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ORIGINAL RESEARCH

On characterisations of the input to state stability properties for conformable fractional order bilinear systems

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Abstract
This paper proposes for the first time the theoretical requirements that a fractional-order bilinear system with conformable derivative has to fulfil in order to satisfy different input-to-state stability (ISS) properties. Variants of ISS, namely ISS itself, integral ISS, exponential integral ISS, small-gain ISS, and strong integral ISS for the general class of conformable fractional-order bilinear systems are investigated providing a set of necessary and sufficient conditions for their existence and then compared. Finally, the correctness of the obtained theoretical results is verified by numerical example.

1 | INTRODUCTION

Over the past centuries, the fractional calculus, dealing with differential equations of fractional order, has attracted a major interest in various fields of science and engineering [4, 15, 21, 24, 25, 32, 36, 50]. During recent decades, many researchers from the control community have been actively exploring the possibility of extension/adaptation of various control methods from the integer setting into the fractional one. For example, combination of Lyapunov theory and the Mittag–Leffler function allowed to stabilise fractional-order nonlinear systems [32]. Several attempts have been also made to generalise the Barbalat-type lemmas in order to analyse the stability of time-varying fractional-order nonlinear systems where it is very difficult to find Lyapunov functions with negative definite derivative [13, 43]. Furthermore, in other research works, several nonlinear control methods have been extended into the fractional dynamic systems such as sliding mode control [34, 47], back-stepping control [38], fuzzy control [20], adaptive control [48].

In the available literature, there exist numerous definitions of non-integer order derivatives, among which the Riemann–Liouville (RL) and Caputo are the most frequently used ones defined via fractional integral [32]. Recently, a simple well-behaved fractional-order derivative so-called conformable derivative which is a natural extension of the usual derivative, was proposed in ref. [19]. Later, this definition was extended to higher orders and a set of interesting properties such as fractional Laplace transform, fractional exponential function, fractional power-series, and fractional chain rule were developed in ref. [1], and the related calculus of variations was also established in ref. [22].

There are weaknesses associated with RL and Caputo definitions that have convinced researchers from different areas, particularly system and control, to adopt conformable derivative as a possible alternative tool [19]. For example,

1) The derivative of a constant function is not zero with the RL derivative [19],
2) The fractional product rule and fractional chain rule are not satisfied either for RL or Caputo derivatives [19],
3) The monotonicity of a function cannot be determined from the sign of its fractional derivative [44].

In addition, since the fractional systems with nonlocal derivatives are infinite-dimensional and their future behavior depends on the entire past history, this gives rise to the problems of fractional systems initialisation [36] and, therefore, using the concept of pseudo-state space instead of state space in control systems [35]. Such limitations, indeed, confine the natural generalisation/adaptation of some important notions and methods from the control theory of integer systems into fractional setting. In other words, the introduction of fractional-order calculus to systems and control at the level of the traditional integer-order calculus is still not an easy task such derivative definitions [27]. For instance, the well-known exact feedback linearisation method [16] cannot be applied to fractional nonlinear systems defined with RL and Caputo derivatives, since the fractional chain rule is not satisfied by these operators [18]. Likewise, the important concepts of the geometric control cannot be naturally generalised into fractional setting defined with nonlocal derivatives as the system-theoretic interpretation of the fractional system through the associated pseudo-state trajectories is significantly different from the integer case due to the nonlocal effect of Caputo-like derivatives and the associated initialisation problems [31].

It is important to mention that the conformable derivative tool allows to overcome some technical problems in investigating fractional-order systems. During recent years, conformable derivative has become one of the most hot topics and several interesting results have been reported in the area of systems and control. For example, the exact linearisation of nonlinear fractional-order systems by state feedback has been addressed in ref. [18]. A version of Barbalat lemma which cannot be adopted by Caputo derivative was developed for conformable fractional-order systems [43]. In ref. [17] a state estimation framework for fractional-order systems under the conformable derivative was presented to address fault detection problem. The conformable derivative has been also interestingly employed to solve fractional infinite-horizon optimal control problems in stabilising and chaos control of fractional-order systems [50]. In ref. [27], a framework in terms of behavioral system theory has been presented to develop a general modelling specification as well as stability conditions for conformable linear systems with a fractional differential order, and the sufficient conditions and tests for stability were provided based on linear matrix inequalities. Furthermore, several well-behaved modelling and control methods have been newly developed under conformable derivative including conformable fractional-order neural sliding-mode control [34], conformable fractional optimal control [22], robust $\mathcal{H}_\infty$ control scheme for nonlinear conformable fractional-order systems [26, 28] and conformable fractional modeling and control of complex biological system [4, 15], to mention a few.

The notion of input-to-state stability, which reflects the robustness of a system against both initial states and external inputs, has been now recognised as a central concept in the theory of nonlinear control systems [42]. The ISS framework was originally introduced in ref. [39], merged the Lyapunov and input-output stability theories and has become the cornerstone of the stability theory for control systems [24, 28, 30, 41, 46]. Indeed, the fractional-order systems are an emerging field of control engineering for which the characterisation of ISS property and its related variants has to be carefully addressed. There are a few research contributions which are dedicated to extend the notion of ISS to the fractional-order field. In ref. [11], the finite-time input-to-state stable theory of fractional-order dynamical system was proposed. Then, based on this theory, a linear feedback controller is derived to achieve synchronisation of smooth chaotic fractional-order systems with indeterminate parameters and external stochastic noise in finite time. Two Lyapunov theorems for the input-to-state practical stability of fractional-order systems were presented in ref. [8] and several adaptive fault estimation and fault accommodation methods for fractional-order non-linear, switched, and interconnected systems under Caputo derivative are proposed. Following this work, the input-to-state practical stability of fractional-order systems was further extended in ref. [9] and the notion of Mittag–Leffler ISS Lyapunov function was proposed in order to design an event-triggered adaptive neural networks controller for fractional systems under Caputo derivative. Moreover, the Lyapunov characterisation of Mittag–Leffler input-to-state stability of the fractional differential equations with exogenous inputs was presented in ref. [37] and employed to investigate the Mittag–Leffler input-to-state stability of a particular class of fractional neutral networks. However, to our best knowledge, all of these works were merely limited to extension of ISS notion to fractional systems and no research study has been dedicated to adapt the other useful variants of ISS namely integral ISS [40], exponential integral ISS, small-gain ISS [10], and strong integral ISS [10] into fractional settings.

Motivated by the above discussions, the focus of this paper is on the development and characterisation of the theoretic requirements that conformable fractional-order nonlinear systems, particularly the conformable fractional bilinear systems, have to satisfy in order to fulfil various ISS properties. That is, several important variants of ISS including integral ISS, exponential integral ISS, small-gain ISS, and strong integral ISS, which have been widely accepted in integer-order systems, are extended into (conformable) fractional-order systems and a set of sufficient and necessary conditions for ISS-type stability analysis of conformable fractional nonlinear systems (in particular, conformable fractional bilinear systems) have been established for the first time.

The remainder of this paper is organised as follows. We start by providing some necessary definitions and useful lemmas in Section 2. The main results are given in Section 3. And finally, an example is presented in Section 4 to show the correctness of the achieved theoretical findings. Concluding remarks are brought up in Section 5. In order to maintain the readability of the presentation, some minor proofs are sketched in the appendices.

1.1 | Notation

We denote $\mathbb{R}_{\geq a} = \{x \in \mathbb{R} : x \geq a\}$, where $\mathbb{R}$ is the set of real numbers. A class $\mathcal{P}$ function is a function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which is continuous and satisfies $\eta(0) = 0$, $\eta(s) > 0$ for all $s \neq 0$. A class $\mathcal{K}$ function is a function $\eta \in \mathcal{P}$, which is strictly increasing. A class $\mathcal{K}_\infty$ function is a function $\eta \in \mathcal{K}$, which is also unbounded. A class $\mathcal{L}$ function is a function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which is continuous, strictly decreasing, and satisfies $\eta(s) \to 0$ as $s \to \infty$, and a class $\mathcal{KL}$ function is a continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\beta(\cdot, t - t_0) \in \mathcal{K}$ for each fixed $t \geq t_0$ and $\beta(s, \cdot) \in \mathcal{L}$, for each fixed $s \geq 0$. We denote $\eta_1(\eta_2(\cdot))$ as the composition of two class $\mathcal{K}$ (respectively, $\mathcal{K}_\infty$) functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$ which is also $\mathcal{K}$ (respectively, $\mathcal{K}_\infty$). Moreover, $\eta^{-1}(\cdot)$ denotes as the inverse of function $\eta(.)$. If $\eta(.)$ is a class $\mathcal{K}$ (respectively, $\mathcal{K}_\infty$) function, then $\eta^{-1}(\cdot)$ is also a class $\mathcal{K}$ (respectively, $\mathcal{K}_\infty$) function. The function $W(\cdot)$ is said to be proper (i.e. radially unbounded), if $W(\|s\|) \to \infty$ as $\|s\| \to \infty$. We denote $\mathcal{U}^m$ as the set of all measurable piecewise continuous functions $u(t)$ from $\mathbb{R}_{\geq t_0}$ to $\mathbb{R}^m$. If $z : \mathcal{Y} \to \mathbb{R}^q$ is a measurable function defined on $[t_0, t]$, then $\|z(t)_{[t_0,t]}\|_\infty$ denotes the (essential) supremum of $z(t)$ and has the following definition: let μ be the Lebesgue measure, thus, $ess\sup\{\|z(t)\| : t \in [t_0, t]\} := \inf\{a \in \mathbb{R} : \mu\{t \in [t_0, t] : z(t) > a\} = 0\}$. When $t = \infty$, we simplify the notation by $\|z\|_\infty$. For $x \in \mathbb{R}^n$, its transpose is denoted by x^T and the Euclidean norm of x is defined by $\|x\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For a given matrix $A \in \mathbb{C}^{n\times n}$, where $\mathbb{C}$ is the set of complex numbers, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of A, respectively. The spectrum of A is defined as $\mathrm{Spec}(A) := \{\lambda \mid \lambda$ is the eigenvalue of $A\}$. We also denote by $\mathrm{Arg}(z)$, the principal argument of $z \in \mathbb{C}$, so that $\mathrm{Arg}(z) \in (-\pi, \pi]$.

2 | PRELIMINARIES

The definition of the fractional conformable derivative and related useful definitions and lemmas are introduced in following section.

2.1 | Conformable fractional derivative and its properties

Definition 1. [1] The (left) conformable fractional derivative of order of $\alpha \in (0, 1]$ starting from t_0 of a function $f : \mathbb{R}_{\geq t_0} \to \mathbb{R}$ is defined by

$$T_{t_0}^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon (t - t_0)^{1-\alpha}\right) - f(t)}{\varepsilon}, \ \forall t > t_0. \quad (1)$$

If $T_{t_0}^{\alpha} f(t)$ exists $\forall t \in (t_0, t_1)$ for some $t_1 > t_0$ and $\lim_{t \to t_0^+} T_{t_0}^{\alpha} f(t)$ exists, then by definition

$$T_{t_0}^{\alpha} f(t_0) = \lim_{t \to t_0^+} T_{t_0}^{\alpha} f(t). \quad (2)$$

When $t_0 = 0$, we write T^α in order to simplify the notation.

It is worth noting that for α equal to 1, the conformable fractional derivative (1) is completely consistent with the classical integer derivative. Moreover, although the definition (1) can be extended for higher orders ($\alpha > 1$) [1], throughout this paper, we only consider $\alpha \in (0, 1)$.

Definition 2. [1] The conformable fractional integral of order $\alpha \in (0, 1]$ of a function $f : \mathbb{R}_{\geq t_0} \to \mathbb{R}$ starting from t_0 is defined by

$$I_{t_0}^{\alpha} f(t) = \int_{t_0}^{t} (\tau - t_0)^{\alpha-1} f(\tau) \mathrm{d}\tau. \quad (3)$$

Lemma 1. *[1] Assume that $f : \mathbb{R}_{\geq t_0} \to \mathbb{R}$ is a continuous function and $\alpha \in (0, 1]$. Then, for all $t > t_0$ we have*

$$T_{t_0}^{\alpha} I_{t_0}^{\alpha} f(t) = f(t). \quad (4)$$

Lemma 2. *[1] Let $f : \mathbb{R}_{\geq t_0} \to \mathbb{R}$ be differentiable function and $\alpha \in (0, 1]$. Then, for all $t > t_0$ we have*

$$I_{t_0}^{\alpha} T_{t_0}^{\alpha} f(t) = f(t) - f(t_0). \quad (5)$$

Lemma 3. *[43] Let $f : \mathbb{R}_{\geq t_0} \to \mathbb{R}$ be a continuous function such that $T_{t_0}^{\alpha} f(t)$ exists on (t_0, ∞), if $T_{t_0}^{\alpha} f(t) \geq 0$ (respectively $T_{t_0}^{\alpha} f(t) \leq 0$), for all $t > t_0$, then function f is increasing (respectively decreasing).*

Lemma 4. *[1] (Fractional chain rule). Assume $f, g : \mathbb{R}_{>t_0} \to \mathbb{R}$ be (left) $\alpha-$differentiable functions, where $\alpha \in (0, 1]$. Let $h(t) = f(g(t))$. Then, $h(t)$ is (left) $\alpha-$differentiable and for all t with $t \neq t_0$ and $g(t) \neq 0$ we have*

$$T_{t_0}^{\alpha} h(t) = T^{\alpha} f(g(t)) \cdot T_{t_0}^{\alpha} g(t) \cdot g(t)^{\alpha-1}. \quad (6)$$

If $t = t_0$, we have

$$T_{t_0}^{\alpha} h(t) = \lim_{t \to t_0^+} T^{\alpha} f(g(t)) \cdot T_{t_0}^{\alpha} g(t) \cdot g(t)^{\alpha-1}. \quad (7)$$

The following lemma establishes a multivariable chain rule in (conformable) fractional setting.

