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QCD at High Energies and Yangian Symmetry

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† Dedicated to the memory of Lev N. Lipatov.

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Abstract: Yangian symmetric correlators provide a tool to investigate integrability features of QCD at high energies. We discuss the kernel of the equation of perturbative Regge asymptotics, the kernels of the evolution equation of parton distributions, Born scattering amplitudes and coupling renormalization.

Keywords: gauge field theories; high-energy scattering; quantum integrable systems

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1. Introduction

In 1993, L.N. Lipatov pointed out the relation of the perturbative Quantum Chromodynamics (QCD) Regge aymptotics [1–5] to integrable quantum spin chains and showed that the contributions of multiple reggeized gluon exchanges can be treated by the methods of quantum integrable systems [6]. This result has great significance far beyond the particular questions of Regge asymptotics. Up to that time, the application of such methods to Quantum field theory was restricted to sophisticated models in $1+1$ dimensions.

Symmetries of physical systems are ordinarily related to Lie algebras with a finite number of generators and a finite rank. Yangian algebras, originally defined by Drinfeld [7], provide an extension of a Lie algebra, where the number of generators $T_{ab}^{[k]}$ and the rank are in general infinite. $T_{ab}^{[1]}$ are the generators of the underlying Lie algebra. We shall be concerned with the Lie algebras of the general and the special linear transformations, $g\ell_n$ and $s\ell_n$. The generators can be collected into the generating function $T(u)_{ab} = 1 + \sum_1^\infty u^{-k} T_{ab}^{[k]}$. The monodromy matrices considered below are particular cases.

Like in the case of finite Lie algebras, the symmetries based on Yangian algebras imply conservation laws (as many as the algebra rank). A number of well-studied integrable models are based on Yangian symmetries. Quantum Chomodynamics (QCD) is not an integrable quantum system; however, starting from L.N. Lipatov's work [6], it turned out that several aspects of its high-energy behaviour are related to such integrable systems.

After Lipatov's paper [6], it was immediately clear that the Bjorken asymptotics [8–13] and the composite operator renormalization have similar features of integrability [14]. In both cases, the Yangian symmetry underlying integrability is based on conformal symmetry, which is broken by loop corrections in the case of QCD. The breaking is suppressed by supersymmetry and is absent in $\mathcal{N}=4$ super Yang–Mills theory. Much work has been devoted in the last two decades to super Yang–Mills under this aspect. Quantum integrability works in the composite operator renormalization in the planar limit to all orders [15,16] and in the computation of scattering amplitudes and Wilson loops [17–21].

As in the cases of the finite Lie algebras, Yangian algebra symmetries manifest themselves in physical quantities transforming in simple ways, first of all in invariants.

Yangian symmetric correlators (YSC) have been proposed as a convenient formulation of the Yangian symmetry of amplitudes. Formally, they are Yangian algebra invariants in particular representations. They are shown to provide a tool of amplitude construction [22,23]. YSC depend on a set of spectral parameters, some combination of which are related to the helicities of scattering particles. The construction method of YSC are related to the method of on-shell graphs in the amplitude construction [24]. In the original form of the latter method [19–21], no such parameters appeared. In a number of papers, the deformation of amplitude expressions by such parameters has been studied and their eventual role for regularization of loop divergencies has been discussed [25–29].

In this contribution, we argue that YSC are useful not only in relation to amplitudes but allow to treat all features of integrability based on Yangian symmetry appearing in QCD at high energies in a uniform way. The notion of YSC is recalled and a few examples of their construction by R operations are given, which appear in the following applications.

We explain how the kernel of the BFKL equation in dipole form [30–32] emerges from a four-point YSC based on $s\ell_2$ symmetry. Based on [33,34], we discuss the relation of the parton evolution kernels and the amplitude of parton splitting [11] to three-point YSC based on $s\ell_2$ symmetry. The relation of the Born level scattering amplitudes of gluons and quarks to four-point YSC with $s\ell_4$ symmetry is considered. Finally, we consider the convolution of two three-point YSC, similar to a gluon self-energy Feynman graph, and show how the leading Gell–Mann–Low coefficient of the coupling renormalization appears. This relation is actually known in terms of the parton splitting amplitude [11] and the momentum sum rules for parton evolution kernels [12] and has been shown recently to extend to arbitrary helicity values [35].

