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Orthogonal and symplectic Yangians

R Kirschner[1]

[1] Institut für Theoretische Physik, Universität Leipzig, 04009 Leipzig, Germany

E-mail: `roland.kirschner@itp.uni-leipzig.de`

Abstract. We consider Yang-Baxter relations with orthogonal or symplectic symmetry, in particular L matrices defining the related Yangian algebra. We study the conditions resulting from the truncation of the expansion of $L(u)$.

1. Introduction

In the case of general linear symmetry the Yang-Baxter (YB) relation of the RLL type, involving the fundamental R matrix and the L operator, is solved by the linear expression $L_{ab}(u) = u\delta_{ab} + M_{ab}$ with no restriction on the Lie algebra representation generated by M_{ab}. This is different in the cases of orthogonal or symplectic symmetries. Here the fundamental R matrix is quadratic in the spectral prameter u [1, 2, 4]. The restriction of the u expansion of $L(u)$ leads to constraints which cannot be fulfilled in all Lie algebra representations. The example of the spinor representation is well known [3]. This and more examples have been pointed out [4, 6]. We summarise some previous results [7, 8] and continue the study of the constraints arising from truncation at second order.

2. Fundamental YB matrices and L operators

We start from the fundamental Yang-Baxter matrix obeying the Yang-Baxter relation in the form

$$R^{a_1a_2}_{b_1b_2}(u)R^{b_1a_3}_{c_1b_3}(u+v)R^{b_2b_3}_{c_2c_3}(v) = R^{a_2a_3}_{b_2b_3}(v)R^{a_1b_3}_{b_1c_3}(u+v)R^{b_1b_2}_{c_1c_2}(u). \tag{1}$$

and symmetric with respect to the orthogonal or symeplectic Lie algebra. The fundamental representation of these Lie algebras is defined by matrices obeying

$$A^d{}_a \, \varepsilon_{db} + \varepsilon_{ad} \, A^d{}_b = 0 \, , \tag{2}$$

where ε_{ab} is a non-degenerate invariant metric

$$\varepsilon_{ab} = \epsilon \, \varepsilon_{ba} \, , \quad \varepsilon_{ab}\varepsilon^{bd} = \delta^d_a \, , \tag{3}$$

which is symmetric $\epsilon = +1$ for $SO(n)$ case and skew-symmetric $\epsilon = -1$ for $Sp(n)$ case.

The well known fundamental R-matrix [1, 2, 3, 4] can be written in the unified form for an arbitrary metrics ε_{ab} (3) as follows (see, e.g., [5])

$$R^{a_1a_2}_{b_1b_2}(u) = u(u + \frac{n}{2} - \epsilon)I^{a_1a_2}_{b_1b_2} + (u + \frac{n}{2} - \epsilon)P^{a_1a_2}_{b_1b_2} - \epsilon \, u \, K^{a_1a_2}_{b_1b_2} \, , \tag{4}$$

where

$$I^{a_1a_2}_{b_1b_2} = \delta^{a_1}_{b_1}\delta^{a_2}_{b_2}\ , \quad P^{a_1a_2}_{b_1b_2} = \delta^{a_1}_{b_2}\delta^{a_2}_{b_1}\ , \quad K^{a_1a_2}_{b_1b_2} = \varepsilon^{a_1a_2}\varepsilon_{b_1b_2}\ , \tag{5}$$

We consider the RLL relation in the form

$$R^{a_1a_2}_{b_1b_2}(u)L^{b_1}_{c_1}(u+v)L^{b_2}_{c_2}(v) = L^{a_2}_{b_2}(v)L^{a_1}_{b_1}(u+v)R^{b_1b_2}_{c_1c_2}(u), \tag{6}$$

with the general form of the L operator expanded in the spectral parameter as

$$L^a_b(u) = \sum_{k=0}^{\infty} \frac{(L^{(k)})^a_b}{u^k}\ , \quad L^{(0)} = I. \tag{7}$$