Lemma 5. *(Fractional multivariable chain rule) Assume $f : \mathbb{R}_{>t_0}^n \to \mathbb{R}$ and $g : \mathbb{R}_{>t_0} \to \mathbb{R}^n$ be (left) $\alpha-$differentiable and $g^{1-\alpha} \in \mathbb{R}^{n\times n}$ is a diagonal matrix defined as $g^{1-\alpha} = diag(g_1(t)^{1-\alpha}, g_2(t)^{1-\alpha}, \ldots, g_n(t)^{1-\alpha})$, where $\alpha \in (0, 1]$. Let $h(t) = f(g(t))$, then $h(t)$ is (left) $\alpha-$differentiable for all t with $t \neq t_0$ and $\det[g^{1-\alpha}] \neq 0$ we have*

$$T_{t_0}^{\alpha} h(t) = \nabla^{\alpha} f(g) \cdot \left(g^{1-\alpha}\right)^{-1} \cdot T_{t_0}^{\alpha} g(t), \quad (8)$$

and, if $t = t_0$ we have

$$T_{t_0}^{\alpha} h(t) = \lim_{t \to t_0^+} \nabla^{\alpha} f(g) \cdot \left(g^{1-\alpha}\right)^{-1} \cdot T_{t_0}^{\alpha} g(t), \quad (9)$$

where $\nabla^\alpha f(g)$ denotes $\alpha-$Gradient of f with respect to g and is defined as follows.

$$\nabla^{\alpha} f(g) = \frac{\partial^{\alpha} f(g)}{\partial g^{\alpha}} = \frac{\partial f(g)}{\partial g} \cdot g^{1-\alpha}. \quad (10)$$

Proof. The proof follows straightforward from Lemma 4.

The fractional fundamental exponential function plays a critical role in expressing the solution of (conformable) fractional-order systems and is defined as follows:

Definition 3. [1] The fractional exponential function is defined by

$$E_\alpha(c, t-t_0) = \exp\left(\frac{c}{\alpha}(t-t_0)^\alpha\right),\ \forall t \geq t_0 \tag{11}$$

where $\alpha \in (0,1]$ and $c \in \mathbb{R}$. If $c=1$, we simply write $E_\alpha(t-t_0)$ [1] .

Lemma 6. *[1] Consider the fractional conformable exponential function $E_\alpha(\omega, t-t_0)$. Then, it has the following convergent fractional power-series expansion*

$$E_\alpha(c, t-t_0) = \sum_{k=0}^{\infty} \frac{c^\alpha (t-t_0)^\alpha}{\alpha^k k!},\ \forall t \geq t_0 \tag{12}$$

where $\alpha \in (0,1]$ and $c \in \mathbb{R}$.

2.2 | Conformable fractional bilinear system

Consider the following conformable fractional−order nonlinear system with exogenous inputs:

$$\begin{cases} T_{t_0}^{\alpha} x(t) = f(x(t), u(t)),\ \ \forall t > t_0, \\ x(t_0) = x_0, \end{cases} \tag{13}$$

(seeking readability, hereafter, the arguments t will be usually omitted) where $T_{t_0}^{\alpha}$ is the conformable derivative of order $\alpha \in (0,1]$, $x \in \mathbb{R}^n$ and $u \in \mathcal{U}^m$ are the state and control input of the system. Moreover, $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is continuously differentiable function satisfying $f(0,0) = 0$. The general fractional-order conformable bilinearised/bilinear system is defined by the following equation

$$\begin{cases} T_{t_0}^{\alpha} x(t) = Ax(t) + \sum_{i=1}^{m} u_i(t) N_i x(t) + Bu(t),\ \ \forall t > t_0, \\ x(t_0) = x_0, \end{cases} \tag{14}$$

where $x \in \mathbb{R}^n$ and $u \in \mathcal{U}^m$ (u_i are the components of the vector u) are the state and control input of the system, and $A, N_i \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are the system constant matrices.

The conformable fractional bilinear systems (CFBS) given by Equation (14) so referred to as non-autonomous bilinear system inspired by ref. [12]. The simplest case of Equation (14) is a fractional bilinear system with the assumption that there is only one input and that therefore there also exist just one N matrix .The focus of this paper is to study various notions of input-to-state stability for fractional-order bilinear systems of the form (14). It is worth noting that the bilinear systems are one of the simplest class of nonlinear control systems that include a linear part as well as nonlinear perturbations characterising as multiplication of input and state due to, e.g. a linear state feedback or a multiplicative perturbation [12]. Nevertheless, they cover wider range of applications in engineering and science including chemical engineering and biology [28], particularly, in tumour-growth modelling [4] and cancer chemotherapy [12]. Furthermore, it is common to linearise the nonlinear systems over an operational steady-state. Such linearised models, however, are not capable of covering the complete range of operation and bilinearisation could be thus an alternative approach to obtain a better approximation of system nonlinearities [12].

Throughout the paper, it is assumed that given any input signal $u \in \mathcal{U}^m$ and $x_0 \in \mathbb{R}^n$, the systems (13) and (14) have a solution. Some sufficient conditions for the existence of solutions for fractional-order conformable differential equations are given in refs. [1, 5–7 14, 33 49].

3 | MAIN RESULTS

3.1 | Stability properties

In this subsection, we provide some definitions for development of input-to-state stability [39] and its variants including integral ISS [40], exponential integral ISS, small-gain ISS [10], and strong integral ISS [10] in conformable fractional setting with emphasis on conformable fractional bilinear systems.

Definition 4. The conformable fractional-order system (13) in the absence of inputs is said to be

i. Globally asymptotically stable (GAS), if there exists function $\beta \in \mathcal{KL}$ such that, for all $x_0 \in \mathbb{R}^n$, its solution satisfies

$$\|x(t)\| \leq \beta(\|x_0\|,\ t-t_0),\ \forall t \geq t_0. \tag{15}$$

ii. Fractional exponentially stable (FES), if there exists function $\eta \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$, its solution satisfies

$$\|x(t)\| \leq \eta(\|x_0\|) E_\alpha(\omega, t-t_0),\ \forall t \geq t_0. \tag{16}$$

for some constant $\omega < 0$.

Definition 5. The conformable fractional-order system (13) is called

i. *Input-to-State Stable (ISS)*, if there exist functions $\beta \in \mathcal{KL}$ (transient term) and $\eta \in \mathcal{K}_\infty$ (asymptotic term) such that, for all $x_0 \in \mathbb{R}^n$ and $u(t) \in \mathcal{U}^m$, its solution satisfies

$$\|x(t)\| \leq \beta(\|x_0\|, t-t_0) + \eta\left(\left\|u(t)_{[t_0,t]}\right\|_\infty\right),\ \forall t \geq t_0. \tag{17}$$

ii. *Integral input-to-state stable (iISS)*, if there exist functions $\beta \in \mathcal{KL}$ and $\eta_1, \eta_2 \in \mathcal{K}$ such that, for all $x_0 \in \mathbb{R}^n$ and $u(t) \in \mathcal{U}^m$, its solution satisfies

$$\|x(t)\| \leq \beta(\|x_0\|, t-t_0) + \eta_1\left(\int_{t_0}^{t} (\tau - t_0)^{\alpha-1} \eta_2(\|u(\tau)\|)\mathrm{d}\tau\right),\ \forall t \geq t_0. \tag{18}$$

iii. *Small-gain input-to-state stable (sg-ISS)*, if there exist an $R > 0$ (input threshold), $\beta \in \mathcal{KL}$, $\eta \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{U}^m$, its solution satisfies

$$\left\| u(t)_{[t_0,t]} \right\|_\infty \leq R \quad \Rightarrow \quad \|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \eta\left(\left\| u(t)_{[t_0,t]} \right\|_\infty \right), \; \forall t \geq t_0. \tag{19}$$

iv. *strong integral input-to-state stable (strong-iISS)*, if it is *sg-ISS* and *strong-iISS*.

Remark 1. By the virtue of the ISS conceptual framework (17), we have the following consequences: if the input vanishes, all solutions of Equation (13) converge to zero (*convergent-input convergent-state stable* (CICS) property [39]). If all inputs are bounded, all of the solutions of Equation (13) are also bounded (*bounded-input bounded-state stable* (BIBS) property [39]). Besides, if system is ISS, then the origin of unperturbed dynamics is said to be GAS defined by Equation (15).

Lemma 7. *Consider the conformable fractional-order system (13). If the system is strong-iISS, then it is also CICS and BIBS.*

Proof. This Lemma is a consequence of Definition 5 (iv). Assume that system (13) is strongly iISS. Thus, it implies that Equations (18) and (19) are fulfilled. Clearly, Equation (18) suggests that for any $u(t) \in \mathcal{U}^m$ there exist solutions to Equation (13). The CICS property is also guaranteed by Equation (18) as a consequence of strong-iISS property. Moreover, we can see from Equation (19) that, the BIBS property is guaranteed for $\{u \in \mathcal{U}^m : \|u_{[t_0,t]}\|_\infty \leq R\}$ in that all the trajectories approach ball of radius related to the magnitude of input. □

Definition 6. The conformable fractional-order system (13) is called

i. *Fractional exponential input-to-state stable (fe-ISS)*, if there exist functions $\eta_1, \eta \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{U}^m$, its solution satisfies

$$\|x(t)\| \leq \eta(\|x_0\|) E_\alpha(\omega, t - t_0) + \eta_1\left(\left\| u(t)_{[t_0,t]} \right\|_\infty \right), \; \forall t \geq t_0. \tag{20}$$

ii. *Fractional exponential integral input-to-state stable (fe-iISS)*, if there exist functions $\eta_1, \eta \in \mathcal{K}_\infty$ and $\eta_2 \in \mathcal{K}$ such that, for all $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{U}^m$, its solution satisfies

$$\eta_1(\|x(t)\|) \leq \eta(\|x_0\|) E_\alpha(\omega, t - t_0) + \int_{t_0}^{t} (\tau - t_0)^{\alpha - 1} \eta_2(\|u(\tau)\|) \mathrm{d}\tau, \; \forall t \geq t_0, \tag{21}$$

for some constant $\omega < 0$.

It is important to mention that, the terms $\eta(\|x_0\|)E_\alpha(\omega, t - t_0)$ and $\eta_1^{-1}(\eta(\|x_0\|)E_\alpha(\omega, t - t_0))$ in Equations (20) and (21) fulfil the requirements of a $\mathcal{KL}$-function. However, Definition 6 establishes stronger ISS and iISS notions compared to Definition 5 (i) and (ii) for system (13), in that, the convergence of solutions in Equation (13) is necessarily bounded by fractional exponential decay.

In many circumstances, it is not straightforward to verify that a fractional-order system (13) satisfies the requirements of the ISS concepts given by the aforementioned conditions as it may be difficult to obtain an explicit solution of the system. Alternatively, we provide Lyapunov-type characterisation of ISS-type properties for conformable fractional systems (13) which is indeed an extension of the Lyapunov stability from the usual sense [16].

Definition 7. A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ that has conformable fractional derivative of order α for all $t > t_0$ is called an ISS–Lyapunov function for system (13), if there exist $\underline{\eta}_1, \overline{\eta}_1 \in \mathcal{K}_\infty$ and $\chi, \eta_1 \in \mathcal{K}$ such that

$$\underline{\eta}_1(\|x\|) \leq V(x) \leq \overline{\eta}_1(\|x\|), \tag{22}$$

$$\|x\| \geq \chi(\|u\|) \; \Rightarrow \; \nabla^\alpha V(x)\left(x^{1-\alpha}\right)^{-1} f(x, u) \leq -\eta_1(\|x\|), \tag{23}$$

for all $t > t_0$, $x \in \mathbb{R}^n$, and $u \in \mathcal{U}^m$.

Remark 2. It is noted that, the estimate Equation (22) quantifies the requirements that V has to be a class $\mathcal{P}$-function and proper.

Definition 8. A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ that has conformable fractional derivative of order α for all $t > t_0$ is called an iISS–Lyapunov function for system (13), if there exist $\underline{\eta}_1, \overline{\eta}_1 \in \mathcal{K}_\infty$ such that

$$\underline{\eta}_1(\|x\|) \leq V(x) \leq \overline{\eta}_1(\|x\|), \tag{24}$$

holds, and there exists $\eta_1 \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ so that following dissipation inequality holds

$$\nabla^\alpha V(x)\left(x^{1-\alpha}\right)^{-1} f(x, u) \leq -\eta_1(\|x\|) + \sigma(\|u\|), \tag{25}$$

for all $t > t_0$, $x \in \mathbb{R}^n$, and $u \in \mathcal{U}^m$.

The following lemma describes the notion of ISS–Lyapunov function for fractional-order system (13) in the dissipation-like characterisation which will be shown that, it is equivalent to concept of ISS–Lyapunov function represented by implication in Definition 7.

Lemma 8. *A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ that has conformable fractional derivative of order α for all $t > t_0$ is called an ISS-Lyapunov function for system (13) if and only if there exist $\underline{\eta}_1, \overline{\eta}_1, \eta_1 \in \mathcal{K}_\infty$ such that*

$$\underline{\eta}_1(\|x\|) \le V(x) \le \overline{\eta}_1(\|x\|), \tag{26}$$

holds, and there exists also $\sigma \in \mathcal{K}$ so that the following dissipation inequality holds

$$\nabla^\alpha V(x)\left(x^{1-\alpha}\right)^{-1} f(x,u) \le -\eta_1(\|x\|) + \sigma(\|u\|), \tag{27}$$

for all $t > t_0$, $x \in \mathbb{R}^n$, and $u \in \mathcal{U}^m$.

Proof. See Appendix A. □

Remark 3. There exists a slight difference between Lemma 8 and the dissipation characterisation for iISS given in Definition 8 as a result of unlike definitions given for $\eta_1(.)$ where in Lemma (8) it is required to be unbounded.