2. Yangian Symmetric Correlators

Consider n Heisenberg canonical pairs, $x_a, \partial_a, a = 1, ..., n$,

$$[\partial_a, x_b] = \delta_{ab}, [x_a, x_b] = 0, [\partial_a, \partial_b] = 0.$$

$L_{ab} = \partial_a x_b$ obey the general linear $g\ell_n$ Lie algebra commutation relations. We define the L operator as a $n \times n$ matrix with these entries with the unit matrix I multiplied by the spectral parameter u added,

$$L(u) = uI + L(0), L(0)_{ab} = L_{ab} = \partial_a x_b.$$

This operator acts on functions of $x_a, a = 1, ..., n$. We consider the case of homogeneous functions with the weight denoted by 2ℓ, e.g., $x_n^{2\ell} \cdot \psi(\frac{x_1}{x_n}, ... \frac{x_{n-1}}{x_n})$. The L operator restricted to such homogeneous functions depends additionally on the weight 2ℓ or on $u^+ = u + 2\ell$,

$$L(u) x_n^{2\ell} \cdot \psi = x_n^{2\ell} L(u^+, u)\psi.$$

This results in $(L(u^+, u))_{ab} = (L(u))_{ab} + \delta_{a,n}(u^+ - u)\frac{x_b}{x_n}$.

We need N copies of the above sets of canonical pairs, $x_{i,a}, \partial_{i,a}$ and of the L operator $L_i(u_i, u_i^+)$, $i = 1, ..., N$ to define the monodromy matrix operator in terms of the matrix product

$$T(\mathbf{u}) = \prod_1^N L_i(u_i^+, u_i).$$

It is convenient to display the parameters in the form

$$\mathbf{u} = \begin{pmatrix} u_1 & ... & u_N \\ u_1^+ & ... & u_N^+ \end{pmatrix}.$$

Furthermore, we consider functions of N points in n dimensional space, homogeneous with respect to the coordinates of each point $\mathbf{x}_i$ of weight $2\ell_i$. We define a Yangian symmetric correlator (YSC) to be such a homogeneous function of N points obeying the eigenvalue relation with the monodromy matrix

$$T(\mathbf{u})\Phi(\mathbf{x}_1, ...\mathbf{x}_N; \mathbf{u}) = E(\mathbf{u})\Phi(\mathbf{x}_1, ...\mathbf{x}_N; \mathbf{u}). \tag{1}$$

The monodromy matrix can be separated into two factors, e.g., $T(\mathbf{u}) = T_1(\mathbf{u}_1)T_2(\mathbf{u}_2)$, where T_1 denotes the product of the first $L_i, i = 1, ..., N_1$ and T_2 the product of the subsequent $L_i, i = N_1 + 1, .., N$. Then, $\Phi = \Phi_1\Phi_2$ solves (1) (with $E = E_1E_2$) if $T_1\Phi_1 = E_1(\mathbf{u}_1)\Phi_1$ and $T_2\Phi_2 = E_2(\mathbf{u}_1)\Phi_2$.

The case of one point is trivial but provides a convenient starting point. We have two obvious one-point YSC, $L(u, u) \cdot 1 = (u + 1)1$, $L(u, u - n)\delta^{(n)}(\mathbf{x}) = u\delta^{(n)}(\mathbf{x})$, and from these the trivial product N-point YSC,

$$\Phi_{K,N}(\mathbf{x}_1, ...\mathbf{x}_N; \mathbf{u}) = \prod_1^K \delta^{(n)}(\mathbf{x}_i), \;\; E(\mathbf{u}) = u_1...u_K(u_{K+1} + 1)...(u_N + 1),$$

$$\mathbf{u} = \begin{pmatrix} u_1 & ... & u_K & u_{K+1} & ... & u_N \\ u_1 - n & ... & u_K - n & u_{K+1} & ... & u_N \end{pmatrix}.$$

The Yang–Baxter RLL relations provide a way to non-trivial YSC. We have

$$R_{12}(u_1 - u_2)\; L_1(u_1^+, u_1)L_2(u_2^+, u_2) = L_1(u_1^+, u_2)L_2(u_2^+, u_1)\; R_{12}(u_1 - u_2),$$

$$R_{21}(u_1^+ - u_2^+)\; L_1(u_1^+, u_1)L_2(u_2^+, u_2) = L_1(u_2^+, u_1)L_2(u_1^+, u_2)\; R_{21}(u_1^+ - u_2^+),$$

where the R operator can be represented by

$$R_{12}(u) = \int \frac{dc}{c^{1+u}} e^{-c(\mathbf{x}_1 \cdot \mathbf{p}_2)} \tag{2}$$

with the integration over a closed contour. If $\Phi(\mathbf{u})$ is a YSC obeying Equation (1), then $\Phi(\mathbf{u}') = R_{i,i+1}(u_i - u_{i+1})\Phi(\mathbf{u})$ obeys the YSC relation with $\mathbf{u}$ replaced by

$$\mathbf{u}' = \begin{pmatrix} u_1 & ... & u_{i+1} & u_i & ... & u_N \\ u_1^+ & ... & u_i^+ & u_{i+1}^+ & ... & u_N^+ \end{pmatrix},$$

i.e., the entries at $i, i + 1$ in the first row are permuted. $\Phi(\mathbf{u}'') = R_{i+1,i}(u_i^+ - u_{i+1}^+)\Phi(\mathbf{u})$ obeys the YSC relation with $\mathbf{u}$ replaced by

