

Orthogonal and symplectic Yangians: oscillator realization

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Abstract. We solve RLL-relation with fundamental orthogonal/symplectic R -matrix examining polynomial with respect to u expressions for L -operator. We use creation-annihilation operators to construct a realization of Yangian generators.

1. Introduction

The simplest and well known mechanism of appearance the projectivity of unitary representations is related to Heisenberg group. Upon passing to central extension in simplest quantization models the Planck constant appears as a measure of the central charge.

In infinite-dimensional case the projective representations appear as the groups of automorphisms of canonical (anti-)commutation relations, i.e. Metaplectic and Metagonal groups.

In other words, if the given group of automorphisms of some algebra (say, Clifford algebra) or group (say, Heisenberg group) maps the Fock representation (fermionic or bosonic) to itself, then in that representation space appears, generally speaking, projective representation of that group of automorphisms.

In finite-dimensional case the projective representation of the orthogonal group by automorphisms of Clifford algebra leads Cartan in 1913 to discovery of spinor group, which provides the twofold covering of the orthogonal group. Later A. Weil shown that S^1 -covering of the symplectic group is reduced to the \mathbb{Z}_2 one [1]. The similarly S^1 -extension of the orthogonal group is Metagonal group. It can be illustrated by the following scheme:

$$MO(2n, \mathbb{R}) \supset Spin(2n, \mathbb{R}) \longrightarrow SO(2n\mathbb{R}),$$

$$Mp(n, \mathbb{R}) \supset Sp_2(n, \mathbb{R}) \longrightarrow Sp(n\mathbb{R}),$$

here the Metagonal group $MO(2n, \mathbb{R})$ includes $Spin(2n, \mathbb{R})$ as S^1 covering, which is in turn is \mathbb{Z}_2 covering of $SO(2n\mathbb{R})$. Similarly the Metaplectic group contains *Spinsymplectic* $Sp_2(n, \mathbb{R})$ and symplectic $Sp(n\mathbb{R})$ groups [2], [3].

So one can conclude from these considerations that the algebra of fermion and boson oscillators is naturally related to orthogonal and symplectic symmetry.



2. Yang-Baxter relations

In this work we shall follow to notations of our previous work.

The fundamental Yang-Baxter equation:

$$\begin{aligned} R_{b_1 b_2}^{a_1 a_2}(u) R_{c_1 b_3}^{b_1 a_3}(u+v) R_{c_2 c_3}^{b_2 b_3}(v) &= R_{b_2 b_3}^{a_2 a_3}(v) R_{b_1 c_3}^{a_1 b_3}(u+v) R_{c_1 c_2}^{b_1 b_2}(u) \Rightarrow \\ R_{12}(u) R_{13}(u+v) R_{23}(v) &= R_{23}(v) R_{13}(u+v) R_{12}(u). \end{aligned} \quad (1)$$

has following solution:

$$R_{b_1 b_2}^{a_1 a_2}(u) = u(u + \frac{n}{2} - \epsilon) I_{b_1 b_2}^{a_1 a_2} + (u + \frac{n}{2} - \epsilon) P_{b_1 b_2}^{a_1 a_2} - \epsilon u K_{b_1 b_2}^{a_1 a_2}, \quad (2)$$

where

$$I_{b_1 b_2}^{a_1 a_2} = \delta_{b_1}^{a_1} \delta_{b_2}^{a_2}, \quad P_{b_1 b_2}^{a_1 a_2} = \delta_{b_2}^{a_1} \delta_{b_1}^{a_2}, \quad K_{b_1 b_2}^{a_1 a_2} = \epsilon^{a_1 a_2} \epsilon_{b_1 b_2}, \quad (3)$$

and the choices $\epsilon = +1$ and $\epsilon = -1$ correspond to the $SO(n)$ and $Sp(n)$ cases respectively. We note that the R -matrix (2) is invariant under the adjoint action of any real form (related to the metric ϵ^{ab}) of the complex groups $SO(n, \mathbb{C})$ and $Sp(n, \mathbb{C})$. Here ϵ_{ab} is a non-degenerate invariant metric in V

$$\epsilon_{ab} = \epsilon \epsilon_{ba}, \quad \epsilon_{ab} \epsilon^{bd} = \delta_a^d, \quad (4)$$

which is symmetric $\epsilon = +1$ for $SO(n)$ case and skew-symmetric $\epsilon = -1$ for $Sp(n)$ case. We denote by ϵ^{bd} (with upper indices) the elements of the inverse matrix ϵ^{-1} . Namely the existence of the invariant tensors ϵ_{ab} leads to the above mentioned problems in $SO(n)$ and $Sp(n)$ cases, e.g., it causes a third term in the corresponding expressions of R -matrices and leads to the dependence on the spectral parameter of second power.