3.2 | Supportive lemmas

In this subsection, we established the following important lemmas needed during the later ISS-type stability proofs of conformable fractional order bilinear systems.

We start with the following lemma where we develop a novel useful fractional comparison lemma [23] by which the asymptotic convergence of the trajectories of conformable fractional-order nonlinear system (13) can be established.

Lemma 9. *Let $\alpha \in (0,1]$. Given any function $\varphi(\cdot) \in \mathcal{P}$, there exists a function $\beta_\varphi(\cdot) \in \mathcal{KL}$ with the following property: Assume that $y(\cdot)$ be a piecewise continuously α-differentiable function for all $t > t_0$ with $y(t) \ge 0$, and also $y(\cdot)$ fulfils the following conformable fractional-order differential inequality*

$$T_{t_0}^\alpha y(t) \le -\varphi(y(t)), \;\; y(t_0) = y_0 \ge 0, \quad \forall t > t_0, \tag{28}$$

then, the following estimate holds

$$y(t) \le \beta_\varphi(y_0, t), \; \forall t > t_0. \tag{29}$$

Proof. See Appendix B. □

By the next lemma, a fractional version of Gronwall–Bellman-type inequality [3] is developed for conformable fractional integrals which will be useful in subsequent stability proofs.

Lemma 10. *(*Fractional Gronwall–Bellman-type Lemma*) Let $\alpha \in (0,1]$, $r(t)$ and $h(t)$ be continuous, nonnegative functions defined for $a \le t \le b$, and $c(t)$ be continuous, positive and nondecreasing function defined for $a \le t \le b$ such that*

$$r(t) \le c(t) + \int_a^t (\tau - a)^{\alpha-1} h(\tau)\, r(\tau) d\tau, \quad a \le t \le b. \tag{30}$$

Then, for all for $a \le t \le b$ we have

$$r(t) \le c(t) \exp\left(\int_a^t (\tau - a)^{\alpha-1} h(\tau) d\tau\right). \tag{31}$$

Proof. See Appendix C. □

In the following lemma, useful conditions for bounding the norm of fractional exponential function are established which will be used for later stability proofs.

Lemma 11. *Let $\alpha \in (0,1]$ and $A \in \mathbb{C}^{n\times n}$. For any $\omega > \max\{Re(\lambda) : \lambda \in Spec(A)\}$,*

- ($\mathcal{B}$1) *there exists a constant $\psi \ge 1$ such that $\|E_\alpha(A, t - t_0)\| \le \psi E_\alpha(\omega, t - t_0)$ for all $t \ge t_0$, and*
- ($\mathcal{B}$2) *if the matrix A such that $|Spec(A)| \ne 0$, $|Arg(Spec(A))| > \lceil\alpha\rceil\frac{\pi}{2}$ does not hold , then $\|E_\alpha(A, t - t_0)\|$ does not converge to zero as $t \to +\infty$.*

Proof. See Appendix D. □

In the following, we establish a matrix lemma in conformable fractional setting that will be employed in the Lyapunov-based stability proofs in next subsection.

Lemma 12. *Let $\alpha \in (0,1]$ and $A \in \mathbb{R}^{n\times n}$. If matrix A such that $|Spec(A)| \ne 0$, $|Arg(Spec(A))| > \lceil\alpha\rceil\frac{\pi}{2}$, then for any $Q \in \mathbb{R}^{n\times n}$ there exists exactly one solution $P \in \mathbb{R}^{n\times n}$ of equation $A^T P + PA = Q$, and if $Q < 0$ then $P > 0$.*

Proof. See Appendix E. □

3.3 | ISS-type stability of conformable fractional bilinear systems

In what follows, we provide some sufficient and necessary conditions for ISS, iISS, sg-ISS, strong-iISS, fe-ISS, and fe-iISS properties in regards to systems (13) with the emphasis on system (14).

We start by showing that the conformable fractional bilinear systems are not, in general, ISS through a counter example.

Remark 4. (*CFBS are neither, in general, ISS nor fe-ISS*). Consider one-dimensional CFBS given by $T_{t_0}^\alpha x = Ax + uNx$ with $|\text{Spec}(A)| \ne 0$, $|\text{Arg}(\text{Spec}(A))| > \lceil\alpha\rceil\frac{\pi}{2}$ for any $\alpha \in (0,1]$. In order to show that the CFBS are not ISS, we give the following counter example: let choose $A = -1$, N=1, and a constant input $u = u_\infty \equiv 2$, then, we have $T_{t_0}^\alpha x = (A + u_\infty N)x$ whose unique solution is given by $x(t) = x_0 E_\alpha(A + u_\infty N, t - t_0)$ [1]. Since $|\text{Arg}(\text{Spec}(A + u_\infty N))| < \lceil\alpha\rceil\frac{\pi}{2}$ for any $0 < \alpha \leq 1$, it thus gives unbounded trajectories for $x_0 = 1$ by using Lemma11. Therefore, we deduce that CFBS given by Equation (14) is not, in general, ISS and not fe-ISS either.

However, we are interested to show that whether the comfortable fractional bilinear system is iISS or not. In the sequel, we will establish sufficient and necessary conditions to show that all conformable fractional-order bilinear systems of the form Equation (14) are iISS.

Proposition 1. *(iISStability of CFBS) Let $\alpha \in (0, 1]$. The fractional-order bilinear system (14) is iISS if and only if the matrix A such that $|Spec(A)| \neq 0$, $|Arg(Spec(A))| > \lceil\alpha\rceil\frac{\pi}{2}$ for every $t > t_0$. In this case, the property (18) is fulfilled by*

$$\beta(s, t - t_0) = \eta_1'(E_\alpha(\omega, t - t_0)\psi s),\ \eta_1(s) = \eta_2'(s),\ \eta_2(s) = K\psi s,$$

where η_1', η_2' are $\mathcal{K}_\infty$ functions defined as $\eta_1'(r) = \frac{1}{2}r^2 + r$, $\eta_2'(r) = \frac{1}{2}(\exp(r) - 1)^2 + b\, r \exp(r)$ for all $b > 0$, and $\psi \geq 1$, $\omega \in \mathcal{R}$ such that $\|E_\alpha(A, t - t_0)\| \leq \psi E_\alpha(\omega, t - t_0)$ for all $t \geq t_0$, and $K > 0$ such that $\|\sum_{i=1}^{m} N_i x(\tau) u_i(\tau)\| \leq K\|u(t)\|\|x(t)\|$.

Proof. See Appendix F. □

The following theorem establishes the relationship between the ISS–Lyapunov function and the ISS property of Equation (13) in fractional setting.

Theorem 1. *The conformable fractional-order system (13) is said to be ISS, if it admits an ISS–Lyapunov function.*

Proof. The proof is inspired by ref. [39]. Let $\underline{\eta}_1, \overline{\eta}_1, \eta_1, \sigma$ be as in Definition 7 and fix a point $x_0 \in \mathbb{R}^n$ and non-zero input function $u \in \mathcal{U}^m$ such that $M = \overline{\eta}_1^{-1}(\chi(\|u(t)\|))$. By the inequality on the right-hand side of Equation (22), we introduce the set $\mathcal{S}_M := \{x \in \mathbb{R}^n : V(x) \leq M\}$.

Claim 1. If there exists $t_0 \geq 0$ such that $x(t_0) \in \mathcal{S}_M$, then $x(t) \in \mathcal{S}_M$ for all $t > t_0$.

Proof of Claim 1. By contradiction, there exist a sufficiently small $\varepsilon > 0$ and some $t \geq t_0$ such that $V(x) > M + \varepsilon$. Let $\tau := \inf\{t \geq t_0 : V(x(t)) \geq M + \varepsilon,$ for all fixed $\varepsilon > 0\}$. It then follows that $\|x(\tau)\| \geq \chi\|u(t)\|$ from which the right-hand side of Equation (23) holds for every t near τ

$$T_{t_0}^\alpha V(x) = \nabla^\alpha V(x)\left(x^{1-\alpha}\right)^{-1} f(x, u) \leq -\eta_1(\|x(t)\|) < 0.$$

Using Lemma 3, we have $V(x(t)) \geq V(x(\tau))$ for some $t \in (t_0, \tau)$. This contradicts the minimality of τ, thus $x(t)$ lies in $\mathcal{S}_M$ for all $t \geq t_0$ as claimed. □

Claim 2. For any x_0 and any bounded input $u \in \mathcal{U}^m$, there exists a time $t_1 > 0$ for state trajectory $x(t)$ such that

($\mathcal{J}1$) the state trajectory of the system lies in $\mathcal{S}_M$ for all $t \geq t_1$, and

($\mathcal{J}2$) $\|x(t)\| \leq \beta(\|x_0\|, t - t_0)$ for all $t < t_1$.

Proof of Claim 2. Let $t_1 = \inf\{t \geq 0 : x(t) \in \mathcal{S}_M\} \leq \infty$. Then, the previous argument shows that $V(x) \leq M$ for all $t \geq t_1$ This in turn implies that $\|x(t)\| \leq \underline{\eta}_1^{-1}(V(x)) \leq \underline{\eta}_1^{-1}(M) = \underline{\eta}_1^{-1}(\overline{\eta}_1(\chi(\|u\|)))$ for all $t \geq t_1$. Setting $\eta(r) = \underline{\eta}_1^{-1}(\overline{\eta}_1(\chi(r)))$ which is a of class $\mathcal{K}_\infty$, we see that, $x(t)$ satisfies

$$\|x(t)\| \leq \eta(\|u\|), \tag{32}$$

for all $t \geq t_1$, as claimed in ($\mathcal{J}1$).

We observe that for all $t < t_1$, $x(t)$ does not belong to $\mathcal{S}_M$ by which it follows that $V(x) > M$, in turn, $\|x(t)\| > \chi\|u(t)\|$. Consequently, by Equation (23)

$$T_{t_0}^\alpha V(x) = \nabla^\alpha V(x)\left(x^{1-\alpha}\right)^{-1} f(x, u) \leq -\eta_1(\|x\|)$$
$$\leq -\eta_1\left(\overline{\eta}_1^{-1}(V(x))\right) < 0.$$

This inequality ensures that the solutions are in fact defined for all $t > t_0 \geq 0$. Further, the function $V(t) = V(x(t))$ is such that

$$T_{t_0}^\alpha V(t) \leq -\eta_1\left(\overline{\eta}_1^{-1}(V(t))\right) \overset{\text{def}}{=} -\tilde{\varphi}(V(t)),$$

for all $t > t_0$. Now, we have the conformable fractional differential inequality of the form Equation (28). By Lemma 9, we observe that, there exists some $\beta_{\tilde{\varphi}} \in \mathcal{KL}$ which merely depends on $\eta_1, \overline{\eta}_1$ such that $V(x(t)) \leq \beta_{\tilde{\varphi}}(V(x(t_0)), t - t_0)$ for all $t \leq t_1$. It then follows from Equation (23) that $\|x(t)\| \leq \underline{\eta}_1^{-1}(\beta_{\tilde{\varphi}}(\overline{\eta}_1(r), t - t_0))$, in which $\underline{\eta}_1^{-1}(\beta_{\tilde{\varphi}}(\overline{\eta}_1(r), t - t_0))$ is again a $\mathcal{KL}$-function. Setting $\beta(\|x_0\|, t - t_0) = \underline{\eta}_1^{-1}(\beta_{\tilde{\varphi}}(\overline{\eta}_1(r), t - t_0))$, we can conclude that

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0), \tag{33}$$

for all $t \in (t_0, t_1)$ as claimed in Equation ($\mathcal{J}2$). □

we now complete the proof of Theorem1 by combining the solutions (32) and (33) by which we obtain the property (17) for all $t > t_0$. Therefore, the system (13) is ISS and this concludes the proof. □

Remark 5. A similar Lyapunov characterisation can be also formulated for sg-ISS. However, there has to exist an input threshold $R > 0$ such that Equation (27) holds for $\{u \in \mathcal{U}^m : \|u\| \leq R\}$.

The following Lyapunov-type theorem formulated in an implication form establishes an stricter condition for ISS property of Equation (13) compared to that of Theorem 1.

Theorem 2. *Consider the fractional-order system (13), and assume that there exist a continuous function* $V:\mathbb{R}^n\to\mathbb{R}$ *that has conformable fractional derivative of order* α *for all* $t>t_0$*, two constants* $q_1,q_2>0$*, and* $\underline{\eta}_1,\overline{\eta}_1\in\mathcal{K}_\infty$ *and* $\chi\in\mathcal{K}$ *functions so that*

($\mathcal{H}1$) $\underline{\eta}_1(\|x\|)=q_1.\|x\|^2\le V(x)\le\overline{\eta}_1(\|x\|)$,

($\mathcal{H}2$) $\|x\|\ge\chi(\|u\|)\ \Rightarrow\ \nabla^\alpha V(x)(x^{1-\alpha})^{-1}f(x,u)\le -q_2V(x)$,

for all $t>t_0$, $x\in\mathbb{R}^n$*, and* $u\in\mathcal{U}^m$*. Then the system is fe-ISS.*

Proof. The proof will be done in the similar way as in Theorem 1.

Let $M=\overline{\eta}_1(\chi(\|u(t)\|))$. Define the set $\mathcal{S}_M:=\{x\in\mathbb{R}^n: V(x)\le M\}$. We make the following two claims where the proof of the first one is analogous to that of Theorem 1 and, therefore, we do not sketch it here.

Claim 1. If there exists $t_0\ge 0$ such that $x(t_0)\in\mathcal{S}_M$, then $x(t)\in\mathcal{S}_M$ for all $t>t_0$.

Claim 2. For any x_0 and any bounded input $u\in\mathcal{U}^m$, there exists a time $t_1>0$ for state trajectory $x(t)$ such that

($\mathcal{A}1$) the state trajectory of the system lies in $\mathcal{S}_M$ for all $t\ge t_1$ and

($\mathcal{A}2$) $\|x(t)\|\le\eta(\|x_0\|)E_\alpha(\omega,t-t_0)$ for all $t<t_1$.