$$\mathbf{u}'' = \begin{pmatrix} u_1 & ... & u_i & u_{i+1} & ... & u_N \\ u_1^+ & ... & u_{i+1}^+ & u_i^+ & ... & u_N^+ \end{pmatrix},$$

i.e., the entries at $i, i + 1$ in the second row are permuted. The resulting YSC are less trivial and, by repeated R operations, we obtain completely connected correlators characterized by the permuted parameter set of the resulting monodromy matrix,

$$\mathbf{u} = \begin{pmatrix} u_{\sigma(1)} & u_{\sigma(2)} & \cdots & u_{\sigma(N)} \\ u_{\bar{\sigma}(1)}^+ & u_{\bar{\sigma}(2)}^+ & \cdots & u_{\bar{\sigma}(N)}^+ \end{pmatrix}. \tag{3}$$

We shall use the abbreviation of writing the indices carried by the parameters only. Notice that by the substitution of parameters $u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(N)} \to v_1, v_2, \ldots, v_N$ applied to both rows the first row can be put into the standard ordering.

One may draw an analogy of Equation (1) to the time-independent Schrödinger equation. In this respect, we have considered so far the position representation. Furthermore, we need the helicity representation. It is defined in the case of even $n, n = 2m$ by the elementary canonical transformation applied to half of the canonical pairs at each point, $a = m + \alpha = m + 1, ..., n$, leaving the first half $a = \dot{\alpha}$, $\dot{\alpha} = 1, ..., m$ unchanged,

$$\begin{pmatrix} x_{i,m+\alpha} \\ \partial_{i,m+\alpha} \end{pmatrix} \to \begin{pmatrix} \partial^{\lambda}_{i,\alpha} \\ -\lambda_{i,\alpha} \end{pmatrix}, \;\; \begin{pmatrix} x_{i,\dot{\alpha}} \\ \partial_{i,\dot{\alpha}} \end{pmatrix} \to \begin{pmatrix} \bar{\lambda}_{i,\dot{\alpha}} \\ \bar{\partial}^{\lambda}_{i,\dot{\alpha}} \end{pmatrix}. \tag{4}$$

The N-point YSC can be written in the link integral form which became a standard of the modern tools of amplitude calculations [19–21]. In the position representation, it is given by

$$\Phi_{K,N}(\mathbf{x}_1, ...\mathbf{x}_N; \mathbf{u}) = \int dc\varphi(\mathbf{c}) \prod_{i=1}^{K} \delta^{(n)}(\mathbf{x}_i - \sum_{j=K+1}^{N} c_{ij}\mathbf{x}_j). \tag{5}$$

K denotes the number of $\delta^{(n)}$ factors.

To change to the helicity representation, we apply the Fourier transformation to the dependence on the components $x_{i,a}, a = m+1, ..., n$ at each point $i = 1, ..., N$. The Fourier variable conjugate to $x_{m+\alpha}$ is denoted by $\lambda_{i,\alpha}, \alpha = 1, ..., m$. The components $x_{i,a}, a = 1, ..., m$ are not changed and merely renamed by $\bar{\lambda}_{i,\dot{\alpha}}$, i.e., $\bar{\lambda}_{i,\dot{\alpha}} = x_{i,\dot{\alpha}}, \dot{\alpha} = 1, ..., m$. Thus, the link integral form of a YSC in the helicity representation is

$$\Phi^{\lambda}_{K,N}(\bar{\lambda}_1, \lambda_1, ...\bar{\lambda}_N, \lambda_N; \mathbf{u}) = \int dc\varphi(\mathbf{c}) \prod_{i=1}^{K} \delta^{(m)}(\bar{\lambda}_i - \sum_{j=K+1}^{N} c_{ij}\bar{\lambda}_j) \prod_{j=K+1}^{N} \delta^{(m)}(\lambda_j + \sum_{i=1}^{K} c_{ij}\lambda_i). \tag{6}$$

Notice that the integrand function φ is the same in both representations. dc abbreviates $dc = \prod_{i=1}^{K} \prod_{j=K+1}^{N} dc_{ij}$.

3. Examples of YSC for N = 2, 3, 4

We present examples of YSC encountered in the applications below and indicate their construction by R operations.

We start with the two-point YSC, $N = 2, K = 1$

$$\Phi_{1,2}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{u}) = R_{21}(u_1^+ - u_2)\delta^{(n)}(\mathbf{x}_1), \tag{7}$$

$$\varphi_{1,2}(c_{12}) = c_{12}^{-1+u_2+n-u_1}, \quad \mathbf{u} = \begin{pmatrix} 1 & 2 \\ 2 & 1^+ \end{pmatrix}.$$

In the second row, 2 stands for u_2 and 1^+ for $u_1 - n = u_1^+$.