Let the index range be $a_1, a_2, \dots = 1, \dots, n$ for the $SO(n)$ case and $a_1, a_2, \dots = -m, \dots, -1, 1, \dots, m$ for the $Sp(n)$ ($n = 2m$) case. For the choice $\epsilon^{a_1 a_2} = \delta^{a_1 a_2}$ in the $SO(n)$ case we have

$$R_{b_1 b_2}^{a_1 a_2}(u) = u(u + \beta) \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} + (u + \beta) \delta_{b_2}^{a_1} \delta_{b_1}^{a_2} - u \delta^{a_1 a_2} \delta_{b_1 b_2}, \quad \beta = (n/2 - 1), \quad (5)$$

and for the choice $\epsilon^{a_1 a_2} = \epsilon_{a_2} \delta^{a_1, -a_2}$ (here $\epsilon_a = \text{sign}(a)$ and $\epsilon^{ab} = -\epsilon_{ab}$) in the $Sp(2m)$ case we have

$$R_{b_1 b_2}^{a_1 a_2}(u) = u(u + \beta) \delta_{b_1}^{a_1} \delta_{b_2}^{a_2} + (u + \beta) \delta_{b_2}^{a_1} \delta_{b_1}^{a_2} - u \epsilon_{a_2} \epsilon_{b_2} \delta^{a_1, -a_2} \delta^{b_1, -b_2}, \quad \beta = (m + 1). \quad (6)$$

3. Linear resolution of $gl(n)$ Yangian

The simplest form fundamental R -matrix acquires in the case of $gl(n)$ symmetry:

$$R_{12}(u) = u I_{12} + P_{12}, \quad (7)$$

and choosing linear ansatz for L -operator: $L(u) = u + G$ one deduces that RLL-relation:

$$R_{12}(u) L_1(u+v) L_2(v) = L_2(v) L_1(u+v) R_{12}(u),$$

consists of contribution, proportional to u :

$$[G_1 + P_{12}, G_2] = 0 = [G_1, G_2 + P_{12}]. \quad (8)$$

Rewriting this relation in matrix indices:

$$[G^{a_1}_{c_1}, G^{a_1}_{c_1}] = \delta^{a_1}_{c_2} G^{a_2}_{c_1} - \delta^{a_2}_{c_1} G^{a_1}_{c_2}, \quad (9)$$

one immediately recognizes $gl(n)$ -algebra.

So in the case of $gl(n)$ symmetry the linear ansatz for L -operator solves RLL-relations with fundamental R -matrix upon restriction that generators G realize $gl(n)$ -algebra.

4. Truncated Yangians of so and sp types

Let \mathcal{G} be the Lie algebra $so(n)$ or $sp(2m)$ ($2m = n$). Then the corresponding Yangian $Y(\mathcal{G})$ is defined [10] as an associative algebra with the infinite number of generators $(L^{(k)})_b^a$ with $(L^{(0)})_b^a = I\delta_b^a$, where I is a unit element in $Y(\mathcal{G})$, and $(L^{(k)})_b^a$ for $k > 0$ satisfy the quadratic defining relations. The generators $(L^{(k)})_b^a \in Y(\mathcal{G})$ are considered as coefficients in the expansion of the L -operator

$$L_b^a(u) = \sum_{k=0}^{\infty} \frac{(L^{(k)})_b^a}{u^k}, \quad L^{(0)} = I, \quad (10)$$

which satisfies to the Yang-Baxter RLL-relations

$$\begin{aligned} R_{b_1 b_2}^{a_1 a_2}(u-v) L_{c_1}^{b_1}(u) L_{c_2}^{b_2}(v) &= L_{b_2}^{a_2}(v) L_{b_1}^{a_1}(u) R_{c_1 c_2}^{b_1 b_2}(u-v) \Leftrightarrow \\ R_{12}(u-v) L_1(u) L_2(v) &= L_2(v) L_1(u) R_{12}(u-v). \end{aligned} \quad (11)$$

which play role of the Yangin $Y(\mathcal{G})$ defining relations.

Here $R_{b_1 b_2}^{a_1 a_2}(u-v)$ is the Yang-Baxter R -matrix (2).