Proof of Claim 2. The proof of Equation ($\mathcal{A}1$) is similar to that of Theorem1. Thus, we skip it here.

As a result of Equation ($\mathcal{A}1$), we observe that $x(t)$ satisfies

$$\|x(t)\|\le\eta(\|u\|),\tag{34}$$

for all $t\ge t_1$. Now, we sketch the proof of part Equation ($\mathcal{A}2$). Note that for all $t<t_1$, $x(t)$ does not belong to $\mathcal{S}_M$ by which it follows that $V(x)>M$, in turn, $\|x(t)\|>\chi\|u(t)\|$. Thus, by using Equation ($\mathcal{H}2$), we have

$$T_{t_0}^{\alpha}V(x(t))=\nabla^\alpha V(x)\left(x^{1-\alpha}\right)^{-1}f(x,u)\le -q_2V(x).$$

Defining $V(t):=V(x(t))$, we have $T_{t_0}^{\alpha}V(t)\le -q_3V(t):=\varphi(V(t))$. From the comparison Lemma 9 applied to the this inequality, we observe that, there exists some $\beta_\varphi\in\mathcal{KL}$ such that $V(t)\le\beta_\varphi(V(t_0),t-t_0)$ for all $t\le t_1$. By setting $\beta_\varphi(V(t_0),t-t_0)=V(t_0)E_\alpha(-q_2,t-t_0)$, we have

$$V(t)\le V(t_0)E_\alpha\left(-q_2,t-t_0\right),\quad \forall t\le t_1,$$

Now, by assumption ($\mathcal{H}1$), we obtain

$$\begin{aligned}\|x\|&\le\left(\frac{1}{q_1}\right)^{\frac{1}{2}}V(t_0)\left\{E_\alpha\left(-q_2,t-t_0\right)\right\}^{\frac{1}{2}}\\&=\eta(\|x_0\|)E_\alpha(\omega,t-t_0),\end{aligned}\tag{35}$$

where $\omega=-\frac{q_2}{2}<0$ for all $t\in(t_0,t_1)$, as claimed in ($\mathcal{A}2$). □

Combining the solutions (34) and (35) gives inequality (20) for all $t>t_0$. Thus, system (13) is fe-ISS. □

It has been shown that the CFBS (14) are iISS, but not ISS (see Remark 4). Now we want to show whether CFBS are sg-ISS or not. To prove this, the following novel theorem in a dissipation-like formulation, inspired by ref. [40], is established in fractional setting. Using this, we will show that all CFBS are fe-iISS as in Equation (21) and, then, we will extend the proof to show that they are also ISS and sg-ISS.

Theorem 3. *Consider the fractional-order system (13), and assume that there exist a continuous function* $V:\mathbb{R}^n\to\mathbb{R}$ *that has conformable fractional derivative of order* α *for all* $t>t_0$*, a constant* $q>0$*, and* $\eta_1,\underline{\eta}_1,\overline{\eta}_1,\sigma\in\mathcal{K}_\infty$ *so that*

($\mathcal{C}1$) $\underline{\eta}_1(\|x\|)\le V(x)\le\overline{\eta}_1(\|x\|)$,

($\mathcal{C}2$) $\nabla^\alpha V(x)(x^{1-\alpha})^{-1}(f(x)+\sum_{i=1}^{m}f_i(x)u_i)\le -(q-\eta_1(\|u\|))V(x)+\sigma(\|u\|)$,

for all $t>t_0$, $x\in\mathbb{R}^n$*, and* $u\in\mathcal{U}^m$*. Then the system is fe-iISS. In this case, the property (21) is satisfies by* $\eta_1(s)=\tilde{\eta}_2^{-1}(\frac{1}{2}\underline{\eta}_1,(s))\in\mathcal{K}_\infty$*,* $\eta_2(s)=\overline{\eta}_1(s)\in\mathcal{K}_\infty$*,* $\eta_3(s)=\Theta(s)\in\mathcal{K}_\infty$*.*

Proof. Assume that there exist a $\mathcal{K}_\infty$ function $\Theta(.)$ be such that $\Theta(s)\ge\eta_1(s)$ and $\Theta(s)\ge\sigma(s)$ for all $s\ge 0$. Now, Suppose that Equation ($\mathcal{C}2$) holds, then inspiring by the approach in ref. [2] for solving the inhomogeneous integer-order systems, we get an estimation of $V(x)$ as follows:

$$\begin{aligned}&\nabla^\alpha V(x(t))\left(x(t)^{1-\alpha}\right)^{-1}f(x,u)\\&\quad\le -\left(q-\eta_1(\|u(t)\|)\right)V\left(x(t)\right)+\sigma(\|u(t)\|),\\&T_{t_0}^{\alpha}V(x(t))\\&\quad\le\left(\Theta(\|u(t)\|)-q\right)V(x(t))+\Theta(\|u(t)\|),\end{aligned}$$

Multiplying both sides by $E_\alpha(q,t-t_0)$, we get

$$\begin{aligned}&E_\alpha\left(q,t-t_0\right)T_{t_0}^{\alpha}V(x(t))\\&\quad\le E_\alpha\left(q,t-t_0\right)(\Theta(\|u(t)\|)-q)V(x(t))\\&\quad+E_\alpha\left(q,t-t_0\right)\Theta(\|u(t)\|),\end{aligned}$$

$$\begin{aligned}&E_\alpha\left(q,t-t_0\right)T_{t_0}^{\alpha}V(x(t))+qV(x(t))E_\alpha\left(q,t-t_0\right)\\&\quad\le E_\alpha\left(q,t-t_0\right)(\Theta(\|u(t)\|)V(x(t)))\\&\quad+E_\alpha\left(q,t-t_0\right)\Theta(\|u(t)\|),\end{aligned}$$

By fractional product rule [19], we have

$$\begin{aligned}T_{t_0}^{\alpha}\left(V(x(t))E_\alpha\left(q,t-t_0\right)\right)&\le E_\alpha\left(q,t-t_0\right)(\Theta(\|u(t)\|)V(x(t)))\\&+E_\alpha\left(q,t-t_0\right)\Theta(\|u(t)\|),\end{aligned}$$

Applying conformable fractional integral on both sides and Lemma 2, we have

$$E_\alpha\big(q,t-t_0\big)V(x(t)) - V(x_0)$$
$$\leq \int_{t_0}^{t} (\tau-a)^{\alpha-1}E_\alpha\big(q,\tau-t_0\big)\Theta\big(\|u(\tau)\|\big)(V(x(\tau))+1)\mathrm{d}\tau,$$
$$E_\alpha\big(q,t-t_0\big)V(x(t))$$
$$\leq V(x_0)+\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}E_\alpha\big(q,\tau-t_0\big)$$
$$\Theta\big(\|u(\tau)\|\big)(V\big(x(\tau)\big)+1)\mathrm{d}\tau,$$

and, we can write

$$E_\alpha\big(q,t-t_0\big)V(x(t)) \leq V(x_0)$$
$$+\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\Theta\big(\|u(\tau)\|\big)E_\alpha\big(q,t-t_0\big)V(x(t))\mathrm{d}\tau$$
$$+\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\Theta\big(\|u(\tau)\|\big)E_\alpha\big(q,\tau-t_0\big)\mathrm{d}\tau.$$

It follows from fractional Gronwall–Bellman-type Lemma (see Lemma 10) that,

$$E_\alpha\big(q,t-t_0\big)V(x(t))$$
$$\leq \left(V(x_0)+\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\Theta(\|u(\tau)\|)E_\alpha\big(q,\tau-t_0\big)\mathrm{d}\tau\right)$$
$$\times\exp\left(\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\Theta(\|u(\tau)\|)\mathrm{d}\tau\right),$$

$$V(x(t)) \leq \big(E_\alpha\big(-q,t-t_0\big)V(x_0)\big)$$
$$\times\exp\left(\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\Theta(\|u(\tau)\|)\mathrm{d}\tau\right)$$
$$+\left(\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\Theta(\|u(\tau)\|)E_\alpha\big(-q,t-t_0\big)E_\alpha\big(q,\tau-t_0\big)\mathrm{d}\tau\right)$$
$$\times\exp\left(\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\Theta(\|u(\tau)\|)\mathrm{d}\tau\right).$$

It is obvious from Definition 3 that, $E_\alpha(-q,t-t_0)E_\alpha(q,\tau-t_0) \leq 1$ for all $\tau \in [t_0,t]$. So, we get

$$V(x(t)) \leq E_\alpha\big(-q,t-t_0\big)V(x_0)$$
$$\times\exp\left(\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\Theta(\|u(\tau)\|)\mathrm{d}\tau\right)$$
$$+\left(\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\Theta(\|u(\tau)\|)\mathrm{d}\tau\right)$$
$$\times\exp\left(\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\Theta(\|u(\tau)\|)\mathrm{d}\tau\right),$$

For the sake of readability, we also define $\xi(t)=\int_{t_0}^{t}(\tau-t_0)^{\alpha-1}\Theta(\|u(\tau)\|)\mathrm{d}\tau$. Thus,

$$V(x(t)) \leq E_\alpha\big(-q,t-t_0\big)V(x_0)\exp\big(\xi(t)\big)+\xi(t)\exp\big(\xi(t)\big),$$

Moreover, by noting the following inequality

$$E_\alpha\big(-q,t-t_0\big)V(x_0)\exp\big(\xi(t)\big) = E_\alpha\big(-q,t-t_0\big)V(x_0)$$
$$+E_\alpha\big(-q,t-t_0\big)V(x_0)\big(\exp\big(\xi(t)\big)-1\big)$$
$$\leq E_\alpha\big(-q,t-t_0\big)V(x_0)+\frac{1}{2}E_\alpha\big(-2q,t-t_0\big)V(x_0)^2$$
$$+\frac{1}{2}\big(\exp\big(\xi(t)\big)-1\big)^2.$$

We can obtain

$$V(x(t)) \leq E_\alpha\big(-q,t-t_0\big)V(x_0)+\frac{1}{2}E_\alpha\big(-2q,t-t_0\big)V(x_0)^2$$
$$+\frac{1}{2}\big(\exp\big(\xi(t)\big)-1\big)^2+\xi(t)\exp\big(\xi(t)\big).$$

Let define $\tilde{\eta}_1,\tilde{\eta}_2 \in \mathcal{K}_\infty$ functions such that $\tilde{\eta}_1(r)=r+\frac{1}{2}r^2$, $\tilde{\eta}_2(r)=\frac{1}{2}(\exp(r)-1)^2+r\exp(r)$ [40]. Then by using assumption ($\mathcal{C}1$), we can get the following estimate:

$$V(x(t)) \leq E_\alpha\big(-q,t-t_0\big)V(x_0)+\frac{1}{2}E_\alpha\big(-2q,t-t_0\big)V(x_0)^2$$
$$+\frac{1}{2}\big(\exp\big(\xi(t)\big)-1\big)^2+\xi(t)\exp\big(\xi(t)\big),$$
$$\underline{\eta}_1(\|x(t)\|) \leq E_\alpha\big(-q,t-t_0\big)V(x_0)+\frac{1}{2}E_\alpha\big(-2q,t-t_0\big)V(x_0)^2$$
$$+\frac{1}{2}\big(\exp\big(\xi(t)\big)-1\big)^2+\xi(t)\exp\big(\xi(t)\big),$$
$$\underline{\eta}_1(\|x(t)\|) \leq \tilde{\eta}_1\big(E_\alpha\big(-q,t-t_0\big)V(x_0)\big)+\tilde{\eta}_2\big(\xi(t)\big),$$
$$\underline{\eta}_1(\|x(t)\|) \leq \tilde{\eta}_1(E_\alpha\big(-q,t-t_0\big)\overline{\eta}_1(\|x_0\|)+\tilde{\eta}_2\big(\xi(t)\big),$$

It is intuitively clear that $\tilde{\eta}_2(r) \geq \tilde{\eta}_1(r)$ for any $r \geq 0$. Using this property, we can write

$$\underline{\eta}_1(\|x(t)\|) \leq \tilde{\eta}_2(E_\alpha\big(-q,t-t_0\big)\overline{\eta}_1(\|x_0\|)+\tilde{\eta}_2\big(\xi(t)\big),$$

Multiplying both sides of this inequality by $\frac{1}{2}$ and applying $\tilde{\eta}_2^{-1}$, using the fact $\tilde{\eta}_2^{-1}(\frac{a_1+a_2}{2}) \leq \tilde{\eta}_2^{-1}(a_1)+\tilde{\eta}_2^{-1}(a_2)$ for all non-negative a_1, a_2 [28], we have

$$\frac{1}{2}\underline{\eta}_1(\|x(t)\|)$$
$$\leq \frac{1}{2}\left(\tilde{\eta}_2(E_\alpha(-q,t-t_0)\overline{\eta}_1(\|x_0\|)+\tilde{\eta}_2(\xi(t))\right),$$
$$\tilde{\eta}_2^{-1}\left(\frac{1}{2}\underline{\eta}_1(\|x(t)\|)\right)$$
$$\leq \tilde{\eta}_2^{-1}\left(\frac{1}{2}\left(\tilde{\eta}_2(E_\alpha(-q,t-t_0)\overline{\eta}_1(\|x_0\|))+\tilde{\eta}_2(\xi(t))\right)\right),$$
$$\tilde{\eta}_2^{-1}\left(\frac{1}{2}\underline{\eta}_1(\|x(t)\|)\right)$$
$$\leq \tilde{\eta}_2^{-1}\left(\tilde{\eta}_2(E_\alpha(-q,t-t_0)\overline{\eta}_1(\|x_0\|))+\tilde{\eta}_2^{-1}(\tilde{\eta}_2(\xi(t)))\right),$$
$$\tilde{\eta}_2^{-1}\left(\frac{1}{2}\underline{\eta}_1(\|x(t)\|)\right)$$
$$\leq E_\alpha(-q,t-t_0)\overline{\eta}_1(\|x_0\|)+\int_{t_0}^{t}(\tau-t_0)^{\alpha-1}\Theta(\|u(\tau)\|)d\tau.$$

Now, we have property (21) satisfied with $\eta_1(s) = \tilde{\eta}_2^{-1}(\frac{1}{2}\underline{\eta}_1(s)) \in \mathcal{K}_\infty$, $\eta_2(s) = \overline{\eta}_1(s) \in \mathcal{K}_\infty$, $\eta_3(s) = \Theta(s) \in \mathcal{K}_\infty$. Therefore, system (13) is fe-iISS. □

It is worth noting that for the case the fractional order α is equal to 1, the results obtained for the fe-iISS of the conformable fractional-order bilinear systems are consistent with those presented for the integer case in ref. [40].

