$$\lim_{n\uparrow\infty} \frac{f(s_n,n)}{n} = 0, \quad \text{whenever} \quad \infty > s_n \geq s_{n+1} \text{ and } s_\infty := \lim_{n\uparrow\infty} s_n > 0. \tag{3.2}$$

The set $\mathcal{H}_{\lambda\downarrow}$ is defined similarly, except we replace (3.2) with the boundary condition

$$\lim_{n\uparrow\infty} \frac{f(s_n,-n)}{n} = 0, \quad \text{whenever} \quad \infty > s_n \geq s_{n+1} \text{ and } s_\infty := \lim_{n\uparrow\infty} s_n > 0. \tag{3.3}$$

In the next result, the function h^* is defined in (1.4).

Theorem 3.1. *Suppose Assumption* 2.1 *holds and the continuous data* $H : \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\sup_{\xi,|\xi|>1} \frac{|H(\xi)|}{\Phi_\lambda(\xi)} < \infty. \tag{3.4}$$

(i) *If* $\int_0^\infty \xi m(d\xi) < \infty$ *and* $\int_{-\infty}^0 |\xi| m(d\xi) = \infty$*, then* $h^* \in \mathcal{H}_{\lambda\uparrow}$*, and there is at most one classical solution* $h \in \mathcal{C}^{1,2}$ *of* (1.2) *in* $\mathcal{H}_{\lambda\uparrow}$*. Moreover,* h^* *is the unique weak solution of* (1.2) *in* $\mathcal{H}_{\lambda\uparrow}$*.*

(ii) *If* $\int_0^\infty \xi m(d\xi) = \infty$ *and* $\int_{-\infty}^0 |\xi| m(d\xi) < \infty$*, then* $h^* \in \mathcal{H}_{\lambda\downarrow}$*, and there is at most one classical solution* $h \in \mathcal{C}^{1,2}$ *of* (1.2) *in* $\mathcal{H}_{\lambda\uparrow}$*. Moreover,* h^* *is the unique weak solution of* (1.2) *in* $\mathcal{H}_{\lambda\downarrow}$*.*

(iii) *If* $\int_0^\infty \xi m(d\xi) < \infty$ *and* $\int_{-\infty}^0 |\xi| m(d\xi) < \infty$*, then* $h^* \in \mathcal{H}_{\lambda\uparrow} \cap \mathcal{H}_{\lambda\downarrow}$*, and there is at most one classical solution* $h \in \mathcal{C}^{1,2}$ *of* (1.2) *in* $\mathcal{H}_{\lambda\uparrow} \cap \mathcal{H}_{\lambda\uparrow}$*. Moreover,* h^* *is the unique weak solution of* (1.2) *in* $\mathcal{H}_{\lambda\uparrow} \cap \mathcal{H}_{\lambda\downarrow}$*.*

Proof of (i). **Step 1/2:** First, because $\Phi_\lambda \in \mathcal{C}$, we see that h^* is continuous on $[0,\infty) \times \mathbb{R}$ if and only if $\mathbb{E}^{\Phi,x}[\frac{H(X_t)}{\Phi_\lambda(X_t)} 1_{t<\zeta}]$ is continuous. We define $\bar{h}(x) := \frac{H(x)}{\Phi_\lambda(x)}$ for $x \in \mathbb{R}$. We let $x_n \to x \in \mathbb{R}$, $t_n \to t \geq 0$, and define the sequence of stopping times $T_n := \inf\{t \geq 0 : X_t = x_n\}$. Blumenthal's 0-1 law gives $\mathbb{P}^{\Phi,x}(\lim_{n\to\infty} T_n = 0) = 1$. Because $\bar{h} \in \mathcal{C}_b$, we have

$$\begin{aligned}\mathbb{E}^{\Phi,x}[\bar{h}(X_{t_n+T_n}) 1_{t_n+T_n<\zeta} 1_{T_n<\zeta}] &= \mathbb{E}^{\Phi,x}[\mathbb{E}^{\Phi,x_n}[\bar{h}(X_{t_n}) 1_{t_n<\zeta}] 1_{T_n<\zeta}] \\ &= \mathbb{E}^{\Phi,x_n}[\bar{h}(X_{t_n}) 1_{t_n<\zeta}] \mathbb{P}^{\Phi,x}(T_n < \zeta),\end{aligned}$$

where the first equality is due to the strong Markov property. Since $\bar{h}$ is bounded, $T_n \to 0$, $\mathbb{P}^{\Phi,x}$-a.s., and $\mathbb{P}^{\Phi,x}(\zeta > 0) = 1$, the dominated convergence theorem gives

$$\mathbb{E}^{\Phi,x}[\bar{h}(X_t) 1_{t<\zeta}] = \lim_{n\to\infty} \mathbb{E}^{\Phi,x_n}[\bar{h}(X_{t_n}) 1_{t_n<\zeta}] \mathbb{P}^{\Phi,x}(T_n < \zeta) = \lim_{n\to\infty} \mathbb{E}^{\Phi,x_n}[\bar{h}(X_{t_n}) 1_{t<\zeta}].$$

This completes the proof of joint continuity. Since $h^*(t,x) := \mathbb{E}^x[H(X_t)]$, the process $\big(h^*(T-t, X_t)\big)_{t\in[0,T]}$ is a $\mathbb{P}^x$-martingale for any $T > 0$, which in turn implies that h^* is a weak solution by Sawyer [42] (see Theorem B.4 in Appendix B for details). The remaining properties needed for $h^* \in \mathcal{H}_{\lambda\uparrow}$ follow from (3.4) and Lemma 2.5.

Step 2/2: Let $h \in \mathcal{H}_{\lambda\uparrow}$ be a classical or weak solution of (1.2).

For a fixed constant $T \in [0,\infty)$, we establish the uniqueness claims by proving the following representation

$$\mathbb{E}^{\Phi,x}\left[\frac{H(X_T)}{e^{-\lambda T}\Phi_\lambda(X_T)} 1_{T<\zeta}\right] = \tilde{h}(T,x), \quad x \in \mathbb{R}. \tag{3.5}$$

In (3.5), the function $\tilde{h}$ is defined as

$$\tilde{h}(t,x) := \frac{h(t,x)}{\Phi_\lambda(x)}, \quad t \geq 0, \quad x \in \mathbb{R}, \tag{3.6}$$

and the Markov kernels $(\mathbb{P}^{\Phi,x})_{x\in\mathbb{R}_\Delta}$ are from Lemma 2.2 with $u := \Phi_\lambda$ where Φ_λ is defined in (3.1). By assumption, the function $\tilde{h}$ in (3.6) satisfies the uniform bound

$$|\tilde{h}(t,x)| \leq K e^{\lambda T}, \quad t \in [0,T], \quad x \in \mathbb{R}. \tag{3.7}$$

Similarly to (2.17), the scale function associated with the diffusion X under $\mathbb{P}^{\Phi,x}$ is defined by

$$s_\Phi(z) := \int_0^z \frac{1}{\Phi_\lambda^2(\xi)} d\xi, \quad z \in \mathbb{R}. \tag{3.8}$$

Because $\lim_{z\downarrow-\infty} s_\Phi(z) > -\infty$ and $\lim_{z\uparrow\infty} s_\Phi(z) < \infty$, the limit of X satisfies $X_\infty \in \{-\infty, \infty\}$, $\mathbb{P}^{\Phi,x}$-a.s., for $x \in \mathbb{R}$ (see, e.g., Proposition 5.5.22(d) in [27]). We also define the stopping times

$$\nu_n := \inf\{t > 0 : |X_t| \geq n\}, \quad n \in \mathbb{N}. \tag{3.9}$$

These stopping times $(\nu_n)_{n\in\mathbb{N}}$ have a limit. Namely, $\lim_{n\uparrow\infty} \nu_n = \zeta$, $\mathbb{P}^{\Phi,x}$-a.s., for all $x \in \mathbb{R}$ where ζ denotes the lifetime of X.