The defining relations (11) are homogeneous in L and allow the redefinition $L(u) \rightarrow f(u) L(u + b_0)$ with any scalar function $f(u) = 1 + b_1/u + b_2/u^2 + \dots$, where b_i are parameters. The Yangian (2), (11) possesses the set of automorphisms

$$L(u) \rightarrow \frac{(u-a)^k}{u^k} L(u), \quad (k = 1, 2, \dots),$$

where a is a constant (in general a is a central element in $Y(\mathcal{G})$). At $k = 1$ the generators $L^{(j)}$ are transforming as

$$L^{(1)} \rightarrow L^{(1)} - aI_n, \quad L^{(2)} \rightarrow L^{(2)} - aL^{(1)}, \quad L^{(3)} \rightarrow L^{(3)} - aL^{(2)}, \quad \dots, \quad (12)$$

Taking $a = \frac{1}{n} \text{Tr}(L^{(1)})$ one can fix $L^{(1)}$ to be traceless. Note that $\text{Tr}(L^{(1)})$ is central element in $Y(\mathcal{G})$.

One has for the fundamental R -matrix (2):

$$\frac{1}{u^2 v^2} R(u-v) = \left(\frac{1}{v} - \frac{1}{u}\right) \left(\frac{1}{v} - \frac{1}{u} + \frac{\beta}{uv}\right) - \left(\frac{1}{uv^2} - \frac{1}{u^2 v} + \frac{\beta}{u^2 v^2}\right) P - \epsilon \left(\frac{1}{uv^2} - \frac{1}{u^2 v}\right) K. \quad (13)$$

here $\beta = (\frac{n}{2} - \epsilon)$. Then the defining relations for the generators $(L^{(k)})_b^a$ of the Yangians $Y(\mathcal{G})$:

$$\begin{aligned} &[L_1^{(k)}, L_2^{(j-2)}] - 2[L_1^{(k-1)}, L_2^{(j-1)}] + [L_1^{(k-2)}, L_2^{(j)}] + \\ &\quad + \beta([L_1^{(k-1)}, L_2^{(j-2)}] - [L_1^{(k-2)}, L_2^{(j-1)}]) + \\ &+ P \left(L_1^{(k-1)} L_2^{(j-2)} - L_1^{(k-2)} L_2^{(j-1)} + \beta L_1^{(k-2)} L_2^{(j-2)} \right) - \\ &\quad - \left(L_2^{(j-2)} L_1^{(k-1)} - L_2^{(j-1)} L_1^{(k-2)} + \beta L_2^{(j-2)} L_1^{(k-2)} \right) P + \\ &+ \epsilon \left(K (L_1^{(k-2)} L_2^{(j-1)} - L_1^{(k-1)} L_2^{(j-2)}) - (L_2^{(j-1)} L_1^{(k-2)} - L_2^{(j-2)} L_1^{(k-1)}) K \right) = 0, \end{aligned} \quad (14)$$

where the operators K, P are given in (3), $\epsilon = +1$ for $\mathcal{G} = so(n)$ and $\epsilon = -1$ for $\mathcal{G} = sp(2m)$.

The defining relations are obviously completed for infinite set of generators, but it can be truncated at finite values as well, which will impose the additional restrictions on generators $(L^{(k)})_b^a$

At $k = 1$ we obtain from (14) the set of relations

$$[L_1^{(1)}, L_2^{(j-2)}] = - \left[(P_{12} - \epsilon K_{12}), L_2^{(j-2)} \right], \quad (\forall j), \quad (15)$$

which in particular lead to the statement that $\text{Tr}(L^{(1)})$ is a central element in $Y(\mathcal{G})$: $[\text{Tr}(L^{(1)}), (L^{(j)})^a_b] = 0$ ($\forall j$). For $j = 3$ we deduce from (15) the defining relations for the Lie algebra generators $G^a_b \equiv -(L^{(1)})^a_b$:

$$[G_1, G_2] = [(P_{12} - \epsilon K_{12}), G_2]. \quad (16)$$

Permutation of indices $1 \leftrightarrow 2$ in this equation gives the consistency conditions and the same conditions are obtained from (14) directly.

$$K_{12} (G_1 + G_2) = (G_1 + G_2) K_{12},$$

Acting on this equation by K_{12} from the left (or by K_{12} from the right) we write it as

$$K_{12} (G_1 + G_2) = \frac{2}{n} \text{Tr}(G) K_{12} = (G_1 + G_2) K_{12}, \quad (17)$$

where we have used

$$K_{12}^2 = \epsilon n K_{12}, \quad K_{12} N_1 K_{12} = K_{12} N_2 K_{12} = \epsilon \text{Tr}(N) K_{12}, \quad (18)$$

Here N is any $n \times n$ matrix.