Next, we give the three-point YSC with $N = 3, K = 1$

$$\Phi_{1,3}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \mathbf{u}) = R_{21}^{++}(u_3^+ - u_2^+) R_{31}^{++}(u_1^+ - u_3^+) \cdot \delta^{(n)}(\mathbf{x}_1), \tag{8}$$

$$\varphi_{1,3} = \left(c_{12}^{1+u_3-u_2} c_{13}^{1+u_1^+-u_3} \right)^{-1}, \quad \mathbf{u} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1^+ \end{pmatrix},$$

and with $N = 3, K = 2$,

$$\Phi_{2,3}(\mathbf{u}) = R_{32}(u_2^+ - u_1^+) R_{31}(u_1^+ - u_3^+)\delta^{(n)}(\mathbf{x}_1)\delta^{(n)}(\mathbf{x}_2), \tag{9}$$

$$\varphi_{2,3} = \left(c_{23}^{1+u_2^+-u_1^+} c_{13}^{1+u_1^+-u_3^+} \right)^{-1}, \quad \mathbf{u} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2^+ & 1^+ \end{pmatrix}.$$

We give also the relevant examples of four-point YSC. Because all of these have coincident values of K and N, $K = 2, N = 4$, other labels like $\Delta, X, ||$ are used to distinguish them.

As an intermediate YSC, from which the following ones are constructed, we write

$$\Phi_{\Delta}(\mathbf{u}) = R_{32}(u_2^+ - u_3^+) R_{41}(u_1^+ - u_4^+)\delta^{(n)}(\mathbf{x}_1)\delta^{(n)}(\mathbf{x}_2), \tag{10}$$

$$\varphi_{\Delta} = \delta(c_{13})\delta(c_{24}) \left(c_{23}^{1+u_2^+-u_3^+} c_{14}^{1+u_1^+-u_4} \right)^{-1}, \quad \mathbf{u} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2^+ & 1^+ \end{pmatrix}.$$

This YSC is incompletely connected with delta distributions in φ. It takes two more steps of R operations to arrive at the completely connected one.

$$\Phi_X(\mathbf{u}) = R_{43}(u_2^+ - u_4^+)R_{21}(u_4^+ - u_3^+)\Phi_\Delta, \tag{11}$$

$$\varphi_X = \left(c_{24}^{1+u_2-u_1}c_{13}^{1+u_4-u_3}(c_{14}c_{23} - c_{23}c_{13})^{1+u_2^+-u_3}\right)^{-1},$$

$$\mathbf{u} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1^+ & 2^+ \end{pmatrix}.$$

The sequence of operations leading to a YSC is not unique. In this case, we may also choose

$$\Phi_X(\mathbf{u}) = R_{12}(u_1 - u_2)R_{21}(u_4^+ - u_3^+)\Phi_\Delta|_{u_1 \leftrightarrow u_2}$$

with the indicated substitution as the last step to obtain the standard ordering in the upper row of u parameters.

The permutation pattern $\mathbf{u}$ fixes the monodromy operator. However, the YSC is not uniquely fixed. For example, the pattern

$$\mathbf{u} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2^+ & 1^+ \end{pmatrix}$$

is the same as for Φ_Δ and is also one of the following two YSC:

$$\Phi_{||1}(\mathbf{u}) = R_{34}(u_3 - u_4)R_{12}(u_1 - u_2)R_{12}(u_1 - u_2)R_{21}(u_4^+ - u_3^+)\Phi_\Delta|_{u_3 \leftrightarrow u_4},$$

$$\varphi_{||1} = \left(c_{24}c_{13}c_{14}^{u_1-u_2+u_3-u_4}(c_{14}c_{23} - c_{23}c_{13})^{1+u_2^+-u_3}\right)^{-1},$$

$$\Phi_{||2}(\mathbf{u}) = R_{43}(u_2^+ - u_1^+)R_{21}(u_3^+ - u_4^+)R_{12}(u_1 - u_2)R_{21}(u_4^+ - u_3^+)\Phi_\Delta|_{u_1 \leftrightarrow u_2},$$

$$\varphi_{||2} = \left(c_{24}c_{13}c_{23}^{1+u_2-u_1+u_4-u_3}(c_{14}c_{23} - c_{23}c_{13})^{1+u_1^+-u_4}\right)^{-1}.$$

There are more YCS with the above parameter pattern. In the following, we shall use $\Phi_{||}(\mathbf{u})$ with

$$\varphi_{||} = \left(c_{24}c_{13}c_{14}^{u_3-u_4}c_{23}^{u_2-u_1}(c_{14}c_{23} - c_{23}c_{13})^{1+u_1^+-u_3}\right)^{-1}. \tag{12}$$

It is the result of the convolution of two YSC of the form Φ_X. At $u_1 = u_2$, it coincides with $\Phi_{||1}$ and at $u_3 = u_4$ it coincides with $\Phi_{||2}$.