To see that (3.5) holds, we note that on $\mathcal{F}_{t\wedge\nu_n}$, we have $\frac{d\mathbb{P}^{\Phi,x}}{d\mathbb{P}} = e^{-\lambda t\wedge\nu_n} \frac{\Phi_\lambda(X_{t\wedge\nu_n})}{\Phi_\lambda(x)}$ for $n \in \mathbb{N}$ and $t \in [0,\infty)$. Because

$$\tilde{h}(T - t\wedge\nu_n, X_{t\wedge\nu_n}) \frac{\Phi_\lambda(X_{t\wedge\nu_n})}{\Phi_\lambda(x)} = h(T - t\wedge\nu_n, X_{t\wedge\nu_n}) \frac{1}{\Phi_\lambda(x)} \tag{3.10}$$

and $\big(h(T-t\wedge\nu_n, X_{t\wedge\nu_n})\big)_{t\in[0,T]}$ is a martingale under $\mathbb{P}^x$, the process $\big(e^{\lambda t\wedge\nu_n}\tilde{h}(T - t\wedge\nu_n, X_{t\wedge\nu_n})\big)_{t\in[0,T]}$ is a martingale under $\mathbb{P}^{\Phi,x}$. This is because when h is a classical solution of (1.2), Itô's lemma gives the local martingale property under $\mathbb{P}^x$. Alternatively, when h is a weak solution of (1.2), the local martingale property under $\mathbb{P}^x$ follows from Sawyer [42] (see also Theorem B.4 in Appendix B). This gives us

$$\begin{aligned} \tilde{h}(T,x) &= \mathbb{E}^{\Phi,x}\big[e^{\lambda T\wedge\nu_n}\tilde{h}(T - T\wedge\nu_n, X_{T\wedge\nu_n})\big] \\ &= \mathbb{E}^{\Phi,x}\big[e^{\lambda T}\tilde{h}(0, X_T) 1_{T<\nu_n}\big] + \mathbb{E}^{\Phi,x}\big[e^{\lambda\nu_n}\tilde{h}(T-\nu_n, X_{\nu_n}) 1_{T\geq\nu_n}\big]. \end{aligned} \tag{3.11}$$

The bound (3.7) allows us to use the dominated convergence theorem in (3.11) when passing $n \uparrow \infty$ to see that

$$\begin{aligned} \tilde{h}(T,x) &= \mathbb{E}^{\Phi,x}\big[e^{\lambda T}\tilde{h}(0, X_T) 1_{T<\zeta}\big] + \lim_{n\uparrow\infty} \mathbb{E}^{\Phi,x}\big[e^{\lambda\nu_n}\tilde{h}(T-\nu_n, X_{\nu_n}) 1_{T\geq\nu_n}\big] \\ &= \mathbb{E}^{\Phi,x}\left[\frac{e^{\lambda T} H(X_T)}{\Phi_\lambda(X_T)} 1_{T<\zeta}\right] + \mathbb{E}^{\Phi,x}\big[\lim_{n\uparrow\infty} e^{\lambda\nu_n}\tilde{h}(T-\nu_n, X_{\nu_n}) 1_{T\geq\nu_n}\big], \end{aligned} \tag{3.12}$$

where the second equality uses the initial condition $\tilde{h}(0,x) = \frac{H(x)}{\Phi_\lambda(x)}$ from (3.6). Therefore, the representation in (3.5) follows as soon as we show

$$\lim_{n\uparrow\infty} \tilde{h}(T-\nu_n, X_{\nu_n}) 1_{T\geq\nu_n} = 0, \quad \mathbb{P}^{\Phi,x}\text{-a.s.} \tag{3.13}$$

First, on the set $(T \geq \nu_n)$, (3.9) gives $X_{\nu_n} \in \{-n, n\}$. Therefore,

(3.14)
$$\begin{aligned}
\varphi_{\lambda\downarrow}(x) &= \lim_{n\uparrow\infty} \mathbb{E}^x\big[e^{-\lambda T\wedge\nu_n}\varphi_{\lambda\downarrow}(X_{T\wedge\nu_n})(1_{T\geq\nu_n} + 1_{T<\nu_n})\big] \\
&\geq \lim_{n\uparrow\infty}\Big(\varphi_{\lambda\downarrow}(-n)\mathbb{E}^x[e^{-\lambda T\wedge\nu_n}1_{T\geq\nu_n}1_{X_{\nu_n}=-n}] \\
&\quad + \mathbb{E}^x\big[e^{-\lambda T\wedge\nu_n}\varphi_{\lambda\downarrow}(X_{T\wedge\nu_n})1_{T<\nu_n}\big]\Big) \\
&= \lim_{n\uparrow\infty}\Big(\frac{\varphi_{\lambda\downarrow}(-n)}{\Phi_\lambda(-n)}\mathbb{E}^x\big[e^{-\lambda T\wedge\nu_n}\Phi_\lambda(X_{T\wedge\nu_n})1_{T\geq\nu_n}1_{X_{\nu_n}=-n}\big] \\
&\quad + \mathbb{E}^x\big[e^{-\lambda T}\varphi_{\lambda\downarrow}(X_T)1_{T<\nu_n}\big]\Big) \\
&=\Phi_\lambda(x)\lim_{n\uparrow\infty}\frac{\varphi_{\lambda\downarrow}(-n)}{\Phi_\lambda(-n)}\mathbb{P}^{\Phi,x}(T\geq\nu_n, X_{\nu_n}=-n) + \mathbb{E}^x\big[e^{-\lambda T}\varphi_{\lambda\downarrow}(X_T)\big] \\
&=\Phi_\lambda(x)\lim_{n\uparrow\infty}\mathbb{P}^{\Phi,x}(T\geq\nu_n, X_{\nu_n}=-n) + \varphi_{\lambda\downarrow}(x).
\end{aligned}$$

The second last equality uses the dominated convergence theorem. The last equality uses the martingale property of $\big(e^{-\lambda t}\varphi_{\lambda\downarrow}(X_t)\big)_{t\in[0,T]}$ from Theorem 2.3 and

$$\lim_{n\uparrow\infty}\frac{\varphi_{\lambda\downarrow}(-n)}{\Phi_\lambda(-n)} = \lim_{n\uparrow\infty}\frac{\varphi_{\lambda\downarrow}(-n)}{\varphi_{\lambda\uparrow}(-n)+\varphi_{\lambda\downarrow}(-n)} = 1$$

because $\lim_{n\uparrow\infty}\varphi_{\lambda\uparrow}(-n) \in [0,\infty)$. The zero limit in (3.14) and the bound (3.7) give

$$\lim_{n\uparrow\infty}\tilde{h}(T-\nu_n, -n)1_{T\geq\nu_n}1_{X_{\nu_n}=-n} = 0, \quad \mathbb{P}^{\Phi,x}\text{-a.s.} \tag{3.15}$$

Second, because the set $(\zeta = T)$ is $\mathbb{P}^{\Phi,x}$-null by Lemma 2.2(iv), the sets $(T\geq\zeta)$ and $(T>\zeta)$ differ only by a $\mathbb{P}^{\Phi,x}$-null set. We can then use the boundary condition (3.2) and the linear growth of $\varphi_{\lambda\uparrow}(\xi)$ and $\Phi_\lambda(\xi)$ as $\xi\uparrow\infty$ to see

$$\begin{aligned}
\lim_{n\uparrow\infty}\tilde{h}(T-\nu_n, n)1_{T\geq\nu_n}1_{X_{\nu_n}=n} &= 1_{T\geq\zeta}1_{X_\zeta=\infty}\lim_{n\uparrow\infty}\tilde{h}(T-\nu_n,n) \\
&= 1_{T>\zeta}1_{X_\zeta=\infty}\lim_{n\uparrow\infty}\tilde{h}(T-\nu_n,n) \\
&= 0,
\end{aligned} \tag{3.16}$$

$\mathbb{P}^{\Phi,x}$-a.s. The two observations (3.15) and (3.16) establish (3.13).

(ii) and (iii): These are similar to (i) and are omitted. □

We note that (3.4) covers continuous data $H:\mathbb{R}\to\mathbb{R}$ of at most linear growth, i.e.,

$$\sup_{\xi,|\xi|>1}\frac{|H(\xi)|}{|\xi|} < \infty. \tag{3.17}$$

When H satisfies (3.17), Theorem 3.10(i) in [16] ensures that h^* in (1.4) is also of at most linear growth. However, our condition (3.4) is more general than (3.17) because Theorem 2.3(ii) shows that when $\int_{-\infty}^0 |\xi| m(d\xi) = \infty$, (3.4) allows for superlinearly growing data H as $\xi\downarrow-\infty$. Similarly, when $\int_0^\infty \xi m(d\xi)=\infty$, (3.4) allows for superlinearly growing data H as $\xi\uparrow\infty$. For example, the logarithm of the two dimensional Bessel process in Example 2.4 allows for superlinearly growing data as $\xi \uparrow \infty$.