Corollary 1. *If there exists a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ that has conformable fractional derivative of order α for all $t > t_0$ and satisfies the corresponding conditions* (C1) *and* (C2)*, then system (13) is also iISS by virtue of Definition 5 (ii) and Definition6 (ii).*

In the following proposition, we show that fractional-order bilinear system (14) is also fe-iISS in the view of Theorem 3. We use Lemma 12 to ensure the existence of $P = P^T > 0$ satisfying $A^T P + PA = -Q$ for any $Q = Q^T > 0$.

Proposition 2. *(*fe-iISStability of CFBS*) Let $\alpha \in (0,1]$. The fractional-order bilinear system (14) is fe-iISS if and only if the matrix A such that $|Spec(A)| \neq 0$, $|Arg(Spec(A))| > \lceil\alpha\rceil\frac{\pi}{2}$.*

Proof. (if). If the matrix A does not satisfies the conditions $|\text{Spec}(A)| \neq 0$, $|\text{Arg}(\text{Spec}(A))| > \lceil\alpha\rceil\frac{\pi}{2}$, then the unforced system is not GAS and consequently not fe-iISS either. (Only if) Suppose that $|\text{Spec}(A)| \neq 0$, $|\text{Arg}(\text{Spec}(A))| > \lceil\alpha\rceil\frac{\pi}{2}$. Then, pick any $P = P^T > 0$ so that $A^T P + PA = -Q$ (see Lemma 12). Let $V(x) = x^T Px$ be a Lyapunov function that has conformable fractional-order derivative of order α for all $t > t_0$, then $T_{t_0}^\alpha V(x(t))$ is given by Lemma 5 as follows: $T_{t_0}^\alpha V(x(t)) = \nabla^\alpha V(x)(x^{1-\alpha})^{-1}T_{t_0}^\alpha x(t)$, where $x^{1-\alpha} = \text{diag}(x_1{}^{1-\alpha}, x_2{}^{1-\alpha}, \ldots, x_n{}^{1-\alpha})$. Suppose that $\det[x^{1-\alpha}] \neq 0$ we can write

$$T_{t_0}^\alpha V(x(t)) = \nabla^\alpha V(x)\left(x^{1-\alpha}\right)^{-1}T_{t_0}^\alpha x(t)$$
$$= 2x^T P\left(x^{1-\alpha}\right)\left(x^{1-\alpha}\right)^{-1}\left(Ax+\sum_{i=1}^{m}u_iN_ix+Bu\right)$$
$$= 2x^T P I\left(Ax+\sum_{i=1}^{m}u_iN_ix+Bu\right)$$
$$= 2x^T PAx+\sum_{i=1}^{m}2u_ix^T PN_ix+2x^T PBu$$
$$= 2x^T PAx+\sum_{i=1}^{m}2u_ix^T PN_ix+2x^T PBu,$$

for all $t > t_0 \geq 0$. We note that, there exists a constant $K > 0$ such that

$$\left\|2x^T P\sum_{i=1}^{m}u_iN_ix\right\| \leq 2\|x\|\|P\|\left\|\sum_{i=1}^{m}uN_ix\right\|$$
$$\leq 2K\lambda_{\max}(P)\|u\|\|x\|^2, \tag{36}$$

and, for all vectors $x \in \mathbb{R}^n$, and $u \in \mathcal{U}^m$, we can write

$$2x^T PBu \leq 2\|x\|\|PB\|\|u\| \leq 2\lambda_{\max}(P)\|B\|\|u\|\|x\| \leq \|x\|^2$$
$$+\lambda_{\max}(P)^2\|B\|^2\|u\|^2, \tag{37}$$

also $2x^T PAx{=}2x^T PAx{=}x^T(A^T P+PA)x \leq -\lambda_{\min}(Q)\|x\|^2)$. Using these all, for all $t > t_0$, we can write

$$T_{t_0}^\alpha V(x(t)) \leq -\lambda_{\min}(Q)\|x\|^2+2K\lambda_{\max}(P)\|u\|\|x\|^2+\|x\|^2$$
$$+\lambda_{\max}(P)^2\|B\|^2\|u\|^2$$
$$\leq (2K\lambda_{\max}(P)\|u\|-\lambda_{\min}(Q)+1)V(x)$$
$$+\lambda_{\max}(P)^2\|B\|^2\|u\|^2$$
$$\leq \left(r2K\lambda_{\max}(P)\|u\|-q\right)V(x)$$
$$+\lambda_{\max}(P)^2\|B\|^2\|u\|^2,\ \forall\lambda_{\min}(Q) > 1,$$

Thus, the requirements from Theorem 3 are satisfied and the conformable fractional-order bilinear systems (14) are, in general, fe-iISS, as desired. □

Remark 6. From the proof of Proposition 2, it is deduced that $V(x) = x^T Px$ is not an iISS–Lyapunov function for CFBS in the view of Definition 8. However, it works to show that the CFBS are fe-iISS and iISS in the view of Theorem 3 and Corollary 1.

Now, we proceed from here to show that CFBS are also sg-ISS. Following proposition provides an approximation of the input threshold $R > 0$ using the same Lyapunov function.

Proposition 3. (sg-ISStability of CFBS) *Let $\alpha \in (0,1]$. The fractional-order bilinear system (14) is sg-ISS, If and only if the matrix A such that $|Spec(A)| \neq 0$, $|Arg(Spec(A))| > \lceil\alpha\rceil\frac{\pi}{2}$. In this case, letting $P = P^T > 0$ satisfying $A^T P + PA = -Q$ for any $Q = Q^T > 0$, an input threshold for Equation (14) is*

$$R = \frac{\lambda_{\min}(Q) - 1}{2K\lambda_{\max}(P)},$$

for $\lambda_{\min}(Q) > 1$, in which $K > 0$ such that $\|\sum_{i=1}^{m} N_i\, x(\tau)u_i(\tau)\| \leq K\|u(t)\|\|x(t)\|$.

Proof. (if). If the matrix A does not satisfies the conditions $|\text{Spec}(A)| \neq 0$, $|\text{Arg}(\text{Spec}(A))| > \lceil\alpha\rceil\frac{\pi}{2}$, then the unforced system is not GAS, and thus, not also fe-iISS. (If). Suppose that $|\text{Spec}(A)| \neq 0$, $|\text{Arg}(\text{Spec}(A))| > \lceil\alpha\rceil\frac{\pi}{2}$. Pick any $P = P^T > 0$ so that $A^T P + PA = -Q$. Let $V(x(t)) = x^T Px$ be a Lyapunov function that has conformable fractional derivative of order α for all $t > t_0$, then $T_{t_0}^{\alpha} V(x(t))$ is given by Lemma 5 as $T_{t_0}^{\alpha} V(x(t)) = \nabla^{\alpha} V(x)(x^{1-\alpha})^{-1} T_{t_0}^{\alpha} x(t)$, where $x^{1-\alpha} = \text{diag}(x_1{}^{1-\alpha}, x_2{}^{1-\alpha}, \dots, x_n{}^{1-\alpha})$. Assuming that $\det[x^{1-\alpha}] \neq 0$ and $\|u\|_\infty \leq R$, we have

$$\begin{aligned} T_{t_0}^{\alpha} V(x(t)) &= \nabla^{\alpha} V(x)\left(x^{1-\alpha}\right)^{-1} T_{t_0}^{\alpha} x(t) \\ &= 2x^T P\left(x^{1-\alpha}\right)\left(x^{1-\alpha}\right)^{-1}\left(Ax + \sum_{i=1}^{m} u_i N_i x + Bu\right) \\ &= 2x^T P I\left(Ax + \sum_{i=1}^{m} u_i N_i x + Bu\right) \\ &= 2x^T PAx + 2x^T P\sum_{i=1}^{m} u_i N_i x + 2x^T PBu, \end{aligned}$$

for all $t > t_0$. Using Equations (36) and (37), and also $x^T(A^T P + PA)x \leq \|x\|\lambda_{\min}(Q)\|x\| \leq -\lambda_{\min}(Q)\|x\|^2$, we can write

$$\begin{aligned} T_{t_0}^{\alpha} V(x(t)) \leq& -\lambda_{\min}(Q)\|x\|^2 + 2K\lambda_{\max}(P)\|u\|\|x\|^2 + \|x\|^2 \\ &+ \lambda_{\max}(P)^2\|B\|^2\|u\|^2 \\ \leq& -\lambda_{\min}(Q)\|x\|^2 + 2K\lambda_{\max}(P)R\|x\|^2 + \|x\|^2 \\ &+ \lambda_{\max}(P)^2\|B\|^2\|u\|^2 \\ \leq& (2K\lambda_{\max}(P)R + 1 - \lambda_{\min}(Q))\|x\|^2 \\ &+ \lambda_{\max}(P)^2\|B\|^2 R^2, \end{aligned}$$

where $\lambda_{\min}(Q) > 1$. By Remark 5 and, thus, property (27) , we have $\eta(s) = (-2K\lambda_{\max}(P)R + \lambda_{\min}(Q) - 1)s^2$ has to be a class $\mathcal{K}_\infty$−function. Therefore, it follows that $(-2K\lambda_{\max}(P)R + \lambda_{\min}(Q) - 1) \geq 0$. That is, $R \leq \frac{\lambda_{\min}(Q)-1}{2K\lambda_{\max}(P)}$, $\forall\lambda_{\min}(Q) > 1$. This concludes the proof. □

Summarising all the above together, we have the following main novel result Concerning the ISS-t the conformable fractional-order bilinear systems.

Theorem 4. *Consider the conformable fractional bilinear system (14). The following statements are equivalent:*

- *(a) The matrix A such that $|Spec(A)| \neq 0$, $|Arg(Spec(A))| > \lceil\alpha\rceil\frac{\pi}{2}$ for all $\alpha \in (0,1]$,*
- *(b) It is iISS,*
- *(c) It is fe-iISS,*
- *(d) It is strong-iISS,*
- *(e) It is sg-ISS,*
- *(f) It is BIBS and CICS.*

Proof. We have: $[a \Leftrightarrow b]$ (see Proposition 1); $[a \Leftrightarrow c]$ (see Proposition 2); $[a \Leftrightarrow e]$ (see Proposition 3); $[c \Rightarrow b]$ (see Definitions 5(ii) and 6(ii) where Equation (21) evidently implies (18); $[d \Rightarrow f]$ (see Lemma 7)); $[a \Leftrightarrow d]$ (since $[d \Rightarrow b \wedge e]$ (see Definition 5(iv)), $[a \Rightarrow b]$ (see Proposition 2), and $[a \Rightarrow e]$ (see Proposition 3), then $[a \Rightarrow d]$. Conversely, If $|\text{Spec}(A)| \neq 0$, $|\text{Arg}(\text{Spec}(A))| > \lceil\alpha\rceil\frac{\pi}{2}$, then, the zero-input system (14) is neither GAS nor SiISS); □

4 | AN ILLUSTRATIVE EXAMPLE

Consider the following multi-inputs fractional-order bilinear system with conformable derivative

$$T^{\alpha}x(t) = Ax(t) + N_1\, x(t)u_1(t) + N_2\, x(t)u_2(t) + Bu(t), t > 0$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T$ is the state of system and $u(t) = (0.5u_s(t) - 0.5u_s(t-4),\ \exp(-t))^T$ is the control input in which $u_s(.)$ denotes unit step function, and

$$A = \begin{pmatrix} -3 & 2 & 1 \\ -4 & -5 & -1 \\ -2 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix},$$

In what follows, the major properties of input-to-state stability for this system is analysed through simulations. By this example we will get a better idea of how strict are the iISS estimations for the conformable fractional bilinear systems

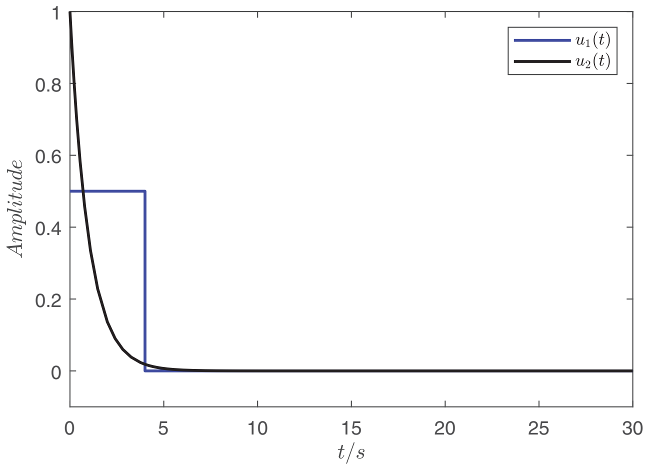

FIGURE 1 Time history of the control inputs

established in Proposition 1. The control inputs are depicted in Figure 1. All simulation results, obtained in Figure 2, are for the initial condition $x_0 = (1, -0.5, 0.8)^T$.