Then, according to the automorphism (12) we redefine the elements $G \rightarrow G - \frac{1}{n} \text{Tr}(G)$ in such a way that for the new generators we have $\text{Tr}(G) = 0$. This leads to the (anti)symmetry conditions for the generators:

$$K_{12} (G_1 + G_2) = 0 = (G_1 + G_2) K_{12} \quad \Rightarrow \quad G^d_a \varepsilon_{db} + \varepsilon_{ad} G^d_b = 0. \quad (19)$$

The equations (16) and (19) for $\epsilon = +1$ and $\epsilon = -1$ define the Lie algebra $\mathcal{G} = so(n)$ and $\mathcal{G} = sp(2m)$ ($2m = n$), respectively. The defining relations (16) and (anti)symmetry condition (19) for the generators $G_{ab} = \varepsilon_{ad} G^d_b$ can be written in the familiar form

$$[G_{ab}, G_{cd}] = \varepsilon_{cb} G_{ad} + \varepsilon_{db} G_{ca} + \varepsilon_{ca} G_{db} + \varepsilon_{da} G_{bc}, \quad G_{ab} = -\epsilon G_{ba}. \quad (20)$$

This means (see [10]) that an enveloping algebra $\mathcal{U}(\mathcal{G})$ of the Lie algebra $\mathcal{G} = so(n), sp(2m)$ is always a subalgebra in the Yangian $Y(\mathcal{G})$.

5. L operators

The Yangian $Y(\mathcal{G})$ (14) with infinite number of generators can be truncated to some set with finite number of ones restricted to some special representation. We will suppose that the Yangian representation space is realized as a space of (matrix) polynomials of So (Sp)-generators G and start with consideration of some general properties of such monomials. Then, thinking generators G and spectral parameter as a dimensionfull variables (their ratio and L -operator are dimensionless) one will describe the Yangian as \mathbb{Z} graded space with finite-dimensional components corresponding to irreducible representations.

1. Linear evaluation of $Y(\mathcal{G})$.

We put equal to zero all generators $L^{(k)} \in Y(\mathcal{G})$ with $k > 1$ and take the L -operator linear by spectral parameter:

$$L_b^a(u) = u\delta_b^a + G_b^a. \quad (21)$$

Then defining RLL-relation takes the form:

$$\begin{aligned} & (u(u + \beta)I_{12} + (u + \beta)P_{12} - u\epsilon K_{12})(u + v + G_1)(v + G_2) = \\ & = (v + G_2)(u + v + G_1)(u(u + \beta)I_{12} + (u + \beta)P_{12} - u\epsilon K_{12}). \end{aligned}$$

This relation can be rewritten as follows:

$$\begin{aligned} & (u + \beta)\left([G_1, G_2] + (G_1 - G_2)P_{12} - \epsilon[K_{12}, G_2]\right) - \epsilon v[K_{12}, G_1 + G_2] - \\ & - \epsilon K_{12}(G_1 - \beta)G_2 + \epsilon G_2(G_1 - \beta)K_{12} = 0, \end{aligned}$$

and has to take place identically by powers of u and v , implying three restrictions on generators G :

$$-v\mathfrak{C}^{(1,1)} = -\epsilon v[K_{12}, G_1 + G_2] = 0, \quad (22)$$

as a coefficient at v , which is takes place upon (22),

$$(u + \beta)\mathfrak{C}^{(1,2)} = (u + \beta)\left([G_1, G_2] + (G_1 - G_2)P_{12} - \epsilon[K_{12}, G_2]\right) = 0, \quad (23)$$

as a coefficient at u and expresses the SO (Sp)-algebra relations. The remaining terms are:

$$-\mathfrak{C}^{(1,3)} = -\epsilon\left(K_{12}(G_1 - \beta)G_2 - G_2(G_1 - \beta)K_{12}\right) = 0. \quad (24)$$

One can summarize constraints following from the Yang-Baxter RLL-relation for linear ansatz (21) as:

$$\mathfrak{C}^{(1,1)}[\epsilon K_{12}, G_1 + G_2] = 0, \quad (25)$$

$$\mathfrak{C}^{(1,2)} = [G_1 + P_{12} - \epsilon K_{12}, G_2] = 0, \quad (26)$$

$$\mathfrak{C}^{(1,3)} = [\epsilon K_{12}, G_1 G_2 + G_2 G_1] = 0. \quad (27)$$

Introducing graded oscillators $c_a = \varepsilon_{ab}c^b$:

$$c^a c^b + \epsilon c^b c^a = \varepsilon^{ab}, \quad c_a c^b + \epsilon c^b c_a = \delta_a^b, \quad c_a c_b + \epsilon c_b c_a = \varepsilon_{ba}, \quad (28)$$

in orthogonal case ($\epsilon = +1$) operators c_a will have Grassmann nature and representation is finite-dimensional, while in symplectic case one has the ordinary oscillators and an infinite-dimensional representation.

Note, that (28) is just "half" of set of oscillators. Indeed, the algebra of automorphisms, corresponding to Bogolyubov transformation of the set of Fermi (Bose) creation-annihilation operators is $O(2n)$ ($Sp(2n)$).