Because $H(\xi) := \xi$ satisfies (3.17), Assumption 2.1 ensures that $\mathbb{E}^x[|X_t|] < \infty$ for $x \in \mathbb{R}$ and $t \geq 0$ (this also follows from Lemma 1 in [31]). However, Example 3.2 shows that real-valued strict local martingales can fail to be integrable in general.

Example 3.2. Let $Y = (Y_t)_{t\geq 0}$ denote the inverse three dimensional Bessel process with dynamics (2.27) and initial value $y \in (0,\infty)$. From, e.g., p.74 in Protter [38], the second moment satisfies $\mathbb{E}^y[Y_t^2] < \infty$ while $\mathbb{E}^y[\langle Y\rangle_t] = \infty$ for $t \in (0,\infty)$. Consequently, the real-valued local martingale

$$X_t := Y_t^2 - \langle Y\rangle_t, \quad t \geq 0, \tag{3.18}$$

is not integrable. In particular, the process X is a strict local martingale too.

Under the following stronger local $\frac{1}{2}$-Hölder-continuity assumption on σ, there is a classical solution to (1.1).

Assumption 3.3. The volatility function $\sigma : \mathbb{R} \to (0,\infty)$ is locally $\frac{1}{2}$-Hölder continuous.

The local $\frac{1}{2}$-Hölder continuity in Assumption 3.3 can be used to upgrade the unique weak solution of (1.1) to a pathwise unique strong solution. To see this, the weaker Engelbert and Schmidt conditions in Assumption 2.1 produce a global weak solution (unique in law). Assumption 3.3 allows us to use Yamada-Watanabe's theorem to prove strong uniqueness for $t \in [0, T_n]$ for a reducing sequence of stopping times $(T_n)_{n\in\mathbb{N}}$. Because $T_n \uparrow \infty$ as $n \uparrow \infty$, this local strong uniqueness extends to global strong uniqueness. Therefore, global strong existence follows.[5]

Theorem 3.4. *Suppose Assumption* 3.3 *holds and assume the continuous data* $H : \mathbb{R} \to \mathbb{R}$ *satisfies* (3.4)*.*

(i) *If* $\int_0^\infty \xi m(d\xi) < \infty$ *and* $\int_{-\infty}^0 |\xi| m(d\xi) = \infty$*, then the function* h^* *in* (1.4) *is the unique classical solution* $h \in \mathcal{C}^{1,2}$ *of* (1.2) *in* $\mathcal{H}_{\lambda\uparrow}$*.*

(ii) *If* $\int_0^\infty \xi m(d\xi) = \infty$ *and* $\int_{-\infty}^0 |\xi| m(d\xi) < \infty$*, then the function* h^* *in* (1.4) *is the unique classical solution* $h \in \mathcal{C}^{1,2}$ *of* (1.2) *in* $\mathcal{H}_{\lambda\downarrow}$

(iii) *If* $\int_0^\infty \xi m(d\xi) < \infty$ *and* $\int_{-\infty}^0 |\xi| m(d\xi) < \infty$*, then the function* h^* *in* (1.4) *is the unique classical solution* $h \in \mathcal{C}^{1,2}$ *of* (1.2) *in* $\mathcal{H}_{\lambda\uparrow} \cap \mathcal{H}_{\lambda\downarrow}$*.*

Proof of (i). **Step 1/2:** In this first step, we consider H positive; that is, $H : \mathbb{R} \to [0,\infty)$. First, we prove that h^* defined in (1.4) is a classical solution of (1.2), h^* is bounded by $Ke^{\lambda t}\Phi_\lambda(x)$, and h^* satisfies (3.2).

Because H satisfies (3.4), we can find two positive constants (r_0, r) such that $H(x) \leq r_0 + r\Phi_\lambda(x)$ for all $x \in \mathbb{R}$. We therefore have the upper bound

$$\begin{aligned}\mathbb{E}^x[H(X_t)] &\leq r_0 + r\mathbb{E}^x[\Phi_\lambda(X_t)]\\ &\leq r_0 + re^{\lambda t}\Phi_\lambda(x),\end{aligned} \tag{3.19}$$

[5] When σ is globally $\frac{1}{2}$-Hölder continuous, the unique strong solution X^x of (1.1) is a martingale. This is because $\sup_{\xi\in\mathbb{R}} \frac{|\sigma(\xi)-\sigma(0)|}{\sqrt{|\xi|}} < \infty$ implies $\int_0^\infty \frac{\xi}{\sigma^2(\xi)}d\xi = \int_{-\infty}^0 \frac{|\xi|}{\sigma^2(\xi)}d\xi = \infty$. Theorem 1 in [31] gives the martingale property of X^x.

where the last inequality follows from $\big(e^{-\lambda t}\Phi_\lambda(X_t)\big)_{t\geq 0}$ being a $\mathbb{P}^x$-supermartingale. Because $\Phi_\lambda > 0$ is uniformly bounded away from zero, the second inequality (3.19) ensures that the function $\tilde{h}$ in (3.6) satisfies the bound (uniform in x)

$$\tilde{h}(t,x) \leq \frac{r_0}{\Phi_\lambda(x)} + re^{\lambda t} \leq \frac{r_0}{\inf_{\xi\in\mathbb{R}}\Phi_\lambda(\xi)} + re^{\lambda t}, \quad t\geq 0, \quad x\in\mathbb{R}. \tag{3.20}$$

The property $\int_0^\infty \xi m(d\xi) < \infty$ and Theorem 2.3 ensure that $\varphi_{\lambda\uparrow}$ satisfies

$$c_0 := \limsup_{y\uparrow\infty} \frac{\varphi_{\lambda\uparrow}(y)}{y} \in (0,\infty). \tag{3.21}$$

For a nonincreasing sequence $(s_n)_{n\in\mathbb{N}} \subset (0,\infty)$ with a positive and finite limit $s_\infty := \lim_{n\uparrow\infty} s_n \in (0,\infty)$, the limit (2.14) in Lemma 2.5(i) gives us the limit in (3.2) because

$$\begin{aligned}
\lim_{n\uparrow\infty} \frac{h(s_n,n)}{n} &\leq \lim_{n\uparrow\infty} \frac{r_0 + r\mathbb{E}^n[\Phi_\lambda(X_{s_n})]}{\varphi_{\lambda\uparrow}(n)} \frac{\varphi_{\lambda\uparrow}(n)}{n} \\
&\leq \lim_{n\uparrow\infty} \frac{r_0 + re^{\lambda s_n}\varphi_{\lambda\downarrow}(n) + r\mathbb{E}^n[\varphi_{\lambda\uparrow}(X_{s_n})]}{\varphi_{\lambda\uparrow}(n)} \frac{\varphi_{\lambda\uparrow}(n)}{n} \\
&\leq c_0 \lim_{n\uparrow\infty} \frac{r_0' + r\mathbb{E}^n[\varphi_{\lambda\uparrow}(X_{s_n})]}{\varphi_{\lambda\uparrow}(n)} \\
&= 0.
\end{aligned} \tag{3.22}$$

In (3.22), the constant c_0 is from (3.21) and r_0' is some positive irrelevant constant.

To see that the PDE (1.2) holds, we change coordinates. To this end, first assume that H satisfies

$$H(\xi) \leq r_0 + r\varphi_{\lambda\uparrow}(\xi), \quad \xi\in\mathbb{R}, \tag{3.23}$$

for positive constants (r_0, r). Due to the continuity of σ, the ODE in (2.7) yields that $\varphi_{\lambda\uparrow}$ is a strictly increasing and strictly convex function with $\varphi_{\lambda\uparrow}, \varphi_{\lambda\uparrow}^{-1} \in \mathcal{C}^2$. The continuous function

$$F(y) := H\big(\varphi_{\lambda\uparrow}^{-1}(y)\big), \quad y > \underline{y} := \lim_{\xi\downarrow-\infty} \varphi_{\lambda\uparrow}(\xi), \tag{3.24}$$

is of at most linear growth and satisfies $\lim_{y\downarrow\underline{y}} F(y) < \infty$. Furthermore, for a fixed constant $T\in(0,\infty)$, we define the process

$$Y_t := e^{\lambda(T-t)}\varphi_{\lambda\uparrow}(X_t), \quad t\in[0,T], \quad \lambda > 0, \tag{3.25}$$

with the local martingale dynamics

$$\begin{aligned}
dY_t &= e^{\lambda(T-t)}\varphi_{\lambda\uparrow}'(X_t)\sigma(X_t)dB_t \\
&= e^{\lambda(T-t)}\varphi_{\lambda\uparrow}'\big(\varphi_{\lambda\uparrow}^{-1}(e^{-\lambda(T-t)}Y_t)\big)\sigma\big(\varphi_{\lambda\uparrow}^{-1}(e^{-\lambda(T-t)}Y_t)\big)dB_t \\
&= \alpha(t,Y_t)dB_t.
\end{aligned} \tag{3.26}$$