The expansion coefficients $(L^{(k)})^a_b$ generate the Yangian algebra. The relation defining this algebra are encoded in the above RLL relation (6). We study the conditions on $(L^{(k)})^a_b$ arising in the case of truncation of this expansion at $k=2$, i.e. if one imposes $(L^{(k)})^a_b = 0, k > 2$. In the study we change the notation to

$$L(u) = u^2 I + uM + N. \tag{8}$$

For this ansatz (6) implies

$$[K_{12}, M_1 + M_2] = 0, \tag{9}$$

$$[M_1, M_2] + [P_{12} - \varepsilon K_{12}, M_2] = 0, \tag{10}$$

$$[M_1, N_2] + [P_{12} - \varepsilon K_{12}, N_2] = 0, \tag{11}$$

$$K_{12}(-\beta M_2 + M_1M_2 + N_1 + N_2) = (-\beta M_2 + M_2M_1 + N_1 + N_2)K_{12}, \tag{12}$$

$$K_{12}(\beta(N_1 - N_2) + M_1N_2 + N_1M_2) = (\beta(N_1 - N_2) + N_2M_1 + M_2N_1)K_{12}, \tag{13}$$

$$[N_1, N_2] - (P_{12} - \varepsilon K_{12})M_1N_2 - N_2M_1(P_{12} - \varepsilon K_{12}) + \beta\varepsilon[K_{12}, N_2] = 0, \tag{14}$$

$$\beta[N_1, N_2] + \beta(P_{12}M_1N_2 - N_2M_1P_{12}) - \varepsilon(K_{12}N_1N_2 - N_2N_1K_{12}) = 0. \tag{15}$$

The relations are written in terms of (operator valued) matrices in the tensor product of two fundamental representation labeled by subscripts. Thus they can be written in components with upper indices a_1, a_2 and lower indices b_1, b_2, In particular we have

$$(M_1)^{a_1,a_2}_{b_1,b_2} = M^{a_1}_{b_1}\delta^{a_2}_{b_2}, \quad (M_2)^{a_1,a_2}_{b_1,b_2} = \delta^{a_1}_{b_1}M^{a_2}_{b_2},$$

$$(P_{12})^{a_1,a_2}_{b_1,b_2} = \delta^{a_1}_{b_2}\delta^{a_2}_{b_1}, \quad (K_{12})^{a_1,a_2}_{b_1,b_2} = \varepsilon^{a_1,a_2}\varepsilon_{b_1,b_2}.$$

Besides of the notations M_1, M_2 we use also

$$(M_+)^{a_1,a_2}_{b_1,b_2} = M^{a_1a_2}\varepsilon_{b_1b_2}, (M_-)^{a_1,a_2}_{b_1,b_2} = \varepsilon^{a_1,a_2}M_{b_1b_2}. \tag{16}$$

We derive easily the following rules

$$KP = PK = \varepsilon K, K^2 = \varepsilon n K, \tag{17}$$

$$KA_1B_2 = \varepsilon(A^tB)_-, A_1B_2K = \varepsilon(AB^t)_+,$$

$$KB_2A_1 = (B^tA)^t_-, B_2A_1K = (BA^t)^t_+,$$

$$KB_2 = B_-, KA_1 = \varepsilon A^t_-, A_1K = A_+, B_2K = \varepsilon B^t_+,$$

$$A_+B_-P = A_+B^t_-, \ PA_+B_- = A^t_+B_-,$$

$$KA_1B_2K = KSp(AB^t) = \varepsilon KSp(A^tB).$$

The first condition (9) means that the algebra valued matrix M has no graded-symmetric contribution,

$$M - \varepsilon M^t = 0, \quad M = c_1 I + M'.$$

M' denotes the graded antisymmetric part of M. The second relation (10) represents the Lie algebra relations. It implies that the matrix elements of M' are the generators of the Lie algebra and c_1 represents a central extension commuting with the generators. The third relation (11) tells that N transforms by the adjoint action of the Lie algebra and c_1 commutes with N.

The remaining conditions express further constraints on M and N. The existence of further constraints on M implies in particular that the truncation is not allowed for arbitrary Lie algebra representations. Our aim is to identify admissable representations.

In the case of the Yangian based on the $s\ell$ type Lie algebras $Y(s\ell_n)$ no further constraints on M appear by truncation, i.e. $Y(s\ell_n)$ has the evaluation representation, where all higher terms in the expansion (7) with $k > 1$ vanish and the matrix elements $L^{(k)})^a_b M_{ab}$ are mapped into the Lie algebra generators. This is well known and can be checked from the conditions in the above formulation (9 - 15), because they reduce to the $s\ell_n$ case by the substitution $K_{12} \to 0$. Indeed, besides of the Lie algebra relations only the conditions (14, 15) remain which constrain N but not M.