Then one can solve RLL-constraints by simple bilinear ansatz

$$G^{(1)a}_b = -c^a c_b + \frac{\epsilon}{2}\delta_b^a = \frac{\epsilon}{2}(c_b c^a - \epsilon c^a c_b) = \epsilon(c_b c^a - \frac{1}{2}\delta_b^a), \quad (29)$$

ensuring symmetry properties of G : expressions with lower and upper indices are $G^{(1)}_{ab} = -c_a c_b + \frac{\epsilon}{2}\varepsilon_{ab}$, $G^{(1)ab} = \epsilon(-c^a c^b + \frac{1}{2}\varepsilon^{ab})$. The overall coefficient is chosen according to (26).

Note that the metric is the unique invariant tensor and the single invariant combination:

$$\varepsilon^{ab}c_ac_b = \frac{1}{2}\varepsilon^{ab}(c_ac_b + \varepsilon c_b c_a) = \frac{n}{2}, \quad (30)$$

can be constructed in the case of one set of oscillators.

2. Quadratic evaluation of $Y(\mathcal{G})$.

Now we put all generators $L^{(k)} \in Y(\mathcal{G})$ with $k > 2$ equal to zero. In this case the L -operator (10), after a multiplication by u^2 , can be written in the form

$$L(u) = u(u - a) + uG' + H. \quad (31)$$

where we introduce

$$L^{(1)} = G' - a = G, \quad L^{(2)} = H, \quad (32)$$

where the constant a is chosen to be $\text{Tr}G'$, so the generator G is traceless.

We are going to consider the general quadratic ansatz

$$L(u) = u(u - a) + uG' + H = u^2 + uG + H, \quad (33)$$

and to solve the restrictions imposed by RLL Yang-Baxter relation.

The defining relation (11) has the form:

$$[u(u + \beta)I_{12} + (u + \beta)P_{12} - \varepsilon uK_{12}]((u + v)^2 + (u + v)G_1 + H_1)(v^2 + vG_2 + H_2) - \quad (34)$$

$$(v^2 - vG_2 + H_2)((u + v)^2 - (u + v)G_1 + H_1)[u(u + \beta)I_{12} + (u + \beta)P_{12} - \varepsilon uK_{12}] = 0.$$

This relation must take place at arbitrary values spectral parameters u and v , i.e. the coefficients at independent monomials $u^k v^r$ ($k + r \leq 4$) must vanish. The l.h.s. of (34) can be represented as a sum of following eight combinations:

$$\mathfrak{C}^{(2,1)} = \varepsilon[K_{12}, G_1 + G_2] = 0, \quad (35)$$

$$\mathfrak{C}^{(2,2)} = [G_1 + P_{12} - \varepsilon K_{12}, G_2] = [G_1, G_2 - P_{12} - \varepsilon K_{12}] = 0, \quad (36)$$

$$\mathfrak{C}^{(2,3)} = [K_{12}, H_1 - \frac{1}{2}(G_1^2 + G_2^2) + H_2] = 0, \quad (37)$$

$$\mathfrak{C}^{(2,4)} = [G_1 + P_{12} - \varepsilon K_{12}, H_2] = 0, \quad (38)$$

$$\mathfrak{C}^{(2,5)} = [H_1, G_2 - P_{12} + \varepsilon K_{12}] = 0, \quad (39)$$

$$\mathfrak{C}^{(2,6)} = [K_{12}, H_1 G_2 + H_2 G_1 + G_1 H_2 + G_2 H_1] = [K_{12}, \{H_1, G_2\} + \{H_2, G_1\}] = 0. \quad (40)$$

$$\mathfrak{C}^{(2,7)} = \left([H_1, H_2] + (G_2 H_1 - H_2 G_1) P_{12} - \frac{\varepsilon}{2} [K_{12}, \{G_1, H_2 - \frac{\varepsilon}{2} G_2\}] \right) = 0, \quad (41)$$

$$\mathfrak{C}^{(2,8)} = \varepsilon \left(K_{12} (H_1 - \beta G_1 + \beta^2) H_2 - H_2 (H_1 - \beta G_1 + \beta^2) K_{12} \right) = 0. \quad (42)$$

Let us now turn to oscillator realization of generators G and H .