In (3.26), we have defined the volatility function

$$\alpha(t,y) := e^{\lambda(T-t)}\varphi_{\lambda\uparrow}'\big(\varphi_{\lambda\uparrow}^{-1}(e^{-\lambda(T-t)}y)\big)\sigma\big(\varphi_{\lambda\uparrow}^{-1}(e^{-\lambda(T-t)}y)\big), \quad y > \underline{y}, \quad t\geq 0. \tag{3.27}$$

Because $\varphi_{\lambda\uparrow}, \varphi_{\lambda\uparrow}^{-1} \in \mathcal{C}^2$, the function $\alpha(t,y)$ is continuous in (t,y) and locally Hölder continuous in y with exponent $\frac{1}{2}$. Theorem 3.2 in Ekström and Tysk [11] guarantees that the function

$$f(t,y) := \bar{\mathbb{E}}^{t,y}[F(Y_T)], \quad t \in [0,T], \quad y > \underline{y}, \tag{3.28}$$

where $\bar{\mathbb{E}}^{t,y}$ denotes the expectation with respect to the law of $(Y_u)_{u\in[t,T]}$ conditional on $Y_t = y$, is a classical solution of the Cauchy problem

$$\begin{cases} f(T,y) = F(y), \quad y > \underline{y}, \\ 0 = f_t(t,y) + \frac{1}{2}\alpha^2(t,y) f_{yy}(t,y), \quad y > \underline{y}, \quad t \in (0,T). \end{cases} \tag{3.29}$$

We then see that the function h^* from (1.4) satisfies the PDE (1.2) because $\varphi_{\lambda\uparrow}$ solves the Sturm-Liouville ODE (2.7) and we have the relation

$$\begin{aligned} f\big(t,y\big) &= \bar{\mathbb{E}}^{t,y}[F(Y_T)] \\ &= \mathbb{E}^{y^\circ(t)}[H(X_{T-t})] \\ &= h^*\big(T-t, \varphi_{\lambda\uparrow}^{-1}(e^{-\lambda(T-t)}y)\big), \end{aligned} \tag{3.30}$$

where we have defined $y^\circ(t) := \varphi_{\lambda\uparrow}^{-1}(e^{-\lambda(T-t)}y)$ for $t \in [0,T]$ and $y > \underline{y}$.

Second, a similar argument but replacing (3.23) with

$$H(\xi) \le r_0 + r\varphi_{\lambda\downarrow}(\xi), \quad \xi \in \mathbb{R}, \tag{3.31}$$

and replacing (3.25) with $Y_t := e^{\lambda(T-t)}\varphi_{\lambda\downarrow}(X_t)$ when changing coordinates, shows that h from (1.4) satisfies the PDE (1.2) again.

Third, by writing

$$H(\xi) = H^1(\xi) + H^2(\xi) - H(0), \quad H^1(\xi) := H(\xi \vee 0), \quad H^2(\xi) := H(\xi \wedge 0), \tag{3.32}$$

and noting that when H satisfies (3.4), the function H^1 satisfies (3.23) and the function H^2 satisfies (3.31). For $i \in \{1,2\}$, the functions

$$h^i(t,x) : = \mathbb{E}^x[H^i(X_t)], \quad t \ge 0, \quad x \in \mathbb{R}, \tag{3.33}$$

satisfy the PDEs

$$\begin{cases} h^i(0,x) = H^i(x), \quad x \in \mathbb{R}, \\ h^i_t(t,x) = \frac{1}{2}\sigma^2(x) h^i_{xx}(t,x), \quad t > 0, \quad x \in \mathbb{R}. \end{cases} \tag{3.34}$$

We therefore have $h(t,x) := h^1(t,x) + h^2(t,x) - H(0)$ is the function in (1.4). Furthermore, by using the PDEs in (3.34), we see that h satisfies (1.2).

Step 2/2: We consider $H : \mathbb{R} \to \mathbb{R}$ and write $H(\xi) = H^+(\xi) - H^-(\xi)$ where $H^+, H^- : \mathbb{R} \to [0,\infty)$ are defined by $H^+(\xi) := H(\xi) \vee 0$ and $H^-(\xi) := -\big(H(\xi) \wedge 0\big)$ for $\xi \in \mathbb{R}$. The first step ensures that the functions

$$h^\pm(t,x) : = \mathbb{E}^x[H^\pm(X_t)], \quad t \ge 0, \quad x \in \mathbb{R}, \tag{3.35}$$

satisfy (uniquely)

$$\begin{cases} h^\pm(0,x) = H^\pm(x), \quad x \in \mathbb{R}, \\ h^\pm_t(t,x) = \frac{1}{2}\sigma^2(x) h^\pm_{xx}(t,x), \quad t \ge 0, \quad x \in \mathbb{R}, \end{cases} \tag{3.36}$$

as well as the limit in (3.2)

$$\lim_{n\uparrow\infty} \frac{h^\pm(s_n,n)}{n} = 0, \quad \text{whenever} \quad \infty > s_n \ge s_{n+1} \text{ and } s_\infty := \lim_{n\uparrow\infty} s_n > 0. \tag{3.37}$$

We then see that the difference $h := h^+ - h^-$ is the function in (1.4). Furthermore, by taking differences in (3.36), we see that h satisfies the PDE (1.2) and the limit in (3.2).

(ii) and (iii): These are similar to (i) and are omitted. □

Based on Theorem 3.4, the value function h^* in (1.4) can exhibit a boundary layer at $t = 0$ in the following sense: Consider the mean $H(\xi) := \xi$, $\xi \in \mathbb{R}$, which satisfies (3.17). Whenever $\int_0^\infty \xi m(d\xi) < \infty$, the value function h^* in (1.4) satisfies (1.5) as $x \uparrow \infty$ because Theorem 3.4(i) gives us

$$\lim_{x\uparrow\infty}\lim_{t\downarrow 0}\frac{h^*(t,x)}{x} = 1, \qquad \lim_{t\downarrow 0}\lim_{x\uparrow\infty}\frac{h^*(t,x)}{x} = 0. \tag{3.38}$$

Similarly, whenever $\int_{-\infty}^0 |\xi| m(d\xi) < \infty$, the value function h^* in (1.4) satisfies (1.5) as $x \downarrow -\infty$ because Theorem 3.4(ii) gives us

$$\lim_{x\downarrow-\infty}\lim_{t\downarrow 0}\frac{h^*(t,x)}{x} = 1, \qquad \lim_{t\downarrow 0}\lim_{x\downarrow-\infty}\frac{h^*(t,x)}{x} = 0. \tag{3.39}$$

3.2. **Examples.** Example 3.5 covers a class of real valued strict local martingales frequently used in finance (see, e.g., Zühlsdorff [45] and Andersen [1]).

Example 3.5. Quadratic normal volatility models use dynamics defined by

$$dX_t := \big(\alpha_0 + \alpha_1 X_t + \alpha_2 X_t^2\big)dB_t, \quad X_0 \in \mathbb{R}, \tag{3.40}$$

and have been widely used in financial economics (see Carr, Fisher, and Ruf [5] for an overview). Depending on the root configuration ($\alpha_0 + \alpha_1\xi + \alpha_2\xi^2 = 0$, $\xi \in \mathbb{R}$) relative to the initial value X_0, the solution to the SDE (3.40) is bounded or unbounded from above and/or below. For example, in a Radner equilibrium model with limited stock-market participation, the following SDE is endogenously derived in Eq. (27) in Basak and Cuoco [2]

$$dX_t = -X_t(1 + X_t)\sigma dB_t, \quad X_0 > 0, \tag{3.41}$$

for a constant $\sigma \in (0, \infty)$. The dynamics (3.41) produce a nonnegative strict local martingale. Another specification of (3.40) is the no-real-root specification used for option pricing in Section 3.6 in Zühlsdorff [45] and Eq. (4.1) in Andersen [1]. This process is exogenously given by the dynamics

$$dX_t = b\Big(1 + \big(\frac{X_t - a}{b}\big)^2\Big)dB_t, \quad X_0 \in \mathbb{R}, \tag{3.42}$$

for constants (a, b) with $b \in (0, \infty)$. The dynamics (3.42) produce a real-valued strict local martingale. Because

$$\int_0^\infty \frac{\xi}{\big(b^2 + (a-\xi)^2\big)^2}d\xi < \infty, \quad \int_{-\infty}^0 \frac{|\xi|}{\big(b^2 + (a-\xi)^2\big)^2}d\xi < \infty, \tag{3.43}$$

we see from Theorem 3.1(iii) that h in (1.4) vanishes as $x \to \pm\infty$ for $t > 0$. In this case, the mean function $H(\xi) := \xi$, $\xi \in \mathbb{R}$, produces a double boundary layer in the sense that for $t > 0$, we have the limits in both (3.38) and (3.39).