The integration variables c_{ij} may be regarded as coordinates on the corresponding Grassmannian variety $\mathcal{G}_{K,N}$ and in the case of completely connected YSC the closed integration contours extend over a maximal Schubert cell. The delta distributions involving the correlator variables $\mathbf{x}$ or $\bar{\lambda}, \lambda$ fix some of these integration variables c_{ij}. In the following applications, we have cases where all are fixed, e.g., in the position representation for $n = 2$ with $(K, N) = (1, 3)$ and $(K, N) = (2, 4)$. The helicity representation has the feature that the delta distributions can be rewritten in a way with the factor $\delta(\sum_1^N \bar{\lambda}_i \lambda_i)$ independent of c_{ij}. The integrations are removed in this representation for $n = 4$ with $(K, N) = (2, 4)$.

4. The BFKL Kernel

YSC may be used as kernels of integral operators. The symmetry of the kernels implies symmetry properties like Yang–Baxter relations for these operators. At $n = 2$, we may change the homogeneous coordinates $\mathbf{x}_i = (x_{i,1}, x_{i,2})$ to the normal coordinates $x_i = \frac{x_{i,1}}{x_{i,2}}$ and find

$$\Phi(\mathbf{x}_1, ..., \mathbf{x}_N) = \prod_1^N x_{i,2}^{2\ell_i} \phi(x_1, ..., x_N).$$

We define the integral operator with a 4-point YSC as kernel,

$$[R\psi](x_1, x_2) = \int dx'_1 dx'_2 \psi(x'_1, x'_2) \phi(x'_1, x'_2, x_1, x_2|\mathbf{u}).$$

In the case of the action on functions on the complex plane, we consider the complex x as $(x, \bar{x})$ and define

$$[R\psi](x_1, x_2) = \int dx'_1 dx'_2 d\bar{x}'_1 d\bar{x}'_2 \psi(x'_1, x'_2, \bar{x}'_1, \bar{x}'_2) \phi(x'_1, x'_2, x_1, x_2|\mathbf{u}) \phi(\bar{x}'_1, \bar{x}'_2, \bar{x}_1, \bar{x}_2|\bar{\mathbf{u}}).$$

We substitute the YSC at $n = 2$ in the normal coordinate form and perform the integrals over $c_{13}, c_{14}, c_{23}, c_{24}$:

$$\phi(x'_1, x'_2, x_1, x_2|\mathbf{u}) = x_{12}^{-2} \varphi(c^*).$$

Provided the substitution $1, 2, 3, 4, \to 1', 2', 1, 2$, we have

$$c^*_{13} = \frac{x_{14}}{x_{34}}, \;\; c^*_{14} = -\frac{x_{13}}{x_{34}}, \;\; c^*_{23} = \frac{x_{24}}{x_{34}}, \;\; c^*_{24} = -\frac{x_{23}}{x_{34}}.$$

We use the abbreviation $x_{ij} = x_i - x_j$ for the difference of normal coordinates (not to be mixed with the component notation $x_{i,1}, x_{i,2}$).

In the case of the 4-point YSC (11), we obtain

$$\phi_X(x_1, x_2, x_3, x_4) = \frac{x_{34}^{1+u_2^+ - u_3}}{x_{23}^{1+u_2-u_1} x_{14}^{1+u_4-u_3} x_{12}^{1+u_1^+ - u_4}} = \frac{x_{12}^{1+2\ell_1-\varepsilon}}{x_{23}^{1+2\ell_1-2\ell_2-\varepsilon} x_{14}^{1-\varepsilon} x_{24}^{1+2\ell_2+\varepsilon}}.$$

In the last step, we have used the relation between the spectral parameters and the weight at each point i, $2\ell_i = u^+_{\bar{\sigma}(i)} - u_{\sigma(i)}$, referring to the parameter permutation pattern (3). In our case (11), we have

$$2\ell_1 = u_3 - u_1, 2\ell_2 = u_4 - u_1, 2\ell_3 = u_1 - n - u_3 = -2\ell_1 - n, 2\ell_4 = u_2 - n - u_4 = -2\ell_2 - n.$$

We introduce $\varepsilon = u_3 - u_4$ and express the exponents in terms of the independent weights $2\ell_1, 2\ell_2$ and the parameter ε, substituting also $n = 2$ for the considered case.