3. The linear evaluation

The effect of truncation can be illustrated by going one further step of truncation, namely imposing $N = 0$ in (9- 15). Besides of the Lie algebra relations only the 4th condition (12) has a non-trivial remainder,

$$K_{12}(-\beta M_2 + M_1 M_2) = (-\beta M_2 + M_2 M_1) K_{12}.$$

It implies the relations for M^2,

$$M'^{\,2} + \beta M' = c_2 I, \tag{18}$$

where $nc_2 = tr(M'^{\,2})$. A representation obeying this constraint is formulated in terms of an undelying algebra generated by $c_a, a = 1, .., n$,

$$[c_a,\, c_b]_\epsilon \;\equiv\; c_a\, c_b + \epsilon\, c_b\, c_a = \varepsilon_{ba} = \epsilon\, \varepsilon_{ab}, \quad c_a\, c^a = \epsilon\; c^a\, c_a = \frac{1}{2}\varepsilon^{ab}. \tag{19}$$

The generators are

$$-M^a_{\;b} \to F^a_{\;b} = \frac{1}{2}(c^a\, c_b - \epsilon\; c_b\, c^a).$$

We find that the constraint (18) is fulfilled,

$$F^a_{\;\,d}\, F^d_{\;\,b} - \beta\, F^a_{\;\,b} = \frac{1}{4}\,(n\epsilon - 1)\,\delta^a_b. \tag{20}$$

In the orthogonal case ($\varepsilon = +1$) this is the spinorial representation and c_a are related to the Dirac Gamma matrices. The relation to the Jordan-Scwinger type representation will be discussed below.

4. The quadratic evaluation

Now we look for solutions of the conditions (9- 15) with $N \neq 0$. We write $M = c_1 I + M'$, $N = c_N I + N' + \tilde{N}$, where M', N' are graded anti-symmetric and $\tilde{N}$ is graded symmetric.

Actually, an important example of a representation resulting in a quadratic u dependence is just the fundamental one. Indeed the expression of the fundamental R matrix (4) can be rewritten in the form of the quadratic ansatz (8) by substituing $M = P - \epsilon K$ and perfoming a shift of u,

$$R(u - \frac{1}{2}\beta) = u^2 I + uM + \frac{1}{2}(M^2 + \beta M) + const\ I.$$

We have here $c_N = 0, N' = 0, M = M'$ and $N = \tilde{N} = \frac{1}{2}(M^2 + \beta M)$. This particular matrix of generators M obeys a cubic relation, since we have

$$(P - \epsilon K)^2 = 1 + \epsilon 2\beta\, K, \quad (P - \epsilon K)^3 = (P - \epsilon K) + (n\epsilon - 1)(n - 2\epsilon)K.$$

It is clear that an example of L quadratic in u is obtained by the product of the L operators of two linear ones. It is instructive to compare with the conditions in this case.

$$L(u) = L_1(u) L_2(u + \delta), \quad L_i(u) = Iu - G_i, \quad G_i^2 - \beta G_i = c_i^{(2)} I.$$

The product is performed in the fundamental representation and the subscript label two copies of representations allowing the first order evaluation, i.e. here the meaning of the subscripts differs from sect 2.

$$L(u) = I(u + \delta) - u(G_1 + G_2) - \delta G_1 + G_1 G_2 = Iu^2 + uM + N,$$

$$-M = G_1 + G_2 - \delta, \quad N = G_1 G_2 - \delta G_1.$$

We assume that G_i are graded-antisymmetric.

$$M = \delta + M', \quad -M' = G_1 + G_2.$$

We separate $G_1 G_2$ into the symmetric and anti-symmetric parts.

$$G_1 G_2 = \frac{1}{2}(G_1 G_2 + G_2 G_1) + \frac{1}{2}(G_1 G_2 - G_2 G_1),$$

$$N' = \frac{1}{2}(G_1 G_2 - G_2 G_1) - \delta G_1, \quad \tilde{N} = \frac{1}{2}(G_1 G_2 + G_2 G_1) = M'^{\,2} + \beta M'.$$

We have to check the consequences of the conditions (9-15) in particular the ones involving the graded anti-symmetric part N' following from (13),

$$2\beta N' = [c_1, c_N]_+ + [c_1, N']_- + [c_1, \tilde{N}]_+ - [c_N, M']_- - [M', N']_+ - [M', \tilde{N}]_-. \tag{21}$$

The second and the fourth term vanish. Further we have

$$[c_1, c_N]_+ + [c_1, \tilde{N}]_+ = -\delta(G_1 G_2 + G_2 G_1),$$

$$[M', \tilde{N}]_- = 0, \qquad [M', N']_+ = -2\beta N' - \delta(G_1 G_2 + G_1 G_2).$$

The consequence (21) is fulfilled.