Example 3.6 is based on the two dimensional Bessel process.

Example 3.6 (Continuation of Example 2.4)**.** Let $X = (X_t)_{t\geq 0}$ be the logarithm of the two dimensional Bessel process (2.11). We claim that

$$\mathbb{E}^x[X_t] = x + \int_{e^x}^{\infty} \frac{1}{r} e^{-\frac{r^2}{2t}} dr, \quad x \in \mathbb{R}, \quad t \geq 0. \tag{3.44}$$

To see this, we define the function

$$h(t,x) := x + \int_{e^x}^{\infty} \frac{1}{r} e^{-\frac{r^2}{2t}} dr, \quad x \in \mathbb{R}, \quad t \geq 0. \tag{3.45}$$

By computing t and x derivatives in (3.45), we see that the PDE in (1.2) holds. Furthermore, for $t > 0$, L'Hopital's rule produces the limit

$$\begin{aligned} \lim_{x\downarrow -\infty} \frac{1}{x} \int_{e^x}^{\infty} \frac{1}{r} e^{-\frac{r^2}{2t}} dr &= -\lim_{x\downarrow -\infty} e^{-\frac{e^{2x}}{2t}} \\ &= -1. \end{aligned} \tag{3.46}$$

For $t > 0$, we therefore see that the function h in (3.45) has the limit in (3.3). Because

$$\int_0^{\infty} \xi e^{2\xi} d\xi = \infty, \quad \int_{-\infty}^{0} |\xi| e^{2\xi} d\xi < \infty, \tag{3.47}$$

we can use the uniqueness part of Theorem 3.1(ii) to see that (3.44) holds. Consequently, the boundary layer limits in (3.39) hold.

As an aside, the limit in (3.3) trivially holds because we have

$$\lim_{x\downarrow -\infty} \mathbb{E}^x[X_t] = \frac{1}{2}\big(\log(2) + \log(t) - \gamma\big) \in \mathbb{R}, \quad t > 0, \tag{3.48}$$

where γ is the Euler-Mascheroni constant ($\gamma \approx 0.57721$).[6]

4. Uniqueness for higher moments

Under Assumption 3.3, we denote by X^x the unique strong solution of (1.1) for $x \in \mathbb{R}$. This section uses the above results to regularize the strict local martingale X^x into a martingale $N = (N_t)_{t\geq 0}$. This regularization allows us to prove uniqueness of classical solutions to an altered PDE when the continuous data H is of at most polynomial growth. In other words, we consider a continuous function $H : \mathbb{R} \to \mathbb{R}$ such that

$$|H(\xi)| \leq c(1 + |\xi|^p), \quad \xi \in \mathbb{R}, \tag{4.1}$$

where $c \in (0,\infty)$ and $p \in (1,\infty)$ are constants (c and p can vary with H). For $T \in (0,\infty)$, we define the martingale

$$\begin{aligned} N_t :&= \mathbb{E}[X_T^x | \mathcal{F}_t] \\ &= h^*(T-t, X_t^x), \quad t \in [0,T], \quad x \in \mathbb{R}, \end{aligned} \tag{4.2}$$

where h^* is from Theorem 3.4. For $t \in [0,T]$, we show in Lemma 4.1 that $x \to h^*(t,x)$ is strictly increasing under the hypothesis of Theorem 3.4. The difficulty in proving this seemingly trivial result stems from X^x being a strict local martingale. Indeed, when X^x is a martingale, $\mathbb{E}[X_t^x] = x$ is trivially strictly increasing. Moreover, X^x being a strict local martingale also implies that we cannot

[6]When X is an inverse three dimensional Bessel process (which is positive), Example 2.2.2 in [8] gives a limit similar to (3.48) with $\lim_{x\uparrow\infty} \mathbb{E}^x[X_t] < \infty$ for $t > 0$.

use strict comparison results for SDEs based on Lipschitz continuity properties like Theorem 33.6 in Kallenberg [24] and Theorem in IX.3.8 in Revuz and Yor [39]. Theorem 1.4 in [37] shows that $x \to h^*(t,x)$ is nondecreasing even when X^x is a strict local martingale. However, this property is insufficient to produce the inverse function $(h^*)^{-1}(t,\cdot)$ we need below.

Lemma 4.1. *Let $F:\mathbb{R}\to\mathbb{R}$ be continuous, satisfy* (3.4)*, and be strictly increasing. Under Assumption* 3.3*, for each $t\geq 0$, the function $\mathbb{R}\ni x\to\mathbb{E}[F(X_t^x)]$ is strictly increasing.*

Proof. Theorem IX.3.8 in [39] or Theorem V.43.1 in [40] ensures that $x<y$ implies $X_t^x\leq X_t^y$, $\mathbb{P}$-a.s., for all $t\geq 0$. We claim that for all $x<y$, we have

$$t_0 := \inf\{t\geq 0 : \mathbb{P}(X_t^x = X_t^y)=1\} = \infty. \tag{4.3}$$

This claim gives us that $\mathbb{R}\ni x\to\mathbb{E}[F(X_t^x)]$ is strictly increasing. We argue by contradiction and assume $t_0\in[0,\infty)$.

Step 1/3: If $t_0=0$, we can find $t_n\downarrow 0$ such that $\mathbb{P}(X_{t_n}^x = X_{t_n}^y)=1$ for all $n\in\mathbb{N}$. The set $\Omega^\circ := \{\omega\in\Omega \mid \forall n\in\mathbb{N}: X_{t_n}^x(\omega)=X_{t_n}^y(\omega)\}$ satisfies $\mathbb{P}(\Omega^\circ)=1$, and so path continuity gives the contradiction

$$\forall\omega\in\Omega^\circ: 0=\lim_{n\to\infty}\big(X_{t_n}^x(\omega)-X_{t_n}^y(\omega)\big)=x-y<0.$$

Step 2/3: If $t_0\in(0,\infty)$, we claim[7]

$$\exists\epsilon\in(0,t_0):\ \mathbb{P}(X_\epsilon^x\leq \tfrac{x+y}{2}, X_\epsilon^y\geq y)>0 \text{ and } \mathbb{P}(X_\epsilon^x\leq x, X_\epsilon^y\geq \tfrac{x+y}{2})>0. \tag{4.4}$$

We argue by contradiction and assume (4.4) fails. In other words, we assume

$$\forall\epsilon\in(0,t_0):\ \mathbb{P}(X_\epsilon^x\leq \tfrac{x+y}{2}, X_\epsilon^y\geq y)=0 \text{ or } \mathbb{P}(X_\epsilon^x\leq x, X_\epsilon^y\geq \tfrac{x+y}{2})=0. \tag{4.5}$$

Equivalently, we assume

$$\forall\epsilon\in(0,t_0):\ \mathbb{P}(X_\epsilon^x> \tfrac{x+y}{2} \text{ or } X_\epsilon^y< y)=1 \text{ or } \mathbb{P}(X_\epsilon^x> x \text{ or } X_\epsilon^y< \tfrac{x+y}{2})=1. \tag{4.6}$$

We define the sequence $\epsilon_n:=\frac{1}{n}$ for $n\in\mathbb{N}$ big enough such that $\frac{1}{n}\in(0,t_0)$. Based on (4.6), there exists a subsequence $(\epsilon'_n)_{n\in\mathbb{N}}\subset(\epsilon_n)_{n\in\mathbb{N}}$ such that at least one of the following two statements holds

$$\forall n\in\mathbb{N}: \mathbb{P}(X_{\epsilon'_n}^x>\tfrac{x+y}{2} \text{ or } X_{\epsilon'_n}^y<y)=1, \tag{4.7}$$

$$\forall n\in\mathbb{N}: \mathbb{P}(X_{\epsilon'_n}^x>x \text{ or } X_{\epsilon'_n}^y<\tfrac{x+y}{2})=1. \tag{4.8}$$