Since the matrix A such that $|\mathrm{Spec}(A)| \neq 0$, $|\mathrm{Arg}(\mathrm{Spec}(A))| > \lceil \alpha \rceil \frac{\pi}{2}$ for $0 < \alpha \leq 1$, then by Proposition 1 we expect that this system is iISS. The trajectories along with the corresponding iISS estimation (18) with respect to the given inputs for two values of α are depicted in Figure 2. Taking in to account that that $\psi \geq 1$ (see Lemma 11), it is observed that the iISS estimate is larger than what is actually needed. Moreover, the transient (overshoot) term $\beta(\cdot)$, obtained when inputs are wiped out, and the asymptotic term $\eta(\cdot)$ of the iISS estimation are obtained as in Proposition 1 and illustrated. It is noted that, at $t = 0$, all the trajectories lies in the ball of radius $\psi\|x_0\| + \frac{1}{2}\psi^2\|x_0\|^2$ (≈ 20 for $\alpha = 0.5$ and ≈ 6.5 for $\alpha = 0.95$) which is quite larger than its actual value($\|x_0\| \approx 1.8$). Clearly, for small $t \leq 4$ s, the $\beta(.)$ term makes the major contribution to amount the transient behavior. On the other hand, for large t, all trajectories approach to the smaller ball of radius $\eta(\cdot)$ since $\beta(\|x_0\|, t)$ converges to zero as $t \to \infty$. In other words, although the inputs fade after $t = 4$ s, the weaker definition of iISS compared to ISS permits the states yet to have some steady-state bias with respect to external inputs. The system is also sg-ISS in the view of Proposition 3. Pick $Q = 2I$, then using Lemma 12 we obtain the solution

$$P = \begin{bmatrix} 0.3116 & -0.1460 & 0.2268 \\ -0.1460 & 0.4260 & -0.5464 \\ 0.2268 & -0.5464 & 2.5424 \end{bmatrix}.$$

Let $K \simeq 0.1$ and $\|B\| = 2$, then we can obtain the input threshold as $R \approx 1.4$ (see Proposition 3). Note that, since $|u_1| \leq 1$ and $|u_2| \leq 0.5$, thus we have $\|u(t)\| \leq 1.4$.

5 | CONCLUSION

In this paper, we have developed several novel input-to-state stability notions namely ISS, iISS, sg-ISS, strong-iISS, fe-ISS, and fe-iISS for (conformable) fractional-order systems with the emphasis on the bilinear systems. That is, a set of Lyapunov-based sufficient conditions has been provided to establish these properties for the nonlinear fractional systems with conformable derivative. We have also established a set of sufficient and necessary conditions particularly for ISStability of the general class of conformable fractional bilinear systems. it is noted that extending the obtained results to the higher-order conformable fractional bilinear systems also represents a worthwhile direction for future research. Furthermore, the notion of ISS-Control-Lyapunov function and its related variants could be generalised for stabilising the control-affine nonlinear conformable fractional systems including conformable fractional bilinear systems can be considered as another research line in the future.

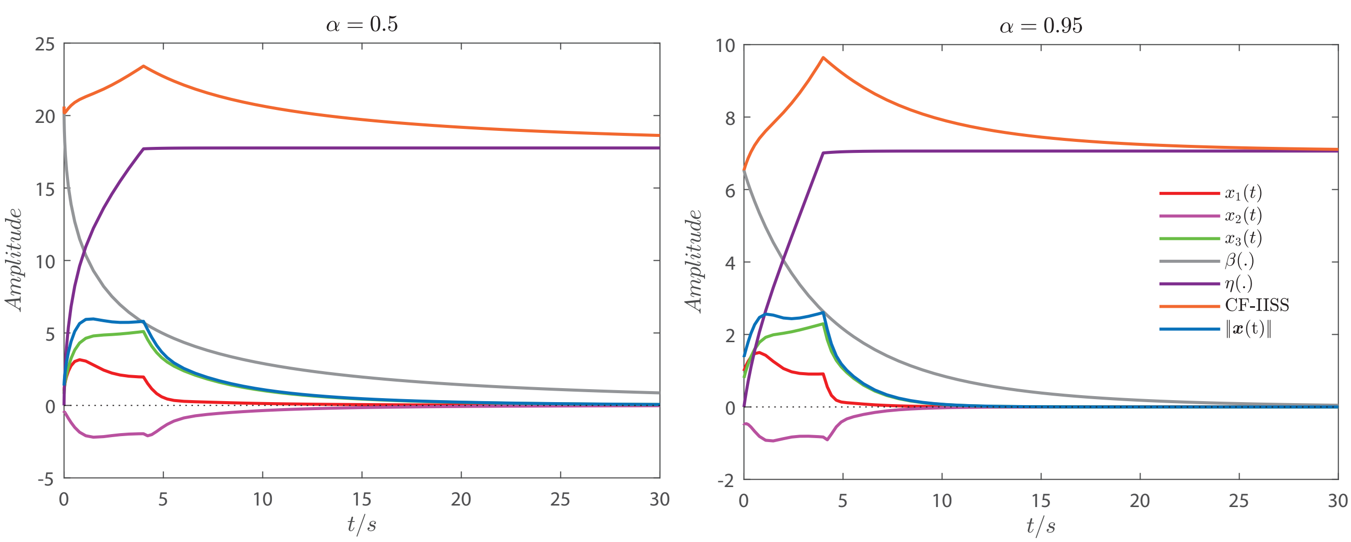

FIGURE 2 Transient and asymptotic behavior using iISS estimation for $\alpha = 0.5$(left) and $\alpha = 0.95$(right).

CONFLICT OF INTEREST

The authors declare no conflict of interest.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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APPENDIX A: PROOF OF LEMMA 8

The idea of proof is inspired by the remarks given in ref. [42]. (If). The proof of implication (27) ⇒ (23) is immediate. (Only if). To prove the opposite implication, suppose that (23) holds and $\|x\| \geq \chi\|u\|$, then Equation (27) holds for any $\sigma(\cdot)$. Let define

$$\overline{\sigma}(r) := \max\left\{ \nabla^{\alpha} V(x)\left(x^{1-\alpha}\right)^{-1} f(x,u) + \eta_1(\chi(\|u\|)) : \|u\| \leq r, \|x\| \leq \chi(r) \right\}.$$

Then, for $\|x\| \leq \chi\|u\|$,

$$\nabla^{\alpha} V(x)\left(x^{1-\alpha}\right)^{-1} f(x,u) \leq -\eta_1(\|x\|) + \overline{\sigma}(\|u\|).$$

Define $\sigma(r) := \max\{0, \overline{\sigma}(r)\}$. Then, $\sigma(\cdot)$ is continuous, nonnegative, $\sigma(0) = 0$, and we can consider it as a $\mathcal{K}_\infty$ function (If $\sigma(\cdot) \notin \mathcal{K}_\infty$, majorise it by a class $\mathcal{K}_\infty$ function) to have inequality (27) satisfied. This completes the proof.

APPENDIX B: PROOF OF LEMMA 9

Let define the following function on $(0, +\infty)$:

$$\delta(s) \overset{\text{def}}{=} -\int_1^s (r-1)^{\alpha-1} \frac{1}{\varphi(r)}\, \mathrm{d}r,$$

for any $0 < \alpha \leq 1$. It follows from Lemma 1 that, $T_1^{\alpha}\delta(s) < 0$ for all $s > 0$ and $0 < \alpha \leq 1$. Using Lemma 3, we deduced that, this is a strictly decreasing $\alpha-$differentiable function on $(0, +\infty)$ for all $0 < \alpha \leq 1$. Suppose, without loss of generality, $\lim_{s\to 0^+} \delta(s) = +\infty$. Otherwise, the following function could be instead considered:

$$\bar{\varphi}(s) \overset{\text{def}}{=} \begin{cases} \min\{s, \varphi(s)\}, & \text{if } 0 \leq s < 1, \\ \varphi(s), & \text{if } 1 \geq s. \end{cases}.$$

The function $\overline{\varphi}(\cdot)$ is also a class $\mathcal{P}$-function that satisfies $\overline{\varphi}(s) \leq \varphi(s)$ for any $s \geq 0$ and $\lim_{s\to 0^+} \int_s^1 (r-s)^{\alpha-1} \frac{1}{\overline{\varphi}(r)} \mathrm{d}r \geq \int_s^1 (r-s)^{\alpha-1} \frac{1}{r}\, \mathrm{d}r \geq \int_s^1 r^{\alpha-1}\frac{1}{r}\, \mathrm{d}r = +\infty$ holds since $\frac{1}{\alpha-1} < 0$. We observe that, if $T_{t_0}^{\alpha} v(t) \leq -\varphi(v(t))$ then also $T_{t_0}^{\alpha} v(t) \leq -\overline{\varphi}(v(t))$, so $\beta_{\overline{\varphi}}$ could be used to bound solutions. Define $0 < a \overset{\text{def}}{=} -\lim_{s\to+\infty} \delta(s)$. Note that the range of δ and thus the domain of δ^{-1} belong to $(-a, +\infty)$(this may be $a = +\infty$). Let define

$$\beta_{\varphi}(s,t) \overset{\text{def}}{=} \begin{cases} 0, & \text{if } s = 0, \\ \delta^{-1}\left(\frac{1}{\alpha}(t-t_0)^{\alpha} + \delta(s)\right), & \text{if } s \geq 0. \end{cases}$$

for all $s \geq 0$, $t > t_0$, and $0 < \alpha \leq 1$.

Claim. Given any $y(.)$ fulfilling all the corresponding conditions in the Lemma 9, then

$$y(t) \leq \beta_{\varphi}(y_0, t). \tag{B.1}$$

for all $t > t_0$ and $0 < \alpha \leq 1$.

Proof of Claim. Assume that Equation (28) holds, then it follows from Lemma 3 that $y(t)$ is nonincreasing and if $y(t_0) = 0$ for some $t_0 \geq 0$, then, we see that $y(t) \equiv 0$ for all $t > t_0$. Now, without loss of generality, suppose that $y(t_0) > 0$. Let $t_0^* \overset{\text{def}}{=} \inf\{t : y(t) = 0\} \leq +\infty$. It is enough to show that the bound in Equation (B.1) holds for $t \in [t_0, t_0^*)$. Since δ is strictly decreasing in the view of Lemma 3, we only need to show that $y(t) \geq \delta^{-1}(\frac{1}{\alpha}(t-t_0)^{\alpha} + \delta(y_0))$ for all $\alpha \in (0,1]$ which can be also written as $\delta(y(t)) \geq \frac{1}{\alpha}(t-t_0)^{\alpha} + \delta(y_0)$. Therefore, by definition, we have $-\int_1^{y(t)} (r-1)^{\alpha-1}\frac{1}{\varphi(r)}\, \mathrm{d}r \geq \frac{1}{\alpha}(t-t_0)^{\alpha} - \int_1^{y_0} (r-1)^{\alpha-1}\frac{1}{\varphi(r)}\, \mathrm{d}r$, which is equivalent to

$$\int_{y(t)}^{y_0} (r-1)^{\alpha-1} \frac{1}{\varphi(r)} \mathrm{d}r \geq \frac{1}{\alpha}(t-t_0)^{\alpha}. \tag{B.2}$$

By Equation (28), one sees that

$$\int_{t_0}^{t} (\tau - t_0)^{\alpha-1} \frac{T_{t_0}^{\alpha} y(\tau)}{\varphi\big(y(\tau)\big)} \mathrm{d}\tau \leq -\int_{t_0}^{t} (\tau - t_0)^{\alpha-1} \mathrm{d}\tau = -\frac{(t-t_0)^{\alpha}}{\alpha}. \tag{B.3}$$

By Lemma 2 and changing variables in Equation (B.3), we get Equation (B.2) by which we obtain the desired result we claimed.

It only remains to prove the claim that β_φ is of class $\mathcal{KL}$. The function β_φ is continuous since both $\delta$and $\delta^{-1}$are continuous in their domains, and $\lim_{r\to+\infty} \delta^{-1}(r) = 0$. It is strictly increasing in s for each fixed t since both δ and δ^{-1} are strictly decreasing. By construction, we see that $\beta_\varphi(s,t) \to 0$ as $t \to +\infty$. Hence, $\beta_\varphi \in \mathcal{KL}$, as claimed. □

APPENDIX C: PROOF OF LEMMA 10

First we remark that the following lemma will be needed in order to prove Lemma 10.

Lemma C.1. *[49]Let $\alpha \in (0,1]$. Assume that $r(t)$ and $h(t)$ be continuous and non-negative functions defined for $a \leq t \leq b$, and c be a non-negative constant such that*

$$r(t) \leq c + \int_a^t (\tau - a)^{\alpha-1} h(\tau)\, r(\tau)\, d\tau, \quad \forall a \leq t \leq b, \tag{C.1}$$

then, for all $a \leq t \leq b$

$$r(t) \leq c.\exp\left(\int_a^t (\tau - a)^{\alpha-1} h(\tau)\, d\tau\right). \tag{C.2}$$

Proof of Lemma 10. Define $r(t) := c(t)\zeta(t)$.Then, by Equation (30), we have

$$\zeta(t) \leq 1 + \int_a^t (\tau - a)^{\alpha-1} h(\tau) \times \frac{c(\tau)\zeta(\tau)}{c(t)} \mathrm{d}\tau \leq 1 + \int_a^t (\tau - a)^{\alpha-1} h(\tau)\, \zeta(\tau)\, \mathrm{d}\tau.$$

By Lemma C.1, we obtain an upper bound for $\zeta(t)$

$$\zeta(t) \leq \exp\left(\int_a^t (\tau - a)^{\alpha-1} h(\tau)\, \mathrm{d}\tau\right). \tag{C.3}$$

Using Equation (C.3) in $r(t) = c(t)\zeta(t)$, we obtain the inequality (31). This concludes the proof. □

APPENDIX D: PROOF OF LEMMA 11

We use 2-norm during this proof. However, what we concludes here also holds when other norms are used as any two norms are equivalent in finite dimensions.