More examples are obtained by fusion from the YB operators obeying an RLL type relation with the spinorial R matrix and acting in the tensor product of the spinorial and another representsation with generators G. The latter is constrained by the condition

$$[G_{[ae}, G_{f)b}]_+ = 0,$$

equivalent to

$$[G_{ae}, G_{fb}]_+ + [G_{ef}, G_{ab}]_+ + [G_{fa}, G_{eb}]_+ = 0. \tag{22}$$

Multiplication by G results in a cubic polynomial relation for the matrix of generators G [7],

$$G^3 + (\varepsilon - n)G^2 + (\varepsilon n - 2)G + \frac{1}{2}tr(G^2)(I_- \varepsilon G) = 0. \tag{23}$$

The Jordan Schwinger type representations discussed below (case $\eta\varepsilon = -1$) obey the condition (22) and provide more examples of quadratic Yangian evaluation with $N' = 0$. The JS type representations in the case $\eta\varepsilon = +1$ reduce to the spinorial representations allowing for the linear evaluation.

5. Jordan-Schwinger representations in metric formulation

Define n canonical pairs $x_a, \partial_a, a = 1, ..., n$, and the metric ε_{ab},

$$\varepsilon_{ab} = \varepsilon\varepsilon_{ba}, \varepsilon_{ac}\varepsilon^{cb} = \delta_a^{\ b} = \varepsilon\delta^b_{\ a} = \varepsilon\delta^a_{\ b},$$

$$[\partial_a, x^b]_\eta = \delta_a^{\ b}, \ [x^a, x^b]_\eta = 0, \ [\partial_a, \partial_b]_\eta = 0. \tag{24}$$

η distinguishes the normal bosonic case $\eta = -1$ from the fermionic/Grassmann case $\eta = +1$. ε distinguishes the parity of the metric.

We define the elementary canonical transformation preserving the Heisenberg commutation relation with metrics.

$$\mathcal{C}[\partial_a, x^b]_\eta \mathcal{C}^{-1} = \delta_a^{\ b}$$

by

$$\mathcal{C}_a \begin{pmatrix} x_a \\ \partial_a \end{pmatrix} \mathcal{C}_a^{-1} = \begin{pmatrix} \partial_a \\ \eta\varepsilon x_a \end{pmatrix}.$$

This implies in particular for the matrices of two versions of JS $g\ell_n$ generators

$$(L^+)_a^{\ b} = \partial_a x^b, \ \ (L^-)_a^{\ b} = \eta\varepsilon x_a \partial^b \tag{25}$$

the duality relation

$$L^+ = \mathcal{C}^{-1} L^- \mathcal{C}.$$

Further, the scalar product defined as

$$(\mathbf{x}^1 \cdot \mathbf{x}^2) = \sum \varepsilon_{ba} x^{1a} x^{2b} = x_{1a} x^{2a}$$

in terms of the dual coordinates, is invariant in the sense

$$[L_1^+ + L_2^-, (\mathbf{x}^1 \cdot \mathbf{x}^2)] = 0.$$

This distinguishes the linear combination of $g\ell_n$ generators defining the subalgebra so or sp,

$$M_{ab} = x_a \partial_b - \varepsilon x_b \partial_a, \quad L^+ + L^- = I - M. \tag{26}$$

in two respects- invariance with respect to the canonical transformation and annihilation of the scalar product, i.e. upon operation on functions of the coordinates this combination is distinguished by the invariance of the scalar product $(\mathbf{x}^1 \cdot \mathbf{x}^2)$ with both coordinate vectors of the same space.

The generators in this metric form are related to the earlier used non-metric form [8] in the following way. In the sp case with the particular metric choice

$$\varepsilon_{ab} = \text{sign}(a)\delta_{a,-b}, \varepsilon^{ab} = -\text{sign}(a)\delta_{a,-b}, a, b = -m, ..., -1, +1, ... + m, n = 2m$$

the previous expression

$$\mathcal{M}_{ab} = x_a\partial_b - \text{sign}(a)\text{sign}(b)x_{-b}\partial_{-a}, \mathcal{M}_{ba} = -\mathcal{M}_{-a,-b}$$

is to be identified with the metric form

$$\mathcal{M}_{ab} \to M^a_{\ b} = x^a\partial_b - \varepsilon x_b\partial^a,$$

with

$$x_a = \varepsilon_{ac}x^c, \ \partial^a = \partial_c\varepsilon^{ca}, \ \varepsilon = -1.$$

The involved symmetry relation of $\mathcal{M}_{ab}$ is mapped into the simple ordinary symmetry relation obeyed by M_{ab}. In the case so and the metric $\varepsilon_{ab} = \delta_{ab}$ the correspondence is trivial.