We substitute $1, 2, 3, 4, \to 1', 2', 1, 2$. At $\ell_{1'} = \ell_{2'} = \ell$, the kernel for the complex plane is

$$\phi(x'_1, x'_2, x_1, x_2) \phi(\bar{x}'_1, \bar{x}'_2, \bar{x}_1, \bar{x}_2) = \frac{|x_{1'2'}|^{2(-1-2\ell-\varepsilon)} |x_{12}|^{2(1+2\ell-\varepsilon)}}{|x_{1'2}|^{2(1-\varepsilon)} |x_{12'}|^{2(1-\varepsilon)}}.$$

In the decomposition in ε, the leading ε^{-1} term corresponds to the kernel of the permutation operator P_{12}. The finite term is

$$|x_{1'2'}|^{2(-1-2\ell)} |x_{12}|^{2(1+2\ell)} \left(|x_{1'2}|^2 \delta^{(2)}(x_{12'}) + |x_{12'}|^2 \delta^{(2)}(x_{1'2}) - \delta^{(2)}(x_{12'}) \delta^{(2)}(x_{1'2}) \int d^2 x_3 |x_{13}|^2 |x_{13}|^2 \right).$$

The subtraction is the appropriately modified + prescription. In the limiting case corresponding to the reggeized gluon exchange $\ell = 0$, the action is defined on functions vanishing at coinciding arguments $x_{1'} = x_{2'}$. The kernel can be better rewritten as

$$|x_{12}|^2 \int \frac{d^2x_3}{|x_{13}|^2|x_{23}|^2} \left(\delta^{(2)}(x_{21'})\delta(x_{2'3}) + \delta^{(2)}(x_{12'})\delta(x_{1'3}) - \delta^{(2)}(x_{12'})\delta(x_{21'})\right).$$

This is the dipole (or Moebius) form of the BFKL kernel [30–32,36].

The other argument line to BFKL starts with noticing that the integral operator obeys the standard RLL relation for intertwining representations with weights ℓ_1, ℓ_2. The x, ∂ operator form of its holomorphic part can be represented in the factorized form [37–39], $R(\varepsilon) = R^1(\varepsilon)R^2(\varepsilon)$,

$$R^1(u^1|v^1, v^2) = \frac{\Gamma(x_{21}\partial_2 + \varepsilon + \ell_1 + \ell_2 + 1)}{\Gamma(x_{21}\partial_2 + 2\ell_2 + 1)}, \quad R^2(u^1, u^2|v^2) = \frac{\Gamma(x_{12}\partial_1 + \varepsilon + \ell_1 + \ell_2 + 1)}{\Gamma(x_{12}\partial_1 + 2\ell_1 + 1)},$$

and then the decomposition in ε results in one of the operator forms of the BFKL operator (compare e.g., [40], Equation (184)). This line of arguments is the one in the original paper by L.N. Lipatov on the relation to the integrable spin chains [6].

5. The Parton Evolution Kernels

In the helicity representation, we have at $n = 2, m = 1$,

$$\Phi(\bar{\lambda}_1, \lambda_1, \ldots, \bar{\lambda}_N, \lambda_N) = \prod_1^N \bar{\lambda}_i^{2h_i} \, \delta(\sum k_i)\phi(k_1, ..., k_N),$$

where $k_i = \bar{\lambda}_i\lambda_i$ have one component only in this case. In the application to parton evolution, these variables k_i have the physical meaning of light-cone momenta and the arguments z of the parton evolution kernels are defined as their ratios. We obtain for the YSC with $N = 3$ and $K = 1$ (8) or $K = 2$ (9)

$$\phi(k_1, k_2, k_3; a_1, a_2, a_3) = (k_1k_2k_3)^{\frac{1}{2}} k_1^{-\eta a_1} k_2^{-\eta a_2} k_3^{-\eta a_3}.$$

Here, $2a_i = 2\ell_1 + 1$ and $\sum_1^3 2a_i = \eta$, with $\eta = \pm 1$. The positive sign $\eta = +1$ corresponds to $K = 1$ (8) and the negative $\eta = -1$ to $K = 2$ (9). The amplitude of parton splitting $3 \to 1 + 2$ is obtained from this three-point YSC by substitutions of k_i in terms of the momentum fraction z and a_i by the parton helicities h_i as follows:

$$Split(h_1, h_2, h_3; z) = \phi(-z, z-1, 1; h_1, h_2, h_3 - \frac{1}{2}\eta).$$

The parton splitting probabilities are calculated as squares of the corresponding splitting amplitudes. The helicities h_i refer to ingoing momenta, i.e., h_1, h_2 are opposite to their physical values in the decay $3 \to 1 + 2$:

$$P_{h_1\ h_3}^{h_2}(z) = (Split(h_1, h_2, h_3; z))^2 = \left(\phi(-z, -1+z, 1; h_1, h_2, h_3 - \frac{1}{2}\eta)\right)^2.$$

The expressions for the leading order parton evolution kernels are reproduced—compare, e.g., [40].