Consider first the case of one set of oscillators. As we prove above the most general expression for G is given then by (29), for which constraints (35-36) take place. Then equation (35) allows to determine ε -symmetric traceless part of H , which is zero, because the only ε -symmetric tensor of second rank constructed from oscillators (28) is metric tensor. Then due to identity:

$$(G^{(1)})_{ab}^2 + \beta G^{(2)}_{ab} = \frac{1}{4}(n\varepsilon - 1)\varepsilon_{ab}, \quad (43)$$

one deduces:

$$H_{ab}^{(1)} = \alpha G_{ab}^{(1)} + \gamma \varepsilon_{ab},$$

where α and γ are arbitrary constants. In other words $H_{ab}^{(1)}$ is an arbitrary tensor, constructed from oscillators (28). Substituting $H^{(1)}$ and $G^{(1)}$ into remaining constraints one obtains identity at $\gamma = -\alpha^2$ and the quadratic ansatz for L -operator will take the form:

$$L(u) = u^2 + (u + \alpha)G^{(1)} - \alpha^2 = (u + \alpha)(u - \alpha + G),$$

which is just shifted linear ansatz up to the overall factor and one comes to:

Proposition *Operator dependence of RLL-relations on single set of oscillators leads to the linear solution.*

Let us now turn to the case of two sets of oscillators.

Oscillators have the natural grading (28). From the other hand in our previous work mentioned that a particular solution corresponding to quadratic ansatz is given by Jordan-Schwinger construction:

$$G_{ab} = x_b \partial_a - \varepsilon x_a \partial_b, \quad (44)$$

where the variables x_a have an opposite grading (even for So - and odd for Sp -case):

$$\partial_a \partial_b - \varepsilon \partial_b \partial_a = 0, \quad x_a x_b - \varepsilon x_b x_a = 0, \quad \partial_a x_b - \varepsilon x_b \partial_a = \varepsilon_{ab}. \quad (45)$$

Labeling oscillators by index $\alpha = 1, 2$:

$$d_a^1 = x_a, \quad d_a^2 = \partial_a, \quad (46)$$

one can rewrite commutation relations (??) in more compact form:

$$d_a^\alpha d_b^\beta - \varepsilon d_b^\beta d_a^\alpha = -\varepsilon^{\alpha\beta} \varepsilon_{ab}, \quad (47)$$

where $\varepsilon^{\alpha\beta}$ is two-dimensional Levi-Civita tensor ($\varepsilon^{12} = 1 = -\varepsilon^{21}$, $\varepsilon^{11} = 0 = \varepsilon^{22}$).

The bilinear ansatz (44) is compatible with cyclic identity:

$$\{G_{ab}, G_{cd}\} + \{G_{ca}, G_{bd}\} + \{G_{bc}, G_{ad}\} = 0, \quad (48)$$

upon substitution $G_{ab} = G_{ab}^{(0)}$ relation (48) becomes identity. The identity (48) is necessary to satisfy the constraints (35-42) following from the RLL-relation.

Then it is not hard to check that along with $G_{c_1 c_2}^{(0)} = \frac{1}{2} \varepsilon_{\alpha\beta} (d_{c_2}^\alpha d_{c_1}^\beta - \varepsilon d_{c_1}^\alpha d_{c_2}^\beta)$ the relation:

$$H = G^2 + \beta G, \quad (49)$$

solves all constraints and provides the quadratic resolution through oscillators (47).

6. Oscillator realization

Let us consider the most general expression for G_{ab} depending on oscillators (28):

$$G_{ab} = \sum_{k=0}^{\infty} G_{ab}^{(k)} = \sum_{k=0}^{\infty} \sum_{e_1, \dots, e_k} A_{ab, e_1 \dots e_k} c^{e_1} \dots c^{e_k}, \quad (50)$$

and try to solve constraints (25-27). In this expansion the operator G_{ab} and the coefficients $A_{ab, e_1 \dots e_k}$ is in one-to-one correspondence if the summation in (50) is ordered, say normally,

when all creation operators c_a with $a > 0$ stand from the left of annihilation ones c_a with $a < 0$. This restriction on summation rule can be lifted if we redefine coefficients A to be ε -antisymmetric:

$$A_{ab,e_1\dots e_i\dots e_j\dots e_k} = -\varepsilon A_{ab,e_1\dots e_j\dots e_i\dots e_k}. \quad (51)$$

Indeed, any tensor $A_{\dots e_i\dots e_j\dots}$ can be represented as the sum of ε -symmetric and ε -antisymmetric parts:

$$A_{\dots e_i\dots e_j\dots} = \frac{1}{2}(A_{\dots e_i\dots e_j\dots} + \varepsilon A_{\dots e_j\dots e_i\dots}) + \frac{1}{2}(A_{\dots e_i\dots e_j\dots} - \varepsilon A_{\dots e_j\dots e_i\dots}) = A'_{\dots e_i\dots e_j\dots} + A''_{\dots e_i\dots e_j\dots}, \quad (52)$$

and upon contraction with oscillators the symmetric part becomes reducible

$$A'_{\dots e_i\dots e_j\dots} c^{e_i} c^{e_j} = \frac{1}{2}(A_{\dots e_i\dots e_j\dots} + \varepsilon A_{\dots e_j\dots e_i\dots}) c^{e_i} c^{e_j} = \frac{1}{2} A_{\dots e_i\dots e_j\dots} \varepsilon^{e_i e_j}.$$