We notice that in the case $\varepsilon\eta = 1$

$$\mathcal{C}(\partial_a \pm x_a)\mathcal{C}^{-1} = (\partial_a \pm x_a)$$

On the other hand with the definition

$$c_a = \frac{1}{2}(\partial_a + x_a), \quad \bar{c}_a = \frac{1}{2}(\partial_a - x_a)$$

the canonical commutation relation imply

$$[c_a, c_b]_\eta = \frac{1}{4}(1 + \varepsilon\eta)\varepsilon_{ab}, \ [\bar{c}_a, \bar{c}_b]_\eta = -\frac{1}{4}(1 + \varepsilon\eta)\varepsilon_{ab}, \ [c_a\bar{c}_b]_\eta = -\frac{1}{4}(1 - \varepsilon\eta)\varepsilon_{ab}.$$

Thus at $\varepsilon\eta = 1$ the Heisenberg algebra separates into independent subalgebras of Clifford form. The so/sp JS form of the Lie algebra also separates in this case. In general we have

$$M_{ab} = [c_a, c_b]_{-\varepsilon} - [\bar{c}_a, \bar{c}_b]_{-\varepsilon} + [\bar{c}_a, c_b]_{+\varepsilon} - [c_a, \bar{c}_b]_{+\varepsilon},$$

and in the case $\eta = \varepsilon$ the last two terms vanish. In this way the JS realisation of the so/sp algebra at $\varepsilon\eta = 1$ appears in two independent subalgebras,

$$M_{ab} = F_{ab} - \bar{F}_{ab}, F_{ab} = [c_a, c_b]_{-\eta}, [F_{ab}, \bar{F}_{cd}]_- = 0.$$

Above we have pointed out that the matrix with the elements F_{ab} obeys the restriction of the linear evaluation (20).

We calculate the squares of the JS generator matrices L^+, L^-.

$$L^{+\ 2} = L^+((\mathbf{xp}) + 1), \quad L^{-\ 2} = L^-(-(\mathbf{xp}) + 1 + \varepsilon\eta n)$$

We calculate also the products.

$$(L^+L^-)_{ab} = \varepsilon\eta\mathbf{x}^2\partial_a\partial_b + (1 - \varepsilon\eta)L^-_{ab}, \quad (L^-L^+)_{ab} = \varepsilon\eta\partial^2 x_a x_b + (1 - \varepsilon\eta)L^+_{ab}.$$

We recall (26) that the sum $L^+ + L^-$ contains the generators of the so/sp subalgebra, $L^+ + L^- = I - M$, and consider its square.

$$(L^+ + L^-)^2 = [(x\partial) + 2 - \eta\varepsilon]L^+ + [\eta\varepsilon n + 2 - \eta\varepsilon - (x\partial)]L^- + \varepsilon\eta[\mathbf{x}^2\partial_a\partial_b + \partial^2 x_a x_b]. \qquad (27)$$

The last term vanishes at $\eta\varepsilon = +1$. Now we proceed with the case $\eta\varepsilon = -1$. $L^+ + L^-$ commutes with $\mathbf{x}^2$ and ∂^2.

$$L^+_{ac}\partial^c\partial_b = [(x\partial)+1]\partial_a\partial_b = \partial_a\partial_b[(x\partial)-1],\ L^-_{ac}\partial^c\partial_b = \varepsilon\eta x_a\partial_b\partial^2,$$
$$L^+_{ac}x^c x_b = \partial_a x_b\mathbf{x}^2,\ L^-_{ac}x^c x_b = x_a x_b[\eta\varepsilon n - 1 - (x\partial)].$$

This is used in the calculation of the third power.

$$(L^+ + L^-)^3 = [(\mathbf{xp})+3][(\mathbf{xp})+1]L^+ + [(\mathbf{xp})+3][-\mathbf{p}^2 x_a x_b + 2L^+]+$$
$$[-n+3-(\mathbf{xp})][-n+1-(\mathbf{xp})]L^- + [-n+3-(\mathbf{xp})][-\mathbf{x}^2\partial_a\partial_b + 2L^-]-$$
$$-\mathbf{x}^2\partial_a\partial_b[(\mathbf{xp})-1] + \mathbf{x}^2 x_a\partial_b\mathbf{p}^2 - \mathbf{p}^2\partial_a x_b\mathbf{x}^2 + \mathbf{p}^2 x_a x_b[n+1+(\mathbf{xp})].$$