If $A \in \mathbb{C}^{n\times n}$, by Jordan Decomposition theorem [2], there exists an invertible $S \in \mathbb{C}^{n\times n}$ such that

$$A = S\,J\,S^{-1} = S\,\mathrm{diag}(J_1, J_2, \dots, J_s)\,S^{-1}) \tag{D.1}$$

with the Jordan block

$$J_l = \begin{pmatrix} & 1 & \\ & \lambda_l & \ddots \\ \lambda_l & \ddots & \lambda_l \end{pmatrix} \in \mathbb{C}^{n_l\times n_l}$$

$l = 1, 2, \dots, s$ and, λ_l is the eigenvalue of matrix A and $\sum_{l=1}^{s} n_l = n$. Substituting Equation (D.1) into $E_\alpha(A, t-t_0)$, we have

$$\begin{aligned} E_\alpha(A, t-t_0) &= E_\alpha\big(SJS^{-1}, t-t_0\big) = S\,E_\alpha(J, t-t_0)S^{-1} \\ &= S\,\mathrm{diag}(E_\alpha(J_1, t-t_0), \dots, E_\alpha(J_s, t-t_0))S^{-1}, \\ &\forall t \geq t_0, \end{aligned} \tag{D.2}$$

with

$$\begin{aligned} E_\alpha(J_l, t-t_0) &= E_\alpha(\lambda_l I_{nl} + Z_{nl}, t-t_0) \\ &= E_\alpha(\lambda_l I_{nl}, t-t_0) E_\alpha(Z_{nl}, t-t_0),\ \forall t \geq t_0, \end{aligned} \tag{D.3}$$

where $Z_{nl} = J_l - \lambda_l I_{nl}$ is an $n_l \times n_l$ nilpotent matrix. Then, $E_\alpha(Z_{nl}, t-t_0)$ can be expressible using Lemma 6 since its fractional series terminates. So,

$$E_\alpha(Z_{nl}, t-t_0) = \sum_{k=0}^{\infty} \frac{{Z_{nl}}^k (t-t_0)^{k\alpha}}{\alpha^k k!},\ \forall t \geq t_0. \tag{D.4}$$

For any sufficiently small $\varepsilon > 0$, we can write

$$\begin{aligned} E_\alpha(J_l, t-t_0) &= E_\alpha(\lambda_l, t-t_0)\left(\sum_{k=0}^{n_l-1} \frac{{Z_{nl}}^k (t-t_0)^{k\alpha}}{\alpha^k k!}\right) \\ &= E_\alpha(\lambda_l + \varepsilon, t-t_0) E_\alpha(-\varepsilon, t-t_0) \times \left(\sum_{k=0}^{n_l-1} \frac{{Z_{nl}}^k (t-t_0)^{k\alpha}}{\alpha^k k!}\right) \\ &= E_\alpha(\lambda_l + \varepsilon, t-t_0) \times \left(\sum_{k=0}^{n_l-1} \frac{E_\alpha(-\varepsilon, t-t_0){Z_{nl}}^k (t-t_0)^{k\alpha}}{\alpha^k k!}\right), \\ &\forall t \geq t_0. \end{aligned} \tag{D.5}$$

Using the fact $\|P\| \leq q \max\{|p_{ij}|\}$, for $P \in \mathbb{C}^{q\times q}$ [45], we obtain

$$\|E_\alpha(J_l, t-t_0)\| \leq n_l |E_\alpha(\lambda_l + \varepsilon, t-t_0)| \overline{\psi}_{k,l}, \quad \forall t \geq t_0, \tag{D.6}$$

where $\overline{\psi}_{k,l}$ is the maximum of the kth term in $\frac{\|Z_{nl}\|^k (t-t_0)^{k\alpha}}{E_\alpha(\varepsilon,t-t_0)\alpha^k k!}$ for any $\varepsilon > 0$ and for all $t \geq t_0$.

Define $\Omega_{k,l}(t) \overset{\text{def}}{=} \frac{\|Z_{nl}\|^k (t-t_0)^{k\alpha}}{E_\alpha(\varepsilon,t-t_0)\alpha^k k!}$ for any $\varepsilon > 0$ and $t \geq t_0$. We observe that, this function has a maximum at a point $t_m > t_0$ for any $\varepsilon > 0$, since $\Omega_{k,l}(t_0) = 0$, $\Omega_{k,l}(t) \geq 0$ for all $t \geq t_0$, and $\Omega_{k,l}(t) \to 0$ $(t \to +\infty)$. Therefore, in order to find this point, we apply the fractional conformable derivative that yields

$$\frac{\|Z_{nl}\|^k (t_m - t_0)^{k\alpha}}{\alpha^k k!} E_\alpha(-\varepsilon, t_m - t_0)(-\varepsilon + \frac{\alpha k}{(t_m - t_0)^\alpha}) = 0,$$

from which we obtain $t_m = (\frac{\alpha k}{\varepsilon})^{\frac{1}{\alpha}} + t_0$. Therefore, we have $\overline{\psi}_{k,l} = \Omega_{k,l}(t_m) = \frac{\|Z_{nl}\|^k k^k}{\exp(k)\varepsilon^k k!}$.

By taking the norm in Equation (D.2) , for every $\omega > \max\{\mathrm{Re}(\lambda) : \lambda \in Spec(A)\}$ we obtain

$$\begin{aligned} \|E_\alpha(A, t-t_0)\| &= \left\|S E_\alpha(J, t-t_0) S^{-1}\right\| \\ &\leq \|S\| \left\|S^{-1}\right\| \left\|E_\alpha(J, t-t_0)\right\| \\ &= \kappa(S) \left\|E_\alpha(J, t-t_0)\right\| \\ &\leq \kappa(S) \max_l \left\{ \| E_\alpha(J_l, t-t_0) \| \right\} \\ &\leq \kappa(S) \max_l \left\{ n_l |E_\alpha(\lambda_l + \varepsilon, t-t_0)| \overline{\psi}_{k,l} \right\} \\ &\leq \overline{n}\, \kappa(S) \max_l \left\{ \overline{\psi}_{k,l} \right\} E_\alpha(\omega, t-t_0) \\ &\leq \psi E_\alpha(\omega, t-t_0), \ \forall t \geq t_0, \end{aligned} \tag{D.7}$$

where $\kappa(S)$ is the condition number of $S \in \mathbb{C}^{n\times n}$ to be $\kappa(S) = \| S \| \|S^{-1}\| \geq \|SS^{-1}\| \geq 1$ by sub-multiplicativity of matrix norm [45] and $\overline{n} = \max(n_1, n_2 \ldots n_s)$. we observe that, $\psi \geq 1$ as desired. Hence, the bound in ($\mathcal{B}$1) holds, as claimed.

We now complete the proof of Lemma 11 by showing that claim ($\mathcal{B}$2) holds. We will do this by studying the elements of matrix $E_\alpha(A, t-t_0)$ in terms of the Jordan blocks: $E_\alpha(A, t-t_0) = S \ \mathrm{diag}(E_\alpha(J_1, t-t_0), \cdots, E_\alpha(J_s, t-t_0)) \ S^{-1}$ where, by Equations (D.4) and (D.5), we observe that the non-zero entries of $E_\alpha(J_l, t-t_0)$ can be expressed uniformly as follows:

$$\frac{1}{(j-1)!}\left(\left(\frac{\partial}{\partial\lambda}\right)^{j-1} E_\alpha(\lambda, t-t_0)\right)_{|\lambda=\lambda_l}, \qquad j = 1, 2, \ldots, n_l.$$

Now, we consider the following separate cases for all $t > t_0$ and $0 < \alpha \leq 1$:

(i) If $\lambda_l = 0$, then

$$\frac{1}{(j-1)!}\left(\left(\frac{\partial}{\partial\lambda}\right)^{j-1} E_\alpha(\lambda, t-t_0)\right)_{|\lambda=\lambda_l} = \frac{(t-t_0)^{(j-1)\alpha}}{\alpha^{j-1}(j-1)!} \to +\infty \ (t \to +\infty),$$

for all $j \geq 2$. So, $\|E_\alpha(J_l, t-t_0)\| \to +\infty$ as $t \to +\infty$, which, in turn, follows that $\|E_\alpha(A, t-t_0)\| \to +\infty$.

(ii) $|\mathrm{Arg}(\lambda_l)| < \lceil\alpha\rceil\frac{\pi}{2}$, then

$$\begin{aligned} &\frac{1}{(j-1)!}\left(\left(\frac{\partial}{\partial\lambda}\right)^{j-1} E_\alpha(\lambda, t-t_0)\right)_{|\lambda=\lambda_l} \\ &= \frac{1}{\alpha^{j-1}(j-1)!}\left((t-t_0)^{(j-1)\alpha} E_\alpha(\lambda_l, t-t_0)\right)_{|\lambda=\lambda_l}, \end{aligned}$$

thus, for all $j \geq 2$

$$\begin{aligned} &\left|\frac{1}{(j-1)!}\left(\left(\frac{\partial}{\partial\lambda}\right)^{j-1} E_\alpha(\lambda, t-t_0)\right)_{|\lambda=\lambda_l}\right| \\ &= \left|\frac{(t-t_0)^{(j-1)\alpha}}{\alpha^{j-1}(j-1)!}\right| \left|E_\alpha\left(|\lambda_l| \cos\left(\frac{\arg(\lambda_l)}{\lceil\alpha\rceil}\right), t-t_0\right)\right|. \end{aligned}$$

Since $|\frac{\mathrm{Arg}(\lambda_l)}{\lceil\alpha\rceil}| < \frac{\pi}{2}$, then, $\cos(\frac{\mathrm{Arg}(\lambda_l)}{\lceil\alpha\rceil}) > 0$. Therefore, $\|E_\alpha(J_l, t-t_0)\| \to +\infty$ as $t \to +\infty$ which, in turn, yields $\|E_\alpha(A, t-t_0)\| \to +\infty$.

(iii) If $|\mathrm{Arg}(\lambda_l)| = \lceil\alpha\rceil\frac{\pi}{2}$, then we first assume that the algebraic and geometric multiplicities of critical eigenvalue λ_l are equal, then by Equation (D.5), we have $E_\alpha(J_l, t-t_0) = E_\alpha(\lambda_l, t-t_0) I_{n_l}$ whose diagonal elements are $|E_\alpha(\lambda_l, t-t_0)| = E_\alpha(|\lambda_l| \cos(\frac{\mathrm{Arg}(\lambda_l)}{\lceil\alpha\rceil}), t-t_0) = 1 (t \to +\infty)$. Therefore, $E_\alpha(J_l, t-t_0)$ and, in turn, $E_\alpha(A, t-t_0)$ are bounded in this case.

On the other hand, assume that the algebraic and geometric multiplicities of critical eigenvalue λ_l are not the same. So, we have

$$\begin{aligned} &\left|\frac{1}{(j-1)!}\left(\left(\frac{\partial}{\partial\lambda}\right)^{j-1} E_\alpha(\lambda, t-t_0)\right)_{|\lambda=\lambda_l}\right| \\ &= \left|\frac{(t-t_0)^{(j-1)\alpha}}{\alpha^{j-1}(j-1)!}\right| \left|E_\alpha\left(|\lambda_l| \cos\left(\frac{\mathrm{Arg}(\lambda_l)}{\lceil\alpha\rceil}\right), t-t_0\right)\right| \\ &= \left|\frac{(t-t_0)^{(j-1)\alpha}}{\alpha^{j-1}(j-1)!}\right| \to +\infty \end{aligned}$$

as $t \to +\infty$ for all $j \geq 2$. Thus, $\|E_\alpha(A, t-t_0)\| \to +\infty$ as $t \to +\infty$.

Putting all above cases together imply that Equation ($\mathcal{B}$2), as desired.

APPENDIX E: PROOF OF LEMMA 12

The proof of Lemma 12 will follow from the subsequent lemma.