So without the loss of generality one can suppose coefficients $A_{ab,e_1\dots e_i\dots e_j\dots e_k}$ ε -antisymmetric and traceless by indices e_i :

$$A_{ab,e_1\dots e_i\dots e_j\dots e_k} \varepsilon^{e_i e_j} = 0. \quad (53)$$

In the first relation:

$$G_{c_1 c_2} + \varepsilon G_{c_2 c_1} = G^b{}_b \varepsilon_{c_1 c_2}, \quad (54)$$

generator G can be chosen traceless without loss of generality. Then substituting expansion (50) into (54) one obtains:

$$A_{c_1 c_2, e_1 \dots e_k} + \varepsilon A_{c_2 c_1, e_1 \dots e_k} = 0, \quad \Rightarrow \quad \varepsilon^{ab} A_{ab, e_1 \dots e_k} = 0. \quad (55)$$

The first two indices a, b are different from the remaining ones $e_1 \dots e_k$. Indeed, generator $G^a{}_b$ should rotate tensor quantities, which determine the possible form of corresponding commutator. In particular, the construction (29) can serve as generator, so we start with the simplest quadratic ansatz (29) and examine the most general expression quadratic by oscillators:

$$[c^{a_1} c_{c_1} - \frac{\varepsilon}{2} \delta_{c_1}^{a_1}, (G^{(k)})^{a_2}{}_{c_2}]. \quad (56)$$

The commutator with $A^a{}_b$ zeroth term of expansion (50) vanishes, because these are c-numbers, not operators. Similarly the first term is vanishes too, because we have no any chosen vector. For quadratic term of expansion one has:

$$\begin{aligned} [c^{a_1} c_{c_1} - \frac{\varepsilon}{2} \delta_{c_1}^{a_1}, (G^{(1)})^{a_2}{}_{c_2}] &= [c^{a_1} c_{c_1} - \frac{\varepsilon}{2} \delta_{c_1}^{a_1}, A^{a_2}{}_{c_2, bd} c^b c^d] = A^{a_2}{}_{c_2, c_1 d} c^{a_1} c^d - A^{a_2}{}_{c_2, bd} \varepsilon^{ba_1} c_{c_1} c^d + \\ &+ A^{a_2}{}_{c_2, bc_1} c^b c^{a_1} - A^{a_2}{}_{c_2, bd} \varepsilon^{da_1} c^b c_{c_1} = 2A^{a_2}{}_{c_2, ed} (\varepsilon^{ea_1} \varepsilon_{c_1 b} - \delta_{c_1}^e \delta_b^{a_1}) c^b c^d = \\ &= (\delta_{c_2}^{a_1} A^{a_2}{}_{c_1, bd} - \delta_{c_1}^{a_2} A^{a_1}{}_{c_2, bd} + \varepsilon^{a_2 a_1} A_{c_1 c_2, bd} - \varepsilon_{c_1 c_2} A^{a_2 a_1}{}_{bd}) c^b c^d. \end{aligned}$$

The only chosen tensor we can operate is metric ε_{ab} , so the second term of expansion is given by:

$$G_{ab}^{(1)} = A_{ab, e_1 e_2} c^{e_1} c^{e_2} = -\frac{1}{2} (\varepsilon_{ae_1} \varepsilon_{be_2} - \varepsilon \varepsilon_{be_1} \varepsilon_{ae_2}) c^{e_1} c^{e_2} = -c_a c_b - \frac{\varepsilon}{2} \varepsilon_{ab}. \quad (57)$$

Following to these arguments one deduces that the higher term of expansion (50) vanish because commutator (56) contains too many terms and one deduces that the linear by oscillators term of the expansion is inappropriate

So the ansatz (29) is the most general in this case.

Suppose now that we have more than one set of oscillators, which are ε -commutative or ε -anticommutative each to other such that the bilinear combinations (57) are commutative:

$$G_{i,ab}G_{j,ab} = G_{j,ab}G_{i,ab}, \quad i \neq j. \quad (58)$$

One can suppose the following commutation relations between oscillators:

$$c_i^a c_j^b + \epsilon c_j^b c_i^a = \delta_{ij} \varepsilon^{ab}, \quad 1 \leq i, j \leq N. \quad (59)$$