We assume (4.8) holds and seek to create a contradiction. Alternatively, when (4.7) is assumed to hold, a similar argument also produces a contradiction. We define the two sets

$$\begin{aligned}\Omega' &:= \{\omega\in\Omega \,|\, \forall n\in\mathbb{N}: X_{\epsilon'_n}^x(\omega)>x \text{ or } X_{\epsilon'_n}^y(\omega)<\tfrac{x+y}{2}\},\\ \Omega'' &:= \{\omega\in\Omega \,|\, \exists N(\omega)\in\mathbb{N}: \forall n\geq N(\omega),\ X_{\epsilon'_n}^x(\omega)>x\}.\end{aligned} \tag{4.9}$$

Eq. (4.8) gives $\mathbb{P}(\Omega')=1$. The path continuity of X^y gives for $\omega\in\Omega$ the limit $\lim_{t\downarrow 0}X_t^y(\omega)=y>\frac{x+y}{2}$. For $\omega\in\Omega'$, we therefore see that $X_{\epsilon'_n}^x(\omega)>x$ for large

[7] In (4.4), we cannot consider intersections like $(X_\epsilon^x\leq x)\cap(X_\epsilon^y\geq y)$ because these sets can be null sets for all $\epsilon>0$. For example, $X_t^x:=x+B_t$ gives the null set $(X_\epsilon^x\leq x)\cap(X_\epsilon^y\geq y)=(B_\epsilon=0)$.

enough values of n. This property gives the set inclusion $\Omega' \subseteq \Omega''$. To show the contradiction $\mathbb{P}(\Omega'') = 0$, we define the sets

$$E_n^\delta := \{\omega \in \Omega \,|\, X^x_{\epsilon'_n}(\omega) \geq x + \delta\}, \quad n \in \mathbb{N}, \quad \delta > 0. \tag{4.10}$$

Because the sets

$$\Omega'''(\delta) := \{\omega \in \Omega \,|\, \exists N(\omega) \in \mathbb{N} : \forall n \geq N(\omega), X^x_{\epsilon'_n}(\omega) \geq x + \delta\}, \quad \delta > 0, \tag{4.11}$$

satisfy $\Omega'''(\delta) \subseteq (E_n^\delta \text{ i.o.})$, $\Omega'''(\delta) \subseteq \Omega'''(\delta')$ for $\delta > \delta' > 0$, and $\cup_{\delta>0}\Omega'''(\delta) = \Omega''$, it suffices to prove $\mathbb{P}(E_n^\delta \text{ i.o.}) = 0$ for each $\delta > 0$ to justify $\mathbb{P}(\Omega'') = 0$. By the Borel-Cantelli lemma, $\mathbb{P}(E_n^\delta \text{ i.o.}) = 0$ is ensured by $\sum^\infty \mathbb{P}(E_n^\delta) < \infty$. To this end, the hitting times $T_{x+\delta} := \inf\{t > 0 : X_t^x = x + \delta\}$ for $x \in \mathbb{R}$ and $\delta > 0$ satisfy

$$\mathbb{P}(E_n^\delta) \leq \mathbb{P}(T_{x+\delta} \leq \epsilon'_n) \leq e\mathbb{E}[e^{-T_{x+\delta}/\epsilon'_n}], \quad n \in \mathbb{N}. \tag{4.12}$$

In (4.12), the last inequality follows from

$$\mathbb{E}[e^{-T_{x+\delta}/\epsilon'_n}] \geq \mathbb{E}[e^{-T_{x+\delta}/\epsilon'_n} 1_{T_{x+\delta} \leq \epsilon'_n}] \geq e^{-1}\mathbb{P}(T_{x+\delta} \leq \epsilon'_n), \quad n \in \mathbb{N}.$$

The proposition on p. 258 in Kotani and Watanabe [32] gives us the limit

$$\lim_{n\to\infty} \sqrt{\epsilon'_n}\big(- \log(\mathbb{E}[e^{-T_{x+\delta}/\epsilon'_n}])\big) = \sqrt{2}\int_x^{x+\delta} |\sigma(\xi)|d\xi \in (0, \infty). \tag{4.13}$$

Consequently, there exists a constant $C^\delta > 0$ (independent of $n \in \mathbb{N}$) such that for large $n \in \mathbb{N}$, we have $-\log(\mathbb{E}[e^{-T_{x+\delta}/\epsilon'_n}]) \geq C^\delta/\sqrt{\epsilon'_n}$. Therefore, the inequality in (4.12) gives

$$\sum^\infty \mathbb{P}(E_n^\delta) \leq e\sum^\infty \mathbb{E}[e^{-T_{x+\delta}/\epsilon'_n}] \leq e\sum^\infty e^{-C^\delta/\sqrt{\epsilon'_n}} \leq e\sum^\infty e^{-C^\delta\sqrt{n}} < \infty.$$

Step 3/3: For ϵ as in (4.4), the Markov property gives

$$f(X^x_\epsilon) = \mathbb{E}[X^x_{t_0}|\mathcal{F}_\epsilon], \quad f(z) := \mathbb{E}[X^z_{t_0-\epsilon}], \quad z \in \mathbb{R}. \tag{4.14}$$

Because $t_0 - \epsilon < t_0$, the definition of t_0 in (4.3) gives $\mathbb{P}(X^x_{t_0-\epsilon} = X^y_{t_0-\epsilon}) < 1$, which combined with $\mathbb{P}(X^x_{t_0-\epsilon} \leq X^y_{t_0-\epsilon}) = 1$ produces the inequality

$$f(x) = \mathbb{E}[X^x_{t_0-\epsilon}] < \mathbb{E}[X^y_{t_0-\epsilon}] = f(y). \tag{4.15}$$

We split $[x, y]$ into $[x, \frac{x+y}{2}]$ and $[\frac{x+y}{2}, y]$. The strict inequality in (4.15) gives us that at least one of the following two inequalities hold

$$f(x) < f(\tfrac{x+y}{2}), \quad f(\tfrac{x+y}{2}) < f(y). \tag{4.16}$$

Both cases in (4.16) are similar and it suffices to consider $f(\frac{x+y}{2}) < f(y)$. We then have

$$f\big(X^x_\epsilon(\omega)\big) \leq f(\tfrac{x+y}{2}) < f(y) \leq f\big(X^y_\epsilon(\omega)\big), \quad \omega \in (X^x_\epsilon \leq \tfrac{x+y}{2}) \cap (X^y_\epsilon \geq y). \tag{4.17}$$

By taking expectations in (4.17) and using (4.4) and $\mathbb{P}(X^x_\epsilon \leq X^y_\epsilon) = 1$, we get

$$\mathbb{E}[f(X^x_\epsilon)] < \mathbb{E}[f(X^y_\epsilon)]. \tag{4.18}$$

On the other hand, we can find $t_n \downarrow t_0$ such that $\mathbb{P}(X^x_{t_n} = X^y_{t_n}) = 1$ for all $n \in \mathbb{N}$. Therefore, path continuity gives $\mathbb{P}(X^x_{t_0} = X^y_{t_0}) = 1$. Consequently, the Markov property in (4.14) produces a contradiction with (4.18):

$$\mathbb{E}[f(X^x_\epsilon)] = \mathbb{E}[X^x_{t_0}] = \mathbb{E}[X^y_{t_0}] = \mathbb{E}[f(X^y_\epsilon)]. \tag{4.19}$$

□

Let h^* be as in (4.2) and denote its inverse by $(h^*)^{-1}(t,\cdot) : \big(\underline{y}(t), \overline{y}(t)\big) \to \mathbb{R}$ where

$$\underline{y}(t) := \lim_{x\downarrow -\infty} h^*(t,x) \in [-\infty,\infty), \quad \overline{y}(t) := \lim_{x\uparrow\infty} h^*(t,x) \in (-\infty,\infty], \quad t\in[0,T].$$