6. Scattering Amplitudes

We consider the four-point YSC at $n = 4$ with $K = 2$ in the helicity representation and do the integrals over the c variables:

$$\Phi_{2,4}^{\lambda} = \varphi(c^*) \;\; \delta^{(4)}(\sum k_i), \tag{13}$$

$$c^*_{13} = \frac{[14]}{[34]},\ c^*_{14} = \frac{[13]}{[43]},\ c^*_{23} = \frac{[24]}{[34]},\ c^*_{24} = \frac{[23]}{[43]}.$$

Here, we denote $(k_i)_{\dot{\alpha},\alpha} = \bar{\lambda}_{i,\dot{\alpha}}\lambda_{i,\alpha}$, $[ij] = \bar{\lambda}_{i,1}\bar{\lambda}_{j,2} - \bar{\lambda}_{i,2}\bar{\lambda}_{j,1}$.

From Equation (13), we obtain the explicit expressions as functions of the independent helicities and the extra parameter ε. As in Section 4, we use the relation between the spectral parameters and the weights referring to the parameter permutation scheme $\mathbf{u}$ (3). The weights are substituted by the helicities as $2\ell_i + 2 = 2h_i$. In case (11), we have

$$2h_1 = u_3 - u_1 + 2 = -2h_3, 2h_2 = u_4 - u_2 + 2 = -2h_4.$$

We introduce $\varepsilon = u_3 - u_4$ and express the exponents in terms of the helicities h_1, h_2 and the parameter ε:

$$\varphi_X(c^*; h_1, h_2, \varepsilon) = \left(\frac{[14][23]}{[12][34]}\right)^{\varepsilon} \frac{[12]^{1+2h_1}[34]^{1-2h_2}}{[14][23]^{1+2h_1-2h_2}}.$$

Thus, in the case of Φ_X, the helicities are related by $h_1 = -h_3,\ \ h_2 = -h_4$. With these conditions, only four of the six helicity configurations of the $2 \to 2$ helicity amplitudes are accessible.

In case (12), we have instead

$$2h_1 = u_4 - u_1 + 2 = -2h_4, 2h_2 = u_3 - u_2 + 2 = -2h_3.$$

We introduce $\varepsilon = u_4 - u_3$ and express the exponents in terms of the helicities h_1, h_2 and the parameter ε:

$$\varphi_{||}(c^*; h_1, h_2, \varepsilon) = \left(\frac{[13][24]}{[12][34]}\right)^{\varepsilon} \frac{[12]^{1+2h_1}[34]^{1-2h_2}}{[14][23][24]^{2h_1-2h_2}}.$$

With $\Phi_{||}$, we cover the remaining cases because here the helicities are related by $h_1 = -h_4,\ \ h_2 = -h_3$. With these conditions, again four helicity configuration cases can be covered, two of them doubling some of the above correlators.

We observe that the gluon tree amplitudes in spinor-helicity form [41,42] are reproduced by these two YSC as

$$M(1,1,-1,-1) = \varphi_X(c^*,1,1,0),\ \ M(-1,-1,1,1) = \varphi_X(c^*,-1,-1,0),$$
$$M(1,-1,-1,1) = \varphi_X(c^*,1,-1,4), M(-1,1,1,-1) = \varphi_X(c^*,-1,1,0),$$
$$M(1,1,-1,-1) = \varphi_{||}(c^*,1,1,0),\ \ M(-1,-1,1,1) = \varphi_{||}(c^*,-1,-1,0),$$
$$M(1,-1,1,-1) = \varphi_{||}(c^*,1,-1,4), M(-1,1,-1,1) = \varphi_{||}(c^*,-1,1,0).$$

The quark tree amplitudes are reproduced as

$$M(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}) = \varphi_X(c^*,\frac{1}{2},\frac{1}{2},0),$$
$$M(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}) = \varphi_X(c^*,\frac{1}{2},-\frac{1}{2},3),\ \ M(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}) = \varphi_X(c^*,-\frac{1}{2},\frac{1}{2},1),$$
$$M(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}) = \varphi_{||}(c^*,\frac{1}{2},\frac{1}{2},0),$$
$$M(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}) = \varphi_{||}(c^*,\frac{1}{2},-\frac{1}{2},3),\ \ M(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}) = \varphi_{||}(c^*,-\frac{1}{2},\frac{1}{2},1).$$

The gluon-quark tree amplitudes are reproduced as

$$M(1,\frac{1}{2},-1,-\frac{1}{2}) = \varphi_X(c^*,1,\frac{1}{2},1),\ \ M(1,-\frac{1}{2},-1,\frac{1}{2}) = \varphi_X(c^*,1,-\frac{1}{2},3),$$

$$M(1,\frac{1}{2},-\frac{1}{2},-1)=\varphi_{||}(c^*,1,\frac{1}{2},1),\;\; M(1,-\frac{1}{2},\frac{1}{2},-1)=\varphi_{||}(c^*,1,-\frac{1}{2},3).$$

7. Asymptotic Freedom

We reconsider the convolution of three-point YSC with a two-particle intermediate state:

$$\int d^n\mathbf{x}_{1'}d^n\mathbf{x}_{2'}\Phi_{1,3}(1,2',1')\,\Phi_{2,3}(1',2',2)=$$

$$\int dc_{1,1'}dc_{12'}\varphi_{1,3}(1,2',1')dc_{2'2}dc_{1'3}\varphi_{2,3}(1',2',2)\delta(\mathbf{x}_1-c_{11'}\mathbf{x}_{1'}-c_{12'}\mathbf{x}_{2'})\,\delta(\mathbf{x}_{1'}-c_{1'2}\mathbf{x}_2)\delta(\mathbf{x}_{2'}-c_{2'2}\mathbf{x}_2).$$

In the function arguments, we have abbreviated the points $\mathbf{x}_i$ by the indices i. The result can be written in the form of a two-point correlator

$$\int d\bar{c}_{12}\bar{\varphi}(\bar{c})\delta(\mathbf{x}_1-\bar{c}_{12}\mathbf{x}_2),$$

where

$$\bar{\varphi}(\bar{c})=\int\frac{dc_{2'2}}{c_{2'2}}dc_{11'}dc_{1'2}\varphi_{1,3}(\frac{\bar{c}_{12}-c_{11'}c_{1'2}}{c_{2'3}},c_{11'})\varphi_{2,3}(c_{2',2},c_{1',2}).$$

We substitute Equations (8) and (9) substituting the variables correspondingly. The spectral parameters of the first factor are denoted by $u_1,u_{2'},u_{1'}$ and the ones of the second factor by $v_{1'},v_{2'},v_2$.

$$\bar{\varphi}(\bar{c})=\int\frac{dc_{2'2}}{c_{2'2}}dc_{11'}dc_{1'2}\left((\bar{c}_{12}-c_{11'}c_{1'2})^{1+u_{1'}-u_{2'}}c_{11'}^{1+u_1^+-u_{1'}}c_{2'2}^{1+v_{2'}^+-v_{1'}+u_{2'}-u_{1'}}c_{1'2}^{1+v_{1'}^+-v_2}\right)^{-1}.$$

We calculate the weights in terms of the spectral parameters marking the weights of the second factor by a prime:

$$2\ell_1=u_{2'}-u_1,2\ell_{2'}=u_{1'}-u_{2'},2\ell_{1'}=u_1^+-u_{1'},$$
$$2\ell'_{1'}=v_2-v_{1'},2\ell'_{2'}=v_{1'}^+-v_{2'},2\ell'_2=v_{2'}-v_2.$$

We impose the weight balance condition,

$$v_{2'}^+-v_{1'}^+=-2\ell'_{2'}-n=2\ell_{2'},v_{1'}^+-v_2=-2\ell'_{1'}-n=2\ell_{1'}.$$

By the change of integration variables $c_{11'}\to c=\frac{c_{11'}c_{1'2}}{\bar{c}_{12}}$, we obtain

$$\bar{\varphi}(\bar{c})=\int\frac{dc_{2'2}dc_{1'2}}{c_{2'2}c_{1'2}}\;B(-2\ell_{1'},-2\ell_{2'})\frac{1}{\bar{c}_{13}^{1+2\ell_{1'}+2\ell_{2'}}}.$$

Thus, the result of the symmetric convolution of three-point YSC is the two-point YSC multiplied by a factor depending on the weights of the intermediate two-particle state,

$$\int d\mathbf{x}_{1'}d\mathbf{x}_{2'}\Phi_{1,3}(1,2',1')\,\Phi_{2,3}(1',2',2)=const\;B(-2\ell_{1'},-2\ell_{2'})\Phi_{1,2}(1,2).$$

We study the resulting dependence on the intermediate state weights entering the Euler beta function factor. We have $2\ell_1+2\ell_{1'}+2\ell_{2'}+n=0$, and we change to the helicities $2\ell+\frac{1}{2}n=2h$:

$$B(-2\ell_{1'},-2\ell_{2'})=B(2h_1+2h_{2'},-2h_{2'}+\frac{1}{2}n).$$

In the interesting case of QCD, where $n=4,h_1=1$, we have with $2h_{2'}\to 2h=2\bar{h}+2\varepsilon$, $2\bar{h}$ integer,

$$B(2h+2,-2h+2)-\frac{1}{12\varepsilon}(2\bar{h}+1)2\bar{h}(2\bar{h}-1)+(-1)^{2\bar{h}}\frac{-12\bar{h}^2+1}{6}+\mathcal{O}(\varepsilon).$$

The finite term results in the contributions to the leading Gell–Mann–Low coefficient with the values $\bar{h} = 1$ for the gluon loop contribution and $\bar{h} = \frac{1}{2}$ for the quark loop contribution.

8. Conclusions

Yangian symmetric correlators provide a tool to treat some features of the high-energy aymptotics of Quantum Chromododynamics in their realation to Quantum Integrable Systems. We have shown this in the examples of the kernel of the equation of perturbative Regge asymptotics, the kernels of the evolution equation of parton distributions, Born scattering amplitudes and coupling renormalization.

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