One can construct the quadratic combinations from our oscillators:

$$C_{ij}^{ab} = c_i^a c_j^b, \quad (60)$$

Consider first N diagonal ones C_{jj}^{ab} : the ε -antisymmetric by group indices combinations are:

$$G_j^{ab} = \epsilon(c_j^a c_j^b - \frac{1}{2} \varepsilon^{ab}), \quad (61)$$

while the ε -symmetric ones are equal to metric tensor. The $N(N-1)/2$ non-diagonal combinations C_{ij}^{ab} $i < j$ can be separated by ε -symmetry:

$$S_{ij}^{ab} = \frac{1}{2}(c_i^a c_j^b + \epsilon c_i^b c_j^a) = \epsilon S_{ij}^{ba} = -S_{ji}^{ab}, \quad (62)$$

$$A_{ij}^{ab} = \frac{\epsilon}{2}(c_i^a c_j^b - \epsilon c_i^b c_j^a) = -\epsilon A_{ij}^{ba} = A_{ji}^{ab}, \quad (63)$$

here we used commutation rule (59). So we see that the remaining $N(N-1)/2$ combinations C_{ij}^{ab} with $i > j$ are expressed by ones with $i < j$. Using (62) one can construct $n(n-1)/2$ scalar combinations

$$s_{ij} = \frac{1}{2} \varepsilon_{ab} S_{ij}^{ab} = \frac{1}{2} \varepsilon_{ab} c_i^a c_j^b, \quad (64)$$

contraction A_{ij}^{ab} with metric vanishes.

Definitions (62) and (63) can be extended to values $i = j$:

$$S_{ii}^{ab} = \frac{1}{2}(c_i^a c_i^b + \epsilon c_i^b c_i^a) = \varepsilon^{ab}, \quad (65)$$

$$A_{ii}^{ab} = \frac{\epsilon}{2}(c_i^a c_i^b - \epsilon c_i^b c_i^a) = G_i^{ab}, \quad (66)$$

Then we again suppose the most general ansatz for generators G , depending on n sets of oscillators :

$$G^{ab} = \sum_{j=1}^n \gamma_j(s_{kl}) G_j^{ab} + \sum_{1 \leq i < j \leq n} \alpha_{ij}(s_{kl}) A_{ij}^{ab} = \sum_{1 \leq i < j \leq n} \alpha_{ij}(s_{kl}) A_{ij}^{ab}, \quad (67)$$

and calculate the following algebra between them:

$$[s_{ij}, G_k^{ab}] = (\delta_{jk} - \delta_{ik}) A_{ij}^{ab}, \quad \Rightarrow \quad [s_{ij}, \sum_{k=1}^n G_k^{ab}] = 0, \quad (68)$$

$$2[s_{ij}, s_{kl}] = \delta_{jk} s_{il} - \delta_{ik} s_{jl} + \delta_{jl} s_{ki} - \delta_{il} s_{kj}, \quad (69)$$

$$2[s_{ij}, A_{kl}^{ab}] = \delta_{jk} A_{il}^{ab} - \delta_{ik} A_{jl}^{ab} + \delta_{jl} A_{ki}^{ab} - \delta_{il} A_{kj}^{ab}, \quad (70)$$

$$2[G_k^{ab}, A_{ij}^{cd}] = \epsilon(\delta_{ik} + \delta_{jk})A_{ij}^{bc,ad} + (\delta_{ik} - \delta_{jk})S_{ij}^{bc,ad}. \quad (71)$$

$$4[A_{ij}^{ab}, A_{kl}^{cd}] = \delta_{jk}S_{il}^{bc,ad} + \delta_{ik}S_{jl}^{bc,ad} - \delta_{jl}S_{ki}^{bc,ad} - \delta_{il}S_{kj}^{bc,ad} + \\ + \epsilon\left(\delta_{jk}A_{il}^{bc,ad} + \delta_{ik}A_{jl}^{bc,ad} + \delta_{jl}A_{ki}^{bc,ad} + \delta_{il}A_{kj}^{bc,ad}\right), \quad (72)$$

where

$$S_{ij}^{bc,ad} = \epsilon^{bc}S_{ij}^{ad} - \epsilon^{ca}S_{ij}^{bd} + \epsilon^{ad}S_{ij}^{bc} - \epsilon^{db}S_{ij}^{ac},$$

and

$$A_{ij}^{bc,ad} = \epsilon^{bc}A_{ij}^{ad} - \epsilon^{ca}A_{ij}^{bd} + \epsilon^{ad}A_{ij}^{bc} - \epsilon^{db}A_{ij}^{ac}.$$

7. Conclusion

The oscillator realization, presented in previous section being the most general way to describe an arbitrary operator dependence, allows to represent generators of the finite resolution of orthogonal or symplectic Yangian. Substituting it into the set of constraints (22-24), (35-42), etc. following from the RLL-relation for the particular truncation. This involving program will allow to construct the complete set of the particular representations compatible with finite resolutions of orthogonal and symplectic Yangians.

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