The martingale N defined in (4.2) has the Markovian dynamics

$$\begin{aligned} dN_t &= h_x^*(T-t, X_t^x)\sigma(X_t^x)dB_t \\ &= \tilde{\sigma}(T-t, N_t)dB_t, \end{aligned} \tag{4.20}$$

where the time-dependent volatility function $\tilde{\sigma}$ is defined as

(4.21)
$$\tilde{\sigma}(u,y) := h_x^*\big(u, (h^*)^{-1}(u,y)\big)\sigma\big((h^*)^{-1}(u,y)\big), \quad u\in[0,T], \quad y \in \big(\underline{y}(u), \overline{y}(u)\big).$$

The proof of the next uniqueness result uses Doob's maximal martingale inequality. Doob's inequality can fail for strict local martingales.[8]

Theorem 4.2. *In the setting of Theorem* 3.4, *suppose* $\mathbb{E}[|X_T^x|^p] < \infty$ *for all* $x\in\mathbb{R}$ *and fixed* $T\in(0,\infty)$ *and* $p>1$. *For continuous data* $H:\mathbb{R}\to\mathbb{R}$ *satisfying* (4.1), *there is at most one classical solution* $g\in\mathcal{C}^{1,2}$ *of*

$$\begin{cases} g(0,y) = H(y), \quad y\in\mathbb{R}, \\ g_t(u,y) = \frac{1}{2}\tilde{\sigma}^2(u,y)g_{yy}(u,y), \quad y\in\big(\underline{y}(u),\overline{y}(u)\big) \quad u\in(0,T], \end{cases} \tag{4.22}$$

satisfying $|g(u,y)| \le c_0(1+|y|^p)$ *for all* $u\in[0,T]$ *and* $y\in\big(\underline{y}(u),\overline{y}(u)\big)$ *for a constant* c_0 *(*c_0 *can vary with* g*).*

Proof. Let g be as in the statement. Itô's lemma and the PDE in (4.22) produce the local martingale dynamics

$$dg(T-t,N_t) = g_y(T-t,N_t)\tilde{\sigma}(T-t,N_t)dB_t, \quad t\in[0,T].$$

For $n\in\mathbb{N}$ with $(h^*)^{-1}(T,y)\in(-n,n)$, we define the passage times $T_n := \inf\{t>0 : |N_t|\ge n\}$. For $n\in\mathbb{N}$, the process $\big(g(T-t\wedge T_n, N_{t\wedge T_n})\big)_{t\in[0,T]}$ is a bounded martingale. We therefore have

$$g(T,y) = \mathbb{E}^{(h^*)^{-1}(T,y)}[g(T-T\wedge T_n, N_{T\wedge T_n})], \quad n\in\mathbb{N}, \quad y\in\big(\underline{y}(T),\overline{y}(T)\big). \tag{4.23}$$

The proof is concluded by justifying that we can pass $n\to\infty$ inside the expectation in (4.23) to produce the representation

$$g(T,y) = \mathbb{E}^{(h^*)^{-1}(T,y)}[H(N_T)], \quad y\in\big(\underline{y}(T),\overline{y}(T)\big). \tag{4.24}$$

The dominated convergence theorem can be used in (4.23) because

(4.25)
$$\begin{aligned} \mathbb{E}^{(h^*)^{-1}(T,y)}\big[\sup_{u\in[0,T]}|g(T-u,N_u)|\big] &\le c_0\big(1+\mathbb{E}^{(h^*)^{-1}(T,y)}\big[\sup_{u\in[0,T]}|N_u|^p\big]\big) \\ &\le c_0\left(1+\Big(\frac{p}{p-1}\Big)^p\mathbb{E}^{(h^*)^{-1}(T,y)}\big[|N_T|^p\big]\right) \\ &= c_0\left(1+\Big(\frac{p}{p-1}\Big)^p\mathbb{E}^{(h^*)^{-1}(T,y)}\big[|X_T|^p\big]\right). \end{aligned}$$

[8] For example, the solution Y of (2.27) has the properties $\mathbb{E}[Y_t^2]<\infty$ and $\mathbb{E}[\sup_{s\in[0,t]}Y_s^2]=\infty$ for $t>0$.

The second inequality in (4.25) applies Doob's maximal martingale inequality to the submartingale $|N_t|$. The last term in (4.25) is finite because $\mathbb{E}^{(h^*)^{-1}(T,y)}\big[|X_T|^p\big] = \mathbb{E}\big[|X_T^{(h^*)^{-1}(T,y)}|^p\big]$ and $\mathbb{E}[|X_T^x|^p] < \infty$ for all $x \in \mathbb{R}$ by assumption. □

When σ is Lipschitz, the martingale X^x has all moments. We leave it open to find conditions on σ ensuring the random variable X_T^x is p integrable when σ is only locally $\frac{1}{2}$-Hölder continuous.

Appendix A. Properties of λ-harmonic functions

This appendix proves properties of λ-harmonic functions which we are unable to find references for.

Proposition A.1. *Suppose Assumption* 2.1 *holds. Let $\lambda > 0$ and $u : \mathbb{R} \to (0, \infty)$ be a λ-harmonic function for the diffusion in* (1.1)*. Then, the following hold:*

(i) *u is strictly convex.*
(ii) *For $x \in \mathbb{R}$, $\lim_{t \downarrow 0} \mathbb{E}^x[e^{-\lambda t} u(X_t)] = u(x)$ and u is λ-excessive.*

Proof. (i) Let $x, y \in \mathbb{R}$ with $x < y$ and $a \in (0, 1)$, and recall the passage times defined in (2.2). The optional stopping theorem produces

$$\begin{aligned}
&u(ax + (1-a)y) \\
&= \mathbb{E}^{ax+(1-a)y}[e^{-\lambda T_x \wedge T_y} u(X_{T_x \wedge T_y})] \\
&< \mathbb{E}^{ax+(1-a)y}[u(X_{T_x \wedge T_y})] \\
&= u(x)\mathbb{P}^{ax+(1-a)y}(X_{T_x \wedge T_y} = x) + u(y)\mathbb{P}^{ax+(1-a)y}(X_{T_x \wedge T_y} = y).
\end{aligned} \tag{A.1}$$

The strict inequality in (A.1) follows from $\mathbb{P}^{ax+(1-a)y}(T_x \wedge T_y > 0) = 1$ and

$$\mathbb{E}^{ax+(1-a)y}[(1-e^{-\lambda T_x \wedge T_y})u(X_{T_x \wedge T_y})] \geq \min\{u(x), u(y)\}\mathbb{E}^{ax+(1-a)y}[1-e^{-\lambda T_x \wedge T_y}] > 0.$$

Because X in (1.1) is on the natural scale, we have $\mathbb{P}^{ax+(1-a)y}(X_{T_x \wedge T_y} = x) = a$ and u's strict convexity follows from (A.1).

(ii) Let $x \in \mathbb{R}$ be arbitrary and let $n \in \mathbb{N}$ be large enough so that $|x| < n$. For T_n defined in (2.2), we have $\mathbb{P}^x(0 < T_n < \infty) = 1$ and $\big(e^{-\lambda t} u(X_t)\big)_{t \geq 0}$ is a $\mathbb{P}^x$-supermartingale. We then have

$$\begin{aligned}
\lim_{t \downarrow 0} \mathbb{E}^x[e^{-\lambda t} u(X_t)] &= \lim_{t \downarrow 0} \mathbb{E}^x[e^{-\lambda t} u(X_t) 1_{t < T_n}] + \lim_{t \downarrow 0} \mathbb{E}^x[e^{-\lambda t} u(X_t) 1_{t \geq T_n}] \\
&= \mathbb{E}^x[\lim_{t \downarrow 0} u(X_t)] + \lim_{t \downarrow 0} \mathbb{E}^x[e^{-\lambda t} u(X_t) 1_{t \geq T_n}] \\
&= u(x) + \lim_{t \downarrow 0} \mathbb{E}^x[e^{-\lambda t} u(X_t) 1_{t \geq T_n}],
\end{aligned}$$

where the last equality is due to the dominated convergence theorem. The last limit is zero because the supermartingale property of $\big(e^{-\lambda t} u(X_t)\big)_{t \geq 0}$ gives

$$\begin{aligned}
\lim_{t \downarrow 0} \mathbb{E}^x[e^{-\lambda t} u(X_t) 1_{t \geq T_n}] &\leq \lim_{t \downarrow 0} \mathbb{E}^x[e^{-\lambda T_n} u(X_{T_n}) 1_{t \geq T_n}] \\
&= \mathbb{E}^x[e^{-\lambda T_n} u(X_{T_n}) \lim_{t \downarrow 0} 1_{t \geq T_n}] = 0.
\end{aligned} \tag{A.2}$$

In (A.2), the first equality uses the dominated convergence theorem and $\mathbb{E}^x[e^{-\lambda T_n} u(X_{T_n})] \leq u(x) < \infty$ and the second equality uses $\mathbb{P}^x(0 < T_n) = 1$.