Lemma E.1. *Let $M, N \in \mathbb{R}^{n\times n}$. Consider the linear map $\mathfrak{F} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ and $\mathfrak{F}(Y) := MY + YN$. If $|Spec(M)| \neq$*

0, $|Arg(Spec(M))| > \lceil\alpha\rceil\frac{\pi}{2}$ *and* $|Spec(N)| \neq 0$, $|Arg(Spec(N))| > \lceil\alpha\rceil\frac{\pi}{2}$ *for all* $\alpha \in (0,1]$, *then* $\mathfrak{T}$ *is invertible.*

Proof. It is enough to show that given any matrix Q there exists some Y so that $MY + YN = Q$. $|\text{Spec}(M)| \neq 0$, $|\text{Arg}(\text{Spec}(M))| > \lceil\alpha\rceil\frac{\pi}{2}$ and $|\text{Spec}(N)| \neq 0$, $|\text{Arg}(\text{Spec}(N))| > \lceil\alpha\rceil\frac{\pi}{2}$, then we have

$$\|E_\alpha(\mu, t-t_0)QE_\alpha(\nu, t-t_0)\| \leq \psi\|Q\|E_\alpha(2\omega, t-t_0),$$

holds for all $t > t_0$ since $|\text{Arg}(\omega)| > \lceil\alpha\rceil\frac{\pi}{2}$ (see Lemma 11).Thus,

$$P := -\int_{t_0}^{\infty} (t-t_0)^{\alpha-1} E_\alpha(\mu, t-t_0) Q\, E_\alpha(\nu, t-t_0)\, \mathrm{d}t$$

is well-defined. Moreover,

$$\begin{aligned} MX + XN &= -\int_{t_0}^{+\infty} (t-t_0)^{\alpha-1} \left[\mu\, E_\alpha(\mu, t-t_0)QE_\alpha(\nu, t-t_0)\right. \\ &\quad \left. + E_\alpha(\mu, t-t_0)QE_\alpha(\nu, t-t_0)N\right]\, \mathrm{d}t \\ &= -\int_{t_0}^{+\infty} (t-t_0)^{\alpha-1} \\ &\quad \times \left(\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\left[E_\alpha(\mu, t-t_0)QE_\alpha(\nu, t-t_0)\right]\right) \mathrm{d}t \\ &= -I_{t_0}^\alpha T_{t_0}^\alpha (E_\alpha(\mu, t-t_0) Q\, E_\alpha(\nu, t-t_0)) \\ &= Q - \lim_{t\to+\infty} (E_\alpha(\mu\,, t-t_0)\, Q\, E_\alpha(\nu, t-t_0)) \\ &= Q, \end{aligned}$$

as desired. □

Proof of Lemma 12. Choose any Q. Since $|\text{Spec}(A)| \neq 0$, $|\text{Arg}(\text{Spec}(A))| > \lceil\alpha\rceil\frac{\pi}{2}$ it also holds $|\text{Spec}(A^T)| \neq 0$, $|\text{Arg}(\text{Spec}(A^T))| > \lceil\alpha\rceil\frac{\pi}{2}$. Hence, by Lemma E.1, there exists a unique solution P for each Q. The explicit formula for P,

$$P := -\int_{t_0}^{\infty} (t-t_0)^{\alpha-1} E_\alpha\left(A^T, t-t_0\right) Q\, E_\alpha(A, t-t_0)\, \mathrm{d}t$$

shows that $P > 0$ if $Q < 0$, since the latter implies $-E_\alpha(A^T, t-t_0)QE_\alpha(A, t-t_0)$ is positive definite matrix for $t > t_0$. □

APPENDIX F: PROOF OF PROPOSITION 1

(Only if). If matrix A does not satisfies the conditions $|\text{Spec}(A)| \neq 0$, $|\text{Arg}(\text{Spec}(A))| > \lceil\alpha\rceil\frac{\pi}{2}$, then the unforced system is not GAS and consequently not iISS either. (Only if). To prove the opposite implication, we inspired by the given approach in ref. [2] for solving the inhomogeneous integer-order systems. We start from Equation (14) and applying conformable fractional integrator on both sides,

$$T_{t_0}^\alpha x(t) = \left(Ax(t) + \sum_{i=1}^{m} N_i\, x(t)u_i(t)\right) + Bu(t)$$

$$\begin{aligned} E_\alpha(-A, t-t_0)T_{t_0}^\alpha x(t) &= E_\alpha(-A, t-t_0)Ax(t) \\ &+ E_\alpha(-A, t-t_0)\sum_{i=1}^{m} N_i\, x(t)u_i(t) + E_\alpha(-A, t-t_0)Bu(t) \end{aligned}$$

$$\begin{aligned} &E_\alpha(-A, t-t_0)T_{t_0}^\alpha x(t) - E_\alpha(-A, t-t_0)Ax(t) \\ &= E_\alpha(-A, t-t_0)\sum_{i=1}^{m} N_i\, x(t)u_i(t) + E_\alpha(-A, t-t_0)Bu(t) \end{aligned}$$

$$\begin{aligned} &T_{t_0}^\alpha\left(E_\alpha(-A, t-t_0)x(t)\right) \\ &= E_\alpha(-A, t-t_0)\sum_{i=1}^{m} N_i\, x(t)u_i(t) + E_\alpha(-A, t-t_0)Bu(t) \end{aligned}$$

$$\begin{aligned} &I_{t_0}^\alpha\left(T_{t_0}^\alpha\left(E_\alpha(-A, t-t_0)x(t)\right)\right) \\ &= I_{t_0}^\alpha\left(E_\alpha(-A, t-t_0)\sum_{i=1}^{m} N_i\, x(t)u_i(t) + E_\alpha(-A, t-t_0)Bu(t)\right), \end{aligned}$$

and, then using Lemma 2, we have

$$\begin{aligned} &E_\alpha(-A, t-t_0)x(t) - x_0 \\ &= \int_{t_0}^{t} (\tau-t_0)^{\alpha-1} E_\alpha(-A, \tau-t_0)\sum_{i=1}^{m} N_i\, x(\tau)u_i\, \mathrm{d}\tau \\ &+ \int_{t_0}^{t} (\tau-t_0)^{\alpha-1} E_\alpha(-A, \tau-t_0)B\, u(\tau)\, \mathrm{d}\tau \end{aligned}$$

$$\begin{aligned} x(t) &= E_\alpha(A, t-t_0)x_0 \\ &+ \int_{t_0}^{t} (\tau-t_0)^{\alpha-1} E_\alpha(A, t-t_0)E_\alpha(-A, \tau-t_0)\sum_{i=1}^{m} N_i\, x(\tau)u_i(\tau)\, \mathrm{d}s \\ &+ \int_{t_0}^{t} (\tau-t_0)^{\alpha-1} E_\alpha(A, t-t_0)E_\alpha(-A, \tau-t_0)B\, u(\tau)\, \mathrm{d}\tau. \end{aligned}$$

Now, we can get the estimate of solution $x(t)$:

$$\begin{aligned} \|x(t)\| &\leq \|E_\alpha(A, t-t_0)x_0\| \\ &+ \left\|\int_{t_0}^{t} (\tau-t_0)^{\alpha-1} E_\alpha(A, t-t_0)E_\alpha(-A, \tau-t_0)\sum_{i=1}^{m} N_i\, x(\tau)u_i(\tau)\mathrm{d}\tau\right\| \\ &+ \left\|\int_{t_0}^{t} (\tau-t_0)^{\alpha-1} E_\alpha(A, t-t_0)E_\alpha(-A, \tau-t_0)B\, u(\tau)\, \mathrm{d}\tau\right\| \end{aligned}$$

$$
\begin{aligned}
&\leq \|E_\alpha(A,t-t_0)\|\|x_0\| \\
&\quad + \int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\|E_\alpha(A,t-t_0)E_\alpha(-A,\tau-t_0)\|\left\|\sum_{i=1}^{m} N_i x(\tau)u_i(\tau)\right\| \mathrm{d}\tau \\
&\quad + \int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\|E_\alpha(A,t-t_0)E_\alpha(-A,\tau-t_0)\|\|B\|\|u(\tau)\| \,\mathrm{d}\tau.
\end{aligned}
$$

We assume that there exists a constant $K > 0$ such that $\|\sum_{i=1}^{m} N_i x(\tau)u_i(\tau)\| \leq K\|u(t)\|\|x(t)\|$. Now, by Lemma 11,

$$
\begin{aligned}
\|x(t)\| &\leq \psi E_\alpha(\omega,t-t_0)\|x_0\| \\
&\quad + \int_{t_0}^{t} (\tau-t_0)^{\alpha-1}K\psi E_\alpha(\omega,t-t_0)E_\alpha(-\omega,\tau-t_0)\|x(\tau)\|\|u(\tau)\| \,\mathrm{d}\tau \\
&\quad + \int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\psi E_\alpha(\omega,t-t_0)E_\alpha(-\omega,\tau-t_0)\|B\|\|u(\tau)\| \,\mathrm{d}\tau.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
&\left\|E_\alpha(-\omega,t-t_0)x(t)\right\| \\
&\quad \leq \left(\psi\|x_0\| + \int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\psi E_\alpha(-\omega,\tau-t_0)\|B\|\|u(\tau)\|\mathrm{d}\tau\right) \\
&\quad + \int_{t_0}^{t} (\tau-t_0)^{\alpha-1}K\psi\left\|E_\alpha(-\omega,\tau-t_0)x(\tau)\right\|\|u(\tau)\|\mathrm{d}\tau.
\end{aligned}
$$

By fractional Gronwall–Bellman Lemma (see Lemma 10), we can write the previous inequality as follows:

$$
\begin{aligned}
&\left\|E_\alpha(-\omega,t-t_0)x(t)\right\| \\
&\quad \leq \left(\psi\|x_0\| + \int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\psi E_\alpha(-\omega,\tau-t_0)\|B\|\|u(\tau)\|\mathrm{d}\tau\right) \\
&\quad \times \exp\left(\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}K\psi\|u(\tau)\|\mathrm{d}\tau\right),
\end{aligned}
$$

then,

$$
\begin{aligned}
\left\|x(t)\right\| &\leq (E_\alpha(\omega,t-t_0)\psi\|x_0\| \\
&\quad + \int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\psi E_\alpha(\omega,t-t_0)E_\alpha(-\omega,\tau-t_0)\|B\|\|u(\tau)\|\mathrm{d}\tau\Big) \\
&\quad \times \exp\left(\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}K\psi\|u(\tau)\|\mathrm{d}\tau\right).
\end{aligned}
$$

Observing that $E_\alpha(\omega,t-t_0)E_\alpha(-\omega,\tau-t_0) \leq 1$ holds (see Definition 3), then, we have

$$
\begin{aligned}
\left\|x(t)\right\| &\leq \left(\psi\|x_0\|E_\alpha(\omega,t-t_0) + \int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\psi\|B\|\|u(\tau)\| \,\mathrm{d}\tau\right) \\
&\quad \times \exp\left(\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}K\psi\|u(\tau)\| \,\mathrm{d}\tau\right) \\
&\leq \left(\psi\|x_0\|E_\alpha(\omega,t-t_0) + \psi\|B\|\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\|u(\tau)\|\mathrm{d}\tau\right) \\
&\quad \times \exp\left(K\psi\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\|u(\tau)\| \,\mathrm{d}\tau\right).
\end{aligned}
$$

Seeking readability, we let $\zeta(t) = \int_{t_0}^{t} (\tau-t_0)^{\alpha-1}\|u(\tau)\| \,\mathrm{d}\tau$

$$
\begin{aligned}
\left\|x(t)\right\| &\leq \left(E_\alpha(\omega,t-t_0)\psi\|x_0\| + \psi\|B\|\zeta(t)\right)\exp\left(K\psi\zeta(t)\right) \\
&\leq E_\alpha(\omega,t-t_0)\psi\|x_0\|\exp\left(K\psi\zeta(t)\right) \\
&\quad + \psi\|B\|\zeta(t)\exp\left(K\psi\zeta(t)\right).
\end{aligned}
$$

We use the fact $2a_1a_2 \leq a_1^2 + a_2^2$ for every $a_1, a_2 > 0$ ref. [41] to convert the first term to a summation as follows:

$$
\begin{aligned}
E_\alpha(\omega,t-t_0)\psi\|x_0\|\exp\left(K\psi\zeta(t)\right) &= E_\alpha(\omega,t-t_0)\psi\|x_0\| \\
&\quad + E_\alpha(\omega,t-t_0)\psi\|x_0\|\left(\exp\left(K\psi\zeta(t)\right)-1\right) \\
&\leq E_\alpha(\omega,t-t_0)\psi\|x_0\| + \frac{1}{2}E_\alpha(2\omega,t-t_0)\psi^2\|x_0\|^2 \\
&\quad + \frac{1}{2}\left(\exp\left(K\psi\zeta(t)\right)-1\right)^2.
\end{aligned}
$$

Now, we can rewrite the estimation of $x(t)$

$$
\begin{aligned}
\left\|x(t)\right\| &\leq E_\alpha(\omega,t-t_0)\psi\|x_0\|\exp\left(K\psi\zeta(t)\right) \\
&\quad + \psi\|B\|\zeta(t)\exp\left(K\psi\zeta(t)\right) \\
&\leq E_\alpha(\omega,t-t_0)\psi\|x_0\| + \frac{1}{2}E_\alpha(2\lambda_m,t-t_0)\psi^2\|x_0\|^2 \\
&\quad + \frac{1}{2}\left(\exp\left(K\psi\zeta(t)\right)-1\right)^2 + \psi\|B\|\zeta(t)\exp\left(K\psi\zeta(t)\right).
\end{aligned}
$$

By defining functions $\eta_1', \eta_2' \in \mathcal{K}_\infty$ such that $\eta_1'(r) = \frac{1}{2}r^2 + r$, $\eta_2'(r) = \frac{1}{2}(\exp(r)-1)^2 + b\,r\exp(r)$ where $b > 0$, we have

$$
\begin{aligned}
\left\|x(t)\right\| &\leq E_\alpha(\omega,t-t_0)\psi\|x_0\| + \frac{1}{2}E_\alpha(2\omega,t-t_0)\psi^2\|x_0\|^2 \\
&\quad + \frac{1}{2}\left(\exp\left(K\psi\zeta(t)\right)-1\right)^2 + \psi\|B\|\zeta(t)\exp\left(K\psi\zeta(t)\right) \\
&\leq E_\alpha(\omega,t-t_0)\psi\|x_0\| + \frac{1}{2}E_\alpha(2\omega,t-t_0)\psi^2\|x_0\|^2 \\
&\quad + \frac{1}{2}\left(\exp\left(K\psi\zeta(t)\right)-1\right)^2 + \frac{\|B\|}{K}(K\psi\zeta(t))\exp\left(K\psi\zeta(t)\right) \\
&\leq \eta_1'(E_\alpha(\omega,t-t_0)\psi\|x_0\|) \\
&\quad + \eta_2'\left(\int_{t_0}^{t} (\tau-t_0)^{\alpha-1}K\psi\|u(\tau)\| \,\mathrm{d}\tau\right).
\end{aligned}
$$

Comparing with Equation (18) and noting that $b = \frac{\|B\|}{K} > 0$, then we obtain

$$\beta(s, t - t_0) = \eta_1'(E_\alpha(\omega, t - t_0)\psi s), \ \eta_1(s) = \eta_2'(s), \ \eta_2(s) = K\psi s\cdot$$

Thus, conformable fractional bilinear systems are generally iISS as claimed.

It is worth highlighting that for the case the fractional order α is equal to 1, the results obtained for the iISS of the conformable fractional-order bilinear systems are consistent with those presented for the integer case in ref. [40].