We have

$$\mathbf{x}^2 x_a\partial_b\mathbf{p}^2 = -\mathbf{x}^2\mathbf{p}^2 L^-_{ab} - 2\mathbf{x}^2\partial_a\partial_b,\quad -\mathbf{p}^2\partial_a\mathbf{x}^2 x_b = -\partial^2\mathbf{x}^2 L^+_{ab} - 2\mathbf{p}^2 x_a x_b$$

This results in

$$(L^+ + L^-)^3 = [(\mathbf{xp})+3)^2 - \mathbf{p}^2\mathbf{x}^2]L^+ + [(n+(\mathbf{xp})-3)^2 - \mathbf{x}^2\mathbf{p}^2]L^- + (n-4)[\mathbf{p}^2 x_a x_b + \mathbf{x}^2\partial_a\partial_b].$$

We use the result (27) for $(L^+ + L^-)^2$ to substitute the last term and obtain

$$(L^+ + L^-)^3 = -(n-1)(L^+ + L^-)^2 + A_+ L^+ + A_- L^-$$

The coefficients of L^+, L^- are equal

$$A_+ = A_- = (x\partial)^2 + (n-2)(x\partial) + n - 3 - \mathbf{x}^2\mathbf{p}^2,$$

where we have used $[\partial^2, \mathbf{x}^2] = 2n + 4(x\partial)$. We insert now the expression for the trace

$$Sp(L^+ + L^-)^2 = 2(\mathbf{xp})^2 + 2(n-2)(\mathbf{xp}) - 2\mathbf{x}^2\partial^2 - n$$

and obtain the polynomial relation in the matrix $(L^+ + L^-)$

$$(L^+ + L^-)^3 + (n-4)(L^+ + L^-)^2 - \frac{1}{2}[Sp(L^+ + L^-)^2 + n - 6)](L^+ + L^-) = 0 \tag{28}$$

For comparison we write $L^+ + L^- = I - M$, (26), and obtain

$$M^3 - (n-1)M^2 - (\frac{1}{2}SpM^2 - n + 2)M + \frac{1}{2}SpM^2\ I. = 0 \tag{29}$$

The resulting polynomial relation for the matrix of generators M compares with the relation for the JS generators G (23) with the substitutions

$$M \to \varepsilon G,\quad n \to \varepsilon n,\quad SpM^2 \to \varepsilon trG^2.$$

We use the relation for the graded anti-symmetric part of M^3 which follows from the Lie algebra relations,

$$M^3 - \varepsilon M^{3t} = 2M^3 - (n-1)(M^2 + \varepsilon M^{2t}) + ISp(M^2),$$

to rewrite the obtained condition as

$$M^3 - \varepsilon M^{3t} - Sp(M^2)M = 0. \tag{30}$$

It shows that the polynomial condition constrains the graded anti-symmetric part of the third power. The graded symmetric part of M^3 can be written in terms of the power powers merely on the basis of the Lie algebra relations.

The calculation of $(L^+ + L^-)^3$ can be repeated in the case $\varepsilon\eta = +1$.

$$(L^+ + L^-)^3 = [(\mathbf{x}\partial)+1]^2 L^+ + [n - (\mathbf{x}\partial) + 1]^2 L^- =$$
$$-\frac{1}{2}[\varepsilon Sp(L^+ + L^-)^2 - n - 2](L^+ + L^-) + (2+n)(L^+ - L^-) + \frac{1}{2}n(n+2)[(L^+ + L^-) - (L^+ - L^-)].$$

In this case there is no analogous polynomial relation in terms of only $(L^+ + L^-)$.

6. Summary

We have discussed how the truncation of the expansion of $L(u)$ in inverse powers of u constrains the Lie algebra representations. We have formulated the condition of second order truncation and investigated their consequences.

The condition of first order evaluation can be formulated as the vanishing of a second order polynomial in the matrix of generators M. It expresses just the vanishing of the graded-symmetric part of M^2. In the second order evaluation this constraint is lifted, instead just this polynomial determines the graded-symmetric part of the second term N. The vanishing of a cubic polynomial in M is a consequence of the conditions in the case of the vanishing of the graded anti-symmetric part of N. It constrains the graded anti-symmetric part of M^3 to be proportional to M.

We have analysed the representations of Jordan-Schwinger type and shown how they result in examples of the first and second order evalations of the orthogonal and symplectic Yangian.

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