□

Appendix B. On weak solutions

This appendix shows that the definition of a weak solution we use coincides with Sawyer's definition in [42].[9] Consider a finite interval $[a,b]\subset\mathbb{R}$ with $a<b$ and a function $f\in\mathcal{C}^2([a,b])$, where the existence and continuity of derivatives at the boundaries are only required to exist from the interior. If $h\in\mathcal{C}^{1,2}$ is a classical solution of (1.2), we integrate by parts twice to get

$$\begin{aligned}\int_a^b h_t(t,x)\frac{2}{\sigma^2(x)}f(x)dx &= \int_a^b h_{xx}(t,x)f(x)dx\\ &= \big\{f(x)h_x(t,x)-f'(x)h(t,x)\big\}\big|_a^b+\int_a^b h(t,x)f''(x)dx,\end{aligned}$$

for $t>0$.

Definition B.1 (Weak solution)**.** $h\in\mathcal{C}$ is a weak solution of (1.2) if (i) $h(0,x)=H(x)$ for all $x\in\mathbb{R}$, and (ii) h satisfies

$$\begin{aligned}\int_a^b \big(h(t_1,x)-h(t_0,x)\big)f(x)m(dx) &= \int_{t_0}^{t_1}dt\int_a^b h(t,x)f''(x)dx\\ &\quad+f'(a)\int_{t_0}^{t_1}h(t,a)dt-f'(b)\int_{t_0}^{t_1}h(t,b)dt,\end{aligned}\tag{B.1}$$

for all $0<t_0<t_1$, all $a<b$, and for all $f\in\mathcal{C}^2([a,b])$ with $f(a)=f(b)=0$.

Because h is continuous in Definition B.1, we can allow $t_0=0$ in (B.1) to produce an equivalent definition.

Lemma B.2. *Let $h\in\mathcal{C}$ satisfy $h(0,x)=H(x)$. We then have that h is a weak solution of* (1.2) *if and only if* (1.3) *holds for all $f\in\mathcal{C}_c^{1,2}$.*

Proof. To see that (B.1) is implied by (1.3), we consider a sequence of functions $(g_n)_{n\in\mathbb{N}}\subset\mathcal{C}^1([t_0,t_1])$ with $g_n(t_i)=g_n'(t_i)=0$ for all $n\in\mathbb{N}$ and $i\in\{0,1\}$ such that for any $u\in\mathcal{C}([t_0,t_1])$, we have

$$\lim_{n\uparrow\infty}\int_{t_0}^{t_1}u(t)g_n'(t)dt=u(t_0)-u(t_1).$$

In other words, $g_n'(t)dt$ converges weakly to the signed Dirac measure $\delta_{t_0}-\delta_{t_1}$. Similarly, we can find another sequence $(f_n)_{n\in\mathbb{N}}\subset\mathcal{C}^2([a,b])$ with $f_n(y)=f_n'(y)=f_n''(y)=0$ for all $n\in\mathbb{N}$ and $y\in\{a,b\}$ such that for any $u\in\mathcal{C}([a,b])$, we have

$$\lim_{n\uparrow\infty}\int_a^b f_n''(x)u(x)dx=\int_a^b f''(x)u(x)dx+f'(a)u(a)-f'(b)u(b).$$

Since $g_nf_n\in\mathcal{C}_c^{1,2}([0,\infty]\times\mathbb{R})$, continuity of h gives the claimed implication.

Conversely, we can reverse the previous argument to conclude that (B.1) implies (1.3) for any function $f=f(t,x)\in\mathcal{D}$, where

$$\mathcal{D}:=\Big\{\sum_{i=1}^j g_i(t)f_i(x): j\geq 1, g_i\in\mathcal{C}_c^1\big((0,\infty)\big)\text{ and } f_i\in\mathcal{C}_c^2(\mathbb{R})\Big\},$$

[9]Sawyer [42] considers locally bounded h (possibly discontinuous). However, to deal with our initial condition $h(0,x)=H(x)$, we restrict Sawyer's definition to $h\in\mathcal{C}$.

and where $\mathcal{C}_c^i$ is the class of i-times continuously differentiable functions with compact support. Since $\mathcal{D}$ is an algebra that separates points, we conclude by the theorem of Stone-Wierstrass and the continuity of h that (1.3) holds for all $f \in \mathcal{C}_c^{1,2}\big((0,\infty)\times\mathbb{R}\big)$. □

Lemma B.3. *Let $h \in \mathcal{C}$ satisfy $h(0,x) = H(x)$. We then have h is a weak solution of* (1.2) *if and only if*

$$\begin{aligned}\int_a^b u^{ab}(x,y)\big(h(t_1,y)-h(t_0,y)\big)m(dy) &= -\int_{t_0}^{t_1} h(t,x)dt \\ &+\frac{b-x}{b-a}\int_{t_0}^{t_1} h(t,a)dt + \frac{x-a}{b-a}\int_{t_0}^{t_1} h(t,b)dt,\end{aligned} \tag{B.2}$$

for all $a<b$, $x\in(a,b)$, and $0<t_0<t_1$. In (B.2), *u^{ab} is the symmetric potential kernel*

$$u^{ab}(x,y) := \frac{(x-a)(b-y)}{b-a}, \qquad x \le y, \quad (x,y)\in(a,b)^2.$$

Proof. The process X^{ab} denotes the solution of (1.1) killed at the first exit time from (a,b). The symmetric potential kernel associated with X^{ab} is $u^{ab}(x,y)$ in the sense that for any nonnegative Borel function g we have (see, e.g., Corollary VII.3.8 in [39])

$$\mathbb{E}^x\left[\int_0^{T_{ab}} g(X_t)dt\right] = U^{ab}g(x), \quad U^{ab}g(x) := \int_a^b u^{ab}(x,y)g(y)m(dy), \quad x\in[a,b].$$

For $g\in\mathcal{C}([a,b])$, we have $f := U^{ab}g \in \mathcal{C}^2([a,b])$ with $f(a)=f(b)=0$. Moreover, we also have

$$\frac{\sigma^2(x)f''(x)}{2} = -g(x), \quad x\in(a,b).$$

The converse also holds: Given $f\in\mathcal{C}^2([a,b])$ with $f(a)=f(b)=0$, there exits $g\in\mathcal{C}([a,b])$ such that $f=U^{ab}g$.[10]

Direct computations show that $f=U^{ab}g$ has the derivative

$$f'(x) = -\frac{1}{b-a}\int_a^x (y-a)g(y)m(dy) + \frac{1}{b-a}\int_x^b (b-y)g(y)m(dy).$$

For $g\in\mathcal{C}([a,b])$, we insert $f=U^{ab}g$ into (B.1) and use the symmetry of u^{ab} to get

$$\begin{aligned}\int_a^b \big(U^{ab}h(t_1,x)-U^{ab}h(t_0,x)\big)g(x)m(dx) &= -\int_{t_0}^{t_1}\int_a^b h(t,x)g(x)m(dx) \\ +\int_a^b \frac{b-y}{b-a}g(y)m(dy)\int_{t_0}^{t_1} h(t,a)dt &+ \int_a^b \frac{y-a}{b-a}g(y)m(dy)\int_{t_0}^{t_1} h(t,b)dt.\end{aligned}$$

Since $g\in\mathcal{C}([a,b])$ is arbitrary and m is absolutely continuous with respect to the Lebesgue measure, we deduce that a continuous function h satisfies (B.1) if and only if (B.2) holds for all $x\in(a,b)$. □

Theorem A2 and Remark 3 in [42] show that h is a weak solution of (1.2) if and only if $\big(h(T-t,X_t)\big)_{t\in[0,T]}$ is a local martingale for all $T>0$.

Theorem B.4 (Sawyer [42])**.** *A function $h\in\mathcal{C}$ is a weak solution of* (1.2) *if and only if $\big(h(T-t,X_t)\big)_{t\in[0,T]}$ is a $\mathbb{P}^x$-local martingale for all $T>0$ and $x\in\mathbb{R}$.*

[10]This can be shown by applying Ito's formula and noticing that $\mathbb{E}^x[f(X_{T_{ab}})]=0